Projective modules over integral group rings and Wall's D2 problem

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I, John Nicholson, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed: .... Date: 27th July 2021

Abstract

In the first part of this thesis, we study the problem of when $P \oplus \mathbb{Z}G \cong Q \oplus \mathbb{Z}G$ implies $P \cong Q$ for projective $\mathbb{Z}G$ modules P, Q where $\mathbb{Z}G$ is the integral group ring of a finite group G. Our main result is a general condition on G under which cancellation holds. This builds upon the results of R. G. Swan and our condition includes all G for which cancellation was previously known to hold.

In the second part of this thesis, we explore applications of these results to Wall's D2 problem which asks whether every cohomologically 2-dimensional finite complex X is homotopy equivalent to a finite 2-complex. The case where $G = \pi_1(X)$ has 4-periodic cohomology has been the source of many proposed counterexamples to Wall's D2 problem and is of special interest due to its implications on the possible cell structures of finite Poincaré 3-complexes. Our main result is a solution to Wall's D2 problem for several infinite families of groups with 4-periodic cohomology, building upon the results of F. E. A. Johnson.

Impact statement

Projective modules over integral group rings are central objects in the homological algebra of cellular chain complexes arising from CW-complexes. In light of this, they are both a useful tool in the classification of CW-complexes and are the target of many topological obstructions such as the finiteness obstructions of C. T. C. Wall and R. G. Swan. The results in Part II of this thesis on Wall's D2 problem, and in particular Theorem C, make essential uses of my results on projective modules and serve as the main intended application.

With suitable modifications and extensions, the results on projective modules also have applications to the homotopy classification of CW-complexes with highly connected universal covers and, in turn, to the homotopy classification of manifolds. I have written a number of articles which contain these applications (see [3], [24], [35], [37]). However, I will not include these results in this thesis.

I am not currently aware of any direct applications of this work outside of academia. This thesis contributes towards the general goal of understanding CW-complexes and manifolds, which are natural models for spaces and shapes which arise in the real world.

Acknowledgements

First and foremost, I would like to acknowledge the guidance and support of my supervisor F. E. A. Johnson. It was through many interesting discussions with him that the main problems addressed in this thesis emerged, and it was through the techniques that he introduced me to early on in my PhD that progress was ultimately able to be made.

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Contents

Introduction

I Projective modules over integral group rings

A ring R is said to have *projective cancellation* if $P \oplus R \cong Q \oplus R$ implies $P \cong Q$ for all (finitely generated) projective R-modules P and Q. In the first part of this thesis, we will be interested in the problem of determining which finite groups G have the property that the integral group ring $\mathbb{Z}G$ has projective cancellation.

Fix a finite group G once and for all, and recall that the real group ring $\mathbb{R}G$ is semisimple and so admits a Wedderburn decomposition

$$\mathbb{R}G \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $n_1, \cdots, n_r$ are integers $\geqslant 1$ and each D_i is one of the real division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $m_{\mathbb{H}}(G)$ be the number of copies of $\mathbb{H} = M_1(\mathbb{H})$ which are factors in this decomposition. We say that $\mathbb{Z}G$ satisfies the *Eichler condition* if $m_{\mathbb{H}}(G) = 0$. By the Jacobinski cancellation theorem [43], this is a sufficient condition for $\mathbb{Z}G$ to have projective cancellation.

It is well-known that $\mathbb{Z}G$ fails the Eichler condition precisely when G has a quotient which is a binary polyhedral group, i.e. a non-cyclic finite subgroup of $\mathbb{H}^\times$. They are the generalised quaternion groups Q_{4n} for $n \geqslant 2$ or one of

$\widetilde{T}$, $\widetilde{O}$, $\widetilde{I}$, the binary tetrahedral, octahedral and icosahedral groups. It was shown by R. G. Swan [46] that, if G is a binary polyhedral group, then $\mathbb{Z}G$ has projective cancellation if and only if G is one of the seven groups

$$Q_8,\ Q_{12},\ Q_{16},\ Q_{20},\ \widetilde{T},\ \widetilde{O},\ \widetilde{I}. \tag{*}$$

It follows from work of A. Fröhlich [16] that, if $\mathbb{Z}G$ has projective cancellation and G has a quotient H, then $\mathbb{Z}H$ has projective cancellation also. In particular, $\mathbb{Z}G$ has non-cancellation whenever G has a quotient which is Q_{4n} for $n \geqslant 6$. Note that this does not yet characterise which groups have projective cancellation; it remains to determine projective cancellation for $\mathbb{Z}G$ when G has a quotient in $(*)$ but none of the form Q_{4n} for $n \geqslant 6$.

Our main result is as follows.

Theorem A. *Let G be a finite group and suppose G has a quotient H such that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ and H is of the form*

$$C_1,\ Q_8,\ Q_{12},\ Q_{16},\ Q_{20},\ \widetilde{T},\ \widetilde{O},\ \widetilde{I} \quad \text{or} \quad \widetilde{T}^n \times \widetilde{I}^m \text{ for } n, m \geqslant 0.$$

Then $\mathbb{Z}G$ has projective cancellation.

Let $C(\mathbb{Z}G)$ denote the projective class group. We say that a class $[P] \in C(\mathbb{Z}G)$ has cancellation if $P_1 \oplus \mathbb{Z}G \cong P_2 \oplus \mathbb{Z}G$ implies $P_1 \cong P_2$ for all projective $\mathbb{Z}G$ modules P_1, P_2 such that $[P_1] = [P] \in C(\mathbb{Z}G)$. In particular, $\mathbb{Z}G$ has projective cancellation if and only if $[P]$ has cancellation for all $[P] \in C(\mathbb{Z}G)$. However, there are groups G with projective $\mathbb{Z}G$ modules P, Q for which $[P]$ has cancellation and $[Q]$ has non-cancellation [46]. We will also consider $D(\mathbb{Z}G) = \operatorname{Ker}(C(\mathbb{Z}G) \to C(\Gamma))$ where $\mathbb{Z}G \subseteq \Gamma \subseteq \mathbb{Q}G$ is a maximal order.

Recall that a group G has *k-periodic cohomology* for some $k \geqslant 1$ if its Tate cohomology groups satisfy $\hat{H}^i(G;\mathbb{Z}) = \hat{H}^{i+k}(G;\mathbb{Z})$ for all $i \in \mathbb{Z}$. Our second result is the following.

Theorem B. *Let G have periodic cohomology and let P be a projective $\mathbb{Z}G$ module. Then*

(i) *If $m_{\mathbb{H}}(G) \leqslant 2$, then $[P]$ has cancellation*

(ii) *If $m_{\mathbb{H}}(G) = 3$, then:*

  (a) *If $\mathrm{Syl}_2(G)$ is cyclic, then $[P]$ has non-cancellation*

  (b) *If $[P] \in D(\mathbb{Z}G)$, then $[P]$ has non-cancellation*

(iii) *If $m_{\mathbb{H}}(G) \geqslant 4$, then $[P]$ has non-cancellation.*

In contrast to (ii) (b), Q_{24} is an example of a group with periodic cohomology and $m_{\mathbb{H}}(Q_{24}) = 3$ but for which there exists $[P] \in C(\mathbb{Z}Q_{24}) \setminus D(\mathbb{Z}Q_{24})$ which has cancellation.

II Applications to Wall's D2 problem

A connected CW-complex X is a Dn complex if $H_i(\widetilde{X}) = 0$ for $i > n$ and $H^{n+1}(X;M) = 0$ for all finitely generated $\mathbb{Z}G$-modules M. In his 1965 paper on finiteness conditions, C. T. C. Wall showed that a finite Dn complex is homotopy equivalent to a finite n-complex for $n > 2$ [47, Theorem E] and this was later proven for $n = 1$ by Stallings-Swan. The case $n = 2$ remains open and is known as Wall's D2 problem [50, Problem D3]:

Wall's D2 problem. *Is every finite* D2 *complex homotopy equivalent to a finite* 2*-complex?*

We is parametrised by finitely presented groups G by saying that G has the *D2 property* if every finite D2 complex X with $\pi_1(X) \cong G$ is homotopy equivalent to a finite 2-complex.

The aim of the second part of this thesis will be to study Wall's D2 problem in the case of groups with 4-periodic cohomology. Recall that a group presentation is *balanced* if it has the same number of generators and relations. Our main result is the following.

Theorem C. *Suppose G has 4-periodic cohomology. Then:*

(i) *If G has the* D2 *property, then G has a balanced presentation*

(ii) *If G has a balanced presentation and $m_{\mathbb{H}}(G) \leqslant 2$, then G has the* D2 *property.*

The question of whether or not groups with 4-periodic cohomology have the D2 property is of particular interest since, as noted by Johnson [21], this would have implications on the open problem of whether every Poincaré 3-complex has a cell structure with a single 3-cell.

In Theorem 4.14, we show that the groups G with 4-periodic cohomology and $m_{\mathbb{H}}(G) \leqslant 2$ are as follows. Here we use the notation of Milnor [32], and each family contains $G \times C_n$ for any G listed and any $n \geqslant 1$ coprime to $|G|$.

(i) C_n for $n \geqslant 1$, the cyclic groups of order n.

(ii) D_{4n+2} for $n \geqslant 1$, the dihedral groups of order $4n+2$.

(iii) $Q_8, Q_{12}, Q_{16}, Q_{20}, \widetilde{T}, \widetilde{O}, \widetilde{I}$.

(iv) $D(2^n, 3)$, $D(2^n, 5)$ for $n \geqslant 3$.

(v) $P'_{8 \cdot 3^n}$ for $n \geqslant 2$.

(vi) P''_{48n} for $n \geqslant 3$ odd.

(vii) $Q(16; m, n)$ for $m > n \geqslant 1$ odd coprime.

By considering which of these groups have balanced presentations, we will show the following. This was previously shown by M. N. Dyer [13] for the groups in (i) and by Johnson [22] for the groups in (ii) and for many of the groups in (iii).

Theorem 6.14. *Suppose G is in* (i)-(v) *or has the form $Q(16; n, 1) \times C_k$ for some $n, k \geqslant 1$ odd coprime. Then G has the* D2 *property.*

Finally, we consider the case $m_{\mathbb{H}}(G) \geqslant 3$. Let Q_{4n} denote the quaternion group of order $4n$, which has 4-periodic cohomology and $m_{\mathbb{H}}(Q_{4n}) = \lfloor n/2 \rfloor$. By combining recent results of W. H. Mannan and T. Popiel [29] with Theorem 5.1, we show:

Theorem 6.21. *Q_{28} has the* D2 *property and $m_{\mathbb{H}}(Q_{28}) = 3$.*

This group was proposed as a counterexample in [1] (see also [30, p23]). We also point out that the example of Mannan-Popiel gives a counterexample to a conjecture of J. M. Cohen [50, p381]. The possibility remains that some group with 4-periodic cohomology does not have a balanced presentation and so would be a counterexample to the D2 problem.

Structure of thesis

We will now give a brief overview of the structure of this thesis as well as a detailed account of where the original content can be found.

Part I.[1] In Chapter 1, we introduce the basic theory of locally free modules

[1] A weaker version of these results can be found in "*A cancellation theorem for modules over integral group rings*" which has been published by Mathematical Proceedings of the Cambridge Philosophical Society [36].

over an order in a semisimple $\mathbb{Q}$-algebra. This includes the case of projective $\mathbb{Z}G$ modules for G finite. Much of this can be found in [45, 46] though we include proofs of Propositions 1.11 and 1.14 as we could not locate them explicitly in the literature.

Chapters 2/3 constitute the technical heart of Part I of this thesis. The main cancellation theorem for locally free modules over orders is Theorem 2.6 and, in combination with Theorem 3.11, this is used to prove Theorem A. The results here are almost entirely original with the two main exceptions being Theorems 2.7 and 3.11 which are due to Swan [46].

Chapter 4 contains a detailed analysis of the quotients and one-dimensional quaternionic representations of groups with periodic cohomology, as well as a proof of Theorem B. This is entirely original with the exception of well-known facts such as Proposition 4.1 and the classification of groups with 4-periodic cohomology.

Part II.[2] The aim of Chapter 5 is to prove Theorems 5.1 and 5.11, which reduce the classification of finite D2 complexes X where $\pi_1(X)$ had 4-periodic cohomology to a cancellation problem projective $\mathbb{Z}G$ modules. These results are mild generalisations of the main result of [22], and the proofs follow a similar structure.

Finally, in Chapter 6, we combine the results from Chapter 5 with Theorem B to prove Theorem C. We also explore applications to the cell structure of Poincaré 3-complexes through observations previously made by Johnson [21] and Wall [49].

Throughout this thesis we will assume, without further mention, that all modules are finitely generated left modules.

[2]These results are contained in "*On CW-complexes over groups with periodic cohoology*" which has been published by Transactions of the American Mathematical Society [38].

Part I

Projective modules over integral group rings

Chapter 1

Preliminaries on projective modules

In this chapter, we will give a brief summary of the theory of locally free modules over orders in finite-dimensional semisimple $\mathbb{Q}$-algebras. As we shall see, this gives a useful framework in which to generalise projective modules over the integral group rings $\mathbb{Z}G$ of a finite group G. Much of this can be found in work of R. G. Swan, such as [43, 45].

1.1 Orders in semisimple algebras

Recall that, for a ring A, a non-zero A-module is *simple* if it contains no submodules other than 0 and itself and is *semisimple* if it is a direct sum of its simple submodules. We say that the ring A itself is semisimple if it is semisimple as a module over itself.

Let K be a field and recall that a K*-algebra* is a ring A equipped with an inclusion of rings $i : K \hookrightarrow A$ whose image is contained in the centre of A.

This can be viewed as a K-vector space in a natural way using multiplication by $K \cong i(K)$. The following was first proven by J. H. M. Wedderburn. See also [22, Section 2] for a modern account.

Proposition 1.1 ([53]). *If A is a finite-dimensional semisimple K-algebra, then there is an isomorphism of rings*

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $n_1, \cdots, n_r$ are integers $\geqslant 1$ and the D_i are division algebras over K. This decomposition is unique up to permutations of the $M_{n_i}(D_i)$ and isomorphisms of the D_i, and is known as the Wedderburn decomposition of A.

If $R \subseteq K$ is an integral domain, then a subring $\Lambda \subseteq A$ is an R-*order* if it is an R-algebra which is finitely generated as an R-module and $K \cdot \Lambda = A$.

One of the central objects in this thesis will be that of a $\mathbb{Z}$-order Λ in a finite-dimensional semisimple $\mathbb{Q}$-algebra A. Two important examples are:

(1) If G is a finite group, then the rational group ring $A = \mathbb{Q}G$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and the integral group ring $\Lambda = \mathbb{Z}G$ is a $\mathbb{Z}$-order in $\mathbb{Q}G$

(2) If $K/\mathbb{Q}$ is a finite field extension, then $A = K$ is a finite-dimensional simple $\mathbb{Q}$-algebra and the ring of integers $\Lambda = \mathcal{O}_K$ is an $\mathbb{Z}$-order in A.

If Λ is a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A, then $\Lambda \otimes \mathbb{R} \cong A \otimes \mathbb{R}$ is a semisimple $\mathbb{R}$-algebra and so has a real Wedderburn decomposition

$$\Lambda \otimes \mathbb{R} \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $n_1, \cdots, n_r$ are integers $\geqslant 1$ and each D_i is one of the real division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, which is unique up to permutation of the pairs (D_i, n_i).

Let $m_{\mathbb{H}}(\Lambda)$ denote the number of copies of $\mathbb{H} = M_1(\mathbb{H})$ which are factors in the decomposition of $A \otimes \mathbb{R}$. We say that Λ satisfies the *Eichler condition* if $m_{\mathbb{H}}(\Lambda) = 0$. In the special case where $\Lambda = \mathbb{Z}G$ for a finite group G, we will write $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(\mathbb{Z}G)$ and we say that the group G satisfies the Eichler condition if $\mathbb{Z}G$ satisfied the Eichler condition. Note that $m_{\mathbb{H}}(G)$ coincides with the number of one-dimensional quaternionic representations of G.

1.2 Milnor patching and the Mayer-Vietoris sequence

Suppose R and S are rings and $f : R \to S$ is a ring homomorphism. We can use this to turn S into an (S, R)-bimodule, with right-multiplication by $r \in R$ given by $x \cdot r = xf(r)$ for any $x \in S$. If M is an R-module, we can define the *extension of scalars* of M by f as the tensor product

$$f_{\#}(M) = S \otimes_R M$$

since S as a right R-module and M as a left R-module. We consider this as a left S-module where left-multiplication by $s \in S$ is given by $s \cdot (x \otimes m) = (sx) \otimes m$ for any $x \in S$ and $m \in M$. This comes with maps of abelian groups

$$f_* : M \to f_{\#}(M)$$

sending $m \mapsto 1 \otimes m$, and defines a covariant functor from R-modules to S-modules [8, p227].

We now recall the following basic properties of the extension of scalars map, which follow from the standard properties of tensor products.

Proposition 1.2 ([26, p145]). *Let $f : R \to S$ and $g : S \to T$ be ring homomorphisms and let M and N be R-modules. Then*

(i) $f_\#(M \oplus N) \cong f_\#(M) \oplus f_\#(N)$

(ii) $f_\#(R) \cong S$

(iii) $(g \circ f)_\#(M) \cong (g_\# \circ f_\#)(M)$.

If $P(R)$ denotes the set of (finitely generated) projective R-modules, then the first two properties show that $f_\#$ induces a map $f_\# : P(R) \to P(S)$ which restricts to each stable class. If $f : R \to S$ and M is an R module, then let $f_* : M \to f_\#(M)$ be the map $m \mapsto 1 \otimes m$.

Recall that, if R, R_1, R_2 and $\bar{R}$ are rings, then a pullback diagram

$$\mathcal{R} = \begin{array}{ccc} R & \xrightarrow{i_2} & R_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_2} \\ R_1 & \xrightarrow{j_1} & \bar{R} \end{array}$$

is a *Milnor square* if either j_1 or j_2 are surjective. If $P_1 \in P(R_1)$, $P_2 \in P(R_2)$ are such that there is a $\bar{R}$-module isomorphism $h : (j_1)_\#(P_1) \to (j_2)_\#(P_2)$, then define an R-module:

$$M(P_1, P_2, h) = \{(x, y) \in P_1 \times P_2 : h((j_1)_*(x)) = (j_2)_*(y)\} \leqslant P_1 \times P_2,$$

where multiplication by $r \in R$ is $r \cdot (x, y) = ((i_1)_*(r)x, (i_2)_*(r)y)$. It was shown by Milnor that $M(P_1, P_2, h)$ is projective [33, Theorem 2.1].

Let $\mathrm{Aut}_R(P)$ denote the set of R-module automorphisms of an R-module P, and we will write this as $\mathrm{Aut}(P)$ when R is understood from the context. The main result on Milnor squares is as follows.

Theorem 1.3 ([33, Section 2]). *Suppose $\mathcal{R}$ is a Milnor square and $P_i \in P(R_i)$ for $i = 1, 2$ are such that $\bar{P} = (j_1)_\#(P_1) \cong (j_2)_\#(P_2)$ as $\bar{R}$-modules. Then there is a one-to-one correspondence*

$$\mathrm{Aut}(P_1)\backslash \mathrm{Aut}(\bar{P})/\mathrm{Aut}(P_2) \leftrightarrow \{P \in P(R) : (i_1)_\#(P) \cong P_1, (i_2)_\#(P) \cong P_2\}$$

given by sending a coset $[h]$ to $M(P_1, P_2, h)$ for any representative h.

Let R be a ring. Recall the following definitions from algebraic K-theory:

(i) Let $K_0(R)$ denote the Grothendieck group of the monoid of isomorphism classes of projective R-modules $P(R)$, i.e. the abelian group generated by $[P]$ for $P \in P(R)$, with relations $[P_1 \oplus P_2] = [P_1] \oplus [P_2]$ for all $P_1, P_2 \in P(R)$

(ii) Let $K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}}$ where $\mathrm{GL}(R) = \bigcup_n \mathrm{GL}_n(R)$ with respect to the inclusions $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$.

It is straightforward to see that K_0 and K_1 are functors from the category of rings to abelian groups. Let $\mathcal{R} = \mathcal{R}(R, R_1, R_2, \bar{R})$ be the pullback diagram defined above for rings R_1, R_2 and $\bar{R}$. The following is referred to as the *Mayer-Vietoris sequence* for a Milnor square $\mathcal{R}$.

Theorem 1.4 ([33, Theorem 3.3]). *If $\mathcal{R}$ is a Milnor square, then there is an exact sequence*

$$K_1(R) \to K_1(R_1) \times K_1(R_2) \to K_1(\bar{R}) \xrightarrow{\partial} K_0(R) \to K_0(R_1) \times K_0(R_2) \to K_0(\bar{R})$$

where, if $x \in K_1(\bar{R})$ is represented by $h \in \mathrm{GL}_n(\bar{R})$, then $\partial(x) = [M(R_1^n, R_2^n, h)] - [R^n]$, and all other maps are functorial.

The following can be found in [9] and gives a convenient way to split a ring R as a pullback. Since all maps are surjective, this is a Milnor square.

Lemma 1.5 ([9, Example 42.3]). *Let I and J be two-sided ideals in a ring R. Then there is a pullback diagram:*

$$\begin{array}{ccc} R/(I \cap J) & \longrightarrow & R/J \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & R/(I+J) \end{array}$$

For example, we obtain the following when $R = \mathbb{Z}G$ be the integral group ring of a finite group by considering the trivially intersecting ideals $I = \mathrm{Ker}(f_* : \mathbb{Z}G \to \mathbb{Z}H) = \mathrm{Ker}(\varepsilon : \mathbb{Z}N \to \mathbb{Z}) \cdot \mathbb{Z}G$ and $J = \Sigma_N \cdot \mathbb{Z}G$ where $\Sigma_N = \sum_{g \in N} g$ is the group norm.

Corollary 1.6. *Let G be a finite group which has quotient $H = G/N$ where N is a normal subgroup of G. Then there is a Milnor square*

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ \mathbb{Z}H & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})[H] \end{array}$$

where $\Lambda = \mathbb{Z}G/\Sigma_N$ and $n = |N|$.

This will be crucial in our later applications since, by combining this with Theorem 1.3, we get a way to compare projective $\mathbb{Z}G$ modules to projective $\mathbb{Z}H$ modules where H is a quotient of G.

1.3 Locally free modules

From now on, we will let Λ be a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A.

Recall that an A-module M is *projective* if there exists an A-module M'

such that $M \oplus M' \cong A^i$ is free for some $i \geqslant 0$. For a prime p, let $\mathbb{Z}_p$ denote the p-adic integers and let $\Lambda_p = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We say a Λ module M is *locally projective* if $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a projective Λ_p module for all primes p. The following is well-known.

Proposition 1.7 ([45, Lemma 2.1])**.** *Let M be a Λ module. Then M is projective if and only if M is locally projective.*

Similarly, we say that M is *locally free (of rank n)* if there exists $n \geqslant 1$ for which M_p is a free Λ_p module of rank n for all p prime. In the special case where $A = \mathbb{Q}G$ and $\Lambda = \mathbb{Z}G$ for G a finite group, we have the following refinement of Proposition 1.7.

Proposition 1.8 ([45, p156])**.** *Let G be a finite group. If M is a $\mathbb{Z}G$ module, then M is projective if and only if M is locally free.*

In light of this, we will now restrict our attention to locally free Λ modules. Define the *locally free class group* $C(\Lambda)$ to be the equivalence classes of locally free Λ modules up to the relation $P \sim Q$ if $P \oplus \Lambda^i \cong Q \oplus \Lambda^j$ for some $i, j \geqslant 0$. By abuse of notation, we write $[P]$ to denote both the class $[P] \in C(\Lambda)$ and, where convenient, the set of isomorphism classes of locally free modules P_0 where $[P_0] = [P]$.

We also define the *class set* $\operatorname{Cls}\Lambda$ as the set of isomorphism classes of rank one locally free Λ-modules, which is finite by the Jordan-Zassenhaus theorem [8, Section 24]. This comes with the stable class map

$$[\,\cdot\,]_\Lambda : \operatorname{Cls}\Lambda \to C(\Lambda)$$

which sends $P \mapsto [P]$. This map is always surjective due to the following. This was proven by A. Fröhlich in [16] using idelic methods, generalising the case $\Lambda = \mathbb{Z}G$ first obtained by Swan [40, Theorem A]. However, it is worth noting that the first part follows already from the cancellation theorems of Bass and Serre [45, Section 2].

Proposition 1.9 ([16, p115])**.** *If M is a locally free Λ module, then:*

(i) *There exists $M_0 \in \operatorname{Cls}\Lambda$ such that $M \cong M_0 \oplus \Lambda^i$ for some $i \geqslant 0$*

(ii) *There exists a left-ideal $I \subseteq \Lambda$ for which $M_0 \cong I$.*

We say that Λ has *locally free cancellation* if $P \oplus \Lambda \cong Q \oplus \Lambda$ implies $P \cong Q$ for all locally free Λ-modules P and Q. By Proposition 1.9, we have that Λ has locally free cancellation if and only if $[\,\cdot\,]_\Lambda$ is bijective, i.e. $\#\operatorname{Cls}\Lambda = \#C(\Lambda)$.

More generally, we say that a class $[P] \in C(\Lambda)$ has cancellation if $P_1 \oplus \Lambda \cong P_2 \oplus \Lambda$ implies $P_1 \cong P_2$ for all $P_1, P_2 \in [P]$. We say that Λ has *stably free cancellation* when $[\Lambda]$ has cancellation, i.e. when every stably free Λ module is free. It will often be convenient to write $\operatorname{Cls}^{[P]}(\Lambda) = [\,\cdot\,]_\Lambda^{-1}([P])$ and $\operatorname{SF}(\Lambda) = \operatorname{Cls}^{[\Lambda]}(\Lambda)$ so that, by Proposition 1.9, a class $[P] \in C(\Lambda)$ has cancellation if and only if $\#\operatorname{Cls}^{[P]}(\Lambda) = 1$.

Recall that $[P] \in C(\Lambda)$ can be represented as a graded tree with vertices the isomorphism classes of non-zero modules $P_0 \in [P]$, edges between each $P_0 \in [P]$ and $P_0 \oplus \Lambda \in [P]$ and with grading from the rank of each locally free Λ module. By Proposition 1.9, this takes the following simple form where the set of minimal vertices corresponds to $\operatorname{Cls}^{[P]}(\Lambda)$ (see Fig. 5.1).

The following is a consequence of a general cancellation theorem of H. Jacobinski which depends on deep results of M. Eichler on strong approximation.

Theorem 1.10 ([20, Theorem 4.1])**.** *If Λ satisfies the Eichler condition, then Λ has locally free cancellation. In particular,* $\operatorname{Cls}^{[P]}(\Lambda) = \{P\}$ *for all* $P \in \operatorname{Cls}(\Lambda)$.

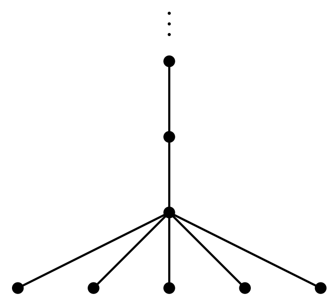

Figure 1.1: Tree structure for $[P] \in C(\Lambda)$

We conclude this section by collecting the basic properties of locally free modules which we will use in this thesis.

Proposition 1.11. *If M is a locally free Λ module, then:*

(i) *$M \otimes \mathbb{Q}$ is a free A module.*

(ii) *$M \otimes (\mathbb{Z}/n\mathbb{Z})$ is a free $\Lambda \otimes (\mathbb{Z}/n\mathbb{Z})$ module for all $n \geqslant 1$.*

The first part is a consequence of the Noether-Deuring theorem and its proof can be found in [15, p. 407]. The second part is well-known [39, Remark 4.9] though we were not able to locate a proof in the literature except in the case $\Lambda = \mathbb{Z}G$ [40, Theorem 7.1]. For convenience, we include a proof below which generalises the proof of [40, Theorem 7.1].

Proof of (ii). First note that $\Lambda/n \cong \Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. In particular, if M is a Λ module, then $M/n \cong M \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ can be viewed as a Λ/n module.

By Proposition 1.9, it suffices to consider the case where M is locally free of rank one. By Proposition 1.7, we have that $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \Lambda_p$ for all p prime. Since $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ induces an isomorphism $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p/p\mathbb{Z}_p$, we have that

$$\begin{aligned} M/p &\cong M \otimes_{\mathbb{Z}} (\mathbb{Z}_p/p\mathbb{Z}_p) \cong (M \otimes_{\mathbb{Z}} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p\mathbb{Z}_p) \\ &\cong \Lambda_p \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p/p\mathbb{Z}_p) \cong \Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}_p/p\mathbb{Z}_p) \cong \Lambda/p. \end{aligned}$$

Let $a_p \in M$ be such that $[a_p] \in M/p$ maps to $1 \in \Lambda/p$ under this isomorphism. As in the proof of Hensel's lemma, we can check that the map $1 \mapsto [a_p]$ also defines an isomorphism $\Lambda/(p^k) \to M/(p^k)$ for all $k \geqslant 1$

If $n = p_1^{k_1} \cdots p_r^{k_r}$ is a factorisation into distinct primes, then the Chinese remainder theorem implies that there exists $\alpha_i \in \mathbb{Z}$ such that $\alpha_i \equiv 1 \bmod p_i$ and $\alpha_i \equiv 0 \bmod p_j$ for $i \neq j$. By the argument above, there exists $a_i \in M$ such that the map $1 \mapsto [a_i]$ gives an isomorphism $\Lambda/p^{k_i} \to M/p^{k_i}$. If $a = \sum_{i=1}^r \alpha_i a_i$, then it is easy to see that the map $1 \mapsto a$ defines an isomorphism $\Lambda/n \to M/n$ as required. □

Corollary 1.12. *Suppose $f : \Lambda \to \bar{\Lambda}$ is a surjective ring homomorphism for a finite ring $\bar{\Lambda}$. If M is a locally free Λ module, then $M \otimes \bar{\Lambda}$ is a free $\bar{\Lambda}$ module.*

Proof. Note that $\bar{\Lambda} = \Lambda/I$ for a two-sided ideal $I \subseteq \Lambda$ and, since Λ is a $\mathbb{Z}$-order in A, we have that $\mathbb{Z} \subseteq \Lambda$. Since $\bar{\Lambda}$ is finite, $I \subseteq \Lambda$ must have finite index and so $I \cap \mathbb{Z} = (n) \subseteq \mathbb{Z}$ for some $n \geqslant 1$. In particular, this implies that $n\Lambda \subseteq I$ and so there is a composition $\Lambda \to \Lambda/n \to \Lambda/I$. Hence, if M is a locally free Λ module, then $M \otimes \bar{\Lambda} \cong (M/n) \otimes \bar{\Lambda}$. Since M/n is a free Λ/n module by Lemma 1.11, we have that $M \otimes \bar{\Lambda}$ is a free $\bar{\Lambda}$ module. □

In particular, this shows that locally free Λ modules cannot be detected on A or on any finite ring quotients of Λ. For example, if $\Lambda = \mathbb{Z}G$, then $M \in \mathrm{Cls}(\mathbb{Z}G)$ has $M \otimes \mathbb{Q} \cong \mathbb{Q}G$ and $M \otimes \mathbb{F}_p \cong \mathbb{F}_p G$. Hence locally free $\mathbb{Z}G$ modules cannot be studied using the usual techniques of representation theory.

For later purposes, we will also need to define the *defect group* $D(\Lambda) = \mathrm{Ker}(i_* : C(\Lambda) \to C(\Gamma))$ where $i : \Lambda \hookrightarrow \Gamma$ and $\Gamma \subseteq A$ is a maximal order. Note that i_* is surjective by [46, Theorem A10]. By [46, Theorem A24], this is independent of the choice of Γ and, if $f : \Lambda_1 \to \Lambda_2$ is a map of $\mathbb{Z}$-orders, then f induces a map $f_* : D(\Lambda_1) \to D(\Lambda_2)$.

We will now specialise the theory presented in Section 1.2 to the special case of locally free Λ modules. Let $K_0^{\mathrm{LF}}(\Lambda)$ denote the subgroup of $K_0(\Lambda)$ generated by $[P]$ for P a locally free Λ-module.

Lemma 1.13 ([45, p157])**.** *There is an isomorphism of abelian groups*

$$K_0^{\mathrm{LF}}(\Lambda) \xrightarrow{\cong} \mathbb{Z} \oplus C(\Lambda)$$

sending $[P] \mapsto (\mathrm{rank}(P), [P])$ *where* $\mathrm{rank}(P)$ *denotes the rank of* P *as a locally free module.*

Let Λ, Λ_1 and Λ_2 be $\mathbb{Z}$-orders in finite-dimensional semisimple $\mathbb{Q}$-algebras A, A_1, A_2 respectively, let $\bar{\Lambda}$ be a finite ring and suppose there is a Milnor square:

$$\mathcal{R} = \begin{array}{ccc} \Lambda & \xrightarrow{i_2} & \Lambda_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_2} \\ \Lambda_1 & \xrightarrow{j_1} & \bar{\Lambda} \end{array}$$

s Since $\bar{\Lambda}$ is a finite ring, we have that $\bar{\Lambda} \otimes \mathbb{Q} = 0$. Since $\mathbb{Q}$ is a flat module, tensoring the above diagram with $\mathbb{Q}$ gives another pullback diagram which implies that the map

$$(i_1, i_2) \otimes \mathbb{Q} : \Lambda \otimes \mathbb{Q} \to (\Lambda_1 \otimes \mathbb{Q}) \times (\Lambda_2 \otimes \mathbb{Q})$$

is an isomorphism, i.e. $A \cong A_1 \times A_2$.

In this context, Theorem 1.4 can be generalised as follows.

Proposition 1.14. *If* $\mathcal{R}$ *is as above, then there is an exact sequence*

$$K_1(\Lambda) \to K_1(\Lambda_1) \times K_1(\Lambda_2) \to K_1(\bar{\Lambda}) \xrightarrow{\partial} C(\Lambda) \to C(\Lambda_1) \times C(\Lambda_2) \to 0$$

where, if $x \in K_1(\bar{\Lambda})$ is represented by $h \in \mathrm{GL}_n(\bar{\Lambda})$, then $\partial(x) = [M(\Lambda_1^n, \Lambda_2^n, h)]$, and all other maps are functorial.

This is essentially proven in [39, 6.2], though the result is stated in a different form. For convenience, we will include a proof below which pieces together the argument in [39, 6.2].

Proof. Consider the Mayer-Vietoris sequence for $\mathcal{R}$ given by Theorem 1.4, and the connecting homomorphism $\partial : K_1(\bar{\Lambda}) \to K_0(\Lambda)$. Using the full hypothesis on $\mathcal{R}$, including that $\bar{\Lambda}$ is a finite ring and so $\bar{\Lambda} \otimes \mathbb{Q} = 0$, we get that $M(\Lambda_1^n, \Lambda_2^n, h)$ is a locally free Λ module of rank n for all $h \in \mathrm{GL}_n(\bar{\Lambda})$. Hence, with respect to the inclusion $C(\Lambda) \subseteq K_0^{\mathrm{LF}}(\Lambda) \subseteq K_0(\Lambda)$ induced by Lemma 1.13, we have that $\mathrm{Im}(\partial) \subseteq C(\Lambda)$.

This implies that we have a sequence

$$K_1(\Lambda) \to K_1(\Lambda_1) \times K_1(\Lambda_2) \to K_1(\bar{\Lambda}) \xrightarrow{\partial} C(\Lambda) \to C(\Lambda_1) \times C(\Lambda_2)$$

which is exact at each of the K_1 terms. By the identification in Lemma 1.13, the map $(i_1, i_2)_* : C(\Lambda) \to C(\Lambda_1) \times C(\Lambda_2)$ is the restriction of the map $(i_1, i_2)_* : K_0(\Lambda) \to K_0(\Lambda_1) \times K_0(\Lambda_2)$, and so the sequence is also exact at $C(\Lambda)$. To see that $(i_1, i_2)_* : C(\Lambda) \to C(\Lambda_1) \times C(\Lambda_2)$ is surjective note that, by the discussion above, $(i_1, i_2) \otimes \mathbb{Q}$ is an isomorphism and so

$$(i_1, i_2)_\# : C(\Lambda) \to C(\Lambda_1 \times \Lambda_2) \cong C(\Lambda_1) \times C(\Lambda_2)$$

is surjective, by Theorem 1.15. ☐

We will now give general conditions under which we can relate cancellation over two orders Λ_1 and Λ_2 when there is a map $f : \Lambda_1 \to \Lambda_2$. The following was shown by Swan and generalises an earlier result of Fröhlich [16, VIII].

Theorem 1.15 ([46, Theorem A10])**.** *Let $f : \Lambda_1 \to \Lambda_2$ be a map of $\mathbb{Z}$-orders in a semisimple $\mathbb{Q}$-algebra A such that the induced map $f_* : \Lambda_1 \otimes_{\mathbb{Z}} \mathbb{Q} \twoheadrightarrow \Lambda_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Then the diagram*

$$\begin{array}{ccc} \mathrm{Cls}(\Lambda_1) & \xrightarrow{f_\#} & \mathrm{Cls}(\Lambda_2) \\ \downarrow [\,\cdot\,]_{\Lambda_1} & & \downarrow [\,\cdot\,]_{\Lambda_2} \\ C(\Lambda_1) & \xrightarrow{f_\#} & C(\Lambda_2) \end{array}$$

is a weak pullback with all maps surjective.

In particular, if $P_1 \in \mathrm{Cls}\,\Lambda_1$ and $P_2 = f_\#(P_1) \in \mathrm{Cls}\,\Lambda_2$, then this implies that the map

$$f_\# : \mathrm{Cls}^{[P_1]}(\Lambda_1) \to \mathrm{Cls}^{[P_2]}(\Lambda_2)$$

is surjective. Hence, if $[P_1]$ has cancellation, then $[P_2]$ has cancellation.

Let G be a finite group with quotient H. By Corollary 1.6, the situation of Theorem 1.15 arises when $\Lambda_1 = \mathbb{Z}G$, $\Lambda_2 = \mathbb{Z}H$ and $f : \mathbb{Z}G \to \mathbb{Z}H$ is induced by the quotient map and is itself surjective. In particular, we have:

Corollary 1.16. *Let G be a finite group which has a quotient H. Then:*

(i) *If $\mathbb{Z}G$ has locally free cancellation, then $\mathbb{Z}H$ has locally free cancellation.*

(ii) *If $\mathbb{Z}G$ has stably free cancellation, then $\mathbb{Z}H$ has stably free cancellation.*

1.4 Central Picard groups

We will now consider the question of when a locally free Λ module can be represented by a two-sided ideal $I \subseteq \Lambda$ and so has the additional structure of a bimodule. This was first considered by Fröhlich [14] and Fröhlich-Reiner-Ullom [15], and we recount the basic theory here for use in Chapter 2.

Recall that, for a ring R, an (R, R)-bimodule M is *invertible* if there exists an (R, R)-bimodule N and bimodules isomorphisms

$$f : M \otimes_R N \to R, \quad g : N \otimes_R M \to R$$

such that the following diagrams commute:

$$\begin{array}{ccc} M \otimes_R N \otimes_R M & \xrightarrow{f \otimes \mathrm{id}} & R \otimes_R M \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes g} & & \Big\downarrow \\ M \otimes_R R & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} N \otimes_R M \otimes_R N & \xrightarrow{g \otimes \mathrm{id}} & R \otimes_R N \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes f} & & \Big\downarrow \\ N \otimes_R R & \longrightarrow & N \end{array}$$

The *central Picard group* $\mathrm{Picent}(R)$ is the group of (R, R)-bimodule isomorphism classes of (R, R)-bimodules M for which $xm = mx$ for all $m \in M$ and all central elements $x \in Z(R)$.

If M is an (R_1, R_2)-bimodule and $f_i : S_i \to R_i$ are ring homomorphisms for $i = 1, 2$, then we write ${}_{f_1}M_{f_2}$ to denote the (S_1, S_2)-bimodule with left action $s \cdot m := f_1(s)m$ for $s \in S_1$, $m \in M$ and with right action $m \cdot s := mf_2(s)$ for $s \in S_2$, $m \in M$. If f_1 is the identity, we write this as M_{f_2} and similarly, if f_2 is the identity, we write ${}_{f_1}M$.

It will be useful to know when two modules in $\mathrm{Picent}(R)$ are actually isomorphic as left R-modules. To determine this, let $\mathrm{Aut}_{\mathbb{Z}}(R)$ denote the group of automorphisms of R as an abelian group, i.e. as a $\mathbb{Z}$-module. Define the set of *central automorphisms* to be

$$\mathrm{Autcent}(R) = \{f \in \mathrm{Aut}_{\mathbb{Z}}(R) : f(c) = c,\ c \in C\}$$

where $C = Z(R)$. Let $\mathrm{In}(R) = \{f \in \mathrm{Autcent}(R) : f(x) = \lambda x \lambda^{-1}, \lambda \in R^\times\}$ denote the subset of inner automorphisms and let $\mathrm{Outcent}(R) = \mathrm{Autcent}(R)/\mathrm{In}(R)$.

If $f \in \mathrm{Autcent}(R)$, then $R_f \in \mathrm{Picent}(R)$. Since $R_f \cong R$ as bimodules for all $f \in \mathrm{In}(R)$, this defines a map $\omega : \mathrm{Outcent}(R) \to \mathrm{Picent}(R)$.

Proposition 1.17 ([9, Theorem 55.12])**.** *Let $X, Y \in \mathrm{Picent}(R)$. Then $X \cong Y$ as left R-modules if and only if $X \cong Y_f$ as (R, R)-bimodules for some $f \in \mathrm{Autcent}(R)$.*

We now consider the special case where $R = \Lambda$ is a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra. Let $I(\Lambda)$ denote the multiplicative group of two-sided ideals $I \subseteq \Lambda$ which are invertible in the sense that there exists a fractional two-sided ideal $J \subseteq \mathbb{Q} \cdot \Lambda$ for which $I \cdot J = \Lambda$. If $I \in I(\Lambda)$, then it follows that I is invertible as a (Λ, Λ)-bimodule and, since I is an ideal, we have that $xm = mx$ for all $m \in I$ and $x \in Z(\Lambda)$. This implies that I represents a class $[I] \in \mathrm{Picent}(\Lambda)$. Moreover, we have:

Proposition 1.18 ([9, Corollary 55.18])**.** *There is an isomorphism of abelian groups:*

$$\mathrm{Picent}(\Lambda) \cong I(\Lambda)/\{\Lambda a : a \in (\mathbb{Q} \cdot C)^\times\}$$

where $C = Z(\Lambda)$ is the centre of Λ.

We will now specialise even further to the case of locally free Λ modules. Define the *locally free Picard group* $\mathrm{LFP}(\Lambda) \subseteq \mathrm{Picent}(\Lambda)$ to be the subgroup consisting of (Λ, Λ)-bimodules M such that $M \in \mathrm{Cls}(\Lambda)$ is locally free as a left Λ-module.

Proposition 1.19 ([9, Proposition 55.29])**.** $\mathrm{LFP}(\Lambda)$ *is the set of two-sided ideals $I \subseteq \Lambda$ for which $I \in \mathrm{Cls}(\Lambda)$. That is, if $I \subseteq \Lambda$ is a two-sided ideal such* *that* $I \in \mathrm{Cls}(\Lambda)$*, then there exists* $J \subseteq \Lambda$ *two-sided with* $J \in \mathrm{Cls}(\Lambda)$ *such that* $I \otimes_\Lambda J \cong \Lambda \cong J \otimes_\Lambda I$ *as* (Λ, Λ)*-bimodules.*

In particular this shows that, if $I \subseteq \Lambda$ be a two-sided ideal such that $I \in \mathrm{Cls}(\Lambda)$, then I induces a bijection $I \otimes_\Lambda - : \mathrm{Cls}(\Lambda) \to \mathrm{Cls}(\Lambda)$.

In this context, we can consider an even stronger notion of local freeness than for left modules. We say that a (Λ, Λ)-bimodule M is *locally free as a bimodule* if there exists $i \geqslant 1$ such that, for all p prime, $M_p \cong \Lambda_p^i$ are isomorphic as (Λ_p, Λ_p)-bimodules. We will now need the following two closely related results.

Proposition 1.20 ([9, Proposition 55.16])**.** *Let* R *be a commutative Noetherian local ring and let* Λ *be a commutative finitely generated* R*-algebra. Then* $\mathrm{Picent}(\Lambda) = 1$*.*

In particular, if Λ is an $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A and $C = Z(\Lambda)$, then C_p is a commutative finitely generated $\mathbb{Z}_p$-algebra and $C_{(p)}$ is a commutative finitely generated $\mathbb{Z}_{(p)}$-algebra. Since $\mathbb{Z}_p$ and $\mathbb{Z}_{(p)}$ are both Noetherian, this implies that $\mathrm{Picent}(C_p) = 1$ and $\mathrm{Picent}(C_{(p)}) = 1$.

The following was shown by Fröhlich (see also [9, Theorem 55.25]). Note that, since $\tau' \circ \tau$ factors $\mathrm{Picent}(C_p)$, the fact that $\tau' \circ \tau = 0$ follows from $\mathrm{Picent}(C_p) = 1$.

Theorem 1.21 ([14, Theorem 6])**.** *For all but finitely many primes* p*, we have* $\mathrm{Picent}(\Lambda_p) = 1$ *and there is an exact sequence*

$$1 \to \mathrm{Picent}(C) \xrightarrow{\tau} \mathrm{Picent}(\Lambda) \xrightarrow{\tau'} \prod_p \mathrm{Picent}(\Lambda_p) \to 1$$

where $C = Z(\Lambda)$ *is the centre of* Λ *and* $\tau(M) = M \otimes_C \Lambda$ *for* $M \in \mathrm{Picent}(C)$*.*

This leads to the following three equivalent characterisations of locally free bimodules. This is presumably well-known, though we were not able to locate a proof in the literature.

Corollary 1.22. *Let $I \subseteq \Lambda$ be a two-sided ideal such that $I \in \mathrm{Cls}(\Lambda)$. Then the following are equivalent:*

(i) *I is generated by central elements*

(ii) *I is locally free as a bimodule, i.e. for all p prime, $I_p \cong \Lambda_p$ are isomorphic as (Λ_p, Λ_p)-bimodules.*

(iii) *For all p prime, $I_{(p)} \cong \Lambda_{(p)}$ are isomorphic as $(\Lambda_{(p)}, \Lambda_{(p)})$-bimodules.*

Proof. By Proposition 1.19, I represents a class $[I] \in \mathrm{Picent}(\Lambda)$. The equivalence of (i) and (ii) now follows from Proposition 1.18 and Theorem 1.21.

Now, (iii) implies (ii) since $\Lambda_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \Lambda_{(p)}$. In order to show that (i) implies (iii), suppose that I is generated by central elements and, for p prime, let $\tau'' : \mathrm{Picent}(\Lambda) \to \mathrm{Picent}(\Lambda_{(p)})$ be the induced map. Then there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Picent}(C) & \xrightarrow{\tau} & \mathrm{Picent}(\Lambda) \\ \downarrow & & \downarrow{\scriptstyle \tau''} \\ \mathrm{Picent}(C_{(p)}) & \longrightarrow & \mathrm{Picent}(\Lambda_{(p)}) \end{array}$$

where all maps are the induced maps. By Proposition 1.20, we have that $\mathrm{Picent}(C_{(p)}) = 1$ and so $\tau'' \circ \tau = 0$. Since I is generated by central elements, $[I] \in \mathrm{Im}(\tau)$ and so $[I_{(p)}] = \tau''(I) = 0 \in \mathrm{Picent}(\Lambda_{(p)})$ which implies that $I_{(p)} \cong \Lambda_{(p)}$ as $(\Lambda_{(p)}, \Lambda_{(p)})$-bimodules. ∎

Chapter 2

Cancellation for modules over orders in semisimple $\mathbb{Q}$-algebras

In this chapter, we will establish Theorem 2.6 which is a new cancellation theorem for orders in semisimple $\mathbb{Q}$-algebras. This constitutes the main technical heart of this part of the thesis.

2.1 Cancellation over fibre squares

Let Λ be a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A and let $A \cong A_1 \times A_2$ be an isomorphism of $\mathbb{Q}$-algebras. For $i = 1, 2$, let Λ_i be the projections onto A_i, which is a $\mathbb{Z}$-orders in A_i. If $\Lambda_1 = \Lambda / I_1$ and $\Lambda_2 = \Lambda / I_2$, then $I_1 \cap I_2 = \{0\}$ and so, by Lemma 1.5, there is a pullback diagram

$$\mathcal{R} = \begin{array}{ccc} \Lambda & \xrightarrow{i_2} & \Lambda_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_2} \\ \Lambda_1 & \xrightarrow{j_1} & \bar{\Lambda} \end{array}$$

where $\bar{\Lambda} = \Lambda/(I_1+I_2)$. Since $(i_1,i_2)\otimes\mathbb{Q}$ induces the isomorphism $A \to A_1 \times A_2$, we must have that $\bar{\Lambda}\otimes\mathbb{Q} = 0$ which implies that $\bar{\Lambda}$ is a finite ring. We will write $\mathcal{R} = \mathcal{R}(\Lambda, A_1, A_2)$ to denote the diagram induced by the splitting $A \cong A_1 \times A_2$.

Consider the maps

$$\mathrm{Cls}_{\mathcal{R}} : \mathrm{Cls}(\Lambda) \to \mathrm{Cls}(\Lambda_1) \times \mathrm{Cls}(\Lambda_2), \quad C_{\mathcal{R}} : C(\Lambda) \to C(\Lambda_1) \times C(\Lambda_2)$$

which are both induced by the extension of scalars maps $((i_1)_\#, (i_2)_\#)$.

Lemma 2.1. *Let $P \in \mathrm{Cls}(\Lambda)$ and let $P_k = (i_k)_\#(P) \in \mathrm{Cls}(\Lambda_k)$ for $k = 1, 2$. Then $\mathrm{Cls}_{\mathcal{R}}$ restricts to a surjection*

$$\mathrm{Cls}_{\mathcal{R}} : \mathrm{Cls}^{[P]}(\Lambda) \twoheadrightarrow \mathrm{Cls}^{[P_1]}(\Lambda_1) \times \mathrm{Cls}^{[P_2]}(\Lambda_2)$$

and $[\,\cdot\,]_\Lambda$ restricts to a surjection

$$[\,\cdot\,]_\Lambda : \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \twoheadrightarrow C_{\mathcal{R}}^{-1}([P_1],[P_2]).$$

Proof. Since $(i,j) : \Lambda \to \Lambda_1 \times \Lambda_2$ is a map of $\mathbb{Z}$-orders in A such that $(i_1,i_2)\otimes\mathbb{Q}$ is an isomorphism, Theorem 1.15 implies that the diagram

$$\begin{array}{ccc} \mathrm{Cls}(\Lambda) & \xrightarrow{\mathrm{Cls}_{\mathcal{R}}} & \mathrm{Cls}(\Lambda_1) \times \mathrm{Cls}(\Lambda_2) \\ \downarrow{\scriptstyle [\,\cdot\,]_\Lambda} & & \downarrow{\scriptstyle [\,\cdot\,]_{\Lambda_1} \times [\,\cdot\,]_{\Lambda_2}} \\ C(\Lambda) & \xrightarrow{C_{\mathcal{R}}} & C(\Lambda_1) \times C(\Lambda_2) \end{array}$$

is a weak pullback in that $\mathrm{Cls}(\Lambda)$ maps onto the pullback of the lower right corner. Hence the fibres of $[\,\cdot\,]_\Lambda$ map onto the fibres of $[\,\cdot\,]_{\Lambda_1} \times [\,\cdot\,]_{\Lambda_2}$ and the fibres of $\mathrm{Cls}_{\mathcal{R}}$ map onto the fibres of $C_{\mathcal{R}}$, as required. □

In order to determine when $[\,\cdot\,]$ is bijective, i.e. when Λ has locally free cancellation, it is therefore useful to give the explicit forms for the fibres of $\mathrm{Cls}_{\mathcal{R}}$ and $C_{\mathcal{R}}$ respectively.

Since j_1 and j_2 are both surjective in our construction above, $\mathcal{R}$ is a Milnor square. We also have that locally free $\bar{\Lambda}$-modules are free since $\bar{\Lambda}$ is finite and hence semilocal [39, Remark 4.9], and so $\bar{\Lambda} \cong (j_1)_\#(P_1) \cong (j_2)_\#(P_2)$ as $\bar{\Lambda}$-modules. Hence we have the following by Theorems 1.3 and 1.4 respectively.

Proposition 2.2. *For $k = 1, 2$, let $P_k \in \mathrm{Cls}(\Lambda_k)$. Then there is an isomorphism of left $\bar{\Lambda}$ modules $\varphi_k : (j_k)_\#(P_k) \to \bar{\Lambda}$ and, for any such φ_k, there is a one-to-one correspondence*

$$\mathrm{Aut}(P_1)\backslash\bar{\Lambda}^\times/\,\mathrm{Aut}(P_2) \leftrightarrow \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)$$

where the maps $\mathrm{Aut}(P_k) \to \bar{\Lambda}^\times$ are defined via the map $\varphi_k(1 \otimes -) : P_k \to \bar{\Lambda}$.

Proposition 2.3. *For $k = 1, 2$, let $P_k \in \mathrm{Cls}(\Lambda_k)$. Then there is a one-to-one correspondence*

$$\frac{K_1(\bar{\Lambda})}{K_1(\Lambda_1) \times K_1(\Lambda_2)} \leftrightarrow C_{\mathcal{R}}^{-1}([P_1], [P_2]).$$

For $i = 1, 2$, Morita equivalence gives us maps

$$\psi_i : \mathrm{Aut}(P_i) = \mathrm{End}(P_i)^\times \to K_1(\mathrm{End}(P_i)) \cong K_1(\Lambda_i).$$

These maps fit into a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Aut}(P_i) & \longrightarrow & K_1(\mathrm{End}(P_i)) & \xrightarrow{\cong} & K_1(\Lambda_i) \\
\downarrow{\scriptstyle h_\#} & & & \searrow{\scriptstyle K_1(h_\#)} & \downarrow{\scriptstyle K_1(h)} \\
\bar{\Lambda}^\times & & \longrightarrow & & K_1(\bar{\Lambda})
\end{array}$$

where $h = j_1$ or j_2 respectively (see, for example, [46, Corollary A17]). We can therefore define a map

$$\Psi_{P_1,P_2} : \mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times / \mathrm{Aut}(P_2) \twoheadrightarrow \frac{K_1(\bar{\Lambda})}{K_1(\Lambda_1) \times K_1(\Lambda_2)}$$

which can be shown to coincide with the map induced by $[\,\cdot\,]$ under the equivalences given in Propositions 2.2 and 2.3. Hence, by the previous discussion, we have that Λ has locally free cancellation if and only if Ψ_{P_1,P_2} is a bijection for all P_1, P_2 such that $P_1 = (i_1)_\#(P)$ and $P_2 = (i_2)_\#(P)$ for some $P \in \mathrm{Cls}(\Lambda)$.

Now consider the constant $K_{\mathcal{R}} = |\frac{K_1(\bar{\Lambda})}{K_1(\Lambda_1)\times K_1(\Lambda_2)}|$ associated to $\mathcal{R}$. It follows from Lemma 2.1 that $[\,\cdot\,]_\Lambda$ is surjective and so $|\mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times / \mathrm{Aut}(P_2)| \geqslant K_{\mathcal{R}}$.

Lemma 2.4. *Let $P_1 \in \mathrm{Cls}(\Lambda_1)$ and $P_2 \in \mathrm{Cls}(\Lambda_2)$ and suppose that*

$$|\mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times / \mathrm{Aut}(P_2)| = K_{\mathcal{R}}.$$

Then $|\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \cap \mathrm{Cls}^{[\tilde{P}]}(\Lambda)| = 1$ for all $[\tilde{P}] \in C_{\mathcal{R}}^{-1}([P_1],[P_2])$.

Proof. By Propositions 2.2 and 2.3, we have that

$$|\mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times / \mathrm{Aut}(P_2)| = |\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)|, \quad K_{\mathcal{R}} = |C_{\mathcal{R}}^{-1}([P_1],[P_2])|$$

and so $|\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)| = |C_{\mathcal{R}}^{-1}([P_1],[P_2])|$ by our hypothesis. By Lemma 2.1, this implies that we have a bijection

$$[\,\cdot\,]_\Lambda : \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \to C_{\mathcal{R}}^{-1}([P_1],[P_2]).$$

The result follows since $[\,\cdot\,]_\Lambda^{-1}([\tilde{P}]) = \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \cap \mathrm{Cls}^{[\tilde{P}]}(\Lambda)$. □

We will now prove the following, which is the main result of this section.

Theorem 2.5. *Let $\mathcal{R}$ be as above, let $P \in \mathrm{Cls}(\Lambda)$ and let $P_k = (i_k)_\#(P) \in \mathrm{Cls}(\Lambda_k)$ for $k = 1, 2$. Suppose that $|\operatorname{Aut}(\widetilde{P}_1)\backslash \bar{\Lambda}^\times / \operatorname{Aut}(\widetilde{P}_2)| = K_{\mathcal{R}}$ for all $\widetilde{P}_1 \in \mathrm{Cls}^{[P_1]}(\Lambda_1)$ and $\widetilde{P}_2 \in \mathrm{Cls}^{[P_2]}(\Lambda_2)$. Then $\mathrm{Cls}_{\mathcal{R}}$ induces a bijection*

$$\mathrm{Cls}^{[P]}(\Lambda) \cong \mathrm{Cls}^{[P_1]}(\Lambda_1) \times \mathrm{Cls}^{[P_2]}(\Lambda_2).$$

Subject to the hypothesis, this implies that $[P] \in C(\Lambda)$ has cancellation if and only if $[P_1] \in C(\Lambda_1)$ has cancellation and $[P_2] \in C(\Lambda_2)$ has cancellation.

Proof. Recall that, by Lemma 2.1, there is a surjection

$$\mathrm{Cls}_{\mathcal{R}} |_{\mathrm{Cls}^{[P]}(\Lambda)} \colon \mathrm{Cls}^{[P]}(\Lambda) \to \mathrm{Cls}^{[P_1]}(\Lambda_1) \times \mathrm{Cls}^{[P_2]}(\Lambda_2)$$

which has fibres $(\mathrm{Cls}_{\mathcal{R}} |_{\mathrm{Cls}^{[P]}(\Lambda)})^{-1}(\widetilde{P}_1, \widetilde{P}_2) = \mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, \widetilde{P}_2) \cap \mathrm{Cls}^{[P]}(\Lambda)$. By Lemma 2.4, we have that $|\mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, \widetilde{P}_2) \cap \mathrm{Cls}^{[P]}(\Lambda)| = 1$ for all $\widetilde{P}_1 \in \mathrm{Cls}^{[P_1]}(\Lambda_1)$ and $\widetilde{P}_2 \in \mathrm{Cls}^{[P_2]}(\Lambda_2)$ and this implies that $\mathrm{Cls}_{\mathcal{R}} |_{\mathrm{Cls}^{[P]}(\Lambda)}$ is a bijection. □

2.2 Main cancellation theorem for orders in semisimple $\mathbb{Q}$-algebras

As before, let Λ be a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A, and let $\mathcal{R} = \mathcal{R}(\Lambda, A_1, A_2)$ denote the fibre square corresponding to a splitting $A \cong A_1 \times A_2$ of $\mathbb{Q}$-algebras.

The main aim of this section will be to prove the following which is our main cancellation theorem for orders in semisimple $\mathbb{Q}$-algebras.

Theorem 2.6. *Let $P \in \mathrm{Cls}(\Lambda)$ and let $P_1 = (i_1)_\#(P) \in \mathrm{Cls}(\Lambda_1)$. Suppose the following conditions are satisfied by $\mathcal{R}$:*

(i) Λ_2 *satisfies the Eichler condition*

(ii) *The map* $\Lambda_1^\times \to K_1(\Lambda_1)$ *is surjective*

(iii) *Every* $\widetilde{P}_1 \in \text{Cls}^{[P_1]}(\Lambda_1)$ *is represented by a two-sided ideal* $I \subseteq \Lambda_1$ *which is generated by central elements.*

Then the map $(i_1)_\# : \text{Cls}^{[P]}(\Lambda) \to \text{Cls}^{[P_1]}(\Lambda_1)$ *is a bijection.*

Subject to the hypothesis, this implies that $[P] \in C(\Lambda)$ has cancellation if and only if $[P_1] \in C(\Lambda_1)$ has cancellation. Note that, by Corollary 1.22, a two-sided ideal $I \subseteq \Lambda_1$ is locally free as a (Λ_1, Λ_1)-bimodule if and only if it is generated by central elements. In particular, hypothesis (iii) is satisfied if I is generated by central elements.

The proof will be given in Section 2.2.4 and will depend on Lemmas 2.9, 2.10 and 2.11 which roughly correspond to the three conditions in Theorem 2.6. The first lemma is due to Swan.

2.2.1 The Eichler condition

The following can be shown by combining [46, Corollary A17, Theorem A18].

Theorem 2.7. *Let* Λ *be a* $\mathbb{Z}$*-order in a semisimple* $\mathbb{Q}$*-algebra, let* $f : \Lambda \twoheadrightarrow \bar{\Lambda}$ *be a ring epimorphism for* $\bar{\Lambda}$ *finite and let* $P \in \text{Cls}(\Lambda)$*. Then* $f(\text{Aut}(P)) \leqslant \bar{\Lambda}^\times$ *is a normal subgroup and the map* $\text{Aut}(P) \to K_1(\bar{\Lambda})$ *induces an isomorphism*

$$\bar{\Lambda}^\times / \text{Aut}(P) \cong K_1(\bar{\Lambda})/K_1(\Lambda).$$

Remark 2.8. By Theorem 2.5, it can be shown that this implies Theorem 1.10.

We can apply this to the case where $\mathcal{R} = \mathcal{R}(\Lambda, A_1, A_2)$ for a splitting of $\mathbb{Q}$-algebras $A \cong A_1 \times A_2$. In the case where either Λ_1 or Λ_2 satisfies the Eichler condition, we will adopt the convention that Λ_2 satisfies the Eichler condition.

Lemma 2.9. *Let $\mathcal{R}$ be as above. If Λ_2 satisfies the Eichler condition then, for all $P_k \in \mathrm{Cls}(\Lambda_k)$ for $k = 1, 2$, there is a bijection $\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \cong \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, \Lambda_2)$.*

Proof. By Theorem 2.7, there are isomorphisms $\bar{\Lambda}^\times / \mathrm{Aut}(P_2) \cong K_1(\bar{\Lambda})/K_1(\Lambda_2) \cong \bar{\Lambda}^\times/\Lambda^\times$. This implies that there is a bijection

$$\mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times / \mathrm{Aut}(P_2) \cong \mathrm{Aut}(P_1)\backslash \bar{\Lambda}^\times/\Lambda_2^\times$$

which is equivalent to $\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \cong \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, \Lambda_2)$ by Proposition 2.2. ☐

2.2.2 Unit representation for K_1

Lemma 2.10. *Let $\mathcal{R}$ be as above and suppose:*

(i) *Λ_2 satisfies the Eichler condition*

(ii) *The map $\Lambda_1^\times \to K_1(\Lambda_1)$ is surjective.*

Then $|\Lambda_1^\times \backslash \bar{\Lambda}^\times/\Lambda_2^\times| = K_{\mathcal{R}}$.

Proof. Since $m_{\mathbb{H}}(\Lambda_2) = 0$, Theorem 2.7 implies that the map $\bar{\Lambda}^\times \to K_1(\bar{\Lambda})$ induces an isomorphism $\bar{\Lambda}^\times/\Lambda_2^\times \cong K_1(\bar{\Lambda})/K_1(\Lambda_2)$. The relevant maps fit into a commutative diagram

$$\begin{array}{ccc} \Lambda_1^\times & \longrightarrow & \bar{\Lambda}^\times/\Lambda_2^\times \\ \downarrow & & \downarrow \cong \\ K_1(\Lambda_1) & \longrightarrow & \dfrac{K_1(\bar{\Lambda})}{K_1(\Lambda_2)} \end{array}$$

and so $\mathrm{Im}(\Lambda_1^\times \to \bar{\Lambda}^\times/\Lambda_2^\times) = \mathrm{Im}(K_1(\Lambda_1) \to K_1(\bar{\Lambda})/K_1(\Lambda_2))$ since the map $\Lambda_1^\times \to K_1(\Lambda_1)$ is surjective. Hence we have $|\Lambda_1^\times \backslash \bar{\Lambda}^\times/\Lambda_2^\times| = K_{\mathcal{R}}$. ☐

2.2.3 Two-sided ideals over orders in semisimple $\mathbb{Q}$-algebras

The main result of this section is as follows. This gives a method of constructing locally free two-sided ideals over Λ from locally free two-sided ideals over the projections Λ_k subject to certain conditions.

Lemma 2.11. *Let $\mathcal{R}$ be as above and, for $k = 1, 2$, suppose $I_k \subseteq \Lambda_k$ is a two-sided ideal such that $I_k \in \mathrm{Cls}(\Lambda_k)$ and which is generated by central elements. Then there exists a two-sided ideal $I \subseteq \Lambda$ with $I \in \mathrm{Cls}(\Lambda)$ such that*

(i) *For $k = 1, 2$, there is a (Λ_k, Λ)-bimodule isomorphism*

$$(i_k)_\#(I) \cong (I_k)_{i_k}$$

(ii) *There is a bijection*

$$I \otimes_\Lambda - : \mathrm{Cls}^{-1}_{\mathcal{R}}(P_1, P_2) \to \mathrm{Cls}^{-1}_{\mathcal{R}}(I_1 \otimes_{\Lambda_1} P_1, I_2 \otimes_{\Lambda_2} P_2).$$

Remark 2.12. This actually holds under the weaker hypothesis that $(I_k)_{(p)} \cong (\Lambda_k)_{(p)}$ are isomorphic as $((\Lambda_k)_{(p)}, (\Lambda_k)_{(p)})$-bimodules for all primes $p \mid |\bar{\Lambda}|$.

We will begin by proving the following embedding result, which can be viewed as a generalisation of [40, Theorem A] to bimodules.

Proposition 2.13. *Let $I \subseteq \Lambda$ be a two-sided ideal generated by central elements such that $I \in \mathrm{Cls}(\Lambda)$. Then, for all $n \neq 0$, there exists a two-sided ideal $J \subseteq \Lambda$ such that $I \cong J$ as (Λ, Λ)-bimodules and $J \cap \mathbb{Z}$ is coprime to (n).*

Note that, by Corollary 1.22, this holds whenever I is locally free as a bimodule. In order to prove this, we will need the following two lemmas.

Lemma 2.14. *Let $n \neq 0$ be an integer and let $I \subseteq \Lambda$ be a two-sided ideal such that $I \in \mathrm{Cls}(\Lambda)$ and, for all $p \mid n$ prime, there is a $(\Lambda_{(p)}, \Lambda_{(p)})$-bimodule isomorphism $I_{(p)} \cong \Lambda_{(p)}$. Then there is a $(\Lambda/n, \Lambda/n)$-bimodule isomorphism $f : \Lambda/n \to I/n$, $1 \mapsto [a]$ for some $a \in Z(\Lambda) \cap I$.*

Proof. For each $p \mid n$ prime, consider the bimodule isomorphism $f : \Lambda_{(p)} \to I_{(p)}$, $1 \mapsto [a_p]$ for some $a_p \in I_{(p)}$. There exists $m \neq 0$ such that $ma_p \in I \subseteq I_{(p)}$ and $f' : \Lambda_{(p)} \to I_{(p)}$, $1 \mapsto [ma_p]$ is still a bimodule isomorphism, and so we can assume that $a_p \in I \subseteq I_{(p)}$. Since f is a bimodule isomorphism, we have that $a_p \in Z(\Lambda_{(p)})$ and so $a_p \in Z(\Lambda) \cap I$.

Now, f induces a bimodule isomorphism $\Lambda_{(p)}/p \cong I_{(p)}/p$. Since $\mathbb{Z}_{(p)}/p \cong \mathbb{Z}/p$, there are bimodule isomorphisms $\Lambda/p \cong \Lambda_{(p)}/p$ and $I/p \cong I_{(p)}/p$ and so there exists a bimodule isomorphism $f_p : \Lambda/p \to I/p$, $1 \mapsto [a_p]$. It is straightforward to check that the map $\widetilde{f}_{p^i} : \Lambda/p^i \to I/p^i$, $1 \mapsto [a_p]$ is also a bimodule isomorphism for all $i \geqslant 1$.

In general, suppose $n = p_1^{n_1} \cdots p_k^{n_k}$ for distinct primes p_i, and integers $n_i \geqslant 1$ and $k \geqslant 1$. By the Chinese remainder theorem, $\mathbb{Z}/n \cong \mathbb{Z}/p_1^{n_1} \times \cdots \times \mathbb{Z}/p_k^{n_k}$. By tensoring with Λ or I, we see that there are bimodule isomorphisms $\Lambda/n \cong \Lambda/p_1^{n_1} \times \cdots \times \Lambda/p_k^{n_k}$ and $I/n \cong I/p_1^{n_1} \times \cdots \times I/p_k^{n_k}$. Hence, by the bimodule isomorphism constructed above, there is a bimodule isomorphism $f : \Lambda/n \to I/n$, $1 \mapsto [a]$ for some $a \in \mathbb{Z} \cdot \langle a_{p_1}, \cdots, a_{p_k} \rangle \subseteq Z(\Lambda) \cap I$. ☐

Lemma 2.15. *Let $n \neq 0$ be an integer, let $I \subseteq \Lambda$ be a two-sided ideal such that $I \in \mathrm{Cls}(\Lambda)$, and let $f : \Lambda/n \to I/n$, $1 \mapsto [a]$ be a $(\Lambda/n, \Lambda/n)$-bimodule isomorphism for some $a \in Z(\Lambda) \cap I$. Then $\Lambda a \cong \Lambda$ as a (Λ, Λ)-bimodule and there exists $m \neq 0$ such that $mI \subseteq \Lambda a$ and $(n, m) = 1$.*

Proof. Since $a \in Z(\Lambda)$, Λa is a bimodule and there is a map of bimodules $\varphi : \Lambda \to \Lambda a$, $x \mapsto xa$. To see that φ is a bimodule isomorphism, note that it is clearly surjective and is injective since f is a bijection.

Since f is an isomorphism, we have $I = \Lambda a + nI$ as ideals in Λ and so there is an equality of finitely generated abelian groups $I/\Lambda a = n \cdot I/\Lambda a$. Hence, as an abelian group, $I/\Lambda a$ is finite of order m where $(n, m) = 1$. Since $m \cdot I/\Lambda a = 0$, we have that $mI \subseteq \Lambda a$. $\square$

Proof of Proposition 2.13. Let $I \subseteq \Lambda$ be a two-sided ideal such that $I \in \mathrm{Cls}(\Lambda)$ and which is generated by central elements. By Corollary 1.22, this implies that $\Lambda_{(p)} \cong I_{(p)}$ are isomorphic as bimodules for all p. By Lemma 2.14, there is a $(\Lambda/n, \Lambda/n)$-bimodule isomorphism $f : \Lambda/n \to I/n$, $1 \mapsto [a]$ for some $a \in Z(\Lambda) \cap I$. By Lemma 2.15, this implies that there is a (Λ, Λ)-bimodule isomorphism $\psi : \Lambda a \to \Lambda$ which sends $xa \mapsto x$ and there exists $m \neq 0$ with $(n, m) = 1$ and $mI \subseteq \Lambda a$. Let $J = \psi(mI) \subseteq \psi(\Lambda a) = \Lambda$, which is a two-sided ideal since ψ is a map of bimodules. Finally, note that the map $I \to J$, $x \mapsto \psi(mx)$ is a (Λ, Λ)-bimodule isomorphism, and $m = \psi(ma) \in \psi(mI) = J$ implies that $J \cap \mathbb{Z} = (m_0)$ where $m_0 \mid m$ and so $(n, m_0) = 1$. $\square$

We will also need the following lemma. In the statement of Lemma 2.11, this shows that part (ii) follows from part (i).

Lemma 2.16. *Let $\mathcal{R}$ be as above and suppose $I \subseteq \Lambda$, $I_k \subseteq \Lambda_k$ are two-sided ideals such that $I \in \mathrm{Cls}(\Lambda)$, $I_k \in \mathrm{Cls}(\Lambda_k)$ and $(i_k)_\#(I) \cong (I_k)_{i_k}$ are isomorphic as (Λ_k, Λ)-bimodules for $k = 1, 2$. Then there is a bijection*

$$I \otimes_\Lambda - : \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2) \to \mathrm{Cls}_{\mathcal{R}}^{-1}(I_1 \otimes_{\Lambda_1} P_1, I_2 \otimes_{\Lambda_2} P_2).$$

Proof. By Proposition 1.19, there exists a two-sided ideal $J \subseteq \Lambda$ such that $J \in \mathrm{Cls}(\Lambda)$ and $I \otimes_\Lambda J \cong \Lambda \cong J \otimes_\Lambda I$ as (Λ, Λ)-bimodules. In particular, I is invertible as a bimodule and determines a bijection $I \otimes_\Lambda - : \mathrm{Cls}(\Lambda) \to \mathrm{Cls}(\Lambda)$ with inverse $J \otimes_\Lambda -$.

Now suppose $P \in \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)$, i.e. that $(i_k)_\#(P) \cong P_k$ are isomorphic as left Λ_k modules for $k = 1, 2$. Then

$$(i_k)_\#(I \otimes_\Lambda P) = \Lambda_k \otimes_\Lambda (I \otimes_\Lambda P) \cong (I_k)_{i_k} \otimes_\Lambda P$$
$$\cong (I_k \otimes_{\Lambda_k} \Lambda_k) \otimes_\Lambda P \cong I_k \otimes_{\Lambda_k} P_k$$

and so $(I \otimes_\Lambda -)(\mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)) \subseteq \mathrm{Cls}_{\mathcal{R}}^{-1}(I_1 \otimes_{\Lambda_1} P_1, I_2 \otimes_{\Lambda_2} P_2)$. Similarly, we can show that $(J \otimes_\Lambda -)(\mathrm{Cls}_{\mathcal{R}}^{-1}(I_1 \otimes_{\Lambda_1} P_1, I_2 \otimes_{\Lambda_2} P_2)) \subseteq \mathrm{Cls}_{\mathcal{R}}^{-1}(P_1, P_2)$. Hence $I \otimes_\Lambda -$ restricts to the required bijection. $\square$

Finally, we will now use Proposition 2.13 to complete the proof of Lemma 2.11.

Proof of Lemma 2.11. By Lemma 2.16, it suffices to prove part (i) only. Let $k = 1$ or 2. By Proposition 2.13 we can assume, by replacing I_k with a bimodule isomorphic two-sided ideal, that $I_k \cap \mathbb{Z}$ is coprime to $|\bar{\Lambda}|$. Let $n \in I_k \cap \mathbb{Z}$ be such that $n \neq 0$ and let $m \in \mathbb{Z}$ be such that $nm \equiv 1 \mod |\bar{\Lambda}|$, which exists since $(|\bar{\Lambda}|, n) = 1$. Consider the left $\bar{\Lambda}$ module homomorphisms

$$\psi_k : \bar{\Lambda} \to \bar{\Lambda} \otimes_{\Lambda_k} I_k, \quad 1 \mapsto m \otimes n, \qquad \varphi_k : \bar{\Lambda} \otimes_{\Lambda_k} I_k \to \bar{\Lambda}, \quad x \otimes y \mapsto x j_k(y)$$

where $x \in \bar{\Lambda}$ and $y \in I_k \subseteq \Lambda_k$. Note that $\varphi_k(\psi_k(1)) = m j_k(n) = mn = 1 \in \bar{\Lambda}$ and $\psi_k(\varphi_k(x \otimes y)) = (x j_k(y) m) \otimes n = xm \otimes yn = xmn \otimes y = x \otimes y$. This shows that ψ_k and φ_k are mutual inverses and so are both bijections.

Now let $M = \{(x_1, x_2) \in I_1 \times I_2 : \varphi_1(1 \otimes x_1) = \varphi_2(1 \otimes x_2)\} \subseteq \Lambda_1 \times \Lambda_2$, which is a left Λ-module under the action $\lambda \cdot (x_1, x_2) = (i_1(\lambda)x_1, i_2(\lambda)x_2)$ for $\lambda \in \Lambda$. This coincides with the standard pullback construction for projective module over a Milnor square $\mathcal{R}$ [33]. However, for the φ_k chosen above, we further have $M = \{(x_1, x_2) \in I_1 \times I_2 : j_1(x_1) = j_2(x_2)\}$ and so M is a (Λ, Λ)-bimodule with action $\lambda \cdot (x_1, x_2) \cdot \mu = (i_1(\lambda) \cdot x_1 \cdot i_1(\mu), i_2(\lambda) \cdot x_2 \cdot i_2(\mu))$ for $\lambda, \mu \in \Lambda$.

Note that $M \subseteq \Lambda_1 \times \Lambda_2 \subseteq \mathbb{Q} \cdot (\Lambda_1 \times \Lambda_2) = \mathbb{Q} \cdot \Lambda$ and so there exists $k \in \mathbb{Z}$ with $k \neq 0$ for which $kM \subseteq \Lambda$. Hence $I = kM$ is a two-sided ideal in Λ which is bimodule isomorphic to M. Now note that $M \in \mathrm{Cls}(\Lambda)$ as a left Λ-module by [39, Lemma 4.4], and so $I \in \mathrm{Cls}(\Lambda)$. Finally note that, by the proof of [33, Theorem 2.3], the map $f : (i_k)_\#(M) \to (I_k)_{i_k}$ which sends $\lambda_k \otimes (x_1, x_2) \mapsto \lambda_k \cdot x_k$ for $\lambda_k \in \Lambda_k$ is a left Λ_k module isomorphism. This is also a right Λ-module isomorphism since

$$\begin{aligned} f((\lambda_k \otimes (x_1, x_2)) \cdot \lambda) &= f(\lambda_k \otimes (x_1 \cdot i_1(\lambda), x_2 \cdot i_2(\lambda))) \\ &= \lambda_k \cdot x_k \cdot i_k(\lambda) = f(\lambda_k \otimes (x_1, x_2)) \cdot i_k(\lambda) \end{aligned}$$

and so $(i_k)_\#(I) \cong (i_k)_\#(M) \cong (I_k)_{i_k}$ are bimodule isomorphic, as required. □

2.2.4 Proof of Theorem 2.6

By Theorem 2.5 and Lemma 2.9, it suffices to show that $\# \mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, P_2) = K_{\mathcal{R}}$ for all $\widetilde{P}_1$. By Lemma 2.9 (ii), there is a bijection $\mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, P_2) \cong \mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, \Lambda_2)$ and, by Lemma 2.10, we have that $\# \mathrm{Cls}_{\mathcal{R}}^{-1}(\Lambda_1, \Lambda_2) = K_{\mathcal{R}}$. Hence it suffices to show that, for all $\widetilde{P}_1$, there is a bijection $\mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, \Lambda_2) \cong \mathrm{Cls}_{\mathcal{R}}^{-1}(\Lambda_1, \Lambda_2)$.

By assumption, there exists a two-sided ideal $I_1 \subseteq \Lambda_1$ such that $I_1 \cong \widetilde{P}_1$ as left Λ_1 modules and such that $(I_1)_p \cong (\Lambda_1)_p$ are isomorphic as bimodules for all primes $p \mid |\bar{\Lambda}|$. By Lemma 2.11, there exists a two-sided ideal $I \subseteq \Lambda$ such that $I \in \mathrm{Cls}(\Lambda)$ and $(i_1)_\#(I) \cong (I_1)_{i_1}$ as (Λ_1, Λ)-bimodules and $(i_2)_\#(I) \cong (\Lambda_2)_{i_2}$ as (Λ_2, Λ)-bimodules. By Lemma 2.16, this induces a bijection

$$I \otimes_\Lambda - : \mathrm{Cls}_{\mathcal{R}}^{-1}(\Lambda_1, \Lambda_2) \to \mathrm{Cls}_{\mathcal{R}}^{-1}(I_1, \Lambda_2),$$

and so there a bijections $\mathrm{Cls}_{\mathcal{R}}^{-1}(\widetilde{P}_1, \Lambda_2) \cong \mathrm{Cls}_{\mathcal{R}}^{-1}(\Lambda_1, \Lambda_2)$, as required.

Chapter 3

Cancellation for modules over integral group rings

The aim of this chapter will be to specialise Theorem 2.6 to the case of integral group rings $\mathbb{Z}G$. We will then prove Theorem A by combining this with an additional cancellation theorem of R. G. Swan which is given in Theorem 3.11.

3.1 Binary polyhedral groups and the Eichler condition

Recall from Proposition 1.8 that, for a finite group G, a $\mathbb{Z}G$ module is projective if and only if it is locally free. In particular, $\mathbb{Z}G$ has projective cancellation if and only if $\mathbb{Z}G$ has locally free cancellation. In order to determine when $\mathbb{Z}G$ has projective cancellation it suffices, by Theorem 1.10, to consider the case where G fails the Eichler condition, i.e. if no copy of $\mathbb{H} = M_1(\mathbb{H})$ appears in the Wedderburn decomposition of the real group ring $\mathbb{R}G$.

We will now determine the finite groups G which fail the Eichler condition.

Firstly note that a *binary polyhedral group* is a non-cyclic finite subgroup of $\mathbb{H}^\times$ where $\mathbb{H}$ is the real quaternions. Since there is a double cover of Lie groups $f : \mathbb{H}^\times \cong S^3 \to SO(3)$, the binary polyhedral groups are the preimages of the non-cyclic finite subgroups of SO(3) which are the dihedral groups D_{2n} of order $2n$ for $n \geqslant 2$ and the symmetry groups of platonic solids.

The double cover of D_{2n} is the quaternion group of order $4n$ for $n \geqslant 2$:

$$Q_{4n} = \langle x, y \mid x^2 = y^2, yxy^{-1} = x^{-1} \rangle.$$

The symmetry groups of platonic solids are the tetrahedral, octahedral and icosahedral groups T, O, I. These have double covers the binary tetrahedral, binary octahedral and binary icosahedral groups, which have presentations:

$$\begin{aligned}
\widetilde{T} &= \langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle \\
\widetilde{O} &= \langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle \\
\widetilde{I} &= \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle.
\end{aligned}$$

The following is well-known [46] though a proof does not appear explicitly in the literature except in the backward direction [9, p305].

Proposition 3.1. *G satisfies the Eichler condition if and only if G has no quotient which is a binary polyhedral group.*

Proof. If G fails the Eichler condition, the Wedderburn decomposition gives a map $G \to \mathbb{H}^\times$. Since G is an $\mathbb{R}$-basis for G, the image must contain an $\mathbb{R}$-basis for $\mathbb{H}$. Since $\mathbb{H}$ is non-commutative, the image must be non-abelian and so a binary polyhedral group. Conversely, a quotient of G into a binary polyhedral group gives a representation $G \to \mathbb{H}^\times$ which does not split over $\mathbb{R}$

or $\mathbb{C}$ since the image is non-abelian. Hence the representation is irreducible and so represents a term in the Wedderburn decomposition. □

Remark 3.2. It is possible to compute $m_{\mathbb{H}}(G)$ in terms of the quotients of G which are binary polyhedral groups. We will consider this in Section 4.2.

3.2 Proof of Theorem A for quaternionic quotients

In order to determine when $\mathbb{Z}G$ has projective cancellation it suffices, by Theorem 1.10, to consider the case where G fails the Eichler condition. By Proposition 3.1, this implies that G has a quotient H which is a binary polyhedral group and so, by Corollary 1.6, there is a Milnor square:

$$\mathcal{R}_{G,H} = \begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ \mathbb{Z}H & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})[H] \end{array}$$

where $\Lambda = \mathbb{Z}G/\Sigma_N$ and $n = |G|/|H|$.

We now aim to prove the following by specialising Theorem 2.6 to $\mathcal{R}_{G,H}$.

Theorem 3.3. *Let G be a finite group and suppose G has a quotient H such that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ and H is of the form*

$$Q_8,\ Q_{12},\ Q_{16},\ Q_{20}.$$

Then $\mathbb{Z}G$ has projective cancellation.

Let G is a finite group with a quotient H such that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ and $H = Q_8$, Q_{12}, Q_{16} or Q_{20}, and suppose conditions (i)-(iii) of Theorem 2.6 hold for $\mathcal{R}_{G,H}$. If $P_G \in \mathrm{Cls}(\mathbb{Z}G)$ and $P_H = (i_1)_\#(P_G) \in \mathrm{Cls}(\mathbb{Z}H)$, then Theorem 2.6 implies that there is a bijection

$$(i_1)_\# : \mathrm{Cls}^{[P_G]}(\mathbb{Z}G) \to \mathrm{Cls}^{[P_H]}(\mathbb{Z}H).$$

It was shown by Swan in [46, Theorem I] that $\mathbb{Z}H$ has projective cancellation for $H = Q_8,\ Q_{12},\ Q_{16},\ Q_{20}$. This implies that $\#\,\mathrm{Cls}^{[P_H]}(\mathbb{Z}H) = 1$ and so $\#\,\mathrm{Cls}^{[P_G]}(\mathbb{Z}G) = 1$ holds for all $P_G \in \mathrm{Cls}(\mathbb{Z}G)$. In particular, this would imply that $\mathbb{Z}G$ has projective cancellation.

Hence, in order to complete the proof of Theorem 3.3, it suffices to verify that conditions (i)-(iii) of Theorem 2.6 hold for $\mathcal{R}_{G,H}$. Since $\mathbb{R}$ is flat, $\mathcal{R}_{G,H} \otimes \mathbb{R}$ is a pullback diagram and so $\mathbb{R}G \cong \mathbb{R}H \times (\Lambda \otimes \mathbb{R})$. Since $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$, this implies that Λ satisfies the Eichler condition and so (i) applies automatically. Conditions (ii) and (iii) will be verified in the following two sections.

3.2.1 Unit representation for quaternion groups

We say that that $K_1(\mathbb{Z}G)$ is *represented by units* when the map $\mathbb{Z}G^\times \to K_1(\mathbb{Z}G)$ is surjective. The problem of when $K_1(\mathbb{Z}G)$ is represented by units for G a finite group was studied in detail by B. Magurn, R. Oliver and L. Vaserstein [27]. In particular, they showed:

Lemma 3.4 ([27, Theorems 7.15 - 7.18])**.**

(i) *$K_1(\mathbb{Z}Q_{4n})$ is represented by units if $n = 2^k$ or if n is prime with $\#C(\mathbb{Z}[\zeta_n])$ odd.*

(ii) *$K_1(\mathbb{Z}Q_{116})$ is not represented by units.*

If $p = 3$ or 5, then it is well known that $\#C(\mathbb{Z}[\zeta_p]) = 1$. Hence $K_1(\mathbb{Z}H)$ is represented by units when $H = Q_8$, Q_{12}, Q_{16} or Q_{20}, which implies that $\mathcal{R}_{G,H}$

satisfies conditions (ii).

3.2.2 Projective modules over quaternion groups

We say that a locally free module $P \in \text{Cls}(\Lambda)$ is *represented by a two-sided ideal* $I \subseteq \Lambda$ if $P \cong I$ are isomorphic as left Λ-modules. The aim of this section will be to prove the following, which implies that $\mathcal{R}_{G,H}$ satisfies condition (iii).

Proposition 3.5. *For* $2 \leqslant n \leqslant 5$, *every* $P \in \text{Cls}(\mathbb{Z}Q_{4n})$ *is represented by a two-sided ideal* $I \subseteq \mathbb{Z}Q_{4n}$ *which is generated by central elements. In particular,* P *is locally free as a* $(\mathbb{Z}Q_{4n}, \mathbb{Z}Q_{4n})$*-bimodule.*

We begin by discussing two families of two-sided ideals which will suffice to represent all projective modules $P \in \text{Cls}(\mathbb{Z}Q_{4n})$ in the case $2 \leqslant n \leqslant 5$. Note that, by [46, Theorem III], we have that $\# \text{Cls}(\mathbb{Z}Q_{4n}) = 2$ for $2 \leqslant n \leqslant 5$.

Swan modules

Let G be a finite group, let $N = \sum_{g \in G} g$ denote the group norm and let $r \in \mathbb{Z}$ with $(r, |G|) = 1$. Then the two-sided ideal $(N, r) \subseteq \mathbb{Z}G$ is projective as a left $\mathbb{Z}G$ module and is known as a *Swan module*. If $r \equiv s \mod |G|$, then $(N, r) \cong (N, s)$ by [41, Lemma 6.1] and so we often write $r \in (\mathbb{Z}/|G|)^\times$. Note that $N, r \in Z(\mathbb{Z}G)$ and so (N, r) is generated by central elements.

By [46, Theorem VI], we have that $[(N, 3)] \neq 0 \in C(\mathbb{Z}Q_{2^n})$ for $n \geqslant 3$ where $(N, 3)$ is a Swan module. Since $\# \text{Cls}(\mathbb{Z}Q_{2^n}) = 2$ for $n = 3, 4$, this implies that:

$$\text{Cls}(\mathbb{Z}Q_8) = \{\mathbb{Z}Q_8, (N, 3)\}, \quad \text{Cls}(\mathbb{Z}Q_{16}) = \{\mathbb{Z}Q_{16}, (N, 3)\}$$

which implies Proposition 3.5 for the groups Q_8 and Q_{16}.

Two-sided ideals of Beyl and Waller

In order to prove Proposition 3.5 for the groups Q_{12} and Q_{20}, we will now consider a family of projective two-sided ideals in $\mathbb{Z}Q_{4n}$ which were first introduced by R. Beyl and N. Waller in [1].

For $n \geqslant 2$, define $P_{a,b} = (a + by, 1 + x) \subseteq \mathbb{Z}Q_{4n}$ for $a, b \in \mathbb{Z}$ such that $(a^2 + b^2, 2n) = 1$ if n is odd and $(a^2 - b^2, 2n) = 1$ if n is even. It follows from [1, Proposition 2.1] that $P_{a,b}$ is a two-sided ideal and is projective as a left $\mathbb{Z}Q_{4n}$ module. For $\alpha \in \mathbb{Z}Q_{4n}$, let $(\alpha) \subseteq \mathbb{Z}Q_{4n}$ denote the two-sided ideal generated by α. If n is odd, then there is a Milnor square

$$\mathcal{R} = \quad \begin{array}{ccc} \mathbb{Z}Q_{4n}/(x^n+1) & \xrightarrow{i_2} & \mathbb{Z}[\zeta_{2n}, j] \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_2} \\ \mathbb{Z}[j] & \xrightarrow{j_1} & (\mathbb{Z}/n)[j] \end{array} \qquad \begin{array}{ccc} x, y & \longmapsto & \zeta_{2n}, j \\ \downarrow & & \downarrow \\ -1, j & \longmapsto & -1, j \end{array}$$

where $\mathbb{Z}Q_{4n}/(x+1) \cong \mathbb{Z}[j]$ and $\mathbb{Z}Q_{4n}/(x^{n-1} - x^{n-2} + \cdots - 1) \cong \mathbb{Z}[\zeta_{2n}, j] \subseteq \mathbb{H}_{\mathbb{R}}$. If $n = p$ is an odd prime then, by Propositions 2.2 and 2.3, we have that

$$\begin{array}{ccc} \operatorname{Cls}_{\mathcal{R}}^{-1}(\mathbb{Z}[\zeta_{2p}, j], \mathbb{Z}[j]) & \xrightarrow{[\,\cdot\,]} & C_{\mathcal{R}}^{-1}([\mathbb{Z}[\zeta_{2p}, j]], [\mathbb{Z}[j]]) \\ \updownarrow & & \updownarrow \\ \dfrac{\mathbb{F}_p[j]^\times}{\mathbb{Z}[j]^\times \times \mathbb{Z}[\zeta_p, j]^\times} & \xrightarrow{\varphi} & \dfrac{K_1(\mathbb{F}_p[j])}{K_1(\mathbb{Z}[j]) \times K_1(\mathbb{Z}[\zeta_{2p}, j])} \end{array}$$

where φ is induced by the map $\mathbb{F}_p[j]^\times \to K_1(\mathbb{F}_p[j])$. It follows from [1, Proposition 2.2] that $P_{a,b} \in \operatorname{Cls}_{\mathcal{R}}^{-1}(\mathbb{Z}[\zeta_{2n}, j], \mathbb{Z}[j])$ with corresponding element $[a + bj] \in \frac{\mathbb{F}_p[j]^\times}{\mathbb{Z}[j]^\times \times \mathbb{Z}[\zeta_p, j]^\times}$.

This allows us to deduce the following, which is an extension of [2, Theorem 3.11] in the case where $n = p$ is an odd prime.

Lemma 3.6. *Let p be an odd prime with $\#C(\mathbb{Z}[\zeta_p])$ odd and let $P_{a,b} = (a +$*

$by, 1+x) \subseteq \mathbb{Z}Q_{4p}$ for $(a^2+b^2, 4p) = 1$. Then:

(i) *$P_{a,b}$ is free if and only if $p \mid a$ or $p \mid b$*

(ii) *$P_{a,b}$ is stably free if and only if a^2+b^2 is a square mod p.*

Proof. By [27, Lemma 7.5], we have that $\mathbb{Z}[\zeta_p, j]^\times = \langle \mathbb{Z}[\zeta_p]^\times, j\rangle$. Furthermore, the map $\mathbb{Z}[\zeta_p]^\times \to \mathbb{F}_p[j]^\times$ sends ζ_p to 1 and so has image $\mathbb{F}_p^\times$ since units of any length are achievable. This implies that:

$$\frac{\mathbb{F}_p[j]^\times}{\mathbb{Z}[j]^\times \times \mathbb{Z}[\zeta_p, j]^\times} \cong \mathbb{F}_p[j]^\times/\mathbb{F}_p^\times \cdot \langle j \rangle$$

and so $P_{a,b}$ is free if and only if $[a+bj] = 1 \in \mathbb{F}_p[j]^\times/\mathbb{F}_p^\times \cdot \langle j\rangle$.

Since $\mathbb{Z}[j]$ is a Euclidean Domain, we have $K_1(\mathbb{Z}[j]) = \mathbb{Z}[j]^\times = \{\pm 1, \pm j\}$ and, since $\mathbb{F}_p[j]$ is a finite and hence semilocal ring, we have $K_1(\mathbb{F}_p[j]) \cong \mathbb{F}_p[j]^\times$. It follows from [27, Lemmas 7.5/7.6] that, if $\#C(\mathbb{Z}[\zeta_p])$ odd, then $\mathrm{Im}(K_1(\mathbb{Z}[\zeta_p, j])) = \langle \mathbb{F}_p^\times, \mathrm{Ker}(N)\rangle$ where $N : \mathbb{F}_p[j]^\times \to \mathbb{F}_p^\times$, $x + yj \mapsto x^2 + y^2$ is the norm on $\mathbb{F}_p[j]$. In particular, there is an isomorphism:

$$N : \frac{K_1(\mathbb{F}_p[j])}{K_1(\mathbb{Z}[j]) \times K_1(\mathbb{Z}[\zeta_{2p}, j])} \cong \frac{\mathbb{F}_p[j]^\times}{\langle \mathbb{Z}[j]^\times, \mathbb{F}_p^\times, \mathrm{Ker}(N)\rangle} \to \mathbb{F}_p^\times/N(\mathbb{F}_p^\times) \cong \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2.$$

Hence the map φ coincides by the map

$$N : \mathbb{F}_p[j]^\times/\mathbb{F}_p^\times \cdot \langle j\rangle \to \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$$

which is induced by $N : \mathbb{F}_p[j]^\times \to \mathbb{F}_p^\times$, $x + yj \mapsto x^2 + y^2$. In particular, $P_{a,b}$ is stably free if and only if $[a^2+b^2] = 1 \in \mathbb{F}_p^\times(\mathbb{F}_p^\times)^2$. The result follows by evaluating these conditions. ∎

If $p = 3$ or 5 then, as noted in Section 3.2.1, we have that $\#C(\mathbb{Z}[\zeta_p]) = 1$. In the case $p = 3$, we have $(1^2+2^2, 12) = 1$ and $3 \nmid 1, 2$ and, in the case $p = 5$, we have $(1^2+4^2, 20) = 1$ and $5 \nmid 1, 4$. Since $\# \operatorname{Cls}(\mathbb{Z}Q_{4p}) = 2$ for $p = 3, 5$, we have:

$$\operatorname{Cls}(\mathbb{Z}Q_{12}) = \{\mathbb{Z}Q_{12}, P_{1,2}\}, \quad \operatorname{Cls}(\mathbb{Z}Q_{20}) = \{\mathbb{Z}Q_{20}, P_{1,4}\}.$$

We will now show that the $P_{a,b}$ are generated by central elements. Our strategy will be to introduce a new family a two-sided ideals which are generated by central elements and show that the $P_{a,b}$ can be expressed in this form.

If $s \in \mathbb{Z}$ is odd, then let $z_s = (-1)^{\frac{s-1}{2}}(x^{-\frac{s-1}{2}} - x^{-\frac{s-3}{2}} + \cdots + x^{\frac{s-1}{2}})$ and let $\widetilde{N} = 1 - x + x^2 - \cdots - x^{2n-1}$. If $t \in \mathbb{Z}$, then let $\alpha_{s,t} = z_s + t \cdot \widetilde{N}y \in \mathbb{Z}Q_{4n}$.

Lemma 3.7. *Let $n \geqslant 2$ and let $r, s, t \in \mathbb{Z}$ where $(s, 2n) = (r, 2n) = 1$. Then $(\alpha_{s,t}, r) \subseteq \mathbb{Z}Q_{4n}$ is projective as a left $\mathbb{Z}Q_{4n}$ module and $\alpha_{s,t} \in Z(\mathbb{Z}Q_{4n})$.*

Proof. It is easy to see that z_s, $\widetilde{N} \in Z(\mathbb{Z}Q_{4n})$ and that $\widetilde{N}x = \widetilde{N}x^{-1}$. Hence we have $y\alpha_{s,t} = \alpha_{s,t}y$ and $x\alpha_{s,t} = z_s x + t(\widetilde{N}x)y = z_s x + t(\widetilde{N}x^{-1})y = \alpha_{s,t}x$, which implies that $\alpha_{s,t} \in Z(\mathbb{Z}Q_{4n})$. Since $r \in (\alpha_{s,t}, r)$ and $(r, 4n) = 1$, we have that $(\alpha_{s,t}, r)$ is a projective $\mathbb{Z}Q_{4n}$ module by [40, Proposition 7.1]. □

Lemma 3.8. *Let $n \geqslant 2$ and let $a, b \in \mathbb{Z}$ be such that $(a^2 - (-1)^n b^2, 2n) = 1$.*

(i) *If $r = (a^2 - (-1)^n b^2)/\gcd(a, b)$, then there exists $a_0, b_0 \in \mathbb{Z}$ such that $a \equiv a_0 \bmod r$, $b \equiv b_0 \bmod r$, $(a_0, 2n) = 1$ and $2n \mid b_0$*

(ii) *$P_{a,b} = (\alpha_{s,t}, r) \subseteq \mathbb{Z}Q_{4n}$ where $s = a_0$ and $t = b_0/2n$.*

In particular, $P_{a,b}$ is generated by central elements.

Proof. Since $(r, 2n) = 1$, there exists $x, y \in \mathbb{Z}$ such that $rx + 2ny = 1$. Then $a_0 = a + rx(1-a)$ and $b_0 = 2nyb$ have the required properties.

Now recall that $P_{a,b} = (a+by, 1+x)$. If $d = \gcd(a, b)$, then $\frac{1}{d}(a-by) \in \mathbb{Z}Q_{4n}$ and so $r = \frac{1}{d}(a-by) \cdot (a+by) \in P_{a,b}$. In particular, since $a \equiv a_0 \bmod r$ and $b \equiv b_0 \bmod r$, this implies that $P_{a,b} = (a+by, 1+x, r) = (a_0 + b_0 y, 1+x, r)$.

Let $s = a_0$ and $t = b_0/2n$. If $e : \mathbb{Z}Q_{4n} \to \mathbb{Z}Q_{4n}$ is the function which evaluates at $x = -1$, then

$$e(\alpha_{s,t}) = e(z_s) + te(\widetilde{N})y = s + t(2n)y = a_0 + b_0y$$

s which implies that $a_0 + b_0y \equiv \alpha_{s,t} \bmod 1 + x$ and so $P_{a,b} = (\alpha_{s,t}, 1 + x, r)$.

Since $s = a_0$ has $(s, 2n) = 1$, we can let $\ell \geqslant 1$ be such that $\ell s \equiv 1 \bmod 2n$. Similarly to the proof of [2, Lemma 1.3], we now define $tz_s = (-x)^{\frac{s-1}{2}} \sum_{i=0}^{\ell-1}(-x^s)^i$ so that $z_s\bar{z}_s \equiv 1 \bmod \widetilde{N}$. This implies that $\bar{z}_s\alpha_{s,t} = \bar{z}_s z_s + \widetilde{N}\bar{z}_s ty \equiv 1 \bmod \widetilde{N}$ since $\widetilde{N} \in Z(\mathbb{Z}Q_{4n})$. Since $(x + 1)\widetilde{N} = 0$, this implies that $(1 + x)\bar{z}_s\alpha_{s,t} = 1 + x$. Hence $1 + x \in (\alpha_{s,t})$ and so $P_{a,b} = (\alpha_{s,t}, r)$. By Lemma 3.7, this implies that $P_{a,b}$ is generated by central elements. □

By Lemma 3.8, this implies that $P_{1,2} \subseteq \mathbb{Z}Q_{12}$ and $P_{1,4} \subseteq \mathbb{Z}Q_{20}$ are generated by central elements. This completes the proof of Proposition 3.5.

We have now shown that, if G is a finite group with a quotient H such that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ and $H = Q_8$, Q_{12}, Q_{16} or Q_{20}, then conditions (i)-(iii) of Theorem 2.6 hold for $\mathcal{R}_{G,H}$. This completes the proof of Theorem 3.3.

Remark 3.9. This argument can also be used to prove Theorem A in the case $H = \widetilde{T}$. However, we will leave this case until the following section.

3.3 Proof of Theorem A for exceptional quotients

The main result of this section is as follows. This generalises [46, Corollary 13.5, Theorem 13.7] which corresponds to the case $G = H$.

Theorem 3.10. *Let G be a finite group and suppose G has a quotient H such that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ and H is of the form*

$$\widetilde{T}, \widetilde{O}, \widetilde{I} \quad \text{or} \quad \widetilde{T}^n \times \widetilde{I}^m \text{ for } n, m \geqslant 0.$$

Then $\mathbb{Z}G$ has projective cancellation.

Let Λ be a $\mathbb{Z}$-order in a semisimple separable $\mathbb{Q}$-algebra A which is finite-dimensional over $\mathbb{Q}$. Then we can write $A \cong A_1 \times \cdots \times A_r \times B$, where the A_i are totally definite quaternion algebras with centres K_i and B satisfies the Eichler condition, i.e. $m_{\mathbb{H}}(B) = 0$. Let Γ_Λ be the projection of Λ onto $A_1 \times \cdots \times A_r$. Let $\mathcal{R} = \mathcal{R}(\Lambda, A_1 \times \cdots \times A_r, B)$ denote the corresponding fibre square.

Suppose that $\Gamma_\Lambda \subseteq A_1 \times \cdots \times A_r$ is maximal, and so of the form $\Gamma_\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ where $\Lambda_i \subseteq A_i$ is a maximal order for $i = 1, \cdots, r$. Then $\overline{\Lambda} = \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_r$ where $\overline{\Lambda}_i$ is the image of Λ_i under the map $\overline{i} : \Gamma_\Lambda \to \overline{\Lambda}$.

It is well-known that there is a finite extension K/K_i for which $A_i \otimes K \cong M_n(K)$ where $n = [K : K_i]$. If $\varphi : A_i \otimes K \to M_n(K)$ is an isomorphism, then we define the *reduced norm* as the map

$$\nu_i : A_i \to K_i$$

given by sending $\lambda \mapsto \det(\varphi(\lambda \otimes 1))$. It can be shown that ν is independent of the choice of K and φ. For an order $\Gamma_i \subseteq A_i$, this restricts to a map $\nu_i : \Gamma_i^\times \to \mathcal{O}_{K_i}^\times$.

Theorem 3.11 ([46, Theorem 13.1])**.** *Let $\mathcal{R}$ be as above and suppose that the projection $\Gamma_\Lambda \subseteq A_1 \times \cdots \times A_r$ is a maximal order. For $i = 1, \cdots, r$ and every maximal $\mathcal{O}_{K_i}$-order $\Gamma_i \subseteq A_i$, suppose that:*

(i) $\nu_i(\Gamma_i^\times) = (\mathcal{O}_{K_i}^\times)^+$

(ii) *There is at most one prime p such that $(\bar{\Lambda}_i)_{(p)} = 0$ and p is ramified in A_i. If p exists, then $(\Gamma_i)_0^\times = \mathrm{Ker}(\nu_i : \Gamma_i^\times \to \mathcal{O}_{K_i}^\times)$ has a subgroup of order $p+1$.*

Then Λ has locally free cancellation if and only if Γ_Λ has locally free cancellation.

For a finite group G, we can write $\mathbb{Q}G \cong A_1 \times \cdots \times A_r \times B$ where the A_i are totally definite quaternion algebras and B satisfies the Eichler condition. As above, let $\Gamma_{\mathbb{Z}G}$ is the projection of $\mathbb{Z}G$ onto $A_1 \times \cdots \times A_r$ and let $r = r(G)$ denote the value of r in the decomposition of $\mathbb{Q}G$ above.

The following was proven in [46, Proposition 4.11] and [46, p84].

Lemma 3.12. *If $G = \widetilde{T}$, $\widetilde{O}$ or $\widetilde{I}$, then $r(G) = 1$ and $\Gamma_{\mathbb{Z}G}$ is a maximal order in $A_{\mathbb{Z}G}$. Furthermore:*

(i) *$A_{\mathbb{Z}\tilde{T}}$ has centre $\mathbb{Z}$, is ramified only at $p = 2$ and $(\Gamma_{\mathbb{Z}\tilde{T}})_0^\times \cong \widetilde{T}$*

(ii) *$A_{\mathbb{Z}\tilde{O}}$ has centre $\mathbb{Z}[\sqrt{2}]$, is finitely unramified and $(\Gamma_{\mathbb{Z}\tilde{O}})_0^\times \cong \widetilde{O}$*

(iii) *$A_{\mathbb{Z}\tilde{I}}$ has centre $\mathbb{Z}[\frac{1}{2}(1+\sqrt{5})]$, is finitely unramified and $(\Gamma_{\mathbb{Z}\tilde{I}})_0^\times \cong \widetilde{I}$.*

By [46, Lemmas 13.8, 13.9], we have that $\Gamma_{\mathbb{Z}[\tilde{T}^n \times \tilde{I}^m]} \cong \Gamma_{\mathbb{Z}\tilde{T}}^n \times \Gamma_{\mathbb{Z}\tilde{I}}^m$ for $n, m \geqslant 0$. The following is an straightforward exercise.

Lemma 3.13. *Let $f : G \twoheadrightarrow H$ have $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$. Then $\mathbb{Q}G \cong \mathbb{Q}H \times B$ where B satisfies the Eichler condition and the projection map $\mathbb{Q}G \to \mathbb{Q}H$ induces an isomorphism $\Gamma_{\mathbb{Z}G} \cong \Gamma_{\mathbb{Z}H}$.*

Proof of Theorem 3.10. By Lemma 3.13 and the discussion above, $\Gamma_{\mathbb{Z}G}$ is of the form $\Gamma_{\mathbb{Z}\tilde{O}}$ or $\Gamma_{\mathbb{Z}\tilde{T}}^n \times \Gamma_{\mathbb{Z}\tilde{I}}^m$ for some $n, m \geqslant 0$. In particular, $\Gamma_{\mathbb{Z}G}$ is a maximal order whose components are maximal orders in $A_{\mathbb{Z}\tilde{T}}$, $A_{\mathbb{Z}\tilde{O}}$ or $A_{\mathbb{Z}\tilde{I}}$.

If $\Gamma = \Gamma_{\mathbb{Z}\tilde{T}}$, $\Gamma_{\mathbb{Z}\tilde{O}}$ or $\Gamma_{\mathbb{Z}\tilde{I}}$, then [46, p84] implies that Γ has projective cancellation and $\#C(\Gamma) = 1$, and so $\Gamma_{\mathbb{Z}G}$ has projective cancellation also. Hence, to show that $\mathbb{Z}G$ has projective cancellation, it suffices to show that the conditions (i), (ii) of Theorem 3.11 hold for maximal orders in $A_{\mathbb{Z}\tilde{T}}$, $A_{\mathbb{Z}\tilde{O}}$ or $A_{\mathbb{Z}\tilde{I}}$.

Firstly note that, if $\Gamma = \Gamma_{\mathbb{Z}\tilde{T}}$, $\Gamma_{\mathbb{Z}\tilde{O}}$ or $\Gamma_{\mathbb{Z}\tilde{I}}$ and $A = A_{\mathbb{Z}\tilde{T}}$, $A_{\mathbb{Z}\tilde{O}}$ or $A_{\mathbb{Z}\tilde{I}}$ respectively, then $\#C(\Gamma) = 1$ implies that every maximal order in A is conjugate to Γ. In particular, it suffices to check (i), (ii) for Γ only.

To show (i) holds, note that $((\mathcal{O}_K)^\times)^2 \subseteq \nu(\Gamma^\times) \subseteq ((\mathcal{O}_K)^\times)^+$ where K is the centre of A. By Lemma 3.12, we have that $K = \mathbb{Q}$, $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\zeta_8 + \zeta_8^{-1})$ or $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_{10} + \zeta_{10}^{-1})$. In each case, we have $C(\mathcal{O}_K) = 1$ and so $(\mathcal{O}_K^\times)^+ = (\mathcal{O}_K^\times)^2$ by, for example, [46, Corollary B24]. Hence $\nu(\Gamma^\times) = ((\mathcal{O}_K)^\times)^+$.

To show (ii) holds, there is nothing to check in the case $\Gamma = \Gamma_{\mathbb{Z}\tilde{O}}$ or $\Gamma_{\mathbb{Z}\tilde{I}}$ since A is finitely unramified by Lemma 3.12. If $\Gamma = \Gamma_{\mathbb{Z}\tilde{T}}$, then A is ramified only at $p = 2$ and $(\Gamma_{\mathbb{Z}\tilde{T}})_0^\times \cong \widetilde{T}$ contains an element of order $p + 1 = 3$. Hence this condition is satified regardless of whether or not $(\bar{\Lambda}_i)_{(p)} = 0$ for $R = \mathbb{Z}G$. $\square$

Remark 3.14. Generalising the remark of Swan in the proof of [46, Corollary 13.5], we note that this argument would also work in the case $H = Q_{12}$.

By combining Theorem 3.10 with Theorem 1.10 and Theorem 3.3, we have now completed the proof of Theorem A from the Introduction.

Chapter 4

Groups with periodic cohomology

In this chapter, we will study groups with periodic cohomology and establish the properties of these groups which are needed to prove Theorem B as a consequence of Theorem A. We will also classify the groups G with 4-periodic cohomology for which $m_{\mathbb{H}}(G) \leqslant 2$.

4.1 Basic definitions and properties

We say that a group G has *k-periodic cohomology* for some $k \geqslant 1$ if its Tate cohomology groups satisfy $\hat{H}^i(G;\mathbb{Z}) = \hat{H}^{i+k}(G;\mathbb{Z})$ for all $i \in \mathbb{Z}$ and that G has *periodic cohomology* if it has *k-periodic cohomology* for some k. We will begin by recalling a few basic facts about groups with periodic cohomology, much of which can be found in [5, Chapter XI].

Firstly, if G has k-periodic cohomology, then G is a finite group and k is even [22, Chapter 7]. Well known examples include cyclic groups C_n of order

$n \geqslant 1$ and the quaternion groups Q_{4n} of order $4n$ for $n \geqslant 2$. By the calculations in [5, p251-254], we have

$$\widehat{H}^i(C_n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & n \text{ even} \\ 1, & n \text{ odd} \end{cases} \qquad \widehat{H}^i(Q_{4n};\mathbb{Z}) = \begin{cases} \mathbb{Z}/4n\mathbb{Z}, & n \equiv 0 \bmod 4 \\ Q_{4n}^{\mathrm{ab}}, & n \equiv 2 \bmod 4 \\ 0, & n \text{ odd} \end{cases}$$

where $Q_{4n}^{\mathrm{ab}} = \mathbb{Z}/4\mathbb{Z}$ when n is odd and $\mathbb{Z}/2\mathbb{Z}^2$ when n is even. Hence C_n has 2-periodic cohomology and Q_{4n} has 4-periodic cohomology. More generally, it can be shown that all binary polyhedral groups have 4-periodic cohomology.

The following gives equivalent criteria for when a group G has periodic cohomology.

Proposition 4.1 ([5, Theorem 11.6])**.** *If G is a finite, then the following are equivalent:*

(i) *G has periodic cohomology*

(ii) *G has no subgroup of the form $C_p \times C_p$ for p prime*

(iii) *The Sylow subgroups of G are cyclic or generalised quaternionic Q_{2^n}.*

Let $\mathrm{SL}_2(\mathbb{F}_p)$ be the special linear group of degree 2 over $\mathbb{F}_p$, let $\mathrm{TL}_2(\mathbb{F}_p)$ be the non-split extension of C_2 by $\mathrm{SL}_2(\mathbb{F}_p)$ [52, Proposition 1.2 (iii)] and recall that $\widetilde{T} \cong \mathrm{SL}_2(\mathbb{F}_3)$, $\widetilde{O} \cong \mathrm{TL}_2(\mathbb{F}_3)$ and $\widetilde{I} \cong \mathrm{SL}_2(\mathbb{F}_5)$. Let $O(G)$ be the unique maximal normal subgroup of odd order. If G has periodic cohomology, then the *type* of G is determined by $G/O(G)$ as follows [52, Corollary 2.6]. For later convenience, we will split II and V into two classes.

Type	I	IIa	IIb	III	IV	Va	Vb	VI
$G/O(G)$	C_{2^n}	Q_8	Q_{2^n}, $n \geqslant 4$	$\widetilde{T}$	$\widetilde{O}$	$\widetilde{I}$	$\mathrm{SL}_2(\mathbb{F}_p)$, $p \geqslant 7$	$\mathrm{TL}_2(\mathbb{F}_p)$, $p \geqslant 5$

Furthermore, the groups within each type can be classified explicitly. We refer the reader to [11, Section 1], [22, Chapter 7] and [52, Sections 1,2] for details. For the purposes of this chapter, it will be useful to recall this classification for the groups of types I and II.

Type I Recall that G has type I if and only if its Sylow subgroups are cyclic, and G has a presentation

$$G = \langle u, v \mid u^m = v^n = 1, vuv^{-1} = u^r \rangle$$

for some $r \in \mathbb{Z}/m$ where $r^n \equiv 1 \bmod m$ and $(n, m) = 1$ [52, Lemma 3.1]. We will write $C_m \rtimes_{(r)} C_n$ to denote this presentation, where $C_n = \langle u \rangle$ and $C_m = \langle v \rangle$. By [22, p165], we can assume that m is odd.

Type II Recall that, if G has type II, then $O(G) \leqslant G$ has cyclic Sylow subgroups and so there exists $n \geqslant 3$ and t, s odd coprime such that

$$G \cong (C_t \rtimes_{(r)} C_s) \rtimes_{(a,b)} Q_{2^n}.$$

Furthermore, if $C_t = \langle u \rangle$, $C_s = \langle v \rangle$ and Q_{2^n} is as above, then Q_{2^n} acts via

$$\varphi_x : u \mapsto u^a, v \mapsto v, \quad \varphi_y : u \mapsto u^b, v \mapsto v$$

for some $a, b \in \mathbb{Z}/t$ with $a^2 \equiv b^2 \equiv 1 \bmod t$ [52, Theorem 3.6]. If $s = 1$, then we will abbreviate this to $C_t \rtimes_{(a,b)} Q_{2^n}$.

4.2 Quaternionic representations

For the rest of this section, we will assume all groups are finite and will write $f : G \twoheadrightarrow H$ to denote a surjective group homomorphism. We will also assume basic facts about quaternion groups; for example, Q_{2^n} has proper quotients C_2 and the dihedral groups D_{2^m} for $1 < m < n$. We begin with the following observation.

Proposition 4.2. *Let $f : G \twoheadrightarrow H$ where G and H have periodic cohomology. If $|H| > 2$, then G and H have the same type.*

Proof. Note that $f(O(G)) \leqslant H$ has odd order and so is contained in $O(H)$. In particular, f induces a quotient $f : G/O(G) \twoheadrightarrow H/O(H)$. Hence it suffices to show that there are no (proper) quotients among groups in the family

$$\mathscr{F} = \{C_{2^n}, Q_{2^m}, \mathrm{SL}_2(\mathbb{F}_p), \mathrm{TL}_2(\mathbb{F}_p) : n \geqslant 2, m \geqslant 3, p \geqslant 3 \text{ prime}\}$$

unless both are cyclic. Firstly, the quotients of Q_{2^n} are D_{2^m} for $1 < m < n$ and C_2 which are not in $\mathscr{F}$. It is easy to verify that the quotients of $\mathrm{SL}_2(\mathbb{F}_3)$ are C_3, A_4 and the quotients of $\mathrm{TL}_2(\mathbb{F}_3)$ are C_2, S_3, S_4, none of which are in $\mathscr{F}$.

For $p \geqslant 5$, it is well known [12] that $\mathrm{SL}_2(\mathbb{F}_p)$ has one (proper) normal subgroup C_2 with quotient $\mathrm{PSL}_2(\mathbb{F}_p)$ and similarly $\mathrm{TL}_2(\mathbb{F}_p)$ has normal subgroups C_2, $\mathrm{SL}_2(\mathbb{F}_p)$ with quotients $\mathrm{PGL}_2(\mathbb{F}_p)$, C_2. These groups are not in $\mathscr{F}$ (see, for example, [52, Proposition 1.3]). □

If G is a finite group, we say that two quotients $f_1 : G \twoheadrightarrow H_1$, $f_2 : G \twoheadrightarrow H_2$ are equivalent, written $f_1 \equiv f_2$, if $\mathrm{Ker}(f_1) = \mathrm{Ker}(f_2)$ are equal as sets. Note that, if $f_1 \equiv f_2$, then $H_1 \cong H_2$ are isomorphic as groups.

For a prime p, let G_p be the isomorphism class of the Sylow p-subgroup of G. It is useful to note that, if $1 \to N \to G \to H \to 1$ is an extension, then there is an extension of abstract groups $1 \to N_p \to G_p \to H_p \to 1$.

Lemma 4.3. *Let $f : G \twoheadrightarrow H$ where G and H have periodic cohomology and $4 \mid |H|$. If $f' : G \twoheadrightarrow H'$ and $|H| = |H'|$, then $f \equiv f'$, i.e. $H \cong H'$ and* $\mathrm{Ker}(f) = \mathrm{Ker}(f')$.

Proof. Let $H = G/N$, $H' = G/N'$ and define $\bar{G} = G/(N \cap N')$. Since there are successive quotients $G \twoheadrightarrow \bar{G} \twoheadrightarrow H$, we have $G_p \twoheadrightarrow \bar{G}_p \twoheadrightarrow H_p$ for all primes p. If G_p is cyclic, then this implies $\bar{G}_p$ is cyclic. If not, then $p = 2$ and $G_2 = Q_{2^n}$ which has proper quotients D_{2^m} for $2 \leqslant m \leqslant n-1$ and C_2. Since H has periodic cohomology, H_2 is cyclic or generalised quaternionic and so $H_2 = Q_{2^n}$ since $4 \mid |H_2|$. Hence $\bar{G}_2 = Q_{2^n}$ since $G_2 \twoheadrightarrow H_2$ factors through $\bar{G}_2$, and so $\bar{G}$ has periodic cohomology.

Now note that $K = N/(N \cap N')$ and $K' = N'/(N \cap N')$ are disjoint normal subgroups of $\bar{G}$ and so $K \cdot K' = K \times K' \leqslant \bar{G}$ by the recognition criteria for direct products. Hence $K \times K' \leqslant \bar{G}$ and, since $\bar{G}$ has periodic cohomology, Proposition 4.1 (ii) implies that $|K|$ and $|K'|$ are coprime. Since $|N| = |N'|$, this implies that $|K| = |K'| = 1$ and so $|N| = |N \cap N'| = |N'|$ and $N = N'$. ☐

Let $\mathcal{B}(G)$ denote the set of equivalence classes of quotients $f : G \twoheadrightarrow H$ where H is a binary polyhedral group. Since $4 \mid |H|$, Lemma 4.3 gives:

Corollary 4.4. *Let G have periodic cohomology and let $f_1, f_2 \in \mathcal{B}(G)$. Then $f_1 \equiv f_2$ if and only if* $\mathrm{Im}(f_1) \cong \mathrm{Im}(f_2)$.

In particular, this shows that $\mathcal{B}(G)$ is in one-to-one correspondence with the isomorphism classes of binary polyhedral groups H which are quotients of G. We will often write $H \in \mathcal{B}(G)$ when there exists $f : G \twoheadrightarrow H$ with $f \in \mathcal{B}(G)$.

In order to determine $\mathcal{B}(G)$, it suffices to determine the set of maximal binary polyhedral quotients $\mathcal{B}_{\max}(G)$, i.e. the subset containing those $f \in \mathcal{B}(G)$ such that f does not factor through any other $g \in \mathcal{B}(G)$. The rest of this section will be devoted to proving the following:

Theorem 4.5. *If G has periodic cohomology, then the type and the number of maximal binary polyhedral quotients $\#\mathcal{B}_{max}(G)$ are related as follows.*

Type	I	IIa	IIb	III	IV	Va	Vb	VI
$\#\mathcal{B}_{\max}(G)$	0,1	1,2,3	1	1	1	1	0	0

Type I Recall from Section 4.1 that, if G has type I, then $G \cong C_m \rtimes_{(r)} C_n$ where

$$C_m \rtimes_{(r)} C_n = \langle u, v \mid u^m = v^n = 1, vuv^{-1} = u^r \rangle$$

for some m odd with $(n, m) = 1$ and some $r \in \mathbb{Z}/m$ with $r^n \equiv 1 \bmod m$.

If G has a binary polyhedral quotient H, then Proposition 4.2 implies that $H = Q_{4a}$ for $a > 1$ odd and $4 \mid n$ since m is odd.

Lemma 4.6. *Let $G = C_m \rtimes_{(r)} C_{4n}$. Then G has a quotient Q_{4a} if and only if $a \mid m$ and $r \equiv -1$ mod a.*

Proof. Recall that $Q_{4a} = C_a \rtimes_{(-1)} C_4$. If $a \mid m$ and and $r \equiv -1 \bmod a$, then $\langle u^a, v^4 \rangle \leqslant G$ is normal since $r^4 \equiv 1 \bmod a$ implies $uv^4u^{-1} = u^{1-r^4}v^4 \in \langle u^a, v^4 \rangle$. This implies that $G/\langle u^a, v^4 \rangle \cong C_a \rtimes_{(r)} C_4 = Q_{4a}$ since $r \equiv -1 \bmod a$.

Conversely, if $f : G \twoheadrightarrow Q_{4a}$, then $Q_{4a} \cong \langle f(u) \rangle \rtimes_{(r)} \langle f(v) \rangle$ and $|\langle f(u) \rangle| \mid m$, $|\langle f(u) \rangle| \mid 4n$. Since Q_{4a} contains a maximal normal cyclic subgroup C_{2a}, and m is odd, we must have $\langle f(u) \rangle \leqslant C_a$. So $a \mid m$, which implies that $(a, 4n) = 1$ and $\langle f(u) \rangle \leqslant C_4$ for some $C_4 \leqslant Q_{4a}$. Hence $\langle f(u) \rangle = C_a$ and $\langle f(v) \rangle = C_4$

since they generate Q_{4a}. As $C_a \leqslant Q_{4a}$ is unique and $C_4 \leqslant Q_{4a}$ is unique up to conjugation, we can write $Q_{4a} \cong \langle f(u) \rangle \rtimes_{(-1)} \langle f(v) \rangle$, i.e. $r \equiv -1 \bmod a$. □

Now suppose G has two maximal binary polyhedral quotients $f_a : G \twoheadrightarrow Q_{4a}$, $f_b : G \twoheadrightarrow Q_{4b}$ for some $a, b > 1$ odd, and we can assume a is maximal. Then Lemma 4.6 implies that $a, b \mid m$ and $r \equiv -1 \bmod a$ and $r \equiv -1 \bmod b$. If $d = \mathrm{lcm}(a, b)$, then $d \mid m$ and $r \equiv -1 \bmod d$ and so there is a quotient $f_d : G \twoheadrightarrow Q_{4d}$ by Lemma 4.6. By Corollary 4.4 (or the proof of Lemma 4.6), f_a and f_b factor through f_d which implies that $a = b = d$ as f_a and f_b are maximal. By Corollary 4.4 again, this implies that f_a and f_b are equivalent. In particular, this shows that $\# \mathcal{B}_{\max}(G) \leqslant 1$.

Type II Recall from Section 4.1 that, if G has type II, then

$$G \cong (C_t \rtimes_{(r)} C_s) \rtimes_{(a,b)} Q_{2^n}$$

where $n \geqslant 3$, t, s are odd coprime and $a, b \in \mathbb{Z}/t$ with $a^2 \equiv b^2 \equiv 1 \bmod t$.

If G has a binary polyhedral quotient H, then the proof of Proposition 4.2 implies that $H/O(H) = Q_{2^n}$ and so $H = Q_{2^n m}$ for some m odd.

Lemma 4.7. *Let $G = (C_t \rtimes_{(r)} C_s) \rtimes_{(a,b)} Q_{2^n}$. Then G has a quotient $Q_{2^n m}$ if and only if $m \mid t$, $r \equiv 1 \bmod m$ and $Q_{2^n m} \cong C_m \rtimes_{(a,b)} Q_{2^n}$.*

Proof. If $m \mid t$ and $r \equiv 1 \bmod m$, then $\langle u^m, v \rangle \leqslant G$ is normal since $uvu^{-1} = u^{1-r}v \in \langle u^m, v \rangle$. This implies that $G/\langle u^m, v \rangle \cong C_t \rtimes_{(a,b)} Q_{2^n}$ which has quotient $C_m \rtimes_{(a,b)} Q_{2^n}$ since $m \mid t$. If $Q_{2^n m} \cong C_m \rtimes_{(a,b)} Q_{2^n}$, then G has quotient $Q_{2^n m}$.

Conversely, suppose $f : G \twoheadrightarrow Q_{2^n m}$. Let $h : G \twoheadrightarrow G/\langle u, v \rangle \cong Q_{2^n}$ and note that, if $g : Q_{2^n m} \twoheadrightarrow Q_{2^n}$, then $\mathrm{Ker}(g \circ f) = \mathrm{Ker}(h) = \langle u, v \rangle$ by Corollary 4.4 and so $\mathrm{Ker}(f) \leqslant \langle u, v \rangle$. By composing g with an element of $\mathrm{Aut}(Q_{2^n})$, we can assume $g \circ f = h$ and so $Q_{2^n m} \cong \mathrm{Ker}(g) \rtimes \langle f(x), f(y) \rangle$. Since $f(v) \in \mathrm{Ker}(g)$ has a trivial action by $\langle f(x), f(y) \rangle \cong Q_{2^n}$, this implies $f(v) = 1$, i.e. $v \in \mathrm{Ker}(f)$. This implies $\mathrm{Ker}(f) = \langle u^\ell, v \rangle$ for some $\ell \mid t$ and we need $\ell = m$ since $\mathrm{Ker}(f) \leqslant G$ has index $2^n m$. Hence $m \mid t$ and, by normality, $uvu^{-1} = u^{1-r}v \in \langle u^m, v \rangle$ and so $r \equiv 1 \bmod m$. Finally, we have $Q_{2^n m} \cong G/\langle u^m, v \rangle \cong C_m \rtimes_{(a,b)} Q_{2^n}$. □

Lemma 4.8. *If $m \geqslant 1$, then $Q_{2^n m} \cong C_m \rtimes_{(a,b)} Q_{2^n}$ if and only if*

$$(a,b) = \begin{cases} (1,-1), (-1,1), (-1,-1), & \text{if } n = 3 \\ (1,-1), & \text{if } n \geqslant 4. \end{cases}$$

Proof. It follows easily from the standard presentation that $Q_{2^n m} \cong C_m \rtimes_{(1,-1)} Q_{2^n}$. If $f : Q_{2^n m} \twoheadrightarrow Q_{2^n}$, then $\mathrm{Ker}(f) = C_m$ is unique by Corollary 4.4. Hence $Q_{2^n m} \cong C_m \rtimes_{(a,b)} Q_{2^n}$ if and only if there exists $\theta \in \mathrm{Aut}(Q_{2^n})$ such that $\varphi_{(a,b)} = \varphi_{(1,-1)} \circ \theta$ where $\varphi_{(i,j)} : Q_{2^n} \to \mathrm{Aut}(C_m) \cong (\mathbb{Z}/m)^\times$ has $\varphi_{(i,j)}(x) = i$, $\varphi_{(i,j)}(y) = j$. This implies that $\mathrm{Im}(\varphi_{(a,b)}) \leqslant \mathrm{Im}(\varphi_{(1,-1)}) = \langle 1, -1 \rangle = \{\pm 1\}$ and so $a, b \in \{\pm 1\}$. If $(a,b) = (1,1)$, then $Q_{2^n m} \cong C_m \times Q_{2^n}$ which is a contradiction unless $m = 1$, in which case $(1,1) = (1,-1)$. In particular, $(a,b) \in \{(1,-1), (-1,1), (-1,-1)\}$.

If $n = 3$, then $\theta_1 : x \mapsto y, y \mapsto x$ satisfies $\varphi_{(1,-1)} \circ \theta_1 = \varphi_{(1,-1)}$ and $\theta_2 : x \mapsto y, y \mapsto xy$ satisfies $\varphi_{(1,-1)} \circ \theta_2 = \varphi_{(-1,-1)}$. Hence all (a,b) are possible. If $n \geqslant 4$, then

$$\mathrm{Aut}(Q_{2^n}) = \{\theta_{i,j} : x \mapsto x^i, y \mapsto x^j y \mid i \in (\mathbb{Z}/2^{n-1})^\times, j \in \mathbb{Z}/2^{n-1}\}$$

and $\varphi_{(1,-1)} \circ \theta_{i,j} = \varphi_{(1,-1)}$ for all i, j and so only $(a,b) = (1,-1)$ is possible. □

Now suppose G has type IIb, i.e. $G/O(G) = Q_{2^n}$ for some $n \geqslant 4$. By combining Lemmas 4.7 and 4.8, we get that G has a quotient $Q_{2^n m}$ if and only if

$m \mid t$, $r \equiv 1 \bmod m$ and $(a,b) = (1,-1) \bmod m$. If G has two distinct maximal binary polyhedral quotients $f_i : G \twoheadrightarrow Q_{2^n m_i}$ for $i = 1, 2$, then $m_1, m_2 \mid t$, $r \equiv 1 \bmod m_1, m_2$ and $(a,b) \equiv (1,-1) \bmod m_1, m_2$. If $m = \operatorname{lcm}(m_1, m_2)$, then this implies that $m \mid t$, $r \equiv 1 \bmod m$ and $(a,b) \equiv (1,-1) \bmod m$ and so $f : G \twoheadrightarrow Q_{2^n m}$. By Corollary 4.4, $m > m_1, m_2$ and f_1 and f_2 must factor through f which is a contradiction. Hence $\# \mathcal{B}_{\max}(G) = 1$.

A similar argument works in the case where G has type IIa, i.e. $G/O(G) = Q_8$. If G has four distinct maximal binary polyhedral quotients $f_i : G \twoheadrightarrow Q_{8m_i}$ for $i = 1, 2, 3, 4$, then Lemmas 4.7 and 4.8 imply there exists i, j for which $(a,b) \equiv (1,-1)$, $(-1,1)$ and $(-1,-1) \bmod m_i, m_j$. By a similar argument to the above, this implies that f_i, f_j factors through $f : G \twoheadrightarrow Q_{8m}$ where $m = \operatorname{lcm}(m_i, m_j)$ which is a contradiction since $m_i \neq m_j$ and f_i, f_j are maximal. Hence $1 \leqslant \# \mathcal{B}_{\max}(G) \leqslant 3$.

If G has quotients Q_{8m_i} and Q_{8m_j}, then this implies that $(a,b) \bmod m_i$ and $(a,b) \bmod m_j$ are distinct which is a contradiction unless $(m_i, m_j) = 1$.

Types III, IV, Va If G has type III, IV or Va, then $G/O(G) = \widetilde{T}$, $\widetilde{O}$ or $\widetilde{I}$. If $f : G \twoheadrightarrow H$ is another binary polyhedral quotient H, then Proposition 4.2 implies that $H \cong G/O(G)$. By Corollary 4.4, f is equivalent to the quotient $G \twoheadleftarrow G/O(G)$. Hence $\# \mathcal{B}_{\max}(G), \#\mathcal{B}(G) = 1$.

Types Vb, VI Suppose G has type Vb or VI. Since no binary polyhedral groups have type Vb or VI, Proposition 4.2 implies that G has no binary polyhedral quotients. Hence $\# \mathcal{B}_{\max}(G), \#\mathcal{B}(G) = 0$. This completes the proof of Theorem 4.5.

Recall that $m_{\mathbb{H}}(G)$ denotes the number of copies of $\mathbb{H}$ in the Wedderburn decomposition of $\mathbb{R}G$ for a finite group G, i.e. the number of one-dimensional quaternionic representations.

Proposition 4.9. *Let $f : G \twoheadrightarrow H$ be a quotient. Then $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$ if and only if every $g \in \mathcal{B}(G)$ factors through f, i.e. if $f^* : \mathcal{B}(H) \to \mathcal{B}(G)$ is a bijection.*

Proof. Firstly note that $m_{\mathbb{H}}(G) \geqslant m_{\mathbb{H}}(H)$ holds in general by lifting quaternionic representations. By looking at the real Wedderburn decomposition, every one-dimensional quaternionic representation of G corresponds to a map $\varphi : G \to \mathbb{H}^\times$ such that the image contains an $\mathbb{R}$-basis for $\mathbb{H}$. In particular, $\operatorname{Im}\varphi$ is a non-abelian finite subgroup of $\mathbb{H}^\times$ and so is a binary polyhedral group.

Hence quaternionic representations of G precisely correspond to lifts of representations from binary polyhedral groups. Since every quotient from G to a binary polyhedral group factors through H, every quaternionic representation over G lifts from one over H. The result now follows. □

For example, this recovers the well-known fact that G satisfies the Eichler condition if and only if G has no quotient which is a binary polyhedral group [9, Theorem 51.3]. It also follows that, if G has a unique maximal binary polyhedral quotient H, then $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$.

We now show how to use this to deduce the following from Theorem 4.5.

Theorem 4.10. *If G has periodic cohomology, then type and $m_{\mathbb{H}}(G)$ are related as follows.*

Type	I	IIa	IIb	III	IV	Va	Vb	VI
$m_{\mathbb{H}}(G)$	$\geqslant 0$	$\geqslant 1$ odd	$\geqslant 2$ even	1	2	2	0	0

Type Vb, VI If G has type Vb or VI, then Theorem 4.5 implies that G has no binary polyhedral quotients and so $m_{\mathbb{H}}(G) = 0$ by Proposition 4.9.

Type IIb, III, IV, Va If G has type IIb, III, IV or Va, then Theorem 4.5 implies that $\# \mathcal{B}_{\max}(G) = 1$, i.e. G has a unique maximal binary polyhedral quotient H. By Proposition 4.9, we must have that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$. By Proposition 4.2, H has the same type as G. Recall that $m_{\mathbb{H}}(Q_{4n}) = \lfloor n/2 \rfloor$ [22, Section 12]. If G has type IIa, then $H = Q_{2^n m}$ for $n \geqslant 4$, $m \geqslant 1$ odd and $m_{\mathbb{H}}(Q_{2^n m}) = 2^{n-3}m \geqslant 2$ is even. If G has type III, IV or Va, then $H = \widetilde{T}$, $\widetilde{O}$ or $\widetilde{I}$ respectively which have $m_{\mathbb{H}}(\widetilde{T}) = 1$, $m_{\mathbb{H}}(\widetilde{O}) = 2$ and $m_{\mathbb{H}}(\widetilde{I}) = 2$.

Type IIa If G has type IIa, then Theorem 4.5 implies that $\# \mathcal{B}_{\max}(G) = 1, 2, 3$. If $b = \# \mathcal{B}_{\max}(G)$, let $f_i : G \twoheadrightarrow Q_{8m_i}$ denote the maximal binary polyhedral quotients for $1 \leqslant i \leqslant b$. By the proof of Theorem 4.5, the m_i are coprime and so the maximal quotient factoring through any two of the f_i is the unique quotient $f : G \twoheadrightarrow Q_8$. Since $m_{\mathbb{H}}(Q_{8m_i}) = m_i$ and $m_{\mathbb{H}}(Q_8) = 1$, it can be shown using real representation theory that

$$m_{\mathbb{H}}(G) = \sum_{i=1}^{b} (m_{\mathbb{H}}(Q_{8m_i}) - 1) + m_{\mathbb{H}}(Q_8) = \begin{cases} m_1, & \text{if } b = 1 \\ (m_1 + m_2) - 1, & \text{if } b = 2 \\ (m_1 + m_2 + m_3) - 2, & \text{if } b = 3 \end{cases}$$

which is odd since the m_i are odd. This completes the proof of Theorem 4.10.

4.3 Proof of Theorem B

The aim of this section will be to prove the following theorem from the Introduction:

Theorem B. *Let G have periodic cohomology and let P be a projective $\mathbb{Z}G$ module. Then*

(i) *If* $m_{\mathbb{H}}(G) \leqslant 2$*, then* $[P]$ *has cancellation*

(ii) *If* $m_{\mathbb{H}}(G) = 3$*, then:*

(a) *If* $\mathrm{Syl}_2(G)$ *is cyclic, then* $[P]$ *has non-cancellation*

(b) *If* $[P] \in D(\mathbb{Z}G)$*, then* $[P]$ *has non-cancellation*

(iii) *If* $m_{\mathbb{H}}(G) \geqslant 4$*, then* $[P]$ *has non-cancellation.*

Our proof will build upon the following, which was proved by Swan in [46].

Lemma 4.12 ([46, Theorem I])**.** *If* G *has a quotient* Q_{4n} *for* $n \geqslant 7$*, then* $[P]$ *has non-cancellation for all projective* $\mathbb{Z}G$ *modules* P*.*

In contrast to (ii), we have the following which follows from [46, Theorem III]. See also the discussion at the end of Section 3.2.2.

Proposition 4.13. *The quaternion group* Q_{24} *of order 24 has* $m_{\mathbb{H}}(Q_{24}) = 3$ *and a non-cyclic sylow 2-subgroup* $\mathrm{Syl}_2(Q_{24}) = Q_8$*. Furthermore,* $[\mathbb{Z}G]$ *has non-cancellation but there exists a projective* $\mathbb{Z}Q_{24}$ *module* P *with* $[P] \notin D(\mathbb{Z}Q_{24})$ *for which* $[P]$ *has cancellation.*

We begin by using the results of Swan [46, Theorems I-III] to show that Theorem B holds in the cases where G is a binary polyhedral group. Recall from Section 4.2 that $m_{\mathbb{H}}(Q_{4n}) = \lfloor n/2 \rfloor$, $m_{\mathbb{H}}(\widetilde{T}) = 1$ and $m_{\mathbb{H}}(\widetilde{O}) = m_{\mathbb{H}}(\widetilde{I}) = 2$. The groups with $m_{\mathbb{H}}(G) \leqslant 2$ have cancellation in every class by [46, Theorem I] and, by the remark following [46, Theorem I], the groups with $m_{\mathbb{H}}(G) \geqslant 4$ have non-cancellation in every class. The only groups with $m_{\mathbb{H}}(G) = 3$ are $G = Q_{24}$ and Q_{28}. In the latter case, $\mathrm{Syl}_2(Q_{28}) = C_4$ and indeed $\mathbb{Z}Q_{28}$ has non-cancellation in every class (by the same remark used previously). In the former cases, the fact that $[P]$ has non-cancellation whenever $[P] \in D(\mathbb{Z}Q_{24})$

follows from close inspection of [46, Theorem III] (see also the discussion at the end of Section 3.2.2).

We will now prove Theorem B by verifying it in three separate cases according to the number of maximal binary polyhedral quotients $\#\mathcal{B}_{\max}(G)$ (see Section 4.2).

Case: $\#\mathcal{B}_{\mathbf{max}}(G) = 0$ By Proposition 4.9, this is equivalent to $m_{\mathbb{H}}(G) = 0$. By Theorem 1.10, we know that $\mathbb{Z}G$ has projective cancellation, i.e. $[P]$ has cancellation for all projective $\mathbb{Z}G$ modules P. This completes the proof of Theorem B in this case.

Case: $\#\mathcal{B}_{\mathbf{max}}(G) = 1$ By Proposition 4.9 again this implies that, if H is the unique maximal binary polyhedral quotient, then $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H)$. If $m_{\mathbb{H}}(G) \leqslant 2$, then $m_{\mathbb{H}}(H) \leqslant 2$ and so H is of the form $Q_8, Q_{12}, Q_{16}, Q_{20}, \widetilde{T}, \widetilde{O}$ or $\widetilde{I}$ by the discussion above. In particular, by Theorem A, $\mathbb{Z}G$ has projective cancellation. This completes the proof of (i) in this case.

Let $f : G \twoheadrightarrow H$ denote the quotient map, let P be a projective $\mathbb{Z}G$ module and let $P' = f_\#(P)$ be the projective $\mathbb{Z}H$ module obtained by the extension of scalars. If $[P']$ has non-cancellation, then $[P]$ has non-cancellation by Theorem 1.15. Hence, in order to verify Theorem B, it suffices to show that $[P']$ fails cancellation in the cases (ii) and (iii).

Suppose that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H) = 3$. If $\mathrm{Syl}_2(G)$ is cyclic, then $\mathrm{Syl}_2(H)$ is a quotient of $\mathrm{Syl}_2(G)$ and so is cyclic. Since Theorem B holds for binary polyhedral groups, this implies that $[P']$ has non-cancellation. If instead we have $[P] \in D(\mathbb{Z}G)$ then, since $f_\#$ induces a map $f_\# : D(\mathbb{Z}G) \to D(\mathbb{Z}H)$, we have $P' = f_\#(P) \in D(\mathbb{Z}H)$. Similarly, this implies that $[P']$ has non-cancellation which completes the proof of (ii). If $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(H) \geqslant 4$, then

$[P']$ has non-cancellation by Lemma 4.12 and this completes the proof of (iii) in this case.

Case: $\#\mathcal{B}_{\mathbf{max}}(G) > 1$ By Theorem 4.5, we know that G necessarily has type IIa and $\#\mathcal{B}_{\max}(G) = 2$ or 3. By the calculations made in the proof of Theorem 4.10, we have that $\mathcal{B}_{\max}(G) = \{Q_{8m_1}, \cdots, Q_{8m_b}\}$ where the m_i are odd coprime and

$$m_{\mathbb{H}}(G) = \begin{cases} (m_1 + m_2) - 1, & \text{if } b = 2 \\ (m_1 + m_2 + m_3) - 2, & \text{if } b = 3 \end{cases}$$

depending on the two cases that arise. We will assume that $m_1 < \cdots < m_b$.

Note that $m_{\mathbb{H}}(G) = 3$ only when $b = 2$, $m_1 = 1$ and $m_2 = 3$. In this case, G has a quotient Q_{24} and so $\mathrm{Syl}_2(G)$ is non-cyclic. Furthermore, if P is a projective $\mathbb{Z}G$ module with $[P] \in D(\mathbb{Z}G)$, then $[P]$ has non-cancellation by Theorem 1.15 since its image lies in $D(\mathbb{Z}Q_{24})$ under the extension of scalars map. In all other cases, we have $m_{\mathbb{H}}(G) \geqslant 5$ and G must have a quotient Q_{8m} for some $m \geqslant 5$. By Lemma 4.12, we have that $\mathbb{Z}Q_{8m}$ has non-cancellation in every class. In particular, by Theorem 1.15 again, $\mathbb{Z}G$ must have non-cancellation in every class. This completes the proof of Theorem B in this case, and ultimately completes the proof of Theorem B.

4.4 Groups with 4-periodic cohomology

The aim of this section will be to classify the groups G with 4-periodic cohomology for which $m_{\mathbb{H}}(G) \leqslant 2$. This will be of particular use in Chapter 6.

We will begin by recalling the classification of groups with 4-periodic cohomology. This can be found in [22], though we will adopt the notation of Milnor [32]. The complete list is as follows where, in addition, we will also assume each family contains $G \times C_n$ for any G listed with $(n, |G|) = 1$.

(i)′ C_n for $n \geqslant 1$, the cyclic group of order n (Type I).

(ii)′ D_{4n+2} for $n \geqslant 1$, the dihedral group of order $4n+2$ (Type I).

(iii)′ Q_{4n} for $n \geqslant 2$ and $\widetilde{T}, \widetilde{O}, \widetilde{I}$ (Type II).

(iv)′ $D(2^n, m) = C_m \rtimes_{(-1)} C_{2^n}$ for $n \geqslant 3$ and $m \geqslant 3$ odd (Type I).

(v)′ $P'_{8 \cdot 3^n} = Q_8 \rtimes_\varphi C_{3^n}$ for $n \geqslant 2$, where $\varphi : C_{3^n} \to \operatorname{Aut} Q_8$ sends the generator $z \in C_{3^n}$ to $\varphi(z) : x \mapsto y, y \mapsto xy$ (Type III).

(vi)′ $P''_{48n} = C_n \cdot \widetilde{O}$ for $n \geqslant 3$ odd, the not-necessary-split extension which has cyclic Sylow 3-subgroup and has action $\widetilde{O} \twoheadrightarrow \widetilde{O}/\widetilde{T} = C_2 \leqslant \operatorname{Aut} C_n$ (Type IV).

(vii)′ $Q(2^n a; b, c) = (C_a \times C_b \times C_c) \rtimes_\varphi Q_{2^n}$ for $n \geqslant 3$ and $a, b, c \geqslant 1$ odd coprime with $b > c$. If $C_a = \langle p \rangle$, $C_b = \langle q \rangle$ and $C_c = \langle r \rangle$, then the action is given by $\varphi(x) : p \mapsto p, q \mapsto q^{-1}, r \mapsto r^{-1}$ $\varphi(y) : p \mapsto p^{-1}, q \mapsto q^{-1}, r \mapsto r$ (Type II).

In the above list, we have also indicated which type each family of groups has. There are, in particular, no groups of type V or VI which have 4-periodic cohomology.

Note that, in the notation of Section 4.2, we have $Q(2^n a; b, c) \cong C_{abc} \rtimes_{(t,s)} Q_{2^n}$ where t and s are such that $(t, s) \equiv (1, -1) \bmod a$, $(t, s) \equiv (-1, -1) \bmod b$ and $(t, s) \equiv (-1, 1) \bmod c$. It is easy to see that $Q(2^n a; b, c)$ has a quotient $Q(2^n a)$.

Theorem 4.14. *The groups G with 4-periodic cohomology for which $m_{\mathbb{H}}(G) \leqslant 2$ are as follows where each family contains $G \times C_n$ for any G listed with $(n, |G|) = 1$.*

(i) C_n *for* $n \geqslant 1$

(ii) D_{4n+2} *for* $n \geqslant 1$

(iii) $Q_8, Q_{12}, Q_{16}, Q_{20}, \widetilde{T}, \widetilde{O}, \widetilde{I}$

(iv) $D(2^n, 3)$, $D(2^n, 5)$ *for* $n \geqslant 3$

(v) $P'_{8\cdot 3^n}$ *for* $n \geqslant 2$

(vi) P''_{48n} *for* $n \geqslant 3$ *odd*

(vii) $Q(16; m, n)$ *for* $m > n \geqslant 1$ *odd coprime.*

Proof. First note that we can ignore the groups of the form $G \times C_n$ for G listed and $(n, |G|) = 1$ since $m_{\mathbb{H}}(G \times C_n) = m_{\mathbb{H}}(G)$ in these cases.

It can be shown that the groups in (i)′, (ii)′ satisfy the Eichler condition [22, Section 12]. For the groups G in (iii)′, we use that $m_{\mathbb{H}}(Q_{4n}) = \lfloor n/2 \rfloor$, $m_{\mathbb{H}}(\widetilde{T}) = 1$, $m_{\mathbb{H}}(\widetilde{O}) = 2$ and $m_{\mathbb{H}}(\widetilde{I}) = 2$ as mentioned previously.

In case (iv)′, suppose G has a binary polyhedral quotient H. Explicit computation shows that $Z(H) = C_2$ and so the quotient map $f : G \twoheadrightarrow H$ must have $f(Z(G)) \subseteq Z(H) = C_2$. If $x \in C_m$ and $y \in C_{2^n}$ are generators, it is easy to see that $Z(D(2^n, m)) = \langle y^2 \rangle = C_{2^{n-1}}$ which has index two subgroup $N = \langle y^4 \rangle$. Hence f factors through $G/\langle y^4 \rangle = C_m \rtimes_{(-1)} C_4 = Q_{4m}$. By Proposition 4.9, we have that $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(Q_{4m}) = (m-1)/2$ since m is odd and so $m_{\mathbb{H}}(G) \leqslant 2$ if and only if $m = 3$ or 5 and any $n \geqslant 3$.

The groups in (v)′ all have quotient $\widetilde{T}$ and so have Type III by Proposition 4.2 and so, by Theorem 4.10, we have $m_{\mathbb{H}}(P'_{8\cdot 3^n}) = 1$. Similarly, the groups in (vi)′ have quotient $\widetilde{O}$ and so have Type IV by Proposition 4.2 and so, by Theorem 4.10, we have $m_{\mathbb{H}}(P''_{48n}) = 2$.

For the groups in (vii)′, suppose $G = Q(2^n a; b, c)$ has $m_{\mathbb{H}}(G) \leqslant 2$ for $a, b, c \geqslant 1$ odd coprime with $b > c$. If $n = 3$, then G has Type IIa and it is easy to see that $G = Q(8a; b, c) \cong Q(8c; a, b) \cong Q(8b; c, a)$ and so has quotients Q_{8a}, Q_{8b} and Q_{8c}. This implies that $m_{\mathbb{H}}(G) \geqslant m_{\mathbb{H}}(Q_{8a}) = a$ which implies $a = 1$ since a is odd, and similarly $b = c = 1$. This is a contradiction since $b > c$. If $n \geqslant 4$, then G has a quotient $Q_{2^n a}$ and so $m_{\mathbb{H}}(G) \geqslant m_{\mathbb{H}}(Q_{2^n a}) = 2^{n-3}a$ and so $n = 4$ and $a = 1$. In particular, if $m_{\mathbb{H}}(G) \leqslant 2$, then we can take G to be $Q(16; m, n)$ for some $m > n \geqslant 1$ odd coprime.

By the remark before the statement of the theorem, we have that $G \cong C_{mn} \rtimes_{(t,s)} Q_{16}$ where t and s are such that $(t, s) \equiv (-1, -1) \mod m$ and $(t, s) \equiv (-1, 1) \mod n$, and so $t = -1$. Since G has Type IIb, Theorem 4.5 implies that G has a unique maximal binary polyhedral quotient and, since G has a quotient Q_{16}, this must be of the form Q_{16k} for some $k \geqslant 1$. By Lemmas 4.7 and 4.8, G has a quotient Q_{16k} if and only if $k \mid mn$ and $(t, s) \equiv (1, -1) \mod k$. Since $t = -1$, this implies $1 \equiv -1 \mod k$ and so $k = 1$ since k is odd. In particlar, Q_{16} is the maximal binary polyhedral quotient and so, by Proposition 4.9, we have $m_{\mathbb{H}}(G) = m_{\mathbb{H}}(Q_{16}) = 2$. □

Remark 4.15. For the groups in the list, the groups in (i)′, (ii)′ have $m_{\mathbb{H}}(G) = 0$, and the groups $Q_8 \times C_n$, $Q_{12} \times C_n$ and $\widetilde{T} \times C_n$ from (iii)′ and the groups in (v)′ all have $m_{\mathbb{H}}(G) = 1$. All other groups in the list have $m_{\mathbb{H}}(G) = 2$.

Part II

Applications to Wall's D2 problem

Chapter 5

Preliminaries on algebraic complexes

Let D2(G) denote the set of polarised homotopy classes of pairs (X, ρ_X) where X is a finite D2 complex and $\rho_X : \pi_1(X) \cong G$ is an isomorphism, i.e. where pairs (X, ρ_X), (Y, ρ_Y) are equivalent if there exists a homotopy equivalence $h : X \to Y$ such that $\rho_Y \circ \pi_1(h) \circ \rho_X^{-1} = \mathrm{id}_G$. This is a graded graph with grading given by the Euler characteristic $\chi(X)$ and edges between between each (X, ρ) and $(X \vee S^2, \rho^+)$ where ρ^+ is induced by the collapse map $X \vee S^2 \to X$.

Let Alg(G, 2) denote the set of chain homotopy classes of algebraic 2-complexes over $\mathbb{Z}G$, which are chain complexes (F_*, ∂_*) of the form

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

where the F_i are free and $H_0(F_*) \cong \mathbb{Z}$ where $\mathbb{Z}$ has trivial G-action. This is a graded graph with grading $\chi(E) = \mathrm{rank}(F_2) - \mathrm{rank}(F_1) + \mathrm{rank}(F_0)$ and edges between each $E = (F_*, \partial_*)$ and $E \oplus \mathbb{Z}G = (F_2 \oplus \mathbb{Z}G \xrightarrow{(\partial_2, 0)} F_1 \xrightarrow{\partial_1} F_0)$.

In this chapter, we will establish the necessary preliminaries which we will need to prove Theorem C. In Section 5.1, we will show that D2(G) and Alg($G, 2$) are isomorphic as graded trees and in Section 5.3 we will show how Alg($G, 2$) can be understood in terms of projective $\mathbb{Z}G$ modules in the case where G has 4-periodic cohomology.

5.1 Polarised homotopy types and algebraic 2-complexes

The aim of this section will be to prove the following. Our proof will be a mild generalisation of the arguments of Johnson in [23].

Theorem 5.1. *Let G be a finite group. Then there exists an isomorphism of graded trees*

$$\widetilde{C}_* : \mathrm{D2}(G) \to \mathrm{Alg}(G, 2)$$

which is the same as the cellular chain map $X \mapsto C_*(\widetilde{X})$ when X is a finite 2-complex.

We begin by noting that every finite D2 complex is a finite D3 complex and so is homotopy equivalent to a finite 3-complex by [47, Theorem E]. We therefore lose no generality in assuming throughout that every finite D2 complex is a finite 3-complex. Let $(X, \rho) \in \mathrm{D2}(G)$ and consider the cellular chain complex

$$C_*(X) = (C_3(\widetilde{X}) \xrightarrow{\partial_3} C_2(\widetilde{X}) \xrightarrow{\partial_2} C_1(\widetilde{X}) \xrightarrow{\partial_1} C_0(\widetilde{X}))$$

where the $C_i(\widetilde{X})$ are free $\mathbb{Z}[\pi_1(X)]$ modules under the monodromy action. We can use ρ to identify this with a chain complex of $\mathbb{Z}G$ modules which we denote $C_*(X, \rho)$. We will now show the following which is also our definition for $\widetilde{C}_*$:

Proposition 5.2. *Let $(X, \rho) \in \mathrm{D2}(G)$. Then $C_*(X, \rho)$ is chain homotopy equivalent to an algebraic 2-complex E over $\mathbb{Z}G$. In particular, we define $\widetilde{C}_*(X, \rho) = E$.*

To prove this, we will need the following two lemmas. Note that, since $\mathrm{Im}(\partial_3) \subseteq \ker(\partial_2)$, there is a well-defined map $\widetilde{\partial}_2 : C_2(\widetilde{X})/\mathrm{Im}(\partial_3) \to C_1(\widetilde{X})$.

Lemma 5.3 ([23, Proposition 6.6])**.** *Let $(X, \rho) \in \mathrm{D2}(G)$. Then there is a chain homotopy equivalence $\varphi : C_*(X, \rho) \to C'_*(X, \rho)$ where*

$$C'_*(X, \rho) = (C_2(\widetilde{X})/\mathrm{Im}(\partial_3) \xrightarrow{\widetilde{\partial}_2} C_1(\widetilde{X}) \xrightarrow{\partial_1} C_0(\widetilde{X})).$$

Lemma 5.4 ([23, Proposition 6.5])**.** *Let $(X, \rho) \in \mathrm{D2}(G)$ and let $C_*(X, \rho) = (C_*(\widetilde{X}), \partial_*)_{0 \leqslant * \leqslant 3}$. Then $C_2(\widetilde{X})/\mathrm{Im}(\partial_3)$ is a stably free $\mathbb{Z}G$ module.*

Proof of Proposition 5.2. By Lemma 5.4, there exists $i, j \geqslant 0$ for which there is an isomorphism $f : C_2(\widetilde{X})/\mathrm{Im}(\partial_3) \oplus \mathbb{Z}G^i \cong \mathbb{Z}G^j$. We can now define a chain homotopy equivalence

$$\begin{array}{c} C'_*(X,\rho) \\ \big\downarrow f_* \\ E \end{array} = \left(\begin{array}{ccccc} C_2(\widetilde{X})/\mathrm{Im}(\partial_3) & \xrightarrow{\widetilde{\partial}_2} & C_1(\widetilde{X}) & \xrightarrow{\partial_1} & C_0(\widetilde{X}) \\ \big\downarrow f \circ (\mathrm{id}, 0) & & \big\downarrow (\mathrm{id}, 0) & & \big\downarrow \mathrm{id} \\ \mathbb{Z}G^j & \xrightarrow{(\partial_2, 0) \circ f^{-1}} & C_1(\widetilde{X}) \oplus \mathbb{Z}G^i & \xrightarrow{(\partial_1, 0)} & C_0(\widetilde{X}) \end{array} \right)$$

where E is an algebraic 2-complex over $\mathbb{Z}G$. Hence, by combining with Lemma 5.3, we obtain a chain homotopy equivalence $f_* \circ \varphi : C_*(X, \rho) \to E$. $\square$

We now turn to the proof of Theorem 5.1. By Proposition 5.2, $\widetilde{C}_*$ is a well-defined map and it is clear that $\widetilde{C}_*(X \vee S^n) \simeq \widetilde{C}_*(X) \oplus \mathbb{Z}G$ and so $\widetilde{C}_*$ gives a map of graded graphs. Note that $\mathrm{Alg}(G, 2)$ is a tree [22, Section 52]. In particular, $\widetilde{C}_*$ is an isomorphism of graded trees if and only if it is bijective.

There are many proofs in the literature that $\widetilde{C}_*$ is surjective (see [48, Theorem 4] or [28, Theorem 2.1]) and so it remains to show that $\widetilde{C}_*$ is injective.

We will now need the following two lemmas. The first is proven in the case where X and Y are finite 2-complexes in [23, Proposition 2.2]. The proof for finite D2 complexes is similar and so will be omitted for brevity.

Lemma 5.5. *Let (X, ρ_X), $(Y, \rho_Y) \in \mathrm{D2}(G)$ be such that $X^{(1)} = Y^{(1)}$. If $\nu : C_*(X, \rho_X) \to C_*(Y, \rho_Y)$ is a chain map, then ν is chain homotopy equivalent to a chain map φ such that $\varphi \mid_{C_i(\tilde{X})} = \mathrm{id}$ for $i \leqslant 1$.*

Let $\mathrm{PHT}(G, 2) \subseteq \mathrm{D2}(G)$ be the subgraph corresponding to the polarised homotopy types of finite 2-complexes.

Lemma 5.6 ([23, Lemma 2.3])**.** *Let (X, ρ_X), $(Y, \rho_Y) \in \mathrm{PHT}(G, 2)$ be such that $X^{(1)} = Y^{(1)}$. If $\varphi : C_*(X, \rho_X) \to C_*(Y, \rho_Y)$ is a chain map such that $\varphi \mid_{C_i(\tilde{X})} = \mathrm{id}$ for $i \leqslant 1$, then there exists a map $f : X \to Y$ such that $f_* = \varphi_*$, $f \mid_{X^{(1)}} = \mathrm{id}$ and $\rho_X = \rho_Y \circ \pi_1(f)$.*

We will now use these lemmas to prove Theorem 5.1. The outline of the argument is taken from [23, Section 6].

Proof of Theorem 5.1. Let (X, ρ_X), $(Y, \rho_Y) \in \mathrm{D2}(G)$ and note that, by the argument of [23, Proposition 2.1], we can assume that $X^{(1)} = Y^{(1)}$ by replacing each space with a polarised homotopy equivalent space. Suppose there is a chain homotopy $\tilde{\nu} : \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$. By Lemma 5.3, this lifts to a chain homotopy $\nu : C_*(X, \rho_X) \to C_*(Y, \rho_Y)$ and, by Lemma 5.5, this is chain homotopy equivalent to a chain homotopy $\varphi : C_*(X, \rho_X) \to C_*(Y, \rho_Y)$ such that $\varphi \mid_{C_i(\tilde{X})} = \mathrm{id}$ for $i \leqslant 1$.

Let $i_X : X^{(2)} \hookrightarrow X$ denote the inclusion and note that this induces a $\mathbb{Z}G$ chain map $(i_X)_* : C_*(X^{(2)}) \to C_*(X)$ where the 2-skeleton $X^{(2)}$ comes equipped with the polarisation $\rho_{X^{(2)}} = \rho_X \circ \pi_1(i_X)$, and similarly for $Y^{(2)}$. Since $(\varphi \circ i_X)_3 = 0$, the composition $\varphi_* \circ (i_X)_* : C_*(X^{(2)}) \to C_*(Y)$ can be viewed as a chain map

$$\varphi_* \circ (i_X)_* : C_*(X^{(2)}) \to C_{*\leqslant 2}(Y) \cong C_*(Y^{(2)}).$$

Since $(\varphi \circ i_X)_* = \text{id}$ for $* \leqslant 1$, Lemma 5.6 implies that there exists a map $f : X^{(2)} \to Y^{(2)}$ such that $f_* = \varphi_* \circ (i_X)_*$, $f\,|_{X^{(1)}} = \text{id}$ and $\rho_{X^{(2)}} = \rho_{Y^{(2)}} \circ \pi_1(f)$. By composing with i_Y, we can assume $f : X^{(2)} \to Y$ which instead has that $\rho_{X^{(2)}} = \rho_Y \circ \pi_1(f)$.

We now claim that f has an extension $F : X \to Y$ such that $F_* = \varphi_* : H_2(\widetilde{X}) \to H_2(\widetilde{Y})$, which is an isomorphism since φ_* is a homology equivalence. Since X and Y are finite D2 complexes, we have that $H_i(\widetilde{X}) = H_i(\widetilde{Y}) = 0$ for $i \neq 2$. This implies that F is a homology equivalence and so is a homotopy equivalence by Whitehead's theorem. Since $F \circ i_X = f$ and $\rho_{X^{(2)}} = \rho_Y \circ \pi_1(f)$, this implies that $\rho_X = \rho_Y \circ \pi_1(F)$ and so F is the required polarised homotopy equivalence from (X, ρ_X) to (Y, ρ_Y).

To find the extension F, first let

$$X = X^{(2)} \cup_{\alpha_1} e_1^3 \cup_{\alpha_2} \cdots \cup_{\alpha_n} e_n^3$$

for 3-cells $e_i^3 \cong D^3$ and attaching maps $\alpha_i \in \pi_2(X^{(2)})$, where such a decomposition exists since X is assumed to be a finite 3-complex.

Using cellular chains, we have that $\partial_3(e_i^3) = \alpha_i$ where we are using the identification $\text{Im}(\partial_3) \subseteq \ker(\partial_2) \cong \pi_2(X^{(2)})$, and so $\alpha_i \in \text{Im}(\partial_3)$ for all $i =$

$1, \dots, n$. Note that there is a commutative diagram

$$\begin{array}{ccc} \pi_2(X^{(2)}) & \xrightarrow{(i_X)_*} & \pi_2(X) \\ \downarrow \cong & & \downarrow \cong \\ \ker(\partial_2) & \xrightarrow{q} & \ker(\partial_2)/\operatorname{Im}(\partial_3) \end{array}$$

where q is the quotient map. This shows that $\operatorname{Im}(\partial_3) = \ker((i_X)_*)$. Consider the composition $f_* = \varphi_* \circ (i_X)_* : \pi_2(X^{(2)}) \to \pi_2(Y)$. Since φ_* is a homology equivalence, this implies that $\ker((i_X)_*) = \ker(f_*)$. By combining with the above two results, we get that $\alpha_i \in \ker(f_*)$ and so the maps $f \circ \alpha_i \in \pi_2(Y)$ are nullhomotopic for all $i = 1, \dots, n$.

By standard homotopy theory, this implies that there exists an extension $F : X \to Y$. In particular, since $f \circ \alpha_i : S^2 \to Y$ is null-homotopic, there is a map $f_i : e_i^3 \to Y$ for which $f_i \circ i = f \circ \alpha_i$ for $i : S^2 = \partial e_i^3 \hookrightarrow e_i^3$ and so we can get a well-defined map $F : X \to Y$ by defining $F\,|_{e_i^3} = f_i$ for each $i = 1, \dots, n$. Finally note that, by the above diagram, $(i_X)_* : \pi_2(X^{(2)}) \to \pi_2(X)$ is surjective. Since $F_* \circ (i_X)_* = \varphi_* \circ (i_X)_*$ for $* \leqslant 2$, this implies that $F_* = \varphi_* : \pi_2(X) \to \pi_2(Y)$ or, equivalently, that $F_* = \varphi_* : H_2(\widetilde{X}) \to H_2(\widetilde{Y})$. □

5.2 Projective chain complexes over integral group rings

The aim of this section will be to recall basic preliminaries on projective chain complexes and the Swan finiteness obstruction. For $\mathbb{Z}G$ modules A and B, we define $\operatorname{Proj}^n_{\mathbb{Z}G}(A, B)$ to be the set of chain homotopy classes of exact sequences of $\mathbb{Z}G$ modules

$$E = (0 \to B \xrightarrow{\iota} P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} A \to 0)$$

where the P_i are projective. By a chain homotopy, we formally mean a chain map which restricts to a chain homotopy equivalence on the middle terms (P_*, ∂_*). For brevity, we will often omit the maps ι and ε in our description of E.

Define the *Euler class* to be $e(E) = \sum_{i=0}^{n-1}(-1)^i[P_i] \in C(\mathbb{Z}G)$, which only depends on the chain homotopy type of E. For a class $\chi \in C(\mathbb{Z}G)$, we define $\text{Proj}^n_{\mathbb{Z}G}(A, B; \chi)$ to be the subset of $\text{Proj}^n_{\mathbb{Z}G}(A, B)$ consisting of those extensions with $\chi(E) = \chi$. Let $\Omega^{\chi}_n(A)$ denote the set of $\mathbb{Z}G$ modules B for which $\text{Proj}^n_{\mathbb{Z}G}(A, B; \chi)$ is non-empty. If $n \geqslant 2$ then, for any $B_0 \in \Omega^{\chi}_n(A)$, we have $\Omega^{\chi}_n(A) = \{B : B \oplus \mathbb{Z}G^i \cong B_0 \oplus \mathbb{Z}G^j, i, j \geqslant 0\}$, i.e. $\Omega^{\chi}_n(A)$ is a stable module [41]. We also define

$$\text{Proj}^n_{\mathbb{Z}G}(A, \Omega^{\chi}_n(A); \chi) = \bigsqcup_{B \in \Omega^{\chi}_n(A)} \text{Proj}^n_{\mathbb{Z}G}(A, B; \chi)$$

which is a graded graph with grading $\chi(E) = \sum_{i=0}^{n-1}(-1)^i \operatorname{rank}(P_i)$ and with edges between each E to the stabilised complex

$$E \oplus \mathbb{Z}G = (P_{n-1} \oplus \mathbb{Z}G \xrightarrow{(\partial_{n-1}, 0)} P_{n-2} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0).$$

We can also define $\Omega^{\chi}_{-n}(B)$ to be the set of $\mathbb{Z}G$ modules A for which $\text{Proj}^n_{\mathbb{Z}G}(A, B; \chi)$ is non-empty and we can similarly define a graded graph $\text{Proj}^n_{\mathbb{Z}G}(\Omega^{\chi}_{-n}(B), B; \chi)$.

Recall that, for a $\mathbb{Z}G$ module A, its dual is defined as $A^* = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$ which is a left $\mathbb{Z}G$ module under the action defined by $(g \cdot \varphi)(x) = \varphi(g^{-1} \cdot x)$ for

$\varphi \in A^*$ and $g \in G$. For $\chi \in C(\mathbb{Z}G)$, we will write $\Omega_n^\chi(A)^* = \{B^* : B \in \Omega_n^\chi(A)\}$. Recall also that a $\mathbb{Z}G$ module A is a $\mathbb{Z}G$ *lattice* if its underlying abelian group is free, i.e. of the form $\mathbb{Z}^n$ for some $n \geqslant 0$. If A is a $\mathbb{Z}G$ lattice, then A^* is a $\mathbb{Z}G$ lattice and there is an isomorphism $(A^*)^* \cong A$ [22, Section 28]. If P is projective, then so is P^* and we also have $(P^*)^* \cong P$ since projective $\mathbb{Z}G$ modules are $\mathbb{Z}G$ lattices.

If $E = (P_*, \partial_*) \in \mathrm{Proj}^n_{\mathbb{Z}G}(A, B)$, then define

$$E^* = (P_0^* \xrightarrow{\partial_1^*} P_1^* \to \cdots \to P_{n-2}^* \xrightarrow{\partial_{n-1}^*} P_{n-1}^*).$$

Lemma 5.7 ([22, Proposition 28.4])**.** *Let A and B be $\mathbb{Z}G$ lattices. Then we have $E^* \in \mathrm{Proj}^n_{\mathbb{Z}G}(B^*, A^*)$ and $(E^*)^* \simeq E$ are chain homotopy equivalent.*

By addition of elementary complexes, we can show that every projective extension E with $e(E) = 0$ is chain homotopy equivalent to an extension with the P_i free provided $n \geqslant 2$. In particular, $\mathrm{Proj}^n_{\mathbb{Z}G}(A, B; 0)$ can be taken to be the set of chain homotopy types of exact sequences E with the P_i free. We also let $\Omega_n(A) = \Omega_n^0(A)$.

It was shown by Swan [41, Theorem 4.1] that a group G has n-periodic cohomology if and only if there exists an exact sequence of the form

$$0 \to \mathbb{Z} \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are projective, i.e. if $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is non-empty. By taking the map $P_0 \to P_{n-1}$ which factors through $\mathbb{Z}$, this can be turned into in n-periodic projective resolution. We say that G has *free period n* if such a resolution exists with the P_i free or, equivalently, if $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}; 0)$ is non-empty.

Define the *Swan map* $S : (\mathbb{Z}/|G|)^\times \to C(\mathbb{Z}G)$ sends $r \mapsto [(N, r)]$, where

(N, r) is the module defined in **??**, and define the *Swan subgroup* to be $T_G = \text{Im}(S)$. If G has n-periodic cohomology and $E \in \text{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$, then we define the *Swan finiteness obstruction* to be the class

$$\sigma_n(G) = [e(E)] \in C(\mathbb{Z}G)/T_G.$$

It was shown by Swan that $\sigma_n(G)$ is independent of the choice of E [41, Lemma 7.3]. Furthermore, we have:

Theorem 5.8 ([41]). *Let G have n-periodic cohomology. Then the following are equivalent:*

(i) *G has free period n.*

(ii) *$\sigma_n(G) = 0 \in C(\mathbb{Z}G)/T_G$.*

(iii) *There is a finite CW-complex X such that $X \simeq S^{n-1}$ and G acts freely on X.*

It took over 20 years until the first example of a group with $\sigma_n(G) \neq 0$ was found by R. J. Milgram [31]. It was later shown by J. F. Davis [10] that the group $Q(16; 3, 1)$ with 4-periodic cohomology has free period 8, which is the example of minimal order. Conversely, we also have:

Proposition 5.9 ([46, Lemma 7.4]). *Let G have n-periodic cohomology and let P_G be a projective $\mathbb{Z}G$ module for which $\sigma_n(G) = [P_G] \in C(\mathbb{Z}G)/T_G$. Then there exists $E \in \text{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ for which $e(E) = [P_G] \in C(\mathbb{Z}G)$.*

The formulation (iii) has the following consequence for finite Poincaré 3-complexes X since, if $\pi_1(X)$ is finite, then $\widetilde{X} \simeq S^3$.

Corollary 5.10. *A finite group G is the fundamental group of a finite Poincaré 3-complex if and only if G has free period 4.*

5.3 Classification of algebraic 2-complexes

This section will largely be dedicated to the proof of the following. From now on, we will assume that G is a finite group.

Theorem 5.11. *Let G have 4-periodic cohomology. Then there is an isomorphism of graded trees*

$$\Psi : \text{Alg}(G, 2) \to [P_G]$$

for any projective $\mathbb{Z}G$ module P_G for which $\sigma_4(G) = [P_G] \in C(\mathbb{Z}G)/T_G$.

A similar statement appears in [22, Theorem 57.4] though, due to a gap in the proof of Theorem 56.9, the argument given only applies in the case of minimal algebraic 2-complexes. Furthermore, the argument that $\{E \in \text{Alg}(G, 2) : \chi(E) = r\}$ and $\{P \in [P_G] : \text{rank}(P) = r\}$ are in one-to-one correspondence for all $r \geqslant 1$ assumes that the former set is non-empty for each r, and this was only subsequently shown when G has free period 4 [22, p234].

Let $\mu_2(G)$ be the minimum value of $\chi(E)$ over all $E \in \text{Alg}(G, 2)$. Since G is finite, we have $\mu_2(G) \geqslant 1$ [42, Corollary 1.3].

Proposition 5.12. *If G has 4-periodic cohomology, then $\mu_2(G) = 1$.*

In order to prove this, we first need the following three lemmas. Recall that the *augmentation ideal* is the module $I = \ker(\varepsilon : \mathbb{Z}G \to \mathbb{Z})$ where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ sends $g \mapsto 1$ for all $g \in G$.

Lemma 5.13. *Let G be a finite group and let $\alpha : \mathbb{Z} \to \mathbb{Z}G^n$ be injective. If* $\text{coker}(\alpha)$ *is a $\mathbb{Z}G$ lattice, then* $\text{coker}(\alpha) \cong I^* \oplus \mathbb{Z}G^{n-1}$.

Proof. The case $n = 1$ follows from the fact that $\alpha = r\Sigma$ for $r \neq 0$ since $\mathbb{Z}G/\Sigma \cong I^*$, and the case $n \geqslant 2$ follows from [22, Proposition 29.2]. □

Lemma 5.14. *Let P be a projective $\mathbb{Z}G$-module, let $r = \text{rank}(P)$ and let I be the augmentation ideal. Then there exists a $\mathbb{Z}G$ lattice J for which*

$$I \oplus P = J \oplus \mathbb{Z}G^r.$$

Proof. Since G is a finite, [40, Theorem A] implies that P is of the form $P = P_0 \oplus \mathbb{Z}G^{r-1}$ for some rank one projective $\mathbb{Z}G$ module P_0 and so it suffices to prove the case $r = 1$. Let $\varphi : P \to \mathbb{Z}$ be the surjection obtained by taking the composition

$$P \to P \otimes \mathbb{Q} \cong \mathbb{Q}G \to \mathbb{Q}$$

whose image is a non-trivial finitely-generated subgroup of $\mathbb{Q}$, and so isomorphic to $\mathbb{Z}$. If $J = \ker(\varphi)$ then, by applying Schanuel's lemma to the exact sequences

$$0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0, \qquad 0 \to J \to P \to \mathbb{Z} \to 0,$$

we get that $I \oplus P \cong J \oplus \mathbb{Z}G$. □

Lemma 5.15 ([51, p514])**.** *Let J be a $\mathbb{Z}G$ lattice. Then* $\text{Ext}^k_{\mathbb{Z}G}(J, \mathbb{Z}G) = 0$ *for all $k \geqslant 1$.*

Proof of Proposition 5.12. Since G has 4-periodic cohomology, the discussion in Section 5.2 implies that there exists an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\alpha} F_3 \to P_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

where P_2 is projective and, by addition of elementary complexes, we can assume the F_i are free. By Lemma 5.13, $\text{coker}(\alpha) \cong I^* \oplus \mathbb{Z}G^r$ where $r =$

$\operatorname{rank}(F_3) - 1$. This gives an exact sequence:

$$0 \to I^* \oplus \mathbb{Z}G^r \to P_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

Now let $\bar{P}_2$ be a projective for which $F_2 = P_2 \oplus \bar{P}_2$ is free. By forming the direct sum with length two exact sequence, we get

$$0 \to I^* \oplus P \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

for $P = \mathbb{Z}G^r \oplus \bar{P}_2$ projective. By dualising the result in Lemma 5.14, we can write $I^* \oplus P = J \oplus \mathbb{Z}G^s$ where $s = \operatorname{rank}(P)$.

Let i denote the injection $i : J \oplus \mathbb{Z}G^s \cong I^* \oplus P \to F_2$, and consider the exact sequences:

$$0 \to J \to F_2/i(\mathbb{Z}G^s) \to F_1 \to F_0 \to \mathbb{Z} \to 0, \quad 0 \to \mathbb{Z}G^s \to F_2 \to F_2/i(\mathbb{Z}G^s) \to 0.$$

The first exact sequence implies that $F_2/i(\mathbb{Z}G^s)$ is a $\mathbb{Z}G$ lattice. By Lemma 5.15, this implies that $\operatorname{Ext}^1_{\mathbb{Z}G}(F_2/i(\mathbb{Z}G^s), \mathbb{Z}G^s) = 0$ and so $F_2 \cong F_2/i(\mathbb{Z}G^s) \oplus \mathbb{Z}G^s$ by the second exact sequence. Hence we get an exact sequence

$$0 \to J \to F_2 \to F_1 \oplus \mathbb{Z}G^s \to F_0 \to \mathbb{Z} \to 0$$

which defines an algebraic 2-complex E. Since $J \cong I^* \cong \mathbb{Z}G^{|G|-1}$ as abelian groups, we have that $\chi(E) = 1$ which completes the proof. □

Recall that a graded tree is a *fork* if it has a single vertex at each non-minimal level (i.e. grade). The following was shown by W. H. Browning. These results were never published, though an alternate proof can be found in [18, Corollary 2.6].

Theorem 5.16 ([4, Theorem 5.4]). *Let G be a finite group. Then* $\mathrm{Alg}(G,2)$ *is a fork.*

On the other hand, if G is a finite group, then [40, Theorem A] implies that every projective $\mathbb{Z}G$ module P is of the form $P = P_0 \oplus \mathbb{Z}G^r$ for some rank one projective $\mathbb{Z}G$ module P_0. This implies that $[P]$ is also a fork.

Hence, in order to prove Theorem 5.11, it suffices to prove that there is a bijection Ψ between $\mathrm{Alg}(G,2)$ and $[P_G]$ at the minimal levels, i.e. that there is a one-to-one correspondence between $\{E \in \mathrm{Alg}(G,2) : \chi(E) = 1\}$ and $\{P \in [P_G] : \mathrm{rank}(P) = 1\}$ (see Fig. 5.1). We now need the following two results.

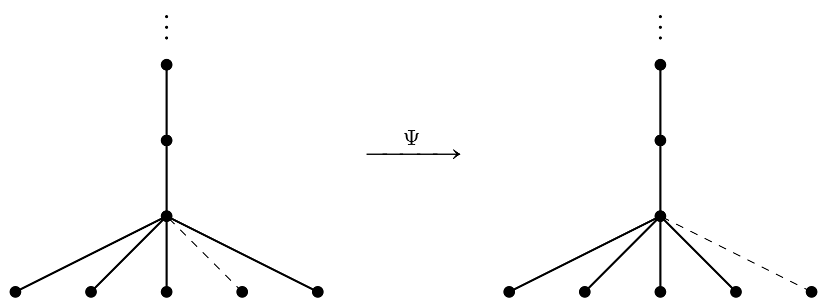

Figure 5.1: Tree structures for $\mathrm{Alg}(G,2)$ and $[P_G]$

Proposition 5.17. *There is an isomorphism of graded trees*

$$\mathrm{Alg}(G,2) \cong \mathrm{Proj}^3_{\mathbb{Z}G}(\mathbb{Z}, \Omega_3(\mathbb{Z}); 0).$$

Proposition 5.18. *Let $\chi = [P] \in C(\mathbb{Z}G)$. Then there is an map of graded trees*

$$\Phi : \mathrm{Proj}^1_{\mathbb{Z}G}(\Omega_3(\mathbb{Z}), \mathbb{Z}; \chi) \to [P]$$

given by $(0 \to \mathbb{Z} \to P_0 \to J \to 0) \mapsto P_0$*, which is a bijection at the minimal level.*

The first is immediate from the discussion in Section 5.2, and the second is a consequence of the following mild extension of [22, Corollary 56.5].

Lemma 5.19. *For $i = 1, 2$, let P_i be projective $\mathbb{Z}G$ modules of rank one and let $\mathcal{E}_i = (0 \to J \to P_i \to \mathbb{Z} \to 0)$ be exact sequences of $\mathbb{Z}G$-modules. Then there is a chain homotopy equivalence $\mathcal{E}_1 \simeq \mathcal{E}_2$ if and only if $P_1 \cong P_2$.*

The following will be useful in comparing Propositions 5.17 and 5.18.

Lemma 5.20. *Let G have 4-periodic cohomology and let P_G be a projective $\mathbb{Z}G$ module for which $\sigma_4(G) = [P_G] \in C(\mathbb{Z}G)/T_G$. If $\chi = [P_G^*] \in C(\mathbb{Z}G)$, then*

$$\Omega_3(\mathbb{Z}) = \Omega_1^{\chi}(\mathbb{Z})^*.$$

Proof. By Proposition 5.9, there exists $E \in \text{Proj}^4_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ with $e(E) = [P_G]$. By addition of elementary complexes, we can assume that

$$E = (P \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0)$$

where the F_i are free, so that $P \in [P_G]$.

Let $J = \ker(\partial_2) = \text{Im}(\partial_3)$. Then $J \in \Omega_3(\mathbb{Z})$ and there are exact sequences

$$\mathcal{E} = (0 \to \mathbb{Z} \xrightarrow{\alpha} P \xrightarrow{\beta} J \to 0), \quad \mathcal{E}^* = (0 \to J^* \xrightarrow{\beta^*} P^* \xrightarrow{\alpha^*} \mathbb{Z} \to 0),$$

where $\mathcal{E}^*$ is exact by Lemma 5.7 since $\mathbb{Z}$ and J are $\mathbb{Z}G$ lattices. Hence $J^* \in \Omega_1^{\chi}(\mathbb{Z})$. Since $(J^*)^* \cong J$, this implies that $J \in \Omega_1^{\chi}(\mathbb{Z})^*$. Hence $\Omega_3(\mathbb{Z}) = \Omega_1^{\chi}(\mathbb{Z})^*$ since two stable modules are equal if they intersect non-trivially. ☐

Recall that, if J is a $\mathbb{Z}G$ module, then an automorphism $\varphi : J \to J$ induces a map $\varphi_* : H^n(G; J) \to H^n(G; J)$. If $J \in \Omega_n(\mathbb{Z})$, then $H^n(G; J) \cong \mathbb{Z}/|G|$ [22, p132]. By fixing this identification, the construction $\varphi \mapsto \varphi_*$ induces a map

$$\nu^J : \mathrm{Aut}_{\mathbb{Z}G}(J) \to (\mathbb{Z}/|G|)^\times.$$

Let $S : (\mathbb{Z}/|G|)^\times \to C(\mathbb{Z}G)$ denote the Swan map, as in Section 5.2. Then:

Lemma 5.21 ([22, Theorems 54.6, 56.10])**.** *Let $\chi \in C(\mathbb{Z}G)$ and $J \in \Omega_n^\chi(\mathbb{Z})$. Then $\mathrm{Im}(\nu^J) \subseteq \ker(S)$ and there is a bijection*

$$\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, J; \chi) \cong \ker(S)/\mathrm{Im}(\nu^J).$$

In particular, $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, J; \chi)$ only depends on J and not on n or χ.

Proof of Theorem 5.11. First note, since the map $P \mapsto P^*$ is an involution on the class of projective $\mathbb{Z}G$ modules, it must induce an isomorphism of graded trees $[P_G] \cong [P_G^*]$. Hence, by Propositions 5.17 and 5.18, it suffices to prove that the graded trees

$$\mathrm{Proj}^3_{\mathbb{Z}G}(\mathbb{Z}, \Omega_3(\mathbb{Z}); 0), \quad \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, \Omega_1^\chi(\mathbb{Z}); \chi)$$

contain the same number of extensions at the minimal level, where $\chi = [P_G^*]$.

To see this, let $J \in \Omega_3(\mathbb{Z})$ be minimal and note that $\mathrm{Aut}_{\mathbb{Z}G}(J) \cong \mathrm{Aut}_{\mathbb{Z}G}(J^*)$ and so there is a bijection $\mathrm{Im}(\nu^J) \cong \mathrm{Im}(\nu^{J^*})$. In particular, we have bijections

$$\mathrm{Proj}^3_{\mathbb{Z}G}(\mathbb{Z}, J; 0) \cong \ker(S)/\mathrm{Im}(\nu^J) \simeq \ker(S)/\mathrm{Im}(\nu^{J^*}) \cong \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, J^*; \chi).$$

By Lemma 5.20, the map $J \mapsto J^*$ induces a bijection $\Omega_3(\mathbb{Z}) \cong \Omega_1^\chi(\mathbb{Z})$. We can now extend the bijection $\mathrm{Proj}^3_{\mathbb{Z}G}(\mathbb{Z}, J; 0) \cong \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, J^*; \chi)$ over all $J \in \Omega_3(\mathbb{Z})$ at the minimal level, and this completes the proof. □

Chapter 6

Wall's D2 problem for groups with 4-periodic cohomology

The aim of this chapter will be to use the results in the previous chapters to study the D2 property for groups with 4-periodic cohomology.

As we shall see, these groups are of particular importance for the D2 problem. In Section 6.2, we will see how the D2 property for groups with 4-periodic cohomology would resolve part of an open problem on the cell structure of finite Poincaré 3-complexes. In addition, these groups were conjectured by J. M. Cohen [6] to be the only candidates for counterexamples to the D2 problem.

6.1 Proof of Theorem C

Recall that, if $\mathcal{P} = \langle s_1, \cdots, s_n \mid r_1, \cdots, r_m \rangle$ is a presentation of a group G, then the *deficiency* of $\mathcal{P}$ is defined to be $\operatorname{def}(\mathcal{P}) = n - m$. We say that $\mathcal{P}$ is a *balanced presentation* if $\operatorname{def}(\mathcal{P}) = 0$, i.e. if $\mathcal{P}$ has the same number of generators and relations.

The aim of this section will be to prove the following theorem from the Introduction:

Theorem C. *Suppose G has 4-periodic cohomology. Then:*

(i) *If G has the* D2 *property, then G has a balanced presentation*

(ii) *If G has a balanced presentation and $m_{\mathbb{H}}(G) \leqslant 2$, then G has the* D2 *property.*

First note the following which follows immediately from Theorem B by noting that, if G has n-periodic cohomology and $\sigma_n(G) = [P_G] \in C(\mathbb{Z}G)/T_G$, then $[P_G] \in D(\mathbb{Z}G)$.

Proposition 6.2. *Let G have n-periodic cohomology and let P_G be a projective $\mathbb{Z}G$ module for which $\sigma_n(G) = [P_G] \in C(\mathbb{Z}G)/T_G$. Then $[P_G]$ has cancellation if and only if $m_{\mathbb{H}}(G) \leqslant 2$.*

We can now use this to establish the following:

Theorem 6.3. *Let G have 4-periodic cohomology. Then* $\mathrm{D2}(G)$ *has cancellation if and only if $m_{\mathbb{H}}(G) \leqslant 2$.*

Proof. Let P_G be such that $\sigma_4(G) = [P_G] \in C(\mathbb{Z}G)/T_G$. By combining Theorems 5.1 and 5.11, we get that there is an isomorphism of graded trees

$$\Psi \circ \widetilde{C}_* : \mathrm{D2}(G) \to [P_G]$$

and so $\mathrm{D2}(G)$ has cancellation if and only if $[P_G]$ has cancellation. By Proposition 6.2, $[P_G]$ has cancellation if and only if $m_{\mathbb{H}}(G) \leqslant 2$. □

Proof of Theorem C. Recall that, as discussed in Section 5.1, G has the D2 property if and only if the induced map $\Psi \circ \widetilde{C}_* : PHT(G,2) \to [P_G]$ is bijective.

Suppose G has the D2 property. Then $\Psi \circ \widetilde{C}_*$ is bijective and, since $[P_G]$ contains a projective $\mathbb{Z}G$ module of rank one, there must exist $(X, \rho) \in PHT(G, 2)$ such that $\chi(X) = 1$. If $\mathcal{P}$ is a presentation such that $X \simeq X_{\mathcal{P}}$, then $\operatorname{def}(\mathcal{P}) = 1 - \chi(X) = 0$ and so $\mathcal{P}$ is a balanced presentation as required.

Now suppose that $m_{\mathbb{H}}(G) \leqslant 2$ and G has a balanced presentation $\mathcal{P}$. By Theorem 6.3, $[P_G]$ has cancellation and so $[P_G] = \{P_0 \oplus \mathbb{Z}G^r : r \geqslant 0\}$ for some projective P_0 of rank one. Let $P_1 = \Psi(\widetilde{C}_*(X_{\mathcal{P}}))$, which has $\operatorname{rank}(P_1) = 1$ since $\chi(X_{\mathcal{P}}) = 1$. Since $P_1 \in [P]$, this implies that $P_1 \cong P_0$. In particular, for all $r \geqslant 0$, we have $\Psi(\widetilde{C}_*(X_{\mathcal{P}} \vee rS^2)) \cong P_0 \oplus \mathbb{Z}G^r$ and so $\Psi \circ \widetilde{C}_*$ is surjective. Since $\Psi \circ \widetilde{C}_*$ is injective, it must be bijective and so G has the D2 property. $\square$

6.2 CW-structures for Poincaré 3-complexes

Recall that an oriented finite Poincaré n-complex is a finite CW-complex with a fundamental class $[X] \in H_n(X; \mathbb{Z})$ such that

$$- \cap [X] \colon C^{n-*}(X; \mathbb{Z}[\pi_1(X)]) \to C_*(X; \mathbb{Z}[\pi_1(X)])$$

is a simple chain homotopy equivalence. By Poincaré duality, every closed topological n-manifold has the structure of a finite Poincaré n-complex, but there exists finite Poincaré n-complexes which are not homotopy equivalent to any closed topological n-manifold [17].

By Morse theory, every closed n-manifold has a cell structure with a single n-cell. In [49], Wall investigated the question of whether or not this is also true for finite Poincaré n-complexes. He firstly noted the following:

Theorem 6.4 ([49, Theorem 2.4]). *Let $n \geqslant 3$. If X is a finite Poincaré n-complex, then there exists a $D(n-1)$ complex K and a map $f : S^{n-1} \to K$ for* *which there is a homotopy equivalence*

$$X \simeq K \cup_f D^n.$$

If K is homotopy equivalent to a finite $(n-1)$-complex, then the splitting $X \simeq K \cup_f D^n$ gives a cell structure on X with a single n-cell. Recall that every Dn complex is homotopy equivalent to a finite n-complex provided $n \geqslant 3$ [47, Theorem E]. In particular, K is homotopy equivalent to a finite $(n-1)$-complex if $n \geqslant 4$. If $n = 3$, then K is homotopy equivalent to a finite 2-complex provided $\pi_1(K)$ has the D2 property. Since $\pi_1(K) \cong \pi_1(X)$, we can state this as:

Theorem 6.5 ([49])**.**

(i) *If $n \geqslant 4$, then every finite Poincaré n-complex has a cell structure with a single n-cell*

(ii) *If $n \geqslant 3$ and G has the* D2 *property, then every finite Poincaré* 3*-complex X with $\pi_1(X) \cong G$ has a cell structure with a single* 3*-cell.*

Since it is not known whether or not every group has the D2 property, this does not imply that every finite Poincaré 3-complex has a cell structure with a single 3-cell. In particular, this remains a significant open problem:

Problem 6.6. *Does every finite Poincaré* 3*-complex have a cell structure with a single* 3*-cell?*

In contrast to all dimensions $\geqslant 4$, not every finitely presented group arises as the fundamental group of a finite Poincaré 3-complex. In particular, the following is well known:

Proposition 6.7. *If X is a Poincaré* 3*-complex, then $\pi_1(X)$ is either infinite or has* 4*-periodic cohomology.*

Remark 6.8. In fact, the finite groups which arise as the fundamental groups of *finite* Poincaré 3-complexes are precisely the groups with free period 4.

By combining this with Theorem 6.5, this gives the following:

Proposition 6.9. *Suppose every group with free period* 4 *has the* D2 *property. Then every finite Poincaré* 3*-complex with finite fundamental group has a cell structure with a single* 3*-cell.*

It is therefore of particular interest to study the D2 property for groups with 4-periodic cohomology. This was first pointed out by Johnson [21] (see also [22, Chapter 11]).

We conclude this section by noting the following consequence of Theorem C for the cell structures of finite Poincaré 3-complexes.

Theorem 6.10. *Suppose G has free period* 4 *and consider the following:*

(i) *G has the* D2 *property*

(ii) *Every finite Poincaré* 3*-complex over G has a cell structure with a single* 3*-cell*

(iii) *Some finite Poincaré* 3*-complex over G has a cell structure with a single* 3*-cell*

(iv) *G has a balanced presentation.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*. If $m_{\mathbb{H}}(G) \leqslant 2$, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)*.*

Proof. We will begin by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Note that (i) $\Rightarrow$ (ii) is proven in Theorem 6.5 and (ii) $\Rightarrow$ (iii) by Remark 6.8, and so it remains to show that (iii) $\Rightarrow$ (iv). First note that, by Remark 6.8, there exists a finite Poincaré 3-complex X with $\pi_1(X) \cong G$. It follows from the definition that there is an isomorphism of abelian groups $H^{3-*}(X) \cong H_*(X)$. By the universal coefficients theorem, we have that $\operatorname{rank}_{\mathbb{Z}}(H_i(X)) = \operatorname{rank}_{\mathbb{Z}}(H^i(X))$ and so:

$$\chi(X) = \sum_{i=0}^{3} (-1)^i \operatorname{rank}_{\mathbb{Z}}(H_i(X)) = 0.$$

If X has a cell structure with a single 3-cell then, with respect to this cell structure, we have $X \simeq X^{(2)} \cup_f D^3$ for some map $f : S^2 \to X^{(2)}$. Note that $K = X^{(2)}$ is a finite 2-complex with $\chi(K) = \chi(X) + 1 = 1$. In particular, if $\mathcal{P}$ is a presentation such that $K \simeq X_{\mathcal{P}}$, then $\mathcal{P}$ is a balanced presentation.

Finally, if $m_{\mathbb{H}}(G) \leqslant 2$, then Theorem C shows that (iv) $\Rightarrow$ (i). Hence, in this case, we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) as required. □

6.3 Balanced presentations for groups with periodic cohomology

If G has periodic cohomology, then $H_2(G; \mathbb{Z}) = 0$ (see, for example, [44]). In particular, by Theorem C, the groups G with 4-periodic cohomology are either counterexamples to the D2 problem or give a new supply of groups with efficient presentations.

This gives some response to comments made by L. G. Kovács [25, p212] and J. Harlander [19, p167] on the scarcity of efficient finite groups. In contrast, we conjecture:

Conjecture 6.11. *If G has periodic cohomology, then G has a balanced presentation.*

By [54, Chapter 6], the groups in (i)′, (iii)′, (iv)′, (v)′ are all finite 3-manifold groups which are well-known to have balanced presentations. This follows, for example, from Theorem 6.10 since 3-manifolds have cell structures with a single

3-cell. The groups in (ii)$'$ also have balanced presentations (see [22, p236]).

Proposition 6.12. *If G is in* (i)$'$-(v)$'$, *then G has a balanced presentation.*

We have not been able to find balanced presentations for any of the groups in (vi), but have succeeded for the following groups in (vii). These groups overlap in the case where $k = n = 1$ with the groups in [34, Theorem 3.1].

Proposition 6.13. *If $n \geqslant 3$ and $a, b, k \geqslant 1$ odd coprime, then:*

$$Q(2^n a; b, 1) \times C_k \cong \langle x, y \mid y^k x^b y^k = x^b, xyx = y^{2^{n-2}a-1} \rangle.$$

By combining Propositions 6.12 and 6.13 with Theorem C, we obtain:

Theorem 6.14. *Suppose G is in* (i)-(v) *or has the form $Q(16; n, 1) \times C_k$ for some $n, k \geqslant 1$ odd coprime. Then G has the* D2 *property.*

This was previously shown by M. N. Dyer [13] for the groups in (i) and by Johnson [22] for the groups in (ii) and for many of the groups in (iii). Note that not all of these groups have free period 4; an example is $Q(16; 3, 1)$ [10].

The simplest groups that we have not been able to find balanced presentations for are $P''_{48 \cdot 3}$ and $Q(16; 3, 5)$. The following is therefore of immediate practical interest:

Question 6.15. *Do $P''_{48 \cdot 3}$ or $Q(16; 3, 5)$ have balanced presentations?*

These correspond to the groups G_{144}^{31}, G_{240}^{22} in GAP's Small Groups Library.

6.4 Potential counterexamples to the D2 problem

Recall that a $\mathbb{Z}G$ module M has the *Swan property* (or is a 'Swan module') if $d_{\mathbb{Z}G}(M) = \max_{p||G|} d_{\mathbb{Z}_pG}(M \otimes \mathbb{Z}_p)$ where $d_R(\cdot)$ is the number of R-module generators.

In 1977, J. M. Cohen proposed the group Q_{32} as a counterexample to the D2 problem [6, Section 4]. More generally, he conjectured the following.

Conjecture 6.16 ([50, Problem D3]). *Let X be a finite* D2 *complex. Then X is homotopy equivalent to a finite 2-complex if and only if $\pi_2(X)$ has the Swan property or $\pi_1(X)$ is infinite.*

The following was proved by Cohen as a consequence of [7, Theorem 3], where I is the augmentation ideal. This gives another reason why the D2 property for groups with 4-periodic cohomology is of particular interest.

Proposition 6.17 ([6, p415]). *Let X be a finite* D2 *complex such that $\pi_2(X)$ does not have the Swan property and $G = \pi_1(X)$ is finite. Then:*

(i) *G has free period 4.*

(ii) *$\chi(X) = 1$.*

(iii) *$\pi_2(X)$ is non-cyclic, i.e. $\pi_2(X) \ncong I^*$.*

We will now show the following as an application of Theorem 6.3. Recall that $\mathbb{Z}G$ modules A and B are $\mathrm{Aut}(G)$-*isomorphic* if there exists a bijection $\varphi : A \to B$ such that, for some $\theta \in \mathrm{Aut}(G)$, $\varphi(g \cdot x) = \theta(x) \cdot \varphi(x)$ for all $x \in A$.

Corollary 6.18. *Suppose the "if" part of Conjecture 6.16 holds. If G does not have the* D2 *property, then G has free period 4 and $m_{\mathbb{H}}(G) \geqslant 3$.*

Proof. By Proposition 6.17, G has free period 4 and there exists a finite D2 complex X with $\pi_1(X) = G$, $\chi(X) = 1$ and $\pi_2(X) \not\cong I^*$. Recall that, in Proposition 5.12, we constructed a finite D2 complex X_G with $\pi_1(X_G) = G$ and $\chi(X_G) = 1$ whenever G has 4-periodic cohomology. Since G has free period 4, we can take $J = I^*$ in the proof of Proposition 5.12 which gives that $\pi_2(X_G) \cong I^*$. In particular, $X \vee rS^2 \simeq X_G \vee rS^2$ for some $r \geqslant 1$. If $X \simeq X_G$, then $\pi_2(X)$ and $\pi_2(X_G)$ would be $\mathrm{Aut}(G)$-isomorphic which is a contradiction since $\pi_2(X)$ is non-cyclic and $\pi_2(X_G) \cong I^* \cong \mathbb{Z}G/\Sigma$. Hence D2$(G)$ has non-cancellation and so $m_{\mathbb{H}}(G) \geqslant 3$ by Theorem 6.3. □

We say that two presentations $\mathcal{P}$ and $\mathcal{Q}$ for a group G are *exotic* if $X_{\mathcal{P}} \not\simeq X_{\mathcal{Q}}$ and $\mathrm{def}(\mathcal{P}) = \mathrm{def}(\mathcal{Q})$ or, equivalently, if $X_{\mathcal{P}} \vee rS^2 \simeq X_{\mathcal{Q}} \vee rS^2$ for some $r \geqslant 0$. The following was shown recently by Mannan and Popiel [29]. This is the only known example of an exotic presentation for a finite non-abelian group.

Theorem 6.19 ([29, Theorem A])**.** *The quaternion group Q_{28} has presentations*

$$\mathcal{P}_1 = \langle x, y \mid x^7 = y^2, xyx = y \rangle, \quad \mathcal{P}_2 = \langle x, \mid x^7 = y^2, y^{-1}xyx^2 = x^3y^{-1}x^2y \rangle$$

such that $\pi_2(X_{\mathcal{P}_1}) \not\cong \pi_2(X_{\mathcal{P}_2})$ are not $\mathrm{Aut}(Q_{28})$-isomorphic. Hence $X_{\mathcal{P}_1} \not\simeq X_{\mathcal{P}_2}$.

We now proceed to point out the following two consequences.

Theorem 6.20. *The "only if" part of Conjecture 6.16 is false.*

Proof. By the remark after [29, Theorem A], we have that $d_{\mathbb{Z}Q_{28}}(\pi_2(X_{\mathcal{P}_1})) = 1$ and $d_{\mathbb{Z}Q_{28}}(\pi_2(X_{\mathcal{P}_1})) \neq 1$. However, $\pi_2(X_{\mathcal{P}_1}) \oplus \mathbb{Z}G^r \cong \pi_2(X_{\mathcal{P}_2}) \oplus \mathbb{Z}G^r$ for some $r \geqslant 0$ and so $\pi_2(X_{\mathcal{P}_1}) \otimes \mathbb{Z}_p \cong \pi_2(X_{\mathcal{P}_2}) \otimes \mathbb{Z}_p$ for $p \mid |G|$ prime since $\mathbb{Z}_pG$ is semisimple by Maschke's theorem. This implies that $d_{\mathbb{Z}_pG}(\pi_2(X_{\mathcal{P}_2}) \otimes \mathbb{Z}_p) = 1$

for all $p \mid |G|$ and so $\pi_2(X_{\mathcal{P}_2})$ does not have the Swan property. This contradicts the "only if" direction of Conjecture 6.16 since $X_{\mathcal{P}_2}$ is a finite 2-complex. $\square$

Theorem 6.21. *Q_{28} has the* D2 *property and $m_{\mathbb{H}}(Q_{28}) = 3$.*

In [1], this is proposed as a counterexample by F. R. Beyl and N. Waller and so this answers their question in the negative (see also [30, p23]).

Proof. Note that Q_{28} is a finite 3-manifold group and so has $\sigma_4(Q_{28}) = 0$. Hence, as in the proof of Theorem C, it suffices to show that the injective map of graded trees $\Psi \circ \widetilde{C}_* : \text{PHT}(Q_{28}, 2) \to [\mathbb{Z}Q_{28}]$ is in fact bijective. By the discussion in Section 5.3, $\text{PHT}(Q_{28}, 2)$ and $[\mathbb{Z}Q_{28}]$ are both forks. By [46, Theorem III], there are two rank one stably free $\mathbb{Z}Q_{28}$ modules and so $[\mathbb{Z}Q_{28}]$ has two vertices at the minimal level. By Theorem 6.19, $PHT(Q_{28}, 2)$ has at least two vertices at the minimal level. Hence the injective map $\Psi \circ \widetilde{C}_*$ must be bijective, and so Q_{28} has the D2 property. $\square$

It should be possible to replicate this proof for other groups with 4-periodic cohomology and $m_{\mathbb{H}}(G) \geqslant 3$. We expect that the quaternion groups contain the main difficulties associated with the case $m_{\mathbb{H}}(G) \geqslant 3$, and so we ask:

Question 6.22. *Does Q_{4n} have the* D2 *property for all $n \geqslant 2$?*

The case Q_{32} is of particular significance since it was the first proposed counterexample to the D2 problem [6, p415].

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