Decentralised Tracking Control for a Class of Nonlinear Interconnected Systems Using Sliding Mode Techniques*

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Abstract— **This paper proposes a decentralised tracking control scheme for a class of nonlinear interconnected systems using sliding mode techniques. The desired output signals are time-varying. Both matched uncertainty and mismatched unknown interconnections are considered. Using geometric transformation, the considered system is transferred to a system with a unique structure to facilitate both the design of the sliding surface and the decentralised controllers. The sliding surface is designed based on the tracking error. A set of conditions are proposed to guarantee that when sliding motion occurs, the tracking errors converge to zero asymptotically while the system states are bounded. Decentralised controllers are then designed so that the interconnected systems' states can be driven to the designed sliding surface. Finally, simulation of a coupled inverted pendulum system demonstrates the results.**

I. INTRODUCTION

With the advancement of technology comes a need to deal with more complex systems, which may be large-scale, to meet practical engineering requirements. Large-scale systems are usually composed of a set of dynamical subsystems which may be distributed over space. The communications between different subsystems may become difficult or expensive due to the transfer of data over large distances. In particular, when the data transformation paths connecting various subsystems are broken or blocked, some data may be lost or, in the worst case, no data from the other systems may be available at all. Centralised control will not work in this case. Conversely, decentralised control needs local information only, and it does not require any of the other subsystems' state information. Thus it provides a practical approach for the control of large-scale interconnected systems.

During the past few decades, many results have been obtained for interconnected systems. A fuzzy controller based on a reduced observer is designed for interconnected descriptor systems using integral sliding mode control in [7]. Mahmoud proposed a decentralised control strategy for interconnected time-delay systems in [10] where the considered system is linear. The finite-time control problem is investigated for nonlinear interconnected systems with dead-zone input in [4], and robust controllers are designed for an interconnected multimachine power system using output feedback sliding mode control in [17]. A decentralised control scheme is proposed for fully nonlinear interconnected system with time delay in [16]. It should be noted that most of these results are focused on stabilisation using either state feedback or output feedback. Compared with the stabilisation problem, the results for tracking control are limited, particularly for the case of large scale nonlinear interconnected systems.

The tracking problem is an important topic in control engineering. Most work related to tracking control is focused on centralised control (see, e.g. [8], [1], [15]). Tracking control for interconnected systems has been studied in [12], [13] where the isolated subsystems are assumed to be linear. Narendra and Zhang studied a class of interconnected systems in [11] where the considered systems are linear, and model reference tracking control is considered. Tracking control for interconnected systems is considered in [9] using integral reinforcement learning. However, it is required that the interconnection terms are matched. Also, Han and Yan propose an observer-based adaptive tracking control of large-scale stochastic nonlinear systems in [2] which increases the dimension of the closed-loop system and thus will increase the computational load required for implementation. This wrk considers both matched and mismatched terms. In most of the existing results considering tracking control for interconnected systems, the developed results are not decentralised and hence not convenient for parctical implementation. Sliding mode control is a popular method due to its high robustness and this has been widely applied to deal with tracking problems (see, e.g. [18], [3], [19]). However, the results on decentralised tracking control using sliding mode techniques for nonlinear interconnected systems are not available when the desired signal is time-varying. Furthermore, compared with adaptive control approaches, there are fewer restrictions on the uncertainty bound when using sliding mode control which means the bound is allowed to be an arbitrary function and not only a constant.

In this paper, a class of nonlinear interconnected systems is considered where both the matched uncertainty in the isolated subsystem and the mismatched unknown interconnections are considered. A nonlinear coordinate transformation is introduced to transfer the nominal isolated subsystem to the required form, facilitating the system analysis and control design. The sliding surface is designed based on the tracking errors, and sliding mode stability is achieved. A decentralised sliding mode control is designed to drive the nonlinear

*This work was supported by the National Natural Science Foundation of China under Grants 61922042 and 62020106003, China Scholarship Council for 3 years' study at the University of Kent, and Qing Lan Project.

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interconnected system to the designed sliding surface in finite time. Finally, the results obtained are applied to a coupled inverted pendulum system, and simulation shows that the method proposed in this paper is effective.

II. SYSTEM DESCRIPTION AND BASIC ASSUMPTIONS

Consider a nonlinear large-scale system formed by N interconnected subsystems as follows:

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_i(x_i)(u_i + \varphi_i(x_i)) + p_i(x_i)\psi_i(x) \\ y_i &= h_i(x_i) \qquad i = 1,2,...,N \end{aligned} \tag{1}$$

where $x = col(x_1, x_2, ..., x_N) \in \Pi$, $x_i \in \Pi_i \subset R^{n_i}$, $u_i \in R$ and $y_i \in R$ are the states, input and output of the ith subsystem respectively, Π_i are neighbourhoods of the origin and $\Pi := \Pi_1 \times ... \times \Pi_N \in R^{\sum_{i=1}^N n_i}$. The terms $\varphi_i(x_i) \in R$ are matched uncertainties and $p_i(x_i)\psi_i(x) \in R^{n_i}$ represent unknown interconnections of the ith subsystem for $i = 1,2,\ldots,N$. All of the nonlinear functions are assumed to be continuous in their arguments to guarantee the existence and uniqueness of the system solutions.

In this paper, the local case will be considered, and the considered domain may not be specified unless necessary.

The objective of this paper is, for a given ideal output signal $y_{id}(t)$, to design a decentralised control such that the output $y_i(t)$ of the controlled system (1) can track the ideal signal $y_{id}(t)$, i.e.

$$\lim_{t\to\infty} |y_i(t) - y_{id}(t)| = 0; \quad i = 1,2,...,N \tag{2}$$

while all the state variables of system (1) are bounded. To deal with the tracking problem stated above, some assumptions on the considered system (1) are required.

Assumption 1. There exist known continuous functions $\rho_i(x_i)$ defined in domain Π_i and positive constants M_i such that for $x_i \in \Pi_i$ with $i = 1,2,...,N$.

(i). $|\varphi_i(x_i)| \leq \rho_i(x_i)$. (ii). $|\psi_i(x)| \leq M_i$.

Remark 1. Assumption 1 implies that all of the uncertainties in the system (1) are required to be bounded, and the bounds are known. The bounds on the uncertainties will be used to design a decentralised control to cancel the effects of the corresponding uncertainties.

Assumption 2. For system (1), the triple (f_i, g_i, h_i) has an uniform relative degree r_i^a in the domain Π_i, the triple (f_i, p_i, h_i) has an uniform relative degree r_i^b in the domain Π_i, and $r_i^a = r_i^b$ for $i = 1,2,\ldots,N$. Furthermore, both distributions generated by the column vectors of function matrices $g_i(x_i)$ and $p_i(x_i)$ respectively, are involutive in the domain Π_i for $i = 1,2,\ldots,n$.

Remark 2. The definition of a uniform relative degree above is available in [5]. The uniform relative degree implies that, for any point $x_i \in \Pi_i$, the system has the same relative degree, which means the relative degree is independent of $x_i \in \Pi_i$. For further discussion about the relative degree, see [5].

Assumption 3. The desired output signals $y_{id}(t)$ and their time derivatives up to the r_i^a-th order are smooth, known and bounded for all $t \in [0,\infty)$.

Remark 3. Assumption 3 requires that the ideal output signals $y_{id}(t)$ are differentiable for a sufficient number of times. This assumption is quite standard and can be satisfied in most cases in reality.

III. SYSTEM STRUCTURE ANALYSIS

Consider the nonlinear interconnected system in (1). Under Assumption 2, it follows from [5] that there exist diffeomorphisms $z_i = T_i(x_i)$ defined in Π_i with $z := col(z_1, z_2, \cdots, z_N)$, described by

$$\begin{bmatrix} x_{i,1} \\ \cdots \\ x_{i,r_i^a} \\ \cdots \\ x_{i,n_i} \end{bmatrix} \xrightarrow{z_i = T_i(x_i)} \begin{bmatrix} z_{i,1} \\ \cdots \\ z_{i,r_i^a} \\ \cdots \\ z_{i,n_i} \end{bmatrix} =: \begin{bmatrix} \xi_{i,1} \\ \cdots \\ \xi_{i,r_i^a} \\ \cdots \\ \eta_{i,n_i} \end{bmatrix} \tag{3}$$

and the feedback transformation

$$u_i = \varpi_i^{-1}(x_i)(-\varsigma_i(x_i) + v_i(t)) \tag{4}$$

where $v_i(t)$ is the new controller to be designed later. The $\varsigma_i(x_i)$ and $\varpi_i(x_i)$ are defined by

$$\varsigma_i(x_i) = L_{f_i}^{r_i^a} h_i(x_i), \ \varpi_i(x_i) = L_{g_i} L_{f_i}^{r_i^a - 1} h_i(x_i) \tag{5}$$

and after a coordinate transformation $z_i = T_i(x_i)$:

$$\alpha_i(z_i) = \varsigma_i(x_i)|_{x_i = T_i^{-1}(z_i)}, \ \beta_i(z_i) = \varpi_i(x_i)|_{x_i = T_i^{-1}(z_i)}$$

Here, the notation $L_{g_i} L_{f_i}^{r_i^a - 1} h_i(x_i)$ denotes the Lie derivative defined in [5]. The new variables $\xi_i := col(\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,r_i^a})$ and $\eta_i := col(\eta_{i,(r_i^a+1)}, \cdots, \eta_{i,n_i})$ are introduced for ease of exposition.

Under the diffeomorphism (3) and the feedback transformation (4), it follows from [5] that in the new coordinates, the system (1) can be described by

$$\begin{aligned} \dot{\xi}_{i,1} &= \xi_{i,2} \\ &\cdots \\ \dot{\xi}_{i,(r_i^a - 1)} &= \xi_{i,r_i^a} \\ \dot{\xi}_{i,r_i^a} &= v_i(t) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \\ \dot{\eta}_{i,(r_i^a+1)} &= q_{i,(r_i^a+1)}(z_i) + \Gamma_{i,(r_i^a+1)}\delta_i(z) \\ &\cdots \\ \dot{\eta}_{i,n_i} &= q_{i,n_i}(z_i) + \Gamma_{i,n_i}\delta_i(z) \end{aligned} \tag{6}$$

where $z_i := col(\xi_i, \eta_i)$, and $z = col(z_1, z_2, \cdots, z_N)$, and

$$\tau_i(z_i) = [\varphi_i(x_i)]|_{x_i = T_i^{-1}(z_i)} \tag{7}$$

$$\gamma_i(z_i) = L_{p_i} L_{f_i}^{r_i^b - 1} h_i(T_i^{-1}(z_i)) \tag{8}$$

$$\delta_i(z) = [\psi_i(x)]|_{x = T^{-1}(z)} \tag{9}$$

The system can be expressed in a compact form as

$$\begin{aligned} \dot{\xi}_i &= A_i\xi_i + B_i[v_i + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z)] \\ \dot{\eta}_i &= q_i(\xi_i, \eta_i) + \Gamma_i(\xi_i, \eta_i)\delta_i(\xi_1, \eta_1, ..., \xi_N, \eta_N) \\ y_i &= C_i\xi_i \qquad i = 1,2,...,N \end{aligned} \tag{10}$$

where $z_i = col(\xi_i, \eta_i)$ with $\xi_i \in R^{r_i^a}$ and $\eta_i \in R^{(n_i - r_i^a)}$, the triple (A_i, B_i, C_i) with appropriate dimensions has a standard *Brunovsky* form as follows:

$$\mathbf{A_i} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{B_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{11}$$

$$\mathbf{C_i} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{12}$$

$q_i(\xi_i, \eta_i)$ and $\Gamma_i(\xi_i, \eta_i)$ are the last $n_i - r_i^a$ rows of the vectors

$$\left[\frac{\partial T_i}{\partial x_i} f_i(x_i)\right]_{x_i = T_i^{-1}(z_i)} \quad \text{and} \quad \left[\frac{\partial T_i}{\partial x_i} p_i(x_i)\right]_{x_i = T_i^{-1}(z_i)}$$

respectively.

From (10), it is clear to see that in this paper it is not required that the nominal subsystem of system (1) is feedback linearizable.

Remark 4. If the relative degree $r_i^a = n_i$, then the system (10) will become

$$\begin{aligned} \dot{\xi}_{i,1} &= \xi_{i,2} \\ &\dots \\ \dot{\xi}_{i,(n_i-1)} &= \xi_{i,n_i} \\ \dot{\xi}_{i,n_i} &= v_i(t) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \\ y_i &= \xi_{i,1} \end{aligned} \tag{13}$$

In this case the nominal isolated subsystem of the interconnected system (10) is completely feedback linearizable and thus the nonlinear part relating to the dynamics of variables η_i in system (10) disappears.

IV. SLIDING MODE BASED OUTPUT TRACKING CONTROL DESIGN

The main results are now presented.

A. Sliding Surface Design

Consider the desired output signal $y_{id}(t)$ satisfying Assumption 3. Then for system (10), the output tracking error e_i is defined by:

$$e_i = y_i(t) - y_{id}(t) \qquad i = 1, 2, ..., N \tag{14}$$

The following sliding function is considered:

$$S_i(\cdot) = e_i^{(r_i^a - 1)} + a_{i,1} e_i^{(r_i^a - 2)} + \dots + a_{i,(r_i^a - 2)} e_i^{(1)} + a_{i,(r_i^a - 1)} e_i^{(0)} \tag{15}$$

where $a_{i,1}, a_{i,2}, ..., a_{i,(r_i^a - 1)}$ are a set of design parameters which is chosen such that the polynomials

$$\lambda^{r_i^a - 1} + a_{i,1}\lambda^{r_i^a - 2} + \dots + a_{i,(r_i^a - 2)}\lambda + a_{i,(r_i^a - 1)} \tag{16}$$

are *Hurwitz* stable for $i = 1, 2, \dots, N$. The corresponding sliding surface can be described by

$$\sum_i : \quad \{S = col(S_1, S_2, \cdots, S_N) \mid \ S_i = 0, \ i = 1, 2, ..., N\} \tag{17}$$

where S_i is defined in (15) above. From the design above, it is clear to see that

$$\lim_{t \to \infty} e_i(t) = 0$$

when $S_i = 0$. This implies that when sliding motion occurs,

$$\lim_{t \to \infty} |y_i(t) - y_{id}(t)| = \lim_{t \to \infty} e_i(t) = 0 \tag{18}$$

i.e. the output $y_i(t)$ of system (1) tracks the ideal signal $y_{id}(t)$ asymptotically for $i = 1, 2, \dots, N$. The following result is now ready to be presented:

Theorem 1: Consider the interconnected system (10). Under Assumption 3, when the system(10) is limited to moving on the sliding surface (17),

i) $\lim_{t \to \infty} |y_i(t) - y_{id}(t)| = \lim_{t \to \infty} |e_i(t)| = 0$,
ii) the states ξ_i in system (10) are bounded

for $i = 1, 2, ..., N$.

Proof: i). This result has been shown above in (18). It remains to show that the result ii) holds.

ii). When the system (10) is constrained to the sliding surface, it follows that: $S_i = 0$, *i.e.*

$$\begin{aligned} S_i &= e_i^{(r_i^a - 1)} + a_{i,1} e_i^{(r_i^a - 2)} + \dots + a_{i,(r_i^a - 2)} e_i^{(1)} + a_{i,(r_i^a - 1)} e_i^{(0)} = 0 \\ e_i^{(r_i^a - 1)} &= -a_{i,1} e_i^{(r_i^a - 2)} - \dots - a_{i,(r_i^a - 2)} e_i^{(1)} - a_{i,(r_i^a - 1)} e_i^{(0)} \end{aligned}$$

Consider the error system as follows and let: $e_{i,1} \triangleq e_i^{(0)}$. Then,

$$\begin{aligned} \dot{e}_{i,1} &= e_i^{(1)} \triangleq e_{i,2} \\ &\dots \\ \dot{e}_{i,(r_i^a - 2)} &= e_i^{(r_i^a - 2)} \triangleq e_{i,(r_i^a - 1)} \\ \dot{e}_{i,(r_i^a - 1)} &= -a_{i,1} e_{i,(r_i^a - 1)} - \dots - a_{i,(r_i^a - 2)} e_{i,2} - a_{i,(r_i^a - 1)} e_{i,1} \end{aligned}$$

The system above can be rewritten in the following compact form:

$$\dot{e}_i = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 \\ -a_{i,(r_i^a - 1)} & -a_{i,(r_i^a - 2)} & -a_{i,(r_i^a - 3)} & \dots & -a_{i,1} \end{bmatrix}}_{E_i} e_i$$

where the error system matrix E_i is in the controllable canonical form and its last row $a_{i,1}, a_{i,2}, ..., a_{i,(r_i^a - 1)}$ forms a *Hurwitz polynomial*. Therefore, the error system is stable and, $\lim_{t \to \infty} e_i(t) = 0$ which implies that

$$\lim_{t \to \infty} \left\| \begin{matrix} \xi_{i,1} - y_{id}^{(0)} \\ \dots \\ \xi_{i,(r_i^a - 1)} - y_{id}^{(r_i^a - 2)} \\ \xi_{i,r_i^a} - y_{id}^{(r_i^a - 1)} \end{matrix} \right\| = 0$$

Further, from Assumption 3, the desired output signal $y_{id}(t)$ and its derivative: $y_{id}^{(1)}, y_{id}^{(2)}, ..., y_{id}^{(r_i^a)}$ are bounded for all $t \in [0, \infty]$. It follows that the state variables ξ_i in system (10) are bounded when limited to moving on the sliding surface (17). Hence the result follows. ■

B. Decentralised Sliding Mode Controller Design

For system (10), the following control law is proposed:

$$v_i = -\dot{S}_i + y_i^{(r_i^a)} - K(z_i)sgn(S_i), \qquad i = 1,2,...,N \quad (19)$$

where the function $K(z_i)$ is the feedback gain to be designed later, $S_i(\cdot)$ is given in (15) and $sgn(\cdot)$ is the sign function. The closed-loop system obtained by applying the control law (19) into system (10) can be described by:

$$\begin{aligned} \dot{\xi}_i &= A_i\xi_i + B_i[-\dot{S}_i + y_i^{(r_i^a)} - K(z_i)sgn(S_i) \\ &\quad +\beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z)] \quad (20) \\ \dot{\eta}_i &= q_i(\xi_i,\eta_i) + \Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N) \quad (21) \\ y_i &= C_i\xi_i \qquad i = 1,2,...,N \quad (22) \end{aligned}$$

With the special structure of the triple (A_i,B_i,C_i) in (10), it follows that

$$\begin{aligned} y_i &= \xi_{i,1} \\ &\cdots \\ y_i^{(r_i^a-1)} &= \xi_{i,r_i^a} \quad (23) \\ y_i^{(r_i^a)} = \dot{\xi}_{i,r_i^a} &= -\dot{S}_i + y_i^{(r_i^a)} - K(z_i)sgn(S_i) \\ &\quad + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \end{aligned}$$

Since $z = T(x)$ is a diffeomorphism, from Assumption 1 and the definition of $\tau_i(z_i)$ and $\delta_i(z)$ in (7) and (9) respectively, it follows that there is a neighbourhood of the origin such that

$$|\tau_i(z_i)| \le \rho_i'(z_i), \ |\delta_i(z)| \le M_i' \quad (24)$$

From the last equation in (23),

$$\dot{S}_i = -K(z_i)sgn(S_i) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \quad (25)$$

Then, from (24) and (25),

$$\begin{aligned} &S^\top \dot{S} = \sum_{i=1}^{N} S_i\dot{S}_i \\ &= \sum_{i=1}^{N}\Big(- K(z_i)|S_i| + \beta_i(z_i)\tau_i(z_i)S_i + \gamma_i(z_i)\delta_i(z)S_i\Big) \\ &\le \sum_{i=1}^{N}\Big(- K(z_i)|S_i| + \parallel \beta_i(z_i) \parallel \rho_i'(z_i)|S_i| + \parallel \gamma_i(z_i) \parallel M_i' \cdot |S_i|\Big) \\ &= \sum_{i=1}^{N}\Big(- K(z_i) + \parallel \beta_i(z_i) \parallel \rho_i'(z_i) + \parallel \gamma_i(z_i) \parallel M_i'\Big)|S_i| \end{aligned} \quad (26)$$

Therefore, choosing a suitable gain:

$$K(z_i) > \parallel \beta_i(z_i) \parallel \rho_i'(z_i) + \parallel \gamma_i(z_i) \parallel M_i' + \sigma_i \quad (27)$$

where σ_i is a positive constant, it follows from (26) and (27) that

$$S^\top \dot{S} < -\sigma \sum_{i=1}^{N} |S_i| \le -\sigma N^{-1/2}\|S\| \quad (28)$$

where $\sigma := min_i\{\sigma_i\} > 0$. This means that the reachability condition holds for the closed-loop interconnected system (20)-(21). Hence the following result is obtained immediately.

Theorem 2: The interconnected system (10) is driven to the sliding surface (15) in finite time by the controller (19) if the control gain $K(z_i)$ in (19) satisfies (27).

Remark 6. Based on the analysis above and from the feedback transformation (19), it follows that the decentralised controller

$$u_i = \varpi_i^{-1}(x_i)(-\varsigma_i(x_i) - \dot{S}_i + y_i^{(r_i^a)} - K'(x_i)sgn(S_i)) \quad (29)$$

where $K'(x_i) = K(z_i)|_{z_i=T_i(x_i)}$, can drive the system (1) to the corresponding sliding surface in finite time.

C. The Boundedness of System States

In this subsection, the boundedness of the closed-loop system (20)-(21) is considered. The following assumption is first introduced:

Assumption 4. The functions $q_i(\xi_i,\eta_i)$ in system (20)-(21) satisfy the *Lipschitz condition* with the *Lipschitz constants* L_{q_i} uniformly for η_i in the considered domain. Moreover, there exists a Lyapunov function $V_{i0}(\eta_i)$ such that:

$$\begin{gathered} \chi_{i1} \parallel \eta_i \parallel^2 \le V_{i0}(\eta_i) \le \chi_{i2} \parallel \eta_i \parallel^2 \\ \frac{\partial V_{i0}}{\partial \eta_i} q_i(0,\eta_i) \le -\chi_{i3} \parallel \eta_i \parallel^2; \ \left\| \frac{\partial V_{i0}}{\partial \eta_i} \right\| \le \chi_{i4} \parallel \eta_i \parallel \end{gathered} \quad (30)$$

where $\chi_{i1}, \chi_{i2}, \chi_{i3}$ and χ_{i4} are positive constants for $i = 1,2,\cdots,N$.

Assumption 5. There exist constants κ_{1j} and κ_{2j} such that:

$$\parallel \Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N) \parallel \le \sum_{j=1}^{N}(\kappa_{1j} \parallel \xi_j \parallel + \kappa_{2j} \parallel \eta_j \parallel) \quad (31)$$

Theorem 3: Under Assumptions 3-5, the states of the closed-loop system (20)-(21) are bounded if the matrix $W^T + W$ is positive definite where the matrix W is defined by

$$W := \begin{bmatrix} \chi_{13} - \chi_{14}\kappa_{21} & -\chi_{14}\kappa_{22} & \cdots & -\chi_{14}\kappa_{2N} \\ -\chi_{24}\kappa_{21} & \chi_{23} - \chi_{24}\kappa_{22} & \cdots & -\chi_{24}\kappa_{2N} \\ \vdots & \vdots & \ddots & \\ -\chi_{N4}\kappa_{21} & -\chi_{N4}\kappa_{22} & \cdots & \chi_{N3} - \chi_{N4}\kappa_{2N} \end{bmatrix} \quad (32)$$

where χ_{ij} and κ_{lj} satisfy the Assumptions 4 and 5.

Proof: From Theorem 1, it follows that the variables $\xi_i = col(\xi_{i1},\xi_{i2},...,\xi_{ir_i^a})$ with $i = 1,2,\ldots,N$ are bounded when sliding motion occurs if Assumption 3 holds. Theorem 2 shows that the interconnected system can be driven to the sliding surface in finite time. From Theorems 1 and 2, it follows that the variables $\xi_i = col(\xi_{i1},\xi_{i2},...,\xi_{ir_i^a})$ with $i = 1,2,\ldots,N$ are bounded. Therefore, there exist constants $C_i > 0$ such that in the considered domain,

$$\parallel \xi_i \parallel \le C_i, \qquad i = 1,2,...,N \quad (33)$$

It remains to prove that η_i is bounded in the closed-loop system (20)-(21).

It should be noted that from (33), the variables ξ_i in the system (21) are bounded and can be considered as parameters defined in a compact set. For this system, consider the candidate Lyapunov function:

$$V(\eta_1,\eta_2,...,\eta_N)=\sum_{i=1}^{N}V_{i0}(\eta_i)$$

where $V_{i0}(\eta_i)$ is defined in Assumption 4. Then, the time derivative of the Lyapunov function $V(\cdot)$ along the trajectories of system (20)-(21) is given by

$$\begin{aligned}
&\dot V(\eta_1,\eta_2,...,\eta_N)\\
&=\sum_{i=1}^{N}\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}[q_i(\xi_i,\eta_i)+\Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N)]\\
&=\sum_{i=1}^{N}[\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}q_i(0,\eta_i)+\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}(q_i(\xi_i,\eta_i)-q_i(0,\eta_i))]\\
&\quad+\sum_{i=1}^{N}\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}[\Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N)]
\end{aligned} \tag{34}$$

Further, from (35) and Assumptions 4 and 5,

$$\begin{aligned}
&\dot V(\eta_1,\eta_2,...,\eta_N)\\
&\le\sum_{i=1}^{N}(-\chi_{i3}\parallel\eta_i\parallel^2+\chi_{i4}L_{qi}C_i\parallel\eta_i\parallel\\
&\quad+\parallel\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}\parallel\parallel\Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N)\parallel)\\
&\le\sum_{i=1}^{N}(-\chi_{i3}\parallel\eta_i\parallel^2+\chi_{i4}L_{qi}C_i\parallel\eta_i\parallel\\
&\quad+\chi_{i4}\parallel\eta_i\parallel\sum_{j=1}^{N}(\kappa_{1j}\parallel\xi_j\parallel+\kappa_{2j}\parallel\eta_j\parallel))\\
&\le\sum_{i=1}^{N}(-\chi_{i3}\parallel\eta_i\parallel^2+\chi_{i4}L_{qi}C_i\parallel\eta_i\parallel+\sum_{j=1}^{N}\chi_{i4}\kappa_{1j}C_i\parallel\eta_i\parallel\\
&\quad+\sum_{j=1}^{N}\chi_{i4}\kappa_{2j}\parallel\eta_i\parallel\parallel\eta_j\parallel)\\
&=-(\sum_{i=1}^{N}\chi_{i3}\parallel\eta_i\parallel^2-\sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\kappa_{2j}\parallel\eta_i\parallel\parallel\eta_j\parallel\\
&\quad-\sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}C_i(L_{qi}+\kappa_{1j})\parallel\eta_i\parallel)\\
&=-\frac{1}{2}\begin{pmatrix}\parallel\eta_1\parallel,...,\parallel\eta_N\parallel\end{pmatrix}(W+W^T)\begin{pmatrix}\parallel\eta_1\parallel\\ \cdots\\ \parallel\eta_N\parallel\end{pmatrix}\\
&\quad+\sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}C_i(L_{qi}+\kappa_{1j})\parallel\eta_i\parallel\\
&\le-\frac{1}{2}\lambda_{min}(W+W^T)\|\eta\|^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}C_i(L_{qi}+\kappa_{1j})\parallel\eta_i\parallel\\
&=-\frac{1}{2}\sum_{i=1}^{N}\left\{\lambda_{min}(W+W^T)\|\eta_i\|-\sum_{j=1}^{N}\chi_{i4}C_i(L_{qi}+\kappa_{1j})\right\}\|\eta_i\|\\
&\le 0
\end{aligned} \tag{35}$$

where $\|\eta\|:=\|(\parallel\eta_1\parallel,\parallel\eta_2\parallel\ ,...,\ \parallel\eta_N\parallel)^T\|$, if

$$\|\eta_i\|\ge\frac{\sum_{j=1}^{N}\chi_{i4}C_i(L_{qi}+\kappa_{1j})}{\lambda_{min}(W)}$$

From Theorem 4.18 in [6], it follows that the variables η_i are bounded for $i=1,2,\dots,N$. Hence the results follow. ■

V. SIMULATION EXAMPLE

Consider two inverted pendulums connected by a spring [14]. Each pendulum is controlled by a torque input u_i applied by a servomotor at its base. It is assumed that both θ_i and $\dot\theta_i$ represent the angular position and velocity respectively which are available for the ith controller for $i=1,2$. The model which describes the motion of the pendulums is given by [14]:

$$\begin{aligned}
\dot x_{1,1}&=x_{1,2}\\
\dot x_{1,2}&=(\frac{m_1gr}{J_1}-\frac{kr^2}{4J_1})sin(x_{1,1})+\frac{kr}{2J_1}(l-b)\\
&\quad+\frac{u_1}{J_1}+\frac{kr^2}{4J_1}sin(x_{2,1})\\
y_1&=x_{1,1}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\dot x_{2,1}&=x_{2,2}\\
\dot x_{2,2}&=(\frac{m_2gr}{J_2}-\frac{kr^2}{4J_2})sin(x_{2,1})-\frac{kr}{2J_2}(l-b)\\
&\quad+\frac{u_2}{J_2}+\frac{kr^2}{4J_2}sin(x_{1,1})\\
y_2&=x_{2,1}
\end{aligned} \tag{37}$$

where $y_1=x_{1,1}=\theta_1$, $y_2=x_{2,1}=\theta_2$ and $x_{1,2}=\dot\theta_1$, $x_{2,2}=\dot\theta_2$.

The parameters m_1 =2kg and m_2 = 2.5kg represent the end masses of the pendulum, J_1 = 0.5 $kg\cdot m^2$ and J_2 = 0.625 $kg\cdot m^2$ are the moments of inertia. g = 9.81 m/s^2 is the gravitational acceleration, k = 100 N/m is the spring constant of the connecting spring, r = 0.5m is the pendulum height and l = 0.5m is the natural length of the spring. The distance between the pendulum hinges is $b=0.5m$, where $b=l$.

Considering (13), it follows that

$$\beta_1=\frac{m_1gr}{J_1}-\frac{kr^2}{4J_1},\ \beta_2=\frac{m_2gr}{J_2}-\frac{kr^2}{4J_2}$$

$$\varphi_1(x_1)=sin(x_{1,1}),\varphi_2(x_2)=sin(x_{2,1});\ \gamma_1(x_1)=\frac{kr^2}{4J_1},\gamma_2(x_2)=\frac{kr^2}{4J_2}$$
$$\psi_1(x)=sin(x_{2,1}),\ \psi_2(x)=sin(x_{1,1})$$

By direct calculation,

$$|\varphi_1(x_1)|=|sin(x_{1,1})|\le\rho_1(x_1)=1$$
$$|\varphi_2(x_2)|=|sin(x_{2,1})|\le\rho_2(x_2)=1$$
$$|\Phi_1(x)|=|sin(x_{2,1})|\le M_1=1,\ |\Phi_2(x)|=|sin(x_{1,1})|\le M_2=1$$

Here, both σ_i for $i=1,2$ are designed as 0.1. It can be verified that the relative degree $r_i^a=r_i^b=2$ for $i=1,2$. The nominal subsystems can be feedback linearized. For simulation purposes, the initial states are chosen as $x_{1,1}(0)=1$ and $x_{2,1}(0)=-0.8$, and the desired output signals $y_{id}(t)$ are chosen as $y_{1d}=0.5\sin(t),\ y_{2d}=5e^{-t}$.

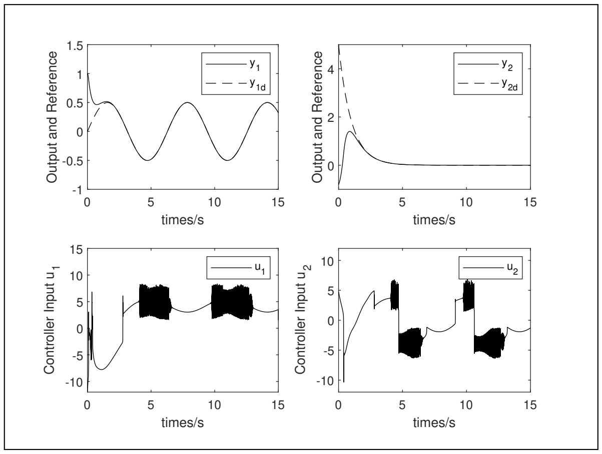

Fig. 1: Time responses of system output and the desired output (upper) and controller inputs of system (36)-(37).

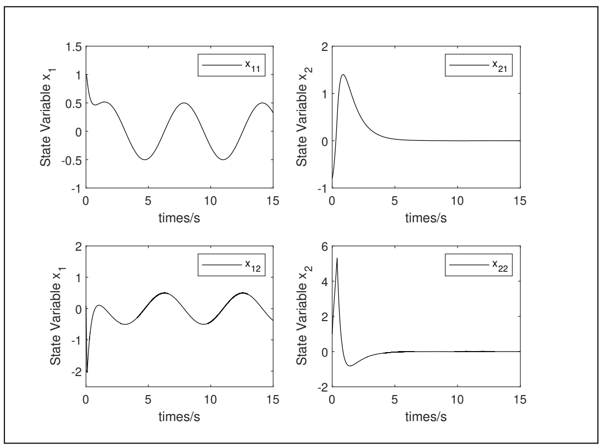

Fig. 2: Evolution of state variables of system (36)-(37).

It is clear that Assumption 3 is satisfied. Let

$$e_i = y_i - y_{id};\ \dot{e}_i = \dot{y}_i - \dot{y}_{id};\ S_i = \dot{e}_i + a_i \cdot e_i,\ i = 1,2. \quad (38)$$

where the sliding function parameters are chosen as $a_1 = 2$ and $a_2 = 3$. Then from (19), the control laws can be described by:

$$v_i = \frac{u_i}{J_i} = -\dot{S}_i + y_i^{(2)} - K(x_i)\text{sgn}(S_i),\ i = 1,2. \quad (39)$$

where, based on (27), the control gain $K(\cdot)$ is chosen by $K(\cdot) = 19.72$. By direct calculation, Assumptions 4-5 are satisfied. The tracking results are shown in Fig.1 where it can be seen that high tracking performance results. Every angular position y_i of the subsystem can track the ideal reference y_{id} after around 2 seconds with the inputs of the designed controller, despite the interactions between the subsystems. The time responses of the states of the system (36)-(37) are shown in Fig.2. It is seen that the system states are bounded. Simulation results demonstrate that the results developed in this paper are effective.

VI. CONCLUSIONS

A sliding mode control scheme for output tracking of a class of nonlinear interconnected systems has been proposed in this paper. The developed results can guarantee asymptotic output tracking while maintaining bounded state variables across the closed-loop system. The designed controllers are decentralised while the desired reference signals are time-varying. It is not required that the isolated subsystems within the interconnected system are linearisable. The developed results can be extended to the case where the isolated subsystems have multiple inputs and multiple outputs. Therefore, the method in this paper is suitable for a wide class of large-scale interconnected systems.

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