Rainbow cliques in randomly perturbed dense graphs

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Abstract. For two graphs G and H, write $G \xrightarrow{\text{rbw}} H$ if G has the property that every *proper* colouring of its edges yields a *rainbow* copy of H. We study the thresholds for such so-called *anti-Ramsey* properties in randomly perturbed dense graphs, which are unions of the form $G \cup \mathbb{G}(n,p)$, where G is an n-vertex graph with edge-density at least d, and d is a constant that does not depend on n.

We determine the threshold for the property $G \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_s$ for every s. We show that for $s \geq 9$ the threshold is $n^{-1/m_2(K_{\lceil s/2 \rceil})}$; in fact, our 1-statement is a supersaturation result. This turns out to (almost) be the threshold for $s = 8$ as well, but for every $4 \leq s \leq 7$, the threshold is lower and is different for each $4 \leq s \leq 7$.

Moreover, we prove that for every $\ell \geq 2$ the threshold for the property $G \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} C_{2\ell-1}$ is n^{-2}; in particular, the threshold does not depend on the length of the cycle $C_{2\ell-1}$. It is worth mentioning that for even cycles, or more generally for any fixed bipartite graph, no random edges are needed at all.

Keywords: Random graphs, Anti-Ramsey, randomly perturbed graphs

1 Introduction

A *random perturbation* of a fixed n-vertex graph G, denoted by $G \cup \mathbb{G}(n,p)$, is a distribution over the supergraphs of G with the latter generated through the addition of random edges sampled from the binomial random graph of edge-desity p, namely $\mathbb{G}(n,p)$. The fixed graph G being *perturbed* or *augmented* in this manner is referred to as the *seed* of the *perturbation* $G \cup \mathbb{G}(n,p)$.

The above model was introduced by Bohman, Frieze, and Martin [6], who allowed the seed G to range over the family of n-vertex graphs with minimum degree at least δn, denoted by $\mathcal{G}_{\delta,n}$. In particular, they discovered the phenomenon that for every $\delta > 0$, there exists a constant $C(\delta) > 0$ such that $G \cup \mathbb{G}(n,p)$ a.a.s. admits a Hamilton cycle, whenever $p := p(n) \geq C(\delta)/n$ and $G \in \mathcal{G}_{\delta,n}$.

Their bound on p undershoots the threshold for Hamiltonicity in $\mathbb{G}(n,p)$ by a logarithmic factor. The notation $\mathcal{G}_{\delta,n} \cup \mathbb{G}(n,p)$ then suggests itself to mean the collection of perturbations arising from the members of $\mathcal{G}_{\delta,n}$ for a prescribed $\delta > 0$.

Several strands of results regarding the properties of randomly perturbed (hyper)graphs can be found in the literature. One prominent such strand can be seen as an extension of the aforementioned result of [6]. Indeed, the emergence of various spanning configurations in randomly perturbed (hyper)graphs was studied, for example, in [3,5,7,8,11,12,15,16,22].

Another prominent line of research regarding random perturbations concerns Ramsey properties of $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$, where here $\mathscr{G}_{d,n}$ stands for the family of n-vertex graphs with edge-density at least $d > 0$, and d is a constant. This strand stems from the work of Krivelevich, Sudakov, and Tetali [17] and is heavily influenced by the now fairly mature body of results regarding the thresholds of various Ramsey properties in random graphs see, e.g. [21,26,27,28].

Krivelevich, Sudakov, and Tetali [17], amongst other things, proved that for every real $d > 0$, integer $t \geq 3$, and graph $G \in \mathscr{G}_{d,n}$, the perturbation $G \cup \mathbb{G}(n,p)$ a.a.s. satisfies the property $G \cup \mathbb{G}(n,p) \to (K_3, K_t)$, whenever $p := p(n) = \omega(n^{-2/(t-1)})$; moreover, this bound on p is asymptotically best possible. Here, the notation $G \to (H_1, \ldots, H_r)$ is used to denote that G has the *asymmetric* Ramsey property asserting that any r-edge-colouring of G admits a colour $i \in [r]$ such that H_i appears with all its edges assigned the colour i.

Recently, the aforementioned result of Krivelevich, Sudakov, and Tetali [17] has been significantly extended by Das and Treglown [10] and also by Powierski [25]. In particular, there is now a significant body of results pertaining to the property $G \cup \mathbb{G}(n,p) \to (K_r, K_s)$ for any pair of integers $r, s \geq 3$, whenever $G \in \mathscr{G}_{d,n}$ for constant $d > 0$. Further in this direction, the work of Das, Morris, and Treglown [9] extends the results of Kreuter [14] pertaining to *vertex Ramsey* properties of random graphs into the perturbed model.

A subgraph $H \subseteq G$ is said to be *rainbow* with respect to an edge colouring ψ, if any two of its edges are assigned different colours under ψ. An edge-colouring ψ of a graph G is said to be *proper* if incident edges are assigned distinct colours under ψ. We write $G \xrightarrow{\text{rbw}} H$, if G has the property that every proper colouring of its edges admits a rainbow copy of H. The first to consider the emergence of small fixed rainbow configurations in random graphs with respect to proper colourings were Rödl and Tuza [29]. The systematic study of the emergence of general rainbow fixed graphs in random graphs with respect to proper colourings was initiated by Kohayakawa, Kostadinidis and Mota [18,19].

In [18] it is proved that for every graph H, there exists a constant $C > 0$ such that $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$, whenever $p \geq Cn^{-1/m_2(H)}$, where here $m_2(H)$ denotes the *maximum* 2*-density* of H, see e.g. [13]. Nenadov, Person, Škorić, and Steger [24] proved, amongst other things, that for $H \cong C_\ell$ with $\ell \geq 7$, and for $H \cong K_r$ with $r \geq 19$, $n^{-1/m_2(H)}$ is, in fact, the threshold for the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$. Barros, Cavalar, Mota, and Parczyk [4] extended the result of [24] for cycles, proving that the threshold of the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} C_\ell$ remains $n^{-1/m_2(C_\ell)}$

also when $\ell \geq 5$. Kohayakawa, Mota, Parczyk, and Schnitzer [20] extended the result of [24] for complete graphs, proving that the threshold of $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_r$ remains $n^{-1/m_2(K_r)}$ also when $r \geq 5$.

For C_4 and K_4 the situation is different. The threshold for the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} C_4$ is $n^{-3/4} = o\left(n^{-1/m_2(C_4)}\right)$, as proved by Mota [23]. For the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_4$, the threshold is $n^{-7/15} = o\left(n^{-1/m_2(K_4)}\right)$ as proved by Kohayakawa, Mota, Parczyk, and Schnitzer [20]. More generally, Kohayakawa, Kostadinidis and Mota [19] proved that there are infinitely many graphs H for which the threshold for the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$ is significantly smaller than $n^{-1/m_2(H)}$.

Lastly, properly edge-coloured triangles are rainbow. Hence, the thresholds for the properties $K_3 \subseteq \mathbb{G}(n,p)$ and $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_3$ coincide so that n^{-1} is the threshold for the latter.

1.1 Our results

For a real $d > 0$, we say that $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. satisfies a graph property $\mathcal{P}$, if $\lim_{n\to\infty} \mathbb{P}[G_n \cup \mathbb{G}(n,p) \in \mathcal{P}] = 1$ holds for *every* sequence $\{G_n\}_{n\in\mathbb{N}}$ satisfying $G_n \in \mathscr{G}_{d,n}$ for every $n \in \mathbb{N}$. We say that $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. does not satisfy $\mathcal{P}$, if $\lim_{n\to\infty} \mathbb{P}[G_n \cup \mathbb{G}(n,p) \in \mathcal{P}] = 0$ holds for at *least* one sequence $\{G_n\}_{n\in\mathbb{N}}$ satisfying $G_n \in \mathscr{G}_{d,n}$ for every $n \in \mathbb{N}$. Throughout, we suppress this sequence-based terminology and write more concisely that $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. satisfies (or does not) a certain property. In particular, given a fixed graph H, we write that a.a.s. $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$ to mean that for every sequence $\{G_n\}_{n\in\mathbb{N}}$, satisfying $G_n \in \mathscr{G}_{d,n}$ for every $n \in \mathbb{N}$, the property $G_n \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$ holds asymptotically almost surely. On the other hand, we write that a.a.s. $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \not\xrightarrow{\text{rbw}} H$ to mean that there exists a sequence $\{G_n\}_{n\in\mathbb{N}}$, satisfying $G_n \in \mathscr{G}_{d,n}$ for every $n \in \mathbb{N}$, for which a.a.s. $G_n \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} H$ does not hold.

A sequence $\widehat{p} := \widehat{p}(n)$ is said to form a *threshold* for the property $\mathcal{P}$ in the perturbed model, if $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. satisfies $\mathcal{P}$ whenever $p = \omega(\widehat{p})$, and if $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. does not satisfy $\mathcal{P}$ whenever $p = o(\widehat{p})$.

For every real $d > 0$ and every pair of integers $s,t \geq 1$, every sufficiently large graph $G \in \mathscr{G}_{d,n}$ satisfies $G \xrightarrow{\text{rbw}} K_{s,t}$; in fact, every proper colouring of G supersaturates G with $\Omega(n^{s+t})$ rainbow copies of $K_{s,t}$. Consequently, the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_{s,t}$ is trivial as no random perturbation is needed for it to be satisfied. The emergence of rainbow copies of non-bipartite prescribed graphs may then be of interest. For odd cycles (including K_3), we prove the following.

Proposition 1. *For every integer $\ell \geq 2$, and every real $0 < d \leq 1/2$, the threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} C_{2\ell-1}$ is n^{-2}.*

Unlike the threshold for the property $\mathbb{G}(n,p) \xrightarrow{\text{rbw}} C_\ell$, established in [4,24], the threshold for the counterpart property in the perturbed model is independent of the length of the cycle.

Our main result concerns the thresholds for the emergence of rainbow complete graphs in properly coloured randomly perturbed dense graphs. From the results of [20,24], one easily deduces that if $r \geq 5$ and $p = o\left(n^{-1/m_2(K_r)}\right)$, then a.a.s. there exists a proper edge-colouring of $\mathbb{G}(n,p)$ admitting no rainbow copy of K_r. Consequently, given a real number $0 < d \leq 1/2$ and an n-vertex bipartite graph G of edge-density d, a.a.s. there exists a proper edge-colouring of $G \cup \mathbb{G}(n,p)$ admitting no rainbow copy of K_{2r-1}, provided that $p = o\left(n^{-1/m_2(K_r)}\right)$. We conclude that $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \overset{\text{rbw}}{\not\longrightarrow} K_{2r}$ and $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \overset{\text{rbw}}{\not\longrightarrow} K_{2r-1}$ hold a.a.s. whenever $p = o\left(n^{-1/m_2(K_r)}\right)$.

For every $r \geq 5$, we prove a matching upper bound for the above construction. Our main result reads as follows.

Theorem 1. *Let a real number $0 < d \leq 1/2$ and an integer $r \geq 5$ be given. Then, the threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_{2r}$ is $n^{-1/m_2(K_r)}$. In fact, $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p)$ a.a.s. has the property that every proper colouring of its edges gives rise to $\Omega\left(p^{2\binom{r}{2}} n^{2r}\right)$ rainbow copies of K_{2r}, whenever $p = \omega(n^{-1/m_2(K_r)})$.*

The following result is an immediate consequence of Theorem 1 and of the aforementioned lower bound.

Corollary 1. *Let a real number $0 < d \leq 1/2$ and an integer $r \geq 5$ be given. Then, the threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_{2r-1}$ is $n^{-1/m_2(K_r)}$.*

Theorem 1 and Corollary 1 establish that for *sufficiently large* complete graphs, i.e., K_s with $s \geq 9$, the threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_s$ is governed by a single parameter, namely, $m_2(K_{\lceil s/2 \rceil})$. This turns out to be true (almost, at least) for $s = 8$ as well, but proving it requires new ideas. For $4 \leq s \leq 7$, this is not the case; here, for each value of s in this range, the threshold is different. Using completely different methods, we prove the following.

Theorem 2. *Let $0 < d \leq 1/2$ be given.*

1. *The threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_4$ is $n^{-5/4}$*
2. *The threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_5$ is n^{-1}.*
3. *The threshold for the property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_7$ is $n^{-7/15}$.*

For K_6 and K_8, we can "almost" determine the thresholds.

Theorem 3. *Let $0 < d \leq 1/2$ be given.*

1. *The property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_6$ holds a.a.s. whenever $p = \omega(n^{-2/3})$.*
2. *For every constant $\varepsilon > 0$ it holds that a.a.s. $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \overset{\text{rbw}}{\not\longrightarrow} K_6$ whenever $p := p(n) = n^{-(2/3+\varepsilon)}$.*

Theorem 4. *Let $0 < d \leq 1/2$ be given.*

1. *The property $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_8$ holds a.a.s. whenever $p = \omega(n^{-2/5})$.*
2. *For every constant $\varepsilon > 0$ it holds that a.a.s. $\mathscr{G}_{d,n} \cup \mathbb{G}(n,p) \stackrel{\text{rbw}}{\not\longrightarrow} K_8$ whenever $p := p(n) = n^{-(2/5+\varepsilon)}$.*

Proofs of all of our results can be found in [1,2].

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