Analytic hypoellipticity of Keldysh operators

Jeffrey Galkowski and Maciej Zworski

Abstract

We consider Keldysh-type operators, $P = x_1 D^2_{x_1} + a(x) D_{x_1} + Q(x, D_{x'})$, $x = (x_1, x')$ with analytic coefficients, and with $Q(x, D_{x'})$ second order, principally real and elliptic in $D_{x'}$ for $x$ near zero. We show that if $Pu = f$, $u \in C^\infty$, and $f$ is analytic in a neighbourhood of 0, then $u$ is analytic in a neighbourhood of 0. This is a consequence of a microlocal result valid for operators of any order with Lagrangian radial sets. Our result proves a generalized version of a conjecture made in (Lebeau and Zworski, Proc. Amer. Math. Soc. 147 (2019) 145–152; Zworski, Bull. Math. Sci. 7 (2017) 1–85) and has applications to scattering theory.

1. Introduction

We consider analytic regularity for generalizations of the Keldysh operator [24],

$$P := x_1 D^2_{x_1} + D^2_{x_2}. \quad (1.1)$$

The operator $P$ has the feature of changing from an elliptic to a hyperbolic operator at $x_1 = 0$. It appears in various places including the study of transsonic flows, see, for instance, Čanić–Keyfitz [8] or population biology — see Epstein–Mazzeo [12]. Our interest in such operators comes from the work of Vasy [31] where the transition at $x_1 = 0$ corresponds to the boundary at infinity for asymptotically hyperbolic manifolds (see [34]), crossing the event horizons of Schwarzschild black holes (see [11, § 5.7]) or the cosmological horizon for de Sitter spaces. The Vasy operator in the asymptotically hyperbolic setting is given by

$$P(\lambda) = 4(x_1 D^2_{x_1} - (\lambda + i) D_{x_1}) - \Delta_h(x_1) + i\gamma(x) \left(2x_1 D_{x_1} - \lambda - i \frac{n-1}{2}\right), \quad (1.2)$$

where $h(x_1)$ is a smooth family of Riemannian metrics in $x'$, $x = (x_1, x') \in \mathbb{R}^n$ and $\gamma \in C^\infty(\mathbb{R}^n)$. The resonant states at resonant frequencies $\lambda$ (see [11, Chapter 5]) are the smooth solutions of $P(\lambda)u = 0$.

For various reasons reviewed in § 1.3, it is interesting to ask if in the case of analytic coefficients the resonant states are real analytic across $x_1 = 0$. That lead to [35, Conjecture 2] which asked if $P(\lambda)u = f$ with $u$ smooth and $f$ analytic near $x_1 = 0$ implies that $u$ is analytic near $x_1 = 0$. For $\gamma(x) \equiv 0$ and $h$ independent of $x_1$, this was shown by Lebeau–Zworski [25] under the assumption that $\lambda \notin -i\mathbb{N}^*$.

The general case was proved by Zuily [32] under the same restriction on $\lambda$. His proof was an elegant adaptation of the work of Baouendi–Goulaouic [1], Bolley–Camus [3] and Bolley–Camus–Hanouzet [4].

In this paper, we prove this result for generalized Keldysh operators with analytic coefficients (1.3). In particular, we do not make any assumptions on lower order terms:

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Figure 1. A comparison of the Keldysh operator \((1.1)\) and the Tricomi operator \((1.5)\). The figures show the cylinder \(\mathbb{R} x_1 \times S^1\) \((\xi_1, \xi_2) = |\xi|(\cos \theta, \sin \theta)\) (this is the boundary of the fibre compactified cotangent bundle \(T^*\mathbb{R}^n\) — see \([11, \S 1.3]\) — with the \(x_2\) variable omitted). The characteristic varieties, \(x_1 \cos^2 \theta + \sin^2 \theta = 0\) and \(\cos^2 \theta + x_1 \sin^2 \theta = 0\), respectively, are shown with the direction of the Hamiltonian flow indicated. In the Keldysh case, the two radial Lagrangians, \(\Lambda^\pm\), correspond to \(\theta = \pi\) and \(\theta = 0\), respectively.

Theorem 1. Suppose that \(U \subset \mathbb{R}^n\) is a neighbourhood of 0,
\[
P := x_1 D_{x_1}^2 + a(x) D_{x_1} + Q(x, D_{x'})\quad (x, x') \in U,
\]
has analytic coefficients, \(Q(x, D_{x'})\) is a second-order elliptic operator in \(D_{x'}\) with a real valued principal symbol. Then there exists a neighbourhood of 0, \(U' \subset U\), such that
\[
Pu \in C^\omega(U), \quad u \in C^\infty(U) \implies u \in C^\omega(U').
\]

We will show in \(\S 1.1\) that this result follows from a more general microlocal result valid for operators of all orders satisfying a natural geometric condition.

Remarks. (1) In the statement of the theorem 0 can be replaced by any point at which \(x_1 \geq 0\) and \(U'\) can be replaced by \(U\) provided we include a bicharacteristic convexity condition. That follows from propagation of analytic singularities — see [26, Theorem 4.3.7] or [22, Theorem 2.9.1]: since there are no singularities near \(x_1 = 0\), there will be no singularities on trajectories hitting \(x_1 = 0\) — see Figure 1.

(2) The result is false for the Tricomi operator
\[
P := D_{x_1}^2 + x_1 D_{x_2}^2.
\]
This can be seen using results about propagation of analytic singularities (unlike \((1.3)\) this operator can be microlocally conjugated to \(D_{y_1}\) — see Figure 1) but is also easily demonstrated by the following example:
\[
u(x) := \int_0^\infty Ai(\tau^{4/3} x_1) e^{i \tau x_2} e^{-\tau} d\tau, \quad Pu = 0, \quad u \in C^\infty(\mathbb{R}^2).
\]
Here, \(Ai\) is the Airy function which satisfies
\[
Ai''(t) + t Ai(t) = 0, \quad |\partial^\ell_t Ai(t)| \leq C(\ell) \frac{\tau^{\ell - \frac{1}{2}}}{\ell!}, \quad t \in \mathbb{R}, \quad \ell \in \mathbb{N}, \quad Ai(0) > 0.
\]
We then have
\[
D_{x_2}^k u(0) = Ai(0) \int_0^\infty \tau^{2k} e^{-\tau} d\tau = Ai(0)(2k)!
\]
and \(u\) is not analytic at 0.
(3) Results similar to (1.4) have been obtained in the setting of other operators. In addition to the works [3, 4] cited above, we mention the work of Baouendi–Sjöstrand [2] who considered a class of Fuchsian operators generalizing

\[ P = |x|^2 \Delta + \mu(x, D_x) + \lambda \]  \hspace{1cm} (1.7)

In the case of (1.7), (1.4) holds for any \( \lambda, \mu \in \mathbb{C} \) and [2] established (1.4) for more general operators satisfying appropriate conditions.

(4) The operators (1.3), (1.5) and (1.7) are not \( \mathcal{C}^\infty \) hypoelliptic, that is, \( Pu \in \mathcal{C}^\infty \not\Rightarrow u \in \mathcal{C}^\infty \). The study of operators which are \( \mathcal{C}^\infty \) hypoelliptic but not analytic hypoelliptic has a long tradition with a simple example [23, §8.6, Example 2] given by

\[ P = D^2_{x_1} + x_1^2 D^2_{x_2} + D^3_{x_3}. \]

For more complicated cases, references, and connections to several complex variables, see Christ [9] and for some recent progress and additional references, Bove–Mughetti [7].

1.1. A microlocal result

We make the following general assumptions. Let \( P \) be a differential operator of order \( m \) with analytic coefficients:

\[ P := \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha_x, \quad a_\alpha \in \mathcal{C}^\omega(U), \quad p(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \]  \hspace{1cm} (1.8)

where \( U \) is an open neighbourhood of \( x_0 \in \mathbb{R}^n \). We make the following assumptions valid in a conic neighbourhood of \((x_0, \xi_0) \in T^* \mathbb{R}^n \setminus 0 \): \( p \) is real valued and there exists a conic Lagrangian submanifold \( \Lambda \), such that

\[ (x_0, \xi_0) \in \Lambda \subset p^{-1}(0), \quad dp|_\Lambda \neq 0, \quad H_p|_\Lambda \parallel \xi \cdot \partial_\xi|_\Lambda. \]  \hspace{1cm} (1.9)

Here \( \parallel \) means that the two vector fields are positively proportional, that is the Lagrangian is radial (the positivity assumptions can be achieved by multiplying \( P \) by \( \pm 1 \)). Except for the analyticity assumption in (1.8), these are the assumptions made in Haber [19] and Haber–Vasy [20].

Theorem 1 follows from the following microlocal result. We denote by \( \text{WF} \) the \( \mathcal{C}^\infty \)-wave front set and by \( \text{WF}_a \) the analytic wave front set — see [23, §8.1] and [23, §8.5,9.3], respectively.

**Theorem 2.** Suppose that \( P \) and \((x_0, \xi_0) \in T^* \mathbb{R}^n \setminus 0 \) satisfy the assumptions (1.8) and (1.9). Then for \( u \in \mathcal{D}'(\mathbb{R}^n) \),

\[ (x_0, \xi_0) \notin \text{WF}(u), \quad (x_0, \xi_0) \notin \text{WF}_a(Pu) \Rightarrow (x_0, \xi_0) \notin \text{WF}_a(u). \]  \hspace{1cm} (1.10)

The proof is based on the theory of microlocal symbolic weights developed by Galkowski–Zworski [14] and based on the work of Sjöstrand — see [29, §2] (and also [21] and [26, §3.5]). With this theory in place we can use escape functions, \( G, H_p G \geq 0 \), which are logarithmically bounded in \( \xi \) (hence the \( \mathcal{C}^\infty \) wave front set assumption on \( u \) allows the use of such weights) and which tend to \( \langle \xi \rangle \) in a neighbourhood of \((x_0, \xi_0)\). The normal form for \( p \) constructed in [19] (following much earlier work of Guillemin–Schaeffer [18] which was based in turn on Sternberg’s linearization theorem [30]) was helpful in the construction of the specific weights needed here. We indicate the method of the proof in §1.2.

**Proof of Theorem 1.** Under the assumptions of Theorem 1, the characteristic set of \( P \) over \( x_1 = 0 \) is given by (in \( T^* \mathbb{R}^n \setminus 0 \))

\[ p^{-1}(0) \cap \{x_1 = 0\} = \{(0, x_2, \xi_1, 0) : \xi_1 \in \mathbb{R} \setminus 0; x_2 \in \text{neigh}_{\mathbb{R}^{n-1}}(0)\} = \Lambda_+ \cup \Lambda_- , \]
where $\pm \xi_1 > 0$ on $\Lambda_\pm$. These two components are Lagrangian and conic and $H_p|_{\Lambda_\pm} = -\xi_1^2 \partial_{\xi_1}|_{\Lambda_\pm}$ is radial. Since $Pu \in C^\omega(U)$, we have $WF_a(Pu) \cap \{x \in U : x_1 = 0\} = \emptyset$ and hence Theorem 2 shows that $WF_a(u) \cap \Lambda_\pm = \emptyset$. On the other hand, [23, Theorem 8.6.1], $WF_a(u) \cap \{x_1 = 0\} \subset \mathcal{P}^{-1}(0) \cap \{x_1 = 0\} = \Lambda_+ \cup \Lambda_-$. Hence $WF_a(u) \cap \{x_1 = 0\} = \emptyset$ and, since $\text{singsupp}_a u = \pi WF_a(u)$, $u$ is analytic near $x_1 = 0$. \hfill \Box

### 1.2. A proof in a special case

To indicate the ideas behind the proof, we consider $P$ given by

$$P = x_1 D_{x_1}^2 + D_{x_2}^2 + aD_{x_1}, \quad a \in \mathbb{C},$$

and a very special $u$:

$$u = e^{i\tau x_2} v(x_1), \quad v \in \mathcal{S}(\mathbb{R}), \quad Pu = e^{i\tau x_2} f(x_1), \quad e^{[\xi_1]_1} \hat{f} \in L^2(\mathbb{R}). \quad (1.11)$$

This assumption is a stronger version of the assumption that $f$ is analytic. We consider a family of smooth functions $G_\epsilon(\xi)$ satisfying

$$0 \leq G_\epsilon(\xi_1) \leq \min(\frac{1}{\epsilon} \log(1 + |\xi_1|), |\xi_1|) \quad (1.12)$$

In view of (1.11),

$$\|v_e\|_{L^2(\mathbb{R})} \leq C_\epsilon, \quad \|f_e\|_{L^2(\mathbb{R})} \leq C_0 \quad v_e := e^{G_\epsilon(D_x)v}, \quad f_e := e^{G_\epsilon(D_x)f}.$$ 

where $C_0$ is independent of $\epsilon$. We then consider

$$P_e := e^{G_\epsilon(D_x)}(x_1 D_{x_1}^2 + aD_{x_1} + \tau^2) e^{-G_\epsilon(D_x)} = x_1 D_{x_1}^2 + iG_\epsilon(D_x)D_{x_1}^2 + aD_{x_1} + \tau^2.$$

We have $P_e v_e = f_e$, and

$$\text{Im} \langle P_e v_e, v_e \rangle_{L^2(\mathbb{R})} = \langle G_\epsilon(D_x)D_{x_1}^2 v_e, v_e \rangle_{L^2(\mathbb{R})} + \langle \text{Im} a + 1 \rangle D_{x_1} v_e, v_e \rangle_{L^2(\mathbb{R})}$$

where we took $d\xi_1/(2\pi)$ as the measure on $L^2(\mathbb{R},\xi_1)$. Let $\chi \in C^\infty(\mathbb{R};[0,1])$ satisfy $\chi|_{\xi_1 \leq 1} = 1, \chi|_{\xi_1 \geq 2} = 0$ and $\chi' \leq 0$. We define

$$G_\epsilon(\xi_1) = (1 - \chi(\xi_1)) \int_0^{\xi_1} (\chi(\epsilon t) + (1 - \chi(\epsilon t))(\epsilon t)^{-1}) dt,$$

which satisfies (1.12) and $G_\epsilon' \geq 0$. Moreover, for $\xi_1 \geq M \geq 2$ and $\epsilon < 1/M,$

$$\xi_1^2 G_\epsilon(\xi_1) \geq \xi_1^2 \chi(\epsilon \xi_1) + \epsilon^{-1} \chi(1 - \chi(\epsilon \xi_1)) \geq M \xi_1.$$

Hence, by taking $M = \max(-\text{Im} a + 1, 2)$, and $\epsilon < 1/M,$

$$\|f_e\|_{\mathcal{H}^1} \geq \|\text{Im} \langle P_e v_e, v_e \rangle \| \geq \| (\xi_1^2 G_\epsilon(\xi_1) + (\text{Im} a + 1) \xi_1) \hat{v}_e, \hat{v}_e \|$$

$$\geq \| \hat{v}_e \|^2 - \| (1 + |\xi_1|(|\text{Im} a| + 1)) \hat{v}_e, \hat{v}_e \| \geq \| \hat{v}_e \|^2 - C_1 \| \hat{v}_e \|,$$

where $C_1 := (|\text{Im} a| + 1)e^{Ad} \| v \|_{\mathcal{H}^1}$ is independent of $\epsilon$. This implies

$$\| \hat{v}_e \| \leq \| f_e \| + C_1 \leq C_0 + C_1.$$

Letting $\epsilon \to 0$ gives $\| e^{[\xi_1]_1} \hat{v}_e, \xi_1 \geq 0 \| \leq C$. A similar argument applies to $\xi_1 \leq 0$ which shows that

$$e^{[\xi_1]_1} \hat{v} \in L^2,$$

and consequently that $u(x) = e^{i\tau x_2} v(x_1)$ is analytic.

In the actual proof, the Fourier transform is replaced by the FBI transform (2.1) and its deformation (2.5) defined using a suitably chosen $G_\epsilon$ satisfying (1.12) (see Lemma 3.1 which is the heart of the argument). One difficulty not present in the simple one-dimensional case
is the localization in other variables. It is here that the $C^\infty$ normal forms of [18, 19, 30] are particularly useful. It is essential that no analyticity is needed in the construction of $G_\epsilon$.

1.3. Applications to scattering theory

As already indicated in [32] analyticity of smooth solution to the Vasy operator (1.2) implies analyticity of resonant states and of their radiation patterns. We review this here and, in Theorem 3, present a slightly stronger result.

For a detailed presentation of scattering on asymptotically hyperbolic manifolds, we refer to [11, Chapter 5]. To state Theorem 3, let $\overline{M}$ be a compact $n+1$ dimensional manifold with boundary $\partial M \neq \emptyset$ and let $M := \overline{M} \setminus \partial M$. We assume that $\overline{M}$ is a real analytic manifold near $\partial M$. A metric $g$ on $M$ is called asymptotically hyperbolic and analytic near infinity if there exist functions $y' \in C^\infty(\overline{M}; \partial M)$ and $y \in C^\infty(\overline{M}; (0,2))$, $y|_{\partial M} = 0$, $dy|_{\partial M} \neq 0$, such that
\[
\overline{M} \ni y^{-1}_1((0,1)) \ni m \mapsto (y_1(m), y'(m)) \in [0,1) \times \partial M
\]
is a real analytic diffeomorphism, and near $\partial M$ the metric has the form,
\[
g|_{y_1 = \epsilon} = \frac{dy_1^2 + h(y_1)}{y_1^2},
\]
where $(0,1) \ni t \mapsto h(t)$, is an analytic family of real analytic Riemannian metrics on $\partial M$.

Let
\[
R_g(\lambda) = (-\Delta_g - \lambda^2 - (n/2)^2)^{-1} : L^2(M, d\text{vol}_g) \to H^2(M, d\text{vol}_g), \quad \text{Im} \lambda > 0.
\]

Mazzeo–Melrose [27] and Guillarmou [17] proved that
\[
R_g(\lambda) : C_c^\infty(M) \to C^\infty(M),
\]
continues to a meromorphic family of operators for $\lambda \in \mathbb{C} \setminus i(-\frac{1}{2} + \mathbb{N})$. In addition, Guillarmou [17] showed that if the metric is even, that is,
\[
g|_{y_1 = \epsilon} = \frac{dy_1^2 + h(y_1^2)}{y_1^2},
\]
(see [11, Theorem 5.6] for an invariant formulation), then $R_g(\lambda)$ is meromorphic in $\mathbb{C}$. In particular, for $\lambda \neq 0$ we have the following Laurent expansion
\[
R_g(\zeta) = \sum_{j=1}^{J(\lambda)} \frac{(-\Delta_g - \lambda^2 - (n/2)^2)^{-1} \Pi(\lambda)}{(\zeta^2 - \lambda^2)^j} + A(\zeta, \lambda), \quad \Pi(\lambda) := \frac{1}{2\pi i} \oint_\lambda R_g(\zeta) 2\zeta d\zeta,
\]
where $\zeta \mapsto A(\zeta, \lambda)$ is holomorphic near $\lambda$. For $\lambda = 0$, we have a Laurent expansions in powers of $\zeta^{-j}$.

The operator $\Pi(\lambda)$ has finite rank and its range consists of generalized resonant states. We then have

**Theorem 3.** Suppose that $(M, g)$ is an even asymptotically hyperbolic manifold (in the sense of (1.16)) analytic near conformal infinity $\partial M$. Then for $\lambda \in \mathbb{C} \setminus \{0\}$,
\[
u \in \Pi(\lambda) C_c^\infty(M) \implies u = y_1^{-i\lambda + \frac{n}{2}} F, \quad F|_{\partial M} \in C^\omega(\partial M).
\]

Moreover, in coordinates of (1.16), $F(y) = f(y_1^2, y')$, $y' \in \partial M$ where $f \in C^\omega((-\delta, \delta) \times \partial M)$.

**Proof.** The metric (1.14) (in the coordinates valid near the boundary) gives the following Laplace operator:
\[-\Delta_g = (y_1 D_{y_1})^2 + i(n + y_1 \gamma_0(y_1', y'))y_1 D_{y_1} - y_1^2 \Delta_{h(y_1)},\]  
(1.18)

\[\gamma_0(t, y') := -\frac{1}{2} \partial_t \bar{h}(t)/\bar{h}(t), \quad \bar{h}(t) := \det h(t), \quad D := \frac{i}{4} \partial_t.\]

Following Vasy [31], we change the variables to \(x_1 = y_1^2, \ x' = y'\) so that

\[y_1^{i\lambda - \frac{n}{2}} (-\Delta_g - \lambda^2 - \left(\frac{n}{4}\right)^2) y_1^{-i\lambda + \frac{n}{2}} = x_1 P(\lambda),\]  
(1.19)

where, near \(\partial M\), \(P(\lambda)\) is given by (1.2). This operator is considered on \(X := (\{\delta, 0\}, \partial M) \sqcup M\). The key fact is that \(P(\lambda)\) is a Fredholm family operators on suitable spaces, \(P(\lambda)^{-1}\) is meromorphic and its poles can be studied using microlocal methods — see [31], [11, Chapter 5] and also [34, § 2] for a short self-contained presentation.

From meromorphy of \(P(\lambda)^{-1}\), we obtain meromorphy of (1.15) using (1.19):

\[R_g(\lambda)f := y_1^{\frac{n}{2} - i\lambda} \left( P(\lambda)^{-1} y_1^{i\lambda - \frac{n+2}{2}} f \right)|_M \in C^\infty(M).\]  
(1.20)

Here we make \(y_1^{i\lambda - \frac{n+2}{2}} f\) into an element of \(C_c^\infty(X)\) by extending it by zero outside of \(M\). Near any \(\lambda, \ P(\zeta)^{-1} = \sum_{k=1}^{\infty} Q_j(\zeta(\zeta - \lambda)^{-j} + Q_0(\zeta, \lambda), \) with \(Q_j(\lambda)\) operators of finite rank and \(\zeta \mapsto Q_0(\zeta, \lambda)\) is analytic near \(\lambda\). We then have

\[\Pi(\lambda) = \frac{1}{2\pi} y_1^{\frac{n}{2} + i\lambda} Q_1(\lambda) y_1^{i\lambda - \frac{n+2}{2}}.\]

Hence, the claim about the range of \(\Pi(\lambda)\) follows from analyticity of functions in the range of \(Q_1(\lambda)\). This follows from Theorem 1. In fact, \(P(\zeta) = P(\lambda) + (\zeta - \lambda)V, \ V := -4Dx_1 + i\gamma(x),\) and hence

\[P(\lambda)Q_k(\lambda) = -VQ_{k+1}(\lambda), \quad Q_{K+1}(\lambda) := 0.\]

Since we already know that the ranges of the operators \(Q_k\) are in \(C^\infty\) (see [11, (5.6.10)]), we inductively conclude that their ranges are in \(C^\infty\).

\[\square\]

**Remark.** Vasy’s adaptation of Melrose’s radial estimates [28] shows that to conclude that \(u \in C^\infty\) when \(P(\lambda)u \in C^\infty\) (see (1.2)), we only need to assume that \(u \in H^{s+1}\) near \(m_0\), where \(s + \frac{1}{2} > -\text{Im} \lambda, \) see [34, § 4, Remark 3].

## 2. Preliminaries on FBI transforms and their deformations

We will use the FBI transform defined in [14] in its \(\mathbb{R}^n\) (rather than \(T^n\)) version. Since the weights we use will be compactly supported in \(x\), the same theory applies. The constructions there are inspired by the works of Boutet de Monvel–Sjöstrand [6], Boutet de Monvel–Guillemin [5], Helffer–Sjöstrand [21] and Sjöstrand [29]. An alternative approach to using the classes of weights we need here was developed independently and in greater generality by Guedesh Bonthonneau–Jézéquel [16].

### 2.1. Deformed FBI transforms

We define

\[ Tu(x, \xi) := h^{-\frac{3n}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{4} \langle x-y, \xi \rangle + \frac{i}{2} \langle \xi(y-y')^2 \rangle} \frac{\xi}{\xi} u(y)dy, \quad u \in C_c^\infty(\mathbb{R}^n), \]  
(2.1)

recalling that the left inverse of \(T\) is given by

\[ Sv(y) = \frac{2\pi h^{-\frac{3n}{2}}}{(2\pi)^{\frac{2n}{2}}} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{4} \langle x-y, \xi \rangle - \frac{i}{2} \langle \xi(y-y')^2 \rangle} \xi \xi (1 + \frac{i}{2} (x - y, \xi / \xi)) v(x, \xi)dx \xi, \]  
(2.2)

see [14, Proposition 2.2].
The first fact we need is the characterization of Sobolev spaces and of the $C^\infty$ wave front set using the FBI transform (2.1). To formulate it we use semiclassical Sobolev spaces $H^s_h$ (see, for instance, [33, § 7.1] or [11, Definition E.18]) but we should in general think of $h$ as being fixed.

**Proposition 2.1.** There exists a constant $C$ such that for $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$
\|u\|_{H^s_h} \leq C\|\langle \xi \rangle^s Tu\|_{L^2(T^*\mathbb{R}^n)} \leq C^2\|u\|_{H^s_h}.
$$

Moreover,

$$(x_0, \xi_0) \notin \text{WF}(u) \Leftrightarrow \exists \chi \in S^0(T^*\mathbb{R}^n), \chi \equiv 1 \text{ in a conic neighbourhood of } (x_0, \xi_0), \forall N \exists C_N \|\langle \xi \rangle^N \chi Tu\|_{L^2(T^*\mathbb{R}^n)} \leq C_N.
$$

**Proof.** This follows from the characterization of the $H^s$ based wave front sets in Gérard [15] as stated in [10, Theorem 1.2]. Since the arguments are similar to the more involved analytic case presented in Proposition 2.3, we omit the details. \(\square\)

As in [29, § 2] and [14, § 3], we introduce a geometric deformation of $\mathbb{R}^{2n}$, $\Lambda = \Lambda_G$:

$$
\Lambda := \{(x - iG_2(x, \xi), \xi + CG_2(x, \xi)) \mid (x, \xi) \in \mathbb{R}^{2n}\} \subset \mathbb{C}^{2n},
$$

$$
\supp G \subset K \times \mathbb{R}^n, \quad K \in \mathbb{R}^n,
$$

$$
sup_{0+|\beta| \leq 2}(\xi^{-1+|\beta|} |\partial_x^\alpha \partial_\xi^\beta G(x, \xi)|) \leq \epsilon_0, \quad \|\langle \xi \rangle^\delta \partial_x^\alpha \partial_\xi^\beta G(x, \xi)| \leq C_{\alpha, \beta}(\xi)^{1-|\beta|},
$$

where $\epsilon_0$ is small and fixed (so that the constructions below remain valid as in [14]). For convenience, we change here the convention from [14]: it amounts to replacing $G$ by $-G$ everywhere.

This provides us with the following new objects: the deformed FBI transform (see [14, § 4]),

$$
T_\Lambda u(x, \xi) := Tu(x - iG_2(x, \xi), \xi + iG_2(x, \xi)), \quad u \in \mathcal{B}_c,
$$

$$
\mathcal{B}_c := \{u \in \mathcal{S}'(\mathbb{R}^n) \mid |\hat{u}(\xi)|^2e^{4\delta|\xi|}d\xi < \infty\},
$$

the spaces $H^s_\Lambda$, defined as in [14, § 4],

$$
H^s_\Lambda := \overline{\mathcal{B}_c}^*_{H^s_h}, \quad \|u\|^2_{H^s_\Lambda} := \int_\Lambda \langle e^{2H_\alpha}\rangle^{2n}|T_\Lambda u(\alpha)|^2e^{-2H(\alpha)/h}d\alpha,
$$

and the orthogonal projector

$$
\Pi_\Lambda : L_\Lambda := L^2(\Lambda, e^{-2H(\alpha)/h}d\alpha) \to T_\Lambda H_\Lambda, \quad H_\Lambda := H^0_\Lambda,
$$

described asymptotically (as $h \to 0$ and as $\xi \to \infty$) in [14, § 5]. The weight $H$ appears naturally in this subject and is given by [14, (3.3),(3.4)], that is, $H(x, \xi) = \xi \cdot G_2(x, \xi) - G(x, \xi)$. The deformed FBI transform $T_\Lambda$ has an exact left inverse $S_\Lambda$ obtained by deforming $S$ in (2.2).

We now prove a slightly modified version of [14, Proposition 6.2]:

**Proposition 2.2.** Suppose that $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is a differential operator with $a_\alpha \in C^\infty_c(\mathbb{R}^n)$ satisfying,

$$
a_\alpha \in C^\infty(U), \quad K \in U,
$$

for an open set $U$ and $K$ as in (2.4). Then

$$
\Pi_\Lambda T_\Lambda h^m PS_\Lambda = \Pi_\Lambda b_\Lambda \Pi_\Lambda + O(h^\infty)_{H^s_\Lambda \to H^s_\Lambda},
$$
where
\[
    b_P(x, \xi) \sim \sum_{j=0}^{\infty} h^j b_j(x, \xi), \quad b_j \in S^{m-j}(\mathbb{R}^{2n}),
\]
\[
b_0 = p|_{\Lambda} := p(x - iG_\xi(x, \xi), \xi + iG_x(x, \xi)).
\]

We remark that the expansion remains valid when \( h \) is fixed. We can use smallness of \( h \) to dominate the lower order terms and then keep it fixed.

**Proof.** The result follows from the analogue of [14, Lemma 6.1] where the operator \( T_\Lambda h^n P S_\Lambda \) is described in the case where the coefficients of \( P \) are globally analytic. Here we point out that the analyticity of the coefficients is only needed in the neighbourhood \( U \) of \( K \subset \mathbb{R}^n \) such that in (2.4) \( \text{supp} \ G \subset K \times \mathbb{R}^n \) and \( \epsilon_0 \) is small enough depending on the size of the complex neighbourhood to which the coefficients extend holomorphically.

In fact, arguing as in the proof of [14, Proposition 6.2] all we need is that for \( a \in C^\infty_c(\mathbb{R}^n) \) and \( a \in C^\omega(U) \), the Schwartz kernel of \( T_\Lambda M_a S_\Lambda \), \( M_a f(x) := a(x)f(x) \), is given by
\[
    K_\alpha(\alpha, \beta) = c_0 h^{-n} e^{i\Psi(\alpha, \beta)} A(\alpha, \beta) + r(\alpha, \beta), \quad \alpha, \beta \in \Lambda = \Lambda_G,
\]
\[
r(\alpha, \beta) \text{ is the kernel of an operator } R = O(h^\infty) : H^{-N}_\Lambda \rightarrow H^N_\Lambda.
\]
The phase in (2.8) is given by
\[
    \Psi(\alpha, \beta) = \frac{i}{2} \langle \alpha_\xi - \beta_\xi \rangle^2 + \frac{i}{2} \langle \beta_\zeta \rangle \langle \alpha_x - \beta_x \rangle^2 + \frac{\langle \beta_\xi \rangle \alpha_x + \langle \alpha_\xi \beta_\zeta \rangle}{\langle \alpha_\xi \rangle + \langle \beta_\zeta \rangle} \cdot (\alpha_x - \beta_x),
\]
and the amplitude satisfies
\[
    A \sim \sum_{j=0}^{\infty} h^j (\alpha_\xi)^{-j} A_j, \quad A_0(\alpha, \alpha) = a|_{\Lambda}(\alpha),
\]
and \( A_j \) are supported in a small conic neighbourhood of the diagonal in \( \Lambda \times \Lambda \). We note that if \( \epsilon_0 \) is small enough, \( a \) extends to some neighbourhood of \( K \subset \mathbb{R}^n \) and hence \( a|_{\Lambda} = a(x - iG_\xi(x, \xi)) \) is well defined.

To see (2.8), we use the definitions of \( T_\Lambda \) and \( S_\Lambda \) to write
\[
    K_\alpha(\alpha, \beta) = c_n \langle \beta_\zeta \rangle \frac{\langle \alpha_\xi \rangle}{N} h^{-\frac{3}{2}} \int e^{h\Phi(z, \zeta, y) + h\Phi(z, \zeta, y) + h\Phi(z, \zeta, y)} a(y)(1 + \langle \beta_x - y, \beta_\xi / \beta_\zeta \rangle) dy,
\]
where
\[
    \varphi_G(\alpha, y) = \Phi(z, \zeta, y) |_{z = \alpha_x, \zeta = \alpha_\xi}, \quad \varphi_G^\zeta(\alpha, y) = -\Phi(z, \zeta, y) |_{z = \alpha_x, \zeta = \alpha_\xi},
\]
\[
    \alpha_x = x - iG_x(x, \xi), \quad \alpha_\xi = \xi + iG_\xi(x, \xi),
\]
\[
    \Phi(z, \zeta, y) = \langle z - y, \zeta \rangle + \frac{1}{2} \langle \zeta \rangle (z - y)^2, \quad \Phi(z, \zeta, y) := \Phi(\bar{z}, \bar{\zeta}, y).
\]

Let \( V, V_1 \) be open such that \( K \subset V_1 \subset V \subset \mathbb{R}^n \). We start by showing that the contribution to \( K_\alpha \) away from the diagonal is negligible. For that let \( \chi \in C^\infty_c(\mathbb{R}) \) with \( \chi \equiv 1 \) near 0. Then for all \( \delta > 0 \) small enough, the operator \( R_1 \) with kernel
\[
    R_1(\alpha, \beta) = K_\alpha(\alpha, \beta) \tilde{\chi}_\delta(\alpha, \beta),
\]
\[
    \tilde{\chi}_\delta(\alpha, \beta) := (1 - \chi(\delta^{-1}|\alpha_x - \beta_x|)) \left( 1 - \chi \left( \frac{|\alpha_\xi - \beta_\xi|}{\delta (|\alpha_\xi - \beta_\xi|)} \right) \right)
\]
satisfies \( R_1 = O_{H^{-N}_\Lambda \rightarrow H^N_\Lambda}(h^\infty) \). This amounts to showing that the operator with kernel
\[
    R_1(\alpha, \beta) e^{h(\Phi(\beta) - \Phi(\alpha))} (\alpha_\xi)^N (\beta_\xi)^N \text{ is bounded on } L^2(\mathbb{R}^{2n}) \text{ with } O(h^\infty) \text{ norm.}
To see this, we first integrate by parts $K$ times in $y$, using that
\[ |\partial_y \Psi| = |\beta_\xi - \alpha_\xi + i(\alpha_\xi(y - \alpha_x) + \langle \beta_\xi \rangle(y - \beta_x))| \geq c(1 + |\alpha_\xi| + |\beta_\xi|) \]
on supp $\tilde{\chi}_\delta$. This reduces the analysis to the case of (2.10) with $a$ replaced by $b(\cdot, \alpha, \beta) \in C^\omega(U) \cap C^\omega_c(\mathbb{R}^n)$ with $|b| \leq h^K(\langle |\alpha_\xi| \rangle + \langle |\beta_\xi| \rangle)^{-K}$.

Next, we choose $\psi \in C^\omega(\mathbb{R}^n; [0, 1])$ with $\psi \equiv 1$ on $V$ and supp $\psi \subset U$, and $\psi_1 \in C^\omega_c(\mathbb{R}^n; [0, 1])$ with $\psi_1 \equiv 1$ on $V_1$ and supp $\psi_1 \subset V$. We then deform the contour
\[ y \mapsto y + i\psi(y) \frac{\beta_\xi - \alpha_\xi}{(|\beta_\xi - \alpha_\xi|)} . \]
This contour deformation is justified since $a \in C^\omega(U)$. The phase in the integrand of (2.10) becomes
\[ \Psi = \langle \alpha_x - y, \alpha_\xi \rangle + \langle y - \beta_x, \beta_\xi \rangle + i\frac{\langle \alpha_\xi \rangle}{2}(\alpha_x - y)^2 + i\frac{\langle \beta_\xi \rangle}{2}(\beta_x - y)^2 \]
\[ + i\psi(y) \frac{|\beta_\xi - \alpha_\xi|^2}{(|\beta_\xi - \alpha_\xi|)} + i\frac{\langle \alpha_\xi \rangle}{2} \left[ 2\psi(y) \frac{\alpha_x - y}{(|\beta_\xi - \alpha_\xi|)} - \epsilon^2 \psi^2(y) \frac{|\beta_\xi - \alpha_\xi|^2}{(|\beta_\xi - \alpha_\xi|)^2} \right] \]
\[ \frac{i\langle \beta_\xi \rangle}{2} \left[ 2\psi(y) \left( \frac{\alpha_x - y}{(|\beta_\xi - \alpha_\xi|)} - \epsilon\psi^2(y) \frac{|\beta_\xi - \alpha_\xi|^2}{(|\beta_\xi - \alpha_\xi|)^2} \right) \right] . \]
In particular, for $y \in V$, and $(\alpha, \beta) \in$ supp $\tilde{\chi}_\delta$, the integrand is bounded by
\[ e^{-c(\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle)(\alpha_x - \beta_x)/h} \]
which is negligible (even after multiplication by $e^{\frac{1}{h^K(\beta_\xi - \alpha_\xi)N(\beta_\xi)}}$).

For the integral over $y \notin V$, we consider three cases. First, if both Re $\alpha_x \in K$ and Re $\beta_x \in K$, then it is easy to see that the integrand is bounded by
\[ e^{-c(\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle)(\alpha_x - \beta_x + |b|)/h} \]
and hence produces a negligible contribution. Next, if Re $\alpha_x \notin K$ and Re $\beta_x \notin K$, then $H(\alpha) = H(\beta) = 0$, $\alpha, \beta$ are real, and integration by parts in $y$ shows that the contribution is negligible.

Finally, we consider the case Re $\alpha_x \in K$, Re $\beta_x \notin K$, (the case Re $\beta_x \in K$ and Re $\alpha_x \notin K$ being similar). In this case, we have $H(\beta) = 0$ and $\beta$ real. Since $y \notin V$, we have that the integrand is bounded by $e^{-c(\langle \alpha_\xi \rangle - \langle \beta_\xi \rangle)/h}$ and hence this term is also negligible.

Since $R$ is negligible, we may assume from now on that
\[ |\alpha_x - \beta_x| \ll 1 \quad \text{and} \quad |\alpha_\xi - \beta_\xi| \ll (\langle |\alpha_\xi| \rangle + \langle |\beta_\xi| \rangle) . \]
In particular, there are three cases: Re $\alpha_x \in K$ and Re $\beta_x \in V_1$, Re $\beta_x \in K$ and Re $\alpha_x \in V_1$, or Re $\alpha_x \notin K$ and Re $\beta_x \notin K$.

The first two cases are similar, so we consider only one of them. Since Re $\alpha_x \in K$ and Re $\beta_x \in V_1$, the contribution from $y \notin V$ is negligible. Therefore, we may deform the contour to
\[ y \mapsto y + \psi(y)\gamma_c(\alpha, \beta), \quad \gamma_c(\alpha, \beta) = \frac{i(\beta_\xi - \alpha_\xi) + \langle \alpha_\xi \rangle\alpha_x + \langle \beta_\xi \rangle\beta_x}{(\alpha_\xi) + (\beta_\xi)} . \]
The proof in this case then follows from the method of complex stationary phase.

When both Re $\alpha_x \notin K$ and Re $\beta_x \notin K$, $\alpha = \text{Re} \alpha$, $\beta = \text{Re} \beta$, and $H(\alpha) = H(\beta) = 0$. In order to handle this situation, we will Taylor expand $a(y)$ around $y = \alpha_x$. For that we first consider (2.10) with $a = O(|y - \alpha_x|^{2N})$. In that case, we consider the integral
\[ K_N(\alpha, \beta) := h^{-N/2} \int e^{\frac{i}{h^K(\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle)(\alpha_\xi)\langle \beta_\xi \rangle(\alpha_x - \beta_x)^2 + \langle \beta_\xi \rangle\beta_x^2}} O(|y - \alpha_x|^{2N}) \langle \alpha_\xi \rangle^N \langle \beta_\xi \rangle^N (1 - \tilde{\chi}_\delta(\alpha, \beta)) dy . \]
\[ (2.12) \]
Changing variables \( y \mapsto y + \alpha_x \),

\[
|K_N(\alpha, \beta)| \leq \int \langle \alpha \xi \rangle^\frac{2n}{N} \frac{h^{N-n}}{\langle \alpha \xi \rangle N} e^{-(\beta_\xi)} (\beta_x - \alpha_x - y)^2 (1 - \chi_\delta) dy \\
\leq C \frac{h^{N-n}}{((\alpha \xi) + (\beta_\xi))^N} e^{-\frac{1}{N}((\alpha \xi) + (\beta_\xi))(\alpha_x - \beta_x)^2} (1 - \chi_\delta(\alpha, \beta)).
\]

Therefore, using the Schur test for boundedness, the operator \( K_N \) with kernel \( K_N(\alpha, \beta) \) satisfies

\[
K_N = O(h^{N-\frac{n}{2}}) : H_A^{-N+\frac{n}{2}+0} \to H_A^{N-\frac{n}{2}-0}
\]

Now, observe that for any \( N > 0 \),

\[
a(y) = a_N(y) + O(||y - \alpha_x||^{2N}),
\]

where \( a_N(y) \) is a polynomial of order \( 2N - 1 \) in \( (y - \alpha_x) \). In particular,

\[
K_a(\alpha, \beta) = K_{aN}(\alpha, \beta) + K_N(\alpha, \beta).
\]

Since \( a_N \) is analytic and the integrand is exponentially decaying in \( y \), we may deform the contour with \( y \mapsto y + y_a(\alpha, \beta) \) in the integral forming the kernel of \( K_{aN} \) and apply complex stationary phase as in the case where \( \Re \alpha_x \in K \) or \( \Re \beta_x \in K \). This finishes the proof of the proposition after taking \( N \) large enough. \( \square \)

### 2.2. Analytic wave front set

We now relate weighted estimates to analyticity.

**Proposition 2.3.** Let \( T \) be the FBI transform defined in (2.1) for some fixed \( h \), and let \( \psi \in S^1(T^*\mathbb{R}^n) \) satisfy

\[
\psi(x, \xi) \geq |\xi|/C, \quad (x, \xi) \in U \times \Gamma,
\]

where \( U \subset \mathbb{R}^n \) and \( \Gamma \subset \mathbb{R}^n \setminus 0 \) is an open cone. Then, for \( u \in H^{-N}(\mathbb{R}^n) \),

\[
e^{\psi}(\xi)^{-N} Tu \in L^2(T^*\mathbb{R}^n) \implies WF_a(u) \cap (U \times \Gamma) = \emptyset.
\]

Conversely, suppose \( u \in H^{-N}(\mathbb{R}^n) \), \( \Gamma_0 \subset \mathbb{R}^n \) is a conic open set such that \( \Gamma_0 \cap S^{n-1} \subset \Gamma \cap S^{n-1}, U_0 \subset U \). Then for any \( \psi \in S^1(\mathbb{R}^n \times \mathbb{R}^n) \) with \( \supp \psi \subset U_0 \times V_0 \),

\[
WF_a(u) \cap (U \times \Gamma) = \emptyset \implies \exists \theta > 0 \quad \langle \xi \rangle^{-N} e^{\theta \psi} Tu \in L^2(T^*\mathbb{R}^n).
\]

**Remark.** Here we do not consider uniformity in \( h \) in the \( L^2 \) bounds. If we demanded that, then we would only need \( \psi \in C^\infty_c(T^*\mathbb{R}^n) \), \( \psi > 0 \) on \( U \times (\Gamma \cap S^{n-1}) \).

The proof is based on the following

**Lemma 2.4.** Let \( T \) and \( S \) be given by (2.1) and (2.2), respectively, with \( h \) fixed. Suppose that \( \chi, \tilde{\chi} \in S^0(\mathbb{R}^n \times \mathbb{R}^n) \) and supp \( \chi \), supp \( \chi_1 \subset K \times \mathbb{R}^n, K \subset \mathbb{R}^n \). Then for any \( a > 0 \), there exists \( b > 0 \) such that

\[
\chi e^{b(\xi)} TS\chi_1 e^{-a(\xi)} = O_N(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}),
\]

for any \( N \).

If in addition \( \chi_1 \equiv 1 \) on a conic neighbourhood of the support of \( \chi \), then there exists \( b > 0 \) such that

\[
\chi e^{b(\xi)} TS(1 - \chi_1)(\xi)_M = O_{N,M}(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}),
\]

for any \( N \).
Proof. We analyse the Schwartz kernel of the operator in (2.16), \( K(x, \xi, y, \eta) \). As in the proofs of [14, Lemma 2.1, Proposition 4.5] (the phase of resulting operator can be computed by completion of squares and is given by [14, (4.10)]) with \( \Lambda = T^* \mathbb{R}^n \), we see that

\[
| (h D_{x, \xi}^\alpha)^n K(x, \xi, y, \eta) | \leq C_\alpha e^{b(|\xi| - a\langle \eta \rangle) - \psi(x, \xi, y, \eta)},
\]

\[
\psi := c((\xi) + \langle \eta \rangle)^{-1}(|\xi - \eta|^2 + \langle \xi \rangle |x - y|^2).
\]

We have

\[
b < \frac{1}{e} \min(a, c) \Rightarrow b\langle \xi \rangle - a\langle \eta \rangle - c((\xi) + \langle \eta \rangle)^{-1}|\xi - \eta|^2 \leq -\frac{1}{2}(b\langle \xi \rangle + a\langle \eta \rangle),
\]

if \( b \) is sufficiently small. (By taking \( b < a/8 \), we can assume that \( |\eta| \leq |\xi|/2 \). But then \( |\xi - \eta| \geq \frac{1}{2}|\xi| \) and \( \langle \xi \rangle + \langle \eta \rangle \leq 2\langle \eta \rangle \).) This proves (2.16) as we can use the Schur criterion.

To see (2.17), we note that we can now assume that \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| > \delta \) or \( |x - y| > \delta \). But then if the kernel of the operator in (2.17) is given by \( K_M(x, \xi, y, \eta) \) where

\[
| (h D_{x, \xi}^\alpha)^n K_M(x, \xi, y, \eta) | \leq C_{\alpha, N} e^{b(|\xi| - M \log \langle \eta \rangle) - \psi(x, \xi, y, \eta)}.
\]

Now, fix \( 0 < \delta < 1 \) small. Then, when \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| > \delta \) or \( |x - y| > \delta \),

\[
|\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2 \geq \frac{\delta^2}{16} (\langle \xi \rangle + \langle \eta \rangle)^2.
\]

To see this, observe that on

\[
|\langle \xi \rangle - \langle \eta \rangle| \leq \frac{\delta}{4},
\]

we have

\[
\frac{\delta}{4} \leq \frac{|\langle \xi \rangle^2 - \langle \eta \rangle^2|}{|\langle \xi \rangle + \langle \eta \rangle|^2} \leq \frac{|\xi - \eta|}{|\langle \xi \rangle + \langle \eta \rangle|}.
\]

On the other hand, when

\[
\frac{|\langle \xi \rangle - \langle \eta \rangle|}{|\langle \xi \rangle + \langle \eta \rangle|} \leq \frac{\delta}{4},
\]

we have

\[
\frac{2\langle \xi \rangle \langle \eta \rangle}{|\langle \xi \rangle + \langle \eta \rangle|^2} = \frac{\langle \xi \rangle + \langle \eta \rangle}{2} - \frac{1}{2} \left( \frac{|\langle \xi \rangle - \langle \eta \rangle|^2}{|\langle \xi \rangle + \langle \eta \rangle|^2} \right) \geq \frac{1}{4} (\langle \xi \rangle + \langle \eta \rangle).
\]

Therefore, if \( |x - y| \geq \delta \), (2.19) follows. If instead, \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| \geq \delta \), then

\[
\frac{|\xi - \eta|}{|\langle \xi \rangle + \langle \eta \rangle|} \geq \frac{1}{2} \left( |\langle \xi \rangle - \eta/\langle \eta \rangle| - \left( \frac{|\xi|}{\langle \xi \rangle} + \frac{|\eta|}{\langle \eta \rangle} \right) |\langle \xi \rangle - \langle \eta \rangle| \right) \geq \frac{\delta}{4},
\]

and (2.19) follows.

From (2.19), we have that there is \( C_{M, \delta} > 0 \) such that if \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| \geq \delta \) or \( |x - y| > \delta \),

\[
b\langle \xi \rangle - c((\xi) + \langle \eta \rangle)^{-1}|\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2 + M \log \langle \eta \rangle
\]

\[
\leq b\langle \xi \rangle - \frac{c^2}{4} (\langle \xi \rangle + \langle \eta \rangle) - \frac{1}{2} c((\xi) + \langle \eta \rangle)^{-1}(\langle \xi - \eta \rangle^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2) + C_{M, \delta},
\]

and the Schur criterion and gives (2.17) for \( b \leq \frac{c^2}{4} \). \( \square \)

Proof of Proposition 2.3. We start by recalling the characterization of the analytic wave front set using the standard FBI/Bargmann–Segal transform:

\[
\mathcal{F} u(x, \xi; h) := c_n h^{-\frac{m}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\xi - y \cdot \xi + \frac{1}{2}(x - y)^2)} u(y) dy, \quad u \in \mathcal{S}'(\mathbb{R}^n).
\]
Then

\[(x_0, \xi_0) \notin \WF_a(u) \quad \iff \quad \exists \delta, U = \text{neigh}(x_0, \xi_0) \quad \text{and} \quad |\mathcal{F} u(x, \xi, h)| \leq C e^{-\delta/h}, \quad (x, \xi) \in U, \quad 0 < h < h_0. \tag{2.20}\]

see [23, Theorem 9.6.3] for a textbook presentation; note the somewhat different convention: \(\mathcal{F} u(x, \xi, h) = e^{-i \frac{h}{\hbar} \xi \cdot T_{1/h} u(x - i \xi)}\).

We first prove (2.14). Hence suppose \((x_0, \xi_0) \in U \times \Gamma\). Let \(\chi \in \mathcal{S}^0\) be supported in a small conic neighbourhood, \(U \times \Gamma, \) of \((x_0, \xi_0)\) and choose \(\chi_1 \in \mathcal{S}^0\) which is supported in \(U \times \Gamma\) and is equal to 1 on a conic neighbourhood of the support of \(\chi\) and \(\chi_2 \in \mathcal{S}^0\) supported in \(U \times \Gamma\) and equal to 1 on a conic neighbourhood of the support of \(\chi_1\). Our assumptions then show that \(e^{a(\xi)/h} \chi_2 T_u \in L^2(\mathbb{R}^{2n})\) for some \(a > 0\). We now write

\[\chi e^{b(\xi)/T_u} = \chi e^{b(\xi)} T_S \left( \chi_1 e^{-a(\xi)} e^{a(\xi)} \chi_2 T_u + (1 - \chi_1) \langle \xi \rangle^{N} \langle \xi \rangle^{-N} T_u \right).\]

Since \(u \in H^{-N}, \ \langle \xi \rangle^{-N} T_u \in L^2(\mathbb{R}^{2n})\) and (2.16), (2.17), now show that \(e^{b(\xi)} \chi T_u \in H^K\) for some \(b > 0\) and any \(K\). By taking \(K > n\) and applying [23, Corollary 7.9.4], we obtain a uniform bound

\[|T u(x, \xi)| \leq C e^{-b(\xi)}, \quad (x, \xi) \in U_0 \times \Gamma_0.\]

Let \(h_1\) be the fixed \(h\) in the definition of \(T\). Then,

\[\mathcal{F} (x, \xi/\langle \xi \rangle; h_1/\langle \xi \rangle) = T u(x, \xi) = O(e^{-b(\xi)}), \quad (x, \xi) \in U_0 \times \Gamma_0. \tag{2.21}\]

Putting \(\omega_0 := \xi_0/\langle \xi_0 \rangle\), it follows that \(\mathcal{F} (x, \omega, h) = O(e^{-h/\hbar})\) for \((x, \omega)\) in a small neighbourhood of \((x_0, \omega_0)\). But then (2.20) shows that \((x_0, \omega_0) \notin \WF_a(u)\). Since \(\WF_a(u)\) is a closed conic set, we conclude that \((x_0, \xi_0) \notin \WF_a(u)\).

Now suppose \(\WF_a(u) \cap (U \times \Gamma) = \emptyset\). Then for \((x, \omega)\) near \(U_0 \times (\Gamma_0 \cap S^{n-1})\) (with \(U_0\) and \(\Gamma_0\), as in the statement of the theorem), \(\mathcal{F} (x, \omega, h) = O(e^{-h/\hbar})\). Reversing the argument in (2.21), we see that

\[|T u(x, \xi)| \leq C e^{-b(\xi)}, \quad (x, \xi) \in U_0 \times \Gamma_0.\]

Now, since \(u \in H^{-N}(\mathbb{R}^{n}), \ \langle \xi \rangle^{-N} T_u \in L^2(\mathbb{R}^{2n})\). In particular, since \(|\psi| \leq C(\langle \xi \rangle)\) and the support of \(\psi\) is contained in \(U_0 \times \Gamma_0, (2.15)\) follows.

The next proposition relates weighted estimates to deformed FBI transform:

**Proposition 2.5.** Suppose that \(H_A, A = A_G\), is defined in [14, (4.7)] with \(G\) satisfying (2.4) with \(\epsilon_0\) chosen as in the definition of \(H_A\).

Then there exists \(\psi \in S^1(\mathbb{T}^*\mathbb{R}^n)\) such that \(T : \mathcal{B}_\delta \to L^2(\mathbb{T}^*\mathbb{R}^n, e^{\delta(\xi)/C\hbar} d\xi d\xi)\) extends to

\[T = \mathcal{O}(1) : H_A \to L^2(\mathbb{T}^*\mathbb{R}^n, e^{\psi(x, \xi)/\hbar} d\xi d\xi), \tag{2.22}\]

and \(S : L^2(\mathbb{T}^*\mathbb{R}^n, e^{-\delta(\xi)/\hbar} d\xi d\xi) \to \mathcal{B}_\delta\), extends to

\[S = \mathcal{O}(1) : L^2(\mathbb{T}^*\mathbb{R}^n, e^{\psi(x, \xi)/\hbar} d\xi d\xi) \to H_A. \tag{2.23}\]

In addition,

\[\psi(x, \xi) = G(x, \xi) + \mathcal{O}(\epsilon_0^2)_{S^1(\mathbb{T}^*\mathbb{R}^n)}. \tag{2.24}\]

For a simpler version of this result in the case of compactly supported weights, see [13, § 8].

**Proof.** The statement (2.22) is equivalent to

\[TS_A = \mathcal{O}(1) : L^2(\mathbb{L}, e^{-2H(\alpha)/\hbar} d\alpha) \to L^2(\mathbb{T}^*\mathbb{R}^n, e^{2\psi(\beta)/\hbar} d\beta)\]
and hence we analyse the kernel of the operator $T S_\lambda$ which is given by

$$K(\alpha, \beta) = c_n h^{-\frac{2n}{n}} \int_{\mathbb{R}^n} e^{\frac{1}{2}(\varphi_0(\alpha y) + \varphi_0^*(\beta y))} (\beta \xi) \frac{1}{\langle \beta \xi \rangle + (\beta \xi)^2} (1 + \frac{1}{2}(\alpha x - y)) dy,$$

where the notation (and also notation for $\Phi$ below) comes from (2.11). The integral in $y$ converges and can be evaluated by a completion of squares as in [14, Proposition 4.4]. That gives the phase (2.9) with $\alpha \in T^*\mathbb{R}^n$ and $\beta \in \Lambda$. The critical point in $y$ is given by

$$y_c(\alpha, \beta) = \frac{1}{\langle \alpha \xi \rangle + \langle \beta \xi \rangle} (\alpha \xi + (\beta \xi) \beta + i(\beta \xi - \alpha \xi)).$$  \hspace{1cm}  (2.25)

We then have (2.22) with

$$\psi(\alpha) := \max_{\beta \in \Lambda} (-\text{Im } \Psi(\alpha, \beta) + H(\beta)).$$ \hspace{1cm}  (2.26)

We have (see [14, (3.3),(3.4)])

$$d_{\beta}(- \text{Im } \Psi(\alpha, \beta) + H(\beta)) = \text{Im}(-\partial_{z,\xi} \Psi(\alpha, (z, \zeta)) - \zeta dz |_\Lambda) \big|_{(z, \zeta) = \beta \in \Lambda}.$$

Now, if $u_c(\alpha, (z, \zeta))$ is the critical point in $y$, then

$$\partial_{z,\xi} \Psi(\alpha, z) = \partial_{z,\xi} \Phi(\alpha, y_c(z, \zeta)) - \Phi(z, \zeta, y_c(z, \zeta))) - \partial_{z,\xi} \Phi |_{y = y_c(z, \zeta)}(z, \zeta)$$

$$= -\zeta \cdot dz + (y_c - z) \cdot d\zeta + i(\zeta)(z - y_c) \cdot d\zeta + \frac{i}{2}(z - y_c)^2 \zeta \cdot d\zeta / \langle \zeta \rangle.$$

For $G = 0$, the critical point (see (2.25)) is given by $\alpha = \beta$. Hence

$$\beta_c = \beta_c(\alpha) = (\alpha_x + O(\epsilon_0)_{S^0}, \alpha_\xi + O(\epsilon_0)_{S^1}),$$ \hspace{1cm}  (2.27)

with $\epsilon_0$ as in (2.4).

Hence we obtain $\psi$ by inserting the critical point $\beta_c$ into the right-hand side of (2.26)

$$\psi(\alpha) = -\text{Im } \Psi(\alpha, \beta_c(\alpha)) + H(\beta_c(\alpha)) \in S^1(T^*\mathbb{R}^n).$$  \hspace{1cm}  (2.28)

(We note that for $G = 0$ the maximum in (2.26) is non-degenerate and unique and it remains such under small symbolic perturbations.) From (2.9), we see that

$$\text{Im } \Psi(\alpha, \beta_c(\alpha)) = \text{Im } \Psi(\alpha, \alpha + O(\epsilon_0)_{S^0 \times S^1}) = \alpha_\xi \cdot G_\xi(\alpha) + O(\epsilon_0^2)_{S^1}.$$

Inserting this into (2.28) and recalling that $H = \xi G_\xi - G$, we obtain (2.24).

To obtain (2.23), we apply the same analysis to $T_\Lambda S$ and we need to show that two weights coincide. That is done as in [13, § 8].

3. Proof of Theorem 2

As already indicated in § 1.2, to prove the theorem we construct a family of weights $G_\epsilon \in S^1$, uniformly bounded in $S^1$, supported in a conic neighbourhood of $\Gamma = \{(0, 0, \xi_1, 0) : \xi_1 > M\}$, $M \gg 1$, and satisfying $0 \leq G_\epsilon \leq C_\epsilon \log \xi_1$. In addition,

$$H_p G_\epsilon \geq 0, \quad G_\epsilon \rightarrow \xi_1 \text{ on } \Gamma \text{ (in } S^{1+}),$$  \hspace{1cm}  (3.1)

with $H_p G_\epsilon \gg \xi_1^{n-1}$ in a suitable sense (see (3.4)) for $\epsilon \ll 1$.

We will then put $\Lambda_\epsilon := \Lambda_{G_\epsilon}$, so that the assumption $u \in C^\infty$ will give $u \in H_{\Lambda_\epsilon}$. On the other hand, the assumption that $\Gamma \cap WF_T(Pu)$ shows that $\|Pu\|_{H_{\Lambda_\epsilon}} \leq C$ with the constant $C$ independent of $\epsilon$. But then [14, Proposition 6.2] and the properties of $G_\epsilon$ show that $\|u\|_{H_{\Lambda_\epsilon}}$ is bounded independently of $\epsilon$. Propositions 2.3 and 2.5 then show that $WF_T(u) \cap \Gamma_0 = 0$. \hspace{1cm} \square
3.1. Construction of the weight

We now construct a family of weights, \( G_\epsilon \), satisfying (3.1). In fact, we need more precise conditions on \( G_\epsilon \) given in the following

**Lemma 3.1.** Suppose that \( p \) satisfies (1.9) at \( \rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0 \) and \( \Gamma \) is an open conic neighbourhood of \( \rho_0 \). Then, there exists \( G_\epsilon \in S^1(T^*\mathbb{R}^n) \), supp \( G_\epsilon \subset \Gamma \), such that

\[
|\partial_p^\beta \partial_{\xi}^\alpha G_\epsilon| \leq C_{\alpha \beta} (\xi^{1-|\beta|}), \quad 0 \leq G_\epsilon \leq C\epsilon^{1} \log(\xi),
\]

\[
G_\epsilon(x,\xi) \bigg|_{1 \leq |\xi| \leq 1/\epsilon} = \Phi(x,\xi)\xi, \quad \Phi \in S^0_{ph\rho}(T^*\mathbb{R}^n), \quad \Phi(x_0,t_0) = 1, \quad t \gg 1,
\]

\[
H \psi G_\epsilon(x,\xi) \geq c_0 (|\xi|^{m}|\partial_\xi G_\epsilon(x,\xi)|^2 + |\xi|^{-2}|\partial_x G_\epsilon(x,\xi)|^2),
\]

\[\forall M_1, \gamma > 0 \exists M_2, K, \epsilon_0 \text{ s.t. } 0 < \epsilon < \epsilon_0, \quad H \psi G_\epsilon \geq M_1 (|\xi|^{m-1}e^\gamma G_\epsilon).
\]

Proof. We use the normal form for \( p \) constructed in [19, § 3]. That means that we take \( x_0 = 0 \) and \( \xi_0 = e_1 := (1, 0, \ldots, 0) \) and can assume that \( p(x,\xi) = -\xi^m \xi_1 \) in a conic neighbourhood of \( \rho = (0, e_1) \). For simplicity, we can assume that \( m = 1 \) as the argument is the same otherwise.

Let \( \chi \in C_\infty_c(\mathbb{R}; [0, 1]) \) satisfy

\[
\text{supp } \chi \subset [-2, 2], \quad \chi_{|t| \leq 1} = 1, \quad t \chi'(t) \leq 0.
\]

and put \( \varphi(t) := \chi(t/\delta) \). Here \( \delta \) will be fixed depending on \( \Gamma \). Using this function we define \( \Phi = \Phi(x,\xi) := \varphi_1 \varphi_2 \varphi_3 \psi \) where

\[
\varphi_1 := \varphi(x_1), \quad \varphi_2 := \varphi(\xi'/\xi_1), \quad \varphi_3 := \varphi(|x'|), \quad \psi := (1 - \varphi((\xi_1_+))).
\]

We choose \( \delta \) small enough so that supp \( \Phi \subset \Gamma \).

We define \( G_\epsilon \) as follows

\[
G_\epsilon(x,\xi) = \Phi(x,\xi)q_\epsilon(\xi_1), \quad q_\epsilon(t) := \int_0^t \left(\chi(\epsilon s) + (1 - \chi(\epsilon s))(se)^{-1}\right)ds.
\]

We check that

\[
\xi_1 \partial_{\xi_1} q_\epsilon \geq \min(\xi_1, e^{-1}),
\]

\[
\xi_1^{\alpha_{\xi_1} + 1/\epsilon} + \epsilon^{-1}(1 + \log(\epsilon \xi_1))^{1/\epsilon} \leq q_\epsilon \leq \xi_1^{1/\epsilon} + \epsilon^{-1}(2 + \log(\epsilon \xi_1))^{1/\epsilon}.
\]

Uniform boundedness of \( G_\epsilon \) in \( S^1 \) means that \( q_\epsilon \) in (3.7) satisfies \( |\partial_{\xi_1} q_\epsilon| \leq C_\epsilon \xi_1^{1-k} \) with functions \( C_\epsilon \) independent of \( \epsilon \). But this is immediate from the definition. We also easily see that \( G_\epsilon \) converges to \( G := \Phi(x,\xi)\xi_1 \) in \( S^1 \) as \( \epsilon \to 0 \). This proves (3.2).

To see (3.3), we first note that, since \( \Phi \geq 0, \Phi \in S^0 \), the standard estimate \( f(z) > 0 \implies |df(z)|^2 \leq Cf(z) \) gives

\[
\Phi(x,\xi) \geq c_1 (\xi_1^2 |\partial_{\xi} \Phi(x,\xi)|^2 + |\partial_x \Phi(x,\xi)|^2).
\]

Note also that we have \( H_p = \xi_1 \partial_{\xi_1} - x_1 \partial_x \), and therefore

\[
H_p \Phi = -x_1 \varphi' \varphi_2 \varphi_3 \psi - (|\xi'|/\xi_1) \varphi' \varphi_1 \varphi_3 \psi - \varphi_1 \varphi_2 \varphi_3 \varphi_1 \varphi' \varphi_1 ((\xi_1_+)) \geq 0.
\]

Since \( q_\epsilon \in S^1 \), \( \xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geq c_2 \xi_1 (\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1))^2 \). We also claim that

\[
\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geq c_2 \xi_1^{-1} q_\epsilon(\xi_1)^2.
\]

In fact, using (3.8) we see that to prove (3.11) it is enough to have

\[
\min(t, \epsilon^{-1}) \geq c_2 t^{-1}(t^{1/\epsilon} + \epsilon^{-1}(2 + \log(\epsilon t))^{1/\epsilon})^2.
\]
This clearly holds (with \( c_2 = 1 \)) for \( t \leq 1/\epsilon \) and for \( t \geq \epsilon \) is equivalent to \( c_2(2 + \log s)^2 \leq s, s = t \epsilon \geq 1 \), which holds with \( c_2 = \frac{1}{4} \). It follows that

\[
\xi_1 \partial_{\xi_1} q_\gamma (\xi_1) \geq c_2 \left( \xi_1^{-1} q_\gamma (\xi_1) \right)^2 + \xi_1 \left( \partial_{\xi_1} q_\gamma (\xi_1) \right)^2,
\]

which combined with (3.9) and (3.10) gives

\[
H_p G_\epsilon = \Phi (\xi_1 \partial_{\xi_1} q_\epsilon) + (H_p \Phi) q_\epsilon
\geq \Phi (\xi_1 \partial_{\xi_1} q_\epsilon) + c_2 (\xi_1^2 |\partial_\xi \Phi|^2 + |\partial_x \Phi|^2) \xi_1^{-1} q_\epsilon^2
\geq c_0 (\xi_1 |\partial_x G_\epsilon|^2 + \xi_1^{-1} |\partial_x G_\epsilon|^2).
\]

Since \( \langle \xi \rangle \sim \xi_1 \) on the support of \( G_\epsilon \), we obtain (3.3).

Finally we prove (3.4). Since by (3.10) we have \( H_p G_\epsilon \geq \Phi H_p q_\epsilon \), we see that (3.4) follows from proving that for any \( M_1 \) we can find \( K, M_2 \) and \( c_0 \) such that for \( \xi_1 \geq 1 \),

\[
\Phi H_p q_\epsilon e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon}.
\]

Using (3.8), we see that for \( \xi_1 \leq 1/\epsilon \) we need \( G_\epsilon e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon} \). This holds for

\[
K = 0, \quad M_2 = 2\gamma^{-1} e^{\gamma M_1 - 1}
\]

since for \( \gamma > 0 \) and \( a \geq 0 \), \( ae^{\gamma a} - M_1 e^{\gamma a} \geq -2\gamma^{-1} e^{\gamma M_1 - 1} \).

For \( \xi_1 \geq 1/\epsilon \), we need to find \( K \) and \( M_2 \) for which

\[
e^{-1} \Phi e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon}.
\]

Using \( ae^{ab} + M_1 e^{M_1 b} \geq M_1 e^{ab} \) with \( a := e^{-1} \Phi \) and

\[
b := \gamma eq_\epsilon \leq \gamma (2 + \log (\epsilon \xi_1)) \leq \gamma (2 + \log \xi_1),
\]

we obtain (3.13) with \( M_2 = M_1 e^{2\gamma M_1} \) and \( K = \gamma M_1 \). Hence we obtain (3.12) proving (3.4).

\[\square\]

### 3.2. Microlocal analytic hypoellipticity

We will have bounds which are uniform in \( \epsilon \) but not in \( h \). We start with the following

**Lemma 3.2.** Suppose that \( P \) is of the form (1.8) with real valued principal symbol \( p \) and suppose that \( \Gamma \subset U \times \mathbb{R}^n \setminus \) is an open cone, \( \Gamma \cap S^{-1} \subset U \times S^{-1} \) and

\[
G \in S^1 (\Gamma; \mathbb{R}), \quad |G| \leq C \log (\xi),
\]

Then for \( \Lambda, \Lambda = \Lambda \theta G \) defined in (2.4) and (2.6), \( h \) and \( \theta \) sufficiently small, and \( u \in H^{N+m}_\Lambda \),

\[
\text{Im} \langle h^m P u, u \rangle_{H^{-N}_\Lambda} \geq \frac{1}{2} \theta (H_p G (\xi)^{-N} T_{\Lambda} u, (\xi)^{-N} T_{\Lambda} u)_{L^2_\Lambda} - M h \| u \|_{H^{N}_{\Lambda} m(N+m-N)}^2,
\]

where \( M \) depends only on \( P \) and the semi-norms of \( G \) in \( S^1 \).

**Proof.** We use Proposition 2.2 and [14, Proposition 6.3] to see that for any \( K > 0 \),

\[
\text{Im} \langle h^m P u, u \rangle_{H^{-N}_\Lambda} = \text{Im} \langle (\xi)^{-2N} T_{\Lambda} h^m P \Lambda T_{\Lambda} u, T_{\Lambda} u \rangle_{L^2_\Lambda}
\]

\[
= \text{Im} \langle (\Pi \Lambda (\xi))^{-2N} T_{\Lambda} h^m P \Lambda T_{\Lambda} u, T_{\Lambda} u \rangle_{L^2_\Lambda}
\]

\[
= \langle (\text{Im} b_{P,N}) T_{\Lambda} u, T_{\Lambda} u \rangle_{L^2_\Lambda} + \mathcal{O}(h^{\infty}) \| u \|_{H^{-N}_\Lambda}^2
\]

\[
\geq \langle (\text{Im} p_{\Lambda}) (\xi)^{-N} T_{\Lambda} u, (\xi)^{-N} T_{\Lambda} u \rangle_{L^2_\Lambda} - M h \| u \|_{H^{N}_{\Lambda} m(N+m-N)}^2.
\]


From (2.7) and (3.14), we obtain
\[ \operatorname{Im} p|_\Lambda = \operatorname{Im} p(x - i\theta \partial_\xi G(x, \xi), \xi + i\theta \partial_\xi G(x, \xi)) = \theta H_p G(x, \xi) + \theta^2 \mathcal{O}((\xi)^m|\partial_\xi G(x, \xi)|^2 + (\xi)^{m-2}|\partial_\xi G(x, \xi)|^2) \geq \frac{1}{2}\theta H_p G(x, \xi), \]
if \( \theta \) is small enough.

The next lemma allows us to use smoothness of \( u \) to obtain weaker weighted estimates:

**Lemma 3.3.** Suppose \( U \subset \mathbb{R}^n \) is an open set,
\[ G \in S^1(T^*\mathbb{R}^n), \quad G \geq 0, \quad \text{supp} \ G \subset K \times \mathbb{R}^n, \quad K \Subset U, \]
and \( T_\Lambda, H_\Lambda, \Lambda = \Lambda_{\theta G} \) are defined in (2.4) and (2.6). Then, there exists \( a > 0 \) such that for every \( \chi, \tilde{\chi} \in S^0 \) with \( \tilde{\chi} \equiv 1 \) in a conic neighbourhood of \( \text{supp} \ \chi \) and every \( K, N > 0 \), there exists \( c, C > 0 \) such that for all \( u \in H^{-N}(\mathbb{R}^n), \)
\[ \|\langle \xi \rangle^K e^{\alpha G/h} \chi T_\Lambda u\|_{L_\Lambda^1} \leq C \|\langle \xi \rangle^K \tilde{\chi} T_\Lambda u\|_{L_\Lambda^1} e^{-c/h} \|\langle \xi \rangle^{-N} Tu\|_{L_\Lambda^2(T^*\mathbb{R}^n)}. \]  
In particular, if \( \chi \equiv 1 \) on \( \text{supp} \ G \), then
\[ \|\langle \xi \rangle^K e^{\alpha G/h} \chi\|_{L_\Lambda^1} + \|\langle \xi \rangle^{-N}(1 - \chi) T_\Lambda u\|_{L_\Lambda^1} \leq C \|\langle \xi \rangle^K \tilde{\chi} T_\Lambda u\|_{L_\Lambda^1(T^*\mathbb{R}^n)} + \|\langle \xi \rangle^{-N} Tu\|_{L_\Lambda^2(T^*\mathbb{R}^n)}. \]  

**Proof.** First, observe that by [14, Lemma 4.5], for any \( \delta > 0, \)
\[ T_\Lambda S = K_\delta + O_{N, \delta}(e^{-c/h}) \langle \xi \rangle^N L_\Lambda^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-N} L_\Lambda^1, \]
and \( K_\delta \) has kernel, \( K_\delta(\alpha, \beta), \) given by
\[ h^{-n} e^{\frac{i}{2} \Psi(\alpha, \beta)} k(\alpha, \beta) |\psi(\delta^{-1} |\Re \alpha_x - \beta_\xi|)\psi(\delta^{-1} |\Re \alpha_\xi - \beta_\xi|)\psi(\delta^{-1} |(\Re \alpha_\xi, (\beta_\xi))^{-1} |\Re \alpha_\xi - \beta_\xi|)), \]
where \( (\alpha, \beta) \in \Lambda \times T^* \mathbb{R}^n \) and \( \Psi \) is as in (2.9), and \( \psi \in C_0^\infty(\mathbb{R}) \) is identically 1 near 0. Therefore, we need to only consider \( K_\delta(\alpha, \beta). \)
To do this, let \( \tilde{\chi} \in S^0 \) be identically 1 on a conic neighborhood of \( \text{supp} \ \chi. \) Then, for \( \delta > 0 \) small enough,
\[ \chi(\Re \alpha) K_\delta(\alpha, \beta)(1 - \tilde{\chi})(\beta) \equiv 0. \]
Therefore,
\[ \chi e^{-a G/h} \langle \xi \rangle^K T_\Lambda S(1 - \tilde{\chi}) = O_{N}(e^{-c/h}) \langle \xi \rangle^N L_\Lambda^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-N} L_\Lambda^1. \]
For the mapping properties
\[ \chi e^{-a G/h} T_\Lambda S \tilde{\chi} : \langle \xi \rangle^{-K} L_\Lambda^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-K} L_\Lambda^2, \]
we consider the operator
\[ \chi e^{-a G/h} e^{-H/h} \langle \xi \rangle^K T_\Lambda S \tilde{\chi} \langle \xi \rangle^{-K} : L_\Lambda^2(T^*\mathbb{R}^n) \rightarrow L_\Lambda^2(\Lambda; dxd\xi). \]
Modulo negligible terms, the kernel of this operator is given by
\[ h^{-n} e^{\frac{i}{2} \Psi((x, \xi), (y, \eta))} \tilde{k}((x, \xi), (y, \eta)) \]
where \( \tilde{k} \in S^0 \) has
\[ \operatorname{supp} \tilde{k} \subset \{ |\xi - \eta| \leq C\delta \langle \xi \rangle \} \cap \{ |x - y| \leq C\delta \}. \]  

(3.19)
and
\[ \varphi = iH(x, \xi) + i\alpha G(x, \xi) + \Psi((x - i\theta G_x, \xi + i\theta G_x(x, \xi)), (y, \eta)), \]
with \( H(x, \xi) = \theta\langle \xi, G(x, \xi) \rangle - \theta G(x, \xi). \) Using (3.19), we have
\[ \text{Im } \varphi = aG + \theta \xi \cdot G_x - \theta G + \frac{(\eta\langle \xi \rangle)}{2(\eta + \langle \xi \rangle)}((x - y)^2 - (\theta G_x)^2) + \frac{(\xi - \eta)^2 - (\theta G_x)^2}{2(\eta + \langle \xi \rangle)} + \theta \xi \cdot G_x + O(\theta|x - y||G_x| + \langle \xi \rangle^{-1}|\xi - \eta||G_x|)) \]
\[ + O(\theta^2(\langle \xi \rangle^{-1}|G_x|^2 + \langle \xi \rangle|G_x|^2)) \]
\[ \geq (a - \theta)G - C\theta^2(\langle \xi \rangle^{-1}(G_x)^2 + \langle \xi \rangle|G_x|^2) + c(\langle \xi \rangle(x - y)^2 + c(\langle \xi \rangle^{-1}(\xi - \eta)^2). \]
In particular, taking a large enough and using that \( G \geq 0, G \in S^1, \) (see the argument for (3.9)), we have
\[ \text{Im } \varphi \geq \frac{a}{2}G(x, \xi) + c\langle \xi \rangle(x - y)^2 + c\langle \xi \rangle^{-1}(\xi - \eta)^2. \]
Therefore, applying the Schur test for \( L^2 \) boundedness completes the proof that
\[ \chi(\xi)K e^{-aG/h\Lambda} S(\xi)^{-K} = O(1) : L^2(T^*\mathbb{R}^n) \to L^2_\Lambda \]
and the lemma follows. \( \square \)

With these two lemmas in place we can prove the main result:

**Proof of Theorem 2.** By multiplying \( u \) by a \( C^\infty_c \)-function which is 1 in a neighbourhood of \( x_0 \), we can assume that \( u \in H^{-N+m} \), for some \( N \), is compactly supported in \( U \) and \( \rho_0 := (x_0, \xi_0) \notin \text{WF}(u) \). By Proposition 2.1, there exists \( \tilde{\chi} \in S^0 \) with \( \tilde{\chi} \equiv 1 \) in an open conic neighborhood, \( \Gamma \), of \( \rho_0 \) such that for any \( K > 0 \),
\[ \|\langle \xi \rangle^K \tilde{\chi}Tu\|_{L^2} \leq C_K. \]  
(3.20)

Also, since \( u \in H^{-N+m} \),
\[ \|\langle \xi \rangle^{-N+m}Tu\|_{L^2} \leq C. \]  
(3.21)

Let \( \Gamma_1 \subseteq \Gamma \) be an open conic neighborhood of \( \rho_0 \) and \( \chi \in S^1 \) with \( \chi \equiv 1 \) on \( \Gamma_1 \) and \( \text{supp } \chi \subseteq \Gamma \).
We choose \( \theta \) small enough so that (2.4) and (3.16) hold. We then fix \( 0 < h \leq 1 \) small enough so that (3.16) holds. From now we neglect the dependence on \( h \) which is considered to be a fixed parameter. We choose for \( G = G_\epsilon \) constructed in Lemma 3.1 and supported in \( \Gamma_1 \). We recall that the estimates depend only on the \( S^1 \) seminorms of \( G \) and these are uniform in \( \epsilon \).
We now claim that
\[ u \in H^{-N+m}_{\Lambda_\epsilon}, \quad \Lambda_\epsilon := \Lambda_{\theta G_\epsilon}. \]
In fact, we can use (3.18) together with (3.20) and (3.21), observing that \( \exp(aG_\epsilon/h) = O(e^{(\langle \xi \rangle C_a/(he))}) \) and taking \( K = C_a/(he) - N + m \).

Next, note that \( Pu \in H^{-N} \) is supported in \( U \) and \( \rho_0 \notin \text{WF}_u(Pu) \). Propositions 2.3 and 2.5 (see (2.15) and (2.23), respectively) then show that for \( G_\epsilon \) satisfying the assumptions of Lemma 3.2 and \( \theta \) sufficiently small \( \|Pu\|_{H^{-N}_{\Lambda_\epsilon}} \leq C_0 \), where \( C_0 \) depends only on \( Pu \) and \( S^1 \)-seminorms of \( \theta G_\epsilon \).
We now apply (3.15) to obtain with \( \Lambda_\epsilon \) as above,
\[ \frac{1}{2}\|u\|^2_{H^{-N+m}_{\Lambda_\epsilon}} + 2C_0^2 \geq \langle(\theta H_pG_\epsilon - M(\langle \xi \rangle^{m-1})(\langle \xi \rangle^{-N-m}T_{\Lambda_\epsilon}u, (\langle \xi \rangle^{-N}T_{\Lambda_\epsilon}u)_{L^2_{\Lambda_\epsilon}}, \]  
(3.22)
Let $a$ be given by Lemma 3.3 (so that (3.17) holds). Then by (3.4), there exist $M_2$ and $K$ such that

$$\theta H_p G_\epsilon + M_2 \langle \xi \rangle^{2K} e^{-a G_\epsilon / \hbar} \geq (M + 1) \langle \xi \rangle^{m-1}.$$  

From (3.17), we have

$$||M_2 x \langle \xi \rangle^K e^{-a G_\epsilon / \hbar \langle \xi \rangle^N T_A u||^2_{L^2_{\Lambda^1}} \leq C(||\langle \xi \rangle^K N T_A u||^2_{L^2_{2(T \cdot \mathbb{R}^n)}} + ||\langle \xi \rangle^N T_A u||^2_{L^2_{2(T \cdot \mathbb{R}^n)}}) \leq C^2 \cdot 1$$  

(3.23)

Therefore, adding (3.23) to (3.22), and using that $\text{supp} G_\epsilon \subset \chi \equiv 1$, we have

$$\frac{d}{d\epsilon}||u||^2_{H_{\Lambda^1}^N} + C_1^2 + 2C_0^2 \geq \langle \chi^2 \langle \xi \rangle^{m-1} \langle \xi \rangle^N T_A u, \langle \xi \rangle^N T_A u \rangle_{L^2_{\Lambda^1}} - \langle M(1 - \chi^2)\langle \xi \rangle^N T_A u, \langle \xi \rangle^N T_A u \rangle_{L^2_{\Lambda^1}} \geq \langle \langle \xi \rangle^{m-1} \langle \xi \rangle^N T_A u, \langle \xi \rangle^N T_A u \rangle_{L^2_{\Lambda^1}} - (M + 1)||u||_{H^{-N+\frac{m-1}{2}}}.$$  

(3.24)

where in the last line we use that $\chi \equiv 1$ on $\text{supp} G_\epsilon$.

Using $m \geq 1$ and rearranging, this yields

$$||u||^2_{H_{\Lambda^1}^N} \leq 2C_1^2 + 4C_0^2 + 2(M + 1)||u||_{H^{-N+\frac{m-1}{2}}}.$$  

where $C_1, C_0$ and $M$ are constants independent of $\epsilon$.

Since $\Lambda^1 \cap \{\xi < 1/\epsilon\} = \Lambda_0 \cap \{\xi < 1/\epsilon\}$ where $G_0 := \Phi[\xi]$, we have that $H_\epsilon |_{\xi < 1/\epsilon} = H_0 |_{\xi < 1/\epsilon}$, where $H_\epsilon = \theta G_\epsilon + \theta G$ is the corresponding weight. Therefore, the monotone convergence theorem implies that $u \in H_{\Lambda_0}$. Since $\Phi(x_0, t\xi_0) = 1$, $t \gg 1$, Proposition 2.3 shows that $(x_0, \xi_0) \not\in WF_a(u)$.  

\begin{thebibliography}{99}
    
    
    
    
    
    
    
    
    
    
    
    
    
    
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Jeffrey Galkowski
Department of Mathematics
University College London
25 Gordon St.
London WC1H 0AY
United Kingdom
j.galkowski@ucl.ac.uk

Maciej Zworski
Department of Mathematics
University of California
970 Evans Hall #3840
Berkeley, CA 94720
USA
zworski@math.berkeley.edu

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