# Recovering Multiple Fractional Orders in Time-Fractional Diffusion in an Unknown Medium* 

Bangti Jin ${ }^{\dagger} \quad$ Yavar Kian ${ }^{\ddagger}$


#### Abstract

In this work, we investigate an inverse problem of recovering multiple orders in a time-fractional diffusion model from the data observed at one single point on the boundary. We prove the unique recovery of the orders together with their weights, which does not require a full knowledge of the domain or medium properties, e.g., diffusion and potential coefficients, initial condition and source in the model. The proof is based on Laplace transform and asymptotic expansion. Further, inspired by the analysis, we propose a numerical procedure for recovering these parameters based on a nonlinear least-squares fitting with either fractional polynomials or rational approximations as the model function, and provide numerical experiments to illustrate the approach for small time $t$.


Key words: order recovery, time-fractional diffusion, multi-order, uniqueness, inverse problem

## 1 Introduction

Let $\Omega \subset \mathbb{R}^{d}(d \geq 2)$ be an open bounded and connected subset with a $C^{2\left\lceil\frac{d}{4}\right\rceil+2}$ boundary $\partial \Omega$ (the notation $\lceil r\rceil$ denotes the smallest integer exceeding $r \in \mathbb{R}$ ). Consider a weak solution (in the sense of Definition 2.1 below) $u$ of the following initial boundary value problem:

$$
\left\{\begin{align*}
\sum_{j=1}^{N} r_{j} \partial_{t}^{\alpha_{j}} u+\mathcal{A} u & =\sigma(t) f(x), \quad \text { in } \Omega \times(0, T),  \tag{1.1}\\
\mathcal{R} u & =0, \quad \text { on } \partial \Omega \times(0, T), \\
u & =u_{0}, \quad \text { in } \Omega \times\{0\} .
\end{align*}\right.
$$

In the model, $\mathcal{A}$ is a second-order elliptic operator on the domain $\Omega$ given by

$$
\begin{equation*}
\mathcal{A} u(x):=-\sum_{i, j=1}^{d} \partial_{x_{i}}\left(a_{i, j}(x) \partial_{x_{j}} u(x)\right)+q(x) u(x), \quad x \in \Omega \tag{1.2}
\end{equation*}
$$

where the potential $q \in C^{2\left\lceil\frac{d}{4}\right\rceil}(\bar{\Omega})$ is strictly positive in $\bar{\Omega}$, and the diffusion coefficient matrix $a:=$ $\left(a_{i, j}\right)_{1 \leq i, j \leq d} \in C^{1+2\left\lceil\frac{d}{4}\right\rceil}\left(\bar{\Omega} ; \mathbb{R}^{d \times d}\right)$ is symmetric and fulfills the following ellipticity condition

$$
\begin{equation*}
\exists c>0, \quad \sum_{i, j=1}^{d} a_{i, j}(x) \xi_{i} \xi_{j} \geq c|\xi|^{2}, \quad \forall x \in \bar{\Omega}, \xi=\left(\xi_{1}, \ldots, \xi_{d}\right) \in \mathbb{R}^{d} \tag{1.3}
\end{equation*}
$$

For a fixed $N \in \mathbb{N}$ and for $j=1, \ldots, N$, we consider the constants $r_{j} \in(0,+\infty), 0<\alpha_{1}<\ldots<\alpha_{N}<1$ and $T \in(0,+\infty)$. In the model 1.1 , the notation $\partial_{t}^{\alpha} u$ denotes the Djrbashian-Caputo fractional derivative of order $\alpha$ in $t$, for $\alpha \in(0,1)$, defined by (cf. [15, p. 92] and [8, Section 2.3.2])

$$
\partial_{t}^{\alpha} u(x, t):=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-s)^{-\alpha} \partial_{s} u(x, s) \mathrm{d} s, \quad(x, t) \in \Omega \times(0, T),
$$

[^0]where the notation $\Gamma(z)=\int_{0}^{\infty} s^{z-1} e^{-s} \mathrm{~d} s, \Re(z)>0$, denotes Euler's Gamma function. In addition, in the model (1.1), the notation $\mathcal{R}$ denotes either the Dirichlet trace $\mathcal{R} u=u_{\mid \partial \Omega \times(0, T)}$ or the normal derivative $\partial_{\nu_{a}}$ associated with the diffusion coefficient matrix $a$ given by
$$
\mathcal{R} u=\partial_{\nu_{a}} u_{\mid \partial \Omega \times(0, T)}=\left.\sum_{i, j=1}^{d} a_{i j} \partial_{x_{j}} u \nu_{i}\right|_{\partial \Omega \times(0, T)},
$$
where $\nu=\left(\nu_{1}, \ldots, \nu_{d}\right) \in \mathbb{R}^{d}$ denotes the unit outward normal vector to the boundary $\partial \Omega$. Throughout, the adjoint trace $\mathcal{R}^{*}$ denotes that $\mathcal{R}^{*}$ is the Dirichlet boundary trace if $\mathcal{R}$ correspond to the Neumann one and $\mathcal{R}^{*}$ is the Neumann boundary trace if $\mathcal{R}$ correspond to the Dirichlet one.

When $N=1$, the model (1.1) reduces to its single-term counterpart, i.e., with $\alpha \in(0,1)$,

$$
\begin{equation*}
\partial_{t}^{\alpha} u-\mathcal{A} u=f, \quad \text { in } \Omega \times(0, T] \tag{1.4}
\end{equation*}
$$

This model has been studied extensively in the engineering, physical and mathematical literature due to its extraordinary capability for describing anomalous diffusion phenomena [26]. It is the fractional analogue of the classical diffusion equation: with $\alpha=1$, it recovers the latter, and thus inherits some of its important analytical properties. However, it also differs considerably from the latter in the sense that, due to the presence of the nonlocal fractional derivative term, it has limited smoothing property in space and slow asymptotic decay in time [8]. The multi-term model (1.1] employs multiple fractional orders to improve the modeling accuracy of the single-term model 1.4 . For example, a two-term fractional-order diffusion model was proposed in [30] for the total concentration in solute transport, in order to describe the mobile and immobile status of the solute. The model with two fractional derivatives appears also naturally when describing subdiffusive motion in velocity fields [27].

This work is interested in the following inverse problem for the model 1.1). Let $u(x, t)$ be the weak solution to the model (1.1) in the sense of Definition 2.1 below. Given the observation $\mathcal{R}^{*} u\left(x_{0}, t\right), t \in(0, T)$, for some $x_{0} \in \partial \Omega$, can one uniquely determine the orders $\left\{\alpha_{i}\right\}_{i=1}^{N}$ and weights $\left\{r_{i}\right\}_{i=1}^{N}$ in the model (1.1)? Physically, the orders $\left\{\alpha_{i}\right\}$ are determined by the inhomogeneity of the media, but it is still unclear which physical law can relate the inhomogeneity to $\left\{\alpha_{i}\right\}_{i=1}^{N}$. Thus, in practice, one natural way is to formulate an inverse problem of determining the parameters from the available data, e.g., $\mathcal{R}^{*} u\left(x_{0}, t\right), 0<t<T$ at a monitoring point $x_{0} \in \bar{\Omega}$. The short answer to the inverse problem is affirmative. To precisely describe the results, we need a proper functional analytic framework. We define an operator $A=\mathcal{A}$ acting on $L^{2}(\Omega)$ with its domain $D(A)$ given by $D(A)=\left\{v \in H^{2}(\Omega): \mathcal{A} v \in L^{2}(\Omega), R v=0\right.$ on $\left.\partial \Omega\right\}$. Moreover, [5], Theorem 2.5.1.1] implies that, for all $\ell=1, \ldots,\left\lceil\frac{d}{4}\right\rceil+1$,

$$
\begin{equation*}
D\left(A^{\ell}\right)=\left\{v \in H^{2 \ell}(\Omega): \mathcal{R} v=\mathcal{R}(\mathcal{A} v)=\ldots=\mathcal{R}\left(\mathcal{A}^{\ell-1} v\right)=0\right\} \tag{1.5}
\end{equation*}
$$

We need the following assumption on the data in the model 1.1 . The space $D\left(A^{s}\right)$ is defined in Section 2.

Definition 1.1. A tuple $\left(\Omega, a, q, f, u_{0}\right)$ is said to be admissible if the following conditions are fulfilled.
(i) $\Omega \subset \mathbb{R}^{d}$ is a $C^{2\left\lceil\frac{d}{4}\right\rceil+2}$ bounded open set, $a:=\left(a_{i, j}\right)_{1 \leq i, j \leq d} \in C^{1+2\left\lceil\frac{d}{4}\right\rceil}\left(\bar{\Omega} ; \mathbb{R}^{d \times d}\right)$ satisfies the ellipticity condition 1.3 , $q \in C^{2\left\lceil\frac{d}{4}\right\rceil}(\bar{\Omega})$ is strictly positive on $\bar{\Omega}$.
(ii) $f \in D\left(A^{r}\right)$ and $u_{0} \in D\left(A^{r+1}\right)$, with $r>\frac{d+3}{2}$.

We shall prove in Section 2 that for any admissible tuple $\left(\Omega, a, q, f, u_{0}\right)$ and $\sigma \in L^{1}(0, T)$, problem 1.1) has a unique weak solution $u \in L^{1}\left(0, T ; C^{1}(\bar{\Omega})\right)$. Further, we have the following affirmative answers to the inverse problem for the cases $u_{0} \equiv 0$ and $f \equiv 0$, respectively; for the detailed proofs, see Section 3 .
Theorem 1.1. Let $\left(\Omega_{k}, a_{k}, q_{k}, f_{k}, 0\right), k=1,2$, be two admissible tuples with $u_{0} \equiv 0, \sigma \in L^{1}(0, T)$ be such that $\sigma \not \equiv 0$, and the constants $r_{1}^{k}, \ldots, r_{N_{k}}^{k} \in(0,+\infty), 0<\alpha_{1}^{k}<\ldots<\alpha_{N_{k}}^{k}<1, N_{k} \in \mathbb{N}, k=1,2$. Let $u^{k}$ be the weak solution of problem (1.1) with $\left(\Omega, a, q, f, u_{0}\right)=\left(\Omega_{k}, a_{k}, q_{k}, f_{k}, 0\right), N=N_{k}, r_{1}=r_{1}^{k}, \ldots, r_{N}=r_{N_{k}}^{k}$, $\alpha_{1}=\alpha_{1}^{k}, \ldots, \alpha_{N}=\alpha_{N_{k}}^{k}$. Assume that there exist $x_{k} \in \partial \Omega_{k}, k=1,2$ such that

$$
\begin{equation*}
\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right) \neq 0, \quad \mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right) \neq 0 \tag{1.6}
\end{equation*}
$$

hold, and that one of the following conditions holds: (i) $\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)$ or (ii) $r_{N_{1}}^{1}=r_{N_{2}}^{2}$. Then the condition

$$
\begin{equation*}
\mathcal{R}_{1}^{*} u^{1}\left(x_{1}, t\right)=\mathcal{R}_{2}^{*} u^{2}\left(x_{2}, t\right), \quad t \in(0, T) \tag{1.7}
\end{equation*}
$$

implies that $N_{1}=N_{2}=N$ and

$$
\begin{equation*}
\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right), \quad \alpha_{1}^{1}=\alpha_{1}^{2}, \ldots, \alpha_{N}^{1}=\alpha_{N}^{2}, \quad r_{1}^{1}=r_{1}^{2}, \ldots, r_{N}^{1}=r_{N}^{2} \tag{1.8}
\end{equation*}
$$

Theorem 1.2. Let $\left(\Omega_{k}, a_{k}, q_{k}, 0, u_{0}^{k}\right), k=1,2$, be two admissible tuples with $f \equiv 0$, and the constants $r_{1}^{k}, \ldots, r_{N_{k}}^{k} \in(0,+\infty), 0<\alpha_{1}^{k}<\ldots<\alpha_{N_{k}}^{k}<1$, for $N_{k} \in \mathbb{N}, k=1,2$. Let $u^{k}$ be the weak solution of problem 1.1) with $\left(\Omega, a, q, f, u_{0}\right)=\left(\Omega_{k}, a_{k}, q_{k}, 0, u_{0}^{k}\right), N=N_{k}, r_{1}=r_{1}^{k}, \ldots, r_{N}=r_{N_{k}}^{k}, \alpha_{1}=\alpha_{1}^{k}, \ldots$, $\alpha_{N}=\alpha_{N_{k}}^{k}$. Assume that there exist $x_{k} \in \partial \Omega_{k}, k=1,2$ such that the condition

$$
\begin{equation*}
\mathcal{R}_{1}^{*} \mathcal{A}_{1} u_{0}^{1}\left(x_{1}\right) \neq 0, \quad \mathcal{R}_{2}^{*} \mathcal{A}_{2} u_{0}^{2}\left(x_{2}\right) \neq 0 \tag{1.9}
\end{equation*}
$$

holds, and that one of the following conditions holds (i) $\mathcal{R}_{1}^{*} \mathcal{A}_{1} u_{0}^{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} \mathcal{A}_{2} u_{0}^{2}\left(x_{2}\right)$ or (ii) $r_{N_{1}}^{1}=r_{N_{2}}^{2}$, where $\mathcal{A}_{k}, k=1,2$, denotes the operator given by 1.2 with $a=a_{k}, q=q_{k}$ and $\Omega=\Omega_{k}$. Then for any $T_{1}, T_{2} \in[0, T]$ satisfying $T_{1}<T_{2}$, the condition

$$
\begin{equation*}
\mathcal{R}_{1}^{*} u^{1}\left(x_{1}, t\right)=\mathcal{R}_{2}^{*} u^{2}\left(x_{2}, t\right), \quad t \in\left(T_{1}, T_{2}\right) \tag{1.10}
\end{equation*}
$$

implies that $N_{1}=N_{2}=N$ and conditions 1.8) and

$$
\begin{equation*}
\mathcal{R}_{1}^{*} u_{0}^{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} u_{0}^{2}\left(x_{2}\right), \quad \mathcal{R}_{1}^{*} \mathcal{A}_{1} u_{0}^{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} \mathcal{A}_{2} u_{0}^{2}\left(x_{2}\right) \tag{1.11}
\end{equation*}
$$

Note that Theorems 1.1 and 1.2 are stated with unknown problem data in the sense that they are stated with $\Omega_{1} \neq \Omega_{2}, a_{1} \neq a_{2}, q_{1} \neq q_{2}, f_{1} \neq f_{2}$ and $u_{0}^{1} \neq u_{0}^{2}$, henceforth the term "unknown medium". Moreover, we mention that in these results the points $x_{k} \in \partial \Omega_{k}, k=1,2$ do not need to coincide, and in fact we can even consider the case $\overline{\Omega_{1}} \cap \overline{\Omega_{2}}=\emptyset$ or the case $\partial \Omega_{1} \cap \partial \Omega_{2}=\emptyset$. The main assumption that we impose on the data is given by condition (1.6) or (1.9), which require that the data $\mathcal{R}^{*} \mathcal{A} u_{0}$ and $\mathcal{R}^{*} f$ do not vanish at the measurement point on $\partial \Omega$. Besides, we do not even need to assume that the values at these points coincides provided that condition (ii) is fulfilled. The proof of the theorems relies on the solution representation in the Laplace domain [8, Section 6.2] and the time analyticity of the measurement. The uniqueness proof is actually constructive, and motivates developing a simple recovery procedure based on the nonlinear leastsquares, and asymptotic expansion at $t=0$, cf. Proposition 4.1. directly inspired by the analysis. In Section 4, we present illustrative numerical experiments to show the feasibility of the recovery using a nonlinear leastsquares procedure (with either fractional polynomials or fractional rational approximations as the regressor), when the measurement at small time is available.

Now we situate the uniqueness results in existing literature. The recovery of fractional orders probably has been extensively studied; see [19] for a survey. However, most existing studies focus on recovering one single order in the model (1.4) [1, 3, 9, 6, 22, 21, 37, sometimes together with other parameters, e.g., diffusion or potential coefficients, given certain observational data. The only works on recovering multiple orders are [16, 20, 33]. Li and Yamamoto [20] proved the unique recovery of multiple orders in two cases: (i) the uniqueness in simultaneously identifying $\left\{\left(\alpha_{i}, r_{i}\right)\right\}_{i=1}^{N}$ when $d=1$ and $u_{0}=\delta\left(x-x^{*}\right)$ (the Dirac delta function concentrated at $x^{*} \in \Omega$ ) by measured data at one endpoint; (ii) the uniqueness in determining $\left\{\left(\alpha_{i}, r_{i}\right)\right\}_{i=1}^{N}$ when $d \geq 1$ and $u_{0} \in L^{2}(\Omega)$ by interior measurement. The analysis is based on the asymptotic behavior of the multinomial Mittag-Leffler functions (cf. Remark 4.1). Li et al [16] proved the unique recovery of orders and several coefficients from data consisting of a suitably defined Dirichlet-to-Neumann map. Sun et al 33 proved the unique recovery of the orders and the potential $q$ (for one-dimensional problem) using the Gel'fand-Levitan theory for Sturm-Liouville problems. All these existing works assume a fully known forward model. This work is also a natural continuation of the authors' recent work [9], where the unique recovery of one single fractional order was proved for the model $\sqrt{1.4}$ ) within an unknown medium (e.g. diffusion coefficient, potential) and scatterer from lateral flux data at one single point (see also [37] in the case of nonself-adjoint elliptic operators.) Note that the analysis 9 relies heavily on the analyticity of the solution at large time, asymptotic of the two-parameter Mittag-Leffler function $E_{\alpha, \beta}(z)$ and the strong maximum principle and Hopf's lemma for elliptic problems. Thus this work differs greatly from [9] in the proof technique and admissible observation data.

The rest of the paper is organized as follows. In Section 2 we recall preliminaries of problem 1.1 , e.g., existence, regularity and analyticity of the solution. Then in Section 3, we prove Theorems 1.1 and 1.2, Last, in Section 4, we present some numerical experiments for recovering the orders. Throughout, the notation $\mathbb{R}_{+}$ denotes the set $(0,+\infty)$, By $\langle\cdot, \cdot\rangle$ we denote the scalar product in $L^{2}(\Omega)$. The notation $C$ denotes a generic constant which may change from one line to the next, but it is always independent of the quantity under analysis, e.g. p.

## 2 Preliminaries

In this section we give several preliminary properties of problem (1.1), e.g., existence of a weak solution and time analyticity. First we recall the concept of weak solutions. Li et al [18] proved the unique existence of a mild solution using multinomial Mittag-Leffler functions. We employ a representation of solutions in terms of inverse Laplace transform [11, 13, 14] (or [8, Section 6.2]).
Definition 2.1. Let $\sigma \in L^{1}(0, T)$ and $u_{0}, f \in L^{2}(\Omega)$. A function $u$ is said to be a weak solution to problem (1.1) if there exists $v \in L_{\text {loc }}^{1}\left(0,+\infty ; L^{2}(\Omega)\right)$ satisfying $u=v_{\mid \Omega \times(0, T)}$ and the following properties:
(i) $\inf \left\{\lambda>0: t \mapsto e^{-\lambda t} v(\cdot, t) \in L^{1}\left(0,+\infty ; L^{2}(\Omega)\right)\right\}=0$,
(ii) for all $p>0$, the Laplace transform $\widehat{v}(\cdot, p):=\int_{0}^{+\infty} e^{-p t} v(\cdot, t) \mathrm{d} t$ of $v$ belongs to $L^{2}(\Omega)$ and solves

$$
\left\{\begin{align*}
\mathcal{A} \widehat{v}(p)+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}} \widehat{v}(p) & =\left(\int_{0}^{T} e^{-p t} \sigma(t) \mathrm{d} t\right) f+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}-1} u_{0}, \quad \text { in } \Omega  \tag{2.1}\\
\mathcal{R} \widehat{v}(p) & =0, \quad \text { on } \partial \Omega
\end{align*}\right.
$$

Throughout, we fix $\theta \in\left(\frac{\pi}{2}, \pi\right), \delta \in \mathbb{R}_{+}$and the contour $\gamma(\delta, \theta)$ in $\mathbb{C}$ defined by $\gamma(\delta, \theta):=\left\{\delta e^{i \beta}: \beta \in\right.$ $[-\theta, \theta]\} \cup \gamma_{ \pm}(\delta, \theta):=\left\{r e^{ \pm i \theta}: r \geq \delta\right\}$ oriented in the counterclockwise direction. Let $\theta_{1} \in\left(0, \frac{2 \theta-\pi}{8}\right)$. For all $z \in D_{\theta_{1}}:=\left\{r e^{i \beta}: r>0, \beta \in\left(-\theta_{1}, \theta_{1}\right)\right\}$, we define two solution operators $S_{1}(z), S_{2}(z) \in \mathcal{B}\left(L^{2}(\Omega)\right)$ by

$$
\begin{align*}
& S_{1}(z) v=\frac{1}{2 i \pi} \int_{\gamma\left(\delta, \theta_{1}\right)} e^{z p}\left(A+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1}\left(\sum_{k=1}^{N} r_{k} p^{\alpha_{k}-1}\right) v \mathrm{~d} p, \quad v \in L^{2}(\Omega)  \tag{2.2}\\
& S_{2}(z) v=\frac{1}{2 i \pi} \int_{\gamma\left(\delta, \theta_{1}\right)} e^{z p}\left(A+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1} v \mathrm{~d} p, \quad v \in L^{2}(\Omega) \tag{2.3}
\end{align*}
$$

The operators $S_{1}(z)$ and $S_{2}(z)$ correspond to the initial data and the right hand side, respectively.
Recall that the spectrum of the operator $A$ consists of a nondecreasing sequence of strictly positive eigenvalues $\left(\lambda_{n}\right)_{n \geq 1}$ repeated with respect to their multiplicity. In the Hilbert space $L^{2}(\Omega)$, we introduce an orthonormal basis of eigenfunctions $\left(\varphi_{n}\right)_{n \geq 1}$ of $A$ associated with the eigenvalues $\left(\lambda_{n}\right)_{n \geq 1}$. For all $s \geq 0$, we denote by $A^{s}$ the fractional power operator defined by

$$
A^{s} g=\sum_{n=1}^{+\infty}\left\langle g, \varphi_{n}\right\rangle \lambda_{n}^{s} \varphi_{n}, \quad g \in D\left(A^{s}\right)=\left\{g \in L^{2}(\Omega): \sum_{n=1}^{+\infty}\left|\left\langle g, \varphi_{n}\right\rangle\right|^{2} \lambda_{n}^{2 s}<\infty\right\}
$$

and in $D\left(A^{s}\right)$, we define the graph norm $\|\cdot\|_{D\left(A^{s}\right)}$ by

$$
\|g\|_{D\left(A^{s}\right)}=\left(\sum_{n=1}^{+\infty}\left|\left\langle g, \varphi_{n}\right\rangle\right|^{2} \lambda_{n}^{2 s}\right)^{\frac{1}{2}}, \quad g \in D\left(A^{s}\right)
$$

Following [12, Lemma 3.4] and [17, Theorem 1.2], we can prove the following result
Lemma 2.1. For all $s \in[0,1]$, the map $z \mapsto S_{j}(z)$ is holomorphic in $D_{\theta_{1}}$ as a map taking values in $\mathcal{B}\left(L^{2}(\Omega) ; D\left(A^{s}\right)\right)$ and there exists $C>0$ depending only on $\mathcal{A}, r_{1}, \ldots, r_{N}, \alpha_{1}, \ldots, \alpha_{N}$ and $\Omega$ such that

$$
\begin{aligned}
\left\|S_{1}(z)\right\|_{B\left(L^{2}(\Omega) ; D\left(A^{s}\right)\right)} \leq C \max \left(|z|^{\alpha_{1}(1-s)-1},|z|^{\alpha_{N}(1-s)-1}, 1\right), \quad z \in D_{\theta_{1}} \\
\left\|S_{2}(z)\right\|_{B\left(L^{2}(\Omega) ; D\left(A^{s}\right)\right)} \leq C \max \left(|z|^{-\alpha_{1} s},|z|^{-s \alpha_{N}}, 1\right), \quad z \in D_{\theta_{1}}
\end{aligned}
$$

In a similar way to [17, Proposition 2.1], one can prove that for $\sigma \in L^{\infty}(0, T)$ and $u_{0}, f \in L^{2}(\Omega)$ problem (1.1) admits a unique weak solution $u \in L^{1}\left(0, T ; D\left(A^{s}\right)\right), s \in[0,1)$ given by

$$
\begin{equation*}
u(\cdot, t)=S_{1}(t) u_{0}+\int_{0}^{t} \sigma(s) S_{2}(t-s) f \mathrm{~d} s, \quad t \in(0, T) \tag{2.4}
\end{equation*}
$$

We claim that the representation (2.4) indeed gives a weak solution of 1.1 in the sense of Definition 2.1 . We consider only the case $u_{0} \equiv 0$. Since $\left\|\left(A+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1}\right\|_{\mathcal{B}\left(L^{2}(\Omega)\right)} \leq C(1+|p|)^{-\alpha_{N}}, p \in \mathbb{C} \backslash(-\infty, 0]$, we can define the operator-valued function

$$
R_{1}(t):=\frac{1}{2 i \pi} \int_{-i \infty}^{+i \infty} e^{t p}(p+1)^{-1}\left(A+\sum_{k=1}^{N} r_{k}(p+1)^{\alpha_{k}}\right)^{-1} \mathrm{~d} p, \quad t \in \mathbb{R}
$$

Following [17, Proposition 2.1], $R_{1} \in L^{\infty}\left(\mathbb{R} ; \mathcal{B}\left(L^{2}(\Omega)\right)\right.$ ) is supported on [0, + $\infty$ ). Moreover, using the argument of [29, Theorem 19.2 and the remark], we deduce that the Laplace transform $\widehat{R_{1}}(p)$ is well defined for $p>0$ and $\widehat{R_{1}}(p)=(p+1)^{-1}\left(A+\sum_{k=1}^{N} r_{k}(p+1)^{\alpha_{k}}\right)^{-1}$. Similarly, following [17, Proposition 2.1], $R_{2}(t):=e^{t} R_{1}(t)$ belongs to the set of tempered distributions supported on $[0,+\infty)$ and taking values in $\mathcal{B}\left(L^{2}(\Omega)\right)$. Let $\tilde{\sigma}$ be the extension of $\sigma$ to $\mathbb{R}$ by zero. Fixing the maps

$$
w(\cdot, t)=\left(R_{2}(t)\right) *(\tilde{\sigma}(t) f)=\int_{0}^{t} \tilde{\sigma}(s) R_{2}(t-s) f \mathrm{~d} s, \quad v(\cdot, t)=\int_{0}^{t} \tilde{\sigma}(s) S_{2}(t-s) f \mathrm{~d} s, \quad t \in \mathbb{R}
$$

and repeating the arguments of [17, Proposition 2.1] give that $\partial_{t} w=v$ in the sense of tempered distributions. Therefore, $\widehat{v}(\cdot, p)=\left(A+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1} \widehat{\tilde{\sigma}}(p) f$ is well defined for $p>0$ and it solves (2.1). Finally, for $u$ given by 2.4, we have $v=u$ on $\Omega \times(0, T)$, and hence the weak solution of (1.1) takes the form (2.4). These arguments show the existence of a weak solution of 1.1 when $\sigma \in L^{\infty}(0, T)$. For the case $\sigma \in L^{1}(0, T)$, it suffices to combine Lemma 2.1 with a density argument (cf. [13, Proposition 6.1]). In a similar way to [17], one can check

$$
S_{j}(z) h=\sum_{n=1}^{+\infty} S_{j, n}(z)\left\langle h, \varphi_{n}\right\rangle \varphi_{n}, \quad z \in D_{\theta_{1}}, j=1,2
$$

with

$$
\begin{aligned}
& S_{1, n}(z)=\frac{1}{2 i \pi} \int_{\gamma\left(\delta, \theta_{1}\right)} e^{z p}\left(\lambda_{n}+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1}\left(\sum_{k=1}^{N} r_{k} p^{\alpha_{k}-1}\right) \mathrm{d} p, \quad z \in D_{\theta_{1}}, n \in \mathbb{N} \\
& S_{2, n}(z)=\frac{1}{2 i \pi} \int_{\gamma\left(\delta, \theta_{1}\right)} e^{z p}\left(\lambda_{n}+\sum_{k=1}^{N} r_{k} p^{\alpha_{k}}\right)^{-1} \mathrm{~d} p, \quad z \in D_{\theta_{1}}, n \in \mathbb{N}
\end{aligned}
$$

In passing, note that the functions $S_{1, n}$ and $S_{2, n}$ can be expressed explicitly via multinomial Mittag-Leffler functions, cf. Remark 4.1. Repeating the arguments of Lemma 2.1, we deduce that for all $n \in \mathbb{N}$ and $j=1,2$, $S_{j, n}$ is holomorphic on $D_{\theta_{1}}$. Moreover, for all $s \in[0,1]$, we have

$$
\begin{align*}
& \left|S_{1, n}(z)\right| \leq C \lambda_{n}^{-s} \max \left(|z|^{-s \alpha_{1}},|z|^{-s \alpha_{N}}, 1\right), \quad z \in D_{\theta_{1}}, \quad n \in \mathbb{N} \\
& \left|S_{2, n}(z)\right| \leq C \lambda_{n}^{-s} \max \left(|z|^{\alpha_{1}(1-s)^{-1}},|z|^{\alpha_{N}(1-s)-1}, 1\right), \quad z \in D_{\theta_{1}}, n \in \mathbb{N} \tag{2.5}
\end{align*}
$$

with $C>0$ depending only on $\mathcal{A}, r_{1}, \ldots, r_{N}, \alpha_{1}, \ldots, \alpha_{N}$ and $\Omega$. Using this result we can prove the following representation of the measured data.

Lemma 2.2. Let $f, u_{0} \in D\left(A^{\frac{d}{4}}\right)$ and $\sigma \in L^{1}(0, T)$. Then the map $t \mapsto S_{1}(t) u_{0}$ and $t \mapsto S_{2}(t) f$ are analytic with respect to $t \in \mathbb{R}_{+}$as a function taking values in $C^{1}(\bar{\Omega})$. Moreover, problem (1.1) admits a unique weak solution $u \in L^{1}\left(0, T ; C^{1}(\bar{\Omega})\right)$ satisfying

$$
\begin{equation*}
\mathcal{R}^{*} u(x, t)=\mathcal{R}^{*}\left[S_{1}(t) u_{0}\right](x)+\int_{0}^{t} \sigma(s) \mathcal{R}^{*}\left[S_{2}(t-s) f\right](x) \mathrm{d} s, \quad t \in(0, T), x \in \partial \Omega \tag{2.6}
\end{equation*}
$$

Proof. Without loss of generality, we only prove the representation for $u_{0} \equiv 0$. In view of the identity (1.5), by interpolation, we deduce that the space $D\left(A^{\frac{d}{4}+\frac{3}{4}}\right)$ embeds continuously into $H^{\frac{d}{2}+\frac{3}{2}}(\Omega)$ and the Sobolev embedding theorem implies that $D\left(A^{\frac{d}{4}+\frac{3}{4}}\right)$ embeds continuously into $C^{1}(\bar{\Omega})$. In addition, applying 2.5 , for all $z \in D_{\theta_{1}}$ and all $m_{1}, m_{2} \in \mathbb{N}, m_{1}<m_{2}$, we have

$$
\begin{align*}
&\left\|\sum_{n=m_{1}}^{m_{2}} S_{2, n}(z)\left\langle f, \varphi_{n}\right\rangle \varphi_{n}\right\|_{C^{1}(\bar{\Omega})} \leq C\left\|\sum_{n=m_{1}}^{m_{2}} S_{2, n}(z)\left\langle f, \varphi_{n}\right\rangle \varphi_{n}\right\|_{D\left(A^{\frac{d}{4}+\frac{3}{4}}\right)}  \tag{2.7}\\
& \leq C \max \left(|z|^{\frac{\alpha_{1}}{4}-1},|z|^{\alpha_{N}}-1\right. \\
&\left.\alpha^{\frac{\alpha_{N}}{4}} 1\right)\left(\sum_{n=m_{1}}^{m_{2}} \lambda_{n}^{\frac{d}{2}}\left|\left\langle f, \varphi_{n}\right\rangle\right|^{2}\right)^{\frac{1}{2}}
\end{align*}
$$

with $C>0$ independent of $z, m_{1}$ and $m_{2}$. Combining this with the condition $f \in D\left(A^{\frac{d}{4}}\right)$ yields that the sequence $\sum_{n=1}^{N} S_{2, n}(z)\left\langle f, \varphi_{n}\right\rangle \varphi_{n}, N \in \mathbb{N}$, converges uniformly with respect to $z$ on any compact set of $D_{\theta_{1}}$ to $S_{2}(z) f$ as a function taking values in $C^{1}(\bar{\Omega})$. This proves that the map $D_{\theta_{1}} \ni z \mapsto S_{2}(z) f$ is holomorphic as a function taking values in $C^{1}(\bar{\Omega})$. This implies the first statement of the lemma. Next, in view of 2.7), we have

$$
\|u(\cdot, t)\|_{D\left(A^{\frac{d}{4}+\frac{3}{4}}\right)} \leq C\|f\|_{D\left(A^{\frac{d}{4}}\right)}\left(\max \left(t^{\frac{\alpha_{1}}{4}-1}, t^{\frac{\alpha_{N}}{4}-1}, 1\right) \mathbb{1}_{(0, T)}\right) *\left(\mid \sigma \mathbb{1}_{(0, T)}\right)(t), \quad t \in(0, T)
$$

where $\mathbb{1}_{(0, T)}$ denotes the characteristic function of $(0, T)$ and $*$ denotes the convolution product. Therefore, applying Young's inequality, we obtain $u \in L^{1}\left(0, T ; D\left(A^{\frac{d}{4}+\frac{3}{4}}\right)\right) \subset L^{1}\left(0, T ; C^{1}(\bar{\Omega})\right)$, showing the second assertion. In the same way, applying 2.7, we deduce 2.6.

Lemma 2.3. The following estimates hold

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n}\left|\left\langle v, \varphi_{n}\right\rangle\right|\left\|\mathcal{R}^{*} \varphi_{n}\right\|_{L^{\infty}(\partial \Omega)} \leq C\|v\|_{D\left(A^{s}\right)}, \forall v \in D\left(A^{s}\right), s>\frac{d}{2}+\frac{3}{2}  \tag{2.8}\\
& \mathcal{R}^{*} v(x)=\sum_{n=1}^{\infty}\left\langle v, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}(x), \quad x \in \partial \Omega, \quad \forall v \in D\left(A^{s}\right), s>\frac{d}{2}+\frac{1}{2} \tag{2.9}
\end{align*}
$$

Proof. Observe that, according to the Weyl's asymptotic formula [36], there exists $C>0$ such that $C^{-1} n^{\frac{2}{d}} \leq$ $\lambda_{n} \leq C n^{\frac{2}{d}}$, for all $n \geq 1$. Thus, we obtain for any $r>\frac{d}{2}$

$$
\begin{equation*}
\sum_{n=1}^{\infty} \lambda_{n}^{-r} \leq C \sum_{n=1}^{\infty} n^{-\frac{2}{d} r}<\infty \tag{2.10}
\end{equation*}
$$

Meanwhile, the Sobolev embedding theorem implies for any $n \in \mathbb{N}$ and $\epsilon>0$,

$$
\left\|\mathcal{R}^{*} \varphi_{n}\right\|_{L^{\infty}(\partial \Omega)} \leq C\left\|\varphi_{n}\right\|_{C^{1}(\bar{\Omega})} \leq C\left\|\varphi_{n}\right\|_{D\left(A^{\left.\frac{d}{4}+\frac{1}{2}+\epsilon\right)}\right.} \leq C \lambda_{n}^{\frac{d}{4}+\frac{1}{2}+\epsilon}
$$

Then the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_{n}\left|\left\langle v, \varphi_{n}\right\rangle\right|\left\|\mathcal{R}^{*} \varphi_{n}\right\|_{L^{\infty}(\partial \Omega)} \leq C \sum_{n=1}^{\infty} \lambda_{n}^{\frac{d}{4}+\frac{3}{2}+\epsilon}\left|\left\langle v, \varphi_{n}\right\rangle\right| \\
= & C \sum_{n=1}^{\infty} \lambda_{n}^{\frac{d}{4}+\frac{3}{2}+\epsilon+\frac{r}{2}}\left|\left\langle v, \varphi_{n}\right\rangle\right| \lambda_{n}^{-\frac{r}{2}} \leq C\left(\sum_{n=1}^{\infty} \lambda_{n}^{\frac{d}{2}+3+2 \epsilon+r}\left|\left\langle v, \varphi_{n}\right\rangle\right|^{2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} \lambda_{n}^{-r}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Combining this with 2.10 and the condition $v \in D\left(A^{s}\right)$ gives 2.8. This argument also shows that $\sum_{n=1}^{N}\left\langle v, \varphi_{n}\right\rangle \varphi_{n}, N \in \mathbb{N}$, converges in $C^{1}(\bar{\Omega})$ to $v$, which directly gives 2.9.

Remark 2.1. The regularity on $v$ in (2.8) can be relaxed to $v \in A^{s}, s>\frac{d}{2}+1$, if $\mathcal{R}^{*}$ is the Dirichlet trace operator, and a similar observation holds for the estimate (2.9).

## 3 Proof of Theorems 1.1 and 1.2

In this section, we give the proof of Theorems 1.1 and 1.2 ,

### 3.1 Proof of Theorem 1.1

Throughout this part, the assumption of Theorem 1.1 is fulfilled and prove that 1.7 implies that $N_{1}=N_{2}$ and 1.8 is fulfilled. Let $S_{2}^{j}$ correspond to 2.3 with $N=N_{j}, r_{k}=r_{k}^{j}, \alpha_{k}=\alpha_{k}^{j}, k=1, \ldots, N_{j}$ and with $A=A_{j}=\mathcal{A}_{j}$, acting in $L^{2}\left(\Omega_{j}\right)$ and with the boundary condition given by $\mathcal{R}=\mathcal{R}_{j}$. We divide the lengthy proof into three steps.
Step 1. In this step, we show that 1.7 implies

$$
\begin{equation*}
\mathcal{R}_{1}^{*}\left[\left(A_{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{-1} f_{1}\right]\left(x_{1}\right)=\mathcal{R}_{2}^{*}\left[\left(A_{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{-1} f_{2}\right]\left(x_{2}\right), \quad p \in \mathbb{R}_{+} . \tag{3.1}
\end{equation*}
$$

By Lemma 2.2, we have

$$
\mathcal{R}_{j}^{*} u_{j}\left(x_{j}, t\right)=\int_{0}^{t} \sigma(s) \mathcal{R}_{j}^{*}\left[S_{2}^{j}(t-s) f_{j}\right]\left(x_{j}\right) \mathrm{d} s, \quad t \in(0, T), j=1,2
$$

Let $v_{j}(t)=\mathcal{R}_{j}^{*}\left[S_{2}^{j}(t) f_{j}\right]\left(x_{j}\right), j=1,2$, as a function in $L^{1}(0, T)$, cf. Lemma 2.2. Therefore, condition 1.7) implies $\int_{0}^{t} \sigma(s)\left[v_{1}(t-s)-v_{2}(t-s)\right] \mathrm{d} s=0$, for $t \in(0, T)$. By Titchmarsh convolution theorem 34, Theorem VII $]$, there exist $T_{1}, T_{2} \in[0, T]$ such that $T_{1}+T_{2} \geq T, \sigma_{\mid\left(0, T_{1}\right)} \equiv 0$ and $\left(v_{1}-v_{2}\right)_{\mid\left(0, T_{2}\right)} \equiv 0$. Meanwhile, since $\sigma \not \equiv 0, T_{1}<T$. Thus, $T_{2}=T_{1}+T_{2}-T_{1} \geq T-T_{1}>0$ and

$$
\mathcal{R}_{1}^{*}\left[S_{2}^{1}(t) f_{1}\right]\left(x_{1}\right)=\mathcal{R}_{2}^{*}\left[S_{2}^{2}(t) f_{2}\right]\left(x_{2}\right), \quad t \in\left(0, T_{2}\right)
$$

The analyticity of the maps $\mathbb{R}_{+} \ni t \mapsto \mathcal{R}_{j}^{*}\left[S_{2}^{j}(t) f_{j}\right]\left(x_{j}\right), j=1,2$, given in Lemma 2.2 implies

$$
\begin{equation*}
\mathcal{R}_{1}^{*}\left[S_{2}^{1}(t) f_{1}\right]\left(x_{1}\right)=\mathcal{R}_{2}^{*}\left[S_{2}^{2}(t) f_{2}\right]\left(x_{2}\right), \quad t \in \mathbb{R}_{+} \tag{3.2}
\end{equation*}
$$

Moreover, applying the properties of the map 2.3 given in Section 2, we have

$$
\widehat{S}_{2}^{j}(t) f_{j}(p)=\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j}
$$

and by the arguments of Lemma 2.2 we obtain

$$
\mathcal{R}_{j}^{*} \widehat{S_{2}^{j}(t) f_{j}}\left(\cdot, x_{j}\right)(p)=\mathcal{R}_{j}^{*}\left[\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j}\right]\left(x_{j}\right)
$$

Combining this with (3.2) leads to (3.1).
Step 2. Now we fix $N=\min \left(N_{1}, N_{2}\right)$ and prove that condition (3.1) implies

$$
\left\{\begin{array}{l}
\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right),  \tag{3.3}\\
\alpha_{N_{1}}^{1}=\alpha_{N_{2}}^{2}, \ldots, \alpha_{N_{1}-N+1}^{1}=\alpha_{N_{2}-N+1}^{2} \\
r_{N_{1}}^{1}=r_{N_{2}}^{2}, \ldots, r_{N_{1}-N+1}^{1}=r_{N_{2}-N+1}^{2}
\end{array}\right.
$$

We prove this result iteratively using the asymptotic properties of

$$
\mathcal{R}_{j}^{*}\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j}\left(x_{1}\right), \quad j=1,2, p \rightarrow+\infty
$$

To this end, we fix $A=A_{j}$ with $\mathcal{A}=\mathcal{A}_{j}, \mathcal{R}=\mathcal{R}_{j}, \Omega=\Omega_{j}$, and denote the non-decreasing sequence of strictly positive eigenvalues of the operator $A_{j}$ by $\left(\lambda_{n}^{j}\right)_{n \geq 1}$ and an $L^{2}\left(\Omega_{j}\right)$ orthonormal basis of eigenfunctions $\left(\varphi_{n}^{j}\right)_{n \geq 1}$ associated with the eigenvalues $\left(\lambda_{n}^{j}\right)_{n \geq 1}$. Then, we have

$$
\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j}=\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)}}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}} \varphi_{n}^{j}, \quad p \in \mathbb{R}_{+}, j=1,2
$$

Since $f_{j} \in D\left(A_{j}^{r}\right), r>\frac{d+3}{2}$, we have $\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j} \in D\left(A_{j}^{r}\right)$. Moreover, following the argument of Lemma 2.3, we obtain

$$
\mathcal{R}_{j}^{*}\left(A_{j}+\sum_{k=1}^{N_{1}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} f_{j}\left(x_{j}\right)=\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}, \quad p \in \mathbb{R}_{+}, j=1,2
$$

Combining this with (3.1) gives

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{\left\langle f_{1}, \varphi_{n}^{1}\right\rangle_{L^{2}\left(\Omega_{1}\right)} \mathcal{R}_{1}^{*} \varphi_{n}^{1}\left(x_{1}\right)}{\lambda_{n}^{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}}=\sum_{n=1}^{\infty} \frac{\left\langle f_{2}, \varphi_{n}^{2}\right\rangle_{L^{2}\left(\Omega_{2}\right)} \mathcal{R}_{2}^{*} \varphi_{n}^{2}\left(x_{2}\right)}{\lambda_{n}^{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}}, \quad p \in \mathbb{R}_{+} \tag{3.4}
\end{equation*}
$$

Meanwhile, since $f_{j} \in D\left(A_{j}^{r}\right), j=1,2$, by the mean value theorem, we deduce

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}} \\
& =\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}-\sum_{n=1}^{\infty} \int_{0}^{1} \frac{\lambda_{n}^{j}\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\left(\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}+s \lambda_{n}^{j}\right)^{2}} \mathrm{~d} s .
\end{aligned}
$$

By Lemma 2.3, there holds

$$
\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)=\sum_{n=1}^{\infty}\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)
$$

and thus we obtain

$$
\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}=\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}-\sum_{n=1}^{\infty} \int_{0}^{1} \frac{\lambda_{n}^{j}\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\left(\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}+s \lambda_{n}^{j}\right)^{2}} \mathrm{~d} s
$$

Then, it follows that, for $p>1$, there holds

$$
\left|\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}-\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}\right| \leq C \frac{\sum_{n=1}^{\infty} \lambda_{n}^{j}\left|\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)\right|}{\left(\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{2}}
$$

Thus, Lemma 2.3 implies

$$
\left|\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}-\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}\right| \leq C \frac{\left\|f_{j}\right\|_{D\left(A_{j}^{r}\right)}}{\left(\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{2}}
$$

Therefore, for $j=1,2$, we have

$$
\sum_{n=1}^{\infty} \frac{\left\langle f_{j}, \varphi_{n}^{j}\right\rangle_{L^{2}\left(\Omega_{j}\right)} \mathcal{R}_{j}^{*} \varphi_{n}^{j}\left(x_{j}\right)}{\lambda_{n}^{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}=\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{\left(\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{2}}\right)
$$

and by combining this with 3.4 , we obtain

$$
\begin{equation*}
\frac{\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)}{\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{\left(\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{2}}\right)=\frac{\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)}{\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{\left(\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{2}}\right) \tag{3.5}
\end{equation*}
$$

Next we use this identity to prove (3.3). We start by proving

$$
\begin{equation*}
\alpha_{N_{1}}^{1}=\alpha_{N_{2}}^{2}, \quad r_{N_{1}}^{1}=r_{N_{2}}^{2}, \quad \mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right) \tag{3.6}
\end{equation*}
$$

Recall that, for $j=1,2$,

$$
\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}=\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{r_{N_{j}}^{j} p^{\alpha_{N_{j}}^{j}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(p^{\alpha_{N_{j}-1}^{j}-2 \alpha_{N_{j}}^{j}}\right) .
$$

Then we deduce from (3.5) that

$$
\frac{\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)}{r_{N_{1}}^{1} p^{\alpha_{N_{1}}^{1}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(p^{\alpha_{N_{1}-1}^{1}-2 \alpha_{N_{1}}^{1}}\right)=\frac{\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)}{r_{N_{2}}^{2} p^{\alpha_{N_{2}}^{2}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(p^{\alpha_{N_{2}-1}^{1}-2 \alpha_{N_{2}}^{2}}\right)
$$

In view of this identity, the fact that $(1.6)$ is fulfilled and the fact that $\alpha_{N_{j-1}}^{j}<\alpha_{N_{j}}^{j}$, we deduce

$$
\alpha_{N_{1}}^{1}=\alpha_{N_{2}}^{2}, \quad \mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)\left(r_{N_{1}}^{1}\right)^{-1}=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)\left(r_{N_{2}}^{2}\right)^{-1}
$$

Combining this with condition (i) or (ii) of Theorem 1.1, we obtain (3.6). Now assume that there exists $\ell \in\{0, \ldots, N-2\}$ such that the following condition is fulfilled

$$
\begin{equation*}
\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right), \alpha_{N_{1}}^{1}=\alpha_{N_{2}}^{2}, \ldots, \alpha_{N_{1}-\ell}^{1}=\alpha_{N_{2}-\ell}^{2}, r_{N_{1}}^{1}=r_{N_{2}}^{2}, \ldots, r_{N_{1}-\ell}^{1}=r_{N_{2}-\ell}^{2} . \tag{3.7}
\end{equation*}
$$

We claim that this condition implies

$$
\begin{equation*}
\alpha_{N_{1}}^{1}=\alpha_{N_{2}}^{2}, \ldots, \alpha_{N_{1}-\ell-1}^{1}=\alpha_{N_{2}-\ell-1}^{2}, \quad r_{N_{1}}^{1}=r_{N_{2}}^{2}, \ldots, r_{N_{1}-\ell-1}^{1}=r_{N_{2}-\ell-1}^{2} . \tag{3.8}
\end{equation*}
$$

Indeed, by letting $Q_{\ell}^{j}(p)=\sum_{k=0}^{\ell} r_{N_{j}-k}^{j} p^{\alpha_{N_{j}-k}^{j}}$, for $j=1$, 2 , we find

$$
\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}}=\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right)}{Q_{\ell}^{j}(p)}-\frac{\mathcal{R}_{j}^{*} f_{j}\left(x_{j}\right) r_{N_{j}-\ell-1}^{j} p^{\alpha_{N_{j}-\ell-1}^{j}}}{Q_{\ell}^{j}(p)^{2}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{p^{\alpha_{N_{j}-\ell-2}^{j}}}{Q_{\ell}^{j}(p)^{2}}\right)
$$

where $\alpha_{0}^{j}=0$. Combining this with (3.5) leads to

$$
\begin{aligned}
& \frac{\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)}{Q_{\ell}^{1}(p)}-\frac{\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right) r_{N_{1}-\ell-1}^{1} p^{\alpha_{N_{1}-\ell-1}^{1}}}{Q_{\ell}^{1}(p)^{2}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{p^{\alpha_{N_{1}-\ell-2}^{1}}}{Q_{\ell}^{1}(p)^{2}}\right) \\
= & \frac{\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)}{Q_{\ell}^{2}(p)}-\frac{\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right) r_{N_{2}-\ell-1}^{2} p^{\alpha_{N_{2}-\ell-1}^{2}}}{Q_{\ell}^{2}(p)^{2}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{p^{\alpha_{N_{2}-\ell-2}^{2}}}{Q_{\ell}^{2}(p)^{2}}\right) .
\end{aligned}
$$

Further, condition (3.7) implies that, for all $p>0, Q_{\ell}^{1}(p)=Q_{\ell}^{2}(p)$ and $\mathcal{R}_{1}^{*} f_{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right)$. Therefore, this identity and conditions (1.6) and (3.1) imply the claim (3.8). Combining the iteration argument with (3.6) implies that (3.3) holds.

Step 3. In this step we assume that 3.3 holds and complete the proof by proving $N_{1}=N_{2}$. Indeed, assuming that $N_{1} \neq N_{2}$, we may assume $N_{1}<N_{2}$. Then, using (1.6), 3.3) and fixing

$$
Q_{N_{1}}(p)=\sum_{k=0}^{N_{1}-1} r_{N_{1}-k}^{1} p^{\alpha_{N_{1}-k}^{1}}=\sum_{k=0}^{N_{1}-1} r_{N_{2}-k}^{2} p^{\alpha_{N_{2}-k}^{2}}, \quad b=\mathcal{R}_{1}^{*} f_{1}\left(x_{2}\right)=\mathcal{R}_{2}^{*} f_{2}\left(x_{2}\right) \neq 0
$$

and, repeating the arguments of Step 2, we deduce that

$$
\frac{b}{Q_{N_{1}}(p)}-\frac{b r_{N_{2}-N_{1}}^{2} p^{\alpha_{N_{2}-N_{1}}^{2}}}{Q_{N_{1}}(p)^{2}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{p^{\alpha_{N_{2}-N_{1}-1}^{2}}}{Q_{N_{1}}(p)^{2}}\right)=\frac{b}{Q_{N_{1}}(p)}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{Q_{N_{1}}(p)^{2}}\right),
$$

with the convention $\alpha_{0}^{2}=0$. Since $b \neq 0, r_{N_{2}-N_{1}}^{2}>0$ and $\alpha_{N_{2}-N_{1}}^{2}>0$, the above identity cannot be true. This leads to a contradiction and $N_{1}=N_{2}$. Therefore, condition (3.3) implies (1.8). This completes the proof of Theorem 1.1.

### 3.2 Proof of Theorem 1.2

Assume that there exist $T_{1}, T_{2} \in[0, T]$, with $T_{1}<T_{2}$, such that 1.10 is fulfilled and we show that $N_{1}=N_{2}$ and conditions 1.8 and 1.11 are fulfilled under the assumptions of Theorem 1.1. Following the argumentation of Section 2.1, we deduce that, for $j=1,2$, we have $u_{j}(\cdot, t)=S_{1}^{j}(t) u_{0}$, with $S_{1}^{j}$ corresponding to 2.2 with $N=N_{j}, r_{k}=r_{k}^{j}, \alpha_{k}=\alpha_{k}^{j}, k=1, \ldots, N_{j}$ and with $A=A_{j}=\mathcal{A}_{j}$, acting in $L^{2}\left(\Omega_{j}\right)$ and with the boundary condition given by $\mathcal{R}=\mathcal{R}_{j}$. Then, condition 1.10 implies

$$
\mathcal{R}_{1}^{*}\left[S_{1}^{1}(t) u_{0}^{1}\right]\left(x_{1}, t\right)=\mathcal{R}_{2}^{*}\left[S_{1}^{2}(t) u_{0}^{2}\right]\left(x_{2}, t\right), \quad t \in\left(T_{1}, T_{2}\right)
$$

This and the analiticity of the maps $\mathbb{R}_{+} \ni t \mapsto \mathcal{R}_{j}^{*}\left[S_{1}^{j}(t) u_{0}^{j}\right]\left(x_{j}\right), j=1,2$, cf. Lemma 2.2 , give

$$
\mathcal{R}_{1}^{*}\left[S_{1}^{1}(t) u_{0}^{1}\right]\left(x_{1}, t\right)=\mathcal{R}_{2}^{*}\left[S_{1}^{2}(t) u_{0}^{2}\right]\left(x_{2}, t\right), \quad t \in \mathbb{R}_{+}
$$

Therefore, repeating the argument in the first step of Theorem 1.1. we deduce that 1.10 implies

$$
\begin{aligned}
& p^{-1} \mathcal{R}_{1}^{*}\left[\left(A_{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{-1}\left(\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right) u_{0}^{1}\right]\left(x_{1}\right) \\
= & \mathcal{R}_{1}^{*}\left[\left(A_{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{-1}\left(\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}-1}\right) u_{0}^{1}\right]\left(x_{1}\right) \\
= & \mathcal{R}_{2}^{*}\left[\left(A_{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{-1}\left(\sum_{k=1}^{N_{1}} r_{k}^{2} p^{\alpha_{k}^{2}-1}\right) u_{0}^{2}\right]\left(x_{2}\right) \\
= & p^{-1} \mathcal{R}_{2}^{*}\left[\left(A_{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{-1}\left(\sum_{k=1}^{N_{1}} r_{k}^{2} p^{\alpha_{k}^{2}}\right) u_{0}^{2}\right]\left(x_{2}\right) .
\end{aligned}
$$

Multiplying both sides of this identity by $p$, we obtain

$$
\begin{align*}
& \mathcal{R}_{1}^{*} u_{0}^{1}\left(x_{1}\right)-\mathcal{R}_{1}^{*}\left[\left(A_{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{-1} A_{1} u_{0}^{1}\right]\left(x_{1}\right)  \tag{3.9}\\
= & \mathcal{R}_{2}^{*} u_{0}^{2}\left(x_{2}\right)-\mathcal{R}_{2}^{*}\left[\left(A_{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{-1} A_{2} u_{0}^{2}\right]\left(x_{2}\right), \quad p \in \mathbb{R}_{+}
\end{align*}
$$

Moreover, repeating the arguments of the preceding section we deduce

$$
\lim _{p \rightarrow+\infty} \mathcal{R}_{j}^{*}\left[\left(A_{j}+\sum_{k=1}^{N_{j}} r_{k}^{j} p^{\alpha_{k}^{j}}\right)^{-1} A_{j} u_{0}^{j}\right]\left(x_{j}\right)=0, \quad j=1,2
$$

Therefore, condition (3.9) implies that $\mathcal{R}_{1}^{*} u_{0}^{1}\left(x_{1}\right)=\mathcal{R}_{2}^{*} u_{0}^{2}\left(x_{2}\right)$ and consequently,

$$
\mathcal{R}_{1}^{*}\left[\left(A_{1}+\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{-1} A_{1} u_{0}^{1}\right]\left(x_{1}\right)=\mathcal{R}_{2}^{*}\left[\left(A_{2}+\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{-1} A_{2} u_{0}^{2}\right]\left(x_{2}\right), \quad p \in \mathbb{R}_{+} .
$$

Combining this identity with the arguments of Step 2 of Theorem 1.1 leads to

$$
\frac{\mathcal{R}_{1}^{*} A_{1} u_{0}^{1}\left(x_{1}\right)}{\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{\left(\sum_{k=1}^{N_{1}} r_{k}^{1} p^{\alpha_{k}^{1}}\right)^{2}}\right)=\frac{\mathcal{R}_{2}^{*} A_{2} u_{0}^{2}\left(x_{2}\right)}{\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}}+\underset{p \rightarrow+\infty}{\mathcal{O}}\left(\frac{1}{\left(\sum_{k=1}^{N_{2}} r_{k}^{2} p^{\alpha_{k}^{2}}\right)^{2}}\right)
$$

Therefore, repeating the arguments used in Steps 2 and 3 of Theorem 1.1, we deduce that $N_{1}=N_{2}$, and conditions 1.8 and 1.11 hold.

Remark 3.1. In view of the t-analyticity of the solution $u\left(x_{0}, t\right)$, the assertion in Theorem 1.2 remains valid, if the time trace $\mathcal{R}^{*} u\left(x_{0}, t\right)$ is only observed at a countable discrete set $t_{1}<t_{2}<\ldots$ with an accumulation point in the open interval $\left(T_{1}, T_{2}\right)$.

Remark 3.2. Theorem 1.2 can also be seen as follows. The solution $u$ of problem (1.1) with $\sigma \equiv 0$ takes the form $u=u_{0}+v$, with $v$ solving

$$
\left\{\begin{aligned}
& \sum_{j=1}^{N} r_{j} \partial_{t}^{\alpha_{j}} v+\mathcal{A} v=-\mathcal{A} u_{0}, \quad \text { in } \Omega \times(0,+\infty), \\
& \mathcal{R} v=0, \\
& v=0, \quad \text { on } \partial \Omega \times(0,+\infty), \\
& v \times\{0\}
\end{aligned}\right.
$$

In terms of Laplace transform, this implies

$$
\hat{u}(p)=\frac{u_{0}}{p}+\hat{v}(p)=p^{-1}\left[u_{0}-\left(A+\sum_{k=1}^{N_{1}} r_{k} p^{\alpha_{k}}\right)^{-1} A u_{0}\right]
$$

which leads directly to formula 3.9 .

## 4 Numerical experiments and discussions

In this section, we illustrate the feasibility of the recovery of orders and the associated weights with a set of numerical experiments, and discuss the potential pitfalls.

### 4.1 Asymptotic expansion

First we develop an algorithm for the numerical recovery, inspired by the analysis. Note that to recover the orders $\alpha_{j}$ and weights $r_{j}$, one classical approach is to apply the regularization method, e.g., Tikhonov regularization [7], which involves a misfit on the measured data (and proper regularization), with the forward map defined implicitly by problem (1.1) (as done recently in [22, 33] for recovering one single order). Unfortunately, this approach does not apply in the setting of this work, since the medium (and thus the forward map) is unknown. Instead, we employ a more direct approach, which is inspired by the uniqueness analysis in Section 3. In the spirit of the classical Karamata-Feller Tauberian theorem 4, Section XIII.5], the asymptotic of the Laplace transform $\hat{u}\left(x_{0}, p\right)$ as $p \rightarrow \infty$ corresponds to the asymptotic of the function $u\left(x_{0}, t\right)$ as $t \rightarrow 0^{+}$. Indeed, the argument for the uniqueness result in Theorem 1.2 is essentially about the asymptotic behavior of the measured data $\mathcal{R}^{*} u\left(x_{0}, t\right)$. We have the following asymptotic expansion, which lays the foundation of the procedure for the numerical recovery.

Proposition 4.1. Let $\left(\Omega, a, q, f, u_{0}\right)$ be an admissible tuple. If $u_{0} \not \equiv 0$ and $f \equiv 0$, the solution $u$ to problem (1.1) satisfies the following asymptotic:

$$
\mathcal{R}^{*} u\left(x_{0}, t\right)=\mathcal{R}^{*} u_{0}\left(x_{0}\right)-\mathcal{R}^{*} A u_{0}\left(x_{0}\right)\left(r_{N} \frac{t^{\alpha_{N}}}{\Gamma\left(\alpha_{N}+1\right)}-\sum_{i=1}^{N-1} \frac{r_{i} t^{2 \alpha_{N}-\alpha_{i}}}{r_{N}^{2} \Gamma\left(2 \alpha_{N}-\alpha_{i}+1\right)}\right)+\mathcal{O}\left(t^{2 \alpha_{N}}\right)
$$

Similarly, if $u_{0} \equiv 0$ and $f \not \equiv 0$ and $\sigma \in L^{1}(0, T)$ with

$$
\hat{\sigma}(p):=\mathcal{L}[\sigma](p)=c_{0} p^{-a-1}+\mathcal{O}\left(p^{-a-2}\right) \quad \text { as } p \rightarrow \infty
$$

for some $a \in[0,1]$, then there holds

$$
\mathcal{R}^{*} u\left(x_{0}, t\right)=c_{0} \mathcal{R}^{*} f\left(x_{0}\right)\left(r_{N} \frac{t^{\alpha_{N}+a}}{\Gamma\left(\alpha_{N}+a+1\right)}-\sum_{i=1}^{N-1} \frac{r_{i} t^{2 \alpha_{N}-\alpha_{i}+a}}{r_{N}^{2} \Gamma\left(2 \alpha_{N}-\alpha_{i}+a+1\right)}\right)+\mathcal{O}\left(t^{2 \alpha_{N}}\right)
$$

Proof. Indeed, the proof of Theorem 1.2 gives the following expansion for $u\left(x_{0}, p\right)$ :

$$
\mathcal{R}^{*} \hat{u}\left(x_{0}, p\right)=\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{N} r_{i} p^{\alpha_{i}-1}}{\lambda_{n}+\sum_{i=1}^{N} r_{i} p^{\alpha_{i}}}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)
$$

$$
=p^{-1} \sum_{n=1}^{\infty}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)-\sum_{n=1}^{\infty} \frac{\lambda_{n} p^{-1}}{\lambda_{n}+\sum_{i=1}^{N} r_{i} p^{\alpha_{i}}}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)
$$

In view of the following identity for large $p$

$$
\begin{aligned}
& \frac{1}{\lambda+\sum_{i=1}^{N} r_{i} p^{\alpha_{i}}}=\frac{1}{r_{N} p^{\alpha_{N}}} \frac{1}{1+\frac{\lambda+\sum_{i=1}^{N-1} r_{i} p^{\alpha_{i}}}{r_{N} p^{\alpha} N}} \\
= & \frac{1}{r_{N} p^{\alpha_{N}}}\left(1-\frac{\lambda+\sum_{i=1}^{N-1} r_{i} p^{\alpha_{i}}}{r_{N} p^{\alpha_{N}}}\right)+\mathcal{O}\left(p^{-3 \alpha_{N}+\alpha_{N-1}}\right) \\
= & \frac{r_{N} p^{\alpha_{N}}-\sum_{i=1}^{N-1} r_{i} p^{\alpha_{i}}}{\left(r_{N} p^{\alpha_{N}}\right)^{2}}+\mathcal{O}\left(p^{-2 \alpha_{N}}\right)
\end{aligned}
$$

The definition of the eigenvalue problem $A \varphi_{n}=\lambda_{n} \varphi_{n}$, integration by parts twice and the completeness of the eigenfunctions $\left(\varphi_{n}\right)_{n=1}^{\infty}$ in $L^{2}(\Omega)$ lead to

$$
\begin{align*}
& \sum_{n=1}^{\infty} \lambda_{n}\left\langle u_{0}, \varphi_{n}\right\rangle \varphi_{n}(x)=\sum_{n=1}^{\infty}\left\langle u_{0}, \lambda_{n} \varphi_{n}\right\rangle \varphi_{n}(x) \\
= & \sum_{n=1}^{\infty}\left\langle u_{0}, A \varphi_{n}\right\rangle \varphi_{n}(x)=\sum_{n=1}^{\infty}\left\langle A u_{0}, \varphi_{n}\right\rangle \varphi_{n}(x)=A u_{0}(x) . \tag{4.1}
\end{align*}
$$

Substituting the last two identities, we obtain

$$
\mathcal{R}^{*} \hat{u}\left(x_{0}, p\right)=p^{-1} \sum_{n=1}^{\infty}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)-\frac{r_{N} p^{\alpha_{N}}-\sum_{i=1}^{N-1} r_{i} p^{\alpha_{i}}}{p\left(r_{N} p^{\alpha_{N}}\right)^{2}} \mathcal{R}^{*} A u_{0}\left(x_{0}\right)+\mathcal{O}\left(p^{-2 \alpha_{N}-1}\right)
$$

Now the Karamata-Feller Tauberian theorem 44, Section XIII. 5] and the standard inverse Laplace transform relation $\mathcal{L}^{-1}\left[p^{-s-1}\right]=\frac{t^{s}}{\Gamma(s)}$ for $s \geq 0$ directly imply the first assertion. Similarly, the proof of Theorem 1.1 and repeating the argument lead to the following expansion as $p \rightarrow \infty$

$$
\begin{aligned}
& \mathcal{R}^{*} \hat{u}\left(x_{0}, p\right)=\sum_{n=1}^{\infty} \frac{\hat{\sigma}(p)}{\lambda_{n}+\sum_{i=1}^{N} r_{i} p^{\alpha_{i}}}\left\langle f, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right) \\
= & \left(\frac{1}{r_{N} p^{\alpha_{N}+a+1}}-\sum_{i=1}^{N-1} \frac{r_{i}}{r_{N}^{2} p^{2 \alpha_{N}-\alpha_{i}+a+1}}\right) \sum_{n=1}^{\infty}\left\langle f, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)+\mathcal{O}\left(p^{-2 \alpha_{N}-a-1}\right) \\
= & \left(\frac{1}{r_{N} p^{\alpha_{N}+a+1}}-\sum_{i=1}^{N-1} \frac{r_{i}}{r_{N}^{2} p^{2 \alpha_{N}-\alpha_{i}+a+1}}\right) \mathcal{R}^{*} f\left(x_{0}\right)+\mathcal{O}\left(p^{-2 \alpha_{N}-a-1}\right) .
\end{aligned}
$$

Then by the inverse Laplace transform and Karamata-Feller theorem [4, Section XIII. 5], we obtain the second expression.

Remark 4.1. The result in Proposition 4.1 can also be seen as follows. Recall the multinomial Mittag-Leffler function $E_{\left(\beta_{1}, \ldots, \beta_{m}\right), \beta_{0}}\left(z_{1}, \ldots, z_{m}\right)$ defined by [24, 23]

$$
E_{\left(\beta_{1}, \ldots, \beta_{m}\right), \beta_{0}}\left(z_{1}, \ldots, z_{m}\right)=\sum_{k=0}^{\infty} \sum_{k_{1}+\ldots+k_{m}=k} \frac{\left(k ; k_{1}, \ldots, k_{m}\right) \prod_{j=1}^{m} z_{j}^{k_{j}}}{\Gamma\left(\beta_{0}+\sum_{j=1}^{m} \beta_{j} k_{j}\right)}
$$

where $0<\beta_{0}<2,0<\beta_{j}<1$ and $z_{j} \in \mathbb{C}, j=1, \ldots, m$, and the notation $\left(k ; k_{1}, \ldots, k_{m}\right)$ denotes the multinomial coefficient

$$
\left(k ; k_{1}, \ldots, k_{m}\right)=\frac{k!}{k_{1}!\ldots k_{m}!}, \quad \text { for } k=k_{1}+\ldots+k_{m}, \quad k_{1}, \ldots, k_{m} \geq 0
$$

This function is a generalization of the classical two-parameter Mittag-Leffler function $E_{\alpha, \beta}(z)$, for $\alpha \in(0,2)$ and $\beta \in \mathbb{R}$ defined by $E_{\alpha, \beta}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(k \alpha+\beta)}$, for $z \in \mathbb{C}$, which is an entire function generalizing the familiar exponential function [8, Section 3.1]. Then assuming $r_{N}=1$, the solution $u$ to problem (1.1) with $f \equiv 0$ is given by (noting $0<\alpha_{1}<\ldots<\alpha_{N}<1$ ) [23, 18]

$$
u(x, t)=\sum_{n=1}^{\infty}\left(1-\lambda_{n} t^{\alpha_{N}} E_{\tilde{\boldsymbol{\alpha}}, 1+\alpha_{N}}\left(-\lambda_{n} t^{\alpha_{N}},-r_{1} t^{\alpha_{N}-\alpha_{1}}, \ldots,-r_{N-1} t^{\alpha_{N}-\alpha_{N-1}}\right)\right)\left\langle u_{0}, \varphi_{n}\right\rangle \varphi_{n}(x)
$$

with $\tilde{\boldsymbol{\alpha}}=\left(\alpha_{N}, \alpha_{N}-\alpha_{1}, \ldots, \alpha_{N}-\alpha_{N-1}\right)$ Thus, for small $t$, the identity 4.1) implies

$$
\begin{aligned}
\mathcal{R}^{*} u\left(x_{0}, t\right)= & \sum_{n=1}^{\infty}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)-\frac{t^{\alpha_{N}}}{\Gamma\left(\alpha_{N}+1\right)} \sum_{n=1}^{\infty} \lambda_{n}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right) \\
& +\sum_{i=1}^{N-1} \frac{r_{i} t^{2 \alpha_{N}-\alpha_{i}}}{\Gamma\left(2 \alpha_{N}-\alpha_{i}+1\right)} \sum_{n=1}^{\infty} \lambda_{n}\left\langle u_{0}, \varphi_{n}\right\rangle \mathcal{R}^{*} \varphi_{n}\left(x_{0}\right)+\mathcal{O}\left(t^{2 \alpha_{N}}\right) \\
= & \sum_{n=1}^{\infty} \mathcal{R}^{*} u_{0}\left(x_{0}\right)-\mathcal{R}^{*} A u_{0}\left(x_{0}\right)\left(\frac{t^{\alpha_{N}}}{\Gamma\left(\alpha_{N}+1\right)}-\sum_{i=1}^{N-1} \frac{r_{i} t^{2 \alpha_{N}-\alpha_{i}}}{\Gamma\left(2 \alpha_{N}-\alpha_{i}+1\right)}\right)+\mathcal{O}\left(t^{2 \alpha_{N}}\right)
\end{aligned}
$$

Thus we have deduced the desired asymptotic expansion in Proposition 4.1 when $r_{N}=1$, and the general case follows by a simple scaling argument. Note that the summation in the bracket actually has the opposite sign of the leading term, and the constants are fully determined by $r_{i}$ and $\alpha_{i}$. Meanwhile, by the KaramataFeller Tauberian theorem [4, Section XIII. 5], the condition on the function $\sigma$ can be restated as $\sigma(t)=$ $\frac{c_{0}}{\Gamma(a+1)} t^{a}+\mathcal{O}\left(t^{a+1}\right)$ as $t \rightarrow \overline{0}^{+}$, and the expression can be derived similarly using the identity

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} t^{\alpha_{N}} E_{\tilde{\boldsymbol{\alpha}}, 1+\alpha_{N}}\left(-\lambda_{n} t^{\alpha_{N}},-r_{1} t^{\alpha_{N}-\alpha_{1}}, \cdots,-r_{N-1} t^{\alpha_{N}-\alpha_{N-1}}\right) \\
= & t^{\alpha_{N}-1} E_{\tilde{\boldsymbol{\alpha}}, \alpha_{N}}\left(-\lambda_{n} t^{\alpha_{N}},-r_{1} t^{\alpha_{N}-\alpha_{1}}, \cdots,-r_{N-1} t^{\alpha_{N}-\alpha_{N-1}}\right),
\end{aligned}
$$

and the fact that the solution $u(x, t)$ to problem (1.1) with $u_{0} \equiv 0$ is given by

$$
u(t)=\sum_{n=1}^{\infty} \int_{0}^{t} s^{\alpha_{N}-1} E_{\tilde{\boldsymbol{\alpha}}, \alpha_{N}}\left(-\lambda_{n} s^{\alpha_{N}},-r_{1} s^{\alpha_{N}-\alpha_{1}}, \ldots,-r_{N-1} s^{\alpha_{N}-\alpha_{N-1}}\right) \sigma(t-s) \mathrm{d} s\left\langle f, \varphi_{n}\right\rangle \varphi_{n}
$$

Motivated by Proposition 4.1, naturally one can develop a procedure for numerically recovering the orders $\alpha_{i}$ and the corresponding weights $r_{i}$. The most direct approach is to fit the fractional powers by a nonlinear least-squares formulation:

$$
\begin{equation*}
J(\mathbf{c}, \boldsymbol{\beta})=\frac{1}{2}\left\|u\left(x_{0}, t\right)-f(\mathbf{c}, \boldsymbol{\beta})\right\|_{L^{2}\left(0, T_{0}\right)}^{2} \tag{4.2}
\end{equation*}
$$

with the regressor / model function $f_{p}(\mathbf{c}, \boldsymbol{\beta})$ given by

$$
f_{p}(\mathbf{c}, \boldsymbol{\beta})=\left\{\begin{aligned}
c_{0}+\sum_{i=1}^{N} c_{i} t^{\beta_{i}}, & \text { if } u_{0} \not \equiv 0, f \equiv 0 \\
\sum_{i=1}^{N} c_{i} t^{\beta_{i}}, & \text { if } u_{0} \equiv 0, f \not \equiv 0
\end{aligned}\right.
$$

provided that the regressor $f(\mathbf{c}, \boldsymbol{\beta})$ approximates $u\left(x_{0}, t\right)$ well. Nonlinear least-squares problems of this type have been employed in statistics [28]. Once the parameters $\mathbf{c}$ and $\boldsymbol{\beta}$ are determined, the orders $\alpha_{i}$ and the weights $r_{i}$ can be easily deduced. Numerical experiments indicate that this condition indeed holds under certain assumptions: either $\alpha_{N}$ is close to one, or $T_{0}$ is sufficiently close to zero, which are however not known a priori and potentially can restrict the range of application. This restriction is severe when $\alpha_{N}$ is close to zero, and probably alternative models are needed. We shall discuss one alternative based on rational approximation.

### 4.2 Numerical results

Now we present numerical results to illustrate the feasibility of recovering the orders and weights from time trace data $g(t)=u\left(x_{0}, t\right)$, when the direct problem 1.1) is equipped with a zero Neumann boundary condition and the medium is unknown. Problem (4.2) uses 100 points uniformly distributed on the interval [0, $T_{0}$ ], for the time horizon $T_{0}$. It is minimized via the stand-alone algorithm L-BFGS-B [2], implemented via MATLAB wrapper from https://www.mathworks.com/matlabcentral/fileexchange/35104-lbfgsb-l-bfgs-b-mex-wrapper (retrieved on May 6, 2021). L-BFGS-B is a quasi-Newton type algorithm that can take care of box constraints on the unknown, and requires computing the value and gradient of the objective function. For the examples presented below, the algorithm converges within tens of iterations (which of course depends strongly on the initial guess), and thus the overall procedure is fairly efficient. Note that problem (4.2) is highly nonconvex for both model functions (i.e., fractional polynomials and rational functions), and thus a good initial guess is needed in order to ensure reasonable recovery.

First we discuss a simple yet illuminating example.
Example 4.1. The domain $\Omega=(0,1), A u=-u^{\prime \prime}+u$ with a zero Neumann boundary condition, $u_{0}=\cos \pi x$, and $f \equiv 0$. The measurement point $x_{0}$ is the left end point $x_{0}=0$. (i) One single-order with $\alpha \in(0,1)$ and (ii) Two terms with $0<\alpha_{1}<\alpha_{2}<1$ and the corresponding weights $r_{1}$ and $r_{2}=1$. Note that $u_{0}$ is actually an eigenfunction of the operator $A$, with the corresponding eigenvalue $\lambda=\pi^{2}+1$, and thus the direct problem essentially reduces to an ODE. The goal is to recover $\boldsymbol{\alpha}$ and $\lambda$ from the data $g(t)=u\left(x_{0}, t\right)$.

Note that for this example, the data $g(t)$ admits a closed form

$$
g(t)=\left\{\begin{aligned}
E_{\alpha, 1}\left(-\lambda t^{\alpha}\right), & \text { case (i) } \\
1-\lambda t^{\alpha_{2}} E_{\left(\alpha_{2}, \alpha_{2}-\alpha_{1}\right), 1+\alpha_{2}}\left(-\lambda t^{\alpha_{2}},-r_{1} t^{\alpha_{2}-\alpha_{1}}\right), & \text { case (ii). }
\end{aligned}\right.
$$

By Proposition 4.1. the asymptotic of $g(t)$ as $t \rightarrow 0^{+}$is given by

$$
g(t)=\left\{\begin{aligned}
1-\lambda \frac{t^{\alpha}}{\Gamma(\alpha+1)}+\mathcal{O}\left(t^{2 \alpha}\right), & \text { case (i) } \\
1-\lambda \frac{t^{\alpha_{2}}}{\Gamma\left(\alpha_{2}+1\right)}+\lambda \frac{r_{1} t^{2 \alpha_{2}-\alpha_{1}}}{\Gamma\left(2 \alpha_{2}-\alpha_{1}+1\right)}+\mathcal{O}\left(t^{2 \alpha_{2}}\right), & \text { case (ii) }
\end{aligned}\right.
$$

Since the feasibility of the formulation 4.2 resides on the accuracy of the fractional polynomial approximation, denoted by $f_{p}(\mathbf{c}, \boldsymbol{\alpha})$ below, we investigate the accuracy of the approximation. We evaluate the function $E_{\alpha, 1}\left(-\lambda t^{\alpha}\right)$ over a small time interval [ $0, T_{0}$ ] by an algorithm from [31, and the multinomial Mittag-Leffler function by summing the power series over $k$ (truncated at $k=100$ ), since the series converges rapidly for a small argument (we are not aware of any algorithm for evaluating the multinomial Mittag-Leffler functions). The numerical results are shown in Figs. 1 and 2 , for the single- and two-term, respectively. Note that the function $f_{p}$ can approximate the target function $g(t)$ over a suitable interval [ $0, T_{0}$ ] accurately, but the size $T_{0}$ depends strongly on the order $\alpha$. In particular, as the order $\alpha$ tends to zero, the size $T_{0}$ should shrink to zero so as to maintain the desired accuracy. Thus, the least-squares approach 4.2 based on the model $f_{p}$ should only make use of data within $\left[0, T_{0}\right.$ ] with a tiny $T_{0}$, in order to handle a broad range of $\alpha$. This observation remains largely valid in the two-term cases, i.e., the $T_{0}$ should be chosen small when both orders are small, cf. Fig. 2

These empirical observations naturally motivate the question whether there is actually a better model than fractional polynomials for the numerical recovery. In the simple setting, the answer is affirmative. One promising model is the lowest-order rational approximation $f_{r}$ given by

$$
f_{r}(t)= \begin{cases}\frac{1}{1+\lambda \frac{t^{\alpha}}{\Gamma(\alpha+1)}}, & \text { case (i) } \\ \frac{1}{1+\lambda \frac{t^{\alpha_{2}}}{\Gamma\left(\alpha_{2}+1\right)}-\lambda \frac{r_{1} t^{2 \alpha_{2}-\alpha_{1}}}{\Gamma\left(2 \alpha_{2}-\alpha_{1}+1\right)}}, & \text { case (ii). }\end{cases}
$$

Actually, it is known that for (i), $f_{r}$ is actually an upper bound on $E_{\alpha, 1}\left(-\lambda t^{\alpha}\right)$ (see e.g., 32] or [8, Theorem 3.6], and empirically observed by Mainardi [25]). Numerically, the model $f_{r}$ is asymptotically tight as $t \rightarrow 0^{+}$,
just as the model $f_{p}$, but it approximates better $g(t)$. We are not aware of the two-term case in the existing literature, but one can glim a hint on the approximation from [35, Lemma 6.1] and fractional polynomial approximation of the resolvent kernel. The numerical results in Figs. 1 and 2 show clearly that the model $f_{r}$ is indeed a more accurate approximation to the data $g(t)$, for both single- and two-term cases, although there is still no rigorous proof of the empirical observation yet. Thus, it is conceived that the rational model $f_{r}$ is better suited as the regressor for the numerical recovery via the nonlinear least-squares approach 4.2 (of course, modulus the challenge of nonlinear optimization).


Figure 1: The Mittag-Leffler function $g(t)=E_{\alpha, 1}\left(-\lambda t^{\alpha}\right)$ and the fractional polynomial approximation $f_{p}$ and rational approximation $f_{r}$ for Example 4.1(i).


Figure 2: The multinomial Mittag-Leffler function $g(t)=1-\lambda t^{\alpha_{2}} E_{\left(\alpha_{2}, \alpha_{2}-\alpha_{1}\right), 1+\alpha_{2}}\left(-\lambda t^{\alpha_{2}},-r_{1} t^{\alpha_{2}-\alpha_{1}}\right)$ and the fractional polynomial approximation $f_{p}$ and rational approximation $f_{r}$ for Example 4.1(ii) with $r_{1}=0.5$.

Next we present results for numerical recovery of the orders for Example 4.1. The numerical results are presented in Tables 1 and 2 for cases (i) and (ii), respectively. It is observed that for a fixed $\alpha=0.7$, the recovered $\alpha$ represents a very accurate approximation to the exact one, when the time horizon $T_{0}$ is sufficiently close to zero, and then both models $f_{p}$ and $f_{r}$ have comparable accuracy. However, when $T_{0}$ increases, the results by the model $f_{r}$ is much more accurate than that by the model $f_{p}$, which agrees with the preceding observation that $f_{r}$ is more broadly valid as an approximation to the target function $g(t)$. Meanwhile, when $T_{0}$ is fixed at $1 \mathrm{e}-6$, the accuracy of the recovered order $\alpha$ deteriorates steadily as the true $\alpha$ decreases towards zero, for either model function, although the results by the model function $f_{r}$ are far more accurate, especially when the exact $\alpha$ is small. This can be attributed to the behavior of $E_{\alpha, 1}\left(-\lambda t^{\alpha}\right)$ : at small $\alpha, E_{\alpha, 1}\left(-\lambda t^{\alpha}\right)$ reaches a quasi-steady state very rapidly, cf. Fig. 1(a), and the fractional polynomial model $f_{p}$ fails to accurately capture the behavior, whereas the rational model $f_{r}$ does so more closely.

The numerical results for the two term case in Example 4.1 (ii) are summarized in Table 2, It is observed that with proper initialization, the method does recover the orders $\left(\alpha_{1}, \alpha_{2}\right)$ to a reasonable accuracy, for a wide range of the time horizon $T_{0}$, with the accuracy of $\alpha_{2}$ being higher than that of $\alpha_{1}$. The latter can be attributed to the asymptotic expansion: the term involving $\alpha_{2}$ is dominating in the expansion and much easier to estimate than the remaining terms. Indeed, if $T_{0}$ is sufficiently small, all other terms are essentially negligible, comparing the results in Figs. 1 and 2, and consequently, the order $\alpha_{1}$ and the weight $r_{1}$ cannot be estimated reliably at all. This can also been seen from the following slightly more refined expansion in

Table 1: Numerical results for Example $4.1(\mathrm{i}), \lambda=\pi^{2}+1 \approx 1.0870$, and the algorithm L-BFGS-B is always initialized to $\alpha=0.5$, and $c$ by the standard linear least-squares method. (a) Fixed $\alpha=0.7$, recovered with different $T_{0}$, and (b) recovered with $T_{0}=1 \mathrm{e}-6$, for different $\alpha$.
(a)

|  | $f_{p}$ |  |  | $f_{r}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}$ | $\alpha$ | $\lambda$ |  | $\alpha$ | $\lambda$ |
| $1 \mathrm{e}-7$ | 0.6993 | 1.074 e 1 |  | 0.6995 | 1.078 e 1 |
| $1 \mathrm{e}-6$ | 0.6998 | 1.083 e 1 |  | 0.7001 | 1.088 e 1 |
| $1 \mathrm{e}-5$ | 0.6989 | 1.071 e 1 |  | 0.7005 | 1.095 e 1 |
| $1 \mathrm{e}-4$ | 0.6947 | 1.022 e 1 |  | 0.7026 | 1.121 e 1 |
| $1 \mathrm{e}-3$ | 0.6742 | 8.532 e 0 |  | 0.7129 | 1.228 e 1 |
| $1 \mathrm{e}-2$ | 0.5856 | 4.779 e 0 |  | 0.7570 | 1.648 e 1 |
| $1 \mathrm{e}-1$ | 0.3455 | 1.660 e 0 |  | 0.8563 | 2.690 e 1 |

(b)

|  | $f_{p}$ |  |  | $f_{r}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha$ | $\alpha$ | $\lambda$ |  | $\alpha$ | $\lambda$ |
| 0.30 | 0.2669 | 5.916 e 0 |  | 0.3034 | 1.157 e 1 |
| 0.40 | 0.3891 | 9.008 e 0 |  | 0.4019 | 1.124 e 1 |
| 0.50 | 0.4969 | 1.032 e 1 |  | 0.5008 | 1.102 e 1 |
| 0.60 | 0.5991 | 1.072 e 1 |  | 0.6003 | 1.092 e 1 |
| 0.70 | 0.6998 | 1.083 e 1 |  | 0.7001 | 1.088 e 1 |
| 0.80 | 0.7995 | 1.079 e 1 |  | 0.7990 | 1.072 e 1 |
| 0.90 | 0.8937 | 9.906 e 0 |  | 0.8909 | 9.521 e 0 |

Table 2: Numerical results for Example 4.1(ii), $\lambda=\pi^{2}+1 \approx 1.0870$, and the algorithm L-BFGS-B, and the corresponding $c$ by the standard linear least-squares method.
(a) $\boldsymbol{\alpha}=(0.6,0.9)$, L-BFGS-B initialized to $\boldsymbol{\alpha}=(0.4,0.8)$

|  | $f_{p}$ |  |  |  |  |  |  |  |  |  |  |  | $f_{r}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\lambda$ | $r_{1}$ |  | $\alpha_{1}$ | $\alpha_{2}$ | $\lambda$ | $r_{1}$ |  |  |  |  |  |  |  |
| $1 \mathrm{e}-7$ | 0.5971 | 0.8985 | 1.056 e 1 | 0 |  | 0.5971 | 0.8985 | 1.056 e 1 | 0 |  |  |  |  |  |  |  |
| $1 \mathrm{e}-6$ | 0.5943 | 0.8972 | 1.032 e 1 | $5.49 \mathrm{e}-12$ |  | 0.5944 | 0.8972 | 1.033 e 1 | $6.61 \mathrm{e}-13$ |  |  |  |  |  |  |  |
| $1 \mathrm{e}-5$ | 0.5887 | 0.8943 | 9.938 e 0 | $1.04 \mathrm{e}-11$ | 0.5890 | 0.8945 | 9.963 e 0 | $1.53 \mathrm{e}-11$ |  |  |  |  |  |  |  |  |
| $1 \mathrm{e}-4$ | 0.5766 | 0.8883 | 9.284 e 0 | $1.59 \mathrm{e}-9$ |  | 0.5794 | 0.8897 | 9.432 e 0 | $1.25 \mathrm{e}-09$ |  |  |  |  |  |  |  |
| $1 \mathrm{e}-3$ | 0.4227 | 0.8955 | 1.015 e 1 | 2.08 e 0 |  | 0.5844 | 0.8922 | 9.830 e 0 | $3.36 \mathrm{e}-01$ |  |  |  |  |  |  |  |
| $1 \mathrm{e}-2$ | 0.3530 | 0.8931 | 9.860 e 0 | 2.82 e 0 |  | 0.5788 | 0.8894 | 9.220 e 0 | 0 |  |  |  |  |  |  |  |

(b) $\boldsymbol{\alpha}=(0.5,0.7)$, L-BFGS-B initialized to $\boldsymbol{\alpha}=(0.2,0.6)$

|  | $f_{p}$ |  |  |  |  |  | $f_{r}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\lambda$ | $r_{1}$ |  | $\alpha_{1}$ | $\alpha_{2}$ | $\lambda$ | $r_{1}$ |  |  |
| $1 \mathrm{e}-7$ | 0.3889 | 0.6944 | 9.616 e 0 | $2.28 \mathrm{e}-12$ |  | 0.3890 | 0.6945 | 9.626 e 0 | $5.67 \mathrm{e}-11$ |  |  |
| $1 \mathrm{e}-6$ | 0.3824 | 0.6912 | 9.129 e 0 | $2.41 \mathrm{e}-11$ |  | 0.3830 | 0.6915 | 9.174 e 0 | $5.53 \mathrm{e}-11$ |  |  |
| $1 \mathrm{e}-5$ | 0.3713 | 0.6856 | 8.466 e 0 | $1.03 \mathrm{e}-9$ |  | 0.3742 | 0.6871 | 8.639 e 0 | $8.70 \mathrm{e}-10$ |  |  |
| $1 \mathrm{e}-4$ | 0.3914 | 0.6959 | 9.954 e 0 | 1.50 e 0 |  | 0.5870 | 0.7010 | 1.160 e 1 | $4.71 \mathrm{e}-1$ |  |  |
| $1 \mathrm{e}-3$ | 0.2804 | 0.6906 | 9.156 e 0 | 2.60 e 0 |  | 0.3625 | 0.6803 | 8.051 e 0 | $9.89 \mathrm{e}-2$ |  |  |
| $1 \mathrm{e}-2$ | 0.5378 | 0.7315 | 1.633 e 1 | 1.41 e 0 | -0.5278 | 0.6860 | 8.261 e 0 | 0 |  |  |  |

fractional polynomials:

$$
1-\lambda t^{\alpha_{2}} E_{\left(\alpha_{2}, \alpha_{2}-\alpha_{1}\right), 1+\alpha_{2}}\left(-\lambda t^{\alpha_{2}},-r_{1} t^{\alpha_{2}-\alpha_{1}}\right)=f_{p}(t)+\lambda^{2} \frac{t^{2 \alpha_{2}}}{\Gamma\left(2 \alpha_{2}+1\right)}+\mathcal{O}\left(t^{3 \alpha_{2}-\alpha_{1}}\right)
$$

This clearly shows the potential pitfalls in the order recovery: the next term can have comparable magnitude with the last term $f_{p}$ when $\alpha_{1}$ is close to zero, and the nonlinear procedure attempts to approximate it with the leading terms, thereby significantly affecting the recovery accuracy. The accuracy of the recovered $\lambda$ is also reasonable, except for fairly large $T_{0}$. However, the accuracy of $r_{1}$ is poor in all cases, due to the aforementioned reasons. Also as the orders decrease, it is becoming increasingly more challenging for the numerical recovery, which agrees with the empirical observation from Example 4.1(i), cf. Table 2(b). This is
attributed to the rapid decay near $t=0$ so that the model functions are not accurate. These results partly confirm the assertion in Theorem 1.2 the recovery is indeed possible, however, numerically this can still be a big challenge, depending on the magnitude of the sought-for orders. It is of great interest to develop further remedies to tackle the numerical issues.

The next example illustrates the feasibility of the approach in the general setting.
Example 4.2. The domain $\Omega=(0,1)^{2}, A u=-\Delta u$ with zero Neumann boundary condition, with a twoterm model with $0<\alpha_{1}<\alpha_{2}<1$ and the corresponding weights $r_{1}$ and $r_{2}=1$. The measurement point $x_{0}$ is the vertex point $x_{0}=(0,0)$. Consider the following two cases: (i) $u_{0}\left(x_{1}, x_{2}\right)=\cos \pi x_{1} \cos \pi x_{2}+$ $\frac{1}{4}\left(\cos 2 \pi x_{1} \cos \pi x_{2}+\cos \pi x_{1} \cos 2 \pi x_{2}\right)+\frac{1}{8} \cos 2 \pi x_{1} \cos 2 \pi x_{2}$ and $f \equiv 0$ and (ii) $u_{0} \equiv 0$ and $f\left(x_{1}, x_{2}, t\right)=$ $\cos \pi x_{1} \cos \pi x_{2}+\frac{1}{2}\left(\cos 2 \pi x_{1} \cos \pi x_{2}+\cos \pi x_{1} \cos 2 \pi x_{2}\right)+\frac{1}{4}\left(\cos 3 \pi x_{1} \cos \pi x_{2}+\cos \pi x_{1} \cos 3 \pi x_{2}\right)$. The goal is to recover the following quantities: $\boldsymbol{\alpha}, r_{1}$ and $A u\left(x_{0}\right)$ or $f\left(x_{0}\right)$ for cases (i) and (ii), respectively, from the data $g(t)=u\left(x_{0}, t\right)$.

Note that for this example, the data $g(t)=u\left(x_{0}, t\right)$ can be generated using the standard Galerkin finite element method in space and convolution quadrature in time 10. Below we employ series expansion using the multinomial Mittag-Leffler function, cf. Remark 4.1, so as to minimize the discretization errors, and like before, it is evaluated by means of series summation (truncated at $k=100$ ). In case (ii), the model functions $f_{p}$ and $f_{r}$ are given respectively by

$$
\begin{aligned}
& f_{p}(t)=f\left(x_{0}\right)\left(\frac{t^{\alpha_{2}}}{\Gamma\left(\alpha_{2}+1\right)}-\frac{r_{1} t^{2 \alpha_{2}-\alpha_{1}}}{\Gamma\left(2 \alpha_{2}-\alpha_{1}+1\right)}\right) \\
& f_{r}(t)=f\left(x_{0}\right)\left(1-\frac{1}{1+\frac{t^{\alpha_{2}}}{\Gamma\left(\alpha_{2}+1\right)}-\frac{r_{1} t^{2 \alpha_{2}-\alpha_{1}}}{\Gamma\left(2 \alpha_{2}-\alpha_{1}+1\right)}}\right)
\end{aligned}
$$

where the latter follows by direct analogy. The numerical results are given in Table 3. The observations from Example 4.1 remain largely valid. In both cases (i) and (ii), the orders can be accurately recovered, provided that the time horizon $T_{0}$ is sufficiently small (so that the model functions are accurate approximations). The accuracy of the two models are largely comparable to each other in either case, and deteriorates steadily as $T_{0}$ increases. Also the accuracy of the recovered $A u\left(x_{0}\right)=5.428 \mathrm{e} 1$ and $f\left(x_{0}\right)=2.500$ is fair. Just as expected, the accuracy of the estimated $r_{1}$ is poor in all cases, irrespective of the time horizon $T_{0}$, which also agrees with preceding observations. These numerical experiments not only confirm the possibility of uniquely recovery as indicated by Theorems 1.1 and 1.2 , but also illustrate the pitfalls in developing practical recovery schemes.

## 5 Conclusions

In this work we have proved the unique recovery of multiple fractional orders and the associated weights in a multi-term time-fractional diffusion model from the observation at one point on the boundary, based on the asymptotics of the solution at small time and the time analyticity of the solution. We have also discussed the numerical recovery based on asymptotic expansion / rational approximation, and demonstrated the feasibility of the least-squares approach for recovering the highest order and the weight.

It is of much interest to study related inverse problems for more complex anomalous diffusion models, e.g., distributed-order or variable-order. It is unclear whether the one-point observation is sufficient for the unique recovery of the order distribution in these models, but a partial determination, e.g., support in the distributed order, might be possible. Further, it remains an outstanding challenge to develop stable and accurate numerical procedures for recovering all fractional orders, by properly overcoming the difficulty with unknown media.

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Table 3: Numerical results for Example 4.2, and the algorithm L-BFGS-B, and the corresponding $c$ by the standard linear least-squares method.
(a) $\boldsymbol{\alpha}=(0.5,0.8)$, L-BFGS-B initialized to $\boldsymbol{\alpha}=(0.3,0.7)$

|  | $f_{p}$ |  |  |  |  | $f_{r}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $A u\left(x_{0}\right)$ | $r_{1}$ |  | $\alpha_{1}$ | $\alpha_{2}$ | $A u\left(x_{0}\right)$ | $r_{1}$ |
| $1 \mathrm{e}-8$ | 0.4992 | 0.7996 | 5.380 e 1 | 0 |  | 0.4992 | 0.7996 | 5.381 e 1 | 0 |
| $1 \mathrm{e}-7$ | 0.4984 | 0.7992 | 5.340 e 1 | 0 |  | 0.4985 | 0.7992 | 5.344 e 1 | 0 |
| $1 \mathrm{e}-6$ | 0.4965 | 0.7983 | 5.261 e 1 | 0 |  | 0.4971 | 0.7985 | 5.284 e 1 | 0 |
| $1 \mathrm{e}-5$ | 0.4913 | 0.7956 | 5.079 e 1 | 0 |  | 0.4946 | 0.7973 | 5.196 e 1 | 0 |
| $1 \mathrm{e}-4$ | 0.4716 | 0.7858 | 4.551 e 1 | 0 |  | 0.4920 | 0.7960 | 5.118 e 1 | 0 |

(b) $\boldsymbol{\alpha}=(0.5,0.7)$, L-BFGS-B initialized to $\boldsymbol{\alpha}=(0.3,0.6)$

|  | $f_{p}$ |  |  |  |  | $f_{r}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $f\left(x_{0}\right)$ | $r_{1}$ |  | $\alpha_{1}$ | $\alpha_{2}$ | $f\left(x_{0}\right)$ | $r_{1}$ |
| $1 \mathrm{e}-8$ | 0.4964 | 0.6982 | 2.393 | 0 |  | 0.4964 | 0.6982 | 2.393 | $1.29 \mathrm{e}-10$ |
| $1 \mathrm{e}-7$ | 0.4941 | 0.6970 | 2.343 | $1.27 \mathrm{e}-10$ |  | 0.4941 | 0.6970 | 2.343 | $9.13 \mathrm{e}-10$ |
| $1 \mathrm{e}-6$ | 0.5023 | 0.7015 | 2.583 | $6.41 \mathrm{e}-1$ |  | 0.4896 | 0.6948 | 2.261 | $2.11 \mathrm{e}-9$ |
| $1 \mathrm{e}-5$ | 0.2083 | 0.6974 | 2.366 | 8.56 e 0 |  | 0.2167 | 0.6975 | 2.367 | 7.54 e 0 |
| $1 \mathrm{e}-4$ | 0.1669 | 0.6969 | 2.343 | 1.23 e 1 |  | 0.1668 | 0.6967 | 2.337 | 1.16 e 1 |

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    ${ }^{\dagger}$ Department of Computer Science, University College London, Gower Street, London WC1E 6BT, UK (b.jin@ucl.ac.uk)
    $\ddagger$ Aix Marseille Université, Université de Toulon, CNRS, CPT, Marseille, France (yavar.kian@univ-amu.fr)

