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APPROXIMATION SCHEMES FOR THE SCATTERING OF SPIN ONE PARTICLES

By

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ABSTRACT

The Wallace high-energy expansion of the scattering amplitude is discussed and generalized to the case of scattering of a spin-one particle from a potential with a tensor spin-orbit coupling. A generating function for the eikonal phase (quantum) corrections is evaluated in closed form.

The first and second Born amplitudes are evaluated for a Gaussian potential-distribution. It is shown that the Wallace-corrections bring the eikonal scattering amplitude closer to its Born counter-part. The tensor structure of the Born amplitude are calculated by developing an SMP program.

The Glauber eikonalization approach is extended to the case of spin-one scattering. Difficulties arise from the properties of spin-one operators as well as the unequal treatment of the initial and final momenta inherent in the eikonal scheme. Different methods of arriving at the Glauber-amplitude, including a diagonalization scheme which enables us to expand the exponential matrix in a closed form, are presented.

For the medium energy deuteron-nucleus scattering, the first order correction is dominant, and is shown to be significant in the measurement of the polarization parameters. This conclusion is supported by a numerical comparison of the eikonal observables with and without corrections, versus the exact observables calculated using a numerical resolution of the Schrödinger equation.

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CHAPTER 1

INTRODUCTION

In this thesis we will discuss different approximation schemes for describing the scattering of massive spin-one particles off spinless nuclear targets[1] within the framework of non-relativistic potential theory.

It is well known that the spin structure of quantum particles does not emerge as a consequence of the symmetrical or the dynamical prerequisites of quantum mechanics. Rather spin is introduced empirically in the theory to account for the magnetic properties of the quantum particles. Classically, magnetic properties of a charged particle arise from its rotation around its centre of mass. However no quantum analog has been formulated that can generate such a classical phenomenon. Nevertheless the intrinsic spin of a quantum particle is described as *spin angular momentum* and is postulated to transform under symmetry operations in the same way as angular momentum.

Special relativity offers a framework in which spin could be understood. In the case of spin-half fermions, the Dirac theory provides an understanding of the spin structure of the particles, which arises from the requirement of invariance of the theory under the Lorentz transformation. However, there is no theory

[1]In this thesis we are only concerned with spin-one incident particles scattering off spinless targets. In what follows we will omit mentioning the spin of the target nucleus and describe our interactions by referring only to the spin of the incident particle.

as simple and elegant as Dirac's in describing spin-one particles. In general the different types of interactions that can occur, because of the spin structure of the particles, are selected using the invariance requirements imposed on the scattering amplitude and hence on any potentials used in their description. In nuclear theory the invariance requirements invoked are those of time reversal and space reflection. For instance, spin-half interactions can occur via a central (spatial) and spin-orbit potentials. The latter couples the spin and orbital momenta of the particle. Both of these momenta being axial-vectors hence their product is a scalar with regard to time reversal and space reflection.

A spin-one particle interacts [1] with a spinless target through a central, a vector spin-orbit potential and three tensor potentials caused by the coupling of the particle's spin to the radial variable, the linear momentum and the orbital angular momentum of the particle. The contribution of any of these potentials to the scattering process depends very much on the energy range at which the interaction is taking place and on the interaction channels that are being studied. Spin-dependent interactions have explained a number of interesting phenomenon in the field of nuclear physics. Perhaps the most famous is the oblate structure of the deuteron which is understood [2] to arise from the tensor spin interaction between the two nucleons of which it is made of.

In the nuclear context spin-one particles, such as the deuteron or ^6Li, can be regarded as made up of spin-half constituents (nucleons). This has led to efforts to construct the phenomenological spin-one potential by folding the sum of the spin-half potentials of the constituent fermions over the internal coordinates of the composite particle. This approach is called the folding model [3]. It is successful in explaining the origin of the radial-spin tensor coupling, mentioned above, as resulting from the nucleon-nucleon tensor interaction averaged over the volume of the non-spherical spin-one composite particle.

Using the folding model [4,5] we can see that the spin-orbit tensor coupling is related to the process of elastic break-up of the composite particle. It is a second-order term which is due to the averaging of the nucleon-target spin-orbit interaction.

The tensor coupling of spin to linear momentum anticipated in the potential is best understood with reference to the low energy process of Pauli-blocking [6], which prohibits the incident nucleon from occupying a spin-state already occupied by a target nucleon with the same energy quantum number.

In this project we are interested in the *Eikonal* approximation and in the way the spin-one structure of the incident particle affects the eikonal scattering amplitude. The eikonal approximation is one of the most successful theories in describing scattering process, especially at small scattering angles and high energies of the incident particle. The theory originates from geometrical optics, where waves are approximated by straight line rays. The basic idea is best understood in analogy with the diffraction of waves through a slit. If the wavelength is small compared to the width of the slit then the passage of the wave through the slit can be studied in terms of the passage of a light-ray through the same slit. In quantum scattering this is equivalent to the high energy condition on the incident particle. This is reflected in the fact that the eikonal phase is obtained by performing a straight line integration along the direction of the incident momentum. Molière [7] and then Fernbach, Serber, and Taylor [8] applied this to nuclear scattering.

The theory acquired its most elegant and comprehensive exposition in Glauber's formalism [9,10]. However in the scattering of a particle through a potential, the notion of a straight line path is true only for small deflection angles of the particle. Glauber improved the angular validity of the theory by choosing the direction onto which the effect of the potential is measured to lie half-way between the initial and final directions of the scattered particle. This redefines the phase-shift of a particle deflected by an angle of θ as equivalent to following a path which is only deflected by $\theta/2$ while travelling through the potential.

Many attempts were made to improve the calculation of the phase function of the particle to account for the bending of the particle's path inside the potential. Saxon and Schiff [11,12] replaced the eikonal phase by the WKB phase, on the ground that the latter includes the eikonal phase plus higher order terms when expanded in powers of the potential strength divided by the product of the wave number and the particle's velocity. Sugar and Blankenbecler [13] systematically developed an eikonal expansion of the scattering matrix. Their work is based upon a series of eikonal approximations to the particle propagator in which the energy denominator is linearized in the momentum by expanding about the initial, final and intermediate wave vectors. However most of these approaches [14] were complicated and did not offer any straightforward prescriptions for calculating the needed corrections.

Wallace [15-18] developed a complete high-energy expansion of the Fourier-Bessel representation of the scattering amplitude. The expansion is not so closely tied to a small angle approximation and hence improves further on the Glauber-type of eikonal approximation, which appears as the leading order term of Wallace's expansion. In one of his papers Wallace [17] demonstrates his method through converting the partial wave sum exactly into a Fourier-Bessel impact parameter representation. In the same paper he shows that the WKB phase contains the Glauber phase as its leading term in an expansion in powers of the potential. Wallace's method was concerned with spherically symmetrical potentials, though he did suggest a possible treatment of spin-dependent interactions. Other attempts followed and were directed at calculating the non-eikonal corrections for spin-half interactions [19,20], but they did not present a simple prescription similar to that of Wallace.

Waxman et al [21] developed Wallace's scheme to incorporate the case of spin-half scattering. In their work they distinguish carefully between the impact parameter dependence due to the linear momentum and the angular momentum variation of the eikonal phases arising from the spin-orbit coupling. This approach is important in the case of spin-dependent interactions which are momentum dependent as well as non-spherical. They also draw attention to the fact that in the case of spin-orbit interactions the Schrödinger equation does not mix states with total angular momentum $j = l + 1/2$, $j = l - 1/2$ which makes it possible to calculate the Wallace corrections unambiguously.

One of the deficiencies of the conventional Glauber amplitude lies with its second-order scattering term [22-24]. To understand this we expand the Glauber amplitude in a power series of the potential strength and as we show in chapter four, the first-order term of the Glauber expansion is identical to the first-order Born term. Higher order terms of the series have the same order of interaction in the potential strength as the corresponding Born terms. However the resulting series is found to alternate between pure real and pure imaginary for terms of different order. For example, the imaginary part of the second-order Born term has its equivalent Glauber term whereas the real part disappears within the Glauber expansion. The straight-line approximation plays a significant role in the vanishing of the real part of the second-order term. In the Wallace amplitude the correction phase contains a major part arising from the correction to the straight-line assumption. A real part of the second-order scattering term is thus generated. Furthermore Swift [22] has demonstrated a method of calculating the non-eikonal corrections via the higher Born-approximation.

In chapter two, we will introduce the spin-one formalism used to develop the general form of the spin-one amplitude. In this chapter also we discuss the spin-one optical potential and its formalism which we relate to the spin-half optical potential via the folding model.

In chapter three we will start with the Born-approximation and develop the series up to its second order term. Since we are interested in the structure of the amplitude we will limit our calculation of the second order term to using a Gaussian potential-distribution which can be resolved analytically. For the first-order term we present the scattering amplitude resulting from using the full spin-one optical potential. In calculating the second-order term and in our numerical calculations (including the first-Born term), we will only use the tensor spin-orbit coupling. This is motivated by the fact that in working out the partial-wave series, which we will carry out later in the thesis, the other two tensor couplings mix states of different orbital angular momentum and hence result in a set of coupled equations which are analytically intractable.

The first-Born amplitude will serve as a tool in determining the proper eikonalization procedure. The second-order term is of such complexity as to render any attempt to calculate it by hand at the least unreliable. We will use a software package called SMP to handle the complicated spin-structure occurring in this calculation. The calculation will be checked using the optical theorem which relates the imaginary part of the second-Born term to the total cross-section in the case of scattering in the forward direction.

In a large part of our work we will use a numerical program called DDTP [25], or the Deuteron Optical Model Program, which calculates the elastic scattering of deuterons by nuclei. The program solves the Schrödinger equation exactly using the partial-wave decomposition. It describes the scattering by an optical model potential which can include the central, spin-orbit and the three tensor couplings. All components of the potential may be complex, and many different options for the radial dependence of each component are available. The version we will employ has been modified to include relativistic kinematics. In chapter three, we will arrive at the second-Born amplitude numerically via the exact DDTP and compare that with our calculation. The numerical calculation of our second-Born term was carried out using the SMP package.

In chapter four we will develop the eikonal amplitude. However, because of the properties of spin-one operators, we arrive at an ansatz which contains non-commuting terms. This makes it very difficult to expand the exponential phase function in powers of the potential and then sum it in a way that decouples the spin-dependence of the phase function as Glauber [9] did for the spin-half eikonal amplitude. To overcome this difficulty we first drop the velocity dependence of the tensor spin-orbit coupling on the grounds that this was done with regard to the linear spin-orbit term. The resultant amplitude fails to reproduce the first-Born term. In a second approach, we rewrite the exponential matrix in terms of three orthogonal matrices which we arrive at by diagonalizing the exponential matrix. This procedure results in an amplitude which has the correct first-Born limit, but which is manifestly asymmetric under time reversal. Although this is a property of the eikonal amplitude in general, nevertheless Glauber's half-angle approximation eliminated this deficiency from the theory in the spinless and spin-half cases. We were not able to achieve this in the spin-one case. Finally at the end of this chapter we conjecture an ansatz which we formulate with the above predicament in mind and show that it does reproduce the correct first-Born term.

In chapter five, Wallace's derivation in the case of spinless interactions is discussed together with Waxman et al's calculation of the spin-half amplitude. From this we generalize the Wallace scheme to include the spin-one case. We then go on to calculate the first quantum corrections to the eikonal phase which, as we demonstrate, improves the second-order eikonal contribution and brings it appreciably closer to its Born counterpart. We also compare the Wallace eikonalized amplitude, with and without correction to the exact DDTP amplitude in the case of d–α scattering using a Gaussian potential. We will also present the observables of the Wallace eikonalized scheme with and without corrections together with the exact DDTP ones, in the case of the scattering of d–^{58}Ni at deuteron incident energies of 400 and 700 Mev. The potentials used are obtained by folding the appropriate nucleon-nucleus optical potentials derived from the Dirac model.

A full discussion of the results obtained, along with the various conclusions that can be drawn, can be found in chapter six.

CHAPTER 2

THE SCATTERING OF SPIN ONE PARTICLES

(2.1) Non-relativistic spin-one operators and eigenfunctions

In this section we will define the spin-one operators, vectors and the scattering matrix. While the spatial coordinates vary continuously, the spin variable can take only a limited number of discrete values. In general [26] the wavefunction of a particle with spin S can be represented in the form of a column vector with $(2s+1)$ components. The spin operators in this case are represented as matrices with $(2s+1)$ rows and columns. The physical space and the spin space for a free particle are independent, hence the wavefunction of the particle can be represented as a product of the physical and spin wavefunctions. This also implies that we can assign to the spin variable the value of the projection of the particle's spin along any physical direction.

$$\Psi_{k_i,s,\nu}(\vec{r}) \;=\; \psi_{k_i}(\vec{r})\phi^s_\nu\,, \qquad (2.1.1)$$

We have defined ϕ^s_ν to be the eigenfunction of the operators for the square of the spin $\vec{S}^2$ and for its component S_z along the direction of the Z-axis[1]. In our case $s = 1$,

$$\vec{S}^2\,\phi^s_\nu \;=\; s(s+1)\,\phi^s_\nu\,, \qquad (2.1.2)$$

[1]In accordance with the Madison convention we will choose the the incident momentum along the Z-axis.

and

$$S_z \phi_\nu^s = \nu \phi_\nu^s . \tag{2.1.3}$$

We define the *Levi-Civita* antisymmetric tensor of rank three[2] by means of,

$$\mathcal{E}_{ijk} = \begin{cases} +1 & \text{if (ijk) is a cyclic permutation of (xyz),} \\ -1 & \text{if (ijk) is a non-cyclic permutation of (xyz),} \\ 0 & \text{otherwise (some or all subscripts are equal).} \end{cases} \tag{2.1.4}$$

A property of these tensors, which we will make use of is,

$$\mathcal{E}_{ijk}\, \mathcal{E}_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} , \tag{2.1.5}$$

where we employ the summation convention.

Spin operators, in general, satisfy the following angular momentum commutation relations,

$$[\, S_i \, , \, S_j \,] = i \mathcal{E}_{ijk}\, S_k \tag{2.1.6}$$

together with the constraint[3],

$$\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = 2\,\mathbf{I}. \tag{2.1.7}$$

They are also Hermitian,

$$S_i = S_i^\dagger . \tag{2.1.8}$$

In matrix form these operators may be represented by

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} ,$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{2.1.9}$$

[2]Cyclic permutation of (xyz) are (xyz), (yzx), and (zxy), while non-cyclic permutation are (xzy), (zyx), and (yxz).

[3]**I** is a 3×3 diagonal unit matrix, with the value of all off-diagonal elements equal to zero, we will omit writing it explicitly except to avoid confusion.

An important property of these matrices is that the product of any three of them can always be reduced to a sum of quadratic products plus that of linear ones [27].

$$S_i S_j S_k \;=\; \frac{i}{2}\,(\mathcal{E}_{ijl} S_k S_l \,+ \mathcal{E}_{ikl} S_l S_j \,+ \mathcal{E}_{jkl} S_i S_l) \,+\, \frac{1}{2}\,(\delta_{ij} S_k \,+ \delta_{jk} S_i)\;. \tag{2.1.10}$$

It follows from this that an arbitrary 3×3 matrix[4], can be build out of the unit matrix U_1, the three spin matrices S_i and quadratic products of spin matrices in the symmetric form,

$$S_{ij} \;=\; \frac{1}{2}\,(S_i S_j \,+\, S_j S_i) \,-\, \frac{2}{3}\delta_{ij}\,. \tag{2.1.11}$$

It is interesting to note that while the orbital angular momentum operator generates rotations of the spatial degrees of freedom of a physical system, the spin angular momentum operator rotates its internal degrees of freedom. The spin operators defined above are related to the generators of the group SO(3) by a similarity transformation [28].

The spin eigenfunctions form a complete set of states,

$$\phi_\nu^{s\,\dagger}\,\phi_{\nu'}^{s} \;=\; \delta_{\nu\nu'}\,. \tag{2.1.12}$$

Spin-one particles can be represented by a three-component spin wavefunction corresponding to the three possible eigenvalues $\nu = +1, 0, -1$ respectively, namely,

$$\phi_{+1}^{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \phi_{0}^{1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \phi_{-1}^{1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1.13}$$

[4] We are describing the reduction of a general second-rank-tensor with nine independent components into a scalar, three independent linear tensors, and a symmetric tensor with zero trace.

(2.2) Scattering matrix

We shall consider the case of scattering of particles with definite spin $s = 1$ by a fixed spinless potential. The Hamiltonian of a spin-dependent interaction is of the form,

$$H = \frac{-1}{2\mu}\left\{\frac{1}{r^2}\frac{\partial}{\partial}(r^2\frac{\partial}{\partial r}) - \frac{\vec{L}^2}{r^2}\right\} + V(\vec{S},\vec{r}). \qquad (2.2.1)$$

The corresponding Schrödinger equation is,

$$H\,\Psi_{k_i,s,\nu}(\vec{r}) = +k^2\Psi_{k_i,s,\nu}(\vec{r}). \qquad (2.2.2)$$

The interaction, $V(\vec{S},r)$, between particles with spin degrees of freedom is in general non-central, which means that it depends not only on the relative distance between the particles but also on the mutual orientation of their spins.

Assuming that the potential tends to zero faster than r^{-1} as $r \to \infty$, a particular solution of the above Schrödinger equation exists [29,30] which satisfies the asymptotic boundary condition

$$\Psi^{out}_{k_i,s,\nu}(\vec{r}) \underset{r\to\infty}{\longrightarrow} N\left[\exp[i\vec{k}_i\cdot\vec{r}]\phi^s_\nu + \frac{e^{ikr}}{r}\sum_{\nu'}\mathcal{F}_{\nu\nu'}(\vec{k}_i,\vec{k}_f)\phi^s_{\nu'}\right]. \qquad (2.2.3)$$

This describes[5] a wavefunction which is the superposition of a plane wave of wave vector $\vec{k}_i$ and an outgoing spherical wave with an *amplitude* $\mathcal{F}_{\nu\nu'}$ depending on θ and ϕ and inversely proportional to r. If the interaction depends on the orientation of the spin[6] (polarized), then there is no azimuthal symmetry and

[5]N is a normalization coefficient and the wave vectors are defined as $\vec{k}_i = (0,0,k_i)$ and $\vec{k}_f = k_f(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

[6]By polarization we mean the non-random orientation of the particle's spin.

the amplitude, in general, depends on ϕ.

The differential cross-section for scattering accompanied by a transition of the particle from an initial state with spin component ν to a final state characterized by a spin component ν' is given by the square of the modulus of the amplitude.

$$\sigma_{\nu \to \nu'}(\vec{k}_i, \vec{k}_f) = |\mathcal{F}_{\nu\nu'}(\vec{k}_i, \vec{k}_f)|^2 . \qquad (2.2.4)$$

If the incident particles are unpolarized and their spin projection after scattering is not fixed, the cross-section must be averaged over the possible values of the spin projection in the initial state and summed over all the possible values in the final state.

$$\sigma(\theta) = \frac{1}{2s+1} \sum_{\nu\nu'} |\mathcal{F}_{\nu\nu'}(\vec{k}_i, \vec{k}_f)|^2 . \qquad (2.2.5)$$

For unpolarized particles there is no preferred direction hence, the averaged cross-section can depend only on the scattering angle θ.

The amplitude defined by eqn (2.2.3) is a 3×3 matrix in spin space and so can be written

$$\mathcal{F}_{\nu\nu'}(\theta) = \mathcal{A} + \vec{S} \cdot \vec{\mathcal{B}} + \sum_{\gamma\beta} S_{\gamma\beta} \mathcal{C}_{\gamma\beta} . \qquad (2.2.6)$$

$\mathcal{A}$, $\vec{\mathcal{B}}$ and $\mathcal{C}_{\gamma\beta}$ are the coefficients of the representation[7]. They depend on the geometry of the collision determined by the vectors $\vec{k}_i$ and $\vec{k}_f$. Nuclear interactions are invariant with respect to rotations, reflections and time reversal. This must be reflected in the structure of the amplitude. Therefore $\mathcal{A}$ must be a scalar function of the above vectors. Also since $\vec{S}$ is an *axial-vector* it follows that the

[7] $S_{\gamma\beta}$ was defined by eqn (2.1.11)

function $\vec{\mathcal{B}}(\vec{k}_i, \vec{k}_f)$, must also be an axial-vector. The only axial-vector we can form using the initial and final wave vectors is,

$$\vec{n} = \vec{k}_i \times \vec{k}_f , \qquad (2.2.7)$$

hence we now get,

$$\vec{\mathcal{B}}(\vec{k}_i, \vec{k}_f) = \mathcal{B}(\theta)\,\hat{n} . \qquad (2.2.8)$$

$\mathcal{C}_{\gamma\beta}$ also has to be a symmetric tensor which is invariant with respect to reflections and time reversal. This in turn can be constructed using the vectors $\vec{k}_i$, $\vec{k}_f$ and $\vec{n}$

$$\mathcal{C}_{\gamma\beta} = \mathcal{C}_n \hat{n}_\gamma \hat{n}_\beta + \mathcal{C}_{k_i} \hat{k}_i^\gamma \hat{k}_i^\beta + \mathcal{C}_{k_f} \hat{k}_f^\gamma \hat{k}_f^\beta + \mathcal{C}_{k_{i,f}} (\hat{k}_i^\gamma \hat{k}_f^\beta + \hat{k}_f^\gamma \hat{k}_i^\beta) , \qquad (2.2.9)$$

where the $\mathcal{C}$'s are scalar coefficients, and we note that part of the θ-dependence is still attached to the unit-vectors. We can form another vector namely,

$$\vec{q} = \vec{k}_i \times \vec{n} . \qquad (2.2.10)$$

This is orthogonal to both $\vec{k}_i$ and $\vec{n}$, and therefore not independent of the combination already employed.

As mentioned earlier the unpolarized amplitude does not depend on the azimuthal angle. If we therefore set it equal to zero, then the three orthogonal vectors $\hat{k}_i$, $\hat{n}$ and $\hat{q}$ will fall on the Z,Y and $(-X)$-directions respectively, also the vector $\hat{k}_f$ will have components in the X-Z plane. Using the commutation relation eqn (2.1.6) we cast the $S_x S_z$ component on the $\hat{n}$-direction leaving only the $S_z S_x$ component. Furthermore we make use of eqn (2.1.7) to limit the other symmetric tensors to S_z^2 and S_y^2. The scattering matrix then takes the form

$$\begin{aligned} \mathcal{F}^i(\theta) &= \mathcal{A}^i(\theta) + \mathcal{B}_1^i(\theta)\vec{S}\cdot\hat{n} + \mathcal{B}_2^i(\theta)S_zS_x \\ &+ \mathcal{C}_{k_i}^i(\theta)\left[(\vec{S}\cdot\hat{k}_i)^2 - \frac{2}{3}\right] + \mathcal{C}_n^i(\theta)\left[(\vec{S}\cdot\hat{n})^2 - \frac{2}{3}\right]. \end{aligned} \qquad (2.2.11)$$

Here the superscript (i) refers to the choice of the Z-axis along the direction of the incident wave vector.

It is hard to impose invariance under $\vec{k}_i \to -\vec{k}_f$ in this frame. To see the consequences of this we will define another reference-frame which is referred to as the average quantization frame. In it the Z-axis lies along the direction of the average momentum, $(\vec{k}_i + \vec{k}_f)/2 = \vec{k}$. We can reach this average-frame by rotating the scattering matrix around the Y-direction through an angle of ($\theta/2$), The transformation is defined by,

$$\begin{aligned} \hat{z} &= \hat{z}'\cos\frac{\theta}{2} - \hat{x}'\sin\frac{\theta}{2} \\ \hat{x} &= \hat{z}'\sin\frac{\theta}{2} + \hat{x}'\cos\frac{\theta}{2}, \end{aligned} \qquad (2.2.12)$$

with the prime standing for the average frame. Applying this to the matrix $\mathcal{F}^i(\theta)$, we arrive at the scattering matrix in the average-quantization frame.

$$\begin{aligned} \mathcal{F}^{av}(\theta) &= \mathcal{A}^{av}(\theta) + \mathcal{B}^{av}(\theta)(\vec{S}\cdot\hat{n}) + \mathcal{C}_k^{av}(\theta)\left[(\vec{S}\cdot\hat{k})^2 - \frac{2}{3}\right] \\ &+ \mathcal{C}_n^{av}(\theta)\left[(\vec{S}\cdot\hat{n})^2 - \frac{2}{3}\right]. \end{aligned} \qquad (2.2.13)$$

Here

$$\mathcal{A}^{av}(\theta) = \mathcal{A}^i(\theta), \qquad (2.2.14)$$

$$\mathcal{B}^{av}(\theta) = \mathcal{B}_1^i(\theta) + \frac{i}{2}\mathcal{B}_2^i(\theta)\,, \tag{2.2.15}$$

$$\mathcal{C}_k^{av}(\theta) = \frac{1}{\sin\theta}\mathcal{B}_2^i(\theta)\,, \tag{2.2.16}$$

$$\mathcal{C}_n^{av} = \mathcal{C}_n^i(\theta) + \frac{\sin^2\theta/2}{\sin\theta}\mathcal{B}_2^i(\theta)\,. \tag{2.2.17}$$

The transformation is only consistent if the following relation holds,

$$\mathcal{B}_2^i(\theta) = \frac{\sin\theta}{\cos\theta}\mathcal{C}_{k_i}^i(\theta)\,. \tag{2.2.18}$$

This shows that there are only four independent amplitudes in (2.2.11).

(2.3) The Optical Potential

In this section we will discuss the optical-potential. The optical model [31] substitutes particle-nucleus interaction by a potential well. It derives its name from the analogy between the particle-nucleus scattering and the scattering of light by a cloudy crystal ball. The model is very useful because of its capacity to fit experimental data phenomenologically. For spin-dependent interactions the potential includes, beside the central term, additional operators which couple the spin to the spatial dynamics of the interaction. As in our discussion preceding the construction of the scattering matrix, in this section we will employ general invariance principles as a tool in selecting the possible forms of interaction between spin-one particles and spinless nuclei.

Conservation of angular momentum requires that the potential be a scalar function under rotation of spatial variables. Also invariance under spatial reflection implies that the potential must have even parity. Again as in the case of the scattering matrix, we write the potential in the general form,

$$V(\vec{S},\vec{r}) \;=\; V_c(\vec{r}) \,+\, V_s(\vec{r})\vec{R}\cdot\vec{S} \,+\, V_n(\vec{r})\sum_{\gamma\beta} S_{\gamma\beta}R^n_{\gamma\beta}\,. \qquad (2.3.1)$$

Here $\vec{R}$ is an operator composed of vectors selected from the spatial operators (coordinate) $\vec{r}$, (momentum) $\vec{p} = -i\vec{\nabla}$ and (orbital angular momentum) $\vec{L} = -i\vec{r}\times\vec{\nabla}$. The spin operators have even parity, which implies that $\vec{R}$ must be also chosen to have even parity. Out of the three spatial vectors listed above only $\vec{L}$ has even parity. Hence only one spin vector is allowed, namely

$$V_s(\vec{r})\vec{L}\cdot\vec{S}\,. \qquad (2.3.2)$$

We have the above three vectors ($n = \vec{r},\vec{p},\vec{L}$) out of which to construct the symmetric spatial tensor[8] $R^n_{\gamma\beta}$. The tensor takes the form,

$$\mathcal{R}_{\gamma\beta} \;=\; \mathcal{C}_r\vec{r}_\gamma\vec{r}_\beta \,+\, \mathcal{C}_p\vec{p}_\gamma\vec{p}_\beta \,+\, \mathcal{C}_L\vec{L}_\gamma\vec{L}_\beta\,. \qquad (2.3.3)$$

Satchler [30,32] introduces the re-coupling relation,

$$\vec{S}_2\cdot\vec{R}_2(\vec{v}_1,\vec{v}_2) \;=\; (\vec{S}\cdot\vec{v}_1)(\vec{S}\cdot\vec{v}_2) \,-\, \frac{i}{2}\vec{S}\cdot(\vec{v}_1\times\vec{v}_2) \,-\, \frac{S^2}{3}(\vec{v}_1\cdot\vec{v}_2)\,. \qquad (2.3.4)$$

This is identical to our symmetric tensor $S_{\gamma\beta}$. To see this we use the commutation relation eqn (2.1.6) in eqn (2.1.11) to arrive at

$$S_{\gamma\beta} \;=\; S_\gamma S_\beta - \frac{i}{2}\mathcal{E}_{\gamma\beta\alpha}S_\alpha \,-\, \frac{2}{3}\delta_{\gamma\beta}\,, \qquad (2.3.5)$$

which is same as Satchler's relation.

In terms of $S_{\gamma\beta}$ Satchler [1] writes the three tensors as,

$$T_r \;=\; \vec{S}_2\cdot\vec{R}_2(\hat{r},\hat{r}) \;=\; (\vec{S}\cdot\hat{r})^2 \,-\, \frac{2}{3}\,,$$

[8] See the discussion preceeding eqn (2.2.9).

$$T_p \;=\; \vec{S}_2 \cdot \vec{R}_2(\vec{p},\vec{p}) \;=\; (\vec{S}\cdot\vec{p})^2 - \frac{2}{3},$$

$$T_L \;=\; \vec{S}_2 \cdot \vec{R}_2(\vec{L},\vec{L}) \;=\; (\vec{S}\cdot\vec{L})^2 + \frac{1}{2}\vec{S}\cdot\vec{L} - \frac{2}{3}L^2 . \qquad (2.3.6)$$

One other notational difference is his choice of a unit vector $\hat{r}$. This is of no consequence since it entails only a change in the definition of the function $V_r(\vec{r}) \to r^2 V_r(\vec{r})$.

The tensor $V_r(\vec{r})T_p$ exhibits an extra complication, arising from the spatial gradient in its argument. This would violate time reversal invariance, once the direction in which the operator acts is reversed [9]. However the symmetric combination, $[V_p(\vec{r})\,T_p \,+\, T_p\,V_p(\vec{r})]$, overcomes this difficulty and guarantees the required time invariance. Finally the most general optical potential describing the interaction of spin-one and spinless particles takes the form,

$$\begin{aligned} V(\vec{S},\vec{r}) \;&=\; V_c(\vec{r}) + V_s(\vec{r})\vec{S}\cdot\vec{L} + V_r(\vec{r})T_r \\ &+\; V_p(\vec{r})T_p + T_pV_p(\vec{r}) + V_L(\vec{r})T_L . \end{aligned} \qquad (2.3.7)$$

(2.4) Deuteron folding model

For completeness, we will present an outline of the folding model. The folding model constructs the deuteron interaction potential by averaging the optical potentials of the nucleons over their internal motion inside the deuteron. Watanabe [3] has shown that, to first order in the potential strength and neglecting the deuteron D-state, the folding model generates a central potential and a vector

[9] For an illustration of this complication in the case of first-Born amplitudes, see eqn (3.2.7)

spin-orbit coupling. If the D-state of the deuteron is included, to first order in the potential strength, this would result in some of the T_r tensor term contributing to the interaction. The inclusion of second order D-state contribution, would induce a small T_L coupling, arising from the nucleon-nucleus spin-orbit potential. However [33], if break-up of the deuteron in-flight is considered, then a considerable T_L contribution will be present. Finally the use of a velocity-dependent (non-local) nucleon-nucleus potential gives rise to T_p type tensor coupling [6]. We shall demonstrate the calculations briefly.

We first define the displacement between the centre of mass of the nucleons to be $\vec{r}_d = \frac{1}{2}(\vec{r}_p + \vec{r}_n)$, and the vector $\vec{r} = \vec{r}_p - \vec{r}_n$. Here $\vec{r}_p$ and $\vec{r}_n$ are the position of the proton and neutron respectively. In terms of the neutron-nucleus (V_n) and proton-nucleus (V_p) potentials, the deuteron-nucleus potential is given by,

$$V(\vec{r}_d, \vec{r}) = V_p(\vec{r}_d + \frac{1}{2}\vec{r}) + V_n(\vec{r}_d - \frac{1}{2}\vec{r}). \qquad (2.4.8)$$

We ignore the proton's Coulomb potential and take the nucleon-nucleus potentials to be of the form,

$$V_i = V_c(\vec{r}_i) + V_s(\vec{r}_i)\vec{\sigma}_i \cdot \vec{L}_i. \qquad (2.4.9)$$

The deuteron wavefunction is expanded in terms of the complete set of neutron-proton wavefunctions,

$$\Psi(\vec{r}_d, \vec{r}) = \Phi_o(\vec{r}_d)\chi_o(\vec{r}) + \int d\vec{k}\, \Phi_k(\vec{r}_d)\chi_k(\vec{r}). \qquad (2.4.10)$$

Here the function $\Phi(\vec{r}_d)$ describes the motion of the deuteron with respect to the target nucleus, and $\chi(\vec{r})$ is the internal wavefunction of the deuteron. It is straightforward to show that the functions Φ_o and Φ_k obey the following Schrödinger equations,

$$\left[\frac{1}{4\mu}\nabla^2_{\vec{r}_d} - \mathcal{U}_o(\vec{r}_d) + (E - \epsilon_o)\right] \Phi_o(\vec{r}_d) = 0 , \qquad (2.4.11)$$

and

$$\left[\frac{1}{4\mu}\nabla^2_{\vec{r}_d} - \mathcal{U}_0(\vec{r}_d) + (E - \epsilon_k)\right] \Phi_k(\vec{r}_d) = < k|V(\vec{r}_d,\vec{r})|0 > \Phi_o(\vec{r}_d) . \qquad (2.4.12)$$

In the first equation we have neglected the break-up term[10], $\int d\vec{k} < 0|V|k > \chi_k(\vec{r})$, and in the second equation we neglected the off diagonal term[11] , $\int d\vec{k'} < k|V|k' > \Phi_{k'}(\vec{r}_d)$. The folded potential U_o is that derived by Watanabe and given by,

$$U_o(\vec{r}_d) = \int d\vec{r} |\chi_o|^2 V(\vec{r}_d,\vec{r}) . \qquad (2.4.13)$$

The outgoing waves are calculated iteratively. This is carried out by assuming that $\Phi_k(\vec{r}_d)$ satisfies the same differential equation as $\Phi_o(\vec{r}_d)$. By repeating this it is straightforward to obtain the formal solution,

$$\Phi_k(\vec{r}_d) = \frac{< k|V(\vec{r}_d,\vec{r})|0 > \Phi_o(\vec{r}_d)}{\epsilon_o - \epsilon_k} . \qquad (2.4.14)$$

Substituting this in the Schrödinger equation we arrive at,

$$U_1(\vec{r}_d) = U_o(\vec{r}_d,\vec{r}) + \int d\vec{k}\frac{< 0|V|k >< k|V|0 >}{\epsilon_o - \epsilon_k} . \qquad (2.4.15)$$

[10]This term would give rise to a T_L type of tensor if the D-state of the deuteron is included in the wavefunction (see eqn (2.4.22) below). However Stamp neglected it on the basis that it is accounted for by the imaginary part of the nucleon-nucleus potential.

[11]These terms can be significant numerically however, since we are only interested in the way the deuteron optical potential is related to the nucleon-optical potentials, we will disregard any unnecessary complications.

The deuteron's internal wavefunction $\chi(\vec{r})$ is in general expressed in terms of the functions $u(r)$ and $w(r)$ standing for the S-state and D-state components of respectively. It takes the form

$$\chi(\vec{r}) = \frac{1}{(4\pi)^{1/2}}\left[\frac{u(r)}{r} + \frac{w(r)}{r\sqrt{8}}\sigma_{np}\right]. \qquad (2.4.16)$$

The spin-operator σ_{np} is derived from the symmetric tensor eqn (2.1.11), with the replacement $\vec{r} \to \hat{r}$, and assuming the deuteron spin to be the vector sum of the neutron and proton spins [12], $\vec{S} = \frac{1}{2}(\vec{\sigma}_n + \vec{\sigma}_p)$. Hence,

$$\begin{aligned}\sigma_{np} &= 6\,T_r^{np} = 3\vec{S}_2(\vec{\sigma}_n, \vec{\sigma}_p)\cdot\vec{R}_2(\vec{r},\vec{r}) \\ &= 3(\vec{\sigma}_n\cdot\hat{r})(\vec{\sigma}_p\cdot\hat{r}) - \vec{\sigma}_n\cdot\vec{\sigma}_p\,. \end{aligned} \qquad (2.4.17)$$

Using the above wave function in eqn (2.4.13), the potential becomes

$$U_o(\vec{r}_d) = U_{oc}(r_d) + U_{os}(r_d)\vec{S}\cdot\vec{L} + U_{oT}(r_d)\vec{S}_2\cdot\vec{R}_2(\hat{r}_d,\hat{r}_d)\,. \qquad (2.4.18)$$

Here $\vec{L} = -i\vec{r}_d \times \vec{\nabla}_{r_d}$ is the orbital angular momentum of the centre of mass of the deuteron. The scalar functions in the above equation are, to leading order [5],

$$U_{oc}(r_d) = \frac{2}{4\pi}\int d\vec{r}\, V_c(|\vec{r}_d + \frac{1}{2}\vec{r}|)\left[\frac{u(r)}{r}\right]^2, \qquad (2.4.19)$$

$$U_{os}(r_d) = \frac{1}{4\pi}\int d\vec{r}\, V_s(|\vec{r}_d + \frac{1}{2}\vec{r}|)\left[\frac{u(r)}{r}\right]^2\left(1 + \frac{\vec{r}\cdot\vec{r}_d}{2r_d^2}\right), \qquad (2.4.20)$$

$$U_{OT}(r_d) = \frac{6\sqrt{2}}{4\pi}\int d\vec{r}\, V_c(|\vec{r}_d + \frac{1}{2}\vec{r}|)\,\frac{u(r)w(r)P_2(\cos\theta)}{r^2}\,. \qquad (2.4.21)$$

[12]The σ_i are Pauli spin matrices. The neutron and proton spin spaces are independent of one another, $[\sigma_n, \sigma_p] = 0$.

Here θ is the angle between the vectors $\vec{r}_d$ and $\vec{r}$. In writing the spin-orbit potential U_{os} we neglected the D-state contribution. However we observe that the tensor folded-potential would vanish if $w(r) = 0$. This potential is a consequence of the D-state of the deuteron.

To calculate U_1, we will assume the deuteron is in a pure S-state [4]. We will also use plane waves for the loop momentum $|k>$ and assume the equality of the neutron and proton radial potential functions $V_s(\vec{r}_n) = V_s(\vec{r}_p)$, so that we can now write

$$< 0|V|k >< k|V|0 > \; = \; \frac{1}{4\pi}\int\int d\vec{r}' d\vec{r} F(r,r') \exp[i(\vec{r} - \vec{r}')\cdot\vec{k}] \left(\vec{S}\cdot\vec{L}\right)^2 \tag{2.4.22}$$

with

$$F(r,r') \; = \; \frac{u(r)\, V_s(|\vec{r}_d + \frac{1}{2}\vec{r}|)\, u(r')\, V_s(|\vec{r}_d + \frac{1}{2}\vec{r}'|)}{rr'}\,. \tag{2.4.23}$$

Substituting this in eqn (2.4.15), assuming ϵ_k is a constant average excitation energy ϵ and hence utilizing the delta-function arising from the integration over the k-space, we arrive at.

$$U_1(\vec{r}_d) \; = \; U_o(\vec{r}_d) + \frac{2\pi^2}{\epsilon}\int d\vec{r}\left[\frac{u(r)V_s(|\vec{r}_d + \frac{1}{2}\vec{r}|)}{r}\right]^2 \left(\vec{S}\cdot\vec{L}\right)^2 . \tag{2.4.24}$$

The second part of the above equation is of the tensor type T_L. Therefore we see that the local nucleon-nucleus potential when folded produces two types of tensor interactions which constitute part of the deuteron's optical potential.

This potential structure is not unique to the deuteron-nucleus scattering, ^6Li-nucleus scattering [34,35] exhibits similar properties. According to the cluster model, the ^6Li wavefunction can be constructed of d+α particle wave functions. In this model it has been shown that the ^6Li-target optical potential may be obtained by folding the sum of the deuteron and α particle potentials, over the ^6Li ground-state wavefunction.

CHAPTER 3

THE BORN APPROXIMATION TO THE SCATTERING AMPLITUDE

In this chapter we will discuss the Born approximation for the scattering of spin-one particles off spin zero targets. In section (3.1) we will outline briefly the basic formulae. In sections (3.2) and (3.3) we will employ a Gaussian potential to calculate the first and second Born amplitudes respectively. In section (3.4) a check on the imaginary part of the second-Born via the first-Born amplitudes is carried out using the optical theorem. In the subsequent section we develop the high energy limit of the Born amplitude. Finally we present the results of our numerical calculations for the second-Born amplitude and compare it with it's equivalent contribution arrived at from the exact calculation using the code DDTP.

(3.1) Introduction

The Born series is in essence a perturbative expansion of the scattering amplitude in powers of the potential strength [29,30]. The rate at which the series converges depends on the strength of the scattering and on the length of time[1] a/v the particle spends within the potential. When this time is short compared

[1] a is the range of interaction, v is the velocity and V_o the strength of the potential

to the time required for the potential to influence the particle $\hbar/V_o$, the scattering is then weak enough and the series converges, i.e.,

$$\frac{V_o a}{\hbar v} \ll 1 \qquad (3.1.1)$$

At high energies, and/or weak potential conditions, the series converges rapidly such that only a few terms are important. In such cases the Born amplitude is a practical method of computing the observables. The approximation can also be used generally to establish certain general properties of the scattering amplitude. In these cases it is sufficient to know only that the series does converge.

In terms of the vectors $\vec{k}_i$, $\vec{k}_f$ representing the initial and final wave vectors respectively, our coordinate system is defined by,

$$\begin{aligned} \vec{q} &= \vec{k}_i - \vec{k}_f\,, \ \ \vec{k} = \frac{1}{2}(\vec{k}_i + \vec{k}_f)\,, \\ \vec{n} &= \vec{k}_f \times \vec{k}_i = \vec{k} \times \vec{q}\,. \end{aligned} \qquad (3.1.2)$$

We begin by writing the scattering amplitude for the scattering of a particle with spin S off a fixed spinless target as an operator in spin-space. If the incident and scattered particles have magnetic quantum numbers ν and ν' respectively, the amplitude has components

$$\mathcal{F}_{\nu\nu'}(\vec{k}_f, \vec{k}_i) = < s, \nu' \mid \mathcal{F}(\vec{k}_f, \vec{k}_i) \mid s, \nu > . \qquad (3.1.3)$$

In this representation we can write,

$$\mathcal{F}(\hat{k}_f \cdot \hat{k}_i) = \frac{-\mu}{2\pi} \int d\vec{r}\ \mathrm{e}^{-i\vec{k}_f \cdot \vec{r}}\, V(\vec{s}, \vec{r})\, \Psi_{ki}^{out}(\vec{r}). \qquad (3.1.4)$$

Here μ is the particle mass, $V(\vec{s},\vec{r})$ is the optical potential defined in section (2.2) and $\Psi^{out}_{ki}(\vec{r})$ is an outgoing wave function, defined by the Lipmann-Schwinger equation as,

$$\Psi^{out}_{ki}(\vec{r}) \; = \; \psi_{ki}(\vec{r}) \, + \, \Psi^{out}_{sc}(\vec{r}) \qquad (3.1.5)$$

Here, $\psi_{ki}(\vec{r}) \; = \; \exp[\vec{k}_i \cdot \vec{r}]$ is an incident plane wave of wave vector $\vec{k}_i$, and

$$\Psi^{out}_{sc}(\vec{r}) \; = \; \int d\vec{r'}\, G^{out}(\vec{r},\vec{r'})\, V(\vec{s},\vec{r'})\, \Psi^{out}_{ki}(\vec{r'}). \qquad (3.1.6)$$

In the above expression the Green-function is defined as

$$G^{out}(\vec{r},\vec{r'}) \; = \; \frac{-2\mu}{(2\pi)^3} \int \frac{\exp[i\vec{k'} \cdot (\vec{r} - \vec{r'})]}{k'^2 - k^2 - i\epsilon}\, dk' \, , \qquad (3.1.7)$$

where for the physical scattering amplitude $k_o \; = \; |k_i| \; = \; |k_f|$. The Born-series is arrived at by iterating eqn (3.1.5) and then substituting in eqn (3.1.4). Iterating once gives the first-Born amplitude, namely

$$\mathcal{F}_{1B}(\vec{k}_f \cdot \vec{k}_i) \; = \; \frac{-\mu}{2\pi} \int d\vec{r}\; \mathrm{e}^{-i\vec{k}_f \cdot \vec{r}}\, V(\vec{s},\vec{r})\, \mathrm{e}^{i\vec{k}_i \cdot \vec{r}}. \qquad (3.1.8)$$

The second iteration results in the second-Born amplitude

$$\mathcal{F}_{2B}(\vec{k}_i, \vec{k}_f) \; = \; \frac{1}{2\pi^2} \int \frac{V(\vec{k}_i, \vec{k}_l)\, V(\vec{k}_l, \vec{k}_f)}{(\, k_o^2 - k_l^2 + i\epsilon)}\, d\vec{k}_l \, , \qquad (3.1.9)$$

where

$$V(\vec{k}_\beta, \vec{k}_\gamma) \; = \; \frac{-\mu}{2\pi} \int d\vec{r}\; \mathrm{e}^{i\vec{k}_\gamma \cdot \vec{r}}\, V(\vec{s},\vec{r})\, \mathrm{e}^{-i\vec{k}_\beta \cdot \vec{r}}. \qquad (3.1.10)$$

This is identical in structure to the first-Born, except that γ, β stand for the initial, final or loop momenta.

(3.2) First-Born Amplitude

In this section and the following one, we will employ a Gaussian potential of the form

$$V_j(r) = V_o^j \exp[\frac{-\alpha_j^2 r^2}{2}], \qquad (3.2.1)$$

to describe the interaction. Here V_o^j is the potential strength and $a = \sqrt{2}/\alpha$ is the *range* of interaction. It is a short-range potential that decays rapidly for $r > a$. Its most appropriate use is in describing a target with a small number of nucleons, where saturation in the middle is negligible, e.g. Helium. However the prime motivation for employing a Gaussian potential in this calculation is its mathematical simplicity which allows us to arrive at analytical expressions of the required amplitudes.

The amplitude, as has been explained before[see eqn (2.1.27)], is constrained by symmetry requirements imposed by parity and time reversal to take the form[2]

$$\begin{aligned} \mathcal{F}_{1B}(\theta) &= \mathcal{A}_{1B}(\theta) + \mathcal{B}_{1B}(\theta)(\vec{s}\cdot\hat{n}) + \mathcal{C}^n_{1B}(\theta)\vec{S}_2\cdot\vec{R}_2(\hat{n},\hat{n}) \\ &+ \mathcal{C}^k_{1B}(\theta)\,\vec{S}_2\cdot\vec{R}_2(\hat{k},\hat{k}) \end{aligned} \qquad (3.2.2)$$

If we define,

$$f_j(q^2) = \frac{-\mu}{2\pi}\int d\vec{r}\, V_j(r)\, e^{i\vec{q}\cdot\vec{r}}, \qquad (3.2.3)$$

then we can write

$$\mathcal{A}_{1B}(\theta) = f_c(q^2) \qquad (3.2.4)$$

[2]The subscript $1B$ will be employed to indicate that this is the first-Born term. The form of the amplitude is general and independent of the order of perturbation.

$$\mathcal{B}_{1B}(\theta) \;=\; 2i \mid n \mid \frac{\partial}{\partial q^2} f_s(q^2)\,. \qquad (3.2.5)$$

The contributions to $\mathcal{C}^n_{1B}(\theta)$ and $\mathcal{C}^k_{1B}(\theta)$ come only from the tensor terms. Using the re-coupling relation eqn (2.2.4) together with eqn (3.1.2) we can write,

$$\vec{S}_2 \cdot \vec{R}_2(\vec{k}_i, \vec{k}_f) \;=\; \vec{S}_2 \cdot \vec{R}_2(\vec{k}, \vec{k}) \;-\; \frac{1}{4}\vec{S}_2 \cdot \vec{R}_2(\vec{q}, \vec{q})\,. \qquad (3.2.6)$$

$$\begin{aligned} T^r_{1B}(\theta) \;&=\; \frac{\mu}{2\pi} \int d\vec{r}\, V_r(r)\, \mathrm{e}^{i\vec{k}_f\cdot\vec{r}}\, \vec{S}_2 \cdot \vec{R}_2(\vec{\nabla}_{\vec{k}_i}, \vec{\nabla}_{\vec{k}_i})\, \mathrm{e}^{i\vec{k}_i\cdot\vec{r}} \\ &=\; -\vec{S}_2 \cdot \vec{R}_2(\vec{\nabla}_{\vec{q}}, \vec{\nabla}_{\vec{q}})\, f(q^2) \\ &=\; 4q^2 \left[\vec{S}_2 \cdot \vec{R}_2(\hat{n}, \hat{n}) \;+\; \vec{S}_2 \cdot \vec{R}_2(\hat{k}_i, \hat{k}_i)\right] f''_r(q^2)\,, \qquad (3.2.7) \end{aligned}$$

where we have defined,

$$f''_j(q^2) \;=\; (\frac{\partial}{\partial q^2})^2\, f_j(q^2)\,. \qquad (3.2.8)$$

Also we have

$$\begin{aligned} T^L_{1B}(\theta) \;&=\; \frac{\mu}{2\pi} \int d\vec{r}\, V_L(r)\, \mathrm{e}^{i\vec{q}\cdot\vec{r}} \left[\vec{S}_2 \cdot \vec{R}_2(\vec{r} \times \vec{k}_i, \vec{r} \times \vec{k}_i) \;-\; i\vec{S}_2 \cdot \vec{R}_2(\vec{k}_i, \vec{r})\right] \\ &=\; -\left[\vec{S}_2 \cdot \vec{R}_2(\vec{k}_i \times \vec{\nabla}_{\vec{q}}, \vec{k}_i \times \vec{\nabla}_{\vec{q}}) \;+\; \vec{S}_2 \cdot \vec{R}_2(\vec{k}_i, \vec{\nabla}_{\vec{q}})\right] f_L(q^2) \\ &=\; \vec{S}_2 \cdot \vec{R}_2(\hat{n}, \hat{n}) \left[\frac{q^2}{2} f'_L(q^2) \;-\; 4n^2\, f''_L(q^2)\right] \\ &\quad \left[+ (2k^2 + \frac{q^2}{2})\vec{S}_2 \cdot \vec{R}_2(\hat{k}_i, \hat{k}_i)\right] f'_L(q^2)\,. \qquad (3.2.9) \end{aligned}$$

Similarly,

$$\begin{aligned} T^p_{1B}(\theta) \;&=\; \frac{\mu}{4\pi} \int d\vec{r}\, \mathrm{e}^{-i\vec{k}_f\cdot\vec{r}} \Big[V_p(r)\vec{S}_2 \cdot \vec{R}_2(\vec{\nabla}_{\vec{r}}, \vec{\nabla}_{\vec{r}}) \\ &+\; \vec{S}_2 \cdot \vec{R}_2(\overleftarrow{\nabla}_{\vec{r}}, \overleftarrow{\nabla}_{\vec{r}})\, V_p(r)\Big]\, \mathrm{e}^{i\vec{k}_i\cdot\vec{r}} \\ &=\; \left[\vec{S}_2 \cdot \vec{R}_2(k^2 - \frac{q^2}{4}) \;-\; \vec{S}_2 \cdot \vec{R}_2(\hat{n}, \hat{n})\frac{q^2}{4}\right] f_p(q^2)\,. \qquad (3.2.10) \end{aligned}$$

Here the arrows indicate the direction in which the operators act. Finally, grouping the coefficients of $\vec{S}_2 \times \vec{R}_2(\hat{k},\hat{k})$ and $\vec{S}_2 \times \vec{R}_2(\hat{n},\hat{n})$, we write

$$\mathcal{C}^n_{1B}(\theta) \; = \; 4q^2 \, f_r''(q^2) + \frac{q^2}{2} \, f_L'(q^2) \, - \, 4n^2 \, f_L''(q^2) \, - \, \frac{q^2}{4} \, f_p(q^2) \qquad (3.2.11)$$

and

$$\mathcal{C}^k_{1B}(\theta) \; = \; 4q^2 \, f_r''(q^2) + (\frac{q^2}{2} + 2k^2) f_L'(q^2) + k^2 f_p(q^2) \qquad (3.2.12)$$

The above expressions for $\mathcal{A}_{1B}, \mathcal{B}_{1B}, \mathcal{C}^n_{1B}, \mathcal{C}^k_{1B}$ are general forms (for any radial potential-density) of the first-Born scattering amplitude for the scattering of spin-one particles off a spinless target.

For a Gaussian potential of the form given by eqn (3.2.1), it is straightforward to show that

$$f_j(q^2) \; = \; \frac{-\mu V_o^j \sqrt{2\pi}}{\alpha^3} \exp[\frac{-q^2}{2\alpha^2}] \, . \qquad (3.2.13)$$

Employing this in the above general expressions, we finally arrive at the following expressions for the scattering amplitudes.

$$\mathcal{A}_{1B}(\theta) \; = \; \frac{-\mu V_o^c \sqrt{2\pi}}{\alpha^3} \exp[\frac{-q^2}{2\alpha^2}] \, . \qquad (3.2.14)$$

$$\mathcal{B}_{1B}(\theta) \; = \; i \frac{\mu V_o^s \sqrt{2\pi}}{\alpha^5} \mid n \mid \exp[\frac{-q^2}{2\alpha^2}] \, . \qquad (3.2.15)$$

$$\mathcal{C}^n_{1B}(\theta) \; = \; \frac{\mu\sqrt{2\pi}}{\alpha^3} \exp[\frac{-q^2}{2\alpha^2}] \left\{ (\frac{q^2}{4\alpha^2} + \frac{n^2}{\alpha^4}) V_o^l - \frac{q^2 V_o^r}{\alpha^4} + \frac{q^2 V_o^p}{4} \right\} . \qquad (3.2.16)$$

$$\mathcal{C}^k_{1B}(\theta) \; = \; \frac{\mu\sqrt{2\pi}}{\alpha^3} \exp[\frac{-q^2}{2\alpha^2}] \left\{ (\frac{q^2}{4\alpha^2} + \frac{k^2}{\alpha^2}) V_o^l - \frac{q^2 V_o^r}{\alpha^4} + \frac{k^2 V_o^p}{4} \right\} . \qquad (3.2.17)$$

(3.3) Second-Born Amplitude

We will now calculate the second-Born amplitude, limiting the tensor contribution to the optical potential to the $V_L(r)\vec{S}_2 \cdot \vec{R}_2(\vec{L},\vec{L})$ term. The calculation is non-trivial in that it includes several hundred terms. We overcome this difficulty by using the computer software SMP (Symbolic Manipulation Program) [36]. Before describing the computer calculation we will put the formulae in a convenient form. We rewrite the potential[3]

$$V(r) \;=\; V_c(r) + V_{sl}(r)(\vec{S}\cdot\vec{L}) + V_Q(r)(\vec{S}\cdot\vec{L})^2 + V_{ll}(r)L^2\,, \qquad (3.3.1)$$

where

$$\begin{aligned} V_{sl}(r) &= V_s(r) + \frac{1}{2}V_l(r) \\ V_Q(r) &= V_l(r) \\ V_{ll}(r) &= -\frac{2}{3}V_l(r) \end{aligned} \qquad (3.3.2)$$

Using eqn (3.3.1) we cast eqn (3.1.10) in the form

$$V(\vec{k}_\beta,\vec{k}_\gamma) \;=\; V_c(\vec{k}_\beta,\vec{k}_\gamma) + V_{sl}(\vec{k}_\beta,\vec{k}_\gamma) + V_Q(\vec{k}_\beta,\vec{k}_\gamma) + V_{ll}(\vec{k}_\beta,\vec{k}_\gamma) \qquad (3.3.3)$$

For a Gaussian potential of the form given by eqn (3.2.1), and in a similar spirit to the first-Born calculation, we first define

$$\begin{aligned} f(q^{\beta\gamma}) &= \frac{-\mu}{2\pi}\int d\vec{r}\, e^{-\alpha^2 r^2/2}\, e^{i\vec{q}_{\beta\gamma}\cdot\vec{r}} \\ &= \frac{-\mu\sqrt{2\pi}}{\alpha^3}\exp[\frac{-q_{\beta\gamma}^2}{2\alpha^2}]\,, \end{aligned} \qquad (3.3.4)$$

[3]It is important to note that with our limitation of the tensor contribution, we are only left with three independent potential terms.

where $\vec{q}_{\beta\gamma} = \vec{k}_\beta - \vec{k}_\gamma$.

Using the definition of the Levi-Civita tensor eqn (2.1.4), we can now write,

$$V_c(\vec{k}_\beta, \vec{k}_\gamma) = V_o^c \, f(\vec{k}_\beta, \vec{k}_\gamma) \,. \qquad (3.3.5)$$

$$V_{sl}(\vec{k}_\beta, \vec{k}_\gamma) = \frac{i}{\alpha^2}(V_o^s + \frac{V_o^l}{2}) f(\vec{k}_\beta, \vec{k}_\gamma) (\vec{S} \cdot \vec{k}^\beta \times \vec{k}^\gamma) \,. \qquad (3.3.6)$$

$$\begin{aligned} V_Q(\vec{k}_\beta, \vec{k}_\gamma) &= \frac{V_o^l}{\alpha^2} f(\vec{k}_\beta, \vec{k}_\gamma) \, \mathcal{E}_{lmn} \, \mathcal{E}_{pvt} S_l S_p \Big[\delta_{vn} \, k_t^\beta q_m^{\beta\gamma} \\ &+ k_v^\beta \, k_m^\beta \delta_{nt} - \frac{1}{\alpha^2} k_v^\beta \, k_m^\beta \, k_t^\gamma \, k_n^\gamma \Big] \,. \end{aligned} \qquad (3.3.7)$$

$$V_{ll}(\vec{k}_\beta, \vec{k}_\gamma) = \frac{-2V_o^l}{3\alpha^2} f(\vec{k}_\beta, \vec{k}_\gamma) \Big[2(\vec{k}^\beta \cdot \vec{k}^\gamma) - \frac{1}{\alpha^2} (\vec{k}^\beta \times \vec{k}^\gamma)^2 \Big] \qquad (3.3.8)$$

Equation (3.3.3) can now be rewritten in the form

$$V(\vec{k}_\beta, \vec{k}_\gamma) = \mathcal{G}_{\beta\gamma} \, f(\vec{k}_\beta, \vec{k}_\gamma) \,. \qquad (3.3.9)$$

Using this in equation (3.1.9), the scattering amplitude can be expressed as,

$$\mathcal{F}_{2B}(q^{if}) = \int d\vec{k}_l \, \mathcal{M} \, \mathcal{G}_{il} \mathcal{G}_{lf} \,, \qquad (3.3.10)$$

where,

$$\mathcal{M} = \frac{f(q^{il}) \, f(q^{lf})}{2\pi^2 (k_l^2 - k_o^2 - i\epsilon)} \,. \qquad (3.3.11)$$

Now,

$$\begin{aligned} f(q^2, k^2) &= \int d\vec{k}_l \, \mathcal{M} \\ &= \frac{1}{k} (\frac{\mu}{\alpha^2})^2 \, exp[-q^2/4\alpha^2] \Big\{ \sqrt{\pi} Q_\alpha(k) + \frac{i\pi}{2} E_\alpha(k) \Big\} \end{aligned} \qquad (3.3.12)$$

Here[4]

$$Q_\alpha(k) = D(\frac{k_o + k}{\alpha}) - D(\frac{k_o - k}{\alpha}) \qquad (3.3.13)$$

[4]Note that $\vec{k} = (0, 0, k_o \cos\theta/2)$.

and

$$E_\alpha(k) \;=\; \mathrm{e}^{-(k_o - k)^2/\alpha^2} \;-\; \mathrm{e}^{-(k_o + k)^2/\alpha^2} \qquad (3.3.14)$$

with the Dawson integral [37] defined as,

$$D(x) \;=\; \mathrm{e}^{-x^2} \int_0^x dt\, \mathrm{e}^{-t^2} \,. \qquad (3.3.15)$$

To calculate $\mathcal{F}_{2B}$ we need to solve integrals of the form

$$F_i \;=\; \int d\vec{k}_l \, k_l^i \, \mathcal{M} \,, \qquad (3.3.16)$$

where i is the i'th component of the momentum. Using parametric differentiation, we first write

$$k_l^i \,\mathcal{M} \;=\; \left[k_i \,+\, \frac{\alpha^2}{2} \frac{\partial}{\partial k_i} \right] \mathcal{M}. \qquad (3.3.17)$$

Hence

$$F_i \;=\; \left[k_i \,+\, \frac{\alpha^2}{2} \frac{\partial}{\partial k_i} \right] f(q^2, k^2) \,. \qquad (3.3.18)$$

Note that the differentiation in the above expressions are with respect to the average momentum defined in eqn (3.1.2). After some algebraic manipulation we obtain

$$F_i \;=\; k_i \,(\, f \,+\, \alpha^2 f^1 \,) \qquad (3.3.19)$$

$$F_{ij} \;=\; k_{ij}(f \,+ 2\alpha^2 f^1 \,+ \alpha^4 f^2) \,+\, \delta_{ij}(\frac{\alpha^2}{2} f \,+\, \frac{\alpha^4}{2} f^1) \qquad (3.3.20)$$

$$\begin{aligned} F_{ijk} \;&=\; k_{ijk}(f \,+ 3\alpha^2 f^1 \,+ 3\alpha^4 f^2 \,+ \alpha^6 f^3) \\ &+\; \mathcal{X}_{ijk}(\frac{\alpha^2}{2} f \,+ \alpha^4 f^1 \,+ \frac{\alpha^6}{2} f^2) \end{aligned} \qquad (3.3.21)$$

$$
\begin{aligned}
F_{ijkl} &= k_{ijkl}(f + 4\alpha^2 f^1 + 6\alpha^4 f^2 + 4\alpha^6 f^3 + \alpha^8 f^4) \\
&+ \mathcal{Z}_{ijkl}(\frac{\alpha^2}{2} f + \frac{3\alpha^4}{2} f^1 + \frac{3\alpha^6}{2} f^2 + \frac{\alpha^8}{2} f^3) \\
&+ \mathcal{Y}_{ijkl}(\frac{\alpha^4}{4} f + \frac{\alpha^6}{2} f^1 + \frac{\alpha^8}{4} f^2) . \qquad (3.3.22)
\end{aligned}
$$

We have defined the tensors

$$
\begin{aligned}
f^n &= (\frac{\partial}{\partial k^2})^n f(q^2, k^2) \\
k_{ijk\cdots} &= k_i\, k_j\, k_k \cdots \\
\mathcal{X}_{ijk} &= k_k \delta_{ij} + k_o \delta_{jk} + k_j \delta_{ik} \\
\mathcal{Y}_{ijkl} &= \frac{\partial}{\partial k_i} \mathcal{X}_{jkl} \\
\mathcal{Z}_{ijkl} &= k_{ij}\delta_{kl} + k_{ik}\delta_{jl} + k_{il}\delta_{jk} \\
&+ k_{jk}\delta_{il} + k_{jl}\delta_{ik} + k_{kl}\delta_{ij} . \qquad (3.3.23)
\end{aligned}
$$

From now on the algebra becomes more complicated due to the large number of terms involved. The idea is to carry out the multiplication implied in the scattering matrix eqn (3.3.10) and then to substitute in the resultant eqns (3.3.19-22). The substitution of the above parametric differentiation formulae is equivalent to integrating eqn (3.3.10) by parts. The expression we now have for the scattering matrix is in terms of the function $f(q^2, k^2)$ and its derivatives multiplying a tensor made of the spin matrices coupled to the momentum vectors in the manner defined by eqns (3.3.6-8) and their products, where the loop-momenta designated by l in eqn (3.3.10) have been replaced by the average momentum via expressions (3.3.19-22). This expression is several hundred terms long. However it is a 3×3 matrix, whose elements can be calculated by summing over the repeated indices. This is all done using SMP [36].

The fundamental feature of this software is its ability to handle variables. This is what we need because for every term of our expression a different combination of indices occurs. To overcome this we construct a chain of commands in SMP which picks up the indices occurring in an expression. If we denote this symbolically as "List" we can formulate its action by

$$'List'\, f(i,j,k)\, g(i,k,j)\, h(k,i) \Rightarrow [i,j,k]\,. \tag{3.3.24}$$

In principle the action of this command can be described as constructing a set formed by the intersection of all the variables in the expression (indices) and the full list of alphabets. In the resultant set no index occurs twice. Equipped with this set we can now sum over its constituents. These are labeled by their position in the set rather than their names.

Finally we write the result in the form defined by eqn (2.1.27)

$$\begin{aligned}
\mathcal{A}_{2B}(\theta) \quad = \quad & V_{oc}^2 f + V_{ol}^2 \left\{ f \left[-\frac{37k^2q^2}{72\alpha^4} + \frac{k^2q^4}{18\alpha^6} - \frac{2k^4q^2}{9\alpha^6} + \frac{k^4q^4}{72\alpha^8} \right.\right. \\
+ \quad & \left. \frac{5k^2}{18\alpha^2} + \frac{4k^4}{9\alpha^4} - \frac{5q^2}{72\alpha^2} + \frac{q^4}{36\alpha^4} \right] + f^1 \left[-\frac{37k^2q^2}{36\alpha^2} + \frac{k^2q^4}{6\alpha^4} \right. \\
+ \quad & \left. -\frac{2k^4q^2}{3\alpha^4} + \frac{k^4q^4}{18\alpha^6} + \frac{5k^2}{18} + \frac{8k^4}{9\alpha^2} - \frac{5q^2}{72} + \frac{q^4}{18\alpha^2} \right] \\
+ \quad & f^2 \left[-\frac{37k^2q^2}{72} + \frac{k^2q^4}{6\alpha^2} - \frac{2k^4q^2}{3\alpha^2} + \frac{k^4q^4}{12\alpha^4} + \frac{4k^4}{9} + \frac{q^4}{36} \right] \\
+ \quad & \left. f^3 \left[\frac{k^2q^4}{18} - \frac{2k^4q^2}{9} + \frac{k^4q^4}{18\alpha^2} \right] + \frac{k^4q^4}{72} f^4 \right\} \\
+ \quad & V_{os}^2 \left\{ f \left[-\frac{k^2q^2}{6\alpha^4} + \frac{2k^2}{3\alpha^2} - \frac{q^2}{6\alpha^2} \right] \right.
\end{aligned}$$

$$+ \quad f^1\left[-\frac{k^2q^2}{3\alpha^2}+\frac{2k^2}{3}-\frac{q^2}{6}\right]-f^2\frac{k^2q^2}{6}\Bigg\} \qquad (3.3.25)$$

$$\begin{aligned}
\mathcal{B}_{2B}(\theta) &= V_{oc}V_{os}\left\{-f\frac{kq}{\alpha^2}-f^1kq\right\}+V_{os}^2\left\{f\frac{kq}{4\alpha^2}+f^1\frac{kq}{4}\right\} \\
&- V_{ol}V_{os}\Bigg\{f\left[-\frac{5kq}{12\alpha^2}+\frac{kq^3}{6\alpha^4}-\frac{2k^3q}{3\alpha^4}+\frac{k^3q^3}{12\alpha^6}\right]+f^1\Bigg[-\frac{5kq}{12} \\
&+ \frac{kq^3}{3\alpha^2}-\frac{4k^3q}{3\alpha^2}+\frac{k^3q^3}{4\alpha^4}\Bigg]+f^2\left[\frac{kq^3}{6}-\frac{2k^3q}{3}+\frac{k^3q^3}{4\alpha^2}+\frac{k^3q^3}{12}\right]\Bigg\} \\
&+ V_{ol}^2\Bigg\{f\left[-\frac{5kq}{16\alpha^2}+\frac{kq^3}{8\alpha^4}-\frac{k^3q}{2\alpha^4}+\frac{k^3q^3}{16\alpha^6}\right] \\
&+ f^1\left[-\frac{5kq}{16}+\frac{kq^3}{4\alpha^2}-\frac{k^3q}{\alpha^2}+\frac{3k^3q^3}{16\alpha^4}\right] \\
&+ f^2\left[\frac{kq^3}{8}-\frac{k^3q}{2}+\frac{3k^3q^3}{16\alpha^2}\right]+f^3\frac{k^3q^3}{16}\Bigg\} \qquad (3.3.26)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{2B}^n(\theta) &= V_{oc}V_{ol}\Bigg\{-f\left[\frac{q^2}{4\alpha^2}+\frac{k^2q^2}{2\alpha^4}\right]-f^2\frac{k^2q^2}{2} \\
&- f^1\left[\frac{q^2}{4}+\frac{k^2q^2}{\alpha^2}\right]\Bigg\}+V_{ol}V_{os}\Bigg\{f\left[\frac{3q^2}{8\alpha^2}+\frac{3k^2q^2}{4\alpha^4}\right] \\
&+ f^1\left[\frac{3q^2}{8}+\frac{3k^2q^2}{2\alpha^2}\right]+f^2\frac{3k^2q^2}{4}\Bigg\}+V_{os}^2\Bigg\{-f\left[\frac{q^2}{8\alpha^2}+\frac{k^2q^2}{4\alpha^4}\right] \\
&- f^1\left[\frac{q^2}{8}+\frac{k^2q^2}{2\alpha^2}\right]-f^2\frac{k^2q^2}{4}\Bigg\}+V_{ol}^2\Bigg\{-f\Bigg[\frac{7q^2}{96\alpha^2} \\
&+ \frac{q^4}{48\alpha^4}-\frac{13k^2q^2}{48\alpha^4}+\frac{7k^2q^4}{96\alpha^6}-\frac{k^4q^2}{4\alpha^6}+\frac{k^4q^4}{48\alpha^8}\Bigg] \\
&- f^1\left[\frac{7q^2}{96}+\frac{q^4}{24\alpha^2}-\frac{13k^2q^2}{24\alpha^2}+\frac{7k^2q^4}{32\alpha^4}-\frac{3k^4q^2}{4\alpha^4}+\frac{k^4q^4}{12\alpha^6}\right] \\
&- f^2\left[\frac{q^4}{48}-\frac{13k^2q^2}{48}+\frac{7k^2q^4}{32\alpha^2}-\frac{3k^4q^2}{4\alpha^2}+\frac{k^4q^4}{8\alpha^4}\right] \\
&- f^3\left[\frac{7k^2q^4}{96}-\frac{k^4q^2}{4}+\frac{k^4q^4}{12\alpha^2}\right]+f^4\frac{k^4q^4}{48}\Bigg\} \qquad (3.3.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{2B}^k(\theta) &= V_{oc}V_{ol}\left\{-f\left[\frac{k^2}{\alpha^2}+\frac{q^2}{4\alpha^2}\right]-f^1\left[k^2+\frac{q^2}{4}\right]\right\} \\
&+ V_{ol}V_{os}\left\{f\left[\frac{3k^2}{2\alpha^2}+\frac{3q^2}{8\alpha^2}\right]+f^1\left[\frac{3k^2}{2}+\frac{3q^2}{8}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad {V_{os}}^2\left\{-f\left[\frac{k^2}{2\alpha^2}+\frac{q^2}{8\alpha^2}\right]-f^1\left[\frac{k^2}{2}+\frac{q^2}{8}\right]\right\} \\
&+ \quad V_{ol}^2\left\{f\left[-\frac{7k^2}{24\alpha^2}+\frac{k^4}{3\alpha^4}-\frac{7q^2}{96\alpha^2}-\frac{q^4}{48\alpha^4}-\frac{k^2q^4}{96\alpha^6}\right.\right. \\
&- \quad \left.\frac{k^4q^2}{24\alpha^6}\right]+f^1\left[-\frac{7k^2}{24}+\frac{2k^4}{3\alpha^2}-\frac{7q^2}{96}-\frac{q^4}{24\alpha^2}-\frac{k^2q^4}{32\alpha^4}\right. \\
&- \quad \left.\frac{k^4q^2}{8\alpha^4}\right]+f^2\left[\frac{k^4}{3}-\frac{q^4}{48}-\frac{k^2q^4}{32\alpha^2}-\frac{k^4q^2}{8\alpha^2}\right] \\
&- \quad \left.f^3\left[\frac{k^2q^4}{96}+\frac{k^4q^2}{24}\right]\right\}. \qquad (3.3.28)
\end{aligned}
$$

(3.4) A check using the optical theorem

The optical theorem relates the total cross-section of an interaction to the imaginary part of the scattering amplitude in the forward direction, namely[5]

$$\sigma_t = \frac{4\pi}{k} Im\mathcal{F}(\theta = 0). \qquad (3.4.1)$$

Now for purely elastic scattering

$$\sigma_t = \int d\Omega \mid \mathcal{F}(\theta) \mid^2 . \qquad (3.4.2)$$

We write the scattering amplitude as

$$\mathcal{F}(\theta) = V_o\mathcal{F}_{1B}(\theta) + V_o^2\mathcal{F}_{2B}(\theta) + V_o^3\mathcal{F}_{3B}(\theta) + \cdots , \qquad (3.4.3)$$

where V_o (the potential strength) is the perturbation parameter.

Substituting this in eqn (3.4.2) results in

$$\sigma_t = V_o^2\sigma_{1B1B} + V_o^3\sigma_{1B2B} + V_o^4(\sigma_{2B2B} + \sigma_{1B3B}) + \cdots \qquad (3.4.4)$$

[5] The extension of expression (3.4.1) to the amplitude $\mathcal{F}_{\nu\nu'}(\theta)$ defined by eqn (3.1.3), is straightforward but we will not need it here, see the paragraph preceeding eqn (3.4.8).

with

$$\sigma_{iBjB} = \int d\Omega\, \mathcal{F}_i \mathcal{F}_j^\dagger \,. \qquad (3.4.5)$$

On comparing (3.4.4) with the right-hand side of eqn (3.4.1), with eqn (3.4.3) substituted for $\mathcal{F}(\theta)$, we can write

$$\sigma_{1B1B} = \frac{4\pi}{k} Im \mathcal{F}_{2B}(\theta = 0) \qquad (3.4.6)$$

$$\sigma_{1B2B} = \frac{4\pi}{k} Im \mathcal{F}_{3B}(\theta = 0) \qquad (3.4.7)$$

etc$\cdots$

In the case of spin interactions there will be an equivalent relation for each of the spin amplitudes, i.e. for each independent matrix element. However because of the condition that θ is equal to zero, only two independent amplitudes will survive, namely $\mathcal{A}(\theta)$ and $\mathcal{C}_k(\theta)$ in our representaion. These two terms contribute to the the check carried out on the trace (Tr) of both sides of eqn (3.4.5).

$$\frac{4\pi}{k} Im \left[Tr\, \mathcal{F}_{2B}(\theta)\right] = \int d\Omega\, Tr \left[\mathcal{F}_{1B}(\vec{k}_i, \vec{k}_l)\mathcal{F}_{1B}(\vec{k}_l, \vec{k}_i)\right] , \qquad (3.4.8)$$

where

$$\mathcal{F}_{1B}^\dagger(\vec{k}_i, \vec{k}_l) = \mathcal{F}_{1B}(\vec{k}_l, \vec{k}_i) \,. \qquad (3.4.9)$$

Now

$$\begin{aligned} \mathcal{F}_{1B}(\vec{k}_i, \vec{k}_l) &= f_c(\vec{k}_i, \vec{k}_l) + \frac{i}{\alpha^2} f_s(\vec{k}_i, \vec{k}_l)(\vec{S} \cdot \vec{k}_i \times \vec{k}_l) - \frac{1}{\alpha^2} f_l(\vec{k}_i, \vec{k}_l) \Big\{ \vec{S}_2 \cdot \vec{R}_2(\vec{k}_i, \vec{k}_l) \\ &+ \frac{1}{\alpha^2} \vec{S}_2 \cdot \vec{R}_2(\vec{k}_i \times \vec{k}_l, \vec{k}_i \times \vec{k}_l) \Big\} \,. \qquad (3.4.10) \end{aligned}$$

If we define $\vec{k}_i = (0, 0, k_o)$, $\vec{k}_l = (k_o \sin\theta \cos\theta, k_o \sin\theta \sin\phi, k_o \cos\theta)$, and make use of the relation $q = 2k_o \sin\theta/2$, then after some algebraic manipulation we obtain

$$
\begin{aligned}
Tr\left[\mathcal{F}_{1B}(\vec{k}_o,\vec{k}_l)\,\mathcal{F}_{1B}(\vec{k}_l,\vec{k}_o)\right] &= 3f_c^2 + \left[\frac{2k_o^2q^2}{\alpha^4} - \frac{q^4}{2\alpha^4}\right] f_s^2 \\
&+ \left[\frac{2k_o^4}{3\alpha^4} + \left(\frac{k_o^2}{2\alpha^6} + \frac{1}{24\alpha^4} + \frac{2k_o^4}{3\alpha^8}\right) q^4 \right. \\
&- \left(\frac{1}{12\alpha^6} + \frac{k_o^2}{3\alpha^8}\right) q^6 \\
&- \left. \left(\frac{k_o^2}{6\alpha^4} + \frac{2k_o^4}{3\alpha^6}\right) q^2 + \frac{q^8}{24\alpha^8}\right] f_l^2\,. \qquad (3.4.11)
\end{aligned}
$$

The integrals on the left-hand side of eqn (3.4.8) are of the form

$$
\begin{aligned}
I_j^n &= \int d\Omega\, f_j^n(q^2) \\
&= (\frac{2\mu\pi V_j}{\alpha^3})^2 \int_0^\pi d\theta\, \sin\theta\, q^{2n}\, \exp[-\frac{q^2}{\alpha^2}] \qquad (3.4.12)
\end{aligned}
$$

$$n = 0,1,2,\cdots$$

It is straightforward to show that,

$$I_j^0 = -2(\frac{\mu\pi V_j}{k_o\alpha^2})^2\,(\exp[-\frac{4k_o^2}{\alpha^2}] - 1)\,. \qquad (3.4.13)$$

Also,

$$I_j^n = -\frac{\partial}{\partial x} I_j^{n-1}\,,\ with\ x = \frac{1}{\alpha^2}\,,\ n > 0 \qquad (3.4.14)$$

Using eqn (3.4.11-14), the left-hand side of eqn (3.4.8) takes the form

$$
\begin{aligned}
L.H.S &= 3I_c^0 + \frac{2k_o^2}{\alpha^4}I_s^1 - \frac{1}{2\alpha^4}I_s^2 + \frac{2k_o^4}{3\alpha^4}I_l^0 \\
&- \left[\frac{k_o^2}{6\alpha^4} + \frac{2k_o^4}{3\alpha^6}\right] I_l^1\,, + \left[\frac{k_o^2}{2\alpha^6} + \frac{1}{24\alpha^4} + \frac{2k_o^4}{3\alpha^8}\right] I_l^2 \\
&- \left[\frac{1}{12\alpha^6} + \frac{k_o^2}{3\alpha^8}\right] I_l^3 + \frac{1}{24\alpha^8} I_l^4\,. \qquad (3.4.15)
\end{aligned}
$$

The trace of the second-Born amplitude is readily calculated from eqn (3.3.25-28)

$$\begin{aligned} Tr\mathcal{F}_{2B}(\theta=0) &= 3V_{oc}^2 f + V_{os}^2\left[\frac{2k^2}{\alpha^2}f + 2k^2 f^1\right] + V_{ol}^2\left\{\frac{4k^4}{3}f^2\right. \\ &+ \left.\left[\frac{5k^2}{6}+\frac{8k^4}{3\alpha^2}\right]f^1 + \left[\frac{5k^2}{6\alpha^2}+\frac{4k^4}{3\alpha^2}\right]f\right\}. \qquad (3.4.16) \end{aligned}$$

From eqn (3.3.12) and the definition of the function $E_\alpha(k)$ eqn (3.3.14)

$$\begin{aligned} Imf(0,k_o^2) &= \frac{\pi}{2k_o}(\frac{\mu}{\alpha^2})^2 E_\alpha(k_o) \\ &= -\frac{\pi}{2k}(\frac{\mu}{\alpha^2})^2\left(\exp[-\frac{4k_o^2}{\alpha^2}]-1\right). \qquad (3.4.17) \end{aligned}$$

Finally if we develop the L.H.S using eqns (3.4.13,14) or the R.H.S using eqns (3.4.16,17), together with the necessary derivatives of f we arrive at the same result, namely

$$\begin{aligned} R.H.S &= L.H.S \\ &= (\mu\pi)^2\left\{-\frac{6}{\alpha^4 k_o}V_{oc}^2\left[\exp[-\frac{4k_o^2}{\alpha^2}]-1\right]\right. \\ &+ V_{os}^2\left[-\frac{2}{k_o^2\alpha^4}+\frac{4}{\alpha^6}+\exp[-\frac{4k_o^2}{\alpha^2}]\left(\frac{2}{k_o^2\alpha^4}+\frac{4}{\alpha^6}\right)\right] \\ &+ V_{ol}^2\left[\frac{7}{6k_o^2\alpha^4}-\frac{7}{3\alpha^6}+\frac{8k_o^2}{3\alpha^8}\right. \\ &- \left.\left.\exp[-\frac{4k_o^2}{\alpha^2}]\left(\frac{7}{6k_o^2\alpha^4}+\frac{7}{3\alpha^6}+\frac{8k_o^2}{3\alpha^8}\right)\right]\right\}. \qquad (3.4.18) \end{aligned}$$

This same procedure can be applied to any element of the scattering matrix. This concludes our check of the second-Born amplitude.

(3.5) High-energy Limit of the Second-Born Amplitude

As was mentioned in the introduction, one of our aims is to compare the corrected Eikonal expansion with the exact perturbation series. To achieve this we must make sure we are describing the same scattering domain. This is arrived at by limiting the Born amplitude to the forward-angle (small-momentum transfer) part of the scattering domain i.e,

$$k_i^2 \gg q^2 . \tag{3.5.1}$$

This leads to

$$k^2 \simeq k_o^2 (1 - \frac{q^2}{4k_o^2}) . \tag{3.5.2}$$

Also from the definition of the Dawson integral eqn (3.3.15) we note that for *large* x we get

$$D(x) \longrightarrow \frac{1}{2x} + \frac{1}{4x^3} \tag{3.5.3}$$

and for *small* x,

$$D(x) \longrightarrow x . \tag{3.5.4}$$

Using eqns (3.5.2-3) in eqns (3.3-13-14) we can write

$$\frac{E_\alpha}{k} = \frac{1}{k_o} (1 + \frac{q^2}{8k_o^2}) \tag{3.5.5}$$

and

$$\frac{Q_\alpha}{k} = \frac{\alpha}{4k_o^2} (1 - \frac{q^2}{2\alpha^2}) . \tag{3.5.6}$$

From the above relations it is straightforward to calculate the high-energy limit of the function f and its derivatives $f^{1 \cdots 5}$.

We can thus write down the *high-energy small angle* limit of the second-Born contribution to the scattering matrix. We present our formulae in terms of the *real* $\Re$ and *imaginary* $\Im$ parts.

$$
\begin{aligned}
\Re\,\mathcal{A}_{2B}(\theta) \quad &\sim \quad \mu^2\sqrt{\pi}\exp[-\frac{q^2}{4\alpha^2}]\Bigg\{ V_{oc}^2\left[\frac{1}{4\alpha^3k_o^2} - \frac{q^2}{8\alpha^5k_o^2}\right] \\
&+ \quad V_{ol}^2\Bigg[\frac{7}{144\alpha^3k_o^2} - \frac{7}{24\alpha^5} - \frac{4k_o^2q^2}{9\alpha^9} + \frac{17k_o^2q^4}{288\alpha^{11}} - \frac{k_o^2q^6}{576\alpha^{13}} \\
&+ \quad \frac{5k_o^2}{9\alpha^7} + \frac{13q^2}{576\alpha^3k_o^4} - \frac{71q^2}{576\alpha^5k_o^2} + \frac{q^2}{288\alpha^7} \\
&+ \quad \frac{25q^4}{3072\alpha^3k_o^6} - \frac{55q^4}{1152\alpha^5k_o^4} + \frac{83q^4}{2304\alpha^7k_o^2} + \frac{23q^4}{576\alpha^9}\Bigg] \\
&+ \quad V_{os}^2\Bigg[-\frac{1}{12\alpha^3k_o^2} + \frac{1}{2\alpha^5} - \frac{q^2}{48\alpha^3k_o^4} + \frac{5q^2}{48\alpha^5k_o^2} - \frac{7q^2}{24\alpha^7} \\
&- \quad \frac{q^4}{256\alpha^3k_o^6} + \frac{5q^4}{192\alpha^5k_o^4} - \frac{q^4}{64\alpha^7k_o^2} + \frac{q^4}{48\alpha^9}\Bigg]\Bigg\}, \qquad (3.5.7)
\end{aligned}
$$

$$
\begin{aligned}
\Im\,\mathcal{A}_{2B}(\theta) \quad &\sim \quad \mu^2\pi\exp[-\frac{q^2}{4\alpha^2}]\Bigg\{ V_{oc}^2\frac{1}{2\alpha^4k_o} + V_{ol}^2\Bigg[\frac{7}{72\alpha^4k_o} - \frac{7k_o}{36\alpha^6} + \frac{7k_oq^2}{144\alpha^8} \\
&- \quad \frac{k_oq^4}{144\alpha^{10}} + \frac{k_oq^6}{1152\alpha^{12}} - \frac{k_o^3q^2}{9\alpha^{10}} + \frac{k_o^3q^4}{144\alpha^{12}} + \frac{2k_o^3}{9\alpha^8} + \frac{13q^2}{288\alpha^4k_o^3} \\
&- \quad \frac{19q^2}{288\alpha^6k_o} + \frac{25q^4}{1536\alpha^4k_o^5} - \frac{7q^4}{288\alpha^6k_o^3} + \frac{7q^4}{384\alpha^8k_o} \\
&+ \quad V_{os}^2\Bigg[-\frac{1}{6\alpha^4k_o} + \frac{k_o}{3\alpha^6} - \frac{k_oq^2}{12\alpha^8} - \frac{q^2}{24\alpha^4k_o^3} \\
&+ \quad \frac{q^2}{24\alpha^6k_o} - \frac{q^4}{128\alpha^4k_o^5} + \frac{q^4}{96\alpha^6k_o^3} - \frac{q^4}{96\alpha^8k_o}\Bigg]\Bigg]\Bigg\}. \qquad (3.5.8)
\end{aligned}
$$

$$
\begin{aligned}
\Re\,\mathcal{B}_{2B}(\theta) \quad &\sim \quad \mu^2\pi\exp[-\frac{q^2}{4\alpha^2}]\Bigg\{ -V_{oc}V_{os}\left[\frac{q}{4\alpha^4k_o^2} - \frac{q}{2\alpha^6} + \frac{q^3}{16\alpha^4k_o^4} - \frac{q^3}{16\alpha^6k_o^2}\right] \\
&- \quad V_{ol}V_{os}\Bigg[-\frac{7q}{48\alpha^4k_o^2} + \frac{7q}{24\alpha^6} - \frac{k_o^2q}{3\alpha^8} + \frac{k_o^2q^3}{24\alpha^{10}} - \frac{5q^3}{96\alpha^4k_o^4} \\
&+ \quad \frac{13q^3}{192\alpha^6k_o^2} - \frac{q^3}{24\alpha^8} - \frac{q^5}{128\alpha^4k_o^6} + \frac{3q^5}{256\alpha^6k_o^4} - \frac{q^5}{96\alpha^8k_o^2} \\
&+ \quad \frac{q^5}{192\alpha^{10}}\Bigg] - V_{ol}^2\Bigg[-\frac{7q}{64\alpha^4k_o^2} + \frac{7q}{32\alpha^6} - \frac{k_o^2q}{4\alpha^8} + \frac{k_o^2q^3}{32\alpha^{10}}
\end{aligned}
$$

$$- \frac{5q^3}{128\alpha^4 k_o^4} + \frac{13q^3}{256\alpha^6 k_o^2} - \frac{q^3}{32\alpha^8} - \frac{3q^5}{512\alpha^4 k_o^6} + \frac{9q^5}{1024\alpha^6 k_o^4}$$

$$+ \frac{q^5}{256\alpha^{10}}\Bigg] + V_{os}^2\left[\frac{q}{16\alpha^4 k_o^2} - \frac{q}{8\alpha^6} + \frac{q^3}{64\alpha^4 k_o^4} - \frac{q^3}{64\alpha^6 k_o^2}\right]\Bigg\}, \quad (3.5.9)$$

$$\Im \mathcal{B}_{2B}(\theta) \sim \mu^2\sqrt{\pi}\exp[-\frac{q^2}{4\alpha^2}]\Bigg\{V_{oc}V_{os}\Bigg[\frac{q}{8\alpha^3 k_o^3} - \frac{3q}{4\alpha^5 k_o} + \frac{q^3}{32\alpha^3 k_o^5}$$

$$- \frac{5q^3}{32\alpha^5 k_o^3} + \frac{q^3}{8\alpha^7 k_o} - \frac{q^5}{64\alpha^5 k_o^5} + \frac{q^5}{64\alpha^7 k_o^3}\Bigg]$$

$$- V_{ol}V_{os}\Bigg[-\frac{7q}{96\alpha^3 k_o^3} + \frac{7q}{16\alpha^5 k_o} - \frac{5k_o q}{6\alpha^7} + \frac{11 k_o q^3}{48\alpha^9}$$

$$- \frac{k_o q^5}{96\alpha^{11}} - \frac{5q^3}{192\alpha^3 k_o^5} + \frac{53q^3}{384\alpha^5 k_o^3} - \frac{5q^3}{96\alpha^7 k_o} - \frac{q^5}{256\alpha^3 k_o^7}$$

$$+ \frac{47q^5}{1536\alpha^5 k_o^5} - \frac{7q^5}{256\alpha^7 k_o^3} + \frac{q^5}{128\alpha^9 k_o}\Bigg]$$

$$+ V_{ol}^2\Bigg[-\frac{7q}{128\alpha^3 k_o^3} + \frac{21q}{64\alpha^5 k_o} - \frac{5k_o q}{8\alpha^7} + \frac{11k_o q^3}{64\alpha^9}$$

$$- \frac{k_o q^5}{128\alpha^{11}} - \frac{5q^3}{256\alpha^3 k_o^5} + \frac{53q^3}{512\alpha^5 k_o^3} - \frac{5q^3}{128\alpha^7 k_o}$$

$$- \frac{3q^5}{1024\alpha^3 k_o^7} + \frac{47q^5}{2048\alpha^5 k_o^5} - \frac{21q^5}{1024\alpha^7 k_o^3} + \frac{3q^5}{512\alpha^9 k_o}\Bigg]$$

$$+ V_{os}^2\Bigg[\frac{q}{32\alpha^3 k_o^3} - \frac{3q}{16\alpha^5 k_o} + \frac{q^3}{128\alpha^3 k_o^5} - \frac{5q^3}{128\alpha^5 k_o^3}$$

$$+ \frac{q^3}{32\alpha^7 k_o} - \frac{q^5}{256\alpha^5 k_o^5} + \frac{q^5}{256\alpha^7 k_o^3}\Bigg]\Bigg\}. \quad (3.5.10)$$

$$\Re \mathcal{C}_{2B}^k(\theta) \sim \mu^2\sqrt{\pi}\exp[-\frac{q^2}{4\alpha^2}]\Bigg\{-V_{oc}V_{ol}\Bigg[\frac{1}{8\alpha^3 k_o^2} - \frac{3}{4\alpha^5}$$

$$+ \frac{3q^2}{64\alpha^3 k_o^4} - \frac{q^2}{4\alpha^5 k_o^2} + \frac{q^2}{8\alpha^7}$$

$$+ \frac{3q^4}{256\alpha^3 k_o^6} - \frac{9q^4}{128\alpha^5 k_o^4} + \frac{q^4}{32\alpha^7 k_o^2}\Bigg]$$

$$+ V_{ol}V_{os}\Bigg[\frac{3}{16\alpha^3 k_o^2} - \frac{9}{8\alpha^5} + \frac{9q^2}{128\alpha^3 k_o^4} - \frac{3q^2}{8\alpha^5 k_o^2}$$

$$+ \frac{3q^2}{16\alpha^7} + \frac{9q^4}{512\alpha^3 k_o^6} - \frac{27q^4}{256\alpha^5 k_o^4} + \frac{3q^4}{64\alpha^7 k_o^2}\Bigg]$$

$$+ \quad V_{os}^2 \left[\frac{1}{16\alpha^3 k_o^2} - \frac{3}{8\alpha^5} + \frac{3q^2}{128\alpha^3 k_o^4} - \frac{q^2}{8\alpha^5 k_o^2}\right.$$

$$+ \quad \left.\frac{q^2}{16\alpha^7} + \frac{3q^4}{512\alpha^3 k_o^6} - \frac{9q^4}{256\alpha^5 k_o^4} + \frac{q^4}{64\alpha^7 k_o^2}\right]$$

$$+ \quad V_{ol}^2 \left[\frac{19}{192\alpha^3 k_o^2} - \frac{19}{32\alpha^5} - \frac{11k^2 q^2}{96\alpha^9} + \frac{k^2 q^4}{192\alpha^{11}}\right.$$

$$+ \quad \left.\left.\frac{5k_o^2}{12\alpha^7} + \frac{21q^2}{512\alpha^3 k_o^4} - \frac{85q^2}{384\alpha^5 k_o^2} + \frac{9q^2}{64\alpha^7}\right]\right\}, \qquad (3.5.11)$$

$$\Im C_{2B}^k(\theta) \sim \mu^2 \pi \exp[-\frac{q^2}{4\alpha^2}] \left\{ V_{oc} V_{ol} \left[\frac{1}{4\alpha^4 k_o} - \frac{k_o}{2\alpha^6} + \frac{3q^2}{32\alpha^4 k_o^3}\right.\right.$$

$$- \quad \left.\frac{q^2}{8\alpha^6 k_o} + \frac{3q^4}{128\alpha^4 k_o^5} - \frac{q^4}{32\alpha^6 k_o^3}\right]$$

$$- \quad V_{ol} V_{os} \left[\frac{3}{8\alpha^4 k_o} - \frac{3k_o}{4\alpha^6} + \frac{9q^2}{64\alpha^4 k_o^3} - \frac{3q^2}{16\alpha^6 k_o}\right.$$

$$+ \quad \left.\frac{9q^4}{256\alpha^4 k_o^5} - \frac{3q^4}{64\alpha^6 k_o^3}\right] + V_{os}^2 \left[\frac{1}{8\alpha^4 k_o} - \frac{k_o}{4\alpha^6}\right.$$

$$+ \quad \left.\frac{3q^2}{64\alpha^4 k_o^3} - \frac{q^2}{16\alpha^6 k_o} + \frac{3q^4}{256\alpha^4 k_o^5} - \frac{q^4}{64\alpha^6 k_o^3}\right]$$

$$+ \quad V_{ol}^2 \left[\frac{19}{96\alpha^4 k_o} - \frac{19k_o}{48\alpha^6} + \frac{kq^2}{24\alpha^8} - \frac{kq^4}{192\alpha^{10}}\right.$$

$$- \quad \frac{k^3 q^2}{48\alpha^{10}} + \frac{k^3}{6\alpha^8} + \frac{21q^2}{256\alpha^4 k_o^3} - \frac{11q^2}{96\alpha^6 k_o}$$

$$+ \quad \left.\left.\frac{3q^4}{128\alpha^4 k_o^5} - \frac{25q^4}{768\alpha^6 k_o^3} + \frac{5q^4}{384\alpha^8 k_o}\right]\right\}. \qquad (3.5.12)$$

$$\Re C_{2B}^n(\theta) \sim \mu^2 \sqrt{\pi} \exp[\frac{-q^2}{4\alpha^2}] \left\{ V_{oc} V_{ol} \left[-\frac{q^2}{16\alpha^3 k_o^4} + \frac{3q^2}{8\alpha^5 k_o^2}\right.\right.$$

$$- \quad \left.\frac{5q^2}{8\alpha^7} - \frac{7q^4}{64\alpha^5 k_o^4} + \frac{3q^4}{32\alpha^7 k_o^2} + \frac{q^4}{16\alpha^9}\right]$$

$$+ \quad V_{ol} V_{os} \left[-\frac{3q^2}{32\alpha^3 k_o^4} + \frac{9q^2}{16\alpha^5 k_o^2} - \frac{15q^2}{16\alpha^7} - \frac{21q^4}{128\alpha^5 k_o^4}\right.$$

$$+ \quad \left.\frac{9q^4}{64\alpha^7 k_o^2} + \frac{3q^4}{32\alpha^9}\right] + V_{os}^2 \left[-\frac{q^2}{32\alpha^3 k_o^4} + \frac{3q^2}{16\alpha^5 k_o^2} - \frac{5q^2}{16\alpha^7}\right.$$

$$- \quad \left.\frac{7q^4}{128\alpha^5 k_o^4} + \frac{3q^4}{64\alpha^7 k_o^2} + \frac{q^4}{32\alpha^9}\right] + V_{ol}^2 \left[\frac{7k^2 q^2}{16\alpha^9}\right.$$

$$- \frac{5k^2q^4}{64\alpha^{11}} + \frac{k^2q^6}{384\alpha^{13}} - \frac{11q^2}{192\alpha^3 k_o^4} + \frac{11q^2}{32\alpha^5 k_o^2}$$
$$- \frac{91q^2}{192\alpha^7} - \frac{343q^4}{1536\alpha^5 k_o^4} + \frac{15q^4}{64\alpha^7 k_o^2} - \frac{13q^4}{96\alpha^9}\Bigg]\Bigg\} , \qquad (3.5.13)$$

$$\begin{aligned}
\Im C_{2B}^{n}(\theta) \sim\ & \mu^2 \pi \exp[\frac{-q^2}{4\alpha^2}] \Bigg\{ V_{oc} V_{ol} \Bigg[-\frac{kq^2}{4\alpha^8} - \frac{q^2}{8\alpha^4 k_o^3} \\
& + \frac{q^2}{4\alpha^6 k_o} + \frac{3q^4}{64\alpha^4 k_o^5} - \frac{3q^4}{32\alpha^6 k_o^3} + \frac{q^4}{16\alpha^8 k_o}\Bigg] \\
& + V_{ol} V_{os} \Bigg[-\frac{3kq^2}{8\alpha^8} - \frac{3q^2}{16\alpha^4 k_o^3} + \frac{3q^2}{8\alpha^6 k_o} \\
& + \frac{9q^4}{128\alpha^4 k_o^5} - \frac{9q^4}{64\alpha^6 k_o^3} + \frac{3q^4}{32\alpha^8 k_o}\Bigg] \\
& + V_{os}^2 \Bigg[-\frac{kq^2}{8\alpha^8} - \frac{q^2}{16\alpha^4 k_o^3} + \frac{q^2}{8\alpha^6 k_o} \\
& + \frac{3q^4}{128\alpha^4 k_o^5} - \frac{3q^4}{64\alpha^6 k_o^3} + \frac{q^4}{32\alpha^8 k_o}\Bigg] \\
& + V_{ol}^2 \Bigg[-\frac{23kq^2}{96\alpha^8} - \frac{3kq^4}{64\alpha^{10}} + \frac{kq^6}{192\alpha^{12}} \\
& + \frac{k^3q^2}{8\alpha^{10}} - \frac{k^3q^4}{96\alpha^{12}} - \frac{11q^2}{96\alpha^4}3 + \frac{11q^2}{48\alpha^6} \\
& + \frac{43q^4}{512\alpha^4 k_o^5} - \frac{43q^4}{256\alpha^6 k_o^3} + \frac{13q^4}{96\alpha^8 k_o}\Bigg]\Bigg\} . \qquad (3.5.14)
\end{aligned}$$

(3.6) <u>Numerical calculations</u>

In this section we present the results of the numerical calculations of the scattering amplitudes eqns (3.3.25-28) together with the second-Born contribution to the exact amplitudes, which were calculated from the program DDTP[6] [25]. As mentioned before, the numerical calculation of the amplitudes (3.3.25-28)

[6] See the introduction page 12-13.

was carried out using the SMP package [36].

The DDTP program calculates the amplitudes to all powers in the potential strength, we therefore have to develop a technique for picking the second Born terms. Now the second-Born term is quadratic in potential strength and hence is the same in the case of an attractive or repulsive potential on the other hand the first Born changes sign. It follows that

$$\mathcal{F}(\theta)_{2B} \sim \left\{\mathcal{F}^{+}(\theta) + \mathcal{F}^{-}(\theta)\right\}_{smallV_o}, \qquad (3.6.1)$$

where the superscript +, − represents attractive and repulsive potential strength respectively. To ensure that no higher-order (even) terms contribute, we repeated the calculation varying the potential strength until the variation in the amplitude corresponded to the square of the variation of the potential. We selected the different coefficients in the amplitudes (3.3.25-28) through a procedure which can be represented as follows, e.g.the case of V_{os}^2 and $V_{oc}V_{os}$,

$$\mathcal{F}_{2B}(\theta)|_{V_{os}^2 contribution} = \frac{1}{2}\left\{\mathcal{F}^{+}_{(V_{ol},V_{oc}=0)} + \mathcal{F}^{-}_{(V_{ol},V_{oc}=0)}\right\}, \qquad (3.6.2)$$

$$\mathcal{F}_{2B}(\theta)|_{V_{oc}V_{os} contribution} = \frac{1}{2}\left\{\mathcal{F}^{+}_{(V_{ol}=0)} - \mathcal{F}^{+}_{(V_{ol},V_{oc}=0)}\right\}, \qquad (3.6.3)$$

and similarly for the other coefficients.

As a model for our Gaussian potential we choose the case of an α-particle. The following are the parameters used; reduced mass $\mu = 6.542$ fm, wave number $k = 2.927\ fm^{-1}$, potential strength $V_o = 0.005\ fm^{-1}$, and range $a = 2.149$ fm. The figures presented in the next few pages show good agreement between our calculations and the DDTP ones. Note however that the DDTP results start showing rounding errors for small values of potential strength.

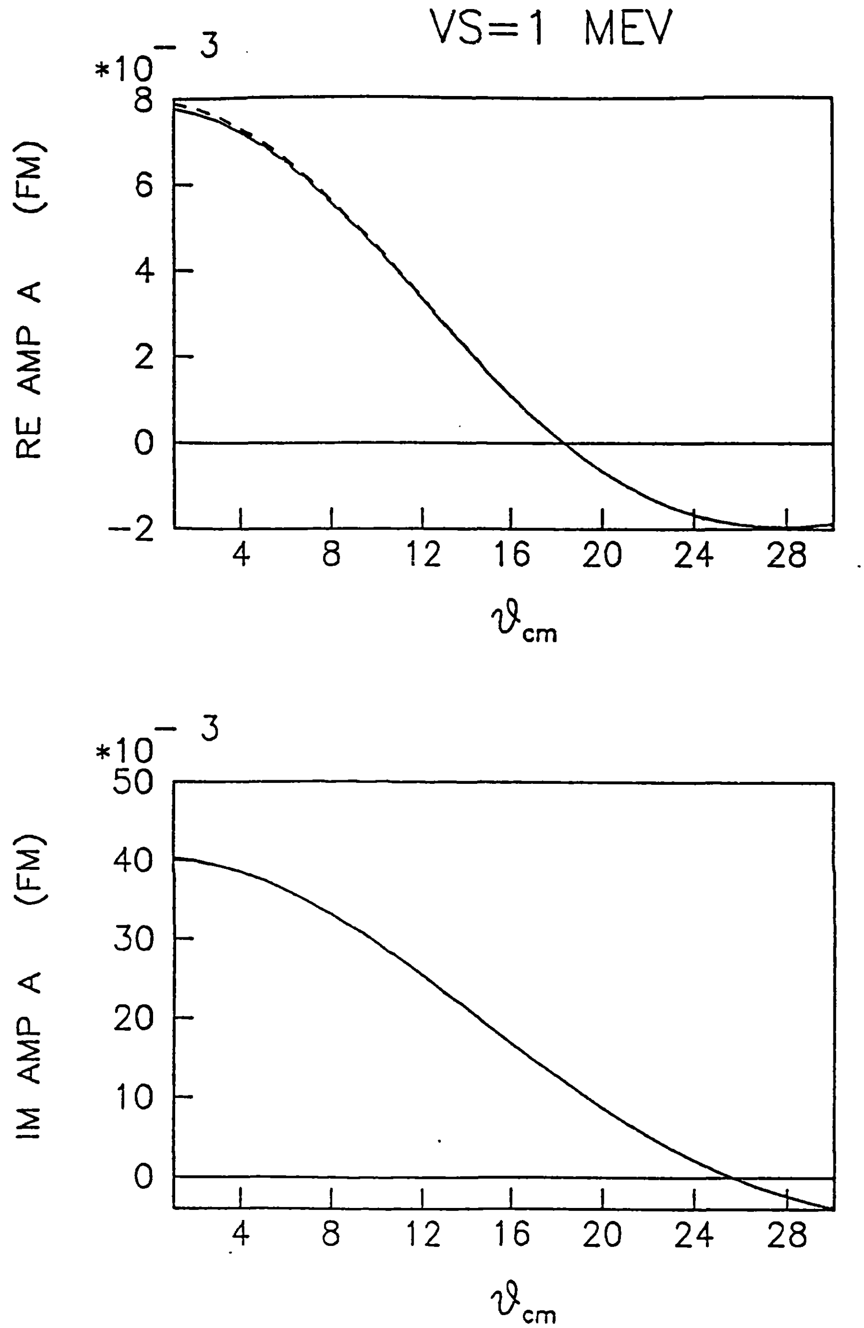

Figure (3.1). The contribution of the coefficient of V_{os}^2 to the scattering amplitude in the case of d–α scattering at $k = 2.927\ fm^{-1}$. The dashed line is the amplitude calculated using the code DDTP and the solid line is the amplitude (3.3.25) calculated numerically using SMP.

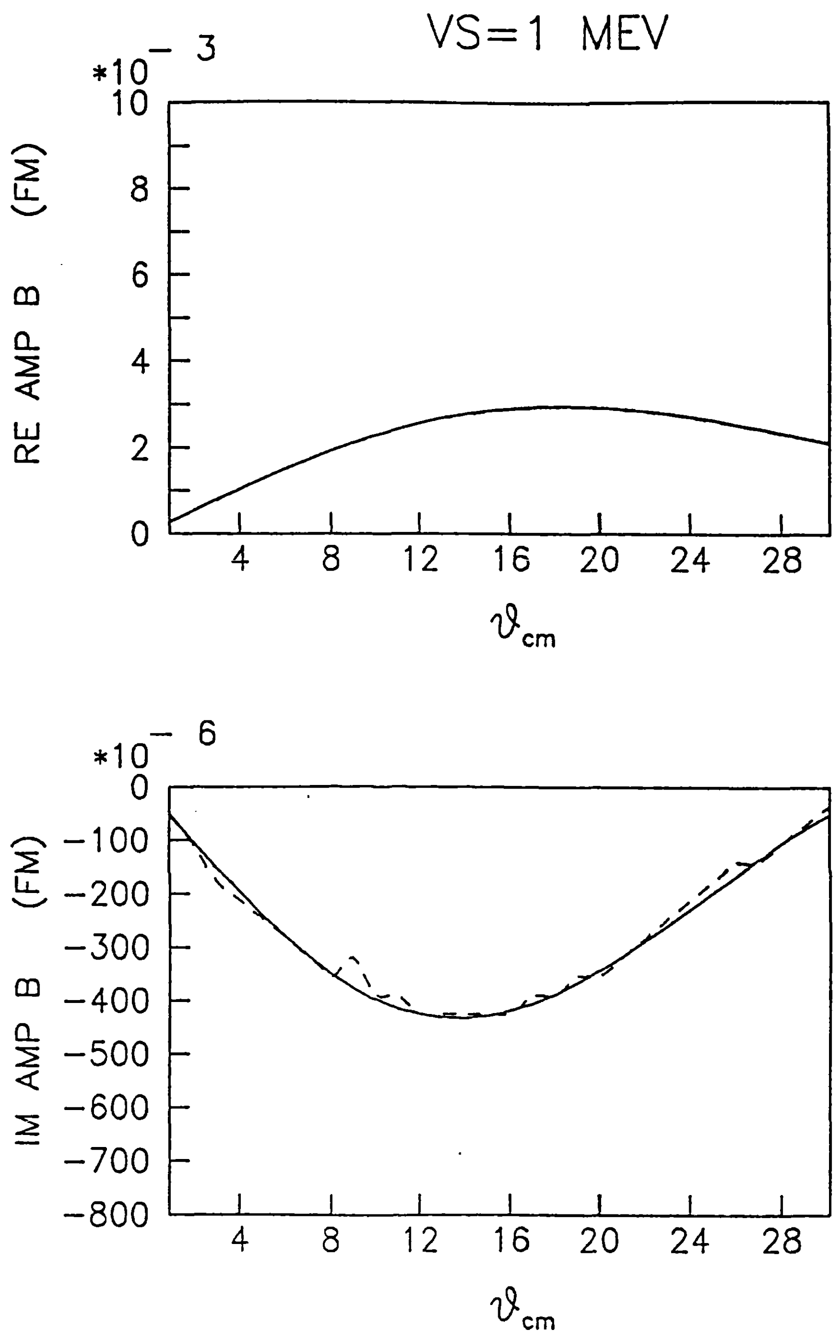

Figure (3.2). Solid & dashed lines represent the calculation of (3.3.26) and the contribution from the exact respectively. Parameters are as for figure (3.1).

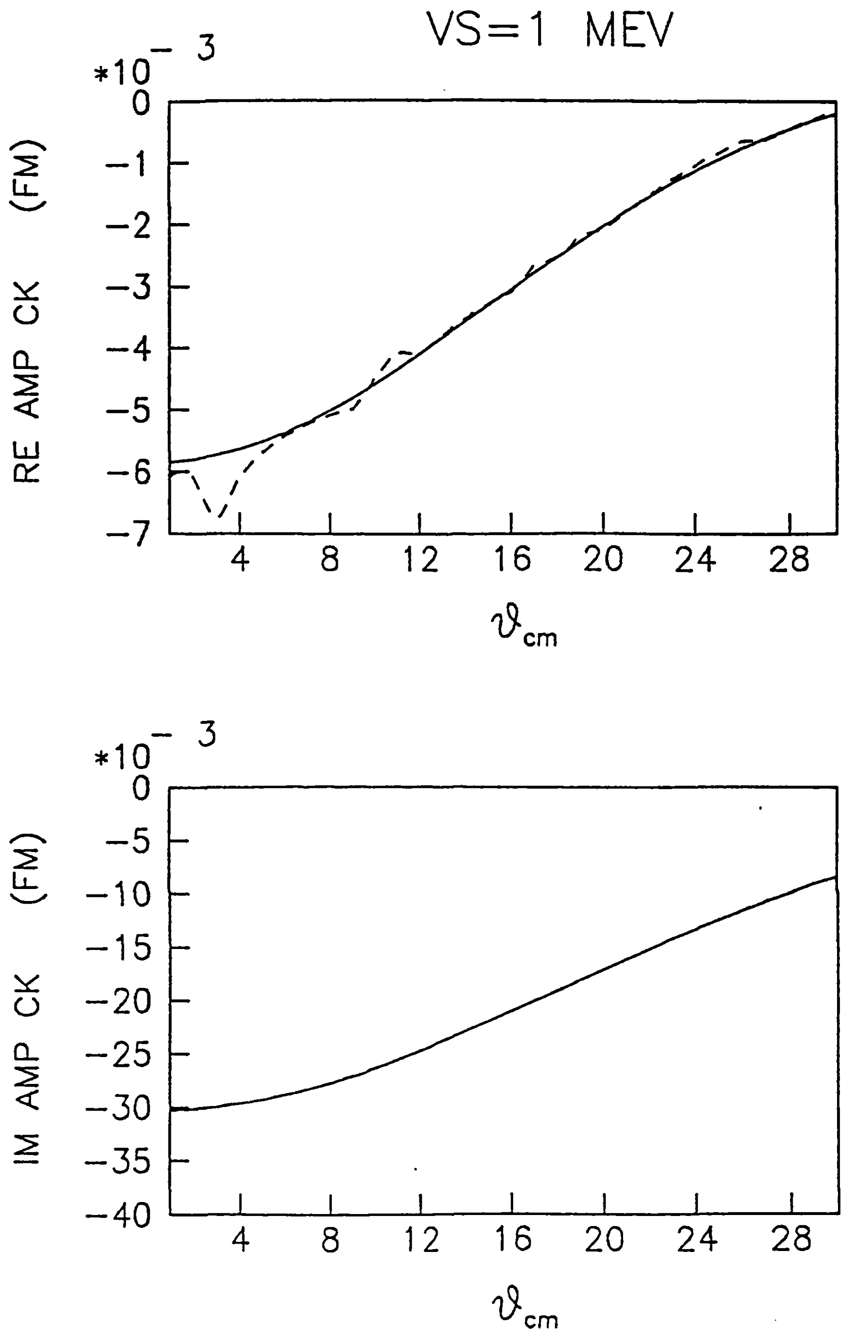

Figure (3.3). Solid & dashed lines represent the calculation of (3.3.27) and the contribution from the exact respectively. Parameters are as for figure (3.1).

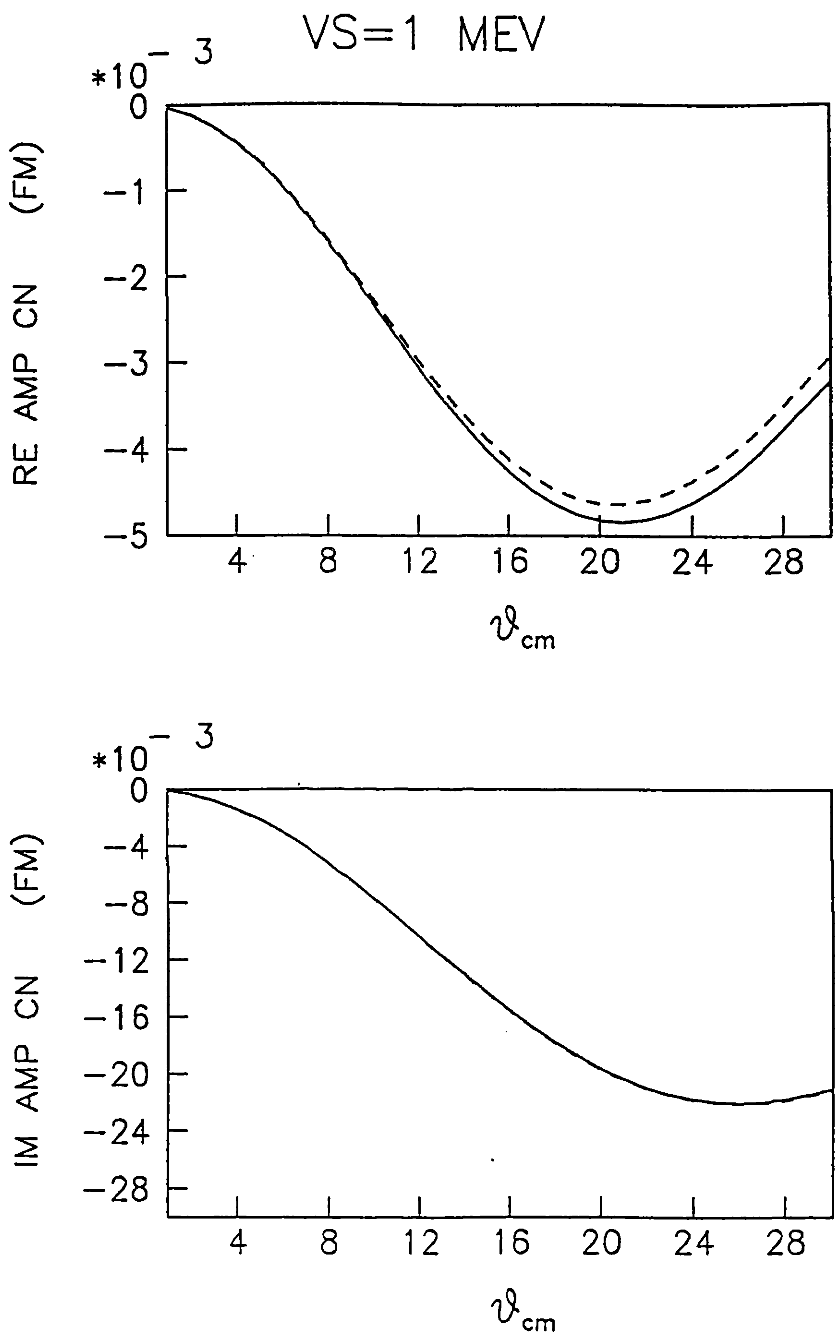

Figure (3.4). Solid & dashed lines represent the calculation of (3.3.28) and the contribution from the exact respectively. Parameters are as for figure (3.1).

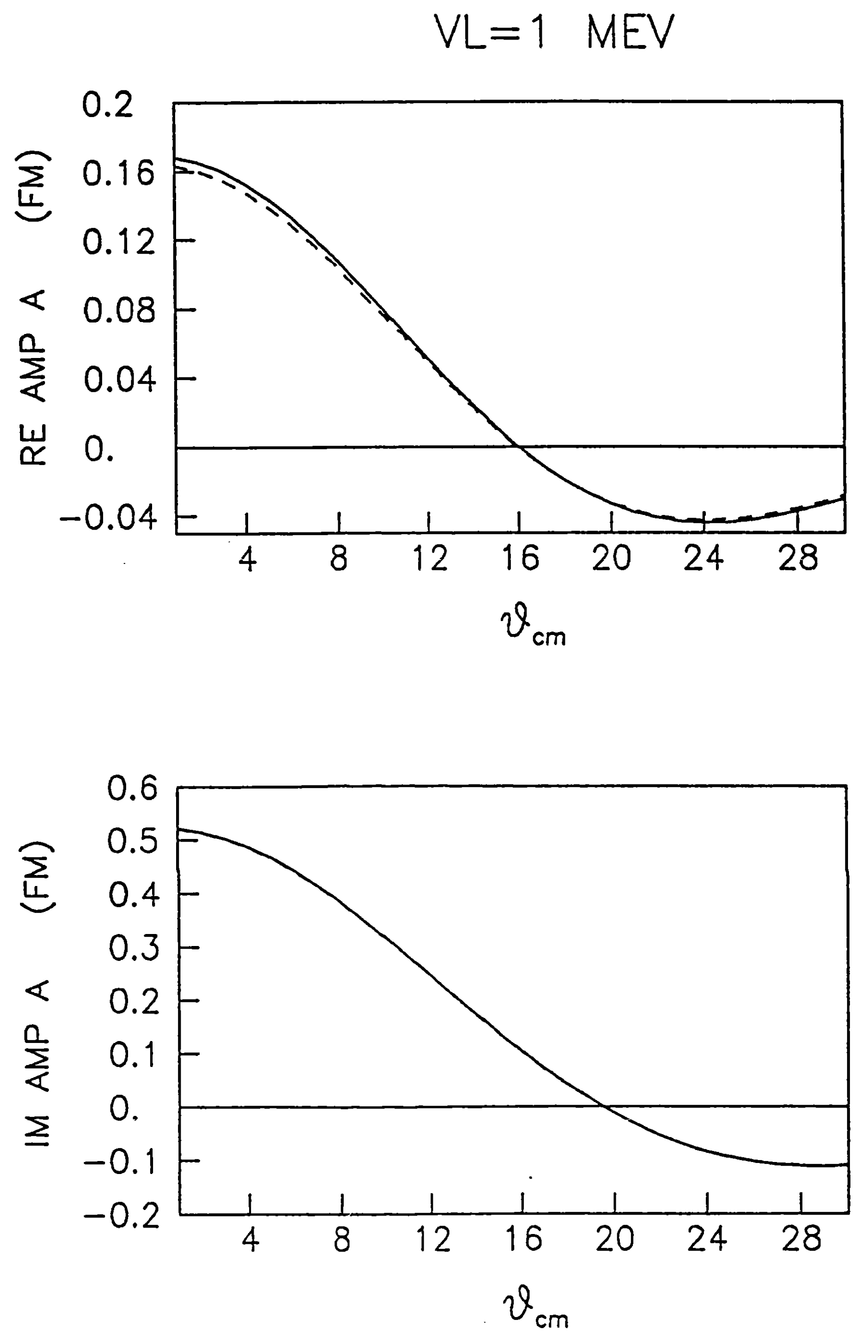

Figure (3.5). The contribution of the coefficient of V_{ol}^2 to the scattering amplitude in the case of d–α scattering at $k = 2.927\ fm^{-1}$. The dashed line is the amplitude calculated using the code DDTP and the solid line is the amplitude (3.3.25) calculated numerically using SMP.

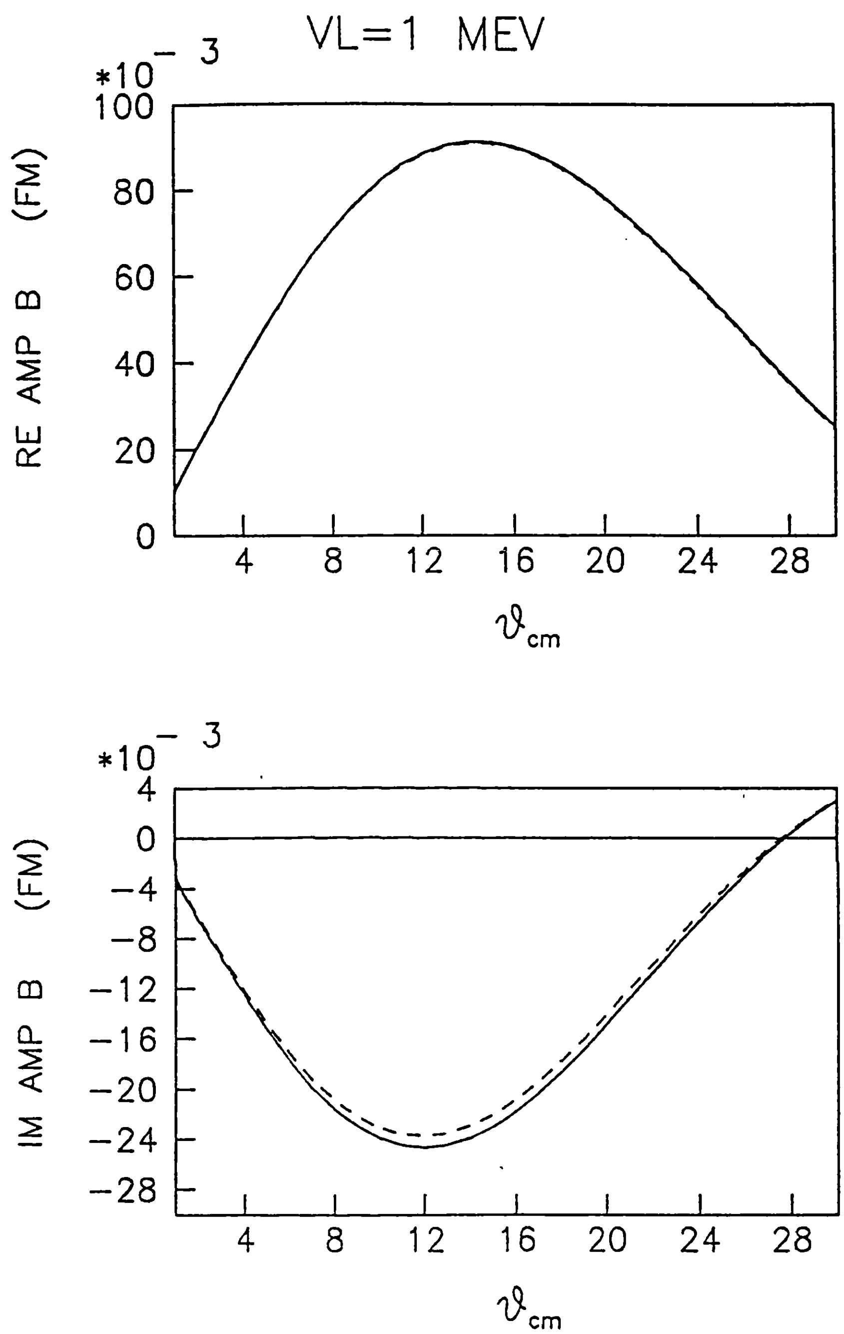

Figure (3.6). Solid & dashed lines represent the calculation of (3.3.26) and the contribution from the exact respectively. Parameters are as for figure (3.5).

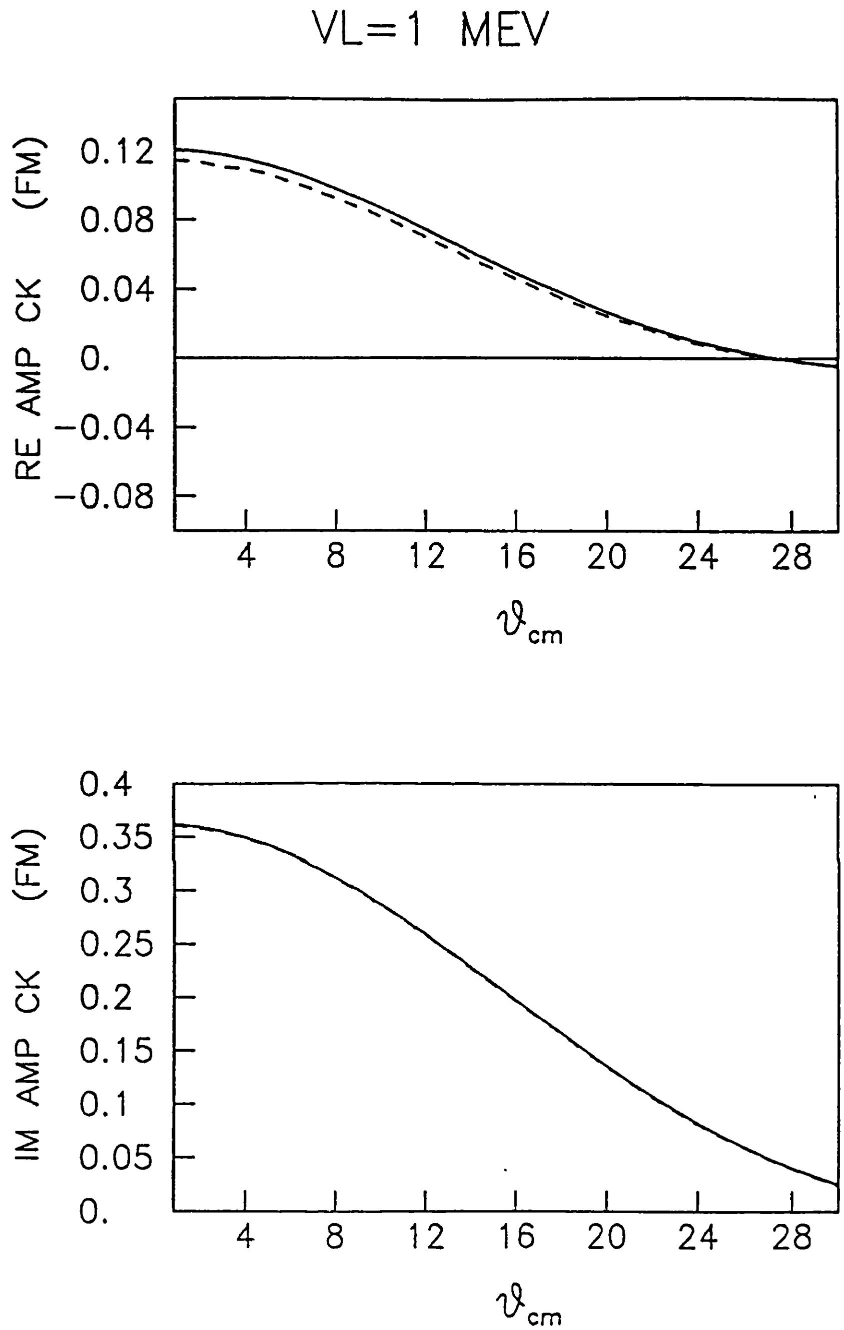

Figure (3.7). Solid & dashed lines represent the calculation of (3.3.27) and the contribution from the exact respectively. Parameters are as for figure (3.5).

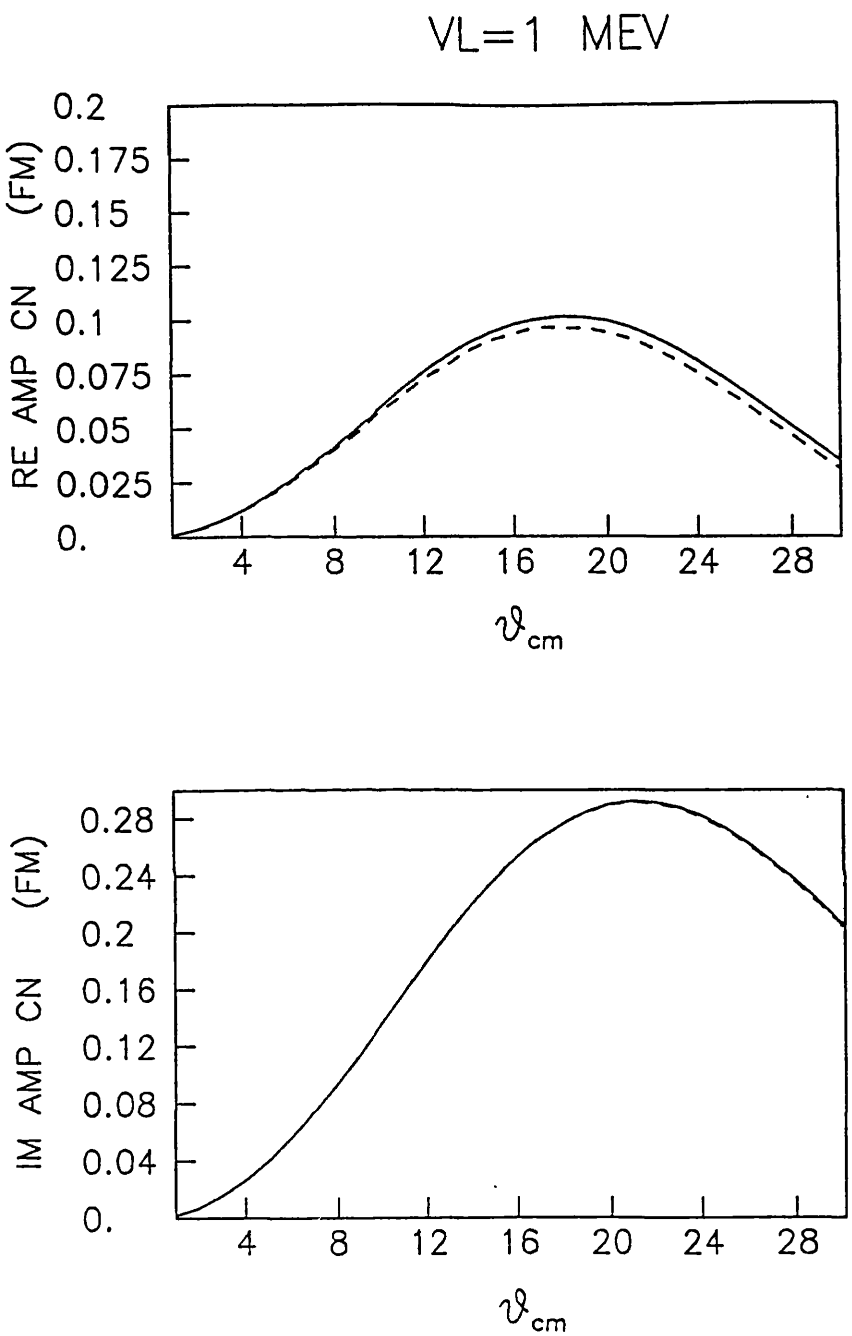

Figure (3.8). Solid & dashed lines represent the calculation of (3.3.28) and the contribution from the exact respectively. Parameters are as for figure (3.5).

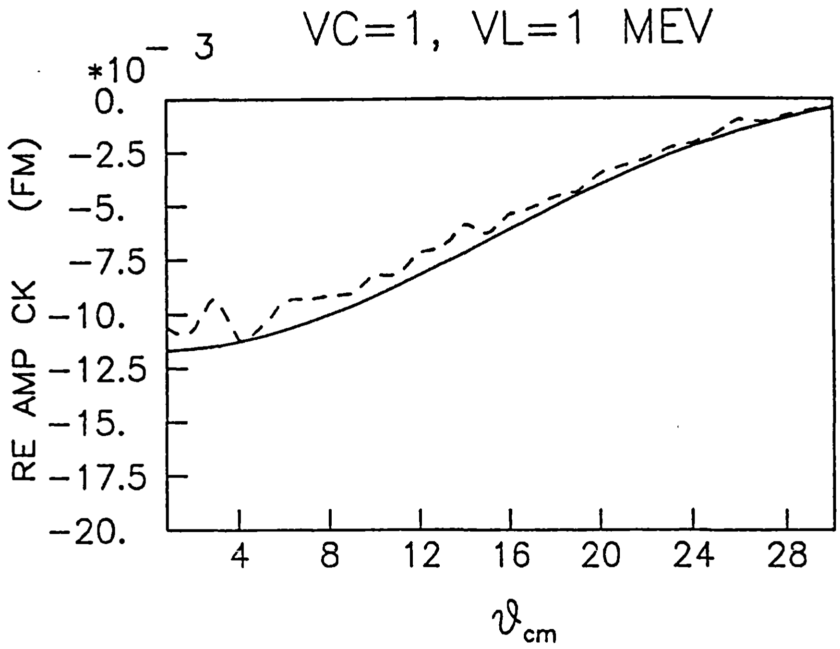

Figure (3.9). The contribution of the coefficient of $V_{oc}V_{ol}$ to the scattering amplitude in the case of d–α scattering at $k = 2.927\ fm^{-1}$. The dashed line is the amplitude calculated using the code DDTP and the solid line is the amplitude (3.3.27) calculated numerically using SMP.

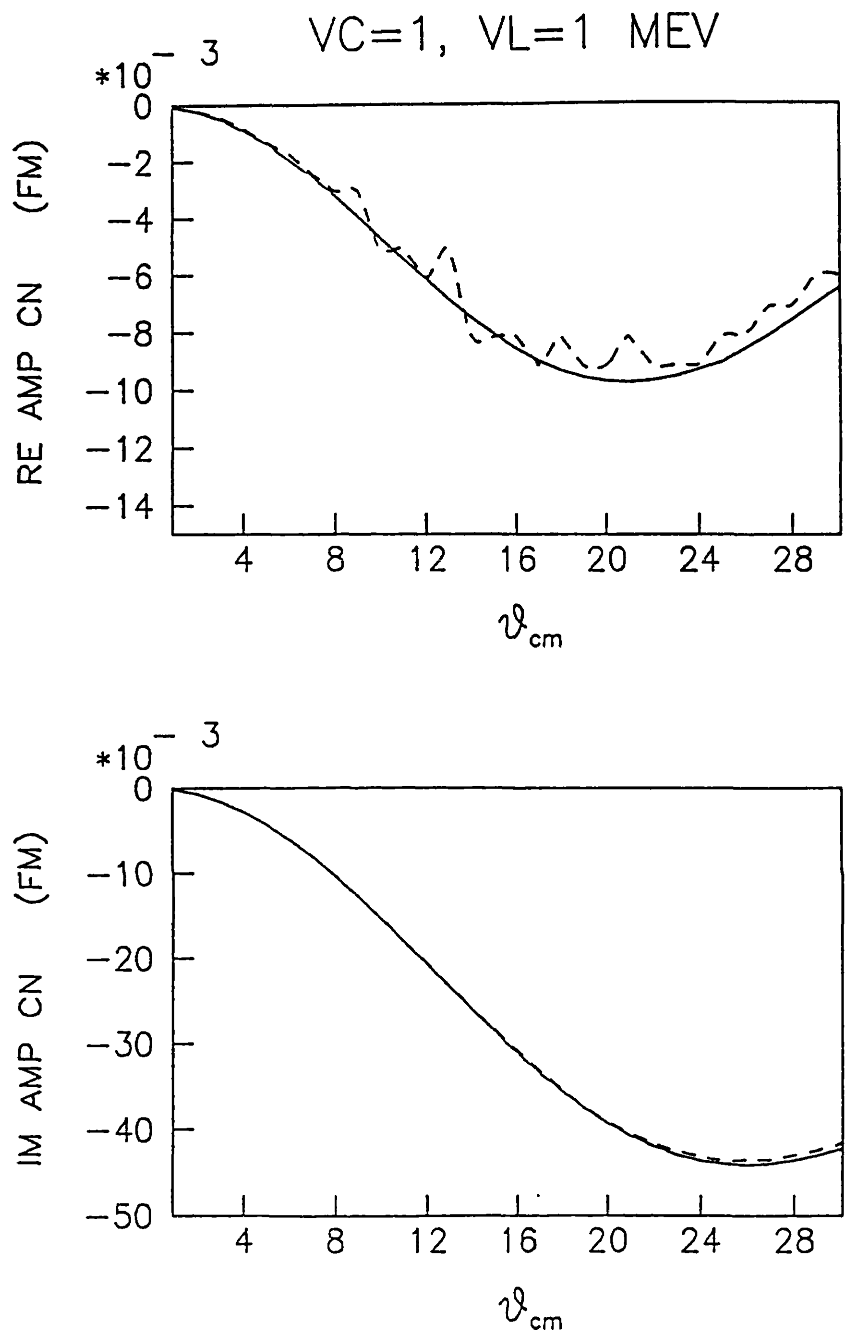

Figure (3.10). Solid & dashed lines represent the calculation of (3.3.28) and the contribution from the exact respectively. Parameters are as for figure (3.9).

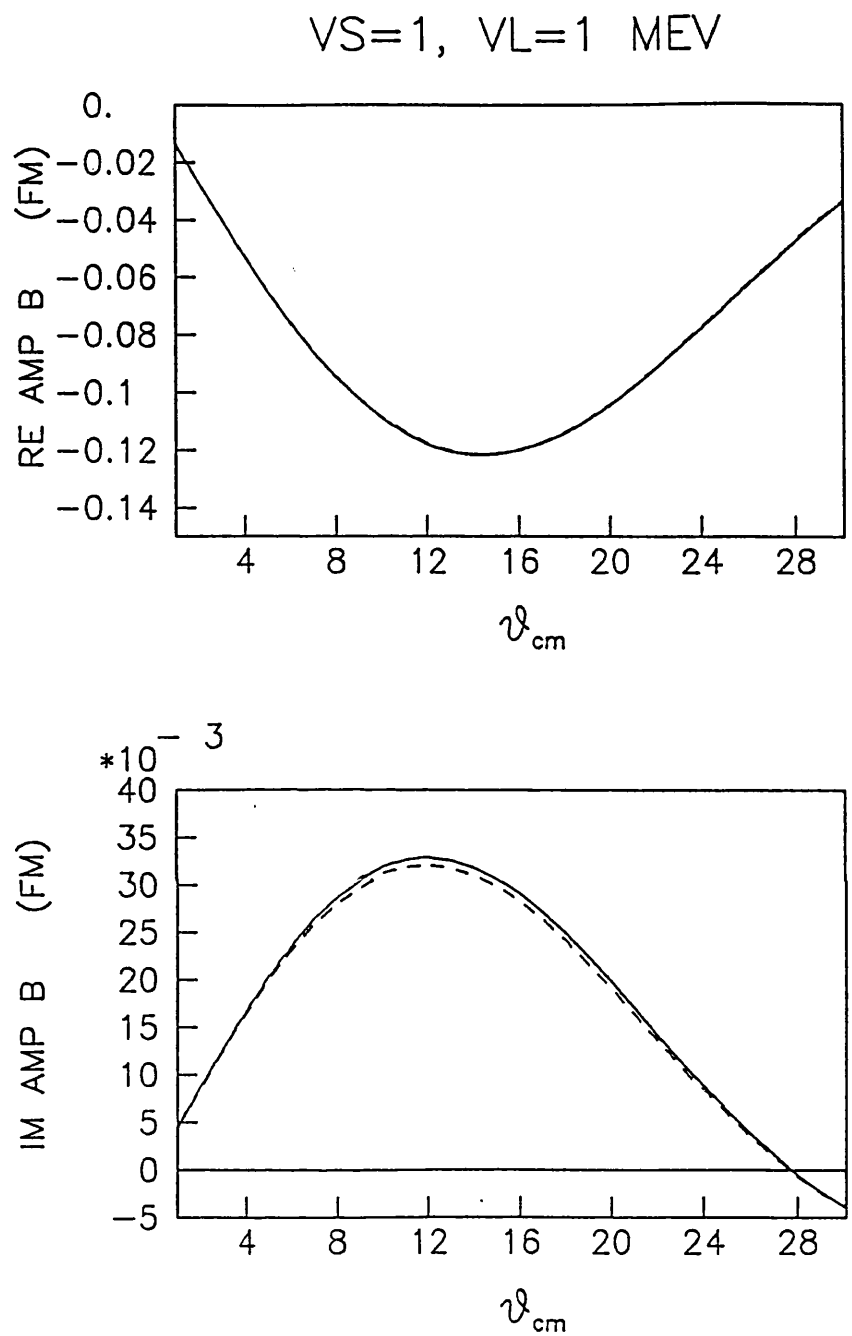

Figure (3.11). The contribution of the coefficient of $V_{os}V_{ol}$ to the scattering amplitude in the case of d–α scattering at $k = 2.927\ fm^{-1}$. The dashed line is the amplitude calculated using the code DDTP and the solid line is the amplitude (3.3.26) calculated numerically using SMP.

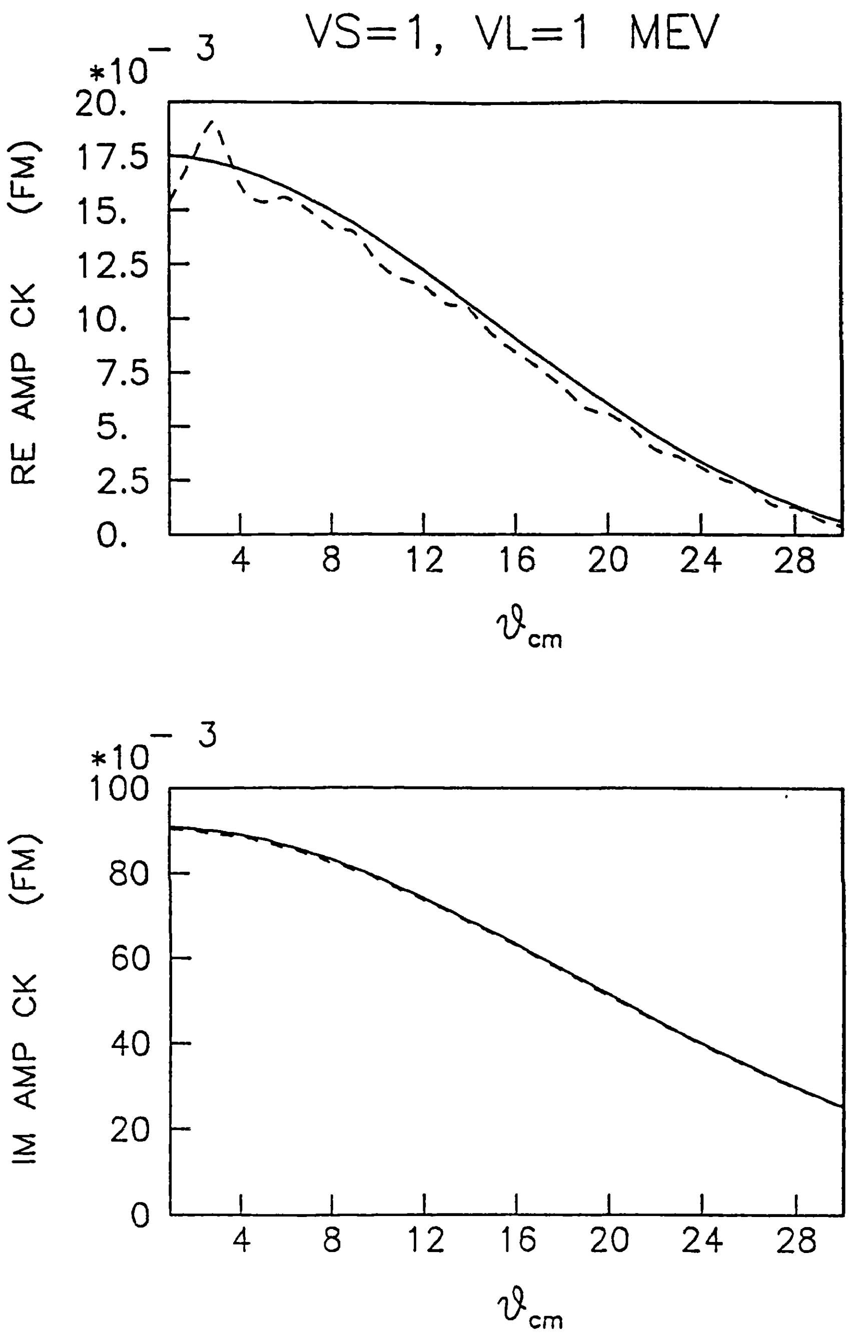

Figure (3.12). Solid & dashed lines represent the calculation of (3.3.27) and the contribution from the exact respectively. Parameters are as for figure (3.11).

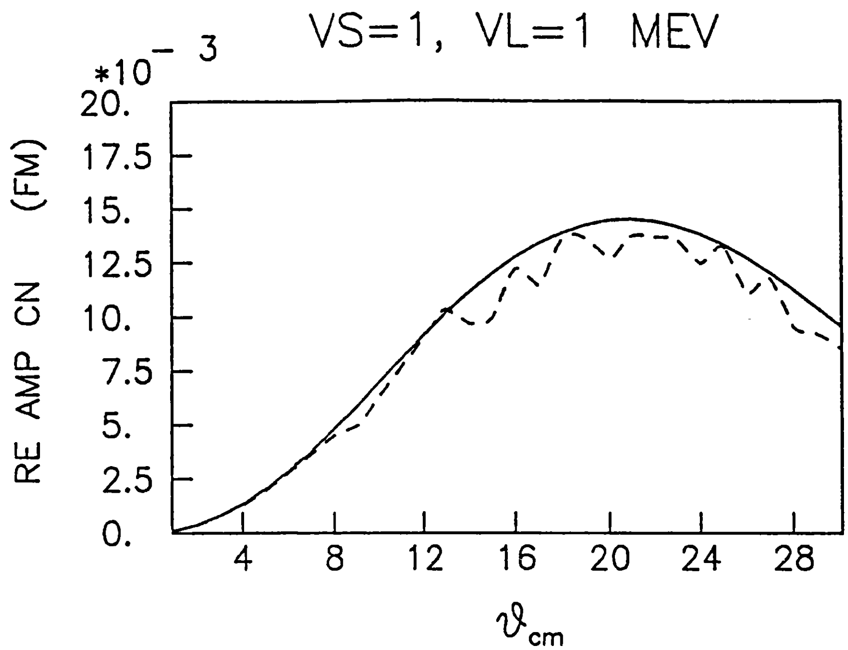

Figure (3.13). Solid & dashed lines represent the calculation of (3.3.28) and the contribution from the exact respectively. Parameters are as for figure (3.11).

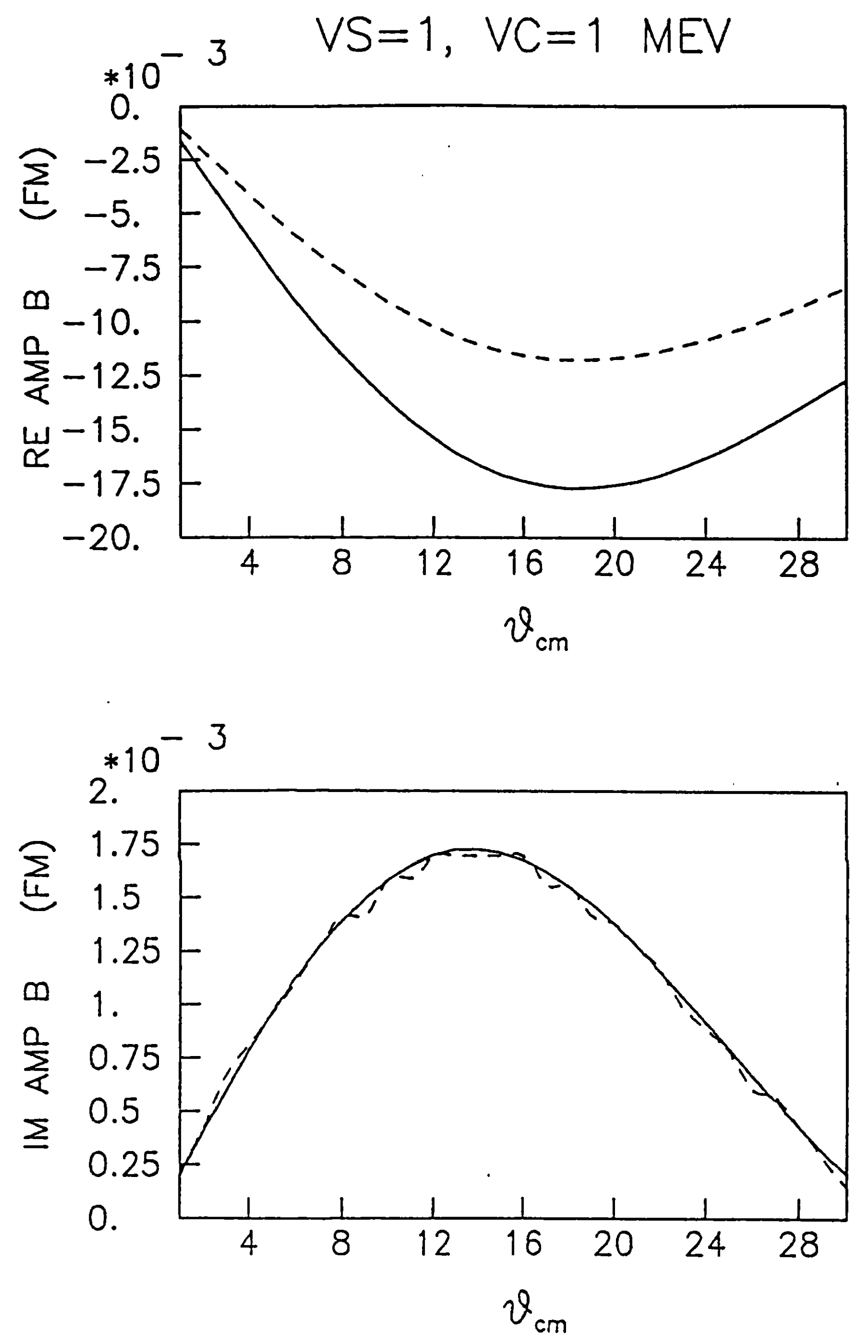

Figure (3.14). The contribution of the coefficient of $V_{oc}V_{os}$ to the scattering amplitude in the case of d–α scattering at $k = 2.927\ fm^{-1}$. The dashed line is the amplitude calculated using the code DDTP and the solid line is the amplitude (3.3.26) calculated numerically using SMP.

CHAPTER 4

THE GLAUBER SCHEME

(4.1) Introduction

The Glauber theory [9,10] provides a very useful approximation to the scattering amplitude in the high energy limit. In this limit, the energy of the incident particle greatly exceeds the magnitude of the interaction potential. The other condition that typifies the Glauber approximation is that the reduced wavelength of the particle is assumed to be much smaller than the potential width (a). In summary one needs $V_o/E \ll 1$ and $ka \gg 1$ and under these conditions the scattering is peaked around the forward direction.

Starting from the Lippmann-Schwinger eqn (3.1.5), the Glauber approximation assumes the separability of the wavefunction into a product of the incident plane wave and a function modulating it,

$$\Psi_{k_i}(\vec{r}) \;=\; \exp[i\vec{k}_i \cdot \vec{r}]\;\Phi(\vec{r})\,, \qquad (4.1.1)$$

$\Phi(\vec{r})$ is a smoothly varying function, which is defined such that it satisfies the condition,

$$|k\;\Phi(\vec{r})| \;\gg\; |\nabla\;\Phi(\vec{r})|\,. \qquad (4.1.2)$$

Substituting eqn (4.1.1) into eqn (3.1.6), we get

$$\Phi^{sc}(\vec{r}) \;=\; \int d\vec{r'}\;\exp[-i\vec{k}_i \cdot \vec{r}]\,G^{out}(\vec{r},\vec{r'})\,V(\vec{S},\vec{r'})\;\exp[i\vec{k}_i \cdot \vec{r'}]\;\Phi(\vec{r'}) \qquad (4.1.3)$$

We will limit the tensor contribution of $V(\vec{S},\vec{r'})$ to $V_L(\vec{r'})\,\vec{S}_2\cdot\vec{R}_2(\bar{L},\bar{L})$. Letting the momentum operators in the potential act on the wave function and using eqn (4.1.2), we arrive at.

$$\Phi^{sc}(\vec{r}) \;=\; \frac{-\mu}{2\pi}\int d\vec{u}\,\frac{1}{u}\exp[ik\,u(1-\omega)]\;\mathring{A}(\vec{r}-\vec{u}) \qquad (4.1.4)$$

where

$$\mathring{A}(\vec{r}) \;=\; \Phi(\vec{r})\,\Big\{V_c(\vec{r}) + V_s(\vec{r})(\vec{S}\cdot\vec{r}\times\vec{k}_i) + V_L(\vec{r})\Big[\vec{S}_2\cdot\vec{R}_2(\vec{r}\times\vec{k}_i,\vec{r}\times\vec{k}_i) - i\,\vec{S}_2\cdot\vec{R}_2(\vec{k}_i,\vec{r})\Big]\Big\} \qquad (4.1.5)$$

and we have made the substitutions.

$$\vec{u} \;=\; \vec{r}-\vec{r'}\;,\;\omega \;=\; \hat{k}_i\cdot\hat{u} \qquad (4.1.6)$$

Integrating over ω by parts and keeping only the leading term,

$$\Phi^{sc}(\vec{r}) \;=\; \frac{-\mu}{2\pi}\int du\,d\phi\left[\frac{\exp[i\,k\,u(1-\omega)]}{i\,k}\;\mathring{A}(\vec{r}-\vec{u})\right]_{-1}^{1} + O(\frac{1}{kd}), \qquad (4.1.7)$$

where ϕ is the azimuthal angle of $\vec{u}$. Note that $\mathring{A}(\vec{r})$ varies appreciably only within the distance, d, which is assumed to be much larger than $1/k$. The main contribution to this integral is when $\omega = 1$. This corresponds to $\vec{k}_i$ and $\vec{u}$ being parallel. In this case

$$\Phi^{sc}(\vec{r}) \;=\; \frac{-i}{v}\int_0^\infty du\,\mathring{A}(\vec{r}-\vec{u}) \qquad (4.1.8)$$

where $v = k/\mu$. If we define the Z-direction to be along that of the initial momentum it follows that $\vec{u} = (0,0,z')$, hence

$$\vec{r} - \vec{u} \;=\; \vec{b} + \hat{k}_i\,(z-z') \qquad (4.1.9)$$

and

$$\Phi^{sc}(\vec{r}) = \frac{-i}{v} \int_{-\infty}^{z} dz' \,\mathring{A}(\vec{b} + z'\hat{k}_i) \qquad (4.1.10)$$

The solution of the subsidiary Lippmann-Schwinger equation for $\Phi(\vec{r})$ is now,

$$\Phi(\vec{r}) = \exp\left[-\frac{i}{v} \int_{-\infty}^{z} dz' \,\mathring{A}(\vec{b} + z'\hat{k}_i)\right] . \qquad (4.1.11)$$

Substituting this in eqn (4.1.1) and using the result in eqn (3.1.4), we arrive at the scattering amplitude

$$\mathcal{F}(\hat{k}_f \cdot \hat{k}_i) = \frac{-\mu}{2\pi} \int d\vec{r} \exp[-i\vec{k}_f \cdot \vec{r}] \,V(\vec{s},\vec{r}) \exp\left[i\vec{k}_i \cdot \vec{r} - \frac{i}{v} \int_{-\infty}^{z} d\acute{z} \,\mathring{A}(\vec{b} + \acute{z}\hat{k}_i)\right] . \qquad (4.1.12)$$

In what follows we will assume that the scattering angle is sufficiently small such that the replacement $(\vec{k}_i - \vec{k}_f) \rightarrow \vec{q} = (q, 0, 0)$ is justified. Carrying out the Z-integration, which is an exact-differential, and using the boundary condition $\Phi(-\infty) = 1$, we get

$$\mathcal{F}(\hat{k}_f \cdot \hat{k}_i) = \frac{ik}{2\pi} \int d\vec{b} \,e^{i\vec{q}\cdot\vec{b}} \left\{1 - \exp\left[-\frac{i}{v} \int_{-\infty}^{\infty} \mathcal{G}(\vec{b} + z\hat{k}_i) \,dz\right]\right\} \qquad (4.1.13)$$

Note that, in carrying through the last calculation, all the terms that are odd in the Z-component (in the function $\mathring{A}(\vec{b} + z\hat{k}_i)$) vanish. Hence,

$$\begin{aligned} \mathcal{G}(\vec{b} + z\hat{k}_i) &= V_c(\vec{r}) + V_s(\vec{r})(\vec{S} \cdot \vec{b} \times \vec{k}_i) + V_L(\vec{r}) \left\{\vec{S}_2 \cdot \vec{R}_2(\vec{b} \times \vec{k}_i, \vec{b} \times \vec{k}_i)\right. \\ &- \left. i\,\vec{S}_2 \cdot \vec{R}_2(\vec{k}_i, \vec{b})\right\} . \end{aligned} \qquad (4.1.14)$$

By implementing the definition of $\Phi(\vec{r})$, eqn (4.1.2), it followed that in effect we have replaced the linear spin-orbit operator by its semiclassical counterpart,

$$\vec{s} \cdot \vec{r} \times \vec{\nabla} \longrightarrow \vec{s} \cdot \vec{r} \times \vec{k}_i \qquad (4.1.15)$$

In so doing we have neglected the velocity-dependent part in that term. However the coupling in the quadratic spin-orbit term, sketched as

$$(\vec{s} \cdot \vec{r} \times \underbrace{\vec{\nabla})(\vec{s} \cdot \vec{r}}\times\vec{\nabla}) \qquad (4.1.16)$$

results in the term $-i\,V_L(\vec{r})\,(\vec{s}\cdot\vec{k}_i)(\vec{S}\cdot\vec{b})$. As we will see later, this gives rise to a contribution to the scattering amplitude which is not invariant under the operation of time reversal except in the forward direction. This is due to the unequal treatment of the initial and final momenta. Our immediate problem however is that it does not commute with the other terms in $\mathcal{G}(\vec{b} + z\hat{k}_i)$, which makes it difficult to expand the exponential matrix in a power series. To overcome this last difficulty we will diagonalise the matrix, allowing us to write it in terms of three orthogonal matrices each corresponding to one of the eigenvalues of the original matrix. The resultant expression can be expanded in a power series. This method will be discussed in section (4.3). In section (4.2) we will follow a much simpler route which is along the lines of neglecting the velocity-dependence, namely we will make the replacement $\nabla^2 \rightarrow k_i^2$. By construction, this ansatz does not contain the first-Born amplitude as its first-order limit (in powers of V_{oj}).

In our final route, discussed in section (4.4), we will replace the integrand of the exponential function (4.1.14) by an average which is given by the matrix-operator and its Hermitian conjugate. This is motivated by the need to ensure that the ansatz is Hermitian. As we will show later the resulting ansatz is simple (with regard to commutivity) to handle and gives rise to an amplitude that contains the first-Born approximation as it's leading term.

Before we proceed, it is important to discuss the connection between the eikonal and the first-Born amplitude. A straightforward examination of eqn (4.1.12) shows that the first order term, in powers of the potential strength, is formally identical to the first-Born amplitude. However, a similar examination of eqn (4.1.13) reveals that the vector $\vec{r}$ that appeared in eqn (4.1.12) has been replaced by the vector $\vec{b}$. Because of the property of cross-products, namely $\vec{b} \times \vec{k}_i = \vec{r} \times \vec{k}_i$, terms of this form in the eikonal ansatz are not affected by the above mentioned replacement. As for the term $-i\, V_L(\vec{r})\, (\vec{S} \cdot \vec{k}_i)(\vec{S} \cdot \vec{b})$, the replacement of $\vec{r}$ by $\vec{b}$ means the loss of the contribution, $-i\, V_L(\vec{r})\, kz\, S_z^2$. This apparent contradiction between eqn (4.1.12) and eqn (4.1.13) stems from the fact that the direction of the incident momentum $\hat{k}_i$ has been treated preferentially in the eikonalization calculation. It should also be noted that the non-invariant contribution we mentioned earlier is also connected to this preferential treatment of $\hat{k}_i$.

phase-function. The eikonal phase-function will be defined as,

$$\chi_j(\vec{b}) = \frac{-1}{v} \int_{-\infty}^{\infty} V_j(\vec{b} + z\hat{k}_i) dz. \qquad (4.1.17)$$

It is equivalent to the change in action which a projectile along the classical path[1] $\vec{r} = \vec{b} + vt\hat{k}_i$ would experience in a complete trajectory. The phase-function represents a picture in which each part of the incident wave passes

[1] t is the time of flight

through the interaction region following a linear path and undergoes a shift of phase characteristic of that path.

In the case of spinless interactions ($V_{os}, V_{ol} \to 0$) the ϕ-dependence is limited to the scalar product $\vec{q} \cdot \vec{b}$, and this gives rise to the familiar result,

$$\mathcal{F}(\hat{k}_f \cdot \hat{k}_i) \;=\; ik \int_0^\infty dbbJ_0(qb)\;[\exp[i\chi(b)] \;-\; 1]\;. \qquad (4.1.18)$$

We have here made use of the integral,

$$\int_0^{2\pi} \exp[iqb\cos\phi]\,d\phi \;=\; 2\pi\, J_0(qb)\,. \qquad (4.1.19)$$

The amplitude arrived at resembles the expressions much discussed in optics describing the diffraction of light by a transparent object.

In the general case the ϕ dependence is complicated by the presence of the spin couplings. However, this does not interfere with the straight-line path of the projectile and all that we have mentioned concerning the phase-function still holds. Using (4.1.17) we will write the spin-one eikonal-function as,

$$\mathcal{E}(\vec{b}) = -\frac{1}{v}\int_{-\infty}^{\infty} \mathcal{G}(\vec{b} + z\hat{k}_i)\,dz\,. \qquad (4.1.20)$$

For future reference we will develop some of our expressions further. First we define $\hat{\tau} = \hat{b} \times \hat{k}_i$, so that

$$\begin{aligned}(\vec{S}\cdot\hat{\tau}) \;&=\; \sin\phi\, S_x \;-\; \cos\phi\, S_y \\ &=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & ie^{-i\phi} & 0 \\ -ie^{i\phi} & 0 & ie^{i\phi} \\ 0 & -ie^{i\phi} & 0 \end{pmatrix}. \end{aligned} \qquad (4.1.21)$$

Also

$$(\vec{S}\cdot\hat{\tau})^2 = \sin^2\phi\, S_x^2 + \cos^2\phi\, S_y^2 - \sin\phi\cos\phi\,(S_xS_y + S_yS_x)$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -e^{-2i\phi} \\ 0 & 2 & 0 \\ -e^{2i\phi} & 0 & 1 \end{pmatrix}. \qquad (4.1.22)$$

Similarly,

$$(\vec{S}\cdot\hat{k}_i)(\vec{S}\cdot\hat{b}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{-i\phi} & 0 \\ 0 & 0 & 0 \\ 0 & -e^{i\phi} & 0 \end{pmatrix}. \qquad (4.1.23)$$

(4.2) Simple-eikonal

In this section we will neglect any velocity-dependence when constructing the exponential ansatz. In addition to the approximation shown in eqn (4.1.15), we will make the replacement $\nabla^2 \to k_i^2$. This can be shown to lead to an expression similar to eqn (4.1.14) without the term $-iV_L(\vec{r})\,(\vec{s}\cdot\vec{k}_i)(\vec{s}\cdot\vec{b})$.

We now write the eikonal ansatz as (subscript 'se' is for simple-eikonal),

$$i\mathcal{E}_{se}(b) = i\left\{\chi_0 + (\vec{s}\cdot\hat{\tau})\chi_1 + (\vec{s}\cdot\hat{\tau})^2\chi_2\right\}, \qquad (4.2.1)$$

where

$$\chi_0 = \chi_c - \frac{2(kb)^2}{3}\chi_l\,,\ \chi_1 = kb(\chi_s + \frac{1}{2}\chi_l)$$
$$\chi_2 = (kb)^2\chi_l\,. \qquad (4.2.2)$$

This ansatz is formed of commuting matrices [2] allowing us to expand each exponential term separately in a power series,

$$e^{i(\vec{s}\cdot\hat{\tau})\chi_1} = 1 + i(\vec{s}\cdot\hat{\tau})\sin(\chi_1) + (\vec{s}\cdot\hat{\tau})^2\left[\cos(\chi_1) - 1\right],$$

[2] We note that $\mathcal{G}$ is non-Hermitian and that by neglecting its velocity dependent contribution we arrived at the above expression which is Hermitian.

$$e^{i(\vec{s}\cdot\hat{\tau})^2\chi_2} = 1 + (\vec{s}\cdot\hat{\tau})^2 \left[e^{i\chi_2} - 1 \right], \qquad (4.2.3)$$

where we have used the properties of spin-one matrices,

$$(\vec{s}\cdot\hat{\tau})^{2n+1} = (\vec{s}\cdot\hat{\tau})\,;\; (\vec{s}\cdot\hat{\tau})^{2n} = (\vec{s}\cdot\hat{\tau})^2. \qquad (4.2.4)$$

Hence

$$\begin{aligned} e^{i\mathcal{E}_{se}(b)} &= e^{i\chi_0}\Big\{ 1 + i(\vec{s}\cdot\hat{\tau})\sin(\chi_1)\, e^{i\chi_2} \\ &+ (\vec{s}\cdot\hat{\tau})^2 \left[\cos(\chi_1)\, e^{i\chi_2} - 1 \right] \Big\}. \end{aligned} \qquad (4.2.5)$$

Finally the scattering amplitude may be written in the form

$$\mathcal{F}(\vec{k}_f, \vec{k}_i) = \frac{ik}{2\pi} \int d\vec{b}\, e^{i\vec{q}\cdot\vec{b}} \left\{ \Gamma_o(b) - i\Gamma_1(b)(\vec{s}\cdot\hat{\tau}) + \Gamma_2(b)(\vec{s}\cdot\hat{\tau})^2 \right\}, \quad (4.2.6)$$

where

$$\begin{aligned} \Gamma_o(b) &= 1 - e^{i\chi_0}\,,\; \Gamma_1(b) = \sin(\chi_1) e^{i(\chi_0+\chi_{II})} \\ \Gamma_2(b) &= \left[1 - \cos(\chi_1) e^{i\chi_{II}} \right] e^{i\chi_0}. \end{aligned} \qquad (4.2.7)$$

The exponential can be expanded as a sum of cylindrical Bessel functions,

$$\exp[i\vec{q}\cdot\vec{b}] = \sum_{m=-\infty}^{\infty} i^m J_m(qb)\, e^{im\phi}. \qquad (4.2.8)$$

Integrating over the azimuthal angle using the integral [3],

$$\int_0^{2\pi} e^{i(m-n)\phi}\, d\phi = 2\pi\, \delta_{m\,n}, \qquad (4.2.9)$$

[3] we also made use of the relation, $J_{-n}(x) = (-1)^n J_n(x)$

and rearranging the resulting expression in the form given by eqn (2.2.11), we finally arrive at

$$A_{se}(\theta) \;=\; i\,k \int_0^\infty b db\, J_0(qb) \left[1 - \frac{1}{3}\left(1 + 2\cos(\chi_1) e^{i\chi_2}\right) e^{i\chi_0}\right] \qquad (4.2.10)$$

$$B_{se}(\theta) \;=\; -i\,k \int_0^\infty b db\, J_1(qb)\, \sin(\chi_1)\, e^{i(\chi_0 + \chi_2)} \qquad (4.2.11)$$

$$C^n_{se}(\theta) \;=\; -i\,k \int_0^\infty b db\, J_2(qb) \left[1 - \cos(\chi_1)\, e^{i\chi_2}\right] e^{i\chi_0} \qquad (4.2.12)$$

$$C^k_{se}(\theta) \;=\; \frac{-i\,k}{2} \int_0^\infty b db\, \left(J_0(qb) \,+\, J_2(qb)\right) \left[1 - \cos(\chi_1)\, e^{i\chi_2}\right] e^{i\chi_0}. \qquad (4.2.13)$$

As noted earlier, our starting ansatz in this section was, by construction, bound to fail in reproducing the first-Born amplitude. This can be readily seen, for example, by expanding the single spin-flip amplitude in powers of the potential strength. To first order this results in

$$B_{se}(\theta)|_{1^{st}order} \;=\; -i\,k^2 \int_0^\infty db\, b^2\, J_1(qb)\,(\chi_s + \chi_l)\,. \qquad (4.2.14)$$

This is seen to depend also on the tensor potential $V_L(\vec{r})$, while the Born calculation depended only on the vector potential.

As before we will assume that the potentials are of the Gaussian form defined in (3.2.1) with equal scattering ranges ($\alpha_l = \alpha_s$). We will make use of the integral [37] ($\Re m > -1, \Re a^2 > 0$),

$$\begin{aligned}
\mathcal{I}_{j,m} &= \int_0^\infty \chi_j(b)\, b^{m+1}\, J_m(qb)\, db \\
&= -\frac{\mu V_o^j \sqrt{2\pi}}{\alpha k} \int_0^\infty \exp[-\alpha^2 b^2/2]\, b^{m+1} J_m(qb)\, db \\
&= -\frac{\mu V_o^j \sqrt{2\pi}}{\alpha k} \left(\frac{q^m}{(\alpha^2)^{m+1}}\right) \exp[-b^2/2\alpha^2]\,, \qquad (4.2.15)
\end{aligned}$$

where χ_j is defined by eqn (4.1.17). This allows us to write

$$\begin{aligned} \mathcal{B}_{se} &= -ik^2 \left(\mathcal{I}_{s,1} + \mathcal{I}_{l,1} \right) \\ &= i\frac{\mu \mid n \mid \sqrt{2\pi}}{\alpha^5} \left(V_o^s + \frac{1}{2} V_o^l \right) \exp[-q^2/2\alpha^2] \, . \end{aligned} \qquad (4.2.16)$$

If we compare the final expression with eqn (3.2.15), we can see that the two single spin-flip amplitudes differ by the term $\frac{1}{2}V_o^l$ in the above equation. The disagreement is a structural one, which does not show itself either in the spin-nonflip amplitude $\mathcal{A}(\theta)$ nor in the double spin-flip ones $\mathcal{C}^k, \mathcal{C}^n$. However the double spin-flip terms agree with the first-Born only in the leading k-terms. This will be discussed in detail in the next section. It is important to remind ourselves that the eikonal amplitude has been calculated on the basis that the spin is quantized along the direction of the initial momentum. In the following section we will rotate to the average frame ($\vec{k} = (\vec{k}_i + \vec{k}_f)/2$), though this will not lead to any significant improvement on the results. In the case at hand it can be said that the coupling we have neglected[4] has partially destroyed one of the basic structural properties of the eikonal approximation, its first order correspondence to the first-Born.

[4]See eqn (4.1.16) and the argument preceeding it

(4.3) Diagonalization of the eikonal ansatz

In this section we will diagonalise the full eikonal-matrix given by eqn (4.1.20). In terms of the definitions developed at the end of section (4.1) we will write

$$i\mathcal{E}(b) \;=\; i\chi_0 \,+\, i\mathcal{E}_{sl}. \qquad (4.3.1)$$

where

$$\mathcal{E}_{sl} \;=\; \left\{(\vec{s}\cdot\hat{\tau})\chi_1 \,+\, (\vec{s}\cdot\hat{\tau})^2\chi_2 \,-\, i(\vec{s}\cdot\hat{k}_i)(\vec{s}\cdot\hat{b})\chi_3\right\}, \qquad (4.3.2)$$

and we have defined the function,

$$\chi_3 \;=\; (kb)\,\chi_l\,. \qquad (4.3.3)$$

In explicit matrix form we have,

$$\mathcal{E}_{sl} \;=\; \begin{pmatrix} \chi_2/2 & [-i\chi_3/\sqrt{2} + i\chi_1/\sqrt{2}]\,\mathrm{e}^{-i\phi} & -\chi_2\mathrm{e}^{-2i\phi}/2 \\ -i\chi_1\mathrm{e}^{i\phi}/\sqrt{2} & \chi_2 & i\chi_1\mathrm{e}^{-i\phi}/\sqrt{2} \\ -\chi_2\mathrm{e}^{2i\phi}/\;2 & [i\chi_3/\sqrt{2} - i\chi_1/\sqrt{2}]\,\mathrm{e}^{i\phi} & \chi_2/2 \end{pmatrix} \qquad (4.3.4)$$

The two matrices in eqn (4.3.1) commute with each other, so that it is $\mathcal{E}_{sl}$ which we will diagonalise, treating χ_0 as a multiplicative factor in the same way as we did in the last section.

From the theory of matrices [38] we know that a matrix (called $\mathcal{R}$), which diagonalizes $\mathcal{E}_{sl}$ may be found by grouping the *eigenvectors* of $\mathcal{E}_{sl}$ into a square matrix. To calculate both the eigenvalues and the eigenvectors, we will solve the *characteristic equation* of the matrix, defined as,

$$\begin{aligned} \mid \mathcal{E}_{sl} - \eta \mid \;&=\; 0 \\ &=\; -\eta^3 + 2\chi_2\eta^2 - (\chi_2^2 + \chi_3\chi_1 - \chi_1^2)\,\eta. \end{aligned} \qquad (4.3.5)$$

This gives three characteristic roots, or *eigenvalues* of the matrix $\mathcal{E}_{sl}$

$$\eta_0 = 0 \, , \, \eta_\pm = \chi_2 \pm \sqrt{\chi_1^2 - \chi_3\chi_1} \, . \qquad (4.3.6)$$

The *eigenvectors* associated with these eigenvalues satisfy the equation

$$(\mathcal{E}_{sl} - \eta_\lambda)\,\xi^\lambda = 0 \, . \qquad (4.3.7)$$

where ξ^λ are the eigenvectors and $\lambda = 0, +, -$ stands for the three eigenvalues. Solving these three simultaneous equations for ξ^λ, grouping the resultant vectors in a matrix form and defining,

$$\mathcal{X} = \frac{\chi_1}{\sqrt{\chi_1^2 - \chi_3\chi_1}} \qquad (4.3.8)$$

we get,

$$\begin{aligned} \mathcal{R} &= (\xi^0, \xi^+, \xi^-) \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & -i\sqrt{2}\mathcal{X}\mathrm{e}^{i\phi} & i\sqrt{2}\mathcal{X}\mathrm{e}^{i\phi} \\ \mathrm{e}^{2\,i\phi} & -\mathrm{e}^{2\,i\phi} & -\mathrm{e}^{2\,i\phi} \end{pmatrix} . \end{aligned} \qquad (4.3.9)$$

This matrix diagonalizes $\mathcal{E}_{sl}$,

$$\mathcal{E}_{sl} = \mathcal{R}\,\Delta\,\mathcal{R}^{-1} \, , \qquad (4.3.10)$$

where the matrix $\mathcal{R}^{-1}$ is the inverse of the non-singular matrix $\mathcal{R}$ and is given explicitly by,

$$\mathcal{R}^{-1} = \begin{pmatrix} 1/2 & 0 & \mathrm{e}^{-2i\phi}/2 \\ 1/4 & i\mathrm{e}^{-i\phi}/2\sqrt{2}\mathcal{X} & -1/4\mathrm{e}^{-2i\phi} \\ 1/4 & -i\mathrm{e}^{-i\phi}/2\sqrt{2} & -1/4\mathrm{e}^{-2i\phi} \end{pmatrix} \qquad (4.3.11)$$

The matrix Δ is defined as

$$\Delta = \eta^0\,\Delta_0 + \eta^+\,\Delta_+ + \eta^-\,\Delta_- \, , \qquad (4.3.12)$$

where

$$\Delta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Delta_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (4.3.13)$$

These matrices are projection operators, $\Delta_j^2 = \Delta_j$, which also commute with one another. This allows us to expand the exponential matrix in a power series,

$$\begin{aligned} \mathrm{e}^{i\,\mathcal{E}_{sl}} &= \mathrm{e}^{\mathcal{R}\,\Delta\,\mathcal{R}^{-1}} \\ &= \mathcal{R}\,\mathrm{e}^{i\Delta}\,\mathcal{R}^{-1}. \end{aligned} \qquad (4.3.14)$$

Furthermore, since $\eta_o = 0$,

$$\mathrm{e}^{i\Delta} = 1 + (\mathrm{e}^{i\eta_+} - 1)\Delta_+ + (\mathrm{e}^{i\eta_-} - 1)\Delta_- . \qquad (4.3.15)$$

Finally we can write,

$$\mathrm{e}^{i\mathcal{E}(b)} = \mathrm{e}^{i\chi_0}\left\{1 + (\mathrm{e}^{i\eta_+} - 1)\mathcal{N}_+ + (\mathrm{e}^{i\eta_-} - 1)\mathcal{N}_-\right\}, \qquad (4.3.16)$$

where

$$\begin{aligned} \mathcal{N}_\pm &= \mathcal{R}\;\Delta_\pm\;\mathcal{R}^{-1} \\ &= \frac{1}{2}(\vec{s}\cdot\hat{\tau})^2 \pm i\frac{1}{2\mathcal{X}}(\vec{s}\cdot\hat{k}_i)(\vec{s}\cdot\hat{b}) \\ &\mp i\frac{\mathcal{X}}{2}(\vec{s}\cdot\hat{b})(\vec{s}\cdot\hat{k}_i) \end{aligned} \qquad (4.3.17)$$

The scattering matrix takes the form

$$\begin{aligned} \mathcal{F}(\hat{k}_f\cdot\hat{k}_i) &= \frac{ik}{2\pi}\int d\vec{b}\,\mathrm{e}^{i\vec{q}\cdot\vec{b}}\left\{\Theta_0(\vec{b}) + \Theta_1(\vec{b})\left[\left(S_xS_z - \frac{1}{\mathcal{X}^2}S_zS_x\right)\cos\phi\right.\right. \\ &+ \left.\left.\left(S_yS_z - \frac{1}{\mathcal{X}^2}S_zS_y\right)\sin\phi\right] + \Theta_2(\vec{b})(\vec{s}\cdot\hat{\tau})^2\right\}. \end{aligned} \qquad (4.3.18)$$

We have defined

$$\begin{aligned}\Theta_0(\vec{b}) &= 1 - \mathbf{e}^{i\chi_0} \\ \Theta_1(\vec{b}) &= -\mathcal{X}\,\sin(\chi_1/\mathcal{X})\,\mathbf{e}^{i(\chi_2+\chi_0)} \\ \Theta_2(\vec{b}) &= \left[1 - \cos(\chi_1/\mathcal{X})\,\mathbf{e}^{i\chi_2}\right]\mathbf{e}^{i\chi_0}\end{aligned} \qquad (4.3.19)$$

Carrying out the ϕ-integration using (4.2.8-9), we arrive at

$$\begin{aligned}\mathcal{F}(\theta) &= \mathcal{A}(\theta) + \mathcal{B}_1(\theta)\vec{S}\cdot\hat{n} + \mathcal{B}_2(\theta)S_zS_x \\ &+ \mathcal{C}^k(\theta)\vec{S}_2\cdot\vec{R}_2(\hat{k}_i,\hat{k}_i) + \mathcal{C}^n(\theta)\vec{S}_2\cdot\vec{R}_2(\hat{n}_i,\hat{n}_i)\,,\end{aligned} \qquad (4.3.20)$$

where,

$$\begin{aligned}\mathcal{A}(\theta) &= ik\int_0^\infty bdb\left[\Theta_0(\vec{b}) + \frac{2}{3}\Theta_2(\vec{b})\right]J_0(qb) \\ \mathcal{B}_1(\theta) &= ik\int_0^\infty bdb\;\Theta_1(\vec{b})\,J_1(qb) \\ \mathcal{B}_2(\theta) &= -k\int_0^\infty bdb\;\Theta_1(\vec{b})\left[1 - \frac{1}{\mathcal{X}^2}\right]J_1(qb) \\ \mathcal{C}^k(\theta) &= -\frac{ik}{2}\int_0^\infty bdb\;\Theta_2(\vec{b})\,[J_0(qb) + J_2(qb)] \\ \mathcal{C}^n(\theta) &= -ik\int_0^\infty bdb\;\Theta_2(\vec{b})\,J_2(qb)\,.\end{aligned} \qquad (4.3.21)$$

Unlike the simple-eikonal, this scattering matrix takes the form of the general matrix defined in section (2.2) eqn (2.2.11). In the limit $\chi_1 \gg \chi_3$ the function $1/\mathcal{X}^2 \to 1$ and we recover the result we arrived at using the simple ansatz.

If we expand the functions $\Theta(\vec{b})$ in powers of the potential strength, retaining terms only up to the second order, we get

$$\Theta_0(\vec{b}) \sim -i\chi_c + \frac{2i}{3}(kb)^2\chi_l + \chi_c^2 - \frac{4}{3}(kb)^2\chi_c\chi_l + \frac{2}{9}(kb)^4\chi_l^2\,,$$

$$\Theta_1(\vec{b}) \sim -(kb)(\chi_s + \frac{1}{2}\chi_l) - i(kb)\left(\chi_c\chi_s + \frac{1}{2}\chi_c\chi_l + \frac{(kb)^2}{3}\chi_s\chi_l + \frac{(kb)^2}{6}\chi_l^2\right),$$

$$\Theta_2(\vec{b}) \sim -i(kb)^2\chi_l + \frac{(kb)^2}{2}\left(2\chi_c\chi_l + \chi_s^2 - \frac{1}{4}\chi_l^2\right) - \frac{(kb)^4}{6}\chi_l^2. \qquad (4.3.22)$$

In a similar way to the discussion at the end of section (4.2) we assume the potential to be of Gaussian form ($\alpha_l = \alpha_s = \alpha_c$). We also generalize the definition of eqn (4.2.15) to[5]

$$\mathcal{I}^r_{j,p,m} = \left(-\frac{\partial}{\partial a_{jp}}\right)^r \int_0^\infty \chi_j(b)\chi_p(b)\, b^{m+1} J_m(qb)db. \qquad (4.3.23)$$

Using this function, together with the above expansions, in eqn (4.3.21) we arrive at

$$\begin{aligned}
\mathcal{A}(\theta)|_{1^{st}+2^{nd}order} &\sim ik\Big[-i\mathcal{I}_{c,0} + \mathcal{I}_{c,c,0} - \frac{2k^2}{3}\mathcal{I}^1_{c,l,0} \\
&+ \frac{k^2}{3}\mathcal{I}^1_{s,s,0} - \frac{k^2}{12}\mathcal{I}^1_{l,l,0} + \frac{k^4}{9}\mathcal{I}^2_{l,l,0}\Big] \\
&\sim \frac{-\mu V_0^c\sqrt{2\pi}}{\alpha^3}\exp[-\frac{q^2}{2\alpha^2}] + i\,\mu^2\pi\exp[-\frac{q^2}{4\alpha^2}] \\
&\times \left\{\left(\frac{V_{oc}^2}{\alpha^4 k} + V_{ol}^2\left[\frac{-k}{12\alpha^6} - \frac{k^3q^2}{9\alpha^{10}} + \frac{k^3q^4}{144\alpha^{12}} + \frac{kq^2}{48\alpha^8}\right.\right.\right. \\
&+ \left.\left.\left.\frac{2k^3}{9\alpha^8}\right] + V_{os}^2\left[\frac{k}{3\alpha^6} - \frac{kq^2}{12\alpha^8}\right] + V_0^cV_o^l\left[\frac{-2k}{3\alpha^6} + \frac{kq^2}{6\alpha^8}\right]\right)\right\}.
\end{aligned}$$

$$(4.3.24)$$

$$\begin{aligned}
\mathcal{C}^k(\theta)|_{1^{st}+2^{nd}order} &\sim -\frac{ik^3}{2}\Big[-i\mathcal{I}^1_{l,0} + \mathcal{I}^1_{c,l,0} + \frac{1}{2}\mathcal{I}^1_{s,s,0} - \frac{1}{8}\mathcal{I}^1_{l,l,0} - i\mathcal{I}_{l,2} \\
&+ \mathcal{I}_{c,l,2} + \frac{1}{2}\mathcal{I}_{s,s,2} - \frac{1}{8}\mathcal{I}_{l,l,2} - \frac{k^2}{6}\left(\mathcal{I}^2_{l,l,0} + \mathcal{I}^1_{l,l,2}\right)\Big] \\
&\sim \frac{\mu\sqrt{2\pi}}{\alpha^5}k^2V_o^l\exp[\frac{-q^2}{2\alpha^2}] + i\,\mu^2\pi\exp[\frac{-q^2}{4\alpha^2}] \\
&\times \left\{\left(V_{ol}^2\left[\frac{k^3}{6\alpha^8} + \frac{k}{16\alpha^6} + \frac{k^3q}{12\alpha^8} - \frac{k^3q^2}{12\alpha^{10}}\right.\right.\right.
\end{aligned}$$

[5] Where; $r = 0, 1, 2.., a_{jp} = \alpha^2, a_{j0} = \alpha^2/2$ and for mathematical convenience we have defined; $\chi_0 = 1, \Rightarrow \mathcal{I}^0_{0,j,m} = \mathcal{I}_{j,m}$.

$$- \frac{k^2q^3}{96\alpha^{10}} + \frac{k^3q^4}{192\alpha^{12}}\Bigg] - V_{os}^2\frac{k}{4\alpha^6} - V_o^cV_o^l\frac{k}{2\alpha^6}\Bigg)\Bigg\} . \tag{4.3.25}$$

Similarly we have,

$$\begin{aligned} C^n(\theta)|_{1^{st}+2^{nd}order} \sim\ & \frac{-\mu\sqrt{2\pi}}{\alpha^7}k^2q^2V_o^l\exp[\frac{-q^2}{2\alpha^2}] + i\,\mu^2\pi\exp[\frac{-q^2}{4\alpha^2}] \\ & \times \Bigg\{\Bigg(V_{ol}^2\left[\frac{kq^2}{32\alpha^8} + \frac{k^3q^4}{6\alpha^8} - \frac{k^3q^2}{48\alpha^{10}}\right] \\ & -V_{os}^2\frac{kq^2}{8\alpha^8} - V_o^cV_o^l\frac{kq^2}{4\alpha^8}\Bigg)\Bigg\} . \end{aligned} \tag{4.3.26}$$

Finally in a similar calculation to that described above we arrive at,

$$\begin{aligned} \mathcal{B}_1(\theta)|_{1^{st}+2^{nd}order} =\ & \Bigg\{i\frac{\mu\sqrt{2\pi}}{\alpha^5}kq\,(V_o^s + V_o^l)\exp[\frac{-q^2}{2\alpha^2}] \\ & - \mu^2\pi\exp[\frac{-q^2}{4\alpha^2}]\left(V_o^cV_o^s\frac{q}{2\alpha^6} + V_o^lV_o^s\left[\frac{k^2q}{3\alpha^8} - \frac{k^2q^3}{24\alpha^{10}}\right]\right) \\ & + \mu\sqrt{2\pi}\frac{kq}{2\alpha^5}V_o^l + i\mu^2\pi\left(V_o^cV_o^l\frac{q}{4\alpha^6} + V_{ol}^2\left[\frac{k^2q}{2\alpha^8} - \frac{k^2q^3}{16\alpha^{10}}\right]\right)\Bigg\}\, S_y , \end{aligned} \tag{4.3.27}$$

and for the first order term of $\mathcal{B}_2(\theta)$

$$\mathcal{B}_2(\theta)|_{1^{st}order} = \frac{\mu\sqrt{2\pi}}{\alpha^5}kq\,V_o^l\exp[\frac{-q^2}{2\alpha^2}]S_zS_x . \tag{4.3.28}$$

As was mentioned at the end of the last section this calculation assumed spin-quantization along the direction of the incident momentum. The average (rotated amplitudes) [6] are related to the ones defined in (4.3.20) by,

$$\mathcal{A}^{av}(\theta) = \mathcal{A}(\theta) , \tag{4.3.29}$$

[6] For the definition of the transformation see eqns (2.2.14-17).

$$\mathcal{B}^{av}(\theta) \;=\; \mathcal{B}_1(\theta) + i\left(\cos^2\frac{\theta}{2}\mathcal{B}_2(\theta) - \frac{1}{2}\sin\theta\,\mathcal{C}_k(\theta)\right), \qquad (4.3.30)$$

$$\mathcal{C}_k^{av}(\theta) \;=\; \mathcal{C}^k(\theta)\cos\theta + \sin\theta\,\mathcal{B}_2(\theta), \qquad (4.3.31)$$

$$\mathcal{C}_n^{av}(\theta) \;=\; \mathcal{C}^n(\theta) - \sin^2\frac{\theta}{2}\mathcal{C}^k(\theta) + \frac{\sin\theta}{2}\mathcal{B}_2(\theta). \qquad (4.3.32)$$

We also get a term that is a coefficient of the matrix $S_x S_z$,

$$\mathcal{B}_3(\theta) \;=\; \cos\theta\,\mathcal{B}_2(\theta) - \sin\theta\,\mathcal{C}^k(\theta). \qquad (4.3.33)$$

It is important to note here that the amplitudes defined in eqn (4.3.21) do not satisfy the reduction relation eqn (2.2.18). Hence the extra spin-flip term in eqn (3.3.20) (the coefficient of $S_z S_x$) does not disappear upon rotating to the average frame. Now from eqns (4.3.24-27) above we can immediately write ($\bar{k} = k\cos\theta$)

$$\mathcal{C}_n^{av}(\theta)|_{1^{st}order} \;=\; \exp[\frac{-q^2}{2\alpha^2}]\frac{\mu\sqrt{2\pi}}{\alpha^5}V_o^l\left[\bar{k}^2 + \frac{1}{2}q^2(\cos\theta/2 - \frac{1}{2})\right]. \qquad (4.3.34)$$

In the limit of θ very small we can see that this has improved the results and approached the first-Born result. In a similar way the unwanted tensor contribution to the spin-flip becomes less significant in the limit of small θ. However it does not vanish completely.

The diagonalization approach has not solved the structural problem of the Glauber eikonalization scheme. There is still a tensor contribution to the single spin-flip amplitude. Nevertheless the diagonalization did highlight the fact that the non-Hermitian tensor contribution has resulted in a non-symmetrical scattering matrix. However in the no-spin-flip amplitude we get complete agreement with the first-Born. In the double spin-flip case we can see that we are missing terms of order ($1/k^2$) in both $\mathcal{C}^n$ and $\mathcal{C}^k$. This is true for both the simple-eikonal and the diagonalized-eikonal.

(4.4) Symmetrical eikonal ansatz

In this section we extend our conjecture, concerning the Hermiticity of the exponential ansatz, a step further by using it as a defining tool for constructing an alternative ansatz. A straightforward way of ensuring the Hermiticity of the ansatz is to average the original matrix together with its Hermitian conjugate. The expression we are after is defined by ('sy' is for symmetrical),

$$\mathcal{E}_{sy}(r) = \frac{\mathcal{E}(r) + \mathcal{E}^{\dagger}(r)}{2} \tag{4.4.1}$$

From eqn (4.1.21) it is straightforward to calculate,

$$\mathcal{E}^{\dagger}(b) = \chi_0 + (\vec{s}\cdot\hat{\tau})\chi_1 + (\vec{s}\cdot\hat{\tau})^2\chi_2 + i(\vec{s}\cdot\hat{b})(\vec{s}\cdot\hat{k}_i)\chi_3 . \tag{4.4.2}$$

Now using the commutation relations defined by eqn (2.1.6) we can write,

$$\left[(\vec{s}\cdot\hat{b}),(\vec{s}\cdot\hat{k}_i)\right] = i(\vec{s}\cdot\hat{\tau}) . \tag{4.4.3}$$

From the definitions (4.2.2) and (4.3.3) we know that,

$$2\chi_1 - \chi_3 = 2(kb)\chi_s . \tag{4.4.4}$$

Hence we can now write

$$\mathcal{E}_{sy}(b) = \chi_0 + (\vec{s}\cdot\vec{\tau})\chi_s + (\vec{s}\cdot\hat{\tau})^2\chi_2 . \tag{4.4.5}$$

In its spin-structure this expression is identical to $\mathcal{E}_{se}(b)$. The difference is in the coefficient of the spin-orbit term. This equivalence allows us to use the results derived in section two with the replacement,

$$\chi_1 \longrightarrow (kb)\chi_s \,. \tag{4.4.6}$$

For example,

$$B_{sy}(\theta) \;=\; -i\,k \int_0^\infty bdb\, J_1(qb)\, \sin(\chi_s)\, e^{i(\chi_c + \chi_l(kb)^2/3)} \,. \tag{4.4.7}$$

Also upon expanding this in the same manner as (4.2.14) we arrive at,

$$B_{sy}(\theta) \;=\; \frac{m \mid n \mid \sqrt{2\pi}}{\alpha^5}\, V_o^s \, \exp[(-q^2/2\alpha^2)] \,. \tag{4.4.8}$$

This is readily seen to agree with eqn (3.2.15) for the corresponding Born amplitude.

CHAPTER 5

THE WALLACE SCHEME

(5.1) Partial-wave analysis

In this section we will develop the partial wave expansion [39] describing the amplitude of a spin-one particle scattering off spinless target. We begin by defining the *generalized* spherical harmonics

$$\mathcal{Y}^M_{jls}(\hat{r}) = \sum_{m\nu} C(lsj, m\nu M)\, Y^m_l(\hat{r})\phi^s_\nu\,, \qquad (5.1.1)$$

where the $C(lsj, m\nu M)$ are *Clebsch-Gordan* coefficients [40,41], and ϕ^s_ν are the normalized eigenvectors of the spin defined by eqn (2.1.13). The above functions are eigenfunctions of both the total angular-momentum operator and its z-component,

$$\vec{J}^2\mathcal{Y}^M_{jls} = j(j+1)\mathcal{Y}^M_{jls}\,, \qquad (5.1.2)$$

$$J_z\mathcal{Y}^M_{jls} = M\mathcal{Y}^M_{jls}\,. \qquad (5.1.3)$$

They are also eigenfunctions of the squares of both the orbital angular-momentum and spin operators,

$$\vec{L}^2\mathcal{Y}^M_{jls} = l(l+1)\mathcal{Y}^M_{jls}\,, \qquad (5.1.4)$$

$$\vec{S}^2\mathcal{Y}^M_{jls} = s(s+1)\mathcal{Y}^M_{jls}\,. \qquad (5.1.5)$$

From the symmetry requirement that the interaction potential be a scalar under rotation, it follows that the Hamiltonian, describing spin-dependent interactions, defined by eqn (2.2.1) is also rotationally invariant. This implies the following commutation relations,

$$\left[\vec{J}_z, H\right] = \left[\vec{J}^2, H\right] = 0\,. \qquad (5.1.6)$$

Consequently, j and M are conserved and we can expand the scattered wave function in terms of the following complete set of states [42],

$$\Psi^{out}_{s,\nu}(\vec{r}) = \sum_{l'ljM} \Psi^{j}_{l',l}(k,r)\, \mathcal{Z}^{jM*}_{ls\nu}(\hat{k}_i)\, \mathcal{Y}^{jM}_{l's}(\hat{r})\,, \qquad (5.1.7)$$

where

$$\mathcal{Z}^{jM}_{ls\nu}(\hat{k}_i) = \mathcal{Y}^{jM}_{ls}(\hat{k}_i) \cdot \phi^{s\dagger}_{\nu}\,. \qquad (5.1.8)$$

Using the orthogonality of the generalized spherical harmonics and defining the potential matrix,

$$\mathcal{V}^{j}_{l',l''}(\vec{r}) = 2\mu \int d\Omega\, \mathcal{Y}^{jM\dagger}_{l's}(\hat{r})\, V(\vec{S},\vec{r}) \mathcal{Y}^{jM}_{l''s}(\hat{r}) \qquad (5.1.9)$$

we obtain the coupled radial wave equations,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{\vec{L}^2}{r^2} + k^2\right] \Psi^{j}_{l',l}(r) = \sum_{l''} \mathcal{V}^{j}_{l',l''}(\vec{r})\, \Psi^{j}_{l'',l}(r)\,. \qquad (5.1.10)$$

The radial functions satisfy

$$\begin{aligned} \Psi^{j}_{l',l} &= j_l(kr)\delta_{ll'} \\ &+ \sum_{l''} \int_0^\infty dr'\, r'^2\, g^{out}_{l'}(r,r') \mathcal{V}^{j}_{l',l''}(r') \Psi^{j}_{l'',l}(r')\,, \end{aligned} \qquad (5.1.11)$$

where $g_l^{out}(r,r')$ is the radial coefficient in the expansion of the Green-function of the outgoing wave function $\Psi_{s\nu}^{out}(\vec{r})$ in a manner similar to eqn (5.1.7).

From the definition $\vec{J} = \vec{L} + \vec{S}$, it follows that,

$$\vec{S}\cdot\vec{L} = \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) \qquad (5.1.12)$$

Inspecting the tensors[1] $T_P(\vec{r})$ and $T_r(\vec{r})$, we note that beside the central and the vector spin-orbit terms, only the tensor[2] $T_L(\vec{r})$ is made up of operators of which the $\mathcal{Y}_{ls}^{JM}(\hat{r})$ are eigenfunctions. Since we are interested in analytical forms which enables us to study the scattering qualitatively rather than quantitatively, we will from this point onwards restrict the tensor contribution of the optical potential to $T_L(\vec{r})$.

Using eqn (5.1.12) the potential is now given by,

$$\begin{aligned} V(\vec{S},\vec{r}) &= V_c(\vec{r}) + \frac{1}{2}\left(V_s + \frac{1}{2}V_L(\vec{r})\right)\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) \\ &+\frac{1}{4}V_L\left[\mathbf{J}^4 + \mathbf{L}^4 + \mathbf{S}^4 - 2\left(\mathbf{J}^2\mathbf{L}^2 + \mathbf{J}^2\mathbf{S}^2 + \mathbf{L}^2\mathbf{S}^2\right) - \frac{8}{3}\mathbf{L}^2\right]. \end{aligned} \qquad (5.1.13)$$

Substituting this in eqn (5.1.9), and making use of eqn (5.1.2-5) together with the orthogonality of the generalized spherical harmonics and the fact that the potential conserves parity, the radial equation decouples into states $j = l\pm1, j = l$ with diagonal potential matrices

$$\mathcal{V}_l^{j=l+1}(\vec{r}) = 2\mu\left\{V_c(\vec{r}) + lV_s(\vec{r}) + \frac{l}{3}\left(l - \frac{1}{2}\right)V_l(\vec{r})\right\} \qquad (5.1.14)$$

[1] See the definition of these tensors eqn (2.3.6)

[2] This follows from the commutation relation, $\left[T_L(\vec{r}), \vec{S}\cdot\vec{L}\right] = 0$, together with relation (5.1.12).

$$\mathcal{V}_l^{j=l-1}(\vec{r}) \;=\; 2\mu\left\{V_c(\vec{r}) - (l+1)V_s(\vec{r}) + \left(\frac{1}{2}+\frac{l}{3}(l+\frac{5}{2})\right)\,V_l(\vec{r})\right\} \tag{5.1.15}$$

$$\mathcal{V}_l^{j=l}(\vec{r}) \;=\; 2\mu\left\{V_c(\vec{r}) - V_s(\vec{r}) + \left(\frac{1}{2}-\frac{2l}{3}(l+1)\right)\,V_l(\vec{r})\right\} \tag{5.1.16}$$

From the asymptotic expansion of the wave function we can write the scattering amplitude as,

$$\mathcal{F}_P(\theta) \;=\; -2\pi i \sum_{ll',jM} \mathcal{Y}_{ls}^{jM\dagger}(\hat{k}_f) T^j_{l,l'} \mathcal{Y}_{l's}^{jM}(\hat{k}_i) \tag{5.1.17}$$

where

$$T^j_{l,l'} \;=\; S^j_{l,l'} - 1\,. \tag{5.1.18}$$

The *S-matrix* elements $S^j_{ls,l's}$ are calculated from the asymptotic form of the radial wave functions,

$$\Psi^j_{l',l}(r) \xrightarrow[r\to\infty]{} \frac{i^{-(l'+1)}}{2kr}\left\{-(-)^l \delta_{ll'} \mathrm{e}^{-ikr} + S^j_{l,l'}\mathrm{e}^{ikr}\right\}, \tag{5.1.19}$$

and are given by

$$\begin{aligned} S^j_{l',l} \;=\; \delta_{ll'} \;&-\; 2ik\sum_{l''}\int_0^\infty dr\, r^2 j_l(kr) \\ &\times\; \mathcal{V}^j_{l',l''}(r)\Psi^j_{l'',l'}(r). \end{aligned} \tag{5.1.20}$$

Now *unitarity* and *time-reversal invariance* of the S-matrix implies that

$$S^j_{l,l'} \;=\; \exp[2i\delta^j_{ll'}] \tag{5.1.21}$$

The functions $\delta^j_{ll'}$ are called the *eigenphaseshifts* and are real for a real potential.

The scattering matrix given by eqn (5.1.17) can be calculated by working out the Clebsch-Gordan coefficients implicit in the equation, together with the fact that the S-matrix is now diagonal, along the same lines as eqn (5.1.14-16). We can obtain the required answer by first defining [39] the projection operators

$$\Pi_{l0} = -\frac{(\vec{S}\cdot\vec{L})^2 + (\vec{S}\cdot\vec{L}) - l(l+1)}{l(l+1)} \tag{5.1.22}$$

$$\Pi_{l+} = \frac{(\vec{S}\cdot\vec{L})^2 + (l+2)(\vec{S}\cdot\vec{L}) + (l+1)}{(l+1)(2l+1)} \tag{5.1.23}$$

$$\Pi_{l}- = \frac{(\vec{S}\cdot\vec{L})^2 + (l-1)(\vec{S}\cdot\vec{L}) - l}{l(2l+1)} \tag{5.1.24}$$

These allow us to write,

$$\mathcal{F}_P(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left\{ T_l^+\,\Pi_{l+} + T_l^-\,\Pi_{l-} + T_l^0\,\Pi_{l0}\right\} P_l(\cos\theta) \tag{5.1.25}$$

Regrouping the terms, we can write

$$\mathcal{F}_P(\theta) = \frac{1}{2ik}\sum_{l=0}\left\{ \mathcal{A}_o + \mathcal{B}_o(\vec{S}\cdot\vec{L}) + \mathcal{C}_o(\vec{S}\cdot\vec{L})^2\right\} P_l(\cos\theta) \tag{5.1.26}$$

where

$$\mathcal{A}_o = T_l^+ - T_l^- + (2l+1)T_l^0 \tag{5.1.27}$$

$$\mathcal{B}_o = \frac{(l+2)}{(l+1)}T_l^+ - \frac{(l-1)}{l}T_l^- - \frac{(2l+1)}{l(l+1)}T_l^0 \tag{5.1.28}$$

$$\mathcal{C}_o = \frac{1}{(l+1)}T_l^+ + \frac{T_l^-}{l} - \frac{(2l+1)}{l(l+1)}T_l^0\,. \tag{5.1.29}$$

To calculate the matrix coefficients of $\mathcal{B}_o$ and $\mathcal{C}_o$, we first define the spherical polar coordinates

$$\begin{aligned}
\hat{\kappa}_f &= (\sin\theta\,\cos\phi\,, \sin\theta\,\sin\phi\,, \cos\theta) \\
\hat{\theta}_f &= (\cos\theta\,\cos\phi\,, \cos\theta\,\sin\phi\,, -\sin\theta) \\
\hat{\phi}_f &= (-\sin\phi\,, \cos\phi\,, 0)
\end{aligned} \tag{5.1.30}$$

The gradient operator in this coordinate frame, can be written as

$$\nabla_{\kappa_f} = \hat{\kappa}_f \frac{\partial}{\partial \kappa_f} + \hat{\theta}_f \frac{1}{\kappa_f} \frac{\partial}{\partial \theta} + \hat{\phi}_f \frac{1}{\kappa_f \sin\theta} \frac{\partial}{\partial \phi} \qquad (5.1.31)$$

Hence

$$\begin{aligned} \vec{S}\cdot\vec{L} &= -i\vec{S}\cdot(\vec{\kappa}_f \times \nabla_{\kappa_f}) \\ &= \frac{i}{\sin\theta}(\vec{S}\cdot\hat{\theta}_f)\frac{\partial}{\partial\phi} - i(\vec{S}\cdot\hat{\phi}_f)\frac{\partial}{\partial\theta} \qquad (5.1.32) \end{aligned}$$

Now

$$(\vec{S}\cdot\vec{L})\, P_l(\cos\theta) = i(\vec{s}\cdot\hat{\phi}_f)\, P_l^1(\cos\theta)\,, \qquad (5.1.33)$$

and similarly (we have used the fact that, $\frac{\partial}{\partial\phi}P_l(\cos\theta) = 0$)

$$\begin{aligned} (\vec{S}\cdot\vec{L})^2\, P_l(\cos\theta) &= \left[(\vec{S}\cdot\hat{\theta}_f)^2\frac{\cos\theta}{\sin\theta} + (\vec{S}\cdot\hat{\theta}_f)(\vec{S}\cdot\hat{k}_f)\right] P_l^1(\cos\theta) \\ &- (\vec{S}\cdot\hat{\phi}_f)^2\left[P_l^2(\cos\theta) - \frac{\cos\theta}{\sin\theta}P_l^1(\cos\theta)\right]. \qquad (5.1.34) \end{aligned}$$

We will choose $\phi = 0$, so that

$$(\vec{S}\cdot\vec{L})\,|_{\phi=0}\; P_l(\cos\theta) = i\, S_y\, P_l^1(\cos\theta)\,, \qquad (5.1.35)$$

and

$$\begin{aligned} (\vec{S}\cdot\vec{L})^2\,|_{\phi=0}\; P_l(\cos\theta) &= \left[(2 - S_z^2)\frac{\cos\theta}{\sin\theta} - S_z S_x\right] P_l^1(\cos\theta) \\ &- S_y^2\, P_l^2(\cos\theta) \qquad (5.1.36) \end{aligned}$$

Using the above relations in eqn (5.1.25), together with the recurrence relation [43],

$$l(l+1)\, P_l(\cos\theta) = \frac{2\cos\theta}{\sin\theta}P_l^1(\cos\theta) - P_l^2(\cos\theta) \qquad (5.1.37)$$

we find

$$\mathcal{F}_P(\theta) = \mathcal{A}^i_P + \mathcal{B}^i_{1,P}(\vec{S}\cdot\hat{n}) + \mathcal{B}^i_{2,P}\left(S_z S_x + \frac{\cos\theta}{\sin\theta}\vec{S}_2\cdot\vec{R}_2(\hat{k}_i,\hat{k}_i)\right) + \mathcal{C}^i_{n,P}\vec{S}_2\cdot\vec{R}_2(\hat{n},\hat{n}) \qquad (5.1.38)$$

with

$$\mathcal{A}^i_P = \frac{1}{6ik}\sum_{l=0}\left\{(2l+3)T^+_l + (2l-1)T^-_l + (2l+1)T^0_l\right\}P_l(\cos\theta) \qquad (5.1.39)$$

$$\mathcal{B}^i_{1,P} = \frac{1}{2k}\sum_{l=0}\left\{\frac{(l+2)}{(l+1)}T^+_l - \frac{(l-1)}{l}T^-_l - \frac{(2l+1)}{l(l+1)}T^0_l\right\}P^1_l(\cos\theta) \qquad (5.1.40)$$

$$\mathcal{B}^i_{2,P} = -\frac{1}{2ik}\sum_{l=0}\left\{\frac{1}{(l+1)}T^+_l + \frac{1}{l}T^-_l - \frac{(2l+1)}{l(l+1)}T^0_l\right\}P^1_l(\cos\theta) \qquad (5.1.41)$$

$$\mathcal{C}^i_{n,P} = -\frac{1}{2ik}\sum_{l=0}\left\{\frac{1}{(l+1)}T^+_l + \frac{1}{l}T^-_l - \frac{(2l+1)}{l(l+1)}T^0_l\right\}P^2_l(\cos\theta). \qquad (5.1.42)$$

The superscript (i) stands for spin-quantization along the direction of the incident beam.

At the end of section (2.1) we defined an average frame, $(\vec{k}_i + \vec{k}_f)/2 = \vec{k}$, through the transformation (2.2.12). The scattering matrix in the average-quantization frame was given by[3]

$$\mathcal{F}^{av}_p(\theta) = \mathcal{A}^{av}_P(\theta) + \mathcal{B}^{av}_P(\theta)(\vec{S}\cdot\hat{n}) + \mathcal{C}^{av}_{k,p}(\theta)\vec{S}_2\cdot\vec{R}_2(\hat{k},\hat{k}) + \mathcal{C}^{av}_{n,P}(\theta)\vec{S}_2\cdot\vec{R}_2(\hat{n},\hat{n}), \qquad (5.1.43)$$

where the transformation from $\mathcal{F}^i$ to $\mathcal{F}^{av}$ was defined by eqns (2.2.14-17). In terms of eqns (5.1.39-42) the transformation leads to

[3]See eqn (2.2.13).

$$\mathcal{A}_P^{av}(\theta) \;=\; \mathcal{A}_P^{i}(\theta)\,. \qquad (5.1.44)$$

$$\begin{aligned}\mathcal{B}_P^{av}(\theta) &= \mathcal{B}_P^{i}(\theta) + \frac{i}{2}\mathcal{B}_{2,P}^{i}(\theta) \\ &= \frac{1}{4k}\sum_{l=0}\left\{\frac{(2l+3)}{(l+1)}T_l^{+} - \frac{(2l-1)}{l}T_l^{-} - \frac{(2l+1)}{l(l+1)}T_l^{0}\right\}P_l^1(\cos\theta)\,.\end{aligned} \qquad (5.1.45)$$

$$\begin{aligned}\mathcal{C}_{k,P}^{av}(\theta) &= \frac{1}{\sin\theta}\mathcal{B}_{2,P}^{i}(\theta) \\ &= -\frac{1}{2ik\sin\theta}\sum_{l=0}\left\{\frac{1}{(l+1)}T_l^{+} + \frac{1}{l}T_l^{-} - \frac{(2l+1)}{l(l+1)}T_l^{0}\right\}P_l^1(\cos\theta)\,.\end{aligned} \qquad (5.1.46)$$

$$\begin{aligned}\mathcal{C}_{n,P}^{av} &= \mathcal{C}_{n,P}^{i}(\theta) + \frac{\sin^2\theta/2}{\sin\theta}\mathcal{B}_{2,P}^{i}(\theta) \\ &= (2\cos\theta + \sin^2\theta/2)\,\mathcal{C}_{k,P}^{av}(\theta) \\ &+ \frac{1}{2ik}\sum_{l=0}\left\{lT_l^{+} + (l+1)T_l^{-} - (2l+1)T_l^{0}\right\}P_l(\cos\theta)\,.\end{aligned} \qquad (5.1.47)$$

(5.2) First-Born amplitude

In this section we will calculate the first-Born amplitude as a limit to the perturbative S-matrix. Iterating eqn (5.1.20) with the help of eqn (5.1.11) together with eqns (5.1.14-16) we see that the first-Born contribution is of the form

$$S_{l_s}^{j} \;=\; 1 \,-\, k\int_0^{\infty} dr\, r^2\,[j_l(kr)]^2\,\mathcal{V}_{l_s}^{j}(r)\,. \qquad (5.2.1)$$

Substituting this in eqns (5.1.44-47) and using the summation formula [36],

$$\sum_{l=0}^{\infty}(2l+1)[j_l(kr)]^2\,P_l(\cos\theta) \;=\; \frac{\sin qr}{qr} \qquad (5.2.2)$$

It is straightforward to arrive at the first-Born amplitudes calculated in section (2.1). For instance,

$$\begin{aligned}
\mathcal{A}_P^{av}|_{1^{st}order} &= \frac{1}{k}\sum_l (2l+1)\delta_c\, P_l(\cos\theta) \\
&= -\frac{2\mu}{q}\int_0^\infty dr\, r\, V_c(r)\sin qr \\
&= \frac{-\mu V_{oc}\sqrt{2\pi}}{\alpha^3}\exp[-q^2/2\alpha^2]\,. \qquad (5.2.3)
\end{aligned}$$

And similarly for all the other amplitudes.

(5.3) Fourier-Bessel expansion of scattering amplitude

In this section we will review briefly Wallace's method [17] for converting the partial-wave sum to a Fourier-Bessel expansion of the scattering amplitude. In describing his method we will, for the sake of clarity, limit ourselves to spinless interactions. This will be generalized to the spin-one case later in this section.

In the absence of spin couplings, it is clear from eqn (5.1.14-16) that

$$\exp[2i\delta_l^+] = \exp[2i\delta_l^-] = \exp[2i\delta_l^0]\,. \qquad (5.3.1)$$

Hence, the scattering amplitude (5.1.25) reduces to,

$$\begin{aligned}
\mathcal{F}(\theta) &= \sum_{l=0}^{\infty}\Gamma_l\, P_l(\cos\theta)\,, \qquad (5.3.2) \\
\Gamma_l &= (-i/k)(l+\frac{1}{2})\,[\exp[2i\delta_l] - 1]\,. \qquad (5.3.3)
\end{aligned}$$

Regge has shown [44,45] that the summation parameter l can be continued from the discrete integer values to the complex angular momentum plane, where the physical angular momenta are only realised when l takes non-negative integer values. It follows that for well behaved potentials, Γ_l can be interpolated to give $\Gamma(l)$ as a function of a continuous variable. Both these functions are analytic over the summation range. For large real values of l the potential remains small, this implies that the phase shift tends to zero as

$$\delta(l) \; = \; O(1/l) \,. \tag{5.3.4}$$

On the basis of Regge's calculation, we can now convert the partial-sum into an integral over real values of l using the Euler summation formula,

$$\mathcal{F}(\theta) \quad = \quad \int_0^\infty dl\,\Gamma(l) P_l(\cos\theta) \; - \; R_1(\theta) \,, \tag{5.3.5}$$

$$R_1(\theta) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n}(1/2) F^{2n-1}(0,\theta) \,. \tag{5.3.6}$$

Here,

$$F^{2n-1}(l,\theta) \; = \; \left(\frac{\partial}{\partial l}\right)^{2n-1} \{\Gamma(l)\, P_l(\cos\theta)\} \,, \tag{5.3.7}$$

and $b_m(x) \; = B_{2m}^{(2x)}(x)$ are the generalized *Bernoulli* polynomials [36] with $b_o(x) \; = 1$ and $b_1(x) \; = -x/6$. Terms similar to $R_1(\theta)$, corresponding to $l \; = \infty$, arise in the Euler Formula but these have vanished because of the constraint (5.3.4) on the phase-shift.

As we have mentioned in our introductory chapter of this thesis, there have been many attempts to derive the impact parameter representation of scattering amplitude. They are approximate procedures valid only near the forward direction where $R_1(\theta)$ is negligible. They assume that the angular momentum is so large as to allow the replacement

$$P_l(\cos\theta) \;\rightarrow\; \sum_{n=0}^{\infty} \frac{\left[-\frac{1}{2}((2l+1)\sin\{\theta/2\})^2\right]^n}{(n!)^2} \;=\; J_0\left[(2l+1)\sin(\theta/2)\right] \;. \quad (5.3.8)$$

Macdonald [46] has developed the leading corrections to the above equation that can be employed to improve on the results. However, Wallace developed an infinite series expansion of the Legendre function which contains the above limit as its leading term. Starting from the power series expansion of the Legendre functions [46],

$$P_l(\cos\theta) \;=\; \sum_{n=0}^{\infty} \frac{\Gamma(n+l+1)}{\Gamma(-n+l+1)} \frac{[-\sin(\theta/2)]^{2n}}{(n!)^2}\,, \qquad (5.3.9)$$

he expanded the ratio of the Γ functions in a power series of $(l+1/2)$, in which the generalized *Bernoulli* polynomials occur as weighting coefficients. This was then developed into the following derivative operator

$$\frac{\Gamma(n+l+1)}{\Gamma(-n+l+1)} \;=\; \sum_{m=0}^{n} \frac{1}{(2m)!} \left\{\frac{\partial}{\partial l}\right\}^{2m} b_m \left\{\frac{1}{4}\frac{\partial}{\partial l}(2l+1)\right\} (l+\frac{1}{2})^{2n}. \quad (5.3.10)$$

Substituting this in eqn (5.3.9) and observing that we can interchange the sums (the Bessel function arising from the sum over n is differentiable continuously to all orders), he finally got

$$P_l(\cos\theta) \;=\; \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left\{\frac{\partial}{\partial l}\right\}^{2m} b_m \left\{\frac{1}{4}\frac{\partial}{\partial l}(2l+1)\right\} J_o[(2l+1)\sin(\theta/2)]\,. \quad (5.3.11)$$

Substituting this into eqn (5.3.5), integrating by parts and rewriting the non-vanishing boundary terms ($l = 0$) as $R_2(\theta)$, we arrive at,

$$\mathcal{F}(\theta) \;=\; \int_0^{\infty} dl J_o[(2l+1)\sin(\theta/2)]\, \mathcal{W}\, \Gamma(l)$$

$$+ \quad R_2(\theta) \; - \; R_1(\theta) \, . \qquad (5.3.12)$$

We have here defined

$$\mathcal{W}(l) \; = \; \sum_{m=0}^{n} \frac{1}{(2m)!} \, b_m \left\{ -\frac{1}{4}(2l+1)\frac{\partial}{\partial l} \right\} \left[\frac{\partial}{\partial l} \right]^{2m} . \qquad (5.3.13)$$

Wallace proved by direct non-trivial calculation that the above remainders are equal and hence cancel each other. The partial-wave sum can then be converted, without approximation, to the integral

$$\mathcal{F}(\theta) \; = \; k \int_0^\infty db \, b J_o(qb) \, S_F(b) \Gamma(b) \, , \qquad (5.3.14)$$

where the operator,

$$S_F(b) \; = \; b^{-1} \, \mathcal{W}(b) \, b \, . \qquad (5.3.15)$$

We have previously used the *semi-classical* identification of the impact parameter[4], $kb \; = \; (l \, + \, \frac{1}{2})$. At intermediate energies ($1/k \neq 0$) it is clear that $S_F(b)$ will introduce derivatives of the phase-function. As Wallace shows, these *unitarity corrections* are of kinematical nature. They are necessary if the Fourier-Bessel representation is to satisfy unitarity.

The phase-shift is in principle determined from the relation (5.1.20). This is achieved in several ways, one of which we illustrated in section (5.2). Alternatively it can be determined by solving the radial equation (5.1.10) or its Lipmann-Schwinger equivalent (5.1.11).

[4]From now on we will use this identification,e.g $\Gamma(l) \rightarrow \Gamma(b)$ also $\mathcal{W}(l) \rightarrow \mathcal{W}(b)$.

(5.4) Spin-one interactions

In what follows we will extend the Fourier-Bessel representation of the scattering amplitude, in a similar spirit to that of the development of the spin-half case by Waxman et al [21] , to spin-one interactions. Using eqn (5.1.18) and the definition of the *S-matrix* elements (5.1.20) then, for example, the generalization of $\Gamma(b)$ eqn (5.3.3) to the case of the amplitude $\mathcal{A}^{av}(\theta)$ eqn (5.1.44), results in

$$\begin{aligned}\Gamma_{\mathcal{A}}(b) &= \frac{(l+1/2)}{3ik}\Big[\Big(\mathrm{e}^{2i\delta^{+}}+\mathrm{e}^{2i\delta^{-}}\Big)+\mathrm{e}^{2i\delta^{0}} \\ &- 3+\frac{1}{(l+1/2)}\Big(\mathrm{e}^{2i\delta^{+}}-\mathrm{e}^{2i\delta^{-}}\Big)\Big],\end{aligned} \qquad (5.4.1)$$

and similarly for the other amplitudes. We will define the following functions,

$$\bar{\chi}(b) = \frac{2}{3}\Big(\delta^{+}+\delta^{-}+\delta^{0}\Big) \qquad (5.4.2)$$

$$\Delta\chi(b) = 2\Big(\delta^{+}-\delta^{-}\Big) \qquad (5.4.3)$$

$$\mathcal{D}\chi(b) = 2\Big(\delta^{+}+\delta^{-}-2\delta^{0}\Big)\,. \qquad (5.4.4)$$

In terms of these functions we can write

$$\begin{aligned}\mathrm{e}^{2i\delta^{+}}+\mathrm{e}^{2i\delta^{-}} &= 2\cos(\frac{\Delta\chi(b)}{2})\exp[i(\bar{\chi}(b)+\frac{\mathcal{D}\chi(b)}{6}],\\ \mathrm{e}^{2i\delta^{+}}-\mathrm{e}^{2i\delta^{-}} &= 2i\sin(\frac{\Delta\chi(b)}{2})\exp[i(\bar{\chi}(b)+\frac{\mathcal{D}\chi(b)}{6}].\end{aligned} \qquad (5.4.5)$$

It is straightforward to arrive at

$$\begin{aligned}\Gamma_{\mathcal{A}}(b) = & -\frac{i}{3}\Big\{\exp[i(\bar{\chi}(b)-\frac{\mathcal{D}\chi(b)}{3})](1+2\cos(\frac{\Delta\chi(b)}{2})\exp[i\frac{\mathcal{D}\chi(b)}{2}]) \\ & -3+\frac{2i}{kb}\sin(\frac{\Delta\chi(b)}{2})\exp[i(\bar{\chi}(b)+\frac{\mathcal{D}\chi(b)}{6})]\Big\}\end{aligned} \qquad (5.4.6)$$

$$\Gamma_{\mathcal{B}}(b) = \frac{1}{2\sin\theta}\Big\{ 2ikb\sin(\frac{\Delta\chi(b)}{2})\exp[i(\bar\chi(b)+\frac{\mathcal{D}\chi(b)}{6}) + \cos(\frac{\Delta\chi(b)}{2})$$
$$\exp[i(\bar\chi(b)+\frac{\mathcal{D}\chi(b)}{6})] - \exp[i(\bar\chi(b)-\frac{\mathcal{D}\chi(b)}{3})]$$
$$-\frac{i}{kb}\sin(\frac{\Delta\chi(b)}{2})\exp[i(\bar\chi(b)+\frac{\mathcal{D}\chi(b)}{6})]\Big\} \quad (5.4.7)$$

$$\Gamma^k_{\mathcal{C}}(b) = -\frac{i}{\sin^2\theta}\Big\{(1-\cos(\frac{\Delta\chi(b)}{2})\exp[i\frac{\mathcal{D}\chi(b)}{2}])\exp[i(\bar\chi(b)-\frac{\mathcal{D}\chi(b)}{3})]$$
$$+\frac{i}{2kb}\sin(\frac{\Delta\chi(b)}{2})\exp[i(\bar\chi(b)+\frac{\mathcal{D}\chi(b)}{6})]\Big\} \quad (5.4.8)$$

To make use of the derivative expansion of the Legendre functions (5.3.11), we introduce the following Legendre recurrence relation[5] [47],

$$P^1_l(\cos\theta) = \frac{l(l+1)}{\sin\theta}\int_{\cos\theta}^{1} P_l(\cos\theta)\, d\cos\theta\,. \quad (5.4.9)$$

Hence we can now write

$$\mathcal{A}^{av}_P(\theta) = k\int_0^\infty b\, db\, J_o(qb)\, S_F(b)\Gamma_{\mathcal{A}}(b) \quad (5.4.10)$$

$$\mathcal{B}^{av}_P(\theta) = 2\sin(\theta/2)\int_0^\infty db\, J_1(qb)\, S_F(b)\,\Gamma_{\mathcal{B}}(b) \quad (5.4.11)$$

$$\mathcal{C}^{av}_{k,P}(\theta) = 2\sin(\theta/2)\int_0^\infty db\, J_1(qb)\, S_F(b)\,\Gamma^k_{\mathcal{C}}(b) \quad (5.4.12)$$

$$\mathcal{C}^{av}_{n,P}(\theta) = (2\cos\theta + \sin^2\theta/2)\mathcal{C}^{av}_{k,P}(\theta)$$
$$-k\sin^2\theta\int_0^\infty b\, db\, J_o(qb)\, S_F(b)\Gamma^k_{\mathcal{C}}(b)\,. \quad (5.4.13)$$

(5.5) A dynamical model for the phase-shift

In this section we will develop a closed form expression for the phase-shift

[5] Note that in ref [47] the definition of the associated Legendre polynomials differs than that of ref [43] by a minus sign. In our work we use the latter definition.

function. First we define the function

$$u_l^j(r) \;=\; r\,\Psi_l^j(r)\,, \qquad (5.5.1)$$

which we can use to cast (5.1.10) in the form

$$\left(\frac{d}{dr^2}\right) u_l^j(r) + \left(k_l^j(r)\right)^2 u_l^j(r) \;=\; 0\,. \qquad (5.5.2)$$

Here we have defined the *local* wave number,

$$k_l^j(r) \;=\; \left[k^2 - \frac{l(l+1)}{r^2} - \mathcal{V}_l^j(r)\right]^{1/2}. \qquad (5.5.3)$$

The corresponding *local* wavelength is $2\pi/k_l^j(r)$. In the case when the kinetic energy of the incident particle is large, the wave number changes little over distances of the order of the *local* wavelength. However, as we approach the classical limit the wavelength gets very long such that $k_l^j \;\to 0$. These are the classical *turning* points, where the radial momentum vanishes. The singularity of the angular momentum barrier at the origin ensures that there will be at least one such turning point for $l > 0$, no matter how high the kinetic energy. We assume that $k_l^j(r)$ has only one zero, i.e. there is only one classical turning point. This distance (r_t) is the distance of closest approach for a particle of angular momentum l. Outside this distance l increases and the variation of $k_l^j(r)$ becomes small so long as the potential is smoothly varying, even though the potential itself may still be appreciable.

The scheme we have just described is the Wentzel-Kramers-Brillouin (WKB) approximation [42]. From the above argument the approximation postulates an exponential ansatz as a solution for the radial function. This solution becomes exact in the limit $k^2 >> V_o$. The WKB phase-shift [29], is given by

$$\delta_j^{WKB}(l) = (l+\frac{1}{2})\frac{\pi}{2} - kr_t - \int_{r_t}^{\infty} \left\{ \left[k^2 - \mathcal{V}_j(l,r) - (l+\frac{1}{2})^2/r^2 \right]^{1/2} - k \right\} dr \,. \tag{5.5.4}$$

For a given incident momentum k and large l the turning point is close to the free case of l/k. The potential[6] remains small over the region $r_o|_{large}$ to infinity. Expanding[7] the integrand in powers of $\mathcal{V}(l,r)/k^2$ and integrating by parts the first order term gives ($kb = (l+1/2)$),

$$\delta_j^{WKB}(l)|_{1^{st}order} \simeq -\frac{1}{2k}\int_{-\infty}^{+\infty} dz\, \mathcal{V}_j(l,r)\,. \tag{5.5.5}$$

This is the Eikonal phase function. It should be emphasized that the impact parameter in this limiting case, is the distance of closest approach of the classical trajectory, which corresponds to the classical turning point. This picture becomes fragile if l approaches zero. In fact it is in the proximity of such point[8] that the expansion parameter is too large to be at all meaningful.

Wallace [15-18] expands eqn (5.5.4) around the parameter $\epsilon \sim V_o/\hbar kv$ and shows that the WKB phase-shift can be expressed as,

$$\delta_j^{WKB}(b) = \sum_{n=0}^{\infty} \delta_n^j(b)\,, \tag{5.5.6}$$

[6]Unlike the spinless case, the potential depends on l. Nevertheless, with some care, the argument still holds.

[7]Note that this is not entirely legitimate since this alters the value of the turning point which is defined in terms of the integrand.

[8]For classical scattering when $\theta = 180^o$, then the kinetic energy$\rightarrow$ 0.

where

$$\delta^j_n(b) \;=\; -\frac{1}{2k(n+1)!}\left\{\left\{\frac{b}{k}\frac{\partial}{\partial b} - \frac{\partial}{\partial k}\right\}\frac{1}{2k}\right\}^n \int_0^\infty dz\,\mathcal{V}_j^{n+1}(b,r)\,. \qquad (5.5.7)$$

To emphasize the separate radial dependence from the spin-coupling one, $l \to b$ the argument of the potential is expressed in terms of (b,r), where $r^2 = b^2 + z^2$. In their work Waxman *et al*, discuss the consequence of the *l*-dependence of the potential with regard to the differential operator in the above sum. The operator is equivalent to,

$$\frac{b}{k}\left\{\frac{\partial}{\partial b}\right\}_k - \left\{\frac{\partial}{\partial k}\right\}_b \;=\; -\left\{\frac{\partial}{\partial k}\right\}_l\,. \qquad (5.5.8)$$

This fixed-*l* form demonstrates the independence of the expansion on the *l-structure* of the potential. It also stresses the dynamical nature of the series. Using the displacement operator they write eqn (5.5.4) as,

$$\delta_j^{WKB}(l) \;=\; \frac{1}{2}\int_{k^2}^\infty dk'^2 \int_0^\infty r\;\; dr\left\{1 - \exp\left[-\mathcal{V}_j(l,r)\left(\frac{\partial}{\partial k'^2}\right)_l\right]\right\}$$
$$\times\;\; (k'^2 r^2 - (l+\frac{1}{2})^2)^{-1/2}\mathcal{H}(k'r - (l+\frac{1}{2}))\,,$$

$$(5.5.9)$$

where $\mathcal{H}(x)$ is the Heavside step function. Expanding the exponential and carrying out the integration over k'^2 it is straightforward to arrive at the Wallace expansion. The first two terms of the series are

$$\delta^j_o(b) \;=\; -\frac{m}{2k}\int_0^\infty dz\,\mathcal{V}_j(r) \qquad (5.5.10)$$

$$\delta^j_1(b) \;=\; -\frac{m^2}{8k^3}\left\{1 + b\frac{\partial}{\partial b} - k\frac{\partial}{\partial k}\right\}\int_0^\infty dz\mathcal{V}_j^2(r)$$

$$(5.5.11)$$

The WKB phase contains the Glauber phase function $\delta_0^j(b)$ as its leading order term in a derivative expansion in powers of the potential. The higher order terms $n \neq 1$ can be thought of as dynamical corrections to the straight line path of the trajectory inside the potential assumed by Glauber. Wallace goes on to show that if higher order corrections to the WKB approximation, such as the Rosen and Yennie ones [48], are included, the extra contribution which is linear in the potential cancels the unitarity corrections to the $S_F(b)$ operator.

(5.6) The eikonal amplitude

If we go back to the three elements ($\mathcal{V}^+, \mathcal{V}^-, \mathcal{V}^0$) of the reduced potential matrix given by eqn (5.1.14-16) and, with the help of eqn (5.5.9), define the *Eikonal phases*

$$\begin{aligned} \chi^j(b) &= 2\,\delta_j^{WKB}(b) \\ &= 2\,(\delta_0^j(b) + \delta_1^j(b) \ldots.)\,. \end{aligned} \tag{5.6.1}$$

Then we can write

$$\chi^+(b) = (\chi_c - \frac{1}{2}\chi_s + \frac{1}{6}\chi_l) + kb(\chi_s - \frac{1}{2}\chi_l) + \frac{1}{3}(kb)^2\chi_l, \tag{5.6.2}$$

$$\chi^-(b) = (\chi_c - \frac{1}{2}\chi_s + \frac{1}{6}\chi_l) - kb(\chi_s + \frac{1}{2}\chi_l) + \frac{1}{3}(kb)^2\chi_l, \tag{5.6.3}$$

$$\chi^0(b) = (\chi_c - \chi_s + \frac{2}{3}\chi_l) - \frac{2}{3}(kb)^2\chi_l\,. \tag{5.6.4}$$

Substituting these into eqn (5.4.2-4) we obtain,

$$\bar{\chi}(b)_0 = \frac{1}{3}(3\chi_c(b) - 2\chi_s(b) + \chi_l(b))\,, \tag{5.6.5}$$

$$\Delta\chi(b)_0 = (kb)(2\chi_s(b) - \chi_l(b))\,, \tag{5.6.6}$$

$$\mathcal{D}\chi(b)_0 = \chi_s(b) - \chi_l(b) + 2(kb)^2\chi_l(b). \qquad (5.6.7)$$

Inserting these into the Fourier-Bessel representation eqns (5.4.10-13), we arrive at the Wallace's Eikonalised scattering amplitude. These amplitudes, as expected, give the correct first-Born limit, as we will demonstrate later.

We first calculate the first-quantum correction using eqn (5.5.11). Remembering that the operator does not act on the combination (kb), we will define, ($\gamma, \beta = c, s, l$),

$$\chi_{\gamma\beta}(b) = -\frac{m^2}{2k^3}\left[1 + b\frac{\partial}{\partial b} - k\frac{\partial}{\partial k}\right]\int_{-\infty}^{+\infty} dz\, V_\gamma(r)V_\beta(r). \qquad (5.6.8)$$

Hence we can write,

$$\begin{aligned}\bar{\chi}(b)_1 &= \frac{2}{9}(kb)^4\chi_{ll} + (kb)^2(\frac{2}{3}\chi_{ss} - \frac{1}{18}\chi_{ll} - \frac{4}{9}\chi_{ls}) \\ &+ \chi_{cc} + \frac{1}{6}\chi_{ll} + \frac{1}{2}\chi_{ss} + \frac{2}{3}\chi_{cl} - \frac{4}{3}\chi_{cs} - \frac{5}{9}\chi_{ls}, \qquad (5.6.9)\\ \Delta\chi(b)_1 &= \frac{4}{3}(kb)^3(\chi_{ls} - \frac{1}{2}\chi_{ll}) - (kb)(2\chi_{ss} + \frac{1}{3}\chi_{ll} + 2\chi_{cl} - \frac{5}{3}\chi_{ls} - 4\chi_{cs})\end{aligned}$$

$$(5.6.10)$$

$$\begin{aligned}\mathcal{D}\chi(b)_1 &= -\frac{2}{3}(kb)^4\chi_{ll} + (kb)^2(2\chi_{ss} + \frac{5}{2}\chi_{ll} + 4\chi_{cl} - \frac{16}{3}\chi_{ls}) \\ &- \frac{3}{2}\chi_{ss} - \frac{5}{6}\chi_{ll} - 2\chi_{cl} + \frac{7}{3}\chi_{ls} + 2\chi_{cs}. \qquad (5.6.11)\end{aligned}$$

Expanding the Γ functions (5.4.6-8) in powers of the potential and retaining terms only up to second-order we get

$$\begin{aligned}S_F(b)\Gamma_{\mathcal{A}}(b) \sim &\Big\{\chi_c + \chi_{cc} + i\chi_c^2 + (kb)^2(\frac{2}{3}\chi_{ss} + \frac{i}{3}\chi_s^2) \\ &+ (-5(kb)^2 + 4(kb)^4)\left[\frac{1}{18}\chi_{ll} + \frac{i}{36}\chi_l^2\right]\Big\}, \qquad (5.6.12)\end{aligned}$$

$$S_F(b)\Gamma_B(b) \sim \frac{1}{2\sin\theta}\Big\{(kb)^2(\frac{1}{2}\chi_s^2 - 2\chi_c\chi_s + 2i\chi_s - \frac{i}{2}\chi_{ss}$$
$$+ 2i\chi_{cs}) + (-\frac{1}{4} + 5(kb)^2 + 4(kb)^4)$$
$$\Big[\frac{i}{6}\chi_{ls} - \frac{i}{8}\chi_{ll} - \frac{1}{6}\chi_l\chi_s + \frac{1}{8}\chi_l^2\Big]\Big\}, \qquad (5.6.13)$$
$$S_F(b)\Gamma_C^k(b) \sim \frac{1}{\sin^2\theta}\Big\{(kb)^2\Big[-\chi_l - \chi_{cl} + \frac{3}{2}\chi_{ls} - \frac{1}{2}\chi_{ss}$$
$$- i\chi_c\chi_l + \frac{3i}{2}\chi_l\chi_s - \frac{i}{2}\chi_s^2\Big]$$
$$- \frac{1}{24}(\frac{1}{4} + 7(kb)^2 - 4(kb)^4)\Big[\chi_{ll} + i\chi_l^2\Big]\Big\}. \qquad (5.6.14)$$

The main difference between these expressions and the equivalent spinless and spin-half cases is that, we needed to keep one term higher in the perturbative operator $S_F(b)$, namely

$$S_F(b) = \Big[1 + \frac{1}{24k^2}(\frac{d}{db})^3 b\Big]. \qquad (5.6.15)$$

This is because of the extra $b(kb)^2$ coefficient that appears in the functions $\Gamma(b)$. For example,

$$\Gamma_B(b)\,|_{1st order} \sim \frac{ib}{\sin\theta}\Big[(kb)^2 - \frac{1}{4}\Big]\chi_s(b). \qquad (5.6.16)$$

From the structure of the operator (5.3.13), it follows that the extra (kb) factor requires us to include the second order term in the $S_F(b)$ series. To first order in potential strength the *Eikonal* condition $|k\chi(b)| \gg |\nabla\chi(b)|$, allows us to neglect all derivatives of the phase function arising[9] from this operator. Hence we get,

$$S_F(b)\Gamma_B(b)|_{1^{st}order} \sim \frac{i(kb)^2}{\sin\theta}\chi_s(b). \qquad (5.6.17)$$

[9]These are the unitarity corrections mentioned earlier, see the discussion following eqn (5.3.13).

Higher order terms, on the other hand, involve both kinematical (unitarity) and dynamical corrections, though the unitarity corrections are of order $(1/k)$ smaller than the dynamical ones. In what follows we will restrict the Eikonal condition, mentioned above, to the unitarity corrections. To compare the Eikonalized with the exact Born amplitude we again assume the potential to be of the Gaussian form given by eqn (2.2.1). In this case the function (5.6.8) simplifies to the form,

$$\chi_{\gamma\beta} = \frac{\alpha\chi_\gamma\chi_\beta}{2k\sqrt{2\pi}}\left((\alpha b)^2 - \frac{1}{2}\right). \qquad (5.6.18)$$

We once more use the function defined by eqn (4.3.23) and, after some algebraic manipulation, arrive at,

$$\begin{aligned}\Re\,\mathcal{A}(\theta)|_{1^{st}+2^{nd}order} &= -\mu\sqrt{2\pi}\exp[-\frac{q^2}{2\alpha^2}]\frac{V_{oc}}{\alpha^3} \\ &+ \mu^2\sqrt{\pi}\exp[-\frac{q^2}{4\alpha^2}]\left\{V_{oc}^2\left[\frac{1}{4\alpha^3k^2} - \frac{q^2}{8\alpha^5k^2}\right]\right. \\ &+ V_{ol}^2\left[-\frac{5}{24\alpha^5} - \frac{2k^2q^2}{9\alpha^9} + \frac{17k^2q^4}{288\alpha^{11}} - \frac{k^2q^6}{576\alpha^{13}} + \frac{5k^2}{9\alpha^7}\right. \\ &+ \left.\left.\frac{35q^2}{288\alpha^7} - \frac{5q^4}{576\alpha^9}\right] + V_{os}^2\left[\frac{1}{2\alpha^5} - \frac{7q^2}{24\alpha^7} + \frac{q^4}{48\alpha^9}\right]\right\}\end{aligned} \qquad (5.6.19)$$

$$\begin{aligned}\Im\,\mathcal{A}(\theta)|_{1^{st}+2^{nd}order} &= \mu^2\pi\exp[-\frac{q^2}{4\alpha^2}]\left\{V_{oc}^2\frac{1}{\alpha^4k} + V_{ol}^2\left[-\frac{5k}{36\alpha^6} + \frac{5kq^2}{144\alpha^8}\right.\right. \\ &- \left.\left.\frac{k^3q^2}{9\alpha^{10}} + \frac{k^3q^4}{144\alpha^{12}} + \frac{2k^3}{9\alpha^8}\right] + V_{os}^2\left[+\frac{k}{3\alpha^6} - \frac{kq^2}{12\alpha^8}\right]\right\}\end{aligned} \qquad (5.6.20)$$

$$\begin{aligned}\Re\,\mathcal{B}(\theta)|_{1^{st}+2^{nd}order} &= \frac{\mu^2\pi}{2\cos\theta/2}\exp[-\frac{q^2}{4\alpha^2}]\left\{+V_{ol}V_{os}\left[\frac{5q}{12\alpha^6} - \frac{2k^2q}{3\alpha^8} + \frac{k^2q^3}{12\alpha^{10}}\right]\right. \\ &+ \left.V_{oc}V_{os}\frac{q}{\alpha^6} + V_{ol}^2\left[-\frac{5q}{16\alpha^6} + \frac{k^2q}{2\alpha^8} - \frac{k^2q^3}{16\alpha^{10}}\right] - V_{os}^2\frac{q}{4\alpha^6}\right]\right\}\end{aligned}$$

$$(5.6.21)$$

$$
\begin{aligned}
\Im\mathcal{B}(\theta)|_{1^{st}+2^{nd}order} &= -\frac{\mu\sqrt{2\pi}}{2\cos\theta/2}\exp[-\frac{q^2}{2\alpha^2}]\frac{\bar{k}q}{\alpha^5} \\
&+ \frac{\mu^2\sqrt{\pi}}{2\cos\theta/2}\exp[-\frac{q^2}{4\alpha^2}]\left\{-V_{oc}V_{os}\left[\frac{q}{2\alpha^5 k}+\frac{q^3}{32\alpha^7 k}\right]\right. \\
&+ V_{ol}V_{os}\left[\frac{q}{96\alpha^3 k^3}+\frac{25q}{48\alpha^5 k}+\frac{5kq}{6\alpha^7}-\frac{23kq^3}{48\alpha^9}\right. \\
&+ \left.\frac{kq^5}{96\alpha^{11}}-\frac{7q^3}{96\alpha^7 k}\right]+V_{ol}^2\left[-\frac{q}{128\alpha^3 k^3}-\frac{21q}{32\alpha^5 k}\right. \\
&- \left.\frac{5kq}{8\alpha^7}+\frac{23kq^3}{64\alpha^9}-\frac{kq^5}{128\alpha^{11}}+\frac{5q^3}{128\alpha^7 k}\right] \\
&+ \left.V_{os}^2\left[-\frac{5q}{16\alpha^5 k}+\frac{q^3}{32\alpha^7 k}\right]\right\} \qquad (5.6.22)
\end{aligned}
$$

$$
\begin{aligned}
\Re\mathcal{C}_k(\theta)|_{1^{st}+2^{nd}order} &= \frac{\mu\sqrt{2\pi}}{\cos^2\theta/2}\exp[-\frac{q^2}{2\alpha^2}]V_o^l\left[\frac{\bar{k}^2}{\alpha^5}+\frac{q^2}{4\alpha^5}\right]-\frac{\mu^2\sqrt{\pi}}{\cos^2\theta/2}\exp[-\frac{q^2}{4\alpha^2}] \\
&\times \left\{V_{oc}V_{ol}\left[\frac{5}{8\alpha^5}-\frac{q^2}{16\alpha^7}\right]-V_{ol}V_{os}\left[\frac{15}{16\alpha^5}-\frac{3q^2}{32\alpha^7}\right]\right. \\
&- V_{ol}^2\left[\frac{1}{384\alpha^3 k^2}-\frac{35}{192\alpha^5}+\frac{7q^2}{384\alpha^7}+\frac{5k^2}{24\alpha^7}-\frac{23k^2q^2}{192\alpha^9}+\frac{k^2q^4}{384\alpha^{11}}\right] \\
&- \left.V_{os}^2\left[\frac{5}{16\alpha^5}-\frac{q^2}{32\alpha^7}\right]\right\} \qquad (5.6.23)
\end{aligned}
$$

$$
\begin{aligned}
\Im\mathcal{C}_k(\theta) &= \frac{\mu^2\pi}{\cos^2\theta/2}\exp[-\frac{q^2}{4\alpha^2}]\left\{-V_{oc}V_{ol}\frac{k}{2\alpha^6}+V_{ol}V_{os}\frac{3k}{4\alpha^6}\right. \\
&- \left.V_{ol}^2\left[\frac{7k}{48\alpha^6}+\frac{k^3}{6\alpha^8}-\frac{k^3q^2}{48\alpha^{10}}\right]-V_{os}^2\frac{k}{4\alpha^6}\right\}. \qquad (5.6.24)
\end{aligned}
$$

We have defined $\bar{k} = k_i\cos\theta/2$, the magnitude of the average initial and final momenta. These three amplitudes are seen to be in complete agreement with the first-order Born amplitudes eqn (3.2.14-17). However calculating the first-order term of the amplitude $\mathcal{C}_n(\theta)$ through eqn (5.4.13) results in

$$\mathcal{C}_n(\theta)|_{1^{st}order} = -\frac{kq(2\cos\theta + \sin^2\theta/2)}{\sin^2\theta}\mathcal{I}_{l,1} + k^3\mathcal{I}^1_{l,0}$$
$$= \frac{\mu\sqrt{2\pi}}{\cos^2\theta/2}\exp[-q^2/2\alpha^2]\,V^l_o\left[\frac{\bar{k}^2q^q}{\alpha^7} - \frac{q^2}{4\alpha^5}\right]. \quad (5.6.25)$$

This is again the same problem we faced in connection with both the simple and averaged Eikonal amplitudes, only this time it is confined to this one term. However if, instead of using the recurrence relation (5.4.9), we used,

$$P^m_n(x) = (1 - x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\,, \qquad (5.6.26)$$

then we would have reached the result,

$$\mathcal{C}_n(\theta) = -\left(\sin^2\theta\frac{d^2}{d\cos^2\theta} + \sin^2\theta/2\frac{d}{d\cos\theta}\right)k\int_0^\infty bdbJ_0(qb)\chi_l$$
$$= \mu\sqrt{2\pi}\exp[-q^2/2\alpha^2]\,V^l_o\left[\frac{\bar{k}^2q^2}{\alpha^7} + \frac{q^2}{4\alpha^5}\right], \qquad (5.6.27)$$

which is identical to the first-Born amplitude. Nevertheless, because of its simpler form, we will use the first derivation when carrying out our numerical calculations, bearing in mind that the difference is only $(\alpha^2/2k^2)$ which can be neglected at high energies.

(5.7) Numerical calculations for a Gaussian potential

To complete our analysis we calculate numerically[10] the Wallace-eikonalized scattering amplitudes eqns (5.4.10-13) without corrections, using the phases defined by eqns (5.6.5-7), with the first-quantum corrections, using eqns (5.6.8-11) substituted in eqn (5.6.1). In the graphs Fig (5.1-4) presented

[10]This calculation was done using the Fortran computer language rather than SMP.

in this section we compare the eikonalized amplitudes with the exact amplitudes which were calculated using the program DDTP [25]. As in section (3.6) we have used a Gaussian distribution for the potential. In this section however, we use complex potentials. The imaginary part of the central potential has been fixed using the optical theorem. The imaginary part of the spin-orbit and tensor potentials are assumed to be equal and fixed to have the same ratio to the complex central term as the ratio between the central and spin-orbit term in nucleon-nucleus scattering [21]. The following are the parameters used; reduced mass $\mu = 6.542$ fm, wave number $k = 2.927\, fm^{-1}$, range $a = 2.92794$, potential strength $V_{oc} = (0.10135 - i\,0.12949)fm^{-1}$, $V_{os} = (0.05067 + i\,0.02746)fm^{-1}$, and $V_{ol} = (0.02533 + i\,0.02746)fm^{-1}$.

The results show that the inclusion of the Wallace corrections improve the amplitudes significantly.

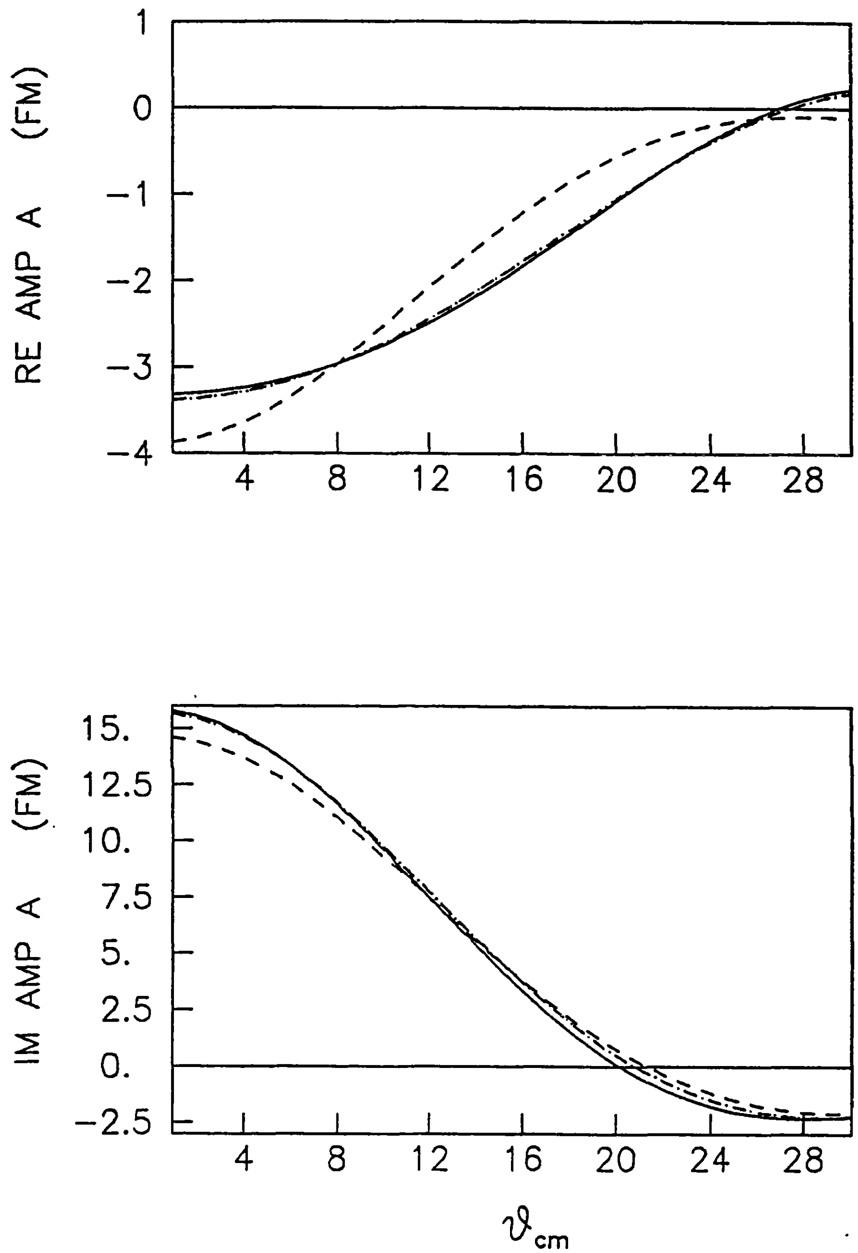

Figure (5.7.1). The amplitude $\mathcal{A}(\theta)$. The dot-dashed & dashed lines represent the eikonal amplitude with and without corrections respectively. The solid line is the amplitude calculated from the code DDTP. Parameters are as defined in section (5.7).

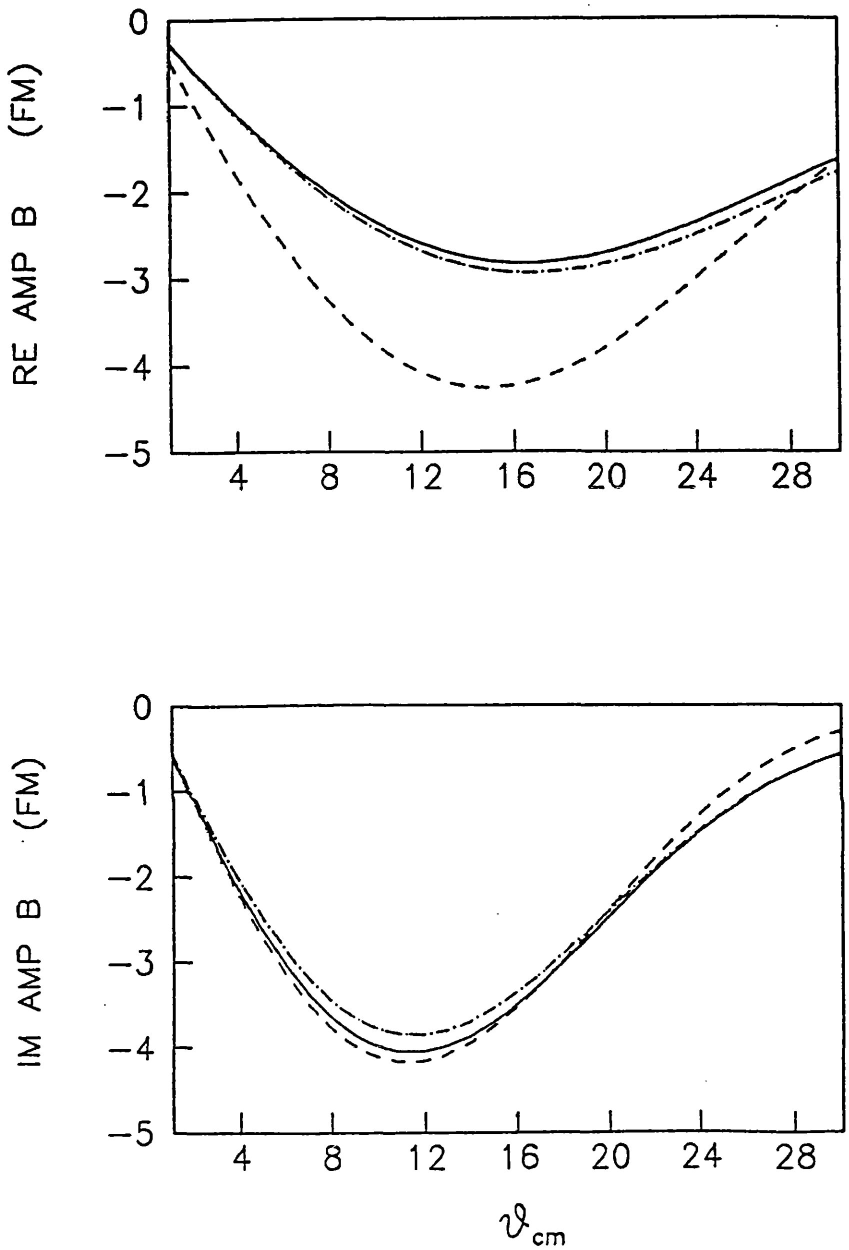

Figure (5.7.2). The amplitude $\mathcal{B}(\theta)$. The dot-dashed & dashed lines represent the eikonal amplitude with and without corrections respectively. The solid line is the amplitude calculated from the code DDTP. Parameters are as defined in section (5.7).

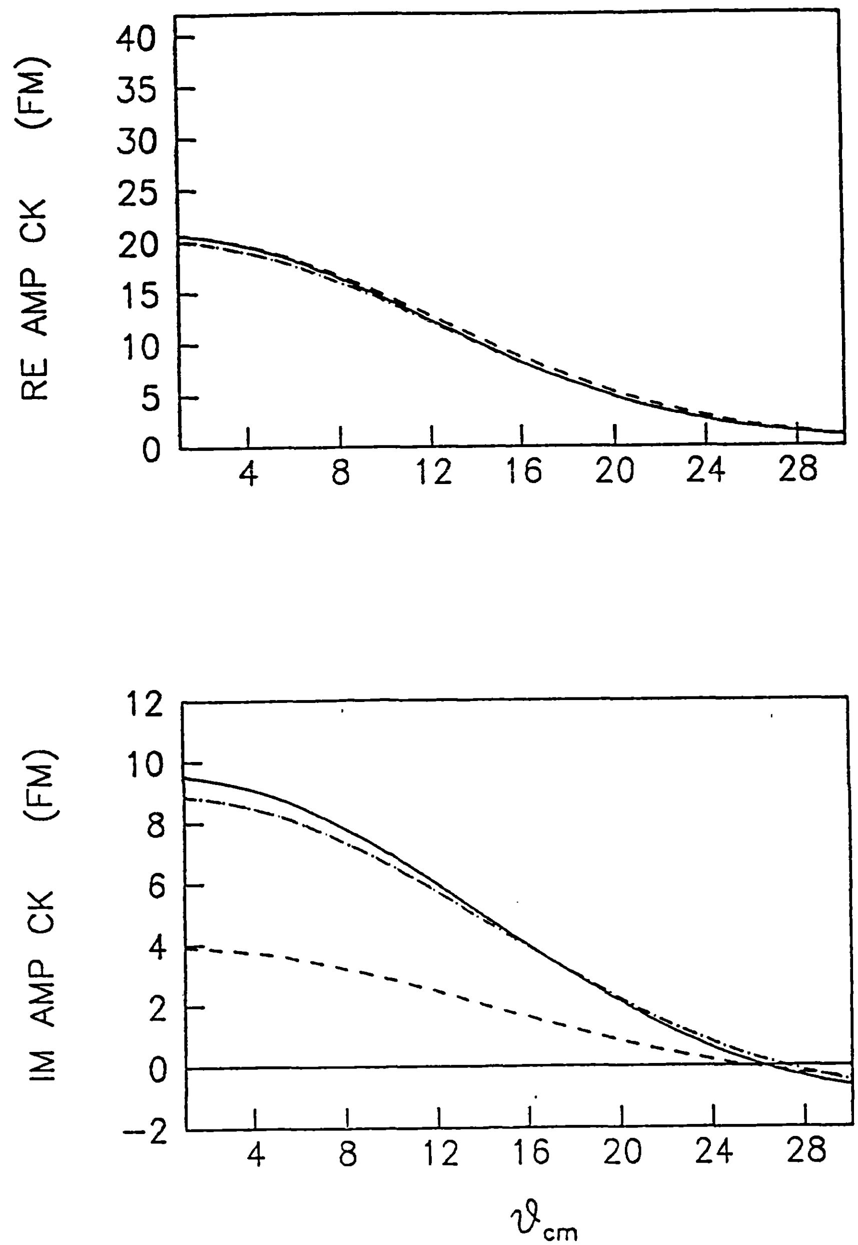

Figure (5.7.3). The amplitude $C_k(\theta)$. The dot-dashed & dashed lines represent the eikonal amplitude with and without corrections respectively. The solid line is the amplitude calculated from the code DDTP. Parameters are as defined in section (5.7).

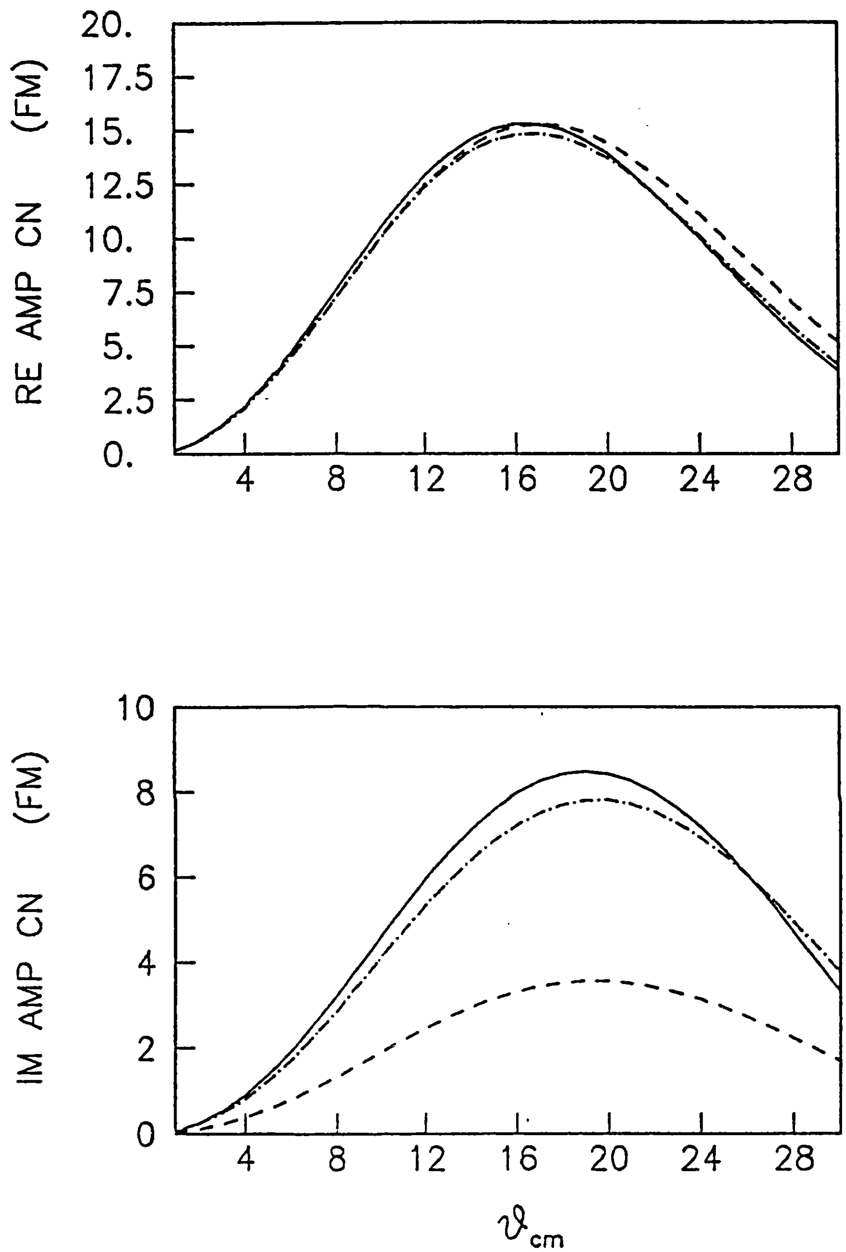

Figure (5.7.4). The amplitude $C_n(\theta)$. The dot-dashed & dashed lines represent the eikonal amplitude with and without corrections respectively. The solid line is the amplitude calculated from the code DDTP. Parameters are as defined in section (5.7).

(5.8) Numerical calculation of the observables for d–^{58}Ni

In this section we will use a numerical distribution for the potential [5]. This phenomenological potential has been calculated by folding the nucleon-nucleus Dirac optical potentials [49-52]. We employ the same distribution for both the spin-orbit and the tensor potentials. Two sets of results for d–^{58}Ni observables are presented, 400 and 700 Mev incident deuteron energies. In both cases we compare the exact (DDTP) observables with those calculated from the Wallace-eikonalised amplitudes, with and without corrections. The graphs clearly demonstrate the significant effect of the corrections in improving the agreement between the exact and the eikonal observables. Finally we present an example of the type of agreement that exists in the case where the tensor-potential $T_L(\vec{r})$ is switched off. From this last graph we can see that the effect of Wallace's dynamical corrections is more significant in the case when the tensor spin-orbit coupling is included in the optical potential. In this calculation we have worked in the reference frame defined by the Madison convention. The tensor analyzing powers together with the transformation from the amplitudes defined by eqns (5.4.10-13) to the set of five amplitudes employed in the DDTP [25] code are both given in the appendix.

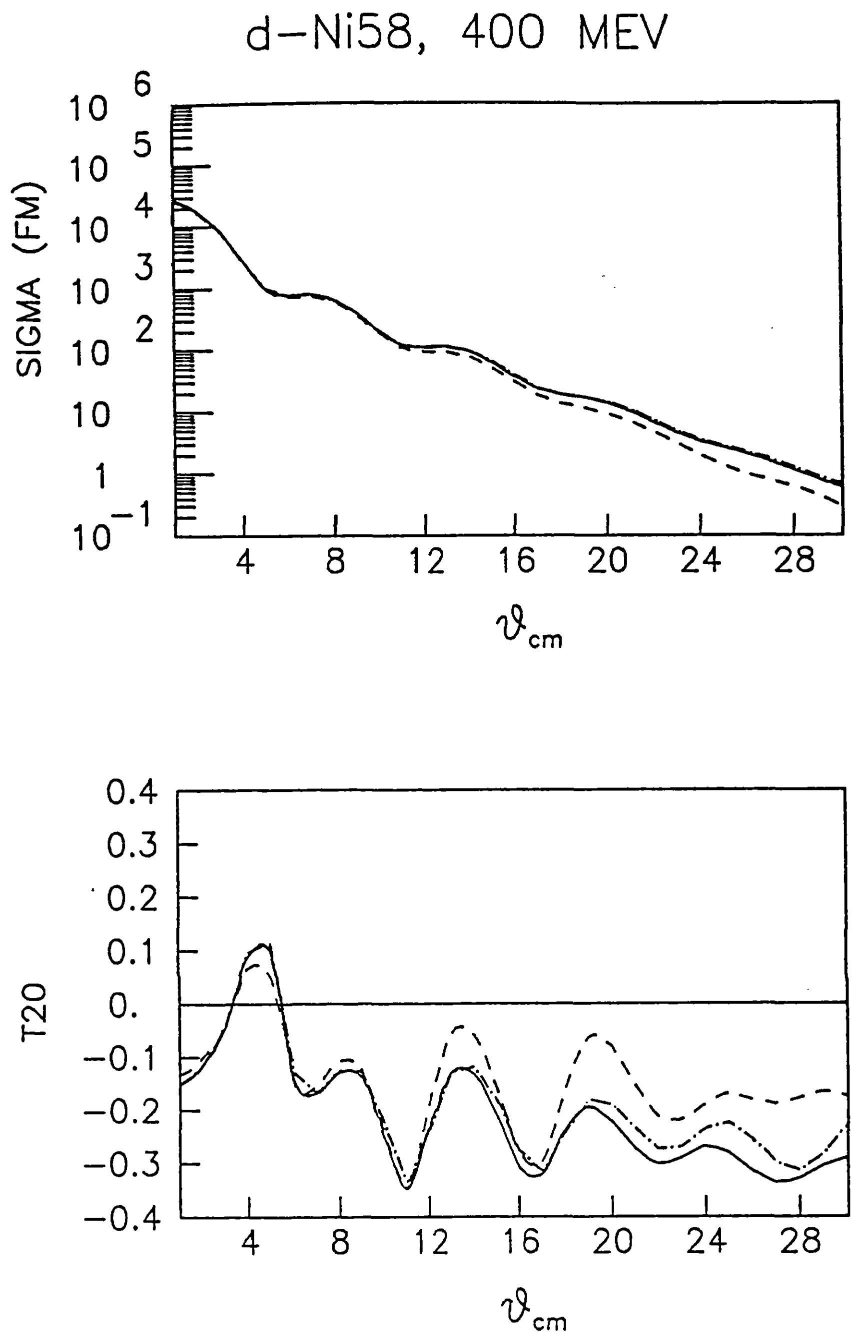

Figure (5.8.1). Differential cross section σ and tensor analyzing power T_{20} for d–^{58}Ni at 400 Mev deuteron incident energy. The dot-dashed and the dashed lines represent the observables calculated using the eikonalized amplitudes with and without corrections respectively. The solid line represents the DDTP calculation.

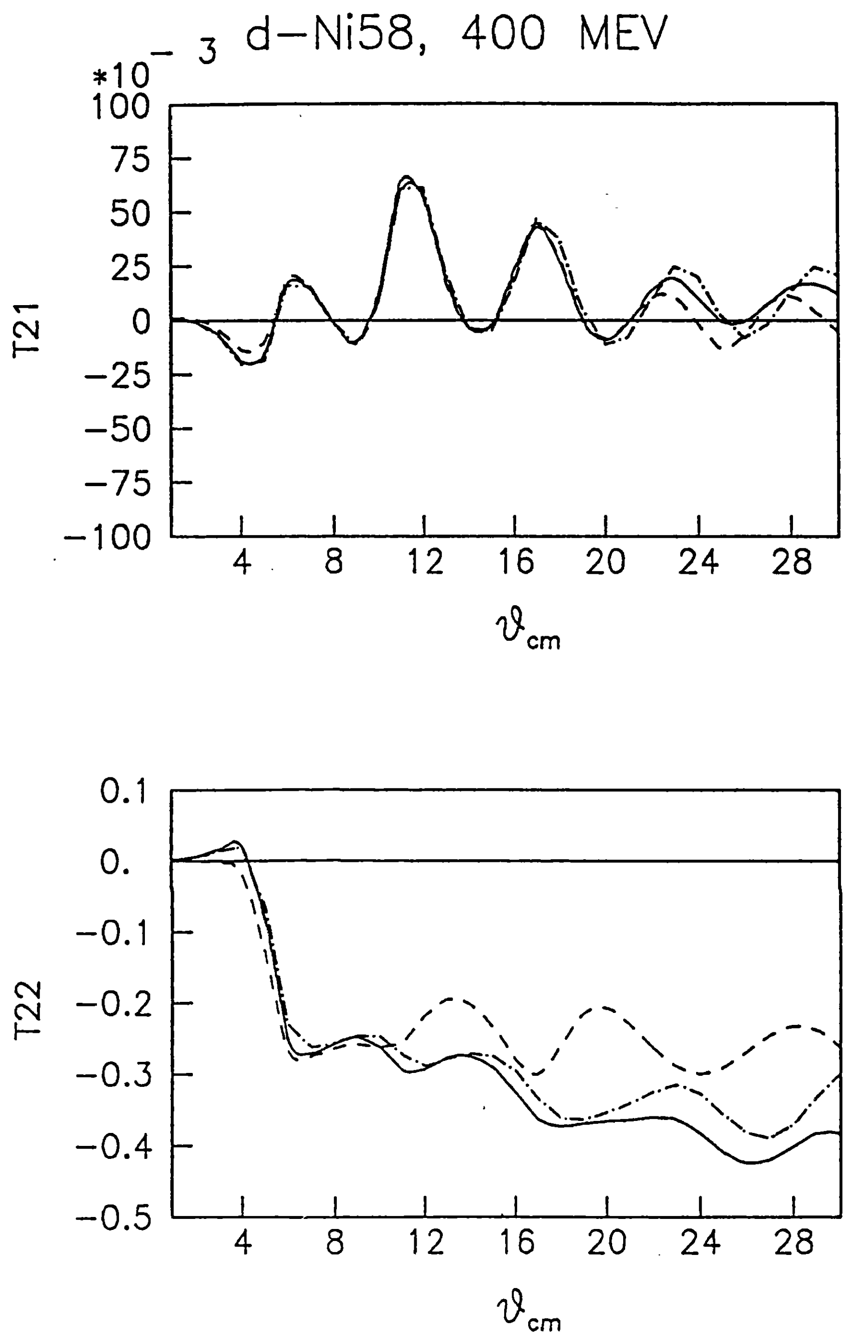

Figure (5.8.2). Tensor analyzing powers T_{21} and T_{22} for d–^{58}Ni at 400 Mev deuteron incident energy. Lines are defined as in figure (5.8.1).

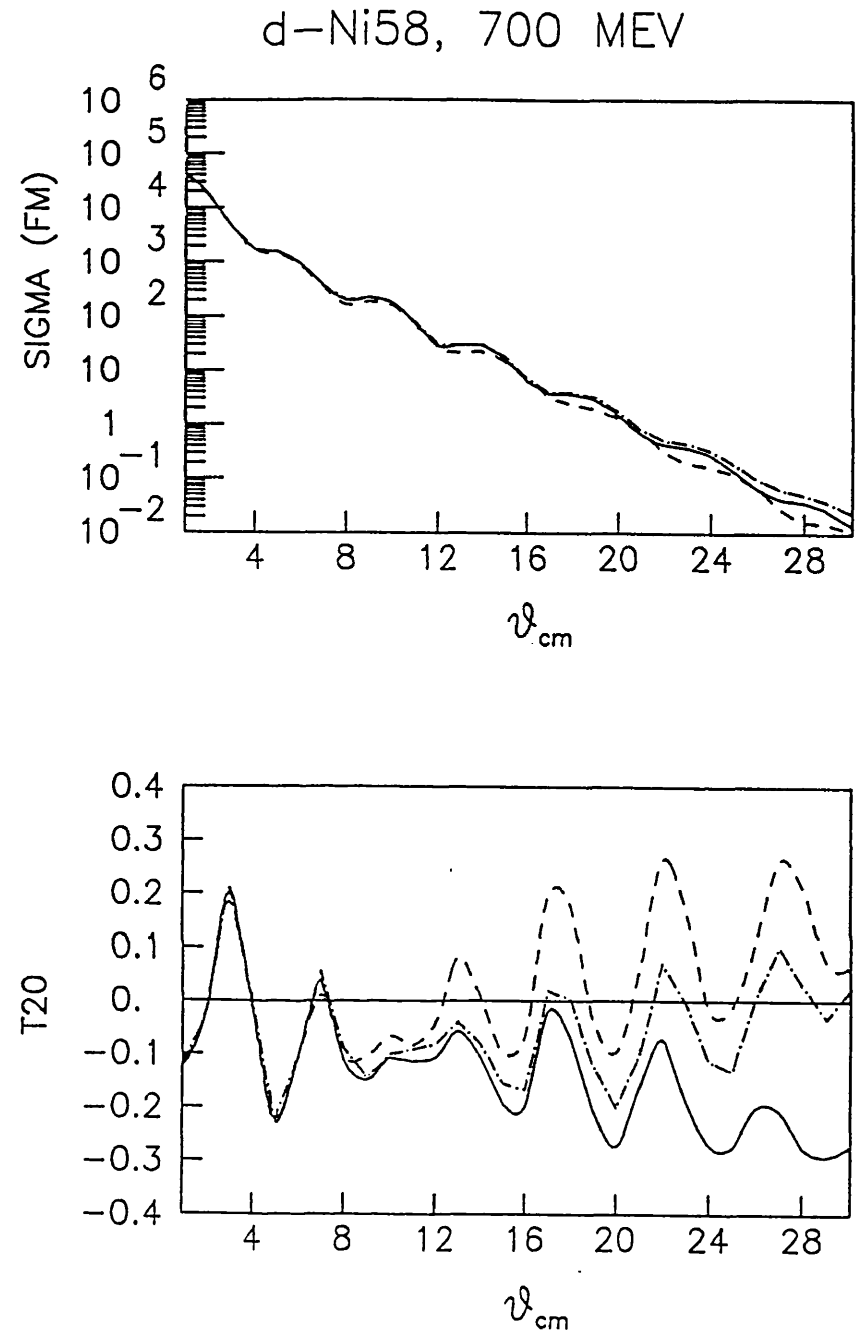

Figure (5.8.3). Differential cross section σ and tensor analyzing power T_{20} for d–^{58}Ni at 700 Mev deuteron incident energy. The dot-dashed and the dashed lines represent the observables calculated using the eikonalized amplitudes with and without corrections respectively. The solid line represents the DDTP calculation.

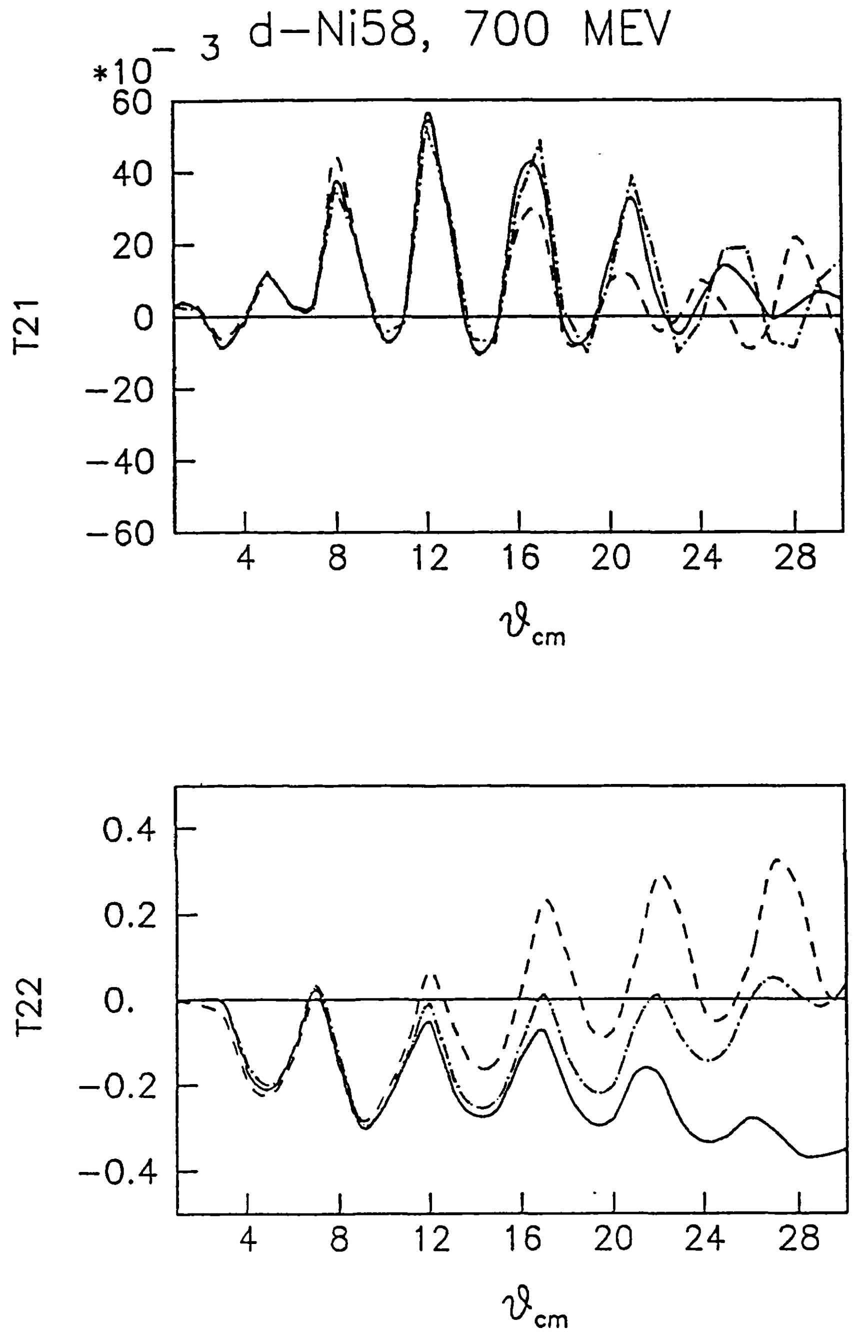

Figure (5.8.4). Tensor analyzing powers T_{21} and T_{22} for d–^{58}Ni at 700 Mev deuteron incident energy. Lines are defined as in figure (5.8.3).

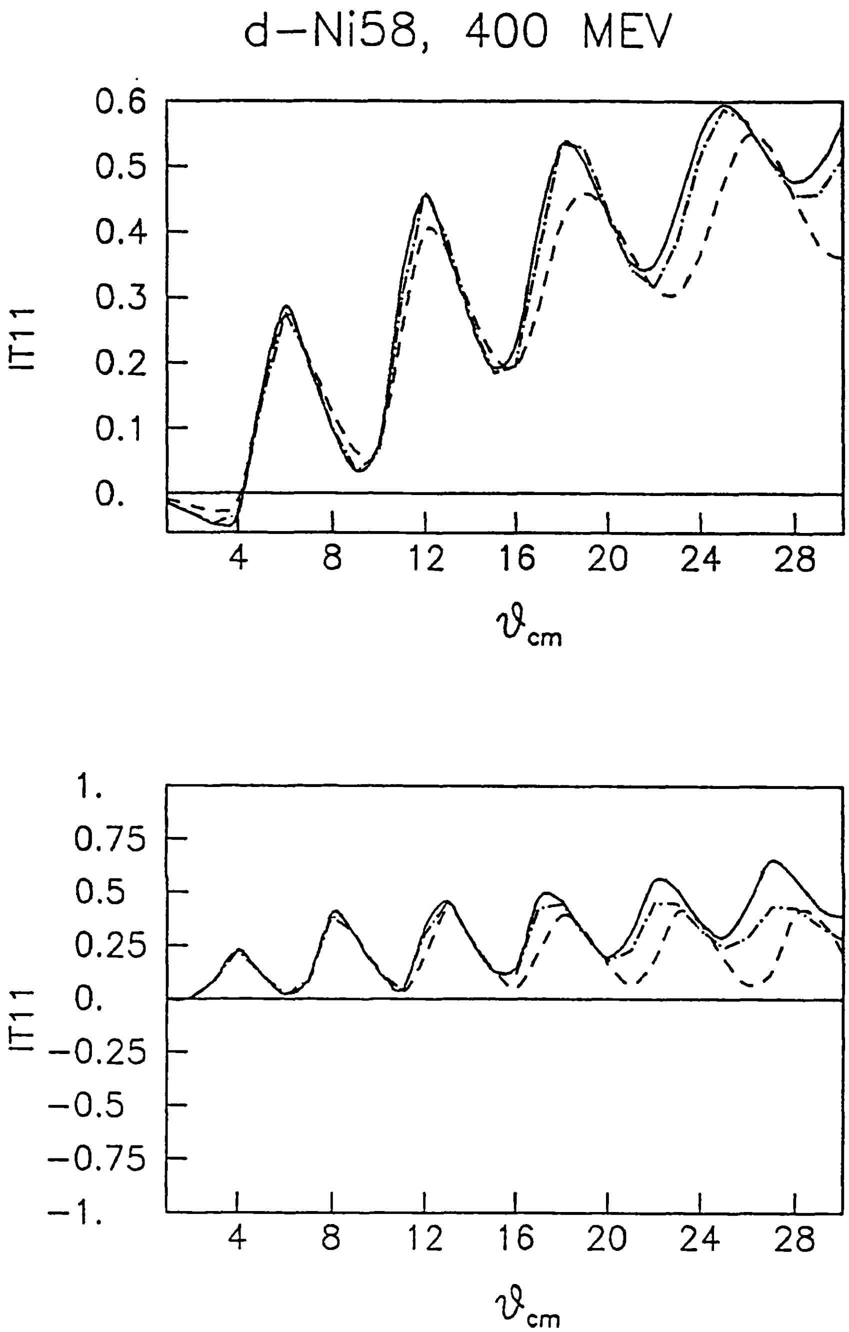

Figure (5.8.5). The top and bottom graphs is iT_{11} for d–^{58}Ni at 400 and 700 mev deuteron incident energy respectively. The dot-dashed, the dashed and the solid lines represent the same calculations as in the previous figures.

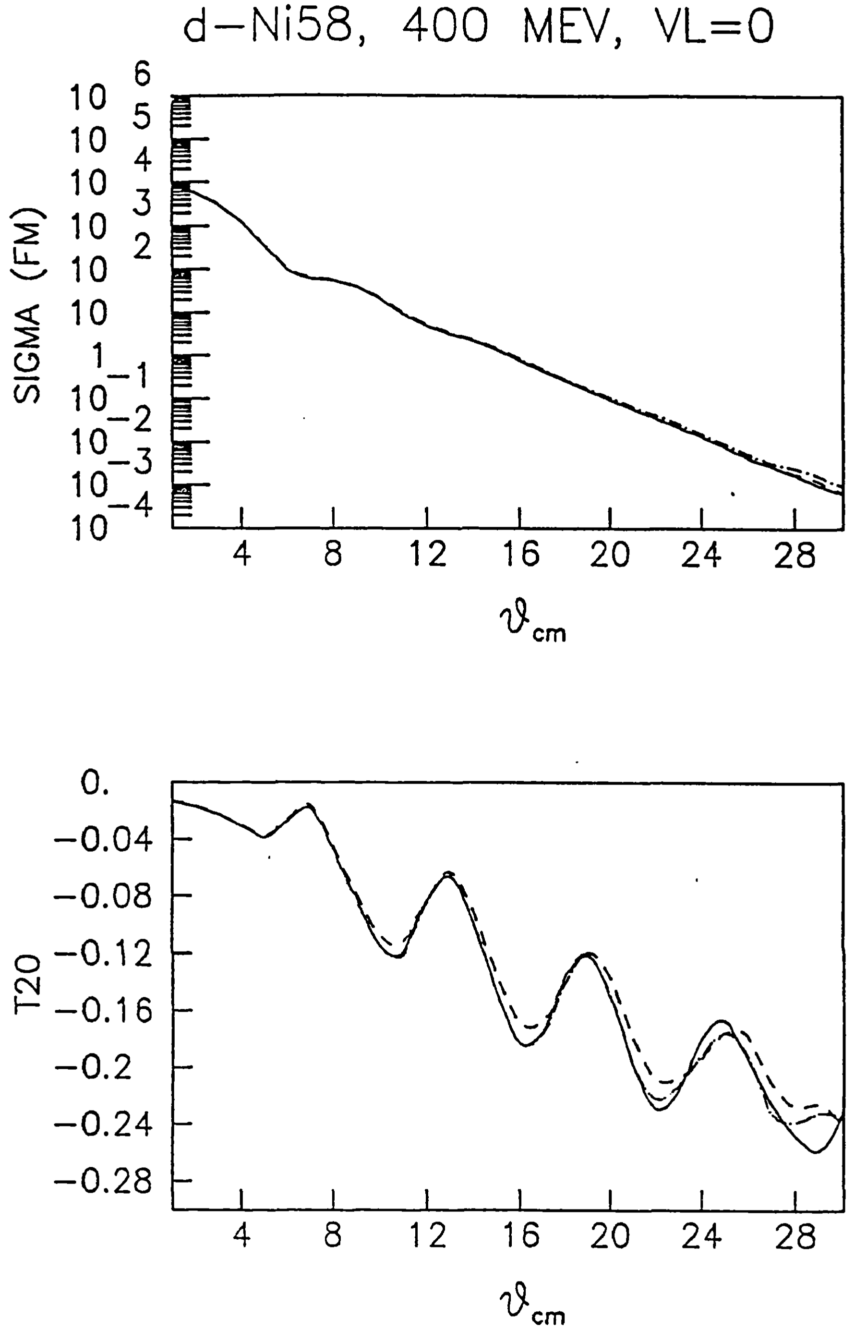

Figure (5.8.6). Differential cross-section and T_{20} in the case the tensor potential is switched off $V_{ol} = 0$ at 400 Mev deuteron incident energy. The lines are as in the previous figures. Note that the difference between the corrected and the uncorrected eikonal-observables has become less significant.

CHAPTER 6

CONCLUSIONS

Previous work [15-19,21] established the connection between the Fourier-Bessel expansion of the scattering amplitude and its eikonal counterpart in the cases of spin-independent and spin-half spin-zero interactions. This project is mainly concerned with generalizing the above mentioned approach to the case of spin-one spin-zero interactions. We have developed a direct connection between the partial-wave and Fourier-Bessel descriptions of the scattering amplitude in the case of spin-one interactions. This connection is very useful and provides some corrections to the Glauber eikonalized amplitudes.

We discussed four different schemes for arriving at the scattering amplitude. The first of these was the perturbative Born-scheme which we developed in chapter three. Limiting the potential distribution to a Gaussian form, we calculated the first and Second Born amplitudes for an interaction involving a central (spin-independent), a vector spin-orbit and a tensor spin-orbit terms. The second Born term involved a large number of terms. In calculating it we made use of the SMP software language. We then calculated the high-energy limit of the Born amplitudes.

By inspecting the formal expression of the eikonal amplitude, we can verify that an expansion of the eikonal scattered wave function in powers of the potential results in a perturbative series that agrees with the Born-series exactly only to first order. The higher order eikonal terms, however, alternate between pure real and pure imaginary, this does not happen in the exact Born-series. Swift [22] has demonstrated that the eikonal amplitude is the high-energy limit of the Born series at fixed momentum transfer. We therefore employed the Born terms first as a tool in deciding on the validity of our various attempts to eikonalize the scattering amplitude and second to compare the eikonal amplitude with and without Wallace's dynamical corrections with the high-energy limit of the Born series.

In our second approach we attempted to eikonalize the scattering amplitude in a manner similar to Glauber's treatment of the spin-half case. Difficulties arise partly from the properties of spin-one operators and partly from the preferential treatment given to the initial momentum in the eikonal scheme. The presence, in the exponential ansatz, of vectors that lie in the scattering plane meant that we could not follow Glauber and replace the initial wave vector by the average vector. In our first attempt to overcome these difficulties, we followed Glauber [9] and dropped the velocity dependence completely from the eikonal exponential ansatz. This eliminated the problem of the non-commuting matrices which were present in the original ansatz, hence allowing us to develop the eikonal amplitude by expanding the exponential matrix in a power series using the properties of spin-one matrices. However this approach is very much equivalent to the Glauber wave vector averaging mentioned above. This meant that the resultant matrix misses the terms that lie in the scattering plane. In fact we ended up with a matrix that resembles a scattering matrix arrived at by defining the Z-axis along the average direction of the incident and final momenta.

In an alternative route we diagonalized the exponential matrix and rewrote it in terms of three orthogonal matrices. This allowed us to expand the exponential matrix without any approximations. The resulting amplitudes show improvement in the agreement with the first Born term, in particular at small scattering angles. However, some structural problems still persist in the single spin-flip amplitude, where a tensor contribution occurs (compared to the first Born). We conjectured that these problems result from the non-Hermitian nature of the exponential ansatz. In our final eikonalization attempt we built an alternative symmetric ansatz by averaging the exponential matrix together with its adjoint. The resulting amplitude agrees with the first Born. The development of this approach to include the second order term would have been very useful in deciding its agreement with the high-energy limit of the Born term. This is a point that we think should be followed in future work.

In our third scheme we generalized Wallace's approach [15,18] to the case of spin-one interactions in a manner similar to that followed by Waxman et al [21]. This approach is very satisfactory in that it deals with all the potential terms without any approximation and results in an amplitude that agrees structurally with the first Born term. Wallace has shown that the eikonal phase function is the first order term in an expansion of the WKB phase function in powers of the parameter $V_o/\hbar kv$. This provided us with a straightforward prescription to calculate dynamical corrections to the eikonal phase. These corrections can be thought of as correcting the straight line path of the particle in the potential by the more realistic curved path.

We find that the corrections reintroduce the missing imaginary part of the second Born term. They also improve on the real part and on comparison with the high-energy limit of the second Born term we find that the corrections improve the agreement significantly.

Throughout our work we checked and enforced our results by carrying out the appropriate numerical calculations. In the case of the second Born term we checked our results in the forward direction using the optical theorem. We also extracted the second Born term from a numerical code which solves the Schrödinger equation exactly (DDTP). The agreement between both results confirmed that our calculations are free from major errors.

We compared the eikonal amplitudes, with and without corrections, to the DDTP calculation, using a Gaussian distribution for the potential. We find that the corrections improve the results significantly. In our final numerical calculation we employed a realistic distributions [52] obtained by folding the Dirac optical potentials for nucleon-nucleus scattering and calculated the cross section and analyzing powers for the case of d–^{58}Ni at 400 and 700 Mev incident deuteron energy. The improvements due to the corrections are very satisfactory and demonstrates clearly the need for the Wallace corrections at intermediate energies.

As mentioned earlier a through investigation of the symmetrical eikonal ansatz would be very instructive and needs to be carried out. It would, also, be very interesting to study the case of including the radial and the momentum tensor couplings in the optical potential. A similar study of the different eikonalization schemes together with Wallace scheme would complete the study of spin-one spin-zero potential scattering case. However, we anticipate that the fact that the radial and the momentum tensor interactions mix spin states of different parities would make it very difficult to arrive at analytical expressions of the phaseshift functions. Nevertheless an approach where these tensor couplings are introduced perturbatively could be considered.

APPENDIX

OBSERVABLES

In this appendix we define the observables which were numerically calculated in section (5.8). First we will relate the scattering amplitudes defined by eqn (2.2.13) to those employed by the DDTP code [25] and widely used in the literature (e.g. see Robson [26]).

In calculating the observables we worked in the Madison reference-frame, which is defined such that the Z-axis is along the direction of the incident momentum $\vec{k}_i$ and the Y-axis is along the direction of $\vec{k}_i \times \vec{k}_f$. In this frame some authors define the scattering matrix to be of the form, ('m' stands for Madison)

$$\mathcal{M}^m = \begin{pmatrix} \mathcal{A}^m & \mathcal{B}^m & \mathcal{C}^m \\ \mathcal{D}^m & \mathcal{E}^m & -\mathcal{D}^m \\ \mathcal{C}^m & -\mathcal{B}^m & \mathcal{A}^m \end{pmatrix} . \qquad (ap.1)$$

Only four of these amplitudes are independent since they satisfy the relation

$$\mathcal{C}^m = (\mathcal{A}^m - \mathcal{E}^m) - \sqrt{2}(\mathcal{B}^m + \mathcal{D}^m)\cot\theta . \qquad (ap.2)$$

These amplitudes are related to the ones defined by eqn (2.2.13) in the following way,

$$\mathcal{A}^m = \frac{1}{3}\left[3\mathcal{A}^{av} - \frac{1}{2}\mathcal{C}_n^{av} + \mathcal{C}_k^{av}\left(\cos(\theta) + \frac{1}{2}\sin^2(\theta/2)\right)\right] \qquad (ap.3)$$

$$\mathcal{B}^m = -\frac{i}{\sqrt{2}}\mathcal{B}^{av} + \frac{\sin\theta}{2\sqrt{2}}\mathcal{C}_k^{av} \qquad (ap.4)$$

$$\mathcal{C}^m = -\frac{1}{2}\left[\mathcal{C}_n^{av} - \mathcal{C}_k^{av}\sin^2(\theta/2)\right] \qquad (ap.5)$$

$$\mathcal{D}^m = \frac{i}{\sqrt{2}}\mathcal{B}^{av} + \frac{\sin\theta}{2\sqrt{2}}\mathcal{C}_k^{av} \qquad (ap.6)$$

$$\mathcal{E}^m = \mathcal{A}^{av} + \frac{1}{3}\mathcal{C}_n^{av} + \frac{1}{3}(\,2\cos\theta + \sin^2(\theta/2)\,)\mathcal{C}_k^{av}\,. \qquad (ap.7)$$

The cross section and analyzing powers are given by [25,26]

$$\sigma(\theta) = \frac{1}{3}Tr(\,\mathcal{M}_m\mathcal{M}_m^{\dagger}\,)$$

$$\mathcal{T}_{kq}(\theta) = \frac{Tr(\,\mathcal{M}_m\tau_{kq}\mathcal{M}_m^{\dagger}\,)}{3\sigma(\theta)}\,. \qquad (ap.8)$$

The matrix elements of the tensor operators τ_{kq} are defined in terms of the Clebsch-Gordon coefficients as (ν is the spin quantum number)

$$< \nu'|\tau_{kq}|\nu > = (2k+1)^{1/2} < s\,\nu'|s\,k\,\nu\,q > \,. \qquad (ap.9)$$

In the case we are studying $s = 1$.

Finally the observables are given by:

$$3\sigma = 2\left[|\mathcal{A}^m|^2 + |\mathcal{B}^m|^2 + |\mathcal{C}^m|^2 + |\mathcal{D}^m|^2\right] + |\mathcal{E}^m|^2 \qquad (ap.10)$$

$$3\sigma\, i\mathcal{T}_{11} = \sqrt{6}\Im\,[\mathcal{B}_m^*(\,\mathcal{A}_m - \mathcal{C}_m\,) + \mathcal{E}_m^*\mathcal{D}_m] \qquad (ap.11)$$

$$3\sigma\,\mathcal{T}_{20} = \sqrt{2}\left[|\mathcal{A}^m|^2 - 2|\mathcal{B}^m|^2 + |\mathcal{C}^m|^2 + |\mathcal{D}^m|^2 - |\mathcal{E}^m|^2\right] \qquad (ap.12)$$

$$3\sigma\,\mathcal{T}_{21} = -\sqrt{6}\Re\,[\mathcal{B}_m^*(\,\mathcal{A}_m - \mathcal{C}_m\,) + \mathcal{E}_m^*\mathcal{D}_m] \qquad (ap.13)$$

$$3\sigma\,\mathcal{T}_{22} = \sqrt{3}\left[2\Re(\,\mathcal{A}_m^*\mathcal{C}_m - |\mathcal{D}_m|^2\right]\,. \qquad (ap.14)$$

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