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Normalisation for Some Quite Interesting Many-Valued Logics

Abstract. In this paper, we consider a set of quite interesting three- and four-valued logics and prove the normalisation theorem for their natural deduction formulations. Among the logics in question are the Logic of Paradox, First Degree Entailment, Strong Kleene logic, and some of their implicative extensions, including RM_3 and RM_3^\supset . Also, we present a detailed version of Prawitz’s proof of Nelson’s logic N4 and its extension by intuitionist negation.

Keywords: natural deduction; normalisation; three-valued logic; four-valued logic; logic of paradox; Nelson logic

1. Introduction: motivation and related work

The Logic of Paradox (LP) is generally accepted to be one of the most important and famous three-valued paraconsistent logics. Its semantics was first introduced by [Asenjo \(1966\)](#) as a logic of antinomies. However, this logic became well-known only after Priest’s comprehensive analysis of its philosophical aspects ([Priest, 1979](#)) (and since then it is known as LP). Our inspiration for the present paper comes from a paper by Neil [Tennat \(2019\)](#) in which he considers Priest’s (2002) natural deduction system for LP and concludes that this

system will be devoid of any meaningful normalization theorem concerning its proofs, which is one of the main motivations for using natural deduction as one’s format for fully regimented proofs.

([Tennat, 2019](#), p. 501)

We shall present a natural deduction system for LP that has a meaningful normalisation theorem. On the way to that result and after having obtained it we shall also consider a number of other logics for which we prove normalisation theorems. The first one will be Nelson’s well-known Constructive Logic with Strong Negation (N4)¹ (Nelson, 1949; Almkudad and Nelson, 1984). The normalisation theorems to be proved will be presented as extensions and modifications of the normalisation theorem for that logic.

We should say that the literature already contains proof systems for LP that are acceptable from the proof-theoretic perspective. Among them are Avron’s (1991) cut-free sequent calculus, *labeled* natural deduction calculi provided by the methods of Baaz, Fermüller and Zach (1993b), of Englander, Haeusler and Pereira (2014)², and of Kaminski and Francez (2021). Let us notice that neither Priest’s natural deduction system for LP nor its modification which we present in this paper are labeled ones.

Tennat (2019) pays special attention to an implication of LP defined as $\neg A \vee B$. This implication is known to invalidate modus ponens. As Tennat observes, it is rather difficult to find proof-theoretically acceptable rules for it. We are not going to present such rules, since we believe that such an implication, despite its classical logic-style definition, is not a good candidate for LP, precisely because of the lack of modus ponens. In our mind, a much better candidate is Ślupecki’s (1939; 1971) and Jaśkowski’s (1948; 1999) implication, which validates modus ponens and is formalised by a proof-theoretically nice set of rules.³ We are not original in choosing Jaśkowski’s implication for LP: Batens (1980) studied such a logic under the name PI^s , Avron (1986) under the name RM_3^{\supset} , Rozonoer (1989) under the name PCont.⁴ Due to

¹ We follow Wansing in naming the system (2001, 422ff). We also thank Heinrich Wansing for comments on this paper.

² In fact, by these methods one can obtain a labeled normalisable natural deduction system for any tabular logic which we consider in this paper. But those logicians who prefer non-labeled natural deduction would not be satisfied with such a result. The methods of (Baaz et al., 1993b) are based on their previous general result (Baaz et al., 1993a) allowing construction of labeled sequent and tableaux calculi for finite-valued logics, including LP.

³ This implication was also studied by D’Ottaviano and da Costa (1970), Asenjo and Tamburino (1975).

⁴ Without any special reason, we choose the dubbing RM_3^{\supset} throughout the paper.

Avron (1991) RM_3^{\supset} has a cut-free sequent calculus. Due to Bolotov and Shangin (2012) it has a linear Jaśkowski-style natural deduction system together with a proof-search procedure. A linear natural deduction system for RM_3^{\supset} (and any other unary and binary truth-functional extension of LP) can be easily obtained by Kooi and Tamminga's (2012) results, but this system has nine rules for implication which is too much in our opinion. Moreover, most of these rules are far from being acknowledged to be proof-theoretically suitable. Consider, e.g., the rule $B \wedge \neg B \vdash ((A \supset B) \wedge \neg(A \supset B)) \vee \neg A$. Better rules may be found in (Petrukhin, 2018), where generalised correspondence analysis is presented, but such a system has eleven rules for implication. We present one with four rules for implication and show that it is normalisable.

Yet another alternative for LP's implication is Sobociński's (1952). Such a logic is known under the name RM_3 from Anderson and Belnap's (1975) school of relevant logic.

Although a presentation of a normalisation theorem for LP fulfils our original plans, we decided to pay attention to some related logics, such as First Degree Entailment FDE, which is one of the most well-known, useful and successful members of the Anderson and Belnap (1975) family of relevant logics. FDE is still actively studied despite its recent forty years anniversary (see Omori and Wansing 2017 on this issue). FDE can be formalised as the fragment of LP without Excluded Middle. Yet another interesting logic is a lesser known system called Par due to Popov (1989) and Be due to Avron (1991) which is the fragment of RM_3^{\supset} without Excluded Middle.

Once we turned to the field of paracomplete logics, it made sense to look at some of those which are not paraconsistent. Among such logics are Strong Kleene logic K_3 (Kleene, 1938) (an extension of FDE by Ex Falso Quodlibet), an implication-free fragment of Łukasiewicz's logic Ł_3 (1920) (for an English translation see (Łukasiewicz, 1970, pp. 87–88)), the first many-valued logic) and Ślupecki, Bryll and Prucnal's (1967) PComp (the name is due to Popov 2009, it is an extension of Par by Ex Falso Quodlibet), which is an improvement of Ł_3 in the sense that it enjoys the deduction theorem.

Cut-free sequent calculi for RM_3 , K_3 , PComp, Ł_3 may be found in the above mentioned paper by Avron (1991) (see also Avron, 1991). A

However, the dubbing PI^s occurs from time to time in the English language literature. The dubbing PCont is the standard one in Russian language literature.

cut-free sequent calculus for Par is presented in (Popov, 1989). See also (Pynko, 1999) for cut-free sequent calculi for FDE, Par and some of their extensions. A natural deduction system for K_3 which we are going to consider is a modification of Priest’s system (Priest, 2002). One may find natural deduction systems (with a huge amount of rules) for any unary/binary tabular extensions of K_3 in (Tamminga, 2014) and (Petrukhin, 2018). A systematic treatment of linear-type natural deduction systems and automatic proof search for (unary and binary) truth-tabular extensions of LP, K_3 and FDE may be found in (Petrukhin and Shangin, 2017, 2019, 2020).

Finally, since most of the logics in question are extensions of Nelson’s logic N4 or its implication-less fragment, which coincides with FDE (see figure 1 for a diagram of their relations), it is reasonable to begin with the normalisation theorem for this logic. Prawitz (1965, Appendix B §2) only observes that the system normalises, but does not carry out a proof. To our knowledge, this is the first detailed, direct proof of normalisation for Prawitz’s formalisation of N4. After that, we prove normalisation for Prawitz’s formalisation for intuitionist logic with strong negation (N3)⁵, which is an extension of N4 by the intuitionist *falsum* constant.

The structure of the paper is as follows. In section 3 we recall Prawitz’s formalisation of N4 and prove normalisation for it. In section 4 we do the same for N3. Section 5 is devoted to normalisation for First Degree Entailment, the Logic of Paradox and Strong Kleene Logic. In sections 6 and 7 we focus on the logics RM_3^{\supset} and RM_3 , respectively. In section 8 we consider the logics Par and PComp. Section 9 contains concluding remarks. Appendix contains the semantics of the logics in question to make our paper self-contained. It also has some additional material regarding proofs of theorems.

We finish this section with some basic definitions regarding normalisation.

2. Terminological preliminaries

The definitions of the languages is standard, except that some of the logics to be considered in this paper have two negations, \neg and \sim .

Deductions in natural deduction have the usual ‘family tree shape’, as Gentzen calls it. The top-most formulas or leaves of the tree are the

⁵ Here, too, we adopt Wansing’s name for the system (Wansing, 2001, 423ff).

assumptions of the deduction, the bottom-most formula or root of the tree is its conclusion. We follow Troestra and Schwichtenberg (2000) in the details.

Assumptions are assigned *assumption classes*. Formula occurrences of different types have different assumption classes, formula occurrences of the same type may belong to the same or to different assumption classes. Every assumption receives a label indicating the assumption class to which it belongs. The *discharge* or *closing* of assumption classes in a deduction is indicated by a square bracket around the formulas in the assumption class and repeating the label of the assumption class to the right of the line marking the inference at which it is discharged or closed. Assumptions that are not discharged or closed are *undischarged* or *open*. Empty assumption classes are permitted to allow for the possibility of *vacuous discharge*, where formulas in an assumption class are discharged that do not occur in the deduction.

Let L be a logic to be considered in this paper. Deductions in L are defined by induction. The basic case is that a formula occurrence A^i , where i marks an assumption class, is a deduction in L of the conclusion A from the open in the assumption class i containing A as its sole member. The induction step is carried out by applying the rules to be specified for each system, which also specify how discharge of assumption classes is effected by their applications.

Two operations on assumption classes are permitted. A label for an assumption class i already present in the deduction may be assigned to new assumptions that are of the same type as the formulas in i , and different assumption classes containing formulas of the same type may be relabelled with the same label. The former corresponds to making the same assumption multiple times; the latter is an operation corresponding to Contraction.

DEFINITION 1. A formula is *atomic* if it contains no connectives. The *degree* of a formula is the number of connectives in A .

Note that \perp , being a 0-place connective, is a formula of degree 1. We call \perp *falsum*.

Many of the rules for the systems we are going to consider come in two kinds: one kind has the format of the usual introduction and elimination rules, where the introduction rules introduce formulas with one connective as the main operator and the elimination rules eliminate such formulas. The other kind concerns the introduction and elimination

of the negations of formulas. Introduction rules will be marked by labels of the format $(*I)$ or $(\neg*I)$, $*$ is a connective, and elimination rules by $(*E)$ or $(\neg*E)$. The major premise of an elimination rule is the formula that mentions the connectives in the general statement of the rules.

We distinguish tree elimination rules governing negation or \perp :

$$(\perp E) \frac{\perp}{B} \quad (\neg\perp) \frac{\neg A \quad A}{\perp} \quad (ECQ) \frac{\neg A \quad A}{B}$$

We may call the first *ex falso quodlibet*, the second *ex contradictione falsum* and the third *ex contradictione quodlibet*.

DEFINITION 2. A *maximal formula* is an occurrence of a formula in a deduction that is the conclusion of an introduction rule and major premise of an elimination rule.

Rules of the kind of disjunction elimination are called *del-rules*: for each logic it will be specified what its del-rules are.

DEFINITION 3 (Segment, Length & Degree of a Segment, Maximal Segment). (a) A *segment* is a sequence of two or more formula occurrences $C_1 \dots C_n$ in a deduction such that C_1 is not the conclusion of a del-rule, C_n is not the minor premise of a del-rule and for every $i < n$, C_i is minor premise of a del-rule and C_{i+1} its conclusion.

(b) The *length* of a segment is the number of formula occurrences of which it consists, its *degree* is their degree.

(c) A segment is *maximal* if and only if its last formula is major premise of an elimination rule.

Notice that we deviate from Prawitz and Troelstra and Schwichtenberg in that maximal segments consist of at least two formulas, while according to their notion, every maximal formula is a maximal segment of length 1. We find keeping the distinction between maximal segments and maximal formulas more perspicuous. This is reflected in the definition of the *rank* of a deduction:

DEFINITION 4 (Rank of a Deduction). The *rank* of a deduction Π is the pair $\langle d, l \rangle$ where d is the highest degree of any maximal formula or maximal segment in Π , and l is the sum of the number of maximal formulas and the sum of the lengths of all maximal segments in Π . If there are no maximal formulas or maximal segments in Π , d and l are both 0.

Ranks are ordered lexicographically: $\langle d, l \rangle < \langle d', l' \rangle$ iff either $d < d'$, or $d = d'$ and $l < l'$.

We follow Troelstra and Schwichtenberg in that our definition of maximal segment is more general than that of Prawitz, who requires the first formula of a maximal segment to be derived by an introduction rule. Our normalisation theorems for N4 and N3 are therefore also slightly more general than the versions Prawitz has in mind.

DEFINITION 5. A deduction is *in normal form* if it contains neither maximal formulas nor maximal segments.

Finally, we need some notions regarding the formulas that occur on deductions.

DEFINITION 6. A deduction Π of a conclusion A from the undischarged assumptions Γ satisfies the *subformula property* iff every formula on the deduction is a subformula either of A or of a formula in Γ .

None of the logics discussed here has the subformula property. A different notion is therefore to be preferred. Many have the following property:

DEFINITION 7. A deduction satisfies the *negation subformula property* iff every formula occurrence on it is either a subformula of an undischarged assumption or of the conclusion or it is the negation of such a formula.

Some only satisfy a weak version thereof:

DEFINITION 8. A deduction satisfies the *weak negation subformula property* iff every formula occurrence on it is either a subformula of an undischarged assumption or of the conclusion or it is the negation of such a formula or it is an occurrence of \perp that is the conclusion of $(\neg\perp)$.

The logics which have Pierce's rule satisfy the specific version of negation and weak negation subformula property.

DEFINITION 9. A deduction Π of a conclusion A from the undischarged assumptions Γ satisfies the *Pierce subformula property* iff every formula on the deduction is a subformula either of A or of a formula in Γ or formulas of the form $B \supset C$ that are discharged by Pierce's Rule and subformulas of formulas C that stand immediately below such formulas.

DEFINITION 10. A deduction Π of a conclusion A from the undischarged assumptions Γ satisfies the *Pierce negation subformula property* iff every formula on the deduction is a subformula either of A or of a formula in Γ or formulas of the form $B \supset C$ that are discharged by Pierce’s Rule and subformulas of formulas C that stand immediately below such formulas or it is the negation of such a formula.

DEFINITION 11. A deduction Π of a conclusion A from the undischarged assumptions Γ satisfies the *Pierce weak negation subformula property* iff every formula on the deduction is a subformula either of A or of a formula in Γ or formulas of the form $B \supset C$ that are discharged by Pierce’s Rule and subformulas of formulas C that stand immediately below such formulas or it is the negation of such a formula or it is an occurrence of \perp that is the conclusion of $(\neg\perp)$.

Proofs that deductions satisfy these notions appeal to Prawitz’s notion of a path in a deduction:

DEFINITION 12. A *path* in a deduction is a sequence of formulas $A_1 \dots A_n$ such that A_1 is an assumption not discharged by a del-rule; (a) if A_i is not major premise of a del-rule, then A_{i+1} is the conclusion of the rule, (b) if A_i is major premise of a del-rule, then A_{i+1} is an assumption discharged by that rule; and A_n is either the conclusion of the deduction or the minor premise of $(\supset E)$, $(\neg\perp)$ or (ECQ) .

Philosophical Comment. The negation subformula is philosophically well motivated, if negation and affirmation are taken as being on a par. Intuitionists tend to give affirmation priority over negation, and define the latter notion in terms of the affirmation of $A \supset \perp$, which of course contains a formula that cannot be affirmed as a subformula. Kürbis (2015) argues that this attempted definition, however, is unsuccessful, and that this provides a motivation for treating negation as equally primitive as affirmation.⁶ N4 is a natural logic for such an approach, provided it is agreed that affirmation and negation run in parallel and do not interact formally. On the other hand, it is natural to consider affirmation and negation to be exclusive, which is captured by $(\neg\perp)$, if \perp is governed by $(\perp E)$ and thereby guaranteed to be an absurdity. This motivates the weak negation subformula property, and consequently, N3 is a natural

⁶ But see (Kürbis, 2019, Ch 5) for critical assessment of this option.

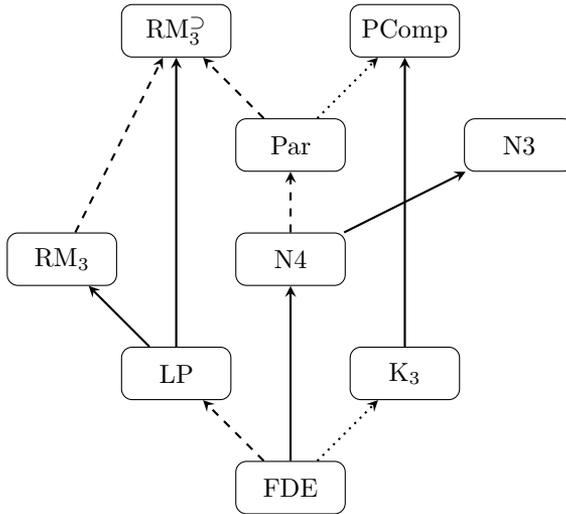


Figure 1. A Hasse-style diagram of the logics in question ordered by inclusion. By dashed lines we indicate that a logic extends another *only* by rules of inference, by ordinary lines we indicate that a logic extends another *also* by connectives, by dotted line we indicate that a logic can be formulated as an extension of another only by rules of inference, but we also consider its version obtained by adding the constant falsum.

logic for this approach to negation and affirmation. The Pierce subformula property and its variations may be more difficult to motivate, but this is not our concern here.

3. Nelson’s Logic N4

Prawitz (1965, 97f) formalises Nelson’s Constructive Logic with Strong Negation (N4) in a system of natural deduction as an extension of intuitionist positive logic:

$$\begin{array}{l}
 (\wedge I) \quad \frac{A \quad B}{A \wedge B} \qquad (\wedge E) \quad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} \\
 \\
 \begin{array}{c} [A]^i \\ \Pi \\ B \end{array} \\
 (\supset I) \quad \frac{\quad}{A \supset B} \quad i \qquad (\supset E) \quad \frac{A \supset B \quad A}{B}
 \end{array}$$

$$\begin{array}{c}
(\vee\text{I}) \quad \frac{A}{A \vee B} \quad \frac{B}{A \vee B} \quad (\vee\text{E}) \quad \frac{A \vee B \quad \frac{[A]^i \quad [B]^j}{\Pi} \quad \frac{\Sigma}{C}}{C} \quad i,j \\
\\
(\neg\wedge\text{I}) \quad \frac{\neg A}{\neg(A \wedge B)} \quad \frac{\neg B}{\neg(A \wedge B)} \\
\\
(\neg\wedge\text{E}) \quad \frac{\neg(A \wedge B) \quad \frac{[\neg A]^i \quad [\neg B]^j}{\Pi} \quad \frac{\Sigma}{C}}{C} \quad i,j \\
\\
(\neg\vee\text{I}) \quad \frac{\neg A \quad \neg B}{\neg(A \vee B)} \quad (\neg\vee\text{E}) \quad \frac{\neg(A \vee B)}{\neg A} \quad \frac{\neg(A \vee B)}{\neg B} \\
\\
(\neg\supset\text{I}) \quad \frac{A \quad \neg B}{\neg(A \supset B)} \quad (\neg\supset\text{E}) \quad \frac{\neg(A \supset B)}{A} \quad \frac{\neg(A \supset B)}{\neg B} \\
\\
(\neg\neg\text{I}) \quad \frac{A}{\neg\neg A} \quad (\neg\neg\text{E}) \quad \frac{\neg\neg A}{A}
\end{array}$$

The del-rules of N4 are $(\vee\text{E})$ and $(\neg\wedge\text{E})$.

Prawitz notes that a normalisation theorem is provable for this logic, by extending the techniques he used for proving the normalisation theorem of intuitionist logic, without giving the details. As it will form the basis of the other logics to be considered in this paper, we begin by proving normalisation for N4.⁷

⁷ Prawitz's formalisation does not contain the rules for negated disjunctions, but he observes that such rules could be added and that disjunction can be defined in the usual way, and then such rules are derivable. One of Kamide and Wansing's (2015, 58f, 68) formalisation of Nelson's logic contains primitive rules for negated disjunctions and is thus the same as ours. They prove normalisation for a version that adds the rules $(\neg\neg\text{I})$, $(\neg\neg\text{E})$ and the DeMorgan Laws mentioned later at the very beginning of section 4.

To prove normalisation for N4, we give reduction steps for removing maximal formulas from deductions and permutative reduction steps for rearranging deductions so as to avoid maximal segments. Following common terminology, we call the former detour conversions, the latter permutation conversions. Repeated application of permutation conversions turns a maximal segment in Prawitz’s narrower sense into a maximal formula, which is to be removed by a detour conversion. The conversions are, *mutatis mutandis*, like those for deductions in intuitionist logic. The rules for double negation introduce new but evident cases.

A deduction Π on top of an open assumption in square brackets $[A]$ of a deduction Σ indicates that each formula in the assumption class to which A belongs is concluded by Π .

(I) Detour Conversions

1. Conjunction

$$\frac{\frac{\frac{\Pi_1}{A_1} \quad \frac{\Pi_2}{A_2}}{A_1 \wedge A_2}}{A_i} \quad \rightsquigarrow \quad \frac{\Pi_i}{\Xi}$$

where $i = 1$ or $i = 2$.

2. Disjunction

$$\frac{\frac{\frac{\Sigma}{A_i}}{A_1 \vee A_2} \quad \frac{\frac{[A_1]^i}{\Pi_1} \quad \frac{[A_2]^j}{\Pi_2}}{C}}{C} \quad i,j}{\Xi} \quad \rightsquigarrow \quad \frac{\frac{\Sigma}{[A_i]} \quad \frac{\Pi_i}{C}}{\Xi}$$

where $i = 1$ or $1 = 2$.

3. Implication

$$\frac{\frac{\frac{[A]^i}{\Pi} \quad \frac{\Sigma}{A}}{A \supset B} \quad i}{B} \quad \rightsquigarrow \quad \frac{\Sigma}{\frac{[A]}{\frac{\Pi}{B}} \Xi}$$

4. Negated Conjunction

$$\frac{\frac{\Sigma}{\neg A_i} \quad \frac{[\neg A_1]^i \quad \Pi_1}{C} \quad \frac{[\neg A_2]^j \quad \Pi_2}{C}}{\frac{\neg(A_1 \wedge A_2)}{C}} \quad i,j \quad \rightsquigarrow \quad \frac{\Sigma}{[\neg A_i]} \quad \frac{\Pi_i}{C} \quad \Xi$$

where $i = 1$ or $1 = 2$.

5. Negated Disjunction

$$\frac{\frac{\Pi_1}{\neg A_1} \quad \frac{\Pi_2}{\neg A_2}}{\frac{\neg(A_1 \vee A_2)}{\neg A_i}} \quad \rightsquigarrow \quad \frac{\Pi_i}{\neg A_i} \quad \Xi$$

where $i = 1$ or $i = 2$.

6. Negated Implication

$$\frac{\frac{\Pi}{A} \quad \frac{\Sigma}{\neg B}}{\frac{\neg(A \supset B)}{A}} \quad \rightsquigarrow \quad \frac{\Pi}{A} \quad \Xi$$

$$\frac{\frac{\Pi}{A} \quad \frac{\Sigma}{\neg B}}{\frac{\neg(A \supset B)}{\neg B}} \quad \rightsquigarrow \quad \frac{\Sigma}{\neg B} \quad \Xi$$

6. Double Negation:

$$\frac{\frac{\Pi}{A}}{\frac{\neg\neg A}{A}} \quad \rightsquigarrow \quad \frac{\Pi}{A} \quad \Xi$$

(II) Permutation Conversions.

These work as in intuitionist logic by permuting the application of the elimination rule to the last formula of the segment upwards. We begin

with give an example familiar from intuitionist logic and then an example for rules specific to N4. The other cases are similar.

$$\frac{A \vee B \quad \frac{[A]^i \quad \Pi}{C \wedge D} \quad \frac{[B]^j \quad \Sigma}{C \wedge D}}{C \wedge D} \quad i,j}{D}$$

\rightsquigarrow

$$\frac{A \vee B \quad \frac{C \wedge D}{D} \quad \frac{C \wedge D}{D}}{D} \quad i,j$$

$$\frac{\neg(A \wedge B) \quad \frac{[\neg A]^i \quad \Pi}{\neg\neg C} \quad \frac{[\neg B]^j \quad \Sigma}{\neg\neg C}}{\neg\neg C} \quad i,j}{C}$$

\rightsquigarrow

$$\frac{\neg(A \wedge B) \quad \frac{\neg\neg C}{C} \quad \frac{\neg\neg C}{C}}{C} \quad i,j$$

This completes the reduction steps.

We now prove that every deduction in N4 can be brought into normal form. In the choice of a suitable maximal formula or segment to which to apply the reduction steps, we follow Prawitz (1965, p. 50).

THEOREM 1. *Any deduction Π of A from open assumptions Γ in N4 can be brought into a deduction in normal form of A from some of Γ .*

PROOF. By induction over the rank of deductions and applying the reduction steps. Take a maximal formula or maximal segment of highest degree such that (i) no maximal formula or segment of highest degree stands above it in the deduction, (ii) no maximal formula or segment of highest degree stands above a minor premise of the elimination rule of which the maximal formula or segment is the major premises, and (iii) no maximal segment of highest degree contains a formula that is minor premise of the elimination rule of which the maximal formula or maximal segment is the major premise. This reduces the rank of the deduction. \dashv

THEOREM 2. *Deductions in normal form in N4 have the negation subformula property.*

PROOF. This follows from two facts. (1) on any path in a deduction in normal form, the major premises of elimination rules precede the conclusions introduction rules: if some conclusion of an introduction rule preceded some major premise of an elimination rule, there'd either be a maximal formula or there'd be a segment beginning with a conclusion of an introduction rule and ending with a major premise of an elimination rule, both of which are excluded by the normality of the deduction. (2) Consider a main path in a deduction be one that ends in the conclusion: inspection of the rules shows that the theorem is fulfilled for main paths. The theorem follows by induction over the order of paths, where a main path has order 0 and a path has order $i+1$ if it ends in the minor premise of (\supset E) the major premise of which is on a path of order i . \dashv

4. Intuitionist Logic with Strong Negation N3

Prawitz (1965, p. 98) suggests adding Nelson's Strong Negation to intuitionist logic. The result is the logic Wansing calls N3. To do so it suffices to add the *falsum* constant \perp to N4, and let it be governed by the rule:

$$(\perp\text{E}) \frac{\perp}{B}$$

where B may be restricted to range over atomic formulas or their strong negations, the general case following by induction over the complexity of formulas.

The intuitionist negation of A , $\sim A$, is defined as usual as $A \supset \perp$. \perp and strong negation are connected by a bridge principle that is also added to N4:

$$(\neg\perp) \frac{\neg B \quad B}{\perp}$$

where B may be restricted to atomic formulas. The latter is established by virtue of the following transformations and an induction over the complexity of formulas:

$$\begin{array}{c} \frac{\frac{\neg\neg B}{\perp} \quad \neg B}{\perp} \quad \rightsquigarrow \quad \frac{\neg B \quad \frac{\neg\neg B}{B}}{\perp} \\ \frac{\frac{\neg(A \supset B)}{\perp} \quad A \supset B}{\perp} \quad \rightsquigarrow \\ \frac{\frac{\neg(A \supset B)}{\neg B} \quad \frac{A \supset B}{B} \quad \frac{\neg(A \supset B)}{A}}{\perp} \\ \frac{\frac{\neg(A \vee B)}{\perp} \quad A \vee B}{\perp} \\ \rightsquigarrow \\ \frac{A \vee B \quad \frac{\frac{\neg(A \vee B)}{\neg A} \quad [A]^i}{\perp} \quad \frac{\frac{\neg(A \vee B)}{\neg B} \quad [B]^j}{\perp}}{\perp} \quad i,j \end{array}$$

The case for conjunction mirrors the one for disjunction. There is obviously no need to allow B to be \perp , as such an application of $(\neg\perp)$ (or $(\perp E)$, for that matter) concludes \perp from \perp and is hence redundant. We have $\neg A \supset \sim A$, but not the converse.

Thus, $N3 = N4 + (\perp E) + (\neg\perp)$, with the added rules subject to the restriction that B be atomic. To show that deductions in N3 normalise, it suffices to add two further conversion to those for deductions in N4. The restrictions on B absolve us from adding detour conversions to handle the cases where a premise of $(\neg\perp)$ is the conclusion of an introduction rule or where the conclusion of $(\perp E)$ is major premise of an elimination rule. However, $(\perp E)$ is effectively also an introduction rule for strong negations of atomic formulas and $(\neg\perp)$ the corresponding elimination rule. Counting applications of these rules in the way just specified and the

premise of $(\neg\perp)$ containing negation to be their major premises means that we can leave definition 2 untouched.

Note that we do not count $(\neg\perp)$ as an introduction rule for \perp : applications of this rule followed by $(\perp E)$ cannot in general be removed from deductions.

Clearly, it is superfluous to derive a premise of $(\neg\perp)$ by $(\perp E)$ or to derive the premise of $(\perp E)$ by $(\neg\perp)$. We continue the numbering of detour conversions of N4:

7. *Falsum*

$$\frac{\frac{\frac{\perp}{\neg B}}{\Sigma} \quad B}{\perp} \quad \Xi \quad \rightsquigarrow \quad \frac{\perp}{\Xi} \quad \Xi$$

By the same reasoning as in the normalisation theorem of N4, we prove:

THEOREM 3. *Any deduction Π of A from open assumptions Γ in N3 can be brought into a deduction in normal form of A from some of Γ .*

PROOF. By extending the proof of the normalisation theorem for N4 with the \perp conversions and noting that they cannot introduce new maximal formulas or maximal segments and that \perp is a formula of degree 1. ⊣

The presence of $(\neg\perp)$ means that deductions in normal form in N3 do not have the negation subformula property: the relevant relation does not hold between the major premise of that rule and its conclusion. However, they do fulfil its weaker version:

THEOREM 4. *Deductions in normal form in N3 have the weak negation subformula property.*

PROOF. Following the pattern of the proof of the negation subformula property for deductions in normal form of N4, but noting that formulas that need to be exempt are conclusion of $(\neg\perp)$ and that these are the sole such formulas. The theorem follows by induction over the order of paths, where, if the major premises of $(\neg\perp)$ is on a path of order i , a path of order $i + 1$ ends with its minor premise, and the following observations: conclusions of $(\perp E)$ can only be premises of introduction rules, by the normal form of the deductions, its minor premise is a subformula of its major premise, and the major premise can only be the conclusion of an elimination rule. ⊣

Digression. It's not needed for the proof, but it is an observation that may be worth pointing out. Let us think about the case where an atomic formula is concluded by (\perp E) and (minor) premises of ($\neg\perp$). It would be possible to remove those formulas, too, by applying the following detour conversion:

$$\frac{\frac{\frac{\Pi}{\neg B} \quad \frac{\Sigma}{\perp}}{B}}{\perp}}{\Xi} \quad \rightsquigarrow \quad \frac{\Sigma}{\perp} \quad \Xi$$

This removes a maximal formula degree 0. Hence to ensure that the conversion reduces the rank of the deduction, we should have to use a little trick in assigning ranks to deductions in N3. The conversion removes not only the atomic formula B , but also its negation $\neg B$, and so we may assign atomic formulas that are the conclusion of (\perp E) and premise of ($\neg\perp$) the degree of the major premise of the latter rule. Then the proof goes through as planned. This additional conversion is not, however, necessary to ensure, for instance, that deductions in normal form in N3 have the subformula property: by the 7. detour conversion, major premises of ($\neg\perp$) can only be either assumptions or derived by an elimination rule, hence are subformulas of assumptions of the deduction.

5. First Degree Entailment, The Logic of Paradox, Strong Kleene Logic

Priest (2002) formalised systems of natural deduction for Belnap (1977) and Dunn's (1976) First Degree Entailment (FDE), Asenjo's (1966) and Priest's (1979) Logic of Paradox (LP), and Strong Kleene (1938) logic (K_3). Later on the same ND systems for LP and K_3 were independently introduced by Kooi and Tamminga (2012) and Tamminga (2014), respectively.

Priest's system for FDE has the rules (\wedge I), (\wedge E), (\vee I), (\vee E), ($\neg\neg$ I) and ($\neg\neg$ E) of N4, and DeMorgan's Laws to govern negated disjunctions and conjunctions:

$$\frac{\neg A \vee \neg B}{\neg(A \wedge B)} \quad \frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg A \wedge \neg B}{\neg(A \vee B)} \quad \frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

His system for LP is an extension of the one for FDE by the law of excluded middle as an axiom, while his system for K_3 extends FDE by the principle of ex contradictione quodlibet (ECQ):

$$\frac{}{A \vee \neg A} \quad \frac{A \quad \neg A}{B}$$

We give each logic formulations more suitable to proof-theoretic investigation. Then we prove a normalisation theorem for our versions of FDE, LP, and K_3 .

5.1. FDE

The four rules of DeMorgan's Laws are derivable in N4, and conversely, given $(\wedge I)$, $(\wedge E)$, $(\vee I)$, $(\vee E)$, the rules $(\neg \wedge I)$, $(\neg \wedge E)$, $(\neg \vee I)$ and $(\neg \vee E)$ may be derived from DeMorgan's Laws. Priest's system for FDE does not have implication. It is therefore equivalent to N4 minus implication, which from now on we'll refer to as our formalisation of FDE. Thus, we can state the following theorems as a trivial consequences of normalisation and negation subformula property for deductions of N4:

THEOREM 5. *Any deduction Π of A from open assumptions Γ in FDE can be brought into a deduction in normal form of A from some of Γ .*

THEOREM 6. *Deductions in normal form in FDE have the negation subformula property.*

5.2. LP

The axiom of the law of excluded middle can be recast into rule form:

$$(EM) \quad \frac{\begin{array}{c} [B]^i \\ \Pi \\ C \end{array} \quad \begin{array}{c} [\neg B]^j \\ \Sigma \\ C \end{array}}{C} \quad i,j$$

The axiom of the law of excluded middle follows from the rule (EM) and conversely, the rule (EM) may be derived from the axiom and disjunction elimination. Thus Priest's formalisation of LP is equivalent to our formalisation of FDE plus (EM), which from now on we'll refer to as LP.

Del-rules of LP are those of N4 and (EM). The formula occurrences C in (EM) are its minor premises; it has no major premises. To prove

normalisation for LP, we need to do something about (EM). Being a del-rule, it may give rise to maximal segments. We could permute them upwards, as usual, thereby shortening or removing the maximal segment. We may, however, also use a different technique that avoids maximal segments arising from applications of (EM) altogether. Applications of (EM) can be permuted downwards, so that any deduction in which that rule is applied can be transformed into one in which the conclusion is derived by a sequence of applications of (EM), and these are the only applications of (EM) in the deduction.

DEFINITION 13. A deduction is (EM) *final* if and only if there are a number of segments all of which are constituted by a sequence of formulas $C_1 \dots C_n$ such that

- (i) for some i , $1 \leq i < n$, C_i is the minor premise and not the conclusion of (EM);
- (ii) there are no applications of (EM) above C_i ;
- (iii) for all j , $i \leq j < n$, C_j is minor premise of (EM) and C_{j+1} is the conclusion of (EM);
- (iv) C_n is the conclusion of the deduction.

LEMMA 1. Any deduction in LP in which (EM) is applied can be transformed into one that is (EM) final.

PROOF. By repeated application of the following transformation:

$$\begin{array}{ccc}
 \begin{array}{c} [B]^i \quad [-B]^j \\ \Pi_1 \quad \Pi_2 \\ \hline C \quad C \\ \Sigma \\ D \end{array} & \rightsquigarrow & \begin{array}{c} [B]^i \quad [-B]^j \\ \Pi_1 \quad \Pi_2 \\ C \quad C \\ \Sigma \quad \Sigma \\ \hline D \quad D \\ D \end{array} \quad \dashv
 \end{array}$$

Begin with an application of (EM) lowest down in the deduction and work your way up.

In a deduction of LP that is (EM)-final, the sub-deductions concluding the premises of the applications of (EM) that are highest up in the deduction are deductions of N4 in which no rules for implication or negated implication are applied (i.e. they are deductions of FDE). To prove normalisation, we apply the conversions to deductions that are (EM)-final: given a deduction that is (EM)-final, it only remains to apply the conversions to the top-most deductions that are as in N4 just described (viz. FDE).

THEOREM 7. *Any deduction Π of A from open assumptions Γ in LP can be brought into a deduction in normal form of A from some of Γ .*

PROOF. If Π is a deduction not containing applications of (EM), the theorem follows by theorem 5. If it contains applications of (EM), apply Lemma 1 and transform it into an (EM)-final deduction. Then apply the reduction steps for maximal formulas and maximal segments as for N4 (viz. FDE). \dashv

Similarly to Theorem 2 we obtain:

THEOREM 8. *Deductions in normal form in LP have the negation subformula property.*

5.3. K_3

To prove normalisation for K_3 , we have two options. Either consider how to handle the rule (ECQ) or we can replace it by (\perp E) and ($\neg\perp$). Choosing the latter option means that we are already done, as all cases have already been considered in previous logics. So let this be our formalisation of K_3 . Similarly to Theorem 5 we obtain:

THEOREM 9. *Any deduction Π of A from open assumptions Γ in K_3 can be brought into a deduction in normal form of A from some of Γ .*

The option of K_3 with (ECQ) instead of (\perp E) and ($\neg\perp$) is considered in the Appendix. The fundamental idea is to treat (ECQ) as a limiting case of a del-rule without minor premises and establish that it need never conclude with a major premise of an elimination rule.

Similarly to Theorem 4 we obtain:

THEOREM 10. *Deductions in normal form in K_3 have the weak negation subformula property.*

6. Avron's Logic RM_3^{\supset}

Although RM_3^{\supset} has first appeared under the name PI^s in (Batens, 1980), we call it RM_3^{\supset} , following (Avron, 1986). It is also known as PCont (Ronzonoer, 1989). According to Avron, RM_3^{\supset} is a three-valued logic which ‘might be considered an optimal paraconsistent logic, since its positive fragment (in the $\{\supset, \wedge, \vee\}$ language) is identical with the classical one. It avoids $\neg A \supset (A \supset B)$, but every proper extension of it (closed under

substitutions) is equivalent to PC' (Avron, 1986, p. 201), PC being the classical propositional calculus. RM_3^\supset has the following axioms:

1. $A \supset (B \supset A)$
2. $A \supset (B \supset C) \supset . A \supset B \supset . A \supset C$
3. $A \supset \neg\neg A$
4. $\neg\neg A \supset A$
5. $(\neg A \supset B) \supset . (A \supset B) \supset B$
6. $A \supset . \neg B \supset \neg(A \supset B)$
7. $\neg(A \supset B) \supset \neg B$
8. $\neg(A \supset B) \supset \neg A$
9. $A \supset (A \vee B)$
10. $B \supset (A \vee B)$
11. $A \supset C \supset . B \supset C \supset . A \vee B \supset C$
12. $\neg(A \vee B) \supset \neg A$
13. $\neg(A \vee B) \supset \neg B$
14. $\neg A \supset . \neg B \supset \neg(A \vee B)$

The rule of inference is *modus ponens*: $A, A \supset B \vdash B$.

The deduction theorem holds for \supset in virtue of axioms 1 and 2 and *modus ponens*, and so $(\supset\text{I})$, $(\supset\text{E})$ are derived rules of RM_3^\supset , and conversely these rules suffice to derive axioms 1 and 2. Axioms 3 and 4 are equivalent to $(\neg\neg\text{I})$, $(\neg\neg\text{E})$; axioms 6, 7 and 8 to $(\neg\supset\text{I})$, $(\neg\supset\text{E})$; axioms 9, 10 and 11 to $(\vee\text{I})$, $(\vee\text{E})$; axioms 12, 13 and 14 to $(\neg\vee\text{I})$, $(\neg\vee\text{E})$. This leaves axiom 5, which is equivalent to (EM) . RM_3^\supset is therefore equivalent to LP without conjunction and extended by \supset , and to N4 without conjunction and extended by (EM) . Avron notes that axioms for conjunction analogous to those for disjunction could be added, but this is not needed, as conjunction is definable in the usual DeMorgan way (Avron, 1986, 207). Nonetheless, we shall do so. So let NRM_3^\supset be N4 with (EM) added. Its del-rules are those of LP, i.e. $(\vee\text{E})$, $(\neg\wedge\text{E})$ and (EM) .

The presence of implication in RM_3^\supset does not affect the proof of Lemma 1, and so we have:

THEOREM 11. *Any deduction Π of A from open assumptions Γ in NRM_3^\supset can be brought into a deduction in normal form of A from some of Γ .*

PROOF. If Π contains applications of (EM) , apply Lemma 1 and turn it into an (EM) -final deduction. Apply detour and permutation conversions as in the proof of normalisation for LP. ⊣

Similarly to Theorem 2 we obtain:

THEOREM 12. *Deductions in normal form in RM_3^\supset have the negation subformula property.*

7. RM_3

As follows from Brady’s paper (Brady, 1982), RM_3 can be axiomatized as an extension of the relevant logic R from (Anderson and Belnap, 1975) by adding the following axioms:

1. $(\neg A \wedge B) \supset (A \supset B)$
2. $A \vee (A \supset B)$

The system is equivalent to the system we call NRM_3 , which is NRM_3^\supset with $(\supset\text{I})$ be replaced with the following rule $(\supset\text{I}')$ and supplied with the rule $(\supset\text{E}')$ (see Appendix for the proof):

$$(\supset\text{I}') \quad \frac{\begin{array}{c} [A]^i \quad [\neg B]^j \\ \Pi_1 \quad \Pi_2 \\ B \quad \neg A \\ \hline A \supset B \end{array}}{A \supset B} \quad i,j \qquad (\supset\text{E}') \quad \frac{A \supset B \quad \neg B}{\neg A}$$

Notice that the rule $(\supset\text{I}')$ blocks the derivation of the axiom $A \supset (B \supset A)$ (as a result RM_3 does not have a standard deduction theorem), despite the fact that we have vacuous discharge.

Deductions in NRM_3 are normalizable in the same way as those in NRM_3^\supset . We need to check the case of $(\supset\text{I}')$ only, and it suffices to add two further detour conversions:

$$\begin{array}{ccc}
 \begin{array}{c} [A]^i \quad [\neg B]^j \\ \Pi_1 \quad \Pi_2 \\ B \quad \neg A \\ \hline A \supset B \end{array} \quad i,j \quad \Sigma & \rightsquigarrow & \begin{array}{c} \Sigma \\ [A] \\ \Pi_1 \\ B \\ \Xi \end{array} \\
 \hline B & & \\
 \Xi & & \\
 \\
 \begin{array}{c} [A]^i \quad [\neg B]^j \\ \Pi_1 \quad \Pi_2 \\ B \quad \neg A \\ \hline A \supset B \end{array} \quad i,j \quad \Sigma & \rightsquigarrow & \begin{array}{c} \Sigma \\ [\neg B] \\ \Pi_2 \\ \neg A \\ \Xi \end{array} \\
 \hline \neg A & & \\
 \Xi & &
 \end{array}$$

It poses no further problem to remove maximal formulas of the form $A \supset B$ from (EM)-final deductions in NRM_3^\supset , and so similarly to Theorem 11 we have:

THEOREM 13. *Any deduction Π of A from open assumptions Γ in NRM_3^\supset can be brought into a deduction in normal form of A from some of Γ .*

Moreover, similarly to Theorem 2 we obtain:

THEOREM 14. *Deductions in normal form in RM_3 have the negation subformula property.*

8. Par and PComp

The logic Par was first introduced in (Popov, 1989) in the form of Hilbert-style and sequent calculi, and later on it has appeared independently in (Avron, 1991) under the name Be. A four-valued semantics for Par was developed in (Pynko, 1999). As follows from (Popov, 1989), Par extends N4 by Peirce’s Law:

$$((A \supset B) \supset A) \supset A.$$

Clearly, the natural deduction system NPar can be defined as an extension of the natural deduction system for N4 by Peirce’s Rule:

$$\text{(P)} \quad \frac{[A \supset B]^i \quad \Pi}{A} i$$

The logic PComp is due to (Słupecki et al., 1967). It was also studied by Avron (1991). Since in both papers there is no name for it, we call it PComp, following (Popov, 2009). As follows from (Avron, 1991), it extends Par by the axiom $A \supset (\neg A \supset B)$. Thus, a natural deduction system for it results by extending the one for NPar by the rule (ECQ).

The presence of Peirce’s rule requires reduction procedures of a new kind. We follow Zimmermann’s (2002) strategy, who proved normalisation for a formalisation of classical logic that results by adding Peirce’s Rule to intuitionist logic.

As [Zimmermann \(2002\)](#) notes, applications of (P) can generate new maximal segments. Thus we extend definition 3 to the effect that maximal segments also arise when the conclusion of (P) is the major premise of an elimination rule. Zimmermann removes them by a procedure he calls *partial implication contractions*. It replaces applications of (P) with ones ‘followed immediately by a lower number of elimination rules than in the removed case’ ([Zimmermann, 2002](#), p. 565). The strategy is comparable to the way maximal segments arising from del-rules are treated: the application of the elimination rule is permuted upwards, so that it concludes with the premise of Pierce’s Rule.

Partial implication contractions for \vee, \wedge, \supset are presented in ([Zimmermann, 2002](#)). Let us look at the cases of the negated formulas that need to be considered in addition in the logic we are considering here. We begin with the case of NPar.

1. Double negation.

$$\begin{array}{c}
 [\neg\neg A \supset B]^i \\
 \Pi \\
 \frac{\neg\neg A}{\neg\neg A} \ i \\
 \frac{A}{A} \\
 \Xi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{[A \supset B]^j}{\frac{B}{\neg\neg A \supset B} \ k} \\
 \frac{[\neg\neg A]^k}{A} \\
 \Pi \\
 \frac{\neg\neg A}{A} \\
 \frac{A}{A} \ j \\
 \Xi
 \end{array}$$

2. Negated Conjunction (it is given on p. 517).

3. Negated Disjunction.

$$\begin{array}{c}
 [\neg(A_1 \vee A_2) \supset B]^l \\
 \Pi \\
 \frac{\neg(A_1 \vee A_2)}{\neg(A_1 \vee A_2)} \ l \\
 \frac{\neg A_i}{\neg A_i} \\
 \Xi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{[\neg(A_1 \vee A_2)]^k}{\neg A_i} \\
 \frac{[\neg A_i \supset B]^j}{\frac{B}{\neg(A_1 \vee A_2) \supset B} \ k} \\
 \Pi \\
 \frac{\neg(A_1 \vee A_2)}{\neg A_i} \\
 \frac{\neg A_i}{\neg A_i} \ j \\
 \Xi
 \end{array}$$

where $i = 1$ or $i = 2$.

$$\begin{array}{c}
 [\neg(A \wedge B) \supset C]^i \\
 \begin{array}{c} \Pi \\ \hline \frac{\neg(A \wedge B)}{\neg(A \wedge B)} \end{array} \quad \begin{array}{c} [\neg A]^n \\ \Sigma_1 \\ \hline D \end{array} \quad \begin{array}{c} [\neg B]^m \\ \Sigma_2 \\ \hline D \end{array} \\
 \hline
 \begin{array}{c} D \\ \Xi \\ \hline \end{array} \quad n,m \\
 \sim \\
 \begin{array}{c} [\neg(A \wedge B)]^k \\ \hline \begin{array}{c} \begin{array}{c} [\neg A]^n \\ \Sigma_1 \\ \hline D \end{array} \quad \begin{array}{c} [\neg B]^m \\ \Sigma_2 \\ \hline D \end{array} \\
 \frac{[D \supset C]^j}{C} \quad \frac{[D \supset C]^j}{C} \\
 \hline C \quad n,m \\
 \hline \frac{C}{\neg(A \wedge B) \supset C} \quad k \\
 \Pi \\
 \frac{\neg(A \wedge B)}{\neg(A \wedge B)} \\
 \hline \frac{D}{D} \quad j \\
 \Xi
 \end{array} \quad n,m
 \end{array}$$

4. Negated Implication.

$$\begin{array}{c}
[\neg(A \supset B) \supset C]^l \\
\Pi \\
\frac{\neg(A \supset B)}{\neg(A \supset B)} \quad l \\
\frac{\quad}{A} \\
\Xi
\end{array}
\rightsquigarrow
\begin{array}{c}
\frac{[A \supset C]^j \quad \frac{[\neg(A \supset B)]^k}{A}}{C} \\
\frac{\quad}{\neg(A \supset B) \supset C} \quad k \\
\Pi \\
\frac{\neg(A \supset B)}{\frac{A}{A} \quad j} \\
\Xi
\end{array}$$

$$\begin{array}{c}
[\neg(A \supset B) \supset C]^l \\
\Pi \\
\frac{\neg(A \supset B)}{\neg(A \supset B)} \quad l \\
\frac{\quad}{\neg B} \\
\Xi
\end{array}
\rightsquigarrow
\begin{array}{c}
\frac{[\neg B \supset C]^j \quad \frac{[\neg(A \supset B)]^k}{\neg B}}{C} \\
\frac{\quad}{\neg(A \supset B) \supset C} \quad k \\
\Pi \\
\frac{\neg(A \supset B)}{\frac{\neg B}{\neg B} \quad j} \\
\Xi
\end{array}$$

We need the following notion.

DEFINITION 14 (Trace; [Zimmermann, 2002](#), p. 565). A *trace* is a sequence of formulas on a path in a deduction such that the first is conclusion of Pierce's Rule and every subsequent formula is the conclusion an elimination rule.

Two things are worth pointing out about traces: (a) traces may contain parts that are segments, (b) being defined in terms of paths, except for the last one, the formulas on a trace are either major premises of elimination rules or minor premises of del-rules, and only the last one can be minor premise of (\supset E). The *length of trace* is the number of formula occurrences of which it consists.

LEMMA 2. *The traces of applications of the rule (P) can be reduced down to 0.*

PROOF. By induction on the length of traces and the degree of the formulas that are the conclusion of the rule (P). It goes through as in ([Zimmermann, 2002](#), Lemma 5), by observing that the additional partial implication contractions needed for deductions in Par do not upset any

properties appealed to in Zimmermann’s proof. As noted there, to reduce traces that have segments as their parts, we need to apply suitable permutative reduction procedures, but this poses no further problem. \dashv

THEOREM 15. *Any deduction Π of A from open assumptions Γ in Par can be brought into a deduction in normal form of A from some of Γ .*

PROOF. Given a deduction Π of A from open assumptions Γ in Par, apply the conversions as in the proof of normalisation for RM_3^\supset and Lemma 2. \dashv

Similarly to Theorem 2 we obtain:

THEOREM 16. *Deductions in normal form in Par have the Pierce negation subformula property.*

In the case of PComp we either need to consider the cases with (ECQ) (this option is considered in Appendix) or proceed as we did in the case of K_3 and replace (ECQ) with the rules ($\perp\text{E}$) and ($\neg\perp$). Then we are almost done. We just need to consider two more cases.

1. ($\perp\text{E}$) and (P).

$$\frac{\frac{\frac{[\perp \supset C]^l}{\Pi}}{\perp} \quad l}{B} \quad \Xi}{\Xi} \rightsquigarrow \frac{\frac{\frac{[\perp]^k}{C}}{\perp \supset C} \quad k}{\Pi}}{\perp} \quad \Xi$$

2. ($\neg\perp$) and (P).

$$\frac{\frac{\frac{[\neg A \supset B]^i}{\Pi}}{\neg A} \quad i \quad \Sigma}{\neg A} \quad A}{\perp} \quad \Xi}{\Xi} \rightsquigarrow \frac{\frac{\frac{[\perp \supset B]^j}{\Pi}}{\neg A \supset B} \quad k \quad \Sigma}{\perp} \quad \frac{[\neg A]^k}{A} \quad \Sigma}{\perp} \quad j}{\perp} \quad \Xi$$

Similarly to Theorem 15 we obtain:

THEOREM 17. *Any deduction Π of A from open assumptions Γ in PComp can be brought into a deduction in normal form of A from some of Γ .*

Moreover, similarly to Theorem 4 we have:

THEOREM 18. *Deductions in normal form in PComp have the Pierce weak negation subformula property.*

9. Conclusion

In the paper, we proved normalisation for some representative many-valued logics: LP, FDE, K_3 , RM_3 , RM_3^\supset , PComp, and Par. All these logics are extensions of Nelson's logic N4 or its implication-less fragment. Moreover, FDE coincides with the implication-less fragment of N4. As a task for future research, one may think about logics which are not extensions of N4 or FDE. In fact, we have already made a first step in this direction. Although RM_3 extends FDE, it does not extend N4, since $A \supset (B \supset A)$ is not RM_3 -valid (as a consequence RM_3 does not enjoy a standard deduction theorem).

However, such logics were not in the center of our attention, we have decided to consider RM_3 just because we were looking for an alternative for an original implication of LP. One may try to prove normalisation for Łukasiewicz's L_3 . Similarly to RM_3 , it extends FDE, but not N4, because $A \supset (B \supset A)$ is not L_3 -valid.

One may think also about investigation of the logics extending intuitionist logic (without strong negation). One of such logics is Heyting-Gödel-Jaśkowski's G_3 . It extends intuitionistic logic by the formula $(\neg A \supset B) \supset (((B \supset A) \supset B) \supset B)$ which was shown by Łukasiewicz (1941) (for an English translation see (Łukasiewicz, 1970, pp. 278–294)).

Besides, one may look at the families of logics of formal undeterminedness and inconsistency which undoubtedly require a separate paper. Last, but not least, five-, six-, ... infinitely-valued logics might be a fruitful area of research.

Appendix

Semantics of the logics in question

The aim of this section is to introduce the semantics of the logics for which we proved normalisation theorems. Our proofs are purely syntactical, so we could deal without any semantics at all. However, we would like to stress the fact that our logics are many-valued ones, and a brief description of the semantics of the logics in question is appropriate for this purpose.

Let us begin with Nelson’s logic N4 (see, e.g., [Nelson, 1949](#); [Almukdad and Nelson, 1984](#)). It is a four-valued logic, but having an intentional implication. Also, its four-valuedness is hidden in the notion of a valuation.

A *Nelson model* is a structure $\langle W, R, v^+, v^- \rangle$ such that W is a non-empty set, R is a binary relation on it, $v^{\natural}: \mathbf{Prop} \rightarrow 2^W$ (where \mathbf{Prop} is the set of all propositional variables) such that for both $\natural \in \{+, -\}$, for all $x, y \in W$ and for all $p \in \mathbf{Prop}$, it holds that if $R(x, y)$, then $x \in v^{\natural}(p)$ implies $y \in v^{\natural}(p)$. In what follows, we write $\mathfrak{M}, x \Vdash^{\natural} p$ for $x \in v^{\natural}(p)$. The truth conditions for the other formulas of N4 in a given Nelson model \mathfrak{M} are as follows:

- $\mathfrak{M}, x \Vdash^+ A \supset B$ iff for all y such that $R(x, y)$, $\mathfrak{M}, y \Vdash^+ A$ implies $\mathfrak{M}, y \Vdash^+ B$,
- $\mathfrak{M}, x \Vdash^- A \supset B$ iff $\mathfrak{M}, x \Vdash^+ A$ and $\mathfrak{M}, x \Vdash^- B$,
- $\mathfrak{M}, x \Vdash^+ A \vee B$ iff $\mathfrak{M}, x \Vdash^+ A$ or $\mathfrak{M}, x \Vdash^+ B$,
- $\mathfrak{M}, x \Vdash^- A \vee B$ iff $\mathfrak{M}, x \Vdash^- A$ and $\mathfrak{M}, x \Vdash^- B$,
- $\mathfrak{M}, x \Vdash^+ A \wedge B$ iff $\mathfrak{M}, x \Vdash^+ A$ and $\mathfrak{M}, x \Vdash^+ B$,
- $\mathfrak{M}, x \Vdash^- A \wedge B$ iff $\mathfrak{M}, x \Vdash^- A$ or $\mathfrak{M}, x \Vdash^- B$,
- $\mathfrak{M}, x \Vdash^+ \neg A$ iff $\mathfrak{M}, x \Vdash^- A$,
- $\mathfrak{M}, x \Vdash^- \neg A$ iff $\mathfrak{M}, x \Vdash^+ A$.

A formula A is true in a Nelson model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ iff $\mathfrak{M}, x \Vdash^+ A$ for any $x \in W$.

If one adds the falsum constant \perp to N4 to express the intuitionist negation $\sim A = A \supset \perp$, then one needs the following truth conditions for it: $\mathfrak{M}, x \not\Vdash^+ \perp$ and $\mathfrak{M}, x \Vdash^- \perp$.

As we have already said, the four-valuedness of N4 is hidden because of the specific of valuations which are used in this logic. However, we can make the secret explicit at least in the case of the $\{\neg, \vee, \wedge\}$ -fragment of N4 by means of the following matrices:

A	\neg
1	0
b	b
n	n
0	1

\vee	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

\wedge	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

The values are interpreted in the Belnapian way: 1 stands for ‘true’, b for ‘both [true and false]’, n for ‘none [neither true and false]’, and 0 for ‘false’. The valuation $v: \mathbf{Prop} \rightarrow \{1, b, n, 0\}$ is extended to all types of formulas according to the matrices presented above. The values 1 and b are considered as designated ones, hence we say that a formula A follows from the set of formulas Γ ($\Gamma \models A$) iff $v(B) \in \{1, b\}$ (for any $B \in \Gamma$) implies $v(A) \in \{1, b\}$. Notice that there are no formula A such that $\emptyset \models A$. One can pluckily add the subscript FDE to \models in this definition, since we have already received Belnap’s (1977) semantics for FDE.

As was shown by Pynko (1999), to obtain the semantics for Par one needs to extend FDE by the following implication:

\supset	1	b	n	0
1	1	b	n	0
b	1	b	n	0
n	1	1	1	1
0	1	1	1	1

Let us note that the extensive survey of various semantics and proof systems for FDE and its extensions is given in (Omori and Wansing, 2017).

If one restrict these matrices to the set $\{1, b, 0\}$, then ones gets the semantics for LP (the case of implication-less language) and the semantics for \mathbf{RM}_3^{\supset} (the case of the $\{\neg, \vee, \wedge, \supset\}$ -language) which implication connective is Jaśkowski’s (1999) implication and is neatly symmetric to conjunction and disjunction. Let us also mention that the restriction of the matrices in question to the set $\{1, n, 0\}$, gives \mathbf{K}_3 by Kleene (1938) (implication-less language) and PComp (full language) by Ślupecki et al. (1967).

In the case of \mathbf{RM}_3 , as follows from (Anderson and Belnap, 1975), one needs to replace Jaśkowski’s implication with the one by Sobociński (1952):

\supset	1	b	0
1	1	0	0
b	1	b	0
0	1	1	1

The equivalence of RM_3 and NRM_3

As we said in the section 7, Brady showed (Brady, 1982) that RM_3 can be axiomatized as an extension of the relevant logic R from (Anderson and Belnap, 1975) by adding the following axioms:

1. $(\neg A \wedge B) \supset (A \supset B)$
2. $A \vee (A \supset B)$

Let us observe the formulation of the logic R (we follow Dunn’s (2000) presentation). It has the following axioms:

1. $A \supset A$
2. $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$
3. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$
4. $(A \supset (A \supset B)) \supset (A \supset B)$
5. $((C \supset A) \wedge (C \supset B)) \supset (C \supset (A \wedge B))$
6. $(A \wedge B) \supset A, (A \wedge B) \supset B$
7. $A \supset (A \vee B), B \supset (A \vee B)$
8. $((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$
9. $(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))$
10. $(A \supset B) \supset (\neg B \supset \neg A)$
11. $\neg\neg A \supset A$
12. $A \supset \neg\neg A$

The rules of inference are *modus ponens*: $A, A \supset B \vdash B$; and *adjunction*: $A, B \vdash A \wedge B$.

THEOREM 19. *The natural deduction system NRM_3 is equivalent to the axiomatic formulation of RM_3 .*

PROOF. The proof consists in two parts: (I) to show that all axioms and rules of the axiomatic system are provable in NRM_3 , and (II) all rules of NRM_3 are derivable in the axiomatic system.

(I). Let us show some the most remarkable cases. *Consider the following proof:*

$$\frac{\frac{[\neg A \wedge B]^1}{B} \quad \frac{[\neg A \wedge B]^1}{\neg A} \quad \frac{\frac{[\neg(A \supset B)]^2}{A}}{\neg \neg A}}{\neg(\neg A \wedge B)} \quad 1,2}{(\neg A \wedge B) \supset (A \supset B)}$$

Consider the following proof:

$$\frac{[A \vee (A \supset B)]^1 \quad \frac{\frac{[\neg(A \vee (A \supset B))]^2}{\neg(A \supset B)}}{A}}{A \vee (A \supset B)} \quad 1,2}{A \vee (A \supset B)}$$

Consider the following proof:

$$\frac{\frac{(*_1) \quad (*_2)}{(C \supset A) \supset (C \supset B)} \quad 2,5 \quad (*_3)}{(A \supset B) \supset ((C \supset A) \supset (C \supset B))} \quad 1,6$$

where $(*_1)$ is as follows:

$$\frac{\frac{[A \supset B]^1 \quad \frac{[C \supset A]^2}{A} \quad [C]^3}{B} \quad \frac{[C \supset A]^2 \quad \frac{[A \supset B]^1 \quad [\neg B]^4}{\neg A}}{\neg C} \quad 3,4}{C \supset B}$$

$(*_2)$ is as follows:

$$\frac{\frac{[\neg(C \supset B)]^5}{C} \quad \frac{[A \supset B]^1}{\neg A} \quad \frac{[\neg(C \supset B)]^5}{\neg B}}{\neg(C \supset A)}$$

$(*_3)$ is given as (I) on p. 526.

Consider the following proof:

$$\frac{\frac{[A \supset (B \supset C)]^1}{B \supset C} \quad \frac{[\neg(A \supset C)]^5}{\neg C}}{\frac{(*_4) \quad \neg B}{B \supset (A \supset C)} \quad 2,5} \quad (*_5) \quad 1,6$$

$$(A \supset (B \supset C)) \supset (B \supset (A \supset C))$$

where $(*_4)$ is given as (II) on p. 526.

(*₅) is as follows:

$$\frac{\frac{\frac{[\neg(A \supset (B \supset C))]^6}{\neg(B \supset C)}}{B} \quad \frac{\frac{[\neg(A \supset (B \supset C))]^6}{A} \quad \frac{[\neg(A \supset (B \supset C))]^6}{\neg(B \supset C)}}{\neg(A \supset C)}}{\neg(B \supset (A \supset C))}$$

Consider the following proof:

$$\frac{\frac{\frac{[A \supset (A \supset B)]^1}{A \supset B} \quad [A]^2}{[A]^2} \quad B}{\frac{A \supset B}{(A \supset (A \supset B)) \supset (A \supset B)}} \quad \begin{matrix} (*_7)_{2,4} \\ (*_8)_{1,3} \end{matrix}$$

where (*₇) is as follows:

$$\frac{\frac{\frac{[A \supset (A \supset B)]^1}{A \supset B} \quad [A]^5}{[A]^4} \quad [\neg B]^4}{\frac{\neg A}{\neg A}} \quad [\neg A]^6 \quad 5,6$$

(*₈) is as follows:

$$\frac{\frac{\neg[A \supset B]^3}{A} \quad [\neg(A \supset B)]^3}{\neg(A \supset (A \supset B))}$$

Consider the following proof:

$$\frac{\frac{\frac{[A \supset B]^1}{\neg A} \quad [\neg B]^2}{\neg B \supset \neg A} \quad \frac{\frac{[A \supset B]^1}{B} \quad \frac{[\neg \neg A]^4}{A}}{\neg \neg B} \quad 2,4}{\frac{(A \supset B) \supset (\neg B \supset \neg A)}}{(*_9)_{1,3}}$$

where (*₉) is as follows:

$$\frac{\frac{[\neg(\neg B \supset \neg A)]^3}{\neg \neg A} \quad \frac{[\neg(\neg B \supset \neg A)]^3}{\neg B}}{\neg(A \supset B)}$$

$$(I)$$

$$\frac{\frac{\frac{[\neg((C \supset A) \supset (C \supset B))]^6}{C \supset A}}{A} \quad \frac{\frac{[\neg((C \supset A) \supset (C \supset B))]^6}{C}}{A} \quad \frac{\frac{[\neg((C \supset A) \supset (C \supset B))]^6}{\neg(C \supset B)}}{\neg B}}{\neg(A \supset B)}$$

$$(II)$$

$$\frac{\frac{\frac{[A \supset (B \supset C)]^1}{B \supset C} \quad \frac{[A]^3}{A}}{C} \quad \frac{[B]^2}{[A \supset (B \supset C)]^1} \quad \frac{\frac{[B]^2}{\neg(B \supset C)} \quad \frac{[\neg C]^4}{\neg(B \supset C)}}{\neg A} \quad 3,4}{A \supset C}$$

Consider the following proofs:

$$\frac{\frac{[A]^1}{\neg\neg A} \quad \frac{[\neg\neg\neg A]^2}{\neg A}}{A \supset \neg\neg A} 1,2 \qquad \frac{\frac{[\neg\neg A]^1}{A} \quad \frac{[\neg A]^2}{\neg\neg\neg A}}{\neg\neg A \supset A} 1,2$$

Consider the following proofs:

$$\frac{\frac{[A]^1}{A \vee B} \quad \frac{[\neg(A \vee B)]^2}{\neg A}}{A \supset (A \vee B)} 1,2 \qquad \frac{\frac{[A \wedge B]^1}{A} \quad \frac{[\neg A]^2}{\neg(A \wedge B)}}{(A \wedge B) \supset A} 1,2$$

The other cases are considered similarly.

(II). The derivability of the rules of NRM_3 in the axiomatic system for RM_3 follows from the completeness of the axiomatic system and the fact that all rules of NRM_3 are sound. \dashv

Yet another proof for K_3

Here we consider the proof for K_3 formulated with (ECQ), but without \perp . The possibility of such a proof was mentioned in section 5.3. The rule (ECQ) can introduce a formula which is deleted by an elimination rule in the next step. Also, it can eliminate formulas of the forms $\neg\neg A$, $\neg(A \wedge B)$, and $\neg(A \vee B)$ which were obtained by an introduction rule in the previous step. Additionally, (ECQ) can introduce some negated formula which is eliminated in the next step by another application of (ECQ). Let us consider all these cases:

1. (ECQ) introduces conjunction.

$$\frac{\frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A}}{\frac{B_1 \wedge B_2}{B_i}} \rightsquigarrow \frac{\Sigma_1 \quad \Sigma_2}{\frac{A}{B_i}} \Xi$$

2. (ECQ) introduces disjunction.

$$\frac{\frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A} \quad \frac{[B_1]^i}{\Pi_1} \quad \frac{[B_2]^j}{\Pi_2}}{\frac{B_1 \vee B_2}{C}} \Xi \rightsquigarrow \frac{\Sigma_1 \quad \Sigma_2}{\frac{A}{C}} \Xi$$

3. (ECQ) introduces negated conjunction.

$$\frac{\frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A} \quad \frac{[\neg B_1]^i \quad [\neg B_2]^j}{\Pi_1 \quad \Pi_2}}{\neg(B_1 \wedge B_2)} \quad C}{C} \quad i,j \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A}}{C} \quad \Xi$$

4. (ECQ) introduces negated disjunction.

$$\frac{\frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A}}{\neg(B_1 \vee B_2)} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A}}{B_i} \quad \Xi$$

5. (ECQ) introduces double negation.

$$\frac{\frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A}}{\neg\neg B} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A}}{B} \quad \Xi$$

6. (ECQ) eliminates negated conjunction.

$$\frac{\Sigma_1 \quad \frac{\Sigma_2}{\neg A_i}}{A_1 \wedge A_2 \quad \neg(A_1 \wedge A_2)} \quad C}{C} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \frac{\Sigma_2}{A_i \wedge A_2}}{A_i \quad \neg A_i}}{C} \quad \Xi$$

In this case $A_1 \wedge A_2$ may become a maximal formula. However, it has a lower degree than $\neg(A_1 \wedge A_2)$. Thus, it can be eliminated.

7. (ECQ) eliminates negated disjunction.

$$\frac{\Sigma_1 \quad \frac{\Sigma_2 \quad \Sigma_3}{\neg A \quad \neg B}}{A \vee B \quad \neg(A \vee B)} \quad C}{C} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \frac{[A]^i \quad \Sigma_2}{A \vee B \quad C} \quad \frac{[B]^j \quad \Sigma_3}{C \quad \neg B}}{C}}{C} \quad i,j \quad \Xi$$

In this case, if C is a maximal formula or a part of a maximal segment, then we make it longer by 1. However, we eliminate $\neg(A \vee B)$.

8. (ECQ) eliminates double negation.

$$\frac{\frac{\Sigma_1}{\neg A} \quad \frac{\Sigma_2}{A}}{\neg\neg A} \quad \rightsquigarrow \quad \frac{\Sigma_2 \quad \Sigma_1}{A \quad \neg A} \quad \frac{B}{\Xi}$$

9. (ECQ) introduces some negated formula and eliminates it.

$$\frac{\Pi \quad \frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A}}{B \quad \neg B} \quad \rightsquigarrow \quad \frac{\Sigma_2 \quad \Sigma_1}{A \quad \neg A} \quad \frac{C}{\Xi}$$

Similarly to Theorem 2 we obtain:

THEOREM 20. *Deductions in normal form in K_3 formulated with (ECQ), but without \perp have the negation subformula property.*

Yet another proof for PComp

Here we consider the proof for PComp formulated with (ECQ), but without \perp . We need to add to the proof for K_3 the cases regarding implication as well as the interaction of (ECQ) and (P).

1. (ECQ) introduces implication.

$$\frac{\frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A}}{B \supset C} \quad \frac{\Sigma_3}{B} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A} \quad \frac{C}{\Xi}$$

2. (ECQ) introduces negated implication.

$$\frac{\frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A}}{\neg(B \supset C)} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A} \quad \frac{B}{\Xi}$$

$$\frac{\frac{\Sigma_1}{A} \quad \frac{\Sigma_2}{\neg A}}{\neg(B \supset C)} \quad \rightsquigarrow \quad \frac{\Sigma_1 \quad \Sigma_2}{A \quad \neg A} \quad \frac{\neg C}{\Xi}$$

3. (ECQ) eliminated negated implication.

$$\frac{\frac{\frac{\Pi}{A \supset B} \quad \frac{\frac{\Sigma_1}{A} \quad \Sigma_2}{\neg(A \supset B)}}{C}}{\Xi} \rightsquigarrow \frac{\frac{\frac{\Sigma_1}{A} \quad \frac{\Pi}{A \supset B}}{B} \quad \frac{\Sigma_2}{\neg B}}{C}}{\Xi}$$

4. (ECQ) and (P).

$$\frac{\frac{\frac{\frac{\Sigma}{A} \quad \frac{\frac{\Pi}{\neg A} \quad i}{\neg A}}{C}}{\Xi} \quad [\neg A \supset B]^i}{\Xi} \rightsquigarrow \frac{\frac{\frac{\frac{\Sigma}{A} \quad \frac{\frac{\frac{\Sigma}{A} \quad [\neg A]^k}{C}}{B}}{\neg A \supset B} \quad k}{\Pi} \quad \frac{\Sigma}{A}}{C} \quad j}{\Xi}}$$

Similarly to Theorem 2 we obtain:

THEOREM 21. *Deductions in normal form in PComp formulated with (ECQ), but without \perp have the Pierce negation subformula property.*

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