Tractably Modelling Dependence in Networks Beyond Exchangeability

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Abstract

We propose a general framework for modelling network data that is designed to describe aspects of non-exchangeable networks. Conditional on latent (unobserved) variables, the edges of the network are generated by their finite growth history (with latent orders) while the marginal probabilities of the adjacency matrix are modeled by a generalization of a graph limit function (or a graphon). In particular, we study the estimation, clustering and degree behavior of the network in our setting. We determine (i) the minimax estimator of a composite graphon with respect to squared error loss; (ii) that spectral clustering is able to consistently detect the latent membership when the block-wise constant composite graphon is considered under additional conditions; and (iii) we are able to construct models with heavy-tailed empirical degrees under specific scenarios and parameter choices. This explores why and under which general conditions non-exchangeable network data can be described by a stochastic block model. The new modelling framework is able to capture empirically important characteristics of network data such as sparsity combined with heavy tailed degree distribution, and add understanding as to what generative mechanisms will make them arise.

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1 Introduction

The major problem facing modern network analysis is representing sufficient network heterogeneity. Classically heterogeneity is not incorporated in popular models because they are assumed exchangeable [10, 38]; e.g., the models are invariant to permutations, and thus have no nodes that are "too extreme". To capture additional heterogeneity research has therefore focused on relaxing models away from standard forms of exchangeability [50, Borgs et al., 15], often modelling edge variables instead of formulating models in terms of the network nodes.

Capturing more network heterogeneity requires us to pose a mechanism for the generation of non-exchangeable networks. In this paper we propose a mechanism that can mimic temporal network growth, and is based on the popular graphon model [10, 38], but is still able to capture additional variability. We shall call our model the 'composite graphon model', a special case of 'latent order non-anticipatory graphs', a dependent network model class we introduce and describe in detail in Section 2. Our understanding is encapsulated by using models of latent dependence, and we explore the performance of standard network algorithms with data produced from such a generative model. These networks exhibit power-law degrees and significant data heterogeneity, typical observed features of non-exchangeability. The common types of data that could require such models include for example citation networks [29], ecological networks [46], technological networks such as the powergrid [40] or communications networks [17].

Mimicking the mechanism of network growth, or network evolution, to produce an output network, is a very general idea. This idea can be said to be the genesis of other very popular frameworks such as the Barabasi–Albert network scheme [6], and temporal evolution underpins the graph processes of Borgs et al. [14]. Borgs et al. [14] clarify the complementary relationship of their models to those of [50], whose modelling framework is nearly identical, even if their achieved results are naturally complementary. Our aim is different: by introducing a model class that has a simple dependence parameter χ_n (rather than a latent Poisson process) which then drives the degree of exchangeability through controlling the probability of a run in consecutively generated edges, and then understanding the effects on estimation in this setting for networks with n nodes.

Furthermore, it is important to study the application of standard network algorithms to non-standard network data, as in practice we often cannot check if conditions of exchangeability are satisfied. Motivated by usage of composite likelihood in classical inference, we develop a model with a parameter that quantifies departure from exchangeability. This is the parameter $\chi_n > 0$, quantifying the dependence strength of the latent variables. By defining and studying the impact of χ_n on estimation, clustering and degree patterns we conclude that when χ_n is small, the exchangability assumption is adequate even if the true model is non-exchangeable, while the assumption is inadequate exactly when χ_n approaches 1.

To be more concrete and granular, the contributions of this paper are fourfold. First, we obtain the minimax estimator of the composite graphon model (as well as the composite version of the stochastic block model) with the \mathcal{L}_2 loss function. Nonparametric regression with stationary/non-stationary time series has already attracted increasing research attention; for example, see [25, 60, 54]. Our result can be considered as a network counterpart of [26].

Second, we investigate spectral clustering for the composite stochastic block model, which is the non-exchangeable counterpart of the existing results such as [43]. We find that the spectral clustering algorithm is robust to certain dependence structure of edges, which answers the question why the algorithm works well when the assumption of conditional independence for the Stochastic Block Model (SBM) fails, for example see [44]. In addition to in network science, spectral clustering has been applied in many scientific fields including image analysis, data mining and speech recognition, see for instance [28] and [7], where we do not know of any latent dependence strength.

Third, we construct a model with a heavy-tailed degree distribution by considering

an unobserved latent order $\omega(\cdot, \cdot)$ and certain dependence structure of the edge variables. This shows a new mechanism resulted from the missing information of correlation between edges that can produce a power law degree distribution that is not preferential attachment model [6], [41] nor the inhomogeneous edge connection probability model [37], or dropping the assumption of array exchangeability [51], [50].

Fourth, we establish a theoretical framework for the analysis of network data with latent time–order non–anticipatory edge structure by developing a concept closely related to the notion of dependence measure ([55]) in the literature of time series analysis in which area a similar framework has been successfully set up to accommodate non-stationarity (see for example [59]). This motivates us to develop many useful mathematical tools in this paper.

The paper is organized as follows. Section 2 introduces our notation and basic model structure. Section 3 introduces a dependence structure for the edges. The composite graphon and its minimax estimator are investigated in Section 4. The composite SBM and the spectral clustering algorithm are studied in Section 6. In Section 7 we investigate via an example the basic behaviour of Latent time-order graphs, especially in the case when the dependence is strong. Finally, the proofs of most results are relegated to the supplementary material.

2 Notation and the Composite Graphon Model

For any set A, let |A| denote the cardinality of A. For a positive integer n, we write $[n] = \{1, 2, ..., n\}$. For n-dimensional random vectors $\mathbf{v} = (v_i, 1 \le i \le n)$ and $\mathbf{u} = (u_i, 1 \le i \le n)$, write $(v_i, 1 \le i \le n) \stackrel{d}{=} (u_i, 1 \le i \le n)$ if \mathbf{v} and \mathbf{u} have the same distribution. For two numbers i, j, denote by $\{i, j\}$ the collection of i and j, e.g. $\{i, j\} = \{j, i\}$ and $\{i, i\} = \{i\}$. Whenever the notation $\{i, j\}$ appears, by default we assume that $i \ne j$. Let $\{[n], [n]\}$ denote the set $\{\{i, j\}, 1 \le i \le n, 1 \le j \le n\}$. Denote by $\mathbf{1}(\cdot)$ the usual indicator function which is one if the corresponding event is true and zero otherwise. Write $a \land b$ for min(a, b), and $a \lor b$ for max(a, b). For a graph with adjacency matrix $A_{i,j}$, its marginal probability is the collection of $\{\mathbb{P}(A_{i,j} = 1), \{i, j\} \in \{[n], [n]\}\}$. Each $A_{i,j}$ indicates the presence of an edge between node i and j, and we refer to it as an edge variable. It is only an edge if $A_{i,j} = 1$. The joint probability of the graph adjacency matrix is $\mathbb{P}(A_{i,j} = a_{i,j}, \{i, j\} \in \{[n], [n]\})$ for $\{a_{i,j}, \{i, j\} \in \{[n], [n]\}\} \in \{0, 1\}^N$ with N = n(n-1)/2. For fixed (i, j), we say $A_{i,j}$ is a edge variable between nodes i and j. When $A_{i,j} = 1$, we say that there is a linked edge between i and j. Let $0^0 = 1$ as per usual. For any vector $\mathbf{V} = (v_1, ..., v_d) \in \mathbb{R}^d$, let $\|\mathbf{V}\| = \sqrt{\sum_{i=1}^d v_i^2}$, and $\|\mathbf{V}\|_{\mathcal{L}_p} = \mathbb{E}((\sum_{i=1}^d v_i^p)^{1/p})$.

With this notation, we will describe the technical framework that we use to quantify a network's departure from exchangeability. We note that lack of exchangeability is a non-property. Non-properties are notoriously hard to quantify, and are in fact not uniquely quantifiable. A network can be non-exchangeable in more than one possible manner. Our method of quantifying departure from exchangeability is merely a possible choice; a non-unique and possibly an imperfect choice. It allows us to quantify that under mild forms of non-exchangeability standard network analysis tools are still applicable and useful. We shall start by proposing a model which allows us to adjust the degree of departure from exchangeability.

Definition 2.1. Composite Graphon Model. We say a network with adjacency matrix A is generated by the composite graphon model $f(\cdot, \cdot)$ with respect to a series of latent *i.i.d.* random variables $\{\xi_i, 1 \le i \le n\}$ if

- (a) $\mathbb{P}(A_{ij} = 1 | \xi_i, \xi_j) = f(\xi_i, \xi_j)$ for some symmetric integrable function $f(\cdot, \cdot) \in [0, 1]$;
- (b) There exists a bijective map $\omega(\cdot, \cdot) : [n], [n] \to [\frac{n(n-1)}{2}]$, such that conditioning on latent variables $\{\xi_i\}, B_s = A_{\omega^{-1}(s)}$ forms an order l Markovian chain for some $l \ge 0$;

where we call l the long memory parameter, and we include the parameter dependence strength χ_n which we discuss in Proposition 4.1 in detail.

The complete and rigorous definition of a composite graphon is provided in Section 4, but we give this intuitive definition here to motivate further developments. When l = 0, the composite graphon reduces to the usual graphon model. We say a network follows a composite stochastic block model (composite SBM or CSBM) if the composite graphon $f(\cdot, \cdot)$ of its adjacency matrix is block-wise constant, just like composite likelihood ignores correlations. In fact, the likelihood of composite graphon/SBM is the composite likelihood [49] of graphon/SBM, which motives the name of model. By varying the parameter χ_n continuously we go from a standard exchangeable network, to one exhibiting increasing dependence between the edge variables.

3 A Graph Sequence Model and a Graph Dependence Measure

Consider a sequence of graphs $\{G_n\}$ with adjacency matrices $\{A_n\}$. For a series of 1-1 corresponding mappings $\omega_n(\{\cdot, \cdot\})$: $\{[n], [n]\} \to [N]$, define the <u>dependence measure</u> of the adjacency matrices w.r.t $\omega_n(\{\cdot, \cdot\})$ to be

$$\Delta_n(k) := \max_{s \in \{0,1\}, i, j} |\mathbb{P}(A_{\omega_n(\{i,j\})} = s | \mathcal{F}_{\omega_n(\{i,j\}) - k}) - \mathbb{P}(A_{\omega_n(\{i,j\})} = s)|, \qquad (3.1)$$

where $\mathcal{F}_i = (B_{-\infty}, ..., B_i)$, and $B_{\omega_n(i,j)} = A_{i,j,n}$.

For convenience, we let $B_s, s \leq 0$ follow an *i.i.d.* Bernoulli(1/2) law and be independent of $\{B_s, 1 \leq s \leq N\}$. In our paper, we call $\{B_s\}_{1 \leq s \leq N}$ the "ordered edge variables with respect to ω ". Note that for a sequence of graphs, their adjacency matrices form an array of dependent Bernoulli random variables, with the n_{th} row of the array corresponding to edge variables of a size n graph ordered by $\omega_n(\{\cdot, \cdot\})$. We call $A_{i,j}$ edge variables, and B_s the ordered edge variables. For each graph G_n , its edge variables behave as a time series indexed by $\omega_n(\{i, j\})$. The quantity produced by (3.1) is closely related to the physical and predictive dependence measure introduced by [55] that quantifies the degree of dependence of outputs on inputs in (nonlinear) physical systems. It is easy to compute for many stochastic process and has been used to quantify the strength of dependence in both stationary and non-stationary time series, see for example [60], [54] among many others. We both introduce the dependence measure to networks, and use it to characterise dependence in our network sequence. The dependence measure (3.1) can be tailored to network data and is easy to calculate due to Bernoulli random variables being bounded by unity.

In Corollary 4.1 we show that for the non-exchangeable network models built in this paper, a uniform $M \ge 0$ and a series χ_n exist, such that $\Delta_n(k) \le M \chi_n^{|k|}$, i.e. for each G_n , the dependence measure for its edge variables is geometrically decaying with respect to $\omega_n(\{\cdot, \cdot\})$, which we refer to as the Geometric Convergence (GC) assumption. We refer to χ_n as the dependence parameter.

Graph sequence models have been well studied in the literature. Among others, for example, [12] studied the metrics for sparse graphs via a graph sequence model; [10] established a graph sequence model with a scaling parameter ρ_n to address the sparsity issues for exchangeable graph model; and [38] approximated the graphon model. The minimax rate of the estimation of sparse graphon sequence model was studied by [32] and others.

4 Latent Non-Anticipatory Order Graphs

Subsequent to this section, we omit the subscript n if this omission produces no potential for ambiguity. We formulate the Composite Graphon Model (CGM) in this section, which corresponds to a special case of a Latent Non–Anticipatory Order Graph, defined as per below. We take inspiration from the non-linear Wold representation used in [55] to build graphs that have latent dependence structure. We can generalize the composite graphon model by the following framework.

Definition 4.1. We say G is a undirected, edge-based and finite memory Latent Non-Anticipatory Order Graphs w.r.t. latent variables $\xi = (\xi_1, ..., \xi_n)$ and mapping $\omega_n(\cdot, \cdot)$ if conditionally on the latent variables ξ , there exists a finite k, such that for all i, the conditional distributions of the edge variables $B_{\omega(i,j)} = A_{i,j}, i \neq j$

$$\mathbb{P}(B_i|B_{i-1},...,B_{-\infty},\xi) = \mathbb{P}(B_i|B_{i-1},...,B_{i-k+1},\xi),$$
(4.1)

where edge variables $\{B_j, j \leq 0\}$ correspond to the burn-in process, which could be chosen as *i.i.d.* Bernoulli(1/2) independent of $B_s, 1 \leq s \leq N$. Let l be the smallest k such that (4.1) holds. Then we say G has a memory parameter l.

The "burn-in process" (e.g. the parameter l) has little impact on the network, rather like the starting values for a time series autoregressive (AR) model. The Latent Non-Anticipatory Order Graph has memory parameter l and has the property that the probability of linking an edge variable relies on the past l-1 network edge variables. Note that there are network models such that the linkage probability of every edge variables depends on *all* the edge variable generated before it. Such models are not Latent Non-Anticipatory Order graphs. An important example is the preferential attachment model ([6]), which we will further discuss in Remark 7.1. Recently the asymptotic normality of the affine preferential attachment network models has been studied by [27].

Consider a latent time-order graph with memory parameter l and w.r.t. the map $\omega_n(\{\cdot,\cdot\})$. Define $U_i = (B_i, ..., B_{i-l+1})^T$. Denote by $\mathcal{X} \in \mathbb{R}^l$ the set of l-dimensional binary

vectors with all entries 0 or 1. The following proposition shows that the dependence between $A_{i,j}$ and $A_{k,l}$ decreases as $|\omega_n(\{i,j\}) - \omega_n(\{k,l\})|$ increases.

Proposition 4.1. Consider a Latent time-order graph with memory parameter l and mapping $\omega_n(\cdot, \cdot)$. Assume $\min_{a,b\in\mathcal{X}\times\mathcal{X}} \mathbb{P}(U_i = a|U_{i-l} = b,\xi) \ge \alpha' > 0$. Then uniformly for i, k and for all $u_i, u_{i-k} \in \mathcal{X}$, we have that for k > l,

$$\mathbb{P}(U_i = u_i | U_{i-k} = u_{i-k}, \xi) - \mathbb{P}(U_i = u_i | \xi) = O(\chi^k \tilde{p}),$$

where $\chi = (1 - 2\alpha')^{1/l}$, and the latent variable ξ is defined in Definition 4.1. Take

$$\tilde{p} = \max_{1 \le s \le l-1} \max_{1 \le i \le n} \max_{a} |\max_{b} \mathbb{P}(U_i = a | U_{i-s} = b, \xi) - \min_{b} \mathbb{P}(U_i = a | U_{i-s} = b, \xi)|.$$

Notice that if given ξ , l = 0 and $\{B_i\}$ is an independent series, then $\tilde{p} = 0$.

This proposition shows the implication of Definition 4.1. When a graph sequence model is considered, α may in fact depend on n. In that case we shall write $U_{i,n}$ for U_i . For each n, we assume $U_{i,n}$ is an order one Markov process. At this time, χ depends on n and we denote it by χ_n when it is important to emphasise the size-dependent relationship. In this article we assume that $\chi_n < 1$. When $\chi_n \to 1$, \tilde{p} will be strictly bounded away from zero. Hence χ_n can be regarded as a proxy of dependence strength. By a straightforward argument using the Markov property, we have the following corollary:

Corollary 4.1. Under the conditions of Proposition 4.1, we have that

$$\Delta_n(k) = \max_{\substack{1 \le i \le N, \\ b_s \in \{0,1\}, \\ 1-k \le s \le i-k}} |\mathbb{P}(B_i = b_i|B_s = b_s, s \le i-k, \xi) - \mathbb{P}(B_i = b_i|\xi)| = O(\chi^k), \quad (4.2)$$

where χ is defined in Proposition 4.1.

If $|\chi| \leq 1 - \epsilon$ for some $\epsilon > 0$, then we say the latent time-order graph sequence is short-range dependent w.r.t. $\omega(\{\cdot, \cdot\})$. In this case equation (4.2) implies a geometric decay of $\mathbb{P}(B_i = b_i | B_s = b_s, s \leq i - k, \xi) - \mathbb{P}(B_i = b_i | \xi)$ in k. This shows a stronger link with an AR(1) process where the term $\mathbb{P}(B_i = b_i | B_s = b_s, s \leq i - k, \xi)$ plays the role of conditional expectation of an observation on another observation k steps ahead from an AR(1) process. **Definition 4.2.** (Composite graphon model) A Latent time-order graph is a composite graphon model with respect to *i.i.d.* latent variables ξ_i if

$$\theta_{i,j} := \mathbb{P}(A_{i,j} = 1 | \xi_1, \dots, \xi_n) = f_n(\xi_i, \xi_j), \ i \neq j$$
(4.3)

for some symmetric function $f_n(\cdot, \cdot)$. If in addition $f_n(\cdot, \cdot)$ is block-wise constant in the \mathbb{R}^2 plane, then we call (4.3) a composite stochastic block model (composite SBM). We shall omit the subscript n when no confusion can arise.

The parameter of interest in (4.3) is independent of the mapping $\omega(\cdot, \cdot)$. This fact is crucial for estimating the composite graphon model without estimating $\omega(\cdot, \cdot)$. Indeed, model (4.3) is quite flexible, including the usual graphon model as its special case. We then present a general pseudo algorithm for constructing the composite graphon model with memory parameter l. For $\omega(\{\cdot, \cdot\})$, define $\omega^{-1} : [N] \to \{[n] \times [n]\}$ as $\omega^{-1}(k) =$ $\{\{i, j\}, \omega(\{i, j\}) = k\}$. For any series $\xi_i, i \in \mathbb{Z}$, denote by $\xi_{\{i, j\}} = \{\xi_i, \xi_j\}$ for short. From the generative pseudo-algorithm Algorithm 1, we see that the joint distribution of the edge variables of the composite graphon model, as well as dependence strength χ_n , is fully determined by the following (infinite dimensional) parameters:

- (i) $f(\omega^{-1}(i)), 1 \le i \le N$,
- (ii) $\mathbb{P}(B_i = 1 | U_i = u_i, \xi)$ for $2 \le i \le N$, $u_i \in \{0, 1\}^{i (i l + 1) \lor 1}$ with constraints (4.4) for $2 \le i \le N$.

In the classic case, the first of these specifications is solely via the graphon function, while specification in (ii) breaks the model exchangeability, and so makes the model more flexible. We recover the classic graphon model when $\mathbb{P}(B_i = 1 | U_i = u_i) = \mathbb{P}(B_i = 1) = f(\omega^{-1}(i))$. For any composite graphon model (4.3), we define its associated composite graph as follows.

Definition 4.3. We say \tilde{G} is a composite graph with respect to a composite graphon model G from (4.3) and with respect to *i.i.d.* latent variables z_i following U(0, 1), if (i) $V(G) = V(\tilde{G})$ where V(G) (or $V(\tilde{G})$) is the vertex set of G (or \tilde{G}), and if (ii) $\mathbb{P}(\tilde{A}_{i,j} =$ $1|\xi) = f(\xi_i, \xi_j)$ where $\tilde{A}_{i,j}$ is the edge variables of \tilde{G} and (iii) conditioning on $\xi_i, 1 \leq i \leq n$, the edge variables $\{\tilde{A}_{i,j}, 1 \leq i < j \leq n\}$ are independently distributed.

Algorithm 1 Generative Pseudo Algorithm

- 1: Generate $\xi_1, ..., \xi_n$. Calculate $f(\xi_i, \xi_j)$.
- 2: For given ω , we first generate B_1 as $\mathbb{P}(B_1 = 1|\xi) = f(\xi_{\omega^{-1}(1)})$.
- 3: Generate B_2 by two parameters $\mathbb{P}(B_2 = 1 | B_1 = 1, \xi)$ and $\mathbb{P}(B_2 = 1 | B_1 = 0, \xi)$, which satisfies that $f(\xi_{\omega^{-1}(2)}) = \mathbb{P}(B_2 = 1, B_1 = 1) + \mathbb{P}(B_2 = 1, B_1 = 0)$, where

$$\mathbb{P}(B_2 = 1, B_1 = 1|\xi) = f(\xi_{\omega^{-1}(1)})\mathbb{P}(B_2 = 1|B_1 = 1, \xi),$$
$$\mathbb{P}(B_2 = 1, B_1 = 0|\xi) = (1 - f(\xi_{\omega^{-1}(1)}))\mathbb{P}(B_2 = 1|B_1 = 0, \xi).$$

By using $\mathbb{P}(B_2 = 0|B_1, \xi) = 1 - \mathbb{P}(B_2 = 1|B_1, \xi)$, in step 2 we have constructed a two-dimensional multivariate Bernoulli model (B_1, B_2) .

- 4: for i = 1 to N 1 do
- 5: Generate B_{i+1} by parameters $\mathbb{P}(B_{i+1} = 1 | U_i = u_i, \xi)$ which satisfy the following constraints:

$$\mathbb{P}(B_{i+1} = 1|\xi) = f(z_{\omega^{-1}(i+1)}) = \sum_{u_i} \mathbb{P}(B_{i+1} = 1|U_i = u_i, \xi) \mathbb{P}(U_i = u_i|\xi), \quad (4.4)$$

for $u_i \in \{0, 1\}^{i-((i-l)\vee 1)}$, where $\mathbb{P}(U_i = u_i | \xi)$ could be obtained by the $(i + 1 - ((i - l + 1) \vee 1))$ -dimensional multivariate Bernoulli model generated in previous iteration.

6: end for with output $B_1, B_2, ..., B_N$.

The composite graphon $f(\cdot, \cdot)$ is the limit of the composite graph in the sense of [35]. For any simple graph F with vertex set V(F), the integral of composite graphon $f(\cdot, \cdot)$ on V(F) corresponds to a homomorphism density (see the definition in Section 5.2.2 of [35]) of F in the composite graph.

Our model connects to the graph limit and convergence in the language of [35] in a marginal way similar to the "composite" concept in the classic statistics literature, see for instance [49]. Furthermore, [15] introduces a latent birth time concept, which is similar to our latent order concept in that it is also temporal. The differences lie in the fact that their latent birth time is for each vertex, while our latent order is for each edge variable, and more fundamentally, lie in the procedure that they drop this latent birth time in their final step to make the model exchangeable such that the labels carry no information while our model is not exchangeable by assuming the information of the labels is missing. As a consequence, [15] involves additional cost for the exchangeability.

Throughout the paper, we shall focus on the composite graphon model given in (4.2). For $\alpha \in (0, 1]$ and a sufficiently large constant M, we define the Hölder class

$$\mathcal{H}_{\alpha}(M) := \{ f : |f(x,y) - f(x',y')| \le M(|x-x'| + |y-y'|)^{\alpha}, x \ge y, x' \ge y', \\ f \text{ symmetric} \},\$$

for all $x \ge y, x' \ge y'$. We consider the following scenarios:

(A) $f(\cdot, \cdot)$ is a block-wise constant symmetric function, or

(B)
$$f \in \mathcal{F}_{\alpha}(M)$$
, where $\mathcal{F}_{\alpha}(M) = \{0 \le f \le 1 : f \in \mathcal{H}_{\alpha}(M)\}.$

Under (A), our model reduces to the composite stochastic block model (composite SBM). Under (B), the composite graphon $f(\cdot, \cdot)$ is smooth and estimable. The smoothness is assumed by for example [26], [32], [38], [1] (which assumes $\alpha = 1$) among others.

4.1 Inhomogeneity of the Composite Graphon Model

In this subsection, we explore the inhomogeneity introduced by conditional dependence via studying examples of composite SBM with memory parameter 1. The memory parameter is given by Definition 4.1. Note that the memory parameter 0 corresponds to the classical stochastic block model.

4.1.1 Inhomogeneity introduced by communities

We first consider the composite SBM with fixed k communities constructed as follows. Define the map $\gamma : [n] \to [k]$, which assigns n nodes into k different groups. Define $\tilde{\gamma} : \{[n] \times [n]\} \to \{[k] \times [k]\}$ as $\tilde{\gamma}(i, j) = \{\gamma(i), \gamma(j)\}$. We shall construct a composite SBM such that the edge variable connection probability depends on its previous edge variable with respect to latent (and unobservable) map ω . For this purpose, let q, l be the numbers such that $\{q, l\} = \tilde{\gamma}(\omega^{-1}(\omega(\{i, j\}) - 1))$, i.e., q, l are the communities of vertices of $B_{\omega(\{i, j\})-1}$. Assume

$$A_{i,j} = B_{w(\{i,j\})} \sim \begin{cases} \text{Bernoulli}(\varrho_{\gamma(i),\gamma(j)}^{0,q,l}) & \text{if } B_{\omega(i,j)-1} = 0\\ \text{Bernoulli}(\varrho_{\gamma(i),\gamma(j)}^{1,q,l}) & \text{if } B_{\omega(i,j)-1} = 1 \end{cases}$$
(4.5)

The probability of linking (i, j) depends on its "parent" edge variable $B_{\omega(i,j)-1}$, and the communities of the four nodes $q, l, \tilde{\gamma}(i, j)$. Conditioning on the latent memberships, (4.5) reduces to an inhomogeneous two-state Markov process. Recently in time series analysis, researchers have developed certain inhomogeneous models to characterise non-stationarity of integer-valued and categorical data, see for example [47]. To define a composite SBM with k groups such that $f(\xi_i, \xi_j) = \theta_{\gamma(i)\gamma(j)}$ for $\frac{k(k+1)}{2}$ connection probabilities $\{\theta_{i,j}, i, j \in [k], \theta_{i,j} = \theta_{j,i}\}$, using Algorithm 1, we specify a composite SBM with the parameters $\{\varrho_{a,b}^{u,c,d}, u \in \{0,1\}, \{a, b, c, d\} \in [k]^4\}$ satisfying the following constraints:

(a) For $1 \le i \le j \le k$,

$$\varrho_{i,j} = \frac{\varrho_{i,j}^{0,i,j}}{1 + \varrho_{i,j}^{0,i,j} - \varrho_{i,j}^{1,i,j}}$$

(b) For $1 \le i \le j \le k$, $1 \le s \le l \le k$, $\{s, l\} \ne \{i, j\}$, $\varrho_{i,j}^{0,s,l}$ and $\varrho_{i,j}^{1,s,l}$ satisfy $\varrho_{i,j} = \varrho_{i,j}^{0,s,l} (1 - \varrho_{s,l}) + \varrho_{i,j}^{1,s,l} \varrho_{s,l}.$

In fact, each sub-chain that maps $(i, j) \rightarrow (i, j)$ describes a homogeneous Markov chain, i.e., if we consider any consecutively generated edge variables which connect the vertices that belong to the same pair of groups (i, j), then these edge variables form a homogeneous Markov Chain with stationary probability $(\varrho_{i,j}, 1 - \varrho_{i,j})$. If k = 1 (corresponding to the scenario of only one group), constraints (a), (b) degenerate to a strictly stationary 2states Markov process. From this point of view, the inhomogeneity is introduced by the specification of communities with stationary probabilities $(\varrho_{1,1}, 1 - \varrho_{1,1})$.

4.1.2 Inhomogeneity Introduced by Individuals

Another source of inhomogeneity is due to the dependence introduced by the latent position $\omega_n(i, j)$. Consider the single group, or k = 1 case. As described by the algorithm under definition 4.2, another composite SBM could be specified by

$$\varrho_{1,1} = \frac{q_{0,i}}{1 - q_{1,i} + q_{0,i}} = \frac{q_{0,j}}{1 - q_{1,j} + q_{0,j}} \quad \text{for } 1 \le i < j \le N,$$
(4.6)

where $q_{0,i} = \mathbb{P}(B_i = 1 | B_{i-1} = 0)$ and $q_{1,i} = \mathbb{P}(B_i = 1 | B_{i-1} = 1)$. It is not hard to see that (4.6) defines an inhomogeneous Markov chain. The conditional connection probabilities of the edge variables depend on the its ordered edge variables' positions in the history of the Markov chain. Since all nodes belong to the same community, the inhomogeneity is evident only at the individual level. This is very different from its Erdős-Rényi counterpart, in which each node is stochastically equivalent.

5 Minimax Rate Estimator of the Composite Graphon

In this section, we discuss the minimax estimator of the composite graphon with respect to squared error loss. Let $\mathcal{Z}_{n,k} = \{z : [n] \to [k]\}$ be the collection of all possible mappings from [n] to [k]. Then for any $\bar{z} \in \mathcal{Z}_{n,k}$, $\{\bar{z}^{-1}(a) : a \in [k]\}$ forms a partition of [n], or equivalently: $\bigcup_{a \in [k]} \bar{z}^{-1}(a) = [n]$ and $\bar{z}^{-1}(a) \cap \bar{z}^{-1}(b) = \emptyset$ for any $a \neq b \in [k]$. In the following, we adopt the notation of [26]. Define

$$\bar{\eta}_{ab}(\bar{z}) = \frac{1}{|\bar{z}^{-1}(a)||\bar{z}^{-1}(b)|} \sum_{i \in \bar{z}^{-1}(a)} \sum_{j \in \bar{z}^{-1}(b)} \eta_{ij} \text{ for } a \neq b \in [k],$$
$$\bar{\eta}_{aa}(\bar{z}) = \frac{2}{|\bar{z}^{-1}(a)|(|\bar{z}^{-1}(a)| - 1)} \sum_{i \in \bar{z}^{-1}(a), j \in \bar{z}^{-1}(a), i < j} \eta_{ij} \text{ for } a \in [k], \ |\bar{z}^{-1}(a)| > 1,$$

where $A = A_{i,j}$ is the adjacency matrix. In this section, let $\theta_{i,j} = \mathbb{P}(A_{i,j} = 1|\xi) = f(\xi_i, \xi_j)$. Define the estimate $\hat{\theta}_{ij} = \hat{Q}_{\hat{z}(i)\hat{z}(j)}$ where

$$(\hat{Q}, \hat{z}) = \underset{Q \in \mathbb{R}^{k \times k}_{sym}, \bar{z} \in \mathcal{Z}_{n,k}}{\operatorname{argmin}} L(Q, \bar{z}),$$
(5.1)

$$L(Q, \bar{z}) = \sum_{a,b \in [k]} \sum_{(i,j) \in \bar{z}^{-1}(a) \times \bar{z}^{-1}(b), i < j} (A_{ij} - Q_{ab})^2.$$
(5.2)

This procedure (5.1) is referred to as minimizing combinatorial least squares (see [26]). The word combinatorial is inserted, as we need to determine group membership (z) which is a combinatorial problem, rather than solely estimating a parameter by weighted averaging. Straightforward calculations show that $\hat{Q}_{ab} = \bar{A}_{ab}(\hat{z})$ for all $a, b \in [k]$. Therefore similarly to [53] we propose a block constant estimates for the composite graphon model. For this purpose, we define

$$\Theta_{k} = \{\{\theta_{i,j}^{\diamond}\} \in [0,1]^{n \times n} : \theta_{i,i}^{\diamond} = 0, \theta_{i,j}^{\diamond} = Q_{ab} = Q_{ba}, \forall (i,j) \in \bar{z}^{-1}(a) \times \bar{z}^{-1}(b),$$

for some $Q_{ab} \in [0,1], \bar{z} \in \mathcal{Z}_{n,k}\}.$

Define the true value on each block by $\{Q_{ab}^*\} \in [0,1]^{k \times k}$, and the oracle assignment $z^* \in \mathcal{Z}_{n,k}$, writing $\theta_{i,j} = Q_{z^*(i)z^*(j)}^*$. For each estimate \hat{z} , define $\tilde{Q}_{ab} \in [0,1]^{k \times k}$ by $\tilde{Q}_{ab} = \bar{\theta}_{ab}(\hat{z})$ and $\tilde{\theta}_{i,j} = \tilde{Q}_{\hat{z}(i)\hat{z}(j)}$ for $i \neq j$. For all $i \in \lfloor n \rfloor$, let $\hat{\theta}_{i,i} = \tilde{\theta}_{i,i} = \theta_{i,i} = 0$ (as we have assumed no self loops). Define $n_a = |\bar{z}^{-1}(a)|$. We first consider the composite SBM model. Recall the definition of χ_n from Corollary 4.1. In this section, we assume that the memory parameter is bounded.

Theorem 5.1. Considering the composite SBM model G_n with k groups. Assume the conditions of (A) hold. For any constant C' > 0, there is a constant C > 0 which only depends on C', such that

$$\frac{1}{n^2} \sum_{i,j \in [n] \times [n]} (\hat{\theta}_{i,j} - \theta_{i,j})^2 \le C \left(\frac{k^2}{n^2} + \frac{\log k}{n}\right) (1 - \chi)^{-2}$$

with probability at least $1 - \exp(-C' n \log k)$ uniformly over $\theta \in \Theta_k$, and

$$\sup_{\theta \in \Theta_k} \mathbb{E}\left\{ (\hat{\theta}_{i,j} - \theta_{i,j})^2 \right\} \le C_1 \left(\frac{k^2}{n^2} + \frac{\log k}{n} \right) (1 - \chi)^{-2}$$

for all $k \in [n]$ with some universal constant $C_1 > 0$.

Proof. This proof proceeds along the lines of [26]. By using the fact that $L(\hat{Q}, \hat{z}) \leq L(Q^*, z^*)$, we have that

$$\|\hat{\theta} - \theta\|^2 \le 2\langle \hat{\theta} - \theta, A - \theta \rangle$$

Direct calculations show that

$$\langle \hat{\theta} - \theta, A - \theta \rangle \leq \|\tilde{\theta} - \hat{\theta}\| \left| \left\langle \frac{\hat{\theta} - \tilde{\theta}}{\|\hat{\theta} - \tilde{\theta}\|}, A - \theta \right\rangle \right| + (\|\tilde{\theta} - \hat{\theta}\|) + \|\hat{\theta} - \theta\|) \left| \left\langle \frac{\tilde{\theta} - \theta}{\|\tilde{\theta} - \theta\|}, A - \theta \right\rangle \right|.$$
 (5.3)

Define for any $\bar{z} \in \mathcal{Z}_{n,k}$,

$$\hat{\theta}(\bar{z}) = \operatorname*{argmin}_{Q \in \mathbb{R}^{k \times k}_{sym}} L(Q, \bar{z}),$$

and $\tilde{\theta}(\bar{z}) = \mathrm{argmin}_{Q \in \mathbb{R}^{k \times k}_{sym}} \tilde{L}(Q, \bar{z}),$ where

$$\tilde{L}(Q, \bar{z}) = \sum_{a, b \in [k]} \sum_{(i,j) \in \bar{z}^{-1}(a) \times \bar{z}^{-1}(b), i < j} (\theta_{ij} - Q_{ab})^2$$

Note that $\tilde{\theta} = \tilde{\theta}(\hat{z})$. Thus, by the property of least squares estimator, we have that $\|\tilde{\theta} - \theta\| \leq \|\hat{\theta} - \theta\|$. As a result, we have

$$\|\tilde{\theta} - \hat{\theta}\| \le 2\|\hat{\theta} - \theta\|.$$

It follows from Lemmas D.2 and D.3 in the supplementary material that the terms

$$\left|\left\langle \frac{\hat{\theta} - \tilde{\theta}}{\|\hat{\theta} - \tilde{\theta}\|}, A - \theta \right\rangle\right|, \left|\left\langle \frac{\tilde{\theta} - \theta}{\|\tilde{\theta} - \theta\|}, A - \theta \right\rangle\right|, \tag{5.4}$$

could be bounded by $\sqrt{k^2 + n \log k} (1 - \chi)^{-1}$ with probability at least $1 - \exp(-C' n \log k)$. Finally, the theorem follows from combining (5.3)–(5.4).

Note that when a graph sequence model is considered, the factor $\chi = \chi_n$ is allowed to depend on n. This is discussed further in Remark 5.1. Regarding the convergence rate, the term $\frac{k^2}{n^2}$ corresponds to the estimation of k^2 unknown parameters with an order of n^2 observations (edge variables), and the term $\frac{\log k}{n}$ corresponds to clustering rate, see for example [26] and [32]. Meanwhile, the term $(1-\chi)^{-2}$ is the effect of the non-exchangeability due to the latent order $\omega(\{\cdot, \cdot\})$ and the conditional dependence between edges given latent variables (ξ_i) . When $p_0 = p_1$, our model reduces to the usual SBM. In this situation, the second part of the convergence rate degenerates to a constant, while the first part agrees with the rate in [26], which has been shown to be rate optimal. Consider the composite graphon model generated from Algorithm 1 and specified by $\{\xi_1, ..., \xi_n, A, \omega(\{\cdot, \cdot\}), l\}$ with symmetric composite graphon $f(\cdot, \cdot)$, i.e., $\theta_{i,j} = f(\xi_i, \xi_j)$, where latent variables $\{\xi_i, 1 \leq i \leq n\}$ are *i.i.d* U(0, 1), ω is the latent order of edge variables and l is the memory parameter. When $f \in \mathcal{F}_{\alpha}(M)$, which is the bounded Hölder's class defined in (B) of Section 4, arguments of Gao et al. (2015) show that there exists an *oracle* ([53]) $z_k^+ \in \mathcal{Z}_{n,k}$ such that for some universal constant C > 0,

$$\frac{1}{n^2} \sum_{a,b \in [k]} \sum_{i \neq j: z_k^+(i) = a, z_k^+(j) = b} (\theta_{i,j} - \bar{\theta}_{a,b}(z_k^+))^2 \le CM^2 (\frac{1}{k^2})^{\alpha \wedge 1}.$$
(5.5)

Consider

$$(\theta^*, z^*) = \underset{Q \in \mathbb{R}^{k \times k}_{sym}, z \in \mathcal{Z}_{n,k}}{\operatorname{argmin}} \tilde{L}(Q, z), \\ \tilde{L}(Q, z) = \sum_{a,b \in [k]} \sum_{(i,j) \in \bar{z}^{-1}(a) \times \bar{z}^{-1}(b), i < j} (\theta_{i,j} - Q_{ab})^2.$$

By choosing $k = n^{\frac{1}{(1+\alpha\wedge 1)}}$, we have the following theorem:

Theorem 5.2. Consider a composite graphon model $G = \{\xi_1, ..., \xi_n, A, \omega(\{\cdot, \cdot\}), l\}$ Assume the conditions of (B) hold. Then there exist constant C, C'

$$\frac{1}{n^2} \sum_{i,j \in [n]} (\hat{\theta}_{i,j} - \theta_{i,j})^2 \le C n^{\frac{-2(\alpha \wedge 1)}{1 + \alpha \wedge 1}} (1 - \chi)^{-2} \log n$$

with probability at least $1 - \exp(-C'n)$, uniformly over $f \in \mathcal{F}_{\alpha}(M)$. Furthermore,

$$\sup_{f \in \mathcal{F}_{\alpha}(M)} \mathbb{E}\left\{\frac{1}{n^2} \sum_{i,j \in \lfloor n \rfloor} (\hat{\theta}_{i,j} - \theta_{i,j})^2\right\} \le C_1 n^{\frac{-2(\alpha \wedge 1)}{1 + \alpha \wedge 1}} (1 - \chi)^{-2} \log n, \tag{5.6}$$

for some constant $C_1 > 0$.

Proof. By similar arguments to those of [26], we have that

$$\|\hat{\theta} - \theta^*\| \le \max\{16(D+C)^2, 4(B+D)E\},$$
(5.7)

where

$$E = \|\tilde{\theta} - \hat{\theta}\| \le 2\|\hat{\theta} - \theta\|,$$
$$B = \left|\left\langle\frac{\hat{\theta} - \tilde{\theta}}{\|\hat{\theta} - \tilde{\theta}\|}, A - \theta\right\rangle\right|, C = \|\theta - \theta^*\|, D = \left|\left\langle\frac{\theta^* - \tilde{\theta}}{\|\theta^* - \tilde{\theta}\|}, A - \theta\right\rangle\right|.$$

Direct calculations show that Lemma D.2 in the supplementary material still holds when replacing $\left|\left\langle \frac{\tilde{\theta}-\theta}{\|\tilde{\theta}-\theta\|}, A-\theta\right\rangle\right|$ with $\left|\left\langle \frac{\theta^*-\tilde{\theta}}{\|\theta^*-\tilde{\theta}\|}, A-\theta\right\rangle\right|$. By Lemma D.2, Lemma D.3 in the supplementary material and (5.5), the theorem follows.

Remark 5.1. Comparing with the results of the usual SBM and the graphon model as discussed in [26], our Theorems 5.1 and 5.2 introduce an extra factor of $(1 - \chi_n)^{-2}$. The convergence rate is therefore slow when χ_n is close to 1, of which the situation indicates the strong conditional dependence between the edge variables on latent variables, see Proposition 4.1. A straightforward calculation using Proposition 4.1 shows that if given ξ_n , l = 0 and $\{B_i\}$ is an independent series, the rate is fully consistent with previous results in the sense that both the model and the rate recovers the known optimal rate of [26].

Remark 5.2. Consider the composite graphon model defined in Theorem 5.2 with ordered edge variables $B_s, 1 \leq s \leq n$. We construct a sparse composite graphon model $\tilde{G} = \{\xi_1, ..., \xi_n, \tilde{A}, \omega(\{\cdot, \cdot\}), l\}$ by $\mathbb{P}(\tilde{U}_i = u_i | \tilde{U}_{i-1} = u_{i-1}, \xi) = \rho_n \mathbb{P}(U_i = u_i | U_{i-1} = u_{i-1}, \xi)$, where $U_i = (B_i, ..., B_{i-l+1})^T$, $\tilde{U}_i = (\tilde{B}_i, ..., \tilde{B}_{i-l+1})^T$, $\tilde{B}_i, 1 \leq i \leq N$ are ordered edge variables of \tilde{A} , and ρ_n is a positive sequence that converge to 0. We therefore represent the "sparsity" by the parameter $\rho_n > 0$. This parameter was used by [12] to uniformly control the success probability across all nodes and uniformly controls the number of edges present. Straightforward calculations show that the upper bound of RHS of (5.6) could be lowered to the order of $\min_k \{\rho_n^2(\frac{1}{k^2})^{\alpha \wedge 1} + \rho_n(\frac{k^2}{n^2} + \frac{\log k}{n})(1-\chi)^2\}$, which coincides with the upper bound of that in [32]. In Section 7, we shall see scenarios of homogeneity that $\frac{\max_{i,j} \mathbb{P}(A_{i,j=1})}{\min_{i,j} \mathbb{P}(A_{i,j=1})} \to \infty$ which is able to produce power law degree distribution. This scenario cannot be captured by scale parameter ρ_n . As a result, we do not focus on the scaled sparse model in detail.

Remark 5.3. Assume the setting of the composite SBM sequence such that $\chi = \chi_n$ is regulated by *n*. Assume that the conditions of Theorem 5.1 hold. By Theorem 5.1, when the number of communities *k* is fixed, a sufficient condition for the consistency of the \mathcal{L}_2 estimator (5.1) is $(\sqrt{n}(1-\chi_n))^{-1} = o(1)$. Theorem 5.2 indicates that the estimator will be inconsistent under strong dependence such that χ_n approaches 1 at a rate faster than $\frac{1}{\sqrt{n}}$. Similarly, Theorem 5.2 implies that when the composite graphon $f \in \mathcal{F}_{\alpha}(M)$ and $k = \lfloor n^{\frac{\alpha \wedge 1}{1+\alpha \wedge 1}} \rfloor$, a sufficient condition for the consistency of the \mathcal{L}_2 estimator (5.1) is $(n^{\frac{\alpha \wedge 1}{1+\alpha \wedge 1}}(1-\chi_n))^{-1} = o(1)$. **Interpretation:** In other words, our estimator is consistent for $1 - \chi_n = \Omega(n^{-a})$, 0 < a < 1/2, where we write $a_n = \Omega(b_n)$ for series a_n , b_n if $b_n = O(a_n)$. By proposition 4.1 this means $\max_{a,b\in\mathcal{X}\times\mathcal{X}} \mathbb{P}(U_i = a|U_{i-l} = b,\xi) \leq 1 - Cn^{-a}$ for some constant C.

Remark 5.4. Theorem 5.2 determines that matrix estimation can be done for this problem, i.e. the sampled graphon can be estimated from an observed adjacency matrix. It does not necessarily relate to the underlying graphon function, unless we derive further results. As noted in [53] the mean square error of the estimate of f(x, y) can be directly related to the matrix mean square error of (5.7). An issue with this statement is that the discretized $p_{ij} = f(\xi_i, \xi_j)$ is still random as ξ_i is random and so statements can be made either marginally or conditionally on ξ_1, \ldots, ξ_n . The added problem of estimating f(x, y), the function, is to use an appropriate metric, and factor out measure preserving transformations.

It is discussed in [26] and [53] that the graphon model is closely related to nonparametric regression with unknown design and *i.i.d.* errors. Consider the one-dimensional regression problem $y_i = f(z_i) + e_i$, where $z_i, 1 \le i \le n$ are *i.i.d.* samples, and e_i are zero mean errors. When $f \in \mathcal{H}_{\alpha}(M)$ and e_i are *i.i.d* normals, the local polynomial estimator achieves the minimax rate $n^{-\frac{2\alpha}{1+\alpha}}$ under the squared error loss $\frac{1}{n} \sum_{i \in [n]} \{\hat{f}(z_i) - f(z_i)\}^2$. When e_i is a short range dependent non-stationary time series for example the piecewise locally stationary time series in [58], Lemma 5 in [60] shows that $\mathbb{E}(e_i e_j) = O(\eta^{|i-j|})$ for some $\eta \in (0, 1)$. It follows from this fact and Proposition 1.13 in [48], that the convergence rate of the local polynomial estimator with non-stationary time series error has the same order as with *i.i.d.* error.

However, under the situation that the design is unknown, an additional difference arises between the time series error and the *i.i.d* error due to the unknown chronological order. Indeed, missing chronological order affects time series but not the *i.i.d.* errors. Surprisingly, for the time series error, the impact of the missing chronological order on the estimation is negligible in terms of order under certain situation. Suppose the edge variables of the graph are short range dependent with respect to the mapping $\omega(\{\cdot, \cdot\})$.

Recall $L(Q, \bar{z})$ in (5.2), and define the new objection $L^0(Q, \bar{z})$ by replacing A with $\theta = \mathbb{E}(A|\xi)$. For given \bar{z} , let $\tilde{Q}(\bar{z})$ and $Q^0(\bar{z})$ be the minimizer of $L(Q, \bar{z})$ and $L^0(Q, \bar{z})$, respectively. In fact, $\tilde{Q}(\bar{z})$ and $Q^0(\bar{z})$ are the average among partitions of adjacency ma-

trix and of true but unknown conditional linkage probability matrix, respectively. Thus $\mathbb{E}(\tilde{Q}(\bar{z})) = Q^0(\bar{z})$. Since Bernoulli random variables are bounded, we show in our paper that under mild conditions and similar to the time series counterpart, the deviations between the average and the mean of the edge variables are bounded uniformly over all possible partitions z as if the edge variables are conditional independent.

The time series structure of our model is important for many real applications. For example, [26] relates link prediction to the graphon model. In real application, links can be modeled by time series in dynamic network, see for example [45]. Despite the convergence rate of the proposed estimator for the composite graphon model is similar to that under usual graphon model, its moment behavior is different under two scenarios. This will further impact on any estimation and hypothesis testing procedure, see for example [11] and [8].

6 Spectral Clustering Algorithm for Composite Stochastic Block Model

In the previous sections we investigate the estimation of the composite graphon model and the composite stochastic block model. In addition to estimation of the linkage probabilities, community detection is another research topic in the network analysis. The connection between estimation and spectral clustering is complicated, and they are not identical problems. A good estimation result for the block heights of a stochastic blockmodel does not necessarily guarantee a good community detection result. For a more detailed discussion of the link between parameter estimation and spectral clustering, we refer to [26]. In the area of community detection, spectral clustering and its variants have already been widely applied ([52]). The consistency of spectral clustering for certain exchangeable network models has been studied by for example [43], [42], [57], [34] among others. In the following, we shall study the performance of spectral clustering for estimating the composite SBM.

6.1 Re-parameterization of the Composite Stochastic Block Model

By choosing a block-wise constant symmetric $f(\cdot, \cdot)$ in Definition 4.2, the composite SBM has the form of $\theta_{i,j} = \mathbb{P}(A_{i,j} = 1 | \xi_i, \xi_j)$, where $\{\xi_i, 1 \leq i \leq n \in \mathbb{R}^k\}$ are *i.i.d.* latent vectors, with one entry equal to one and all other entries equal to zero. Let θ be an $n \times n$ matrix, with $\theta_{i,j}$ its $(i, j)_{th}$ entry. Then θ could be parameterized as

$$\theta = Z B^{\dagger} Z^T,$$

where $B^{\dagger} \in [0,1]^{k \times k}$ is full rank and symmetric, and $Z \in \mathbb{R}^{n \times k}$ is a matrix with *ith* row ξ_i such that it has one 1 in each row and at least one 1 in each column. For each node i, we say it belongs to group j if $\xi_{i,j}$, which is the j_{th} element of ξ_i , equals 1. With the re-parameterization, we are able to define the graph Laplacian, which is essential for the spectral clustering algorithm. Define diagonal matrices D and \overline{D} with diagonal elements $D_{i,i}$ and $\{\overline{D}_{i,i}\}, i = 1...n$, respectively, where

$$D_{i,i} = \sum_{k=1}^{n} A_{i,k}, \quad \bar{D}_{i,i} = \sum_{k=1}^{n} \theta_{i,k}.$$

Define L and \overline{L} for the Laplacian of A and θ , respectively, as

$$L = D^{-1/2} A D^{-1/2}, \bar{L} = \bar{D}^{-1/2} \theta \bar{D}^{-1/2}.$$

Note that \overline{L} is the population version of L since the former is the Laplacian of θ and the latter is the Laplacian of adjacency matrix A. Both L and \overline{L} depend on the number of nodes n. Let $c_i = \overline{D}_{i,i}/n$ and $\tau_n = \min_{i=1,...,n} c_i$. We shall write L as $L^{(n)}$, \overline{L} as $\overline{L}^{(n)}$ and τ as τ_n when we need to emphasise the sample size. In the remainder of this section, we assume Z is unknown but fixed (unless specified). After obtaining L, the spectral clustering algorithm is given by:

- 1. Compute the eigenvectors $u_1, ..., u_k$ w.r.t. the first k largest eigenvalues of L.
- 2. Run a k-means algorithm on vectors $y_1, ..., y_n$, $\{y_i\}_{1 \le i \le n} \in \mathbb{R}^{1 \times k}$ to cluster them into clusters $C_1, ..., C_k$, where y_i is the *i*th row of matrix U, an $n \times k$ matrix such that the *j*th column of U is u_k .

Then node *i* is in class *g* if y_i is assigned to C_q .

6.2 Properties of Mis-clustered Nodes

For simplicity of exploration and the ease of comparison, we will use the notion of [43]. In order to discuss the property of mis-clustered nodes, we first introduce the following notation. In addition we define

$$P_n = \max_{j=1,\dots,k} (Z^T Z)_{j,j}.$$

We then give two properties of composite SBM. The validity of the properties could be shown similarly to Lemma 3.1 and Lemma 3.2 of [43], and so we omit the proof for the sake of brevity.

- (a) There exists a matrix $V_1 \in \mathbb{R}^{k \times k}$ such that the columns of ZV_1 are the eigenvectors of \overline{L} which correspond to the nonzero eigenvalues. In addition, $z_iV_1 = z_jV_1$ if and only if $z_i = z_j$.
- (b) Let $V_2 \in \mathbb{R}^{n \times k}$ be a matrix whose orthonormal columns are the eigenvectors which correspond to the ordered largest k eigenvalues of L (in absolute value). Let $c_i, 1 \leq i \leq n$ be the centroid corresponding to the i_{th} row of V_2 . Let the columns of $U, \bar{U} \in \mathbb{R}^{n \times k}$ be k orthonormal eigenvectors of LL and $\bar{L}\bar{L}$ (recall L and \bar{L} are symmetric matrix) which correspond to the first k largest eigenvalues of the two matrices in absolute value, respectively. Define matrices O_1 and O_2 with the singular decomposition $\bar{U}^T U = O_1 \Sigma O_2^T$, where O_1, O_2 are orthonormal matrices and Σ is a diagonal matrix. Let $O = O_1 O_2^T$. Then $\|c_i z_i V_1 O\| < \frac{1}{\sqrt{2P_n}}$ if and only if $\|c_i z_i V_1 O\| < \|c_i z_j V_1 O\|$ for any $z_i \neq z_j$.

Under conditions of Theorem 6.1 below, the Davis-Kahan Theorem [23] shows that $||V_2 - ZV_1O||_F = o(1)$ almost surely, which leads to that the corresponding eigenvectors of the observed graph Laplacian L is close to that of the population graph Laplacian \bar{L} ; see [43] for a detailed introduction of the Davis-Kahan Theorem. As a result, by (a), (b), we define the set of mis-clustered nodes as

$$\mathcal{M} = \left\{ i : \|c_i - z_i V_1 O\| \ge \frac{1}{\sqrt{2P_n}} \right\},\,$$

since similarly to the argument in [43], we can show that if any node $i \notin \mathcal{M}$, then *i* will be correctly clustered by spectral clustering algorithm. For any symmetric matrix M, define

 $\lambda(M)$ to be the eigenvalues of M. For any interval $S \in \mathbb{R}$, define $\lambda_S(M) = \{\lambda(M) \cap S\}$, Let $\bar{\iota}_1 \geq ... \geq \bar{\iota}_n$ be the elements of $\lambda(\bar{L}\bar{L})$, and $\iota_1 \geq ... \geq \iota_n$ be the elements of $\lambda(LL)$. Define $G(\chi, N, u) = \sum_{r=0}^{N} r^u \chi^{r/2}$, where χ is defined in Proposition 4.1.

Before stating Theorem 6.1 regarding the performance spectral clustering for estimating the composite SBM, we present the following Proposition 6.1 which studies the tail probability of $||LL - \bar{L}\bar{L}||_F$. The latter is the difference between the population version of and the usual graph Laplacian. The graph Laplacian plays a central role in the spectral clustering, therefore the difference $||LL - \bar{L}\bar{L}||_F$ is key to study the asymptotic behavior of the corresponding clusters. The proof of Proposition 6.2 and Proposition 6.1 are inspired by [59], as well as [43] and [59], respectively. The proof of Theorem 6.1 rests on the the following Proposition 6.1.

Proposition 6.1. Under conditions of Theorem 6.1, there exist sufficiently large positive constants η_0, η_1, M' such that if $n \ge M'$

$$\mathbb{P}(\|LL - \bar{L}\bar{L}\|_F \ge \frac{\log n}{\tau^2 n^{1/2}} G^{1/2}(\chi, N, 3)(1 - \chi)^{-1/2}) \le \zeta(n),$$

where

$$\zeta(n) = \frac{\eta_0}{n \log^4 n} + \eta_1 n^{-2}.$$

Theorem 6.1. Consider a size *n* composite SBM with a fixed unknown mapping $\omega(\{\cdot, \cdot\})$. Denote by k_n the number of groups of nodes, and by n_{k_n} the corresponding group size. Let $|\lambda_1| > ... > |\lambda_{k_n}|$ be the absolute values of ordered k_n largest absolute and nonzero eigenvalues of \overline{L} . Assume that $n^{-1/2}(\log n)^2 G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2} = O(\lambda_{k_n}^2)$, and $\tau_n^2 > M/\log n$ for a sufficiently large constant M. Then we have that the number of miss-specified nodes has the order of

$$|\mathcal{M}| = o\left(\frac{P_n \log^2 n}{\lambda_{k_n}^4 \tau_n^4 n} G(\chi, N, 3)(1-\chi)^{-1}\right), \quad a.s.$$

provided that $G(\chi, N, 3)(1 - \chi) \log n$ is sufficiently large such that $\zeta(n)$ (defined as in Proposition 6.1 below) is summable.

The conditions on the eigenvalues and on τ are similar to those of [43] that ensure the eigengap of $\overline{L}\overline{L}$ and the smallest nonzero eigenvalues of \overline{L} cannot be too small. Hence we omit the discussion here for the sake of brevity.

Proof of Theorem 6.1. Since $\zeta(n)$ in Proposition 6.1 is summable, by the Borel-Cantelli Lemma, we have that

$$\|LL - \bar{L}\bar{L}\|_F = o\left(\frac{\log n}{\tau^2 n^{1/2}}G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}\right), \quad a.s.,$$

where $\|\cdot\|_F$ represents the Frobenius norm. Thus we have that

$$\max_{1 \le i \le n} |\iota_i - \bar{\iota}_i| = o\left(\frac{\log n}{\tau^2 n^{1/2}} G^{1/2}(\chi, N, 3)(1 - \chi)^{-1/2}\right), \quad a.s.$$
(6.1)

Define $S_n = [\lambda_{k_n}^2/2, 2]$, and

$$\delta_n = \inf\{|l-s|; l \in \lambda(\bar{L}\bar{L}), l \notin S_n, s \in S_n\},\$$

$$\delta'_n = \inf\{|l-s|; l \in \lambda_{S_n}(\bar{L}\bar{L}), s \notin S_n\}.$$

Then $\delta_n = \delta'_n = \lambda_{k_n}^2/2$. The quantity δ_n measures the distance between spectrum (eigenvalue) of $\bar{L}\bar{L}$ outside S_n and S_n . δ'_n measures the how the S_n separates the eigenvalues of $\bar{L}\bar{L}$. They are needed for the application of the Davis-Kahan Theorem ([43]). Together with (6.1), by assumption, the number of elements in $\lambda_{S_n}(\bar{L}\bar{L})$ will be equal to the number of elements in $\lambda_{S_n}(LL)$. By the definition of \mathcal{M} , as well as the properties (a) & (b), and the definition of a centroid, we have that

$$|\mathcal{M}| \le 8P_n \|V_2 - ZV_1 O\|_F^2 = o\left(\frac{P_n \log^2 n}{\lambda_{k_n}^4 \tau_n^4 n} G(\chi, N, 3)(1-\chi)^{-1}\right), \quad a.s$$

The last equality follows from the Davis-Kahan theorem, the results of Theorem 2.2, Lemmas 3.1 and 3.2, and the proof of Theorem 3.1 in [43]. \Box

Theorem 6.1 shows the consistency of the spectral clustering algorithm for composite SBM under regularity conditions. With χ gets closer to 1, the dependence between edge variables becomes stronger and the theoretically guaranteed convergence rate deteriorates. On the other hand, the requirement that $\tau_n^2 > M/\log n$ is almost as restrictive as the requirement of at least linearly growing expected degree for all nodes. The following proposition is key to study the tail probability of $\|LL - \bar{L}\bar{L}\|_F$ (Proposition 6.1), which controls the mis-clustering rate in Theorem 6.1. The proposition is of general interest. It reveals the covariance structure of the latent time-order graph. **Proposition 6.2.** Let C > 0 be a sufficiently large constant. Consider the latent timeorder graph with fixed memory parameter $l < \infty$. Let $h \ge 1$ be an integer. Define $\{\tilde{B}_j\}_{1 \le j \le N}$ to be $\{B_j - \mathbb{E}(B_j|\xi)\}_{1 \le j \le N}$. Then for any 2l different integers $\{i_s, 1 \le s \le 2h\} \in [N]^{2h}$ and for a sufficiently large positive constant C which may depend on h, we have that (a)

$$|Cov(\Pi_{u=1}^{h}\tilde{B}_{i_{u}},\Pi_{v=h+1}^{2h}\tilde{B}_{i_{v}}|\xi)| = O(\chi^{\Lambda(i_{s},1\leq s\leq 2h)/2}(1-\chi)^{-1}),$$

where $\Lambda(i_s, 1 \le s \le 2h) = \min_{1 \le s \le h} (\min_{h+1 \le u \le 2h} |i_s - i_u|)$. In addition, we have (b)

$$|Cov(\Pi_{u=1}^{h}\tilde{B}_{i_{u}},\Pi_{v=h+1}^{2h}\tilde{B}_{i_{v}}|\xi)| = O(\chi^{\max_{1\leq s\leq 2l}\iota(i_{s})/2}(1-\chi)^{-1}),$$

where $\iota(i_s) = \min_{1 \le j \le 2l, j \ne s} (|i_s - i_j|), Cov(A, B|\xi) = \mathbb{E}(AB|\xi) - \mathbb{E}(A|\xi)\mathbb{E}(B|\xi).$

Remark 6.1. Proposition 6.2 shows that for the composite SBM, the upper bound of the covariance between the product of two groups of ordered edge variables, $\{B_{j_s}, 1 \leq s \leq l\}$ and $\{B_{j_s}, l+1 \leq s \leq 2l\}$, is determined by (a) $\min_{1 \leq s \leq l}(\min_{l+1 \leq u \leq 2l} |i_s - i_u|)$ and (b) $\max_{1 \leq s \leq 2l} \iota(i_s)$. $Cov(\prod_{u=1}^{h} \tilde{B}_{i_u}, \prod_{v=h+1}^{2h} \tilde{B}_{i_v}|\xi)$ will become smaller when the terms described by (a) and (b) become larger. The term in (a) is large if the two groups of labels are far away from each other, i.e. , the smallest distance between two labels, one from the $j_s, 1 \leq s \leq l$ and the other from $j_s, l+1 \leq s \leq 2l$ is large. The quantity (b) is large if there is a label far way from all other labels.

Recently, many complex models have been proposed based on the SBM to capture additional and important graph structure. For instance, the general SBM proposed in [18] allows for a portion of arbitrary outliers, where the majority of nodes are generated from a fixed SBM. As a comparison, all nodes from the composite SBM in this paper differ from the SBM when edge variables are conditionally dependent on the latent membership. Another prominent model that can generate arbitrary degree inhomogeneity is the degree corrected stochastic block model (DC-SBM) (see [31]). For this model, consistency of community detection has been studied (see for example [57]), and corresponding spectral clustering algorithms have been proposed (for example see [42]). Also, SBM has been generalized to a mixed membership (for example [1]), and the K-median approach ([56]). A tensor approach ([4]) have been proposed to address the mixed-membership. In this paper, we have built up a general framework for non-exchangeable graphs, and investigate the spectral clustering algorithm for composite SBM in detail. **Remark 6.2.** The key concepts of a "composite graph" and composite SBM in Sections 4 and 6 are closely related to notion of composite likelihood. Composite likelihood inference is a popular and successful tool for statistical research when the joint likelihood is hard to evaluate, see [49] among others for a comprehensive review. In the literature of network analysis, the idea of analyzing pseudo or approximate likelihood has been proposed to tackle the complex and computational-infeasible joint likelihood of graph models, see for example ([2], [3] and [9] among others.)

Remark 6.3. We shall assume that $k_n \equiv k$. Assume that there exists $\zeta > 0$ such that $|n_k| \geq \lfloor \zeta n \rfloor$. Straightforward calculations show that $G(\chi, N, 3)$ is of the order $(1 - \chi_n^{1/2})^{-4}$. As a result, the condition $n^{-1/2}(\log n)^2 G^{1/2}(\chi, N, 3)(1 - \chi)^{-1/2} = O(\lambda_k^2)$ reduces to

$$(1 - \chi_n)(1 - \chi_n^{1/2})^{-4} \log^4 n = O(n), \tag{6.2}$$

which yields the weak consistency of clustering in the sense of [57], i.e, the mis-clustering rate in Theorem 6.1 is therefore simplified to $|\mathcal{M}| = o(1)$. A straightforward calculation shows that a sufficient condition for (6.2) is that

$$1 - \chi_n = \Omega(n^{-1/5} \log n^{-4/5}).$$

As a comparison, Remark 5.3 shows that the estimation error of Theorem 5.1 is negligible if $1 - \chi_n = \Omega(n^{-1/2})$.

6.3 Simulation Study of the Mis-clustering Rate

In this section, we examine the performance of the spectral clustering algorithm used on the composite SBM. We consider two simulation scenarios: two groups and three groups where the *n* nodes are partitioned into. A latent order $\omega_1(\{\cdot, \cdot\})$ representing a strong dependence and a latent order $\omega_2(\{\cdot, \cdot\})$ representing a weak dependence are considered. The orders (and these are literal orderings, not orders of magnitude) are constructed in a way such that their corresponding marginal edge variables linkage probabilities are the identical. Their forms are deferred to and discussed in detail in Sections 7.1.1 and 7.1.2. The other detailed parameters of the considered two and three group composite SBMs can be found in Section A of the supplemental material.

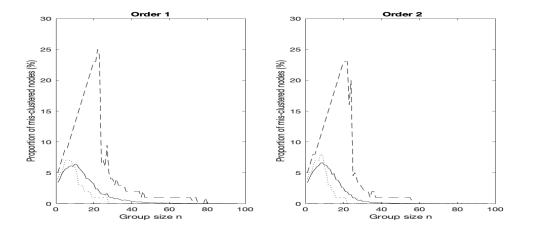


Figure 1: Exploring the mis-clustering under different latent orders for two groups. Left: the number of mis-clustered nodes under order $\omega_1(\{\cdot, \cdot\})$. Right: the number of misclustered nodes under order $\omega_2(\{\cdot, \cdot\})$. We display the mean (solid line), median (dotted line) and upper/lower frequency band (dashed line) of the mis-clustering number over 1000 simulations.

The simulation results are displayed in Figures 1 for two groups and which show that there are fewer mis-clustered nodes under order $\omega_2(\{\cdot, \cdot\})$ than under $\omega_1(\{\cdot, \cdot\})$. This reinforces the message of Section 6, namely that stronger conditional dependence between edge variables, as introduced by the order $\omega_1(\{\cdot, \cdot\})$, tends to increase the mis-clustering rate. These simulation results also support the consistency of the spectral clustering algorithm for the composite SBM. The clustering results for the three group scenario are similar to that of the two group cases and have been shown in Figure A.4 of the supplemental material.

7 Example: Marginally Edge Constant Latent Time– Order Graph

In this section, we study the effect of the posited conditional dependence by studying the given model of the marginally edge constant latent time-order graph sequence model (MECLTG). The MECLTG sequence model is defined as a composite graphon model with $f(\cdot, \cdot) \equiv c_n$ where c_n is a function of n, and $f(\cdot)$ is defined in (4.3). For each fixed n, $\mathbb{P}(A_{i,j} = 1) = \mathbb{P}(A_{k,l} = 1) = c_n$ for every $(i, j), (k, l), i \neq j, k \neq l$. It is also a composite SBM with only one group, (as the SBM with one group corresponds to the Erdös-Rényi model). Meanwhile, edge variables are correlated with respect to the latent order $\omega_n(\{\cdot, \cdot\})$. Via studying the MECLTG, we can investigate the effect of the (conditional) dependence separately from the effect of inhomogeneous (marginal) edge variables' linkage probabilities. In the following arguments, for simplicity we omit the subscript n of $\omega_n(\{\cdot, \cdot\}), p_{1,n}, p_{0,n}$ and p_n if no confusion arises.

Consider the Markov process of MECLTG, which we write as $CG(V, \omega, p_0, p_1)$:

$$\mathbb{P}(B_i|B_{i-1}) = (p_1^{B_i}(1-p_1)^{1-B_i})^{B_{i-1}}(p_0^{B_i}(1-p_0)^{1-B_i})^{1-B_{i-1}},$$
(7.1)
where $B_i = A_{\omega^{-1}(i)}.$

From (7.1), if $B_{i-1} = 1$ then B_i is distributed as $Bernoulli(p_1)$, otherwise it is distributed as $Bernoulli(p_0)$. By the fundamental theorem of Markov Chains, the edge variables of $CG(V, \omega, p_0, p_1)$ have a limiting distribution

$$p := \mathbb{P}(B_{\infty} = 1) = p_0/(1 + p_0 - p_1) \in [p_0 \land p_1, p_0 \lor p_1].$$
(7.2)

We then consider the stationary scenario, i.e.,

$$\mathbb{P}(B_j) = p \quad \forall 1 \le j \le N.$$
(7.3)

This is because when the total number of the edge variables is large, the majority of the edge variables of $CG(V, \omega, p_0, p_1)$ have marginal linkage probabilities close to p.

Definition 7.1. We say that the graph $CG(V, \omega, p_0, p_1)$ is a first order homogeneous MECLTG with vertices V, driven by the order $\omega(\{\cdot, \cdot\})$ if (7.1) and (7.3) hold.

Notice that when $p_0 = p_1 = p$, then the latent structure is not active. As a result, $CG(V, \omega, p_0, p_1)$ reduces to the standard Erdős-Rényi graph G(|V|, p). The following corollary explicitly calculates the conditional probability of B_i given B_{i-k} for $k \ge 2$:

Corollary 7.1. Consider $CG(V, \omega, p_0, p_1)$. Define $\mathbb{P}_k(a|b) = \mathbb{P}(B_j = a|B_{j-k} = b)$ for $a, b \in \{0, 1\}$. Then we have that

$$\mathbb{P}_{k}(1|0) = \frac{p_{0}(1 - (p_{1} - p_{0})^{k})}{1 - p_{1} + p_{0}}; \qquad \mathbb{P}_{k}(1|1) = \frac{p_{0} + (1 - p_{1})(p_{1} - p_{0})^{k}}{1 - p_{1} + p_{0}}, \\ \mathbb{P}_{k}(0|1) = \frac{(1 - p_{1})(1 - (p_{1} - p_{0})^{k})}{1 - p_{1} + p_{0}}; \qquad \mathbb{P}_{k}(0|0) = \frac{(1 - p_{1}) + p_{0}(p_{1} - p_{0})^{k}}{1 - p_{1} + p_{0}}.$$

If we let $p = \frac{p_0}{1-p_1+p_0}$, then we have that

$$\mathbb{P}_{k}(1|0) = p - p(p_{1} - p_{0})^{k}, \quad \mathbb{P}_{k}(1|1) = p + (1 - p)(p_{1} - p_{0})^{k}$$
$$\mathbb{P}_{k}(0|1) = (1 - p) - (1 - p)(p_{1} - p_{0})^{k}, \quad \mathbb{P}_{k}(0|0) = (1 - p) + p(p_{1} - p_{0})^{k}$$

The results show that the dependence between edge variables, or equivalently $\Delta_n(k)$ defined in (3.1) decays at the geometric rate $(p_1 - p_0)^k$. We discuss the phase transition of MECLTG in Section C of the online supplementary material. In the remaining of the paper, we focus on the degree distribution of the MECLTG.

7.1 Degree Distribution

In previous sections we find that the ordering $\omega_n(\{\cdot, \cdot\})$ has an asymptotically negligible impact on graphon estimation and on community detection under weak dependence (between edge variables), i.e. $\chi = \chi_n \in (0, 1)$. Any $\omega_n(\{\cdot, \cdot\})$ will yield a consistent estimator of the graphon or detection for communities when $0 \leq \chi = \chi_n < 1$, where χ is defined in Proposition 4.1. In this subsection, by investigating simple examples, we shall see that i) different ordering $\omega_n(\{\cdot, \cdot\})$ have different impact on the network structure when $\chi_n \to 1$, so the impact of missing information of $\omega_n(\{\cdot, \cdot\})$ is no longer asymptotic negligible; and ii) our model is flexible enough to produce networks both with and without a heavy-tailed degree distribution. The idea is that when $\mathbb{P}(B_i = 1|B_{i-1} = 1) \to 1$, we can design latent orders such that incident edge variables (e.g., A_{ij} and A_{ik}) are strongly correlated (or weakly correlated), and hence their summation, or corresponding degrees, cannot (can) be well approximated by sums of independent Bernoulli random variables.

To illustrate this, assume that $p_0 = \lambda_0/n$ and also $p_1 \ge p_0$. Recall the homogeneous probability $p = \frac{p_0}{1-p_1+p_0}$. Obviously, p and p_0 are of the same order if either p_1 is a constant or goes to 0. If $p_1 = 1 - n^{-1}g_n$ for a sequence of positive real numbers g_n , then p still goes to 0 as long as $g_n \to \infty$.

7.1.1 Examples of MECLTG with heavy tail degree distribution

Let $\varpi_1(i,j) = n(i-1) - i(i-1)/2 + j - i$ for $1 \le i < j \le n$, and $\omega_1(\{i,j\}) = \varpi(i \land j, i \lor j)$, $i \ne j$. Consider the first order homogeneous process MECLTG $CG(V, \omega_1, p_0, p_1)$. In

particular we choose the ordering $\omega_1(\{i, j\})$ where the edge variables are generated as follows:

$$A_{1,2}, A_{1,3}, \dots, A_{1,n}, A_{2,3}, \dots, A_{2,n}, A_{3,4}, \dots, A_{3,n}, \dots, A_{n-1,n}$$

In other words, a node generates its edge variables after all the node labels before it have generated their edge variables. We shall see that the considered ordering is able to generate a heavy-tailed degree distribution if we set $p_1 \rightarrow 1$, since a connected edge variables $A_{a,b}$ will lead to a high chance that the next edge variable $A_{a,b+1}$ is connected, where the two edge variables have the same vertex. In such a way our model is able to produce a larger number of high degree nodes than the Erdös-Rényi model.

We now study the empirical degree distribution $n^{-1}\sum_{i=1}^{n} I(d_i = k)$ for $1 \leq k \leq n$, where d_i is the degree of node i. When the nodes have a homogeneous degree distribution, for example in the Erdös-Rényi graph, $n^{-1}\sum_{i=1}^{n} I(d_i = k)$ is an unbiased estimator of $\mathbb{E}d_i$, see for example [39]. Meanwhile, inhomogeneity introduced by strong dependence will distort the empirical degree distribution, i.e., as we shall show among a wide range of k, the expectation of $n^{-1}\sum_{i=1}^{n} I(d_i = k)$ of the graph decays with k at a polynomial rate. In this way, the graph displays the power law degree distribution. Different from the Erdös-Rényi model, $\mathbb{P}(d_i = k)$ for MECLTG is heterogeneous in i instead of remaining constant in i.

Theorem 7.1. (Heavy-tailed Degree Distribution) Consider the first order homogeneous MECLTG $CG(V, \omega_1, p_0, p_1)$ with |V| = n. Suppose $p_0 = \frac{\lambda_0}{n}$ with $\lambda_0 \ge 1$, and $p_1 = 1 - \lambda_1 n^{-c}$, 0 < c < 1/2. For any $\gamma > 1$, $\mu > 0$, define $M_{\gamma} : \sum_{k=1}^{n} M_{\gamma} \frac{1}{k^{\gamma}} = 1$, and $M_{\gamma,\mu} : \sum_{k=1}^{n} M_{\gamma,\mu} \frac{1}{k^{\gamma}} \exp(-\mu k) = 1$. Let $A_{n,\gamma} = \{k : n^{-1} \sum_{i=1}^{n} \mathbb{P}(d_i = k) \ge M_{\gamma} k^{-\gamma}\},$ $B_{n,\gamma,\mu} = \{k : n^{-1} \sum_{i=1}^{n} \mathbb{P}(d_i = k) \ge M_{\gamma,\mu} k^{-\gamma} \exp(-\mu k)\}$. Then there exist $a_0, b_0, c_0, d_0 > 0$ (which may depend on γ), such that

$$\left\{k: \lfloor a_0 n^{\frac{2c}{1+\gamma}}\rfloor \le k \le \lfloor b_0 n^c \log n\rfloor\right\} \subset A_{n,\gamma},$$
$$\left\{k: \lfloor c_0 \log n\rfloor \le k \le \lfloor d_0 n\rfloor\right\} \subset B_{n,\gamma,\mu}.$$

Proof. See section F of Appendix .

Theorem 7.1 shows that, within a wide range of values of k, the tail of the distribution of the degrees of the MECLTG $CG(V, \omega_1, p_0, p_1)$ model behaves similarly to the

power law distribution (or to a power law degree distribution with exponential cutoff, see [36]). Consider the usual Erdös-Rényi graph G(V, p), where $p = \frac{p_0}{1-p_1+p_0}$, so that the marginal linkage probabilities of edge variables are the same as the first order homogeneous MECLTG $CG(V, \omega_1, p_0, p_1)$. Let $C_n = \{k : k \ge g(n)\}$ where $g(n) \to \infty$ arbitrarily slowly. By proposition F.2 (a Poisson approximation) in the supplementary supplement and the large deviation theorem (see the proof of Lemma 7.1 in the supplementary material), it follows that there exist constants c, d such that both $C_n \cap A_{n,\gamma}$ and $C_n \cap B_{n,\gamma,\mu}$ are subset of $\{k : \lfloor an^c \rfloor \le k \le \lfloor bn^c \rfloor\}$ when n is large enough. Thus, the first order homogeneous MECLTG $CG(V, \omega_1, p_0, p_1)$ has much larger $|C_n \cap A_{n,\gamma}|$ and $|C_n \cap B_{n,\gamma,\mu}|$ than SRG G(V, p).

7.1.2 Examples of MECLTG with light-tailed degree distribution

In this section, we construct a first order homogeneous MECLTG $CG(V, \omega_2, p_0, p_1)$ which has similar $|C_n \cap A_{n,\gamma}|$ and $|C_n \cap B_{n,\gamma,\mu}|$ to that of Erdös-Rényi graph $G(V, p = \frac{p_0}{1-p_1+p_0})$. The order ω_2 we consider is $\omega_2(\{i, j\}) = \varpi_2(i \wedge j, i \vee j), i \neq j$, where $\varpi_2(i, j) = i + \frac{(2n-(j-i))(j-i-1)}{2}$ for $1 \leq i < j \leq n$. In particular, our choices of ordering follow which the edge variables $A_{i,j}, i < j$ are generated are as follows

$$A_{1,2}, A_{2,3}, \dots, A_{n-1,n}, A_{1,3}, \dots, A_{n-2,n}, A_{1,4}, \dots, A_{n-3,n}, \dots, A_{1,n}$$

Observe that the edge variables are generated in increasing order of j - i. Among edge variables with equal j - i, the edge variables with smaller i are generated earlier. We now study the expectation of $n^{-1} \sum_{i=1}^{n} I(d_i = k)$ to show the characteristic of A.

Lemma 7.1. Consider the first order homogeneous MECLTG Graph

 $CG(V, \omega_2, p_0, p_1)$ where $p_0 = \frac{\lambda_0}{n}, \lambda_0 \ge 1$ and $p_1 = 1 - \lambda_1 n^{-c}, c \in (0, 1/2)$ are defined in Theorem 7.1. Let Y_n follow Poisson(p) for $p = \frac{p_0}{1-p_1+p_0}$. Let d_i be the degree of node *i*. Let g(n) be a series of real numbers which diverges but may increase at an arbitrarily slow rate. The we have for some $\iota > c, \, \iota + c < 1$,

(i)
$$\sum_{k=0}^{\infty} |\mathbb{P}(d_i = k) - \mathbb{P}(Y_n = k)| = O(n^{\iota + c - 1}),$$

(ii) $\mathbb{P}(d_i = k) \le \exp(-0.5(\iota - c)k\log n)$ for $\lfloor n^{\iota}g(n) \rfloor \le k \le n.$ (7.4)

Proof. See supplementary material, Appendix Y.

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Lemma 7.1 shows that the behavior of the MECLTG $CG(V, \omega_2, p_0, p_1)$ is similar to an Erdös-Rényi graph G(n, p), in the sense that both of their degree distributions can be mimicked by a Poisson $(\frac{\lambda_0}{\lambda_1}n^c)$ random variable. Equation (7.4) also shows that the tail of the degree distribution decays very rapidly. Together with Theorem 7.1, we find that simply multiplying a scale parameter to the marginal edge variable linkage probabilities (for example in [10] which models the sparse graphon as $\rho_n f(\cdot, \cdot)$) is not able to capture the heteroskedasticity in the probability of linkage, in the sense that $\frac{\max_{i,j} \mathbb{P}(A_{ij}=1)}{\min_{i,j} \mathbb{P}(A_{ij}=1)}$ stays unchanged under this parameterization. This shows the greater flexibility and rich structure of our model class. We illustrate this property in Figure B.5 of the supplementary material. We discuss the images from left to right in Figure B.5. Figure B.5 shows typical graphs generated from MECLTGs $CG(V, \omega_1, p_0, p_1)$, $CG(V, \omega_2, p_0, p_1)$ and Erdős-Rényi G(V,p), respectively with $p_0 = 0.01$, $p_1 = 1 - \frac{1}{n^{1/3}}$, $p = \frac{p_0}{1 - p_1 + p_0}$ with |V| = 100. From the figure, we see that the first network is very inhomogeneous: it has the most hubs among the three networks. The second network is less inhomogeneous than the first network, but is more inhomogeneous than the third network. Notice that we construct the three networks in such a way that the marginal connection probability is $n^{-2/3}$, where n is the size of the network. This is larger than the connectivity threshold $\frac{\log n}{n}$. However, all the three networks in Figure B.5 have some isolated nodes just like the models of [15]. For the third network, this is because the sample size is not large enough, so $100^{-2/3}$ is very close to $\log 100/100$. We observed that the first and second network have more components, which is the price we pay for the inhomogeneity. Since the marginal connection probability in our experiment is controlled, the expected total edges of the three networks are fixed. As a result, the structure with more hubs will also tend to have more small degree nodes, and also more isolated nodes. The edge variables are distributed according to the dependence structures ω_1 and ω_2 in the first and second networks, and purely randomly distributed in the third network.

Remark 7.1. A familiar model for networks with power law degree distributions is the preferential attachment (PA) model, where the network is growing sequentially node by node. In PA, a node can not affect the relationship among earlier nodes. This shares some features with out model. Thus, the generating order (or history) of the edge variables of

PA could be written as

$$A_{1,2}, A_{1,3}, A_{2,3}, A_{1,4}, A_{2,4}, A_{3,4}, \dots, A_{1,n}, \dots, A_{n-1,n}.$$

with the associated ordering $\omega(\{i, j\}) = \varpi(i \land j, i \lor j)$, where $\varpi(i, j) = \frac{(j-1)(j-2)}{2} + i$, $1 \le i < j \le n$. The linkage probability of an edge variable is determined by the popularity of its earlier (more popular) verteces. Hence PA is not a latent time-order graph since the required properties fail to hold for any finite k. The well known heavy-tailed degree distribution of PA is contributed by infinite memory, order $\omega(\{,\})$, and inhomogeneous edge variables linkage probabilities.

Recently [Borgs et al.] proposed a class of normalized *unbounded* graphon model. Given latent positions $(\xi_i = x_i)_{1 \le i \le n}$, the edge variables are independently connected with probabilities $\mathbb{P}(A_{ij} = 1 | \xi_i = x_i, \xi_j = x_j) = \min(1, \rho W(x_i, x_j))$, where ρ is the target density and W is a (possibly) unbounded graphon. For detailed definition of ρ and W we refer to [13]. Due to the inhomogeneous conditional connection probabilities and the unboundedness of W, their model is allowed to have a large portion of high degree nodes and therefore the feature of heavy-tailed degree distribution under some circumstances.

In contrast to the aforementioned models, the MECLTG model $CG(V, \omega_1, p_1, p_0)$ has homogeneous (marginal) edge variable linkage probabilities. Hence the power law degree distribution is a consequence of the order $\omega_1(\{\cdot, \cdot\})$ and the strength of dependence which is determined by p_0, p_1 . As a comparison, our construction $CG(V, \omega_2, p_1, p_0)$ does not have a power law distribution though it has the same strength of dependence as $CG(V, \omega_1, p_1, p_0)$. The only difference between the two MECLTGs is the order function. The unobserved ordering $\omega_i, i = 1, 2$ introduces correlation, this increasing the probability of an edge variable between nodes adjacent in the ordering. This reveals the complex nature of the latent time-order graph. We display the adjacency matrix of typical $CG(V, \omega_1, p_0, p_1)$ and $CG(V, \omega_2, p_0, p_1)$ with network size $n = 100, p_0 = \frac{1}{n}, p_1 = 1 - \frac{1}{n^{1/3}}$ in left and right panels of Figure 2. Notice that d_i is the sum of i_{th} row of the adjacency matrix. The figure shows that $CG(V, \omega_1, p_0, p_1)$ generates high degree nodes with greater frequency than $CG(V, \omega_2, p_0, p_1)$. Recall that degrees are calculated by averaging along rows or columns, whilst diagonal structure does not aggregate to form larger degrees.



Figure 2: Adjacency matrix of $CG(V, \omega_1, p_0, p_1)$ (left) and $CG(V, \omega_2, p_0, p_1)$ (right).

7.2 Simulation Results of Degree Distributions

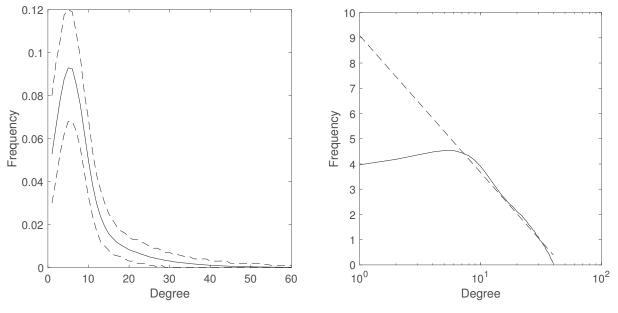
In this section we generate MECLTGs with n nodes. The orderings are discussed in Section 7.1. Let $p_0 = \lambda_0/n$, $p_1 = 1 - \lambda_1/n^c$ for $\lambda_0 = \lambda_1 = 1$. We calculate and plot the empirical degree distribution $n^{-1} \sum_{i=1}^n I(d_i = k)$, k = 0, ...n, and fit a power law to the degree distribution as follows: let $k' = \operatorname{argmax}_{0 \le k \le \sqrt{n}} n^{-1} \sum_{i=1}^n I(d_i = k)$, and $k'' = \max\{k : n^{-1} \sum_{i=1}^n I(d_i = k) > 0\}$. We then fit the regression

$$\log \sum_{i=1}^{n} I(d_i = k) = \gamma_0 + \gamma_1 \log k,$$

for $k' \leq k \leq k''$ to estimate γ_1 , and use $\hat{\gamma}_1$ as the estimate of the power law index. We draw the power line together with the degree distribution in the log scale plot. The empirical distribution is generated by 1000 replications in each simulation study, and the corresponding 95% confidence interval is provided by the simulated 0.025 and 0.975 quantiles of the simulated samples, respectively. In figure 3 we show the degree distribution for n = 1000, c = 0.3 with latent order ω_1 . We also examine scenarios with various c's, n's for latent order ω_1 and ω_2 in figures B.6–B.12 shown in the supplementary material. Those figures indicate that $\hat{\gamma}_1$ increases as c increases. Also we observe that

$$\hat{\gamma}_1(G(V,\omega_1,p_0,p_1)) > \hat{\gamma}_1(G(V,\omega_2,p_0,p_1)) \approx \hat{\gamma}_1(SRG).$$

When modelling the network via either the composite graphon model or the composite SBM, we have demonstrated in Sections 4 and 6 that the usual methods are still valid when



(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -2.4$ (dashed) & degree distribution (solid).

Figure 3: The degree distribution for n = 1000, and c = 0.3.

the dependence is not strong, and the consequences of the inhomogeneous degree pattern is negligible. Our simulation results support this, which coincides with the conclusion in [57]: comparing with the SBM, the estimation from DC-SBM only improves a little when the variation of degrees among nodes is not large. On the other hand, when the dependence is strong and the consequences of inhomogeneous degree pattern of network is significant, more advanced approaches are required for modelling and statistically analysing the network.

8 Discussion

As social media data sets, and other types of relational observations (networks) have become prevalent, so unsurprisingly the mathematical treatment of data taking the form of relationships between entities has become increasingly important. The analysis of networks has been the focus of considerable efforts where the properties of estimators for popular models have now been established. Following on from the understanding of correctly specified parametric models is the usage of non-parametric and incorrectly specified models. For example, our understanding of classical approaches can be found to extend when considering dense exchangeable arrays, see for example [53, 37, 21, 34].

Unfortunately the world contains many data sets that cannot be assumed to be exchangeable, despite how innocuous the assumption may seem, rather like stationarity for time series. For that reason we introduce the composite graphon model, and finite memory latent time-order graphs. By focusing on the latent variables in the model directly we can build a continuum of types of networks that are exchangeable, or strongly nonexchangeable, all tuned explicitly in terms of the dependence strength. This helps us to understand data of this form, and when we can apply regular network tools to novel types of data, and understand the consequences of that choice.

Non-exchangeable networks produce many challenges. The presence of strong powerlaws in the degrees and further heterogeneity in the graphon function itself are still challenging researchers. It is not unreasonable to believe that these features reflect how the network was formed. By assuming that the network formed sequentially we are able to both define a parameter that tunes its degree of exchangeability, and thus we may understand standard tools when applied to such data. Our understanding of this mechanism simultaneously give glimpses into the formation of non-exchangeability, and provide a gray-scale understanding of networks, letting us see how the mechanism allows us to gradually "dial away" from exchangeability as a consequence of evolution and growth.

A number of developments have sought to understand greater heterogeneity by modelling edge variables directly rather than relationships between nodes [22], this allowing a more natural and direct treatment of edge sparsity than some competing models. Others have concerned developing the practical application of work by Kallenberg's constructions [30], such as [15, 14]. The two key aspects of the latter construction is to use a latent Poisson construction and a latent time. We also used a latent variable which is uniform rather than Poisson. We correlate the latent uniforms directly, and show how the correlation of the latent variables drive the degree of non-exchangeability directly and quantitatively. The advantage of our framework is that it naturally straddles the model space between strong heterogeneity to the standard exchangeable graph model, with a direct tuning of its degree of non-regularity. If the correlation is not too strong, then standard methods apply for estimating the graphon model, rather like in time-series analysis with stationary errors when estimating polynomial trends. As the correlation becomes very strong, the observations exhibit more strong heterogeneity, and standard tools like the stochastic blockmodel approximation of the underlying graphon model will become increasingly problematic.

A number of questions remain unanswered. In parts this falls back to the difficulty of understanding a non-property, which has already haunted both non- stationary and non-linear time series (there are many ways to be non-stationary or non-linear, but only one to be stationary). In parts it falls back to understanding non-exchangeability itself, as one property rather than several real-life observed consequences thereof. By providing this framework, we can better see the limitations of exchangeable models, and how exchangeability can fail to materialize as a consequence of dependence.

Acknowledgements

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Supplemental material for 'Tractably Modelling Dependence in Networks Beyond Exchangeability'

Abstract

The structure of the supplementary material is organized as follows. Section A provided arameters in simulation models for spectral clustering algorithm in Section 6.3 of the main article. Section B contains additional simulation results for section 7 of the main article. Section C contains a discussion of the phase transition for connectivity and giant component of MECLTG. Section D provides the detailed proof of results in Section 4 of the main article. Section E provides the detailed proof of results in Section 6 of the main article. Finally, Section F provides the detailed detailed proof of results in Section 7 of the main article. Notice that N = n(n-1)/2.

A Parameters in simulation models for spectral clustering algorithm in Section 6.3 of the main article

For two groups case, we choose the parameter listed in Tables 1 and 2, respectively, where we refer the notation ρ to Section 4.1.1 of the main article. The corresponding results are shown in Figure 1 in the main article.

Table 1: The tables for parameters $\varrho_{index}^{0,index}$

index	11	12	22
value	0.1	0.01	0.2

Table 2: The tables for parameters $\varrho_{down}^{1,up}$

		down			
		11	12	22	
	11	0.4	0.05	0.3	
up	12	0.3	0.1	0.1	
	22	0.2	0.03	0.6	

We now show parameters for three groups case in Tables 3 and 4, respectively. The clustering result is presented in Figure A.4.

Table 3: The tables for parameters $\varrho_{index}^{0,index}$

• 1	11	10	10		0.0	
index	11	12	13	22	23	- 33
value	0.3	0.01	0.02	0.3	0.06	0.3

B Simulation results for degree distribution

		down					
		11	12	13	22	23	33
	11	0.5	0.02	0.05	0.3	0.02	0.04
	12	0.3	0.1	0.05	0.2	0.02	0.04
	13	0.2	0.1	0.08	0.05	0.02	0.04
up	22	0.15	0.02	0.02	0.6	0.02	0.04
	23	0.15	0.1	0.05	0.01	0.1	0.04
	33	0.2	0.01	0.02	0.05	0.02	0.7

Table 4: The tables for parameters $\varrho_{down}^{1,up}$

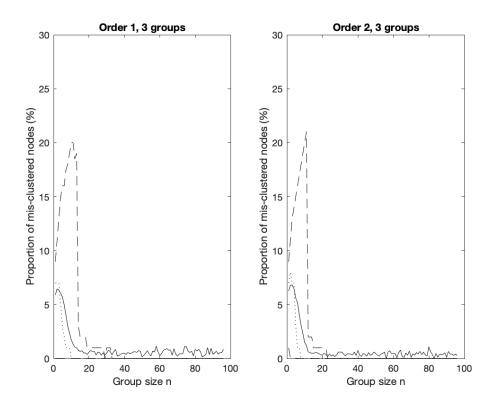


Figure A.4: Exploring the mis-clustering under different latent orders and three groups. Left: The number of mis-clustered nodes under order $\omega_1(\{\cdot, \cdot\})$. Right: The number of mis-clustered nodes under order $\omega_2(\{\cdot, \cdot\})$. We display the Mean (solid), median (dot) and upper/lower frequency band (dashed) of the mis-clustering number over 1000 simulations.

In this section we display Figure B.5 which shows typical graphs generated from MECLTGs $CG(V, \omega_1, p_0, p_1)$, $CG(V, \omega_2, p_0, p_1)$ and Erdős-Rényi G(V, p), respectively with $p_0 = 0.01$, $p_1 = 1 - \frac{1}{n^{1/3}}$, $p = \frac{p_0}{1-p_1+p_0}$ with |V| = 100. Then we show the degree distribution of various MECLTGs in Section 7.2 of the main article with different sizes, different values of c and two choices of latent order ω , i.e., ω_1 and ω_2 . Results are shown in Figures B.6–B.12. The related analysis are concluded in Section 7 of the main article.

C Phase Transition for Connectivity and Giant Component

In this section we discuss the phase transition of connectivity and of the giant component for the first order homogeneous MECLTG defined in Section 7 of the main article. It is well known that the threshold probability for connectivity in simple random graph (SRG) G(n,p) is $p = \frac{\log n}{n}$ (e.g.[24]). The threshold probability for the emergence of a giant component is $p = \frac{1}{n}$ (e.g. [24]). However, the traditional method of finding the threshold probability relies heavily on the assumptions of edge variables independence that does not hold for MECLTG. For example, to calculate the threshold probabilities for connectivity, many traditional methods need to evaluate $Cov(I_i, I_j)$, where $I_i = 1$ if node i is isolated and 0 otherwise. To calculate the threshold for the emergence of giant component in the SRG, techniques based on branching process, random walk, or depth first search algorithm are proposed. However, due to the complicated dependence structure and the unobserved latent order $\omega_n(\{\cdot, \cdot\})$, these techniques are not directly applicable to the first order homogeneous MECLTG. The threshold probabilities of the connectivity and of the emergence of a giant component are discussed for example [24], [33]. In the following theorem, we show by construction that under some circumstances, the first order homogeneous MECLTG possess the threshold properties similar to SRG.

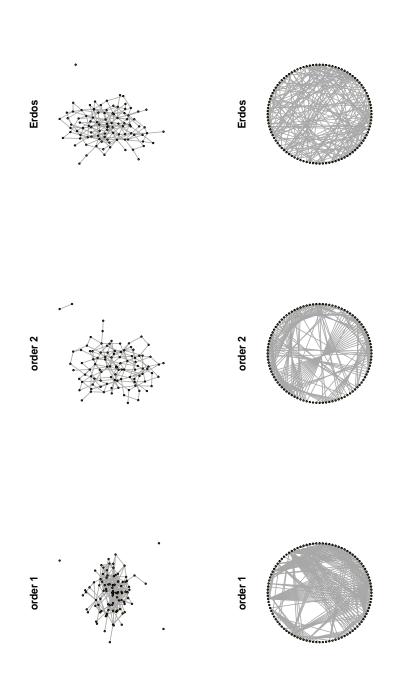
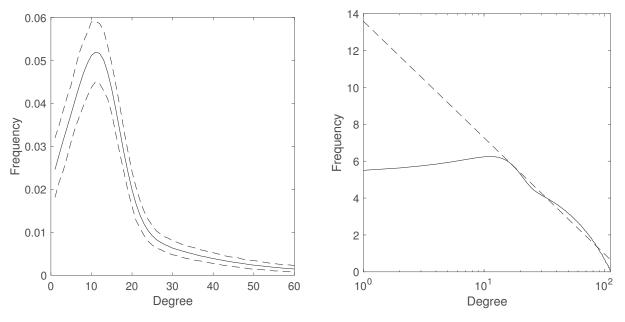


Figure B.5: Left and Middle: typical visualization of the size n = 100 first order homogeneous MECLTG with ordering ω_1 and ω_2 , and $p_0 = 0.01$, $p_1 = 1 - \frac{1}{n^{1/3}}$, respectively. Right: Size n = 100 Erdős-Rényi Graph with the $p = \frac{p_0}{1 - p_1 + p_0}$.



(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -2.74$ (dashed) & degree distribution (solid).

Figure B.6: The degree distribution for n = 10,000, and c = 0.3. with latent order ω_1 .

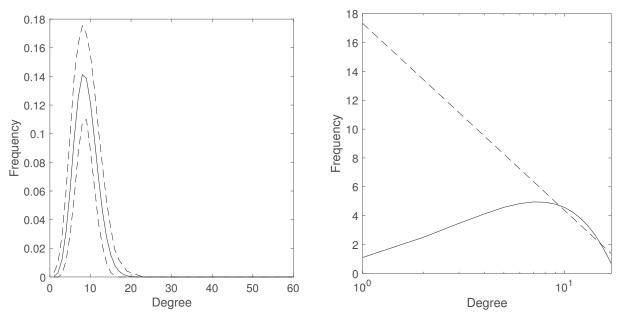
Theorem C.1. Let $G_n = CG(V_n, \omega, p_{0,n}, p_{1,n})$ be a series of first order homogeneous MECLTG, where V_n is the set of vertices such that $|V_n| = n$, and $p_{w,n} = \frac{\lambda_w \log n}{n}$, with $w \in \{0, 1\}$ and $\lambda_0, \lambda_1 > 0$ are positive constants.

- (a) Denote by \mathcal{A}_n the event that G_n is connected. Then we have i) if $\max\{\lambda_0, \lambda_1\} < 1$, then $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}_n) = 0$; ii) if $\min\{\lambda_0, \lambda_1\} > 1$ then $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}_n) = 1$.
- (b) Let $|S_i|$ be the size of the component that contains *i*. As $n \to \infty$, we have: i) if $\max\{\lambda_0, \lambda_1\} < 1$, then for all (sufficiently large) a > 0, $\lim_{n\to\infty} \mathbb{P}(\max_{1\le i\le n} |S_i| \ge a \log n) = 0$; ii) if $\min\{\lambda_0, \lambda_1\} > 1$, then there exists c > 0, such that

 $\mathbb{P}(\text{There exists a component with size} \geq c\sqrt{n}) \to 1$

as $n \to \infty$.

To avoid tedious computation due to the latent order $\omega_n(\{\cdot, \cdot\})$, and comply with the complex dependence structure, we prove the theorem via studying an algorithm that generates the SRG and the first order MECLTG simultaneously. The algorithm reveals the connection between the two constructions.



(a) Mean distribution (solid) and upper/lower frequency band (dashed).

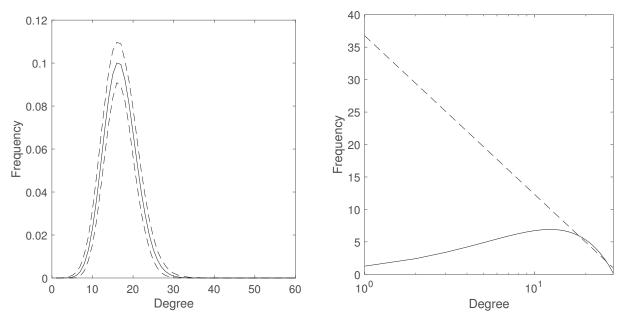
(b) Powerlaw $\gamma_1 = -5.6$ (dashed) & degree distribution (solid).

Figure B.7: The degree distribution for n = 1,000, and c = 0.3, with latent order ω_2 .

Proof. Let p_a be a pre-specified number, and ω be the latent order. Consider the following algorithm, which construct MECLTG by a series of random variables \tilde{A}_i and \tilde{B}_i , $1 \leq i \leq N$.

- (a) Generate $U \sim \text{Uniform}(0, 1)$. Set $\tilde{B}_1 = 1$ if $U \leq p$, where p is a function of $p_0 = p_{0,n}$, $p_1 = p_{1,n}$ The function is defined in equation (7.2) of the main article. Set $\tilde{A}_1 = 1$ if $U \leq p_a$.
- (b) At steps $i \ge 2$, generate independently a new $U \sim \text{Uniform}(0, 1)$. If $\tilde{B}_{i-1} = 1$, set $\tilde{B}_i = 1$ if $U \le p_1$. If $\tilde{B}_{i-1} = 0$, set $\tilde{B}_i = 1$ if $U \le p_0$. Set $\tilde{A}_i = 1$ if $U \le p_a$.

Thus, $\{B_{\omega^{-1}(i)}, 1 \leq i \leq N\} = \{\tilde{B}_i, 1 \leq i \leq N\}$ forms the ordered edge variables of a first order homogeneous MECLTG $CG(V, \omega, p_0, p_1)$, while $\{A_{\omega^{-1}(i)}, 1 \leq i \leq N\} = \{\tilde{A}_i, 1 \leq i \leq N\}$ forms the edge variables of a first order homogeneous MECLTG $CG(V, \omega, p_a, p_a)$. By our definition, $CG(V, \omega, p_a, p_a)$ is the simple random graph $G(|V|, p_a)$. Write $\bar{p}_1 = \max\{p_1, p_0\}, \bar{p}_0 = \min\{p_1, p_0\}$. Note that $p \in [\bar{p}_0, \bar{p}_1]$.

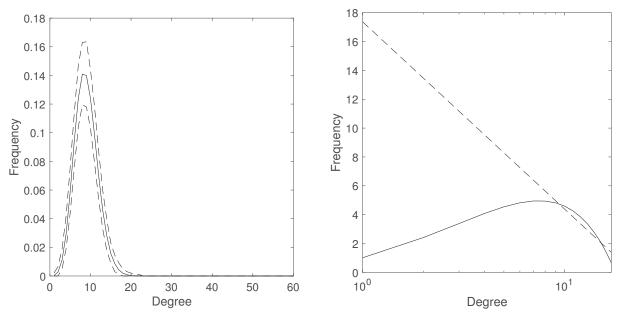


(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -10.2$ (dashed) & degree distribution (solid).

Figure B.8: The degree distribution for n = 10,000, and c = 0.3, with latent order ω_2 .

From the construction, when $\bar{p}_1 \leq p_a$, then in each step $i, 1 \leq i \leq N, A_{\omega^{-1}(i)} = 0$ implies $B_{\omega^{-1}(i)} = 0$. This is because if $A_{\omega^{-1}(i)} = 0$ (this means $\tilde{A}_i = 0$), then $U > p_a \geq \bar{p}_1$ in step (i). This will make $\tilde{B}_i = 0$, and consequently $B_{\omega_n^{-1}(i)} = 0$. The above fact shows that when $\bar{p}_1 \leq p_a$, if node i is isolated in the constructed $CG(V, \omega, p_a, p_a)$, the corresponding node is also isolated in the constructed $CG(V, \omega, p_a, p_a)$, the corresponding node is also isolated in the constructed $CG(V, \omega, p_0, p_1)$. As a result, we have that $|S_i^A| \geq |S_i^B|$ where $|S_i^A|$ is the size of the component contains node i in $CG(V, \omega, p_a, p_a)$, and $|S_i^B|$ is the size of the component contains node i in $CG(V, \omega, p_a, p_a)$, and $|S_i^B|$ is the size of the threshold probabilities of the connectivity and the emergence of a giant component for Erdös-Rényi Graph (e.g.[24]). When $\bar{p}_0 \geq p_a$, the theorem follows from a similar argument, and the proof is completed.



(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -5.6$ (dashed) & degree distribution (solid).

Figure B.9: The degree distribution for n = 1,000, and c = 0.3, for simple random graph.

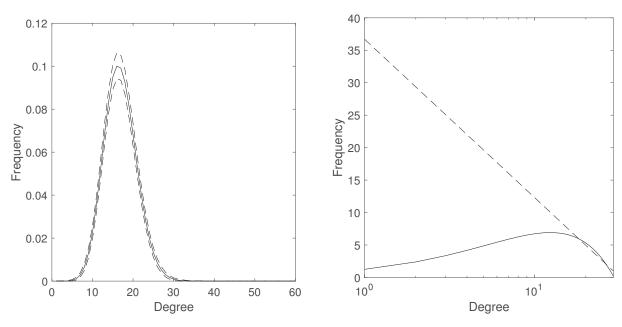
D Proof of auxiliary results for Theorem 5.1 and Theorem 5.2 in the main article

Proof of Proposition 4.1. By assumption $\min_{a,b\in\chi\times\chi} \mathbb{P}(U_i = a | U_{i-l} = b) \ge \alpha' > 0$, By definition, $\alpha' \le \frac{1}{2}$. The key to proof the proposition is to show that for $k \ge l$,

$$\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = b)$$

$$\leq (1 - 2\alpha') [\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b)].$$
(D.1)

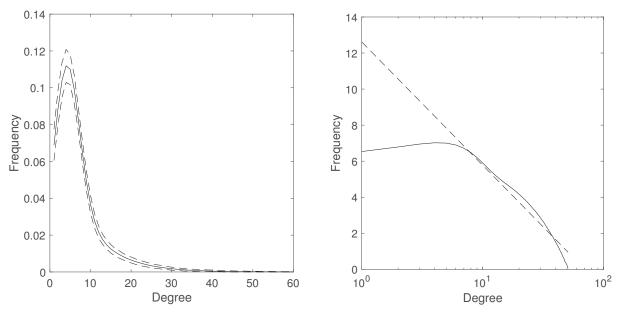
If equation (D.1) holds then for k = lq + s, $0 \le s \le l - 1$, we have by iteratively applying



(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -10.6$ (dashed) & degree distribution (solid).

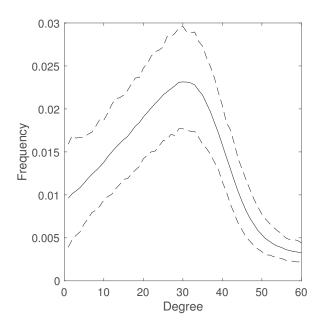
Figure B.10: The degree distribution for n = 10,000, and c = 0.3, for simple random graph.



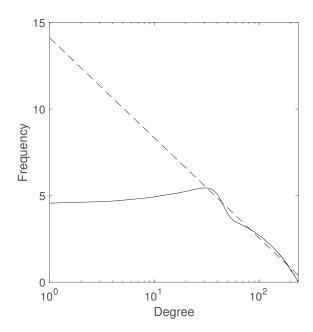
(a) Mean distribution (solid) and upper/lower frequency band (dashed).

(b) Powerlaw $\gamma_1 = -2.97$ (dashed) & degree distribution (solid).

Figure B.11: The degree distribution for n = 10,000, and c = 0.2, for latent order ω_1 .



(a) Mean distribution (solid) and upper/lower frequency band (dashed).



(b) Powerlaw $\gamma_1 = -2.52$ (dashed) & degree distribution (solid).

Figure B.12: The degree distribution for n = 10,000, and c = 0.4, for latent order ω_1 .

(D.1), the following inequality

$$\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = b)$$

$$\leq (1 - 2\alpha')^{q} [\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-s} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-s} = b)]$$

$$\leq M(1 - 2\alpha')^{k/l} \tilde{p}$$
(D.2)

holds, where $M = (1 - 2\alpha')^{-\frac{l-1}{l}} \leq (1 - 2\alpha')^{-1}$. Then the proposition follows from equation (D.2). It remains to show (D.1). Let $b_0 = \operatorname{argmin}_b \mathbb{P}(U_i = u_i | U_{i-k+l} = b)$. Notice that

$$\begin{split} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = u_{i-k}) \\ &= \sum_{b \in \mathbb{R}^{l}} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \mathbb{P}(U_{i-k+l} = b | U_{i-k} = u_{i-k}) \\ &\leq \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b_{0}) \mathbb{P}(U_{i-k+l} = b_{0} | U_{i-k} = u_{i-k}) \\ &+ \sum_{b \neq b_{0}} (\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b)) \mathbb{P}(U_{i-k+l} = b | U_{i-k} = u_{i-k}) \\ &= \max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \\ &- \mathbb{P}(U_{i-k+l} = b_{0} | U_{i-k} = u_{i-k}) \times \left(\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \right) \\ &\leq \max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \alpha' \left(\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \right) \end{split}$$

By taking maximum on both side of (D.3), we have

$$\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k} = b) \leq \max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \alpha' \left(\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \right) + \alpha' \left(\max_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_{i} = u_{i} | U_{i-k+l} = b) \right)$$

Similarly

$$\min_{b} \mathbb{P}(U_i = u_i | U_{i-k} = b) \ge \min_{b} \mathbb{P}(U_i = u_i | U_{i-k+l} = b) + \alpha' \left(\max_{b} \mathbb{P}(U_i = u_i | U_{i-k+l} = b) - \min_{b} \mathbb{P}(U_i = u_i | U_{i-k+l} = b) \right)$$

The (D.1) follows from (D.4) and (D.5).

Recall χ of Proposition 4.1 of the main article in the following arguments.

Lemma D.1. Assume that the conditions of Theorem 5.1 holds. Recall $Z_{n,k}$ defined in Section 5. Then for all $z \in \mathbb{Z}_{n,k}$, there exist constant C_1 (which does not depend on n) such that for $a, b \in [k]$,

$$\mathbb{E}\left(\exp\left(\left(\frac{C_1\left(1-\chi\right)}{\sqrt{n_a n_b}}\sum_{i\in\bar{z}^{-1}(a),j\in\bar{z}^{-1}(b)}(A_{i,j}-\theta_{i,j})\right)^2\right)\right) \le 2.$$

Proof. Consider the case of $a \neq b$. The case of a = b follows *mutatis mutandis*. Recall the map $\omega(\cdot, \cdot)$ and define $B_i, i \in [N]$ such that $B_{\omega(\{i,j\})} = A_{i,j}$. Define a series of integers $u_j, 1 \leq j \leq n_a n_b$ as

$$\{u_j, 1 \le j \le n_a n_b\} = \{\omega(\{i, j\}), i \in \bar{z}^{-1}(a), j \in \bar{z}^{-1}(b)\}$$

such that $u_s < u_l$ for s < l. Consider the filtration $\mathcal{F}_i = (\{B_{u_j}, j \leq i\}, z_1, ..., z_n)$, where $z := \{z_i, 1 \leq i \leq n\}$ are latent U(0, 1) variables. Let $B_{u_s} = B_s$ for $s \leq 0$. Define the projection operator for $j \in \mathbb{Z}$,

$$\mathcal{P}_j(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_j, z) - \mathbb{E}(\cdot|\mathcal{F}_{j-1}, z).$$

By our construction of B_i , $i \leq 0$ in Definition 4.1 of the main article, we get

$$\mathcal{P}_j(B_k) = 0 \text{ for } j \le 0, \quad k > 0.$$

It follows from the fact that $\theta_{i,j} = \mathbb{E}(A_{i,j}|z)$,

$$\sum_{i\in\bar{z}^{-1}(a),j\in\bar{z}^{-1}(b)} \{A_{i,j} - \mathbb{E}(A_{i,j}|z)\} = \sum_{i=1}^{n_a n_b} \sum_{s=0}^{\infty} \mathcal{P}_{i-s} B_{u_i} = \sum_{s=0}^{\infty} \sum_{i=1}^{n_a n_b} \mathcal{P}_{i-s} B_{u_i}.$$
 (D.6)

Note that $\mathcal{P}_{i-s}B_{u_i}$ forms a martingale difference w.r.t $\{\mathcal{F}_{i-1}, z_1, ..., z_n\}$. By Burkholder inequality, for v > 0, we have that

$$\|\sum_{i=1}^{n_a n_b} \mathcal{P}_{i-s} B_{u_i}\|_{\mathcal{L}_v}^2 \le C v \sum_{i=1}^{n_a n_b} \|\mathcal{P}_{i-s} B_{u_i}\|_{\mathcal{L}_v}^2,$$

where C is a constant independent of v, n_a , n_b and n, and $||X||_{\mathcal{L}_v} := \{\mathbb{E}(|X|^v)\}^{1/v}$. By Corollary 4.1 in the main article and the fact that $u_i - u_{i-s} \ge s$, we have

$$|\mathcal{P}_{i-s}B_{u_i}| \le C'\chi^s.$$

Here the above equation does not depend on i due to Corollary 4.1 which only involves the distant of indices. Thus we obtain

$$\left\|\sum_{i=1}^{n_a n_b} \mathcal{P}_{i-s} B_{u_i}\right\|_{\mathcal{L}_v}^2 \le C'' v n_a n_b \chi^{2s}.$$

Combining with (D.6) and by triangle inequality, we have that

$$\left\| \sum_{i \in \bar{z}^{-1}(a), j \in \bar{z}^{-1}(b)} (A_{i,j} - \theta_{i,j}) \right\|_{\mathcal{L}_v} \le \frac{C_1 \sqrt{n_a n_b v}}{1 - \chi},\tag{D.7}$$

where C', C'' and C_1 are constants independent of n and v. By using Taylor expansion, equation (D.7) shows that there exists a small $\eta \in (0, 1/2)$ such that

$$\mathbb{E}\left(\exp\left(\eta\left(\frac{(1-\chi)}{C_1\sqrt{n_a n_b}}\sum_{i\in\bar{z}^{-1}(a),j\in\bar{z}^{-1}(b)}(A_{i,j}-\theta_{i,j})\right)^2\right)\right) \le 1+\sum_{u=1}^{\infty}\frac{u^u\eta^u}{u!}\le 2.$$
(D.8)

The last inequality use the fact that $u! > (\frac{u}{e})^u$ and we take η small such that $\eta e \leq 1/2$. \Box

Corollary D.1. Assume that the conditions of Theorem 5.1 holds. Consider real numbers $\{\gamma_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq n, i \neq j}$ satisfies $\sum_{i,j} \gamma_{i,j}^2 = 1$. Then we have that there exists a constant C_2 such that

$$\mathbb{E}\left(\exp\left(\left(C_2(1-\chi)\sum_{i,j}\gamma_{i,j}(A_{i,j}-\theta_{i,j})\right)^2\right)\right) \le 2.$$

Proof. Recall the proof of Lemma D.1 and the filtration $\mathcal{F}_i = (\{B_j, j \leq i\}, z)$ defined there, associate with the projection operator $\mathcal{P}_j(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_j, z) - \mathbb{E}(\cdot | \mathcal{F}_{j-1}, z)$ in Lemma D.1. Define a series γ' such that $\gamma'_{\omega\{i,j\}} = \gamma_{i,j}$ for i < j such that $\sum_{j=1}^N \gamma'^2_j = 1/2$. Thus we have

$$\sum_{i>j} \gamma_{i,j} \left(A_{i,j} - \theta_{i,j} \right) = \sum_{s=0}^{\infty} \sum_{i=1}^{N} \gamma'_i \mathcal{P}_{i-s} B_i.$$

By Burkholder's inequality and the triangle inequality, a similar argument to that of Lemma D.1 yields that for $v \ge 0$,

$$\left\|\sum_{i>j}\gamma_{i,j}\left(A_{i,j}-\theta_{i,j}\right)\right\|_{\mathcal{L}_{v}} \leq C_{2}\sqrt{v}/(1-\chi),$$

where C_2 is a constant independent of n and v. The corollary follows from the same argument of (D.8).

Lemma D.2. Assume that the conditions of Theorem 5.1 hold. Then for any C' > 0, there exists C > 0 such that

$$\left| \left\langle \frac{\tilde{\theta} - \theta}{\|\tilde{\theta} - \theta\|}, A - \theta \right\rangle \right| \le C\sqrt{n \log k} (1 - \chi)^{-1}$$

with probability $1 - \exp(-C' n \log k)$.

Proof. By lemma 4.2 of Gao et al. (2015), we have that

$$\left| \left\langle \frac{\tilde{\theta} - \theta}{\|\tilde{\theta} - \theta\|}, A - \theta \right\rangle \right| \le \max_{z \in \mathcal{Z}_{n,k}} \left| \sum_{i,j} \gamma_{i,j}(z) (A_{i,j} - \theta_{i,j}) \right|$$

where $\gamma_{i,j}(z) \propto \sum_{a,b \in [k]} \bar{\theta}_{a,b}(z) \mathbf{1}((i,j) \in z^{-1}(a) \times z^{-1}(b)) - \theta_{i,j}$, satisfying $\sum_{i,j} \gamma_{i,j}^2(z) = 1$. Then Markov's inequality, union bound and Corollary D.1 lead to

$$\mathbb{P}\left((1-\chi)\max_{z\in\mathcal{Z}_{n,k}}|\sum_{i,j}\gamma_{i,j}(z)(A_{i,j}-\theta_{i,j})|>t\right)\leq C_1\exp(-C_0t^2+n\log k).$$
 (D.9)

Then the lemma follows from letting $t = M\sqrt{n \log k}$ in (D.9) for some large constant M.

Lemma D.3. Assume that the conditions of Theorem 5.1 hold. Then for any constant C' > 0, there exists a constant C > 0 only depending on C', such that

$$\left| \left\langle \frac{\hat{\theta} - \tilde{\theta}}{\|\hat{\theta} - \tilde{\theta}\|}, A - \theta \right\rangle \right| \le C\sqrt{k^2 + n\log k}(1 - \chi)^{-1}$$

with probability at least $1 - \exp(-C' n \log k)$.

Proof. Define an *n*-dimensional ball $\mathcal{B} \subset \left\{ a \in \mathbb{R}^{n \times n} : \sum_{ij} a_{ij}^2 \leq 1 \right\}$ with the following property

if
$$a, b \in \mathcal{B}$$
, then $\frac{a-b}{\|a-b\|} \in \mathcal{B}$. (D.10)

Let \mathcal{B}' be a 1/2-net of \mathcal{B} such that $|\mathcal{B}'| \leq \mathcal{N}(1/2, \mathcal{B}, \|\cdot\|)$ where $\mathcal{N}(1/2, \mathcal{B}, \|\cdot\|)$ is the covering number of 1/2 net of set \mathcal{B} . By Lemma A.1. of Gao et al. (2015), we have that:

$$\sup_{a \in \mathcal{B}} \left| \sum_{i,j} a_{i,j} (A_{i,j} - \theta_{i,j}) \right| \le 2 \max_{b \in \mathcal{B}'} \left| \sum_{i,j} b_{i,j} (A_{i,j} - \theta_{i,j}) \right|.$$

As a consequence, by Corollary D.1 and the union bound, we have

$$\mathbb{P}\left(\left(1-\chi\right)\sup_{a\in\mathcal{B}}\left|\sum_{i,j}a_{i,j}(A_{i,j}-\theta_{i,j})\right| \geq t\right) \\
\leq \mathbb{P}\left(2(1-\chi)\max_{b\in\mathcal{B}'}\left|\sum_{i,j}b_{i,j}(A_{i,j}-\theta_{i,j})\right| \geq t\right) \\
\leq |\mathcal{B}'|C_1\exp(-C_0t^2), \quad (D.11)$$

for some constants $C_0 \in \mathbb{R}^+$ and $C_1 \in \mathbb{R}^+$. For $\overline{z} \in \mathcal{Z}_{n,k}$, define

$$\mathcal{B}_{\bar{z}} = \{\{c_{i,j}\} : c_{i,j} = Q_{ab} \text{ if } (i,j) \in \bar{z}^{-1}(a) \times \bar{z}^{-1}(b) \text{ for some } Q_{ab},$$

and
$$\sum_{i,j} c_{i,j}^2 \leq 1, c_{i,j} = c_{j,i}, c_{i,i} = 0, 1 \leq i \leq n\}.$$

Notice that $\mathcal{B}_{\bar{z}}$ satisfy (D.10), and $\mathcal{N}(1/2, \mathcal{B}_{\bar{z}}, \|\cdot\|) \leq \exp(C_2 k^2)$ for some constant $C_2 \in \mathbb{R}^+$. By the proof of Corollary D.1 and the union bound, we have that

$$\mathbb{P}\left(\left(1-\chi\right)\left|\left\langle\frac{\hat{\theta}-\tilde{\theta}}{\|\hat{\theta}-\tilde{\theta}\|}, A-\theta\right\rangle\right| > t\right) \tag{D.12}$$

$$\leq \mathbb{P}\left(\left(1-\chi\right)\max_{z\in\mathcal{Z}_{n,k}}\sup_{c\in\mathcal{B}_{z}}\left|\sum_{ij}c_{i,j}\left(A_{i,j}-\theta_{i,j}\right)\right| > t\right) \\
\leq C_{0}\exp\left(-C_{0}t^{2}+C_{1}n\log k+C_{2}k^{2}\right)$$

for some positive constants C_0 , C_1 , C_2 . The last inequality requires (D.11). Then the lemma follows from taking $t = M\sqrt{n\log k + k^2}$ for sufficiently large M in (D.12).

E Proof of auxiliary results for Theorem 6.1

For latent variable z and random variable X, let $||X||_{\mathcal{L}_2} = \sqrt{\mathbb{E}(X^2|z)}$. We need the following proposition to show Proposition 6.2 of the main article.

Proposition E.1. Consider the latent undirected Markov model with memory parameter $h < \infty$. Let \mathcal{F}_i be the filtration generated by $\{B_s\}_{s \leq i}$, where $\{B_s\}_{s \leq i}$ is defined in Definition 4.1 in the main article. Let $\delta(k)$: $\delta(k) = (1 - 2\alpha')^{-1/2}\chi^{k/2}$ for $k \geq 0$, χ and α' defined in Proposition 4.1 in the main article, and 0 if $k \leq -1$. Then for any integers $i_1 \leq ... \leq i_l$ for some fixed constant l, we have that

$$\|\mathcal{P}_{a}(\Pi_{s=1}^{l}B_{i_{s}})|z\|_{\mathcal{L}_{2}} \leq 2\sum_{s=1}^{l}\delta(i_{s}-a)\mathbf{1}(i_{s-1}$$

where $\mathcal{P}_j(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_j, z) - \mathbb{E}(\cdot | \mathcal{F}_{j-1}, z), \ \mathcal{F}_i = (B_{-\infty}, ..., B_i) \text{ and } i_0 = -\infty.$

Proof. Without loss of generality, let $i_s \leq i_{s+1}$ for $s \leq s+1$. Then by direct calculations, we obtain that

$$\begin{aligned} \|\mathcal{P}_{a}(\Pi_{s=1}^{l}B_{i_{s}})|z\|_{\mathcal{L}_{2}}^{2} &= \mathbb{E}\left(\mathbb{E}\left(\mathcal{P}_{a}^{2}(\Pi_{s=1}^{l}B_{i_{s}})|\mathcal{F}_{a-1},z\right)|z\right) \\ &= \mathbb{E}(\mathbb{E}^{2}(\Pi_{s=1}^{l}B_{i_{s}}|\mathcal{F}_{a},z)|z) - \mathbb{E}(\mathbb{E}^{2}(\Pi_{s=1}^{l}B_{i_{s}}|\mathcal{F}_{a-1},z)|z) := \mathbb{E}(AB|z), \end{aligned}$$

where

$$A = \mathbb{E}(\Pi_{s=1}^{l} B_{i_s} | \mathcal{F}_a, z) + \mathbb{E}(\Pi_{s=1}^{l} B_{i_s} | \mathcal{F}_{a-1}, z),$$

$$B = \mathbb{E}(\Pi_{s=1}^{l} B_{i_s} | \mathcal{F}_a, z) - \mathbb{E}(\Pi_{s=1}^{l} B_{i_s} | \mathcal{F}_{a-1}, z).$$

As a result we have that

$$\|\mathcal{P}_{a}(\Pi_{s=1}^{l}B_{i_{s}})|z\|_{\mathcal{L}_{2}} \leq (\|A|z\|_{\mathcal{L}_{v}}\|B|z\|_{\mathcal{L}_{v}})^{1/2}.$$

By the boundedness of Bernoulli random variable, we have that $|A| \leq 2$. For $a \geq i_l + 1$, it is easy to see that $||\mathcal{P}_a(\Pi_{s=1}^l B_{i_s})|z|| = 0$. Recall the definition of $U_i = (B_i, ..., B_{i-h+1})'$ below the Definition 4.1 in the main article. Let $u_i = (b_i, ..., b_{i-h+1})' \in \{0, 1\}^h$. Note that for $a \leq i_1 - 1$ and v = a, a - 1,

$$\mathbb{E}(\Pi_{s=1}^{l}B_{i_{s}}|\mathcal{F}_{v},z) = \sum_{\substack{u_{i}=(b_{i},\dots,b_{i-h+1})',,\\b_{j}=\{0,1\},i_{1}\leq j\leq i_{l}+h,\\j\neq i_{s},1\leq s\leq l}} \Pi_{i=i_{1}+h}^{i_{l}+h} \mathbb{P}(U_{i}=u_{i}|U_{i-1}=u_{i-1},z)\mathbb{P}(U_{i_{1}+h-1}=u_{i_{1}+h-1}|\mathcal{F}_{v},z)\Big|_{b_{i_{s}}=1,s=1,\dots,l}.$$

Notice that

$$\sum_{\substack{u_i = (b_i, \dots, b_{i-h+1})',, \\ b_j = \{0,1\}, i_1 \le j \le i_l + h, \\ j \ne i_s, 1 \le s \le l}} \prod_{i=i_1+h}^{i_l+h} \mathbb{P}(U_i = u_i | U_{i-1} = u_{i-1}, z) \Big|_{b_{i_s} = 1, s = 1, \dots, l}$$

Using this fact, Proposition 4.1 in the main article and the upper bound of M in equation (D.2) we have that $B \leq 2(1 - 2\alpha')^{-1}\chi^{i_1 - a}$. Consequently, we have that

$$\|\mathcal{P}_{a}(\Pi_{s=1}^{l}B_{i_{s}})|z\|_{\mathcal{L}_{2}} \le 2\delta(i_{1}-a).$$
 (E.1)

Observe that (E.1) still holds for $a = i_1$. For $i_u + 1 \le a \le i_{u+1}$, $1 \le u \le l-1$, we have that for v = a, a - 1,

$$\mathbb{E}(\Pi_{s=1}^{l}B_{i_s}|\mathcal{F}_{v},z) = (\Pi_{w=1}^{u}B_{i_w})\mathbb{E}(\Pi_{s=u+1}^{l}B_{i_s}|\mathcal{F}_{v},z).$$

Therefore we have that

$$\mathcal{P}_a(\Pi_{s=1}^l B_{i_s}) = (\Pi_{w=1}^u B_{i_w}) \mathcal{P}_a(\Pi_{s=u+1}^l B_{i_s} | \mathcal{F}_v, z).$$

It follows from the boundedness of $B'_i s$ and a similar argument to the $a < i_1$ case that

$$\|\mathcal{P}_{a}(\Pi_{s=1}^{l}B_{i_{s}})|z\|_{\mathcal{L}_{2}} \leq 2\delta(i_{u+1}-a),$$

from which the proposition follows.

Proof of Proposition 6.2 in the main article. Recall the definition of $\delta(k)$ in Proposition E.1. Notice that by the orthogonality of $\mathcal{P}'_k s$ defined in Proposition E.1 and Fubini's

theorem, we have that

$$\begin{aligned} |Cov(\Pi_{u=1}^{l}\tilde{B}_{i_{u}},\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}}|z)| \\ &= |Cov(\Pi_{u=1}^{l}\tilde{B}_{i_{u}} - \mathbb{E}(\Pi_{u=1}^{l}\tilde{B}_{i_{u}}|z),\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}} - E(\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}}|z)|z)| \\ &= \left| \mathbb{E}\left(\sum_{a=-\infty}^{N} \mathcal{P}_{a}(\Pi_{u=1}^{l}\tilde{B}_{i_{u}})\sum_{b=-\infty}^{N} \mathcal{P}_{b}(\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}})\Big|z\right) \right| \\ &= \left| \mathbb{E}\left(\sum_{a=-\infty}^{N} \mathcal{P}_{a}(\Pi_{u=1}^{l}\tilde{B}_{i_{u}})\mathcal{P}_{a}(\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}})\Big|z\right) \right| \\ &\leq \sum_{a=-\infty}^{N} \|\mathcal{P}_{a}(\Pi_{u=1}^{l}\tilde{B}_{i_{u}})|z\|_{\mathcal{L}_{2}}\|\mathcal{P}_{a}(\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}})|z\|_{\mathcal{L}_{2}}. \end{aligned}$$
(E.2)

By Proposition E.1, we have

(E.2)
$$\leq 4 \sum_{a=-\infty}^{N} \left(\sum_{s=1}^{l} \delta(i_s - a) \mathbf{1}(i_{s-1} < a) \sum_{u=l+1}^{2l} \delta(i_u - a) \mathbf{1}(i_{u-1} < a) \right)$$

which shows (a) by straightforward calculations. For (b), we shall show that

$$|Cov(\Pi_{u=1}^{l}\tilde{B}_{i_{u}},\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}}|z)| \le 12l(1-2\alpha')^{-1}\chi^{\iota(i_{s})/2}(1-\chi)^{-1},$$
(E.3)

for s = 1, ..., 2l. Without loss of generality, we only show the case for s = 1. The other cases follow similar arguments. Direct calculations show that

$$|Cov(\Pi_{u=1}^{l}\tilde{B}_{i_{u}},\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}}|z)|$$

= $|\mathbb{E}(\Pi_{u=1}^{2l}\tilde{B}_{i_{u}}|z) - \mathbb{E}\left(\Pi_{u=1}^{l}\tilde{B}_{i_{u}}|z\right)\mathbb{E}\left(\Pi_{v=l+1}^{2l}\tilde{B}_{i_{v}}|z\right)| \leq I + II,$

where

$$I := |\mathbb{E}(\Pi_{u=1}^{2l} \tilde{B}_{i_u} | z)| = |Cov(\tilde{B}_{i_1}, \Pi_{u=2}^{2l} \tilde{B}_{i_u} | z)|$$

$$II := |\mathbb{E}\left(\Pi_{u=1}^l \tilde{B}_{i_u} | z\right) \mathbb{E}\left(\Pi_{v=l+1}^{2l} \tilde{B}_{i_v} | z\right)| \le |Cov(\tilde{B}_{i_1}, \Pi_{u=2}^l \tilde{B}_{i_u} | z)|.$$
(E.4)

The second equality of I is due to $\mathbb{E}\tilde{B}_{i_1} = 0$, and the second equality of II is due to $\mathbb{E}\tilde{B}_{i_1} = 0, \mathbb{E}\tilde{B}_{i_{l+1}} = 0$. By similar arguments to equation E.2. we have that for a sufficiently large constant C > 0,

$$I \le 4 \sum_{a=-\infty}^{N} \left(\delta(i_1 - a) \sum_{u=2}^{2l} \delta(i_u - a) \right),$$
$$II \le 4 \sum_{a=-\infty}^{N} \left(\delta(i_1 - a) \sum_{u=2}^{l} \delta(i_u - a) \right)$$

Then (E.3) follows from (E.4) and straightforward calculations, which finishes the proof. \Box

Proof of Proposition 6.1 in the main article

Let C be a sufficiently large positive generic constant which varies from line to line. Recall the re-parameterization θ in Section (6.1) of the main article. Let $\theta_{i,j}$ be the (i, j)-entry of θ . Define normalized Laplacian $\tilde{L} = \bar{D}^{-1/2} A \bar{D}^{-1/2}$. We shall show (i): there exists a sufficiently large positive constant η_0 such that

$$\mathbb{P}\left(\|\tilde{L}\tilde{L} - \bar{L}\bar{L}\|_F \ge \frac{\log n}{\tau^2 n^{1/2}} G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}\right) \le \frac{\eta_0}{n \log^4 n},$$

and (ii): there exists a set E_n with its complement E_n^c and sufficiently large positive constants η_1 and M', such that when $n \ge M'$

$$\mathbb{P}\left(\|\tilde{L}\tilde{L} - LL\|_F \ge \frac{\log n}{\tau^2 n^{1/2}} G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}, E_n\right) = 0,$$
(E.5)

$$\mathbb{P}(E_n^c) \le \eta_1 n^{-2}. \tag{E.6}$$

where C_0 is a sufficiently small positive constant. The LHS of equation (E.5) means the probability of intercept of events E_n and $\{\|\tilde{L}\tilde{L} - LL\|_F \ge \frac{M\log n}{\tau^2 n^{1/2}}G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}\}$. For (i), direct calculation shows that

$$|\tilde{L}\tilde{L} - \bar{L}\bar{L}|_{ij} = \frac{1}{n^2\sqrt{c_ic_j}} \sum_{k=1}^n \left(\frac{A_{i,k}A_{k,j} - \theta_{i,k}\theta_{k,j}}{c_k}\right).$$

By definition of τ , we only need to show that

$$\mathbb{P}\left(\sum_{i,j}^{n} \left(\sum_{k=1}^{n} \left(\frac{A_{i,k}A_{k,j} - \theta_{i,k}\theta_{k,j}}{c_k}\right)\right)^2 \ge \frac{n^3 \log^2 n}{\tau^2} G(\chi, N, 3)(1-\chi)^{-1}\right) \le \frac{\eta_0}{n \log^4 n}.$$
 (E.7)

Write $\tilde{A}_{i,j} = A_{i,j} - \theta_{i,j}$ for the centered adjacency matrix. Then we have

$$A_{i,k}A_{k,j} - \theta_{i,k}\theta_{k,j} = \tilde{A}_{i,k}\tilde{A}_{k,j} + \theta_{k,j}\tilde{A}_{i,k} + \theta_{i,k}\tilde{A}_{k,j}.$$

As a result, we get

$$\sum_{i,j} \left(\sum_{k=1}^{n} \left(\frac{A_{i,k}A_{k,j} - \theta_{i,k}\theta_{k,j}}{c_k} \right) \right)^2$$

= $\sum_{i,j,k,l} \left(\left(\frac{\tilde{A}_{i,k}\tilde{A}_{k,j} + \theta_{k,j}\tilde{A}_{i,k} + \theta_{i,k}\tilde{A}_{k,j}}{c_k} \right) \times \left(\frac{\tilde{A}_{i,l}\tilde{A}_{l,j} + \theta_{l,j}\tilde{A}_{i,l} + \theta_{i,l}\tilde{A}_{l,j}}{c_l} \right) \right)$
= $I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9,$

where

$$I_{1} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\tilde{A}_{k,j}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}, I_{2} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\tilde{A}_{k,j}\tilde{A}_{i,l}\theta_{l,j}}{c_{k}c_{l}}, I_{3} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\tilde{A}_{k,j}\theta_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}, I_{4} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\theta_{k,j}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}, I_{5} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\theta_{k,j}\tilde{A}_{i,l}\theta_{l,j}}{c_{k}c_{l}}, I_{6} = \sum_{i,j,k,l} \frac{\tilde{A}_{i,k}\theta_{k,j}\theta_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}, I_{7} = \sum_{i,j,k,l} \frac{\theta_{i,k}\tilde{A}_{k,j}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}, I_{8} = \sum_{i,j,k,l} \frac{\theta_{i,k}\tilde{A}_{k,j}\tilde{A}_{i,l}\theta_{l,j}}{c_{k}c_{l}}, I_{9} = \sum_{i,j,k,l} \frac{\theta_{i,k}\tilde{A}_{k,j}\theta_{i,l}\tilde{A}_{l,j}}{c_{k}c_{l}}.$$

We will show that, there exist constants M_s , $1 \le s \le 9$ such that

$$\mathbb{P}\left(I_s \ge \frac{n^3 \log^2 n}{\tau^2} G(\chi, N, 3)(1-\chi)^{-1}\right) \le \frac{\eta_{0,s}}{n \log^4 n}.$$
(E.8)

By taking $\eta_0 = \sum_{s=1}^{9} \eta_{0,s}$, we shall show (E.7). For the sake of brevity, we only show the case that s = 1, and the situation that $2 \le s \le 9$ follows from a similar argument. Further define $\bar{A}_{(i,k),(k,j)} = \tilde{A}_{i,k}\tilde{A}_{k,j} - \mathbb{E}(\tilde{A}_{i,k}\tilde{A}_{k,j})$. Notice that

$$\mathbb{E}\left(\bar{A}_{(i,k)(k,j)}\tilde{A}_{i,l}\tilde{A}_{l,j}\right) = Cov(\bar{A}_{(i,k)(k,j)}\tilde{A}_{i,l},\tilde{A}_{l,j}) = Cov(\bar{A}_{(i,k)(k,j)},\tilde{A}_{i,l}\tilde{A}_{l,j}).$$

By the similar arguments to the proof of Proposition 6.2, we have that

$$\left| \mathbb{E}\left(\frac{\bar{A}_{(i,k)(k,j)}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_k c_l} \right) \right| \le \frac{40(1-2\alpha')^{-1}}{\tau^2} \chi^{\Xi(i,k,j,l,\omega)/2} (1-\chi)^{-1},$$

where we have defined

$$\Xi(i,k,j,l,\omega) = \max\{\max_{3 \le s \le 4} \min_{1 \le j \le 4, j \ne s} (|i_s - i_j|), \min(|i_1 - i_3|, |i_1 - i_4|, |i_2 - i_3|, |i_2 - i_4|)\}, \ldots, \omega\}$$

 χ and α' are defined in defined in Proposition 4.1 in the main article, and integers $i_1 = \omega(\{i,k\}), i_2 = \omega(\{k,j\}), i_3 = \omega(\{i,l\}), i_4 = \omega(\{l,j\})$. We then argue that for any $\omega(\{\cdot,\cdot\})$

and given i, k, j and r, the possible number of l such that $\Xi(i, k, j, l, \omega) = r$ is at most 8r. We also use the following argument and its analog. If $i_5 = \omega(\{u, v\})$ has k possible different values, then the number of different values of u and v should be both smaller than k + 1. This can be easily seen by contradiction. When i, j, k are fixed, i_1, i_2 are fixed. For any real number a and integer r > 0, denote by $B_a(r)$ the interval [a - r, a + r]. Then one of the i_3 and i_4 must fall into $B_{i_1}(r) \cup B_{i_2}(r)$. Otherwise $\min(|i_1 - i_3|, |i_1 - i_4|, |i_2 - i_3|, |i_2 - i_4|) > r$, which implies $\Xi(i, k, j, l, \omega) > r$. Since $\omega(\{\cdot, \cdot\})$ is a 1 - 1 map, when i, j, k are fixed, the number of possible choices of l is less than or equal to both the numbers of possible choices of i_3 and i_4 . The total number of integers in the interval $B_{i_1}(r) \cup B_{i_2}(r)$ are is most $(2r+1) \times 2 = 4r+2$, which implies that the possible number of l such that $\Xi(i, k, j, l, \omega) = r$ is at most 8r + 5. Thus,

$$\sum_{i,j,k,l} \left| \mathbb{E}\left(\frac{\bar{A}_{(i,k)(k,j)}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_k c_l}\right) \right| \le \frac{40(1-2\alpha')^{-1}}{\tau^2} (1-\chi)^{-1} \sum_{i,j,k} \sum_{r=0}^N (8r+5)\chi^{r/2} \le \frac{Cn^3}{\tau^2} G(\chi, N, 1)(1-\chi)^{-1}.$$
(E.9)

Similarly, we get

$$\left| \mathbb{E}\left(\frac{\mathbb{E}(\tilde{A}_{i,k}\tilde{A}_{k,j})\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_k c_l} \right) \right| \le \frac{16(1-2\alpha')^{-1}}{\tau^2} \chi^{|\omega(\{i,l\})-\omega(\{j,l\})|/2} (1-\chi)^{-1},$$

Then similar but easier arguments of (E.9) (by fixing i, l, k) lead to

$$\sum_{i,j,k,l} \left| \mathbb{E}\left(\frac{\mathbb{E}(\tilde{A}_{i,k}\tilde{A}_{k,j})\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_k c_l} \right) \right| \le \frac{Cn^3}{\tau^2} G(\chi, N, 1)(1-\chi)^{-1},$$

which together with (E.9) shows that

$$\mathbb{E}I_1 \le \frac{Cn^3}{\tau^2} G(\chi, N, 1)(1-\chi)^{-1}.$$
(E.10)

On the other hand, Proposition 6.2 leads to that

$$Cov\left(\sum_{i,j,k,l}\frac{\tilde{A}_{i,k}\tilde{A}_{k,j}\tilde{A}_{i,l}\tilde{A}_{l,j}}{c_kc_l},\sum_{u,w,v,p}\frac{\tilde{A}_{u,v}\tilde{A}_{w,v}\tilde{A}_{u,p}\tilde{A}_{w,p}}{c_vc_p}\right)$$
$$\leq \frac{112(1-2\alpha')^{-1}\sum_{i,j,k,l}\sum_{u,w,v,p}\chi^{\frac{1}{2}\max(\Lambda(i_s,1\leq s\leq 8),\max_{1\leq s\leq 8}\iota(i_s))}(1-\chi)^{-1}}{\tau^4},$$

where $i_1 = \omega(\{i,k\}), i_2 = \omega(\{i,l\}), i_3 = \omega(\{j,k\}), i_4 = \omega(\{j,l\}), i_5 = \omega(\{u,v\}), i_6 = \omega(\{w,v\}), i_7 = \omega(\{u,p\}), i_8 = \omega(\{w,p\})$. For given i, j, k, l and r, we shall show that the total number of possible pairs of integers u, w, v, p such that $\max(\Lambda(i_s, 1 \le s \le 8), \max_{1\le s\le 8} \iota(i_s)) = r$ is at most $(8r+5)^2(12r+7)n$. Recall the definition of $\Lambda(c)$ and $\iota(\cdot)$. On one hand, $\Lambda(i_s, 1 \le s \le 2l) = \min_{1\le s\le l}(\min_{l+1\le u\le 2l} |i_s - i_u|)$ is the smallest distance between 2 indices, of which one is from $i_1, ..., i_l$ and the other is from $i_{l+1}, ..., i_{2l}$. On the other hand, $\iota(i_s) = \min_{1\le j\le 2l, j\ne s}(|i_s - i_j|)$ measures the smallest distance between index i_s and other indices. We now calculate the total number of possible pairs of integers u, w, v, p such that $\max(\Lambda(i_s, 1 \le s \le 8), \max_{1\le s\le 8} \iota(i_s)) = r$.

First note that at least one of i_5 , i_6 , i_7 , i_8 should belong to the interval $B_{i_1}(r) \cup B_{i_2}(r) \cup B_{i_3}(r) \cup B_{i_4}(r)$. Otherwise all distances between two indices, such that one indices is from i_1, i_2, i_3, i_4 and the other is from the group of i_5, i_6, i_7, i_8 are large than r, so $\Lambda(i_s, 1 \le s \le 8) > r$. Without loss of generality, consider the case that $i_5 = \omega(\{u, v\}) \in B_{i_1}(r) \cup B_{i_2}(r) \cup B_{i_3}(r) \cup B_{i_4}(r)$, then the possible numbers of i_5 is at most $4 \times (2r + 1) = 8r + 4$, which implies the possible numbers of different values of u, v are both at most 8r + 5. For each pair of (u, v), define $I = B_{i_1}(r) \cup B_{i_2}(r) \cup B_{i_3}(r) \cup B_{i_4}(r) \cup B_{r_5}(r)$. Then if (a) $i_6 \in I$, the total possible number of different values of w is no larger than one plus the possible number of p is n. So for (a) the total possible number of pairs (u, w, v, p) is $(8r + 5)^2(10r + 6)n$.

If (b) $i_6 \notin I$, then w has at most n possibilities. However at least one of $i_7 = \omega(\{u, p\})$ and $i_8 = \omega(\{w, p\})$ should fall into $B_{r_6}(r)$, otherwise $\iota(i_6) > r$. Since $i_7 = \omega(\{u, p\})$ and $i_8 = \omega(\{w, p\})$, the possible number of different values of p is bounded by one plus the minimal possible numbers of i_7 and i_8 Thus, p has at most 2r + 1 choices, and for case (b) the total possible number of pairs (u, w, v, p) is $(8r + 5)^2(2r + 1)n$. Combining (a), (b), we find that the total possible number of pairs (u, w, v, p) is $(8r + 5)^2(12r + 7)n$. The explanation of (b) is in the scanned figure. In conclusion, we have that

$$\frac{112(1-2\alpha')^{-1}\sum_{i,j,k,l}\sum_{u,w,v,p}\chi^{\frac{1}{2}\max(\Lambda(i_s,1\leq s\leq 8),\max_{1\leq s\leq 8}\iota(i_s))}(1-\chi)^{-1}}{\tau^4} \leq \frac{112(1-2\alpha')^{-1}n^4}{\tau^4}\sum_{r=0}^N(8r+5)^2(12r+7)n\chi^{r/2}(1-\chi)^{-1} \leq \frac{Cn^5}{\tau^4}G(\chi,N,3)(1-\chi)^{-1}.$$

Equation (E.10) with Markov's inequality show (E.8) (with s = 1). As a result, (i) follows from similar arguments to $I_v, 2 \le v \le 9$. For (ii), define

$$E_n = \cap_i \{ D_{ii} \in \bar{D}_{ii}(l(n), u(n)) \}$$
 (E.11)

with l(n) = 1 - b(n), u(n) = 1 + b(n), where we have chosen

$$b(n) = \frac{\log n}{16\sqrt{n}} G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}$$

We take \mathfrak{D} such that

$$\mathfrak{D} := \left| D_{ss}^{-1} (D_{ii} D_{jj})^{-1/2} - \bar{D}_{ss}^{-1} (\bar{D}_{ii} \bar{D}_{jj})^{-1/2} \right| \le \frac{1}{\tau^2 n^2} \left| \frac{\bar{D}_{ss} (\bar{D}_{ii} \bar{D}_{jj})^{1/2}}{D_{ss} (D_{ii} D_{jj})^{1/2}} - 1 \right|.$$

On the other hand, on set E such that for all $i, D_{ii} \in \overline{D}_{ii}(l(n), u(n))$,

$$\frac{\bar{D}_{ss}(\bar{D}_{ii}\bar{D}_{jj})^{1/2}}{D_{ss}(D_{ii}D_{jj})^{1/2}} - 1 \in ((1+b(n))^{-2} - 1, (1-b(n))^{-2} - 1).$$

Notice that $b(n) \to 0$. Let M' be the integer that b(n) < 0.5 if $n \ge M'$, then mean value theorem leads to

$$\mathfrak{D} \le \frac{16b(n)}{\tau^2 n^2}.$$

By the boundedness of a Bernoulli random variable, we have that when $n \ge M'$,

$$|LL - \tilde{L}\tilde{L}|_{i,j} \leq \sum_{s=1}^{n} |A_{i,s}A_{s,j}| |D_{ss}^{-1}(D_{ii}D_{jj})^{-1/2} - \bar{D}_{ss}^{-1}(\bar{D}_{ii}\bar{D}_{jj})^{-1/2}|$$
$$\leq \frac{16b(n)}{n^{2}\tau^{2}} \sum_{s=1}^{n} |A_{i,s}A_{s,j}| \leq \frac{16(n)}{n\tau^{2}}.$$

As a result, we have that when $n \geq M'$

$$\mathbb{P}\left(\|\tilde{L}\tilde{L} - LL\|_F \ge \frac{\log n}{\tau^2 n^{1/2}} G^{1/2}(\chi, N, 3)(1-\chi)^{-1/2}, E_n\right) = 0.$$

which further implies (E.5). It remains to show (E.6). The definition of set E_n in equation (E.11) yields that

$$\mathbb{P}(E_n^c) \le \sum_i \mathbb{P}\left(\frac{|D_{i,i} - \bar{D}_{i,i}|}{\sqrt{n}} \ge \frac{\bar{D}_{i,i}b_n}{\sqrt{n}}\right) \le \sum_i \mathbb{P}\left(\frac{|D_{i,i} - \bar{D}_{i,i}|}{\sqrt{n}} \ge \tau\sqrt{n}b_n\right). \quad (E.12)$$

Moreover it follows from the proof of Lemma D.1, that there exists a sufficiently small constant C_0 which is independent of *i* such that

$$\mathbb{E}\left(\exp\left(\left(\frac{C_0(1-\chi)}{\sqrt{n}}(D_{i,i}-\bar{D}_{i,i})\right)^2\right)\right) \le 2.$$

The above expression together with (E.12), Markov's inequality and the fact that $\tau_n^2 \log n > M$ for sufficiently large M implies (E.6). Thus (ii) follows, which together with (i) completes the proof.

F Proof of Results in Section 7

We need the following proposition to show Corollary 7.1 in the main article.

Proposition F.1. Define $W_k(B_j)$, $U_k(B_j)$ for $j \ge 0$ iteratively as follows:

$$W_{1}(B_{j}) = p_{1}^{B_{j}}(1-p_{1})^{1-B_{j}}, \quad U_{1}(B_{j}) = p_{0}^{B_{j}}(1-p_{0})^{1-B_{j}},$$

$$W_{k}(B_{j}) = p_{1}W_{k-1}(B_{j}) + (1-p_{1})U_{k-1}(B_{j}), \quad k \ge 2$$

$$U_{k}(B_{j}) = p_{0}W_{k-1}(B_{j}) + (1-p_{0})U_{k-1}(B_{j}), \quad k \ge 2.$$
(F.1)

Then the dependence between two ordered edge variables in $CG(V, \omega, p_0, p_1)$ could be represented by $U_k(B_j)$ and $W_k(B_j)$ as follows. For $j - 1 \ge k \ge 2$, we have

$$\mathbb{P}(B_j|B_{j-k}) = W_{k-1}(B_j)p_1^{B_{j-k}}p_0^{1-B_{j-k}} + U_{k-1}(B_j)(1-p_1)^{B_{j-k}}(1-p_0)^{1-B_{j-k}}, \quad (F.2)$$

where $CG(V, \omega, p_0, p_1)$ is defined in definition 7.1 in the main article.

Proof. We shall use mathematical induction to prove the proposition. For k = 2, we have that by Markov property of the sequence $\{B_j\}$ and $j \ge 2$,

$$\mathbb{P}(B_j|B_{j-2}) = \sum_{B_{j-1}=0}^{1} \mathbb{P}(B_j|B_{j-1})\mathbb{P}(B_{j-1}|B_{j-2})$$

= $p_1^{B_j}(1-p_1)^{1-B_j}p_1^{B_{j-2}}p_0^{1-B_{j-2}} + p_0^{B_j}(1-p_0)^{1-B_j}(1-p_1)^{B_{j-2}}(1-p_0)^{1-B_{j-2}}.$

Suppose that for $k = l, l \ge 2$, the equation (F.2) holds. Then for k = l + 1,

$$\mathbb{P}(B_j|B_{j-l-1}) = \sum_{B_{j-l}=0}^{1} \mathbb{P}(B_j|B_{j-l})\mathbb{P}(B_{j-l}|B_{j-l-1})$$

= $(W_{l-1}(B_j)p_1 + U_{l-1}(B_j)(1-p_1))p_1^{B_{j-l-1}}p_0^{1-B_{j-l-1}}$
+ $(W_{l-1}(B_j)p_0 + U_{l-1}(B_j)(1-p_0))(1-p_1)^{B_{j-l-1}}(1-p_0)^{1-B_{j-l-1}}$
= $W_l(B_j)p_1^{B_{j-l-1}}p_0^{1-B_{j-l-1}} + U_l(B_j)(1-p_1)^{B_{j-l-1}}(1-p_0)^{1-B_{j-l-1}},$

where the equality is due to (F.1). By mathematical induction the proposition follows. \Box

Proof of Corollary 7.1 in the main article

Note that (F.1) in Proposition F.1 implies that

$$\begin{pmatrix} W_k(B_j) \\ U_k(B_j) \end{pmatrix} = \begin{pmatrix} p_1 & 1-p_1 \\ p_0 & 1-p_0 \end{pmatrix} \begin{pmatrix} W_1(B_{j-1}) \\ U_1(B_{j-1}) \end{pmatrix},$$

which by further iteration results in

$$\begin{pmatrix} W_k(B_j) \\ U_k(B_j) \end{pmatrix} = \begin{pmatrix} p_1 & 1-p_1 \\ p_0 & 1-p_0 \end{pmatrix}^{k-1} \begin{pmatrix} W_1(B_j) \\ U_1(B_j) \end{pmatrix}.$$

Define the matrix $\mathbf{P} = \begin{pmatrix} p_1 & 1-p_1 \\ p_0 & 1-p_0 \end{pmatrix}$. Direct calculations show that the eigenvalues of \mathbf{P} are $\lambda_1 = 1$ and $\lambda_2 = p_1 - p_0$. Let \mathbb{I} be the 2 × 2 identity matrix. The by Cayley-Hamilton Theorem, we have the following formula Let notation \mathbf{P}^k represent the n_{th} power of matrix \mathbf{P} .

$$\mathbf{P}^{k} = \frac{\lambda_{1}^{k} - \lambda_{2}^{k}}{\lambda_{1} - \lambda_{2}} \mathbf{P} - \lambda_{1} \lambda_{2} \frac{\lambda_{1}^{k-1} - \lambda_{2}^{k-1}}{\lambda_{1} - \lambda_{2}} \mathbb{I},$$

which by simple calculations leads to

$$\mathbf{P}^{k} = \frac{1}{1 - p_{1} + p_{0}} \begin{pmatrix} p_{0} + (1 - p_{1})(p_{1} - p_{0})^{k} & (1 - p_{1})(1 - (p_{1} - p_{0})^{k}) \\ p_{0}(1 - (p_{1} - p_{0})^{k}) & (1 - p_{1}) + p_{0}(p_{1} - p_{0})^{k} \end{pmatrix}$$

Then the corollary is a consequence of the fact that $W_1(1) = p_1, U_1(1) = p_0, W_1(0) = 1 - p_1, U_1(0) = 1 - p_0$, Proposition F.1 and straightforward calculations.

The results show that the dependence between edge variables, or equivalently $\Delta_n(k)$ defined in (3.1) decays at the geometric rate $(p_1 - p_0)^k$. We also observe that by Proposition F.1, for all $k \geq 2$, the following bounds hold for the conditional probabilities that defined in Corollary 7.1:

$$\begin{aligned} \mathbb{P}_k(0|1) &\in [1 - \bar{p}_1, 1 - \bar{p}_0], \\ \mathbb{P}_k(1|0) &\in [\bar{p}_0, \bar{p}_1], \end{aligned} \qquad \qquad \mathbb{P}_k(0|0) &\in [1 - \bar{p}_1, 1 - \bar{p}_0], \\ \mathbb{P}_k(1|1) &\in [\bar{p}_0, \bar{p}_1], \end{aligned}$$

where $\bar{p}_1 = p_0 \vee p_1$ and $\bar{p}_0 = p_0 \wedge p_1$, where \vee represents 'max' and \wedge represents 'min'. We then introduce an auxiliary proposition about the rate of Poisson approximation to Binomial distribution, which has been used to compare MECLTG $CG(V, \omega_1, p_0, p_1)$ with Erdös-Rényi graph G(V, p) below Theorem 7.1 of the main article.

Proposition F.2. Suppose $k^2 = o(n)$, Let Z_n follow a Bernoulli $(n, \lambda_0/n)$ distribution with $\lambda_0 := \lambda_0(n)$ satisfying $\lambda_0 k = o(n)$. Let Y follow a Poisson (λ_0) distribution. Then it follows that

$$\mathbb{P}(Z_n = k) = \mathbb{P}(Y = k) \left(1 + O\left(\frac{k^2 + \lambda_0^2}{n}\right) \right).$$

Proof. The proof proceeds by direct calculation. Note that

$$\mathbb{P}(Z_n = k) = \binom{n}{k} \left(\frac{\lambda_0}{n}\right)^k \left(1 - \frac{\lambda_0}{n}\right)^{n-k}$$
$$= \frac{\lambda_0^k n!}{k! (n-k)!} (n-\lambda_0)^{-k} \exp(-\lambda_0) \left(1 + O\left(\frac{\lambda_0^2}{n}\right)\right)$$
$$= \frac{\lambda_0^k}{k!} \exp(-\lambda_0) \left(1 + O\left(\frac{\lambda_0^2}{n}\right)\right) U(\lambda_0, n, k),$$

where $U(\lambda_0, n, k) = \frac{\prod_{s=1}^k (n-s+1)}{(n-\lambda_0)^k}$. Direct calculations shows that

$$U(\lambda_0, n, k) \in [U_1(\lambda_0, n, k), U_2(\lambda_0, n, k)],$$

where

$$U_1(\lambda_0, n, k) = \left(1 + \frac{\lambda_0 - k + 1}{n - \lambda_0}\right)^k = 1 + O\left\{\frac{k(\lambda_0 - k)}{n}\right\},$$
$$U_2(\lambda_0, n, k) = \left(1 + \frac{\lambda_0}{n - \lambda_0}\right)^k = 1 + O\left(\frac{\lambda_0 k}{n}\right).$$

which completes the proof. The last step uses that $k^2/n \to 0$,

$$\left(1+\frac{k}{n}\right)^{k} = \exp\left\{k\ln\left(1+\frac{k}{n}\right)\right\} = \exp\left\{\frac{k^{2}}{n} + O\left(\frac{k^{3}}{n^{2}}\right)\right\} = 1 + O\left(\frac{k^{2}}{n}\right).$$

In fact, this proposition implies that the distribution of a Bernoulli random variable with a small probability of a successful trial is close to that of a Possion random variable, and quantifies the difference between the two distributions by their parameters. \Box

The following proposition is required for proving Theorem 7.1 in the main article.

Proposition F.3. Considering the MECLTG $CG(V, \omega_1, p_0 = \frac{\lambda_0}{n}, p_1 = 1 - \frac{\lambda_1}{n^c})$ define in Definition 7.1 of the main article with $\lambda_0 > 1$. Let n'' be an integer which is smaller than $\lfloor \epsilon n \rfloor$ for some $\epsilon > 0$, and $D_{n''} = \sum_{j=1}^{n-n''} I(A_{n'',n''+j} = 1)$. Then for any integer $0 \le k \le n - n'' - 2$, there exist strictly positive constants C_1, C_2 independent of k and n, such that

$$\mathbb{P}(D_{n''}=k) \ge C_1 n^{-c} \left(\frac{n'-k-1}{n'}\right) \exp\left(-C_2\left(\frac{k}{n^c}\right)\right).$$

Proof. Write n' = n - n''. To simplify the notation, we set $\binom{n}{k} = 0$ if k < 0 or k > n.

The k = 0 case is obvious. For $1 \le k \le n' - 2$ we have

$$\mathbb{P}(D_{n''}=k) = p \sum_{d=2}^{k} \binom{n'-k-1}{d-2} \binom{k-1}{d-1} p_{1}^{k-d} p_{0}^{d-1} (1-p_{1})^{d-1} (1-p_{0})^{n'-k-d+1} + (1-p) \sum_{d=1}^{k} \binom{n'-k-1}{d-1} \binom{k-1}{d-1} p_{1}^{k-d} p_{0}^{d} (1-p_{1})^{d-1} (1-p_{0})^{n'-k-d} + p \sum_{d=1}^{k} \binom{n'-k-1}{d-1} \binom{k-1}{d-1} p_{1}^{k-d} p_{0}^{d-1} (1-p_{1})^{d} (1-p_{0})^{n'-k-d} + (1-p) \sum_{d=1}^{k} \binom{n'-k-1}{d} \binom{k-1}{d-1} p_{1}^{k-d} p_{0}^{d} (1-p_{1})^{d} (1-p_{0})^{n'-k-d-1} := I + II + III + IV,$$
(F.3)

where $p = \frac{p_0}{1-p_1+p_0}$, as discussed in (7.2) of the main article. The latter equations define the fewer known I,II, III & IV. The first term is correspond to the situation that $(A_{n'',n''+1}, A_{n'',n}) = (1,1)$. The second term is correspond to the case of (0,1), The third is correspond to (1,0), and the last term is correspond to the case of (0,0). We only discuss the quantity I in (F.3) since II, III, IV follow mutatis mutandis. To this end, we need to calculate for each k,

$$\mathbb{P}(A_{n'',n''+1} = a_1, ..., A_{n'',n} = a_{n'})$$

where $\{a_i \in \{0,1\}, 1 \leq i \leq n': \sum_{i=1}^{n'} a_i = k, a_1 = 1, a_{n'} = 1\}$. If $a_1 = 1, a_{n'} = 1$, and in the series of $a_i, 1 \leq i \leq n'$, there are $d, 2 \leq d \leq n'$ pieces of consecutive 1's, then we have that, starting with 1, the series will have k - d of 1's following 1, d - 1 of 0's following 1, d - 1 of 1's following 0, and n' - (k - d + d - 1 + d - 1) - 1 = n' - k - d + 1 of 0's following 0, where the last term of -1 in the LHS of the above equation is due to $a_1 = 1$. Define $U_{d,k} = \{\{a_i, 1 \leq i \leq n'\} \in \{0, 1\}^{n'}: \sum_{i=1}^{n'} a_i = k$, there are d strings of 1's, $a_1 = 1, a_{n'} = 1\}$. Here "a strings of 1" means a sub-series only containing 1. Then we have that

$$\mathbb{P}(A_{n'',n''+1} = a_1, \dots, A_{n'',n} = a_{n'}, \{a_i\}_{i=1}^{n'} \in U_{d,k}) = pp_1^{k-d}(1-p_1)^{d-1}p_0^{d-1}(1-p_0)^{n'-k-d+1}.$$

Now we study $|U_{d,k}|$ as follows. Since there are n' - k of 0's, we first choose d - 2 out of n' - k - 1 positions to place strings of 1's, where d - 2 is due to $a_1 = 1$ and $a_{n'} = 1$. Then we assign the length of each string of 1. This leads to choose d - 1 out of k - 1. Thus we

obtain

$$\sum_{\{a_i,1\leq i\leq n'\}\in U_{d,k}} \mathbb{P}(A_{n'',n''+1} = a_1, \dots, A_{n'',n} = a_{n'})$$

= $\binom{n'-k-1}{d-2} \binom{k-1}{d-1} pp_1^{k-d} (1-p_1)^{d-1} p_0^{d-1} (1-p_0)^{n'-k-d+1},$

Also, we have $\mathbb{P}(D_{n''} = k, A_{n'',n''+1} = 1, A_{n'',n} = 1) = \sum_{d=2}^{k} \sum_{\{a_i,1 \le i \le n'\} \in U_{d,k}} \mathbb{P}(A_{n'',n''+1} = a_1, \dots, A_{n'',n} = a_{n'})$. Similar arguments apply to the $(A_{n'',n''+1}, A_{n'',n}) = (0, 1), (1, 0)$, and the (0, 0) cases. Notice that I, II, III and IV are all positive. In particular, IV is greater than its first term, which is

$$IV \ge (1-p)\binom{n'-k-1}{1}\binom{k-1}{0}p_1^{k-1}p_0(1-p_1)(1-p_0)^{n'-k-2}$$
$$= \frac{(1-p_1)^2}{1-p_1+p_0}p_1^{k-1}\binom{n'-k-1}{1}p_0(1-p_0)^{n'-k-2}.$$
(F.4)

Since $p_0 = \frac{\lambda_0}{n}$, it follows from straightforward calculations that for sufficiently large n' there exists a positive constant $\eta_0 < 1$ such that

$$\binom{n'-k-1}{1} p_0 (1-p_0)^{n'-k-2} \ge \eta_0 \frac{(n'-k-1)\lambda_0}{n'} \exp(-\lambda_0).$$
(F.5)

Since $p_1 = 1 - \frac{\lambda_1}{n^c}$, expression (F.5) yields that if n' is sufficiently large

$$(F.4) \geq \eta_1 \eta_0 \frac{n'-k-1}{n'} \lambda_0 \exp(-\lambda_0) \frac{\lambda_1}{n^c} (1-\frac{\lambda_1}{n^c})^{k-1}$$
$$\geq \eta_2 \eta_0 \frac{n'-k-1}{n'} \lambda_0 \exp(-\lambda_0) \frac{\lambda_1}{n^c} \exp(-\frac{(k-1)\lambda_1}{n^c}),$$

for positive constants $0 < \eta_2 < \eta_1 < 1$, which completes the proof.

Proof of Theorem 7.1 in the main article

Let n'' be an integer which is smaller than $\lfloor \epsilon n \rfloor$ for some fixed $\epsilon \in (0, 1)$. We consider node n''. Denote by $d_{n''}$ the degree of the node n''. Notice that for node n'', the edge variables are generated in the following order

$$A_{1,n''}, A_{2,n''}, \dots, A_{n''-2,n''}, A_{n''-1,n''}, A_{n'',n''+1}, \dots, A_{n'',n}.$$

As a result, the degree has the following decomposition

$$d_{n''} = D_{n''} + W_{n''}$$
, where $D_{n''} = \sum_{j=1}^{n-n''} I(A_{n'',n''+j} = 1)$, $W_{n''} = \sum_{j=1}^{n''-1} I(A_{j,n''} = 1)$ (F.6)

For $k \in \mathbb{N}$,

$$\mathbb{P}(d_{n''}=k) = \sum_{s=0}^{k} \mathbb{P}(D_{n''}=k-s|W_{n''}=s)\mathbb{P}(W_{n''}=s).$$
(F.7)

Recall the definition of ω_1 in Section 7.1.1 of the main article, i.e., $\omega_1(\{i, j\}) = \varpi(i \land j, i \lor j)$ for $i \neq j$, where $\varpi_1(i, j) = n(i-1) - i(i-1)/2 + j - i$ for $1 \leq i < j \leq n$. Direct calculations based on the form of ω_1 show that $\omega_1(\{n'' - k, n''\}) - \omega_1(\{n'' - k - 1, n''\}) = n - n'' + k$ for $1 \leq k \leq n'' - 2$. Also we have that $\omega_1(\{n'', n'' + 1\}) - \omega_1(\{n'' - 1, n''\}) = n - n'' + 1$. Thus it follows from from (F.6) that

$$\mathbb{P}(D_{n''} = k - s | W_{n''} = s) = \sum_{\substack{\sum_{l=1}^{n-n''} a_{n'',n''+l} = k-s, \\ \{a_{n'',n''+l}\}_{l=1}^{n-n''} \in \{0,1\}^{n-n''}}} \Pi_{l=2}^{n-n''} \mathbb{P}(A_{n'',n''+l} = a_{n'',n''+l} | A_{n'',n''+l-1} = a_{n'',n''+l-1}) \times \mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1} | W_{n''} = s).$$
(F.8)

On the other hand, by Markov property and theory of total probability, we obtain that

$$\mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1}, W_{n''} = s)
= \sum_{\substack{\sum_{l=1}^{n''-1} a_{l,n''} = s, \\ \{a_{l,n''}\}_{l=1}^{n''-1} \in \{0,1\}^{n''-1}}} \mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1} | A_{n''-1,n''} = a_{n''-1,n''})
\times \prod_{l=1}^{n''-2} \mathbb{P}(A_{n''-l,n''} = a_{n''-l,n''} | A_{n''-l-1,n''} = a_{n''-l-1,n''}) \mathbb{P}(A_{1,n''} = a_{1,n''}). \quad (F.9)$$

By using Corollary 7.1 in the main article, we get

$$\mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1} | A_{n''-1,n''} = a_{n''-1,n''})$$

= $\mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1})(1 + O(n^{1-c}(p_1 - p_0)^{n-n''+1}))$
= $\mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1})(1 + O(n^{1-c}\exp(-n^{\lambda_1(1-c)})))$ (F.10)

if n is sufficiently large. Then by (F.10), (F.9) can be simplified to equal to

$$(F.9) = \mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1})\mathbb{P}(W_{n''} = s)\{1 + O(n^{1-c}\exp(-n^{\lambda_1(1-c)}))\},\$$

which further implies that

$$\mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1} | W_{n''} = s) = \mathbb{P}(A_{n'',n''+1} = a_{n'',n''+1})\{1 + O(n^{1-c}\exp(-n^{\lambda_1(1-c)}))\}$$

Therefore (F.7), (F.8) and (F.11) result in

$$\mathbb{P}(d_{n''}=k) = \sum_{s=0}^{k} \mathbb{P}(D_{n''}=k-s) \mathbb{P}(W_{n''}=s) \{1 + O(n^{1-c} \exp(-n^{\lambda_1(1-c)}))\}.$$

Similar arguments using the fact $\omega_1(\{n''-k,n''\}) - \omega_1(\{n''-k-1,n''\}) = n - n'' + k$ yield that

$$\mathbb{P}(W_{n''} = s) = \left(\sum_{\substack{\sum_{l=1}^{n''-1} a_{l,n''} = s, \\ \{a_{l,n''}\}_{l=1}^{n''-1} \in \{0,1\}^{n''-1} \\ \in \mathbb{P}(Y_{n''} = s)(1 + O(n^{2-c}\exp(-n^{\lambda_1(1-c)}))), \right) \{1 + O(n^{2-c}\exp(-n^{\lambda_1(1-c)}))\}$$

$$= \mathbb{P}(Y_{n''} = s)(1 + O(n^{2-c}\exp(-n^{\lambda_1(1-c)}))), \quad (F.12)$$

where $Y_{n''} \sim Binomial(n''-1,p)$, for $p = \frac{p_0}{1-p_1+p_0}$ as discussed in equation (7.2) in the main article. By Proposition F.3, there exist constants positive C_1 , C_2 such that for $0 \le k \le n - n'' - 2$

$$\mathbb{P}(D_{n''}=k) \ge C_1 n^{-c} \left(\frac{n'-k-1}{n'}\right) \exp\left(-C_2\left(\frac{k}{n^c}\right)\right).$$

Let $k^* = \lfloor k/p \rfloor$. Then we have for $0 \le k \le (n - n'' - 2)$,

$$\begin{split} & \mathbb{E}\left(\frac{\sum_{n''=1}^{n}I(d_{n''}=k)}{n}\right) \geq \frac{\sum_{n''=1}^{k^* \wedge n} \mathbb{P}(d_{n''}=k)}{n} \\ &= \frac{1}{n}\sum_{n''=1}^{k^* \wedge n}\sum_{s=0}^{k} \mathbb{P}(W_{n''}=k-s)\mathbb{P}(D_{n''}=s)\{1+O(n^{1-c}\exp(-n^{\lambda(1-c)}))\} \\ &\geq \frac{1}{n}\sum_{n''=1}^{k^* \wedge n}\sum_{s=0}^{k} \mathbb{P}(W_{n''}=k-s)C_1n^{-c}\exp\left(-C_2\left(\frac{k}{n^c}\right)\right)\{1+O(n^{1-c}\exp(-n^{\lambda(1-c)}))\} \\ &= \frac{1}{n}\sum_{n''=1}^{k^* \wedge n}\sum_{s=0}^{k} \mathbb{P}(Y_{n''}=k-s)C_1n^{-c}\exp\left(-C_2\left(\frac{k}{n^c}\right)\right) \\ &\times\{1+O(n^{1-c}\exp(-n^{\lambda(1-c)}))\}\{1+O(n^{2-c}\exp(-n^{\lambda(1-c)}))\} \\ &\geq \frac{1}{n}\sum_{n''=1}^{k^* \wedge n}\sum_{s=0}^{k} \mathbb{P}(Y_{n''}=k-s)C_1n^{-c}\exp\left(-C_2\left(\frac{k}{n^c}\right)\right) \end{split}$$

for some positive constant \tilde{C}_1 , where the equality is due to equation (F.12). Furthermore, observe that by the basic property of a Binomial random variable

$$\sum_{n''=1}^{\lfloor k/p \rfloor \wedge n} \mathbb{P}(Y_{n''} \le k) \ge \zeta((k/p) \wedge n)$$

for some constant ζ , which leads to

$$\frac{1}{n} \sum_{n''=1}^{k^* \wedge n} \sum_{s=0}^k \mathbb{P}(Y_{n''} = k - s) C_1 n^{-c} \exp\left(-C_2\left(\frac{k}{n^c}\right)\right)$$
$$= \frac{1}{n} \sum_{n''=1}^{\lfloor k/p \rfloor} \mathbb{P}(Y_{n''} \le k) C_1 n^{-c} \exp\left(-C_2\left(\frac{k}{n^c}\right)\right)$$
$$\ge \check{C}_1 \frac{1}{n^c} \left(\frac{k}{n^c} \wedge 1\right) \exp\left(-C_2\left(\frac{k}{n^c}\right)\right)$$

for some sufficiently small positive constant \check{C}_1 . Define $M_{\gamma} = (\sum_{k=1}^n \frac{1}{k^{\gamma}})^{-1}$ for $\gamma > 1$, $M_{\gamma,\mu} = (\sum_{k=1}^n \frac{1}{k^{\gamma}} \exp(-\mu k))^{-1}$ for $\gamma > 1$, $\mu > 0$. Denote $J_{n''} = [1, n - n'' - 2] \cap \mathbb{Z}$, and

$$\tilde{A}_{n,\gamma} := \left\{ k \in J_{n''} : \check{C}_1 \frac{1}{n^c} \left(\frac{k}{n^c} \wedge 1 \right) \exp\left(-C_2 \left(\frac{k}{n^c} \right) \right) \ge M_\gamma \frac{1}{k^\gamma} \right\},\$$
$$\tilde{B}_{n,\gamma,\mu} = \left\{ k \in J_{n''} : \check{C}_1 \frac{1}{n^c} \left(\frac{k}{n^c} \wedge 1 \right) \exp\left(-C_2 \left(\frac{k}{n^c} \right) \right) \ge M_{\gamma,\mu} k^{-\gamma} \exp(-\mu k) \right\}.$$

Direct calculations show that there exist a_0, b_0, c_0, d_0 such that for sufficiently large n,

$$\left\{ k : \lfloor a_0 n^{\frac{2c}{1+\gamma}} \rfloor \le k \le \lfloor b_0 n^c \log n \rfloor \right\} \subset \tilde{A}_{n,\gamma} \subset A_{n,\gamma}, \\ \left\{ k : \lfloor c_0 \log n \rfloor \le k \le \lfloor d_0 n \rfloor \right\} \subset \tilde{B}_{n,\gamma,\mu} \subset B_{n,\gamma,\mu},$$

which completes the proof.

Proof of Lemma 7.1 in the main article

Proof. Without loss of generality, we assume that n is even. The case of n odd can be shown *mutatis mutandis.* Consider node k such that $2 \le k \le n/2$. Note that for node k, the order of the corresponding edge variables are generated in the following order:

$$\mathcal{A}_k := (A_{k-1,k}, A_{k,k+1}, A_{k-2,k}, A_{k,k+2}, \dots, A_{k-a,k}, A_{k,k+a}, \dots, A_{1,k}, A_{k,2k-1}, A_{k,2k}, A_{k,2k+1}, \dots, A_{k,n}),$$

where \mathcal{A}_k is a (n-1)-vector, with $\mathcal{A}_k^{(l)}$ its l_{th} element. Recall that $\omega_2(\{i, j\}) = i + \frac{(2n-(j-i))(j-i-1)}{2}$. Straightforward calculations show that i): if $|a-b| < |c-d|, \omega_2(\{a, b\}) < \omega_2(\{c, d\})$ and ii): if |a-b| = |c-d|, then $\omega_2(\{a, b\}) < \omega_2(\{c, d\})$ when $\min(a, b) < \min(c, d)$. Let $\mathcal{B}_i \in \mathbb{R}^{1 \times (n-1)}$ with $\mathcal{B}_i^{(l)}$ as its l_{th} element. Define for each finite $k, 1 \le k \le n/2$, the real series $\{a_i(k)\}_{i=1}^{n-2}$ as

$$\begin{cases} a_{2s+1}(k) = s+1, & 0 \le s \le k-2\\ a_{2s}(k) = n-2s-1, & 1 \le s \le k-2\\ a_l(k) = n+k-1-l, & 2k-2 \le l \le n-2 \end{cases}$$

Let $\mathcal{B}_k^{(1)} = B_1$, and $\mathcal{B}_k^{(s)} = B_{1+\sum_{u=1}^{s-1} a_u(k)}$ for $s \ge 2$, where $\{B_k\}_{1\le k\le N}$ is the ordered edge variables in (7.1) of the main article with $\mathbb{P}(B_i = 1) = \frac{p_0}{1-p_1+p_0}$. Then by the stationary assumption and direct calculation, we can show that

$$\mathcal{A}_k \stackrel{d}{=} \mathcal{B}_k$$

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where $\stackrel{d}{=}$ means the equivalence in distribution. For node k with $\frac{n}{2} + 1 \le k \le n$, define similarly

$$\begin{cases} a_{2s+1}(k) = s+1, & 0 \le s \le n-k-1 \\ a_{2s}(k) = n-2s-1, & 1 \le s \le n-k \\ a_l(k) = 2n-k-1-l, & 2n-2k+1 \le l \le n-2. \end{cases}$$

Let $\mathcal{B}_k^{(s)} = B_{1+\sum_{u=1}^{s-1} a_u(k)}$. By our construction, we have

$$\mathcal{A}_k \stackrel{a}{=} \mathcal{B}_k.$$

Recall c in the definition of $p_1 = 1 - \lambda_1 n^{-c}$. Let ι be a constant such that $c - 1 + \iota < 0$. For the purpose of evaluating $\mathbb{P}(d_i = k)$ we define

$$d_{i,1} = \sum_{j=1}^{\lfloor n^{\iota} \rfloor} \mathcal{B}_i^{(j)}, d_{i,2} = \sum_{\lfloor n^{\iota} \rfloor + 1}^{n - \lfloor n^{\iota} \rfloor - 1} \mathcal{B}_i^{(j)}, d_{i,3} = \sum_{n - \lfloor n^{\iota} \rfloor}^n \mathcal{B}_i^{(j)}.$$

This therefore splits up $\mathbb{P}(d_i = k)$ into three parts. Then by similar arguments to the proof of (F.8) of Theorem 7.1, we have that

$$\mathbb{P}(d_i = k) = \sum_{s,u} \mathbb{P}(d_{i,3} = u | d_{i,2} = k - s - u) \mathbb{P}(d_{i,2} = k - s - u | d_{i,1} = s) \mathbb{P}(d_{i,1} = s)$$
$$= \sum_{s,u} \mathbb{P}(d_{i,3} = u) \mathbb{P}(\bar{d}_{i,2} = k - s - u) \mathbb{P}(d_{i,1} = s) (1 + O(n^{2-c} \exp(-n^{\lambda_1(\iota - c)}))) \mathbb{P}(1)$$

where $\bar{d}_{i,2}$ follows a $Binomial(n - 2\lfloor n^t \rfloor - 1, p)$ distribution with $p = \frac{p_0}{1-p_1+p_0}$. Without loss of generality, we consider node 1. For node $i, 2 \le i \le n$, the equations follow *mutatis mutandis*. By similar arguments to the discussion of (F.13), we then have

$$\mathbb{P}(d_1 = k) = \sum_u \mathbb{P}(d_{1,3} = u | d_{1,2} + d_{1,1} = k - u) \mathbb{P}(d_{1,2} + d_{1,1} = k - u)$$
$$= \sum_u \mathbb{P}(d_{1,3} = u) \mathbb{P}(\tilde{D}_{1,1} = k - u) (1 + O(n^{2-c} \exp(-n^{\iota - c}))), \quad (F.14)$$

where $\tilde{D}_{1,1}$ follows a $Binomial(n - \lfloor n^{\iota} \rfloor, p)$ distribution. Let V_1 follow a $Poisson(n^{\iota}p)$ distribution. Let M be a sufficiently large constant. By Theorem 4.1 of [20] or Theorem 1 of [5], we have that for a sufficiently large positive constant M,

$$\frac{1}{2}\sum_{i=0}^{\infty} |\mathbb{P}(V_1 = i) - \mathbb{P}(d_{1,3} = i)| = M(b_1 + b_2 + b_3),$$

where we choose

$$b_{1} = \sum_{i=1}^{\lfloor n^{\iota} \rfloor} p^{2} \leq M n^{\iota + 2c - 2}, b_{2} = 0,$$

$$b_{3} = \sum_{i=1}^{\lfloor n^{\iota} \rfloor} \mathbb{E}|\mathbb{E}(B_{i}|B_{j}, j \neq i) - p(B_{i} = 1)| \leq 2p\lfloor n^{\iota} \rfloor \leq M n^{\iota + c - 1}.$$
(F.15)

where for the bound of b_3 , we have used the property of conditional expectation and the non-negativity of the Binomial random variable. On the other hand, let W_i follow a $Poisson((n - \lfloor n^{\iota} \rfloor)p)$ distribution. By [19], we have

$$\sum_{k=0}^{\infty} |\mathbb{P}(W_i = k) - \mathbb{P}(\tilde{D}_{1,1} = k)| \le M n^{c-1}.$$
 (F.16)

By the property of sum of independent Poisson random variables, (F.15) and (F.16), (i) follows. It remains to show (ii). As per usual, define the entropy function

$$H(a, p) = a \log(\frac{a}{p}) + (1 - a) \log(\frac{1 - a}{1 - p}).$$

For any Binomial(n, p) random variable S_n , we estimate its tail behavior as follows. For $0 \le p < a \le 1$, we have that for $\beta > 0$,

$$\mathbb{P}(S_n \ge an) \le \exp(-\beta an) \mathbb{E}(\exp(\beta S_n)) = (\exp(-\beta an) (1 - p + p \exp(\beta)))^n .$$
(F.17)

In equation (F.17) set $\beta = \log\left(\frac{a(1-p)}{p(1-a)}\right) > 0$, then we have that

$$\mathbb{P}(S_n \ge an) \le \exp(-nH(a,p)). \tag{F.18}$$

In the remainder of the proof, we consider node 1. The other nodes follow *mutatis mutandis*. Let V follow $Binomial(n - \lfloor n^{\iota} \rfloor, p)$. By (F.14) and the property of Binomial distribution (F.18), we have that for $k \geq \lfloor n^{\iota}g(n) \rfloor$, where g(n) is a series of real numbers which go to infinity arbitrarily slowly,

$$\mathbb{P}(d_1 = k) \leq \mathbb{P}(V \geq k - \lfloor n^{\iota} \rfloor)(1 + O(n^{2-c} \exp(-n^{\lambda_1(\iota-c)})))$$

$$\leq \exp\left(-(n - \lfloor n^{\iota} \rfloor)H(a, p)\right)(1 + O(n^{2-c} \exp(-n^{\lambda_1(\iota-c)}))),$$

where $a = \frac{k - \lfloor n^{\iota} \rfloor}{n - \lfloor n^{\iota} \rfloor}$. Since $k \ge \lfloor n^{\iota}g(n) \rfloor$ and $p \asymp n^{c-1}$, when n large enough, $\log(a/p) \ge (\iota - c) \log n$. Therefore it follows from Taylor expansion that

$$H(a,p) \ge (\iota - c)a\log n + (1-a)\log(1-a) + O(n^{c-1}).$$
(F.19)

Notice that

$$\lim_{a \to 0} \frac{(1-a)\log(1-a)}{a} = -1.$$
 (F.20)

Expression (F.19), (F.20) and the definition of a imply that

$$\{n - \lfloor n^{\iota} \rfloor\} H(a, p) \ge \delta(\iota - c)k \log n \text{ for some } \delta \in (0.5, 1)$$

when n is sufficiently large, which completes the proof.

References

- Airoldi, E. M., Costa, T. B., and Chan, S. H. (2013). Stochastic blockmodel approximation of a graphon: Theory and consistent estimation. In Advances in Neural Information Processing Systems, pages 692–700.
- [2] Amini, A. A., Chen, A., Bickel, P. J., and Levina, E. (2013). Pseudo-likelihood methods for community detection in large sparse networks. *The Annals of Statistics*, 41(4):2097–2122.
- [3] Amini, A. A. and Levina, E. (2018). On semidefinite relaxations for the block model. The Annals of Statistics, 46(1):149–179.
- [4] Anandkumar, A., Ge, R., Hsu, D., and Kakade, S. M. (2014). A tensor approach to learning mixed membership community models. *The Journal of Machine Learning Research*, 15(1):2239–2312.
- [5] Arratia, R., Goldstein, L., Gordon, L., et al. (1989). Two moments suffice for poisson approximations: the chen-stein method. *The Annals of Probability*, 17(1):9–25.
- [6] Barabási, A.-L. and Albert, R. (1999). Emergence of scaling in random networks. *science*, 286(5439):509-512.
- [7] Berkhin, P. (2006). A survey of clustering data mining techniques. In *Grouping multidimensional data*, pages 25–71. Springer.

- [8] Bhattacharyya, S. and Bickel, P. J. (2015). Subsampling bootstrap of count features of networks. *The Annals of Statistics*, 43(6):2384–2411.
- [9] Bickel, P., Choi, D., Chang, X., and Zhang, H. (2013). Asymptotic normality of maximum likelihood and its variational approximation for stochastic blockmodels. *The Annals of Statistics*, 41(4):1922–1943.
- [10] Bickel, P. J. and Chen, A. (2009). A nonparametric view of network models and newman-girvan and other modularities. *Proceedings of the National Academy of Sciences*, 106(50):21068–21073.
- [11] Bickel, P. J., Chen, A., and Levina, E. (2011). The method of moments and degree distributions for network models. *The Annals of Statistics*, 39(5):2280–2301.
- [12] Bollobs, B. and Riordan, O. (2009). Metrics for sparse graphs, page 211288. London Mathematical Society Lecture Note Series. Cambridge University Press.
- [13] Borgs, C., Chayes, J. T., Cohn, H., and Ganguly, S. (2020+). Consistent nonparametric estimation for heavy-tailed sparse graphs. *The Annals of Statistics*.
- [14] Borgs, C., Chayes, J. T., Cohn, H., and Holden, N. (2017). Sparse exchangeable graphs and their limits via graphon processes. *The Journal of Machine Learning Research*, 18(1):7740– 7810.
- [15] Borgs, C., Chayes, J. T., Cohn, H., Veitch, V., et al. (2019). Sampling perspectives on sparse exchangeable graphs. *The Annals of Probability*, 47(5):2754–2800.
- [Borgs et al.] Borgs, C., Chayes, J. T., Cohn, H., and Zhao, Y. An l^p theory of sparse graph convergence i: limits, sparse random graph models, and power law distributions. *Trans. Amer. Math. Soc.*, (372):3019–3062.
- [17] Broido, A. D. and Clauset, A. (2019). Scale-free networks are rare. Nature communications, 10(1):1017.
- [18] Cai, T. T. and Li, X. (2015). Robust and computationally feasible community detection in the presence of arbitrary outlier nodes. *The Annals of Statistics*, 43(3):1027–1059.
- [19] Chen, L. H. (1974). On the convergence of poisson binomial to poisson distributions. The Annals of Probability, pages 178–180.
- [20] Chen, L. H., Röllin, A., et al. (2013). Approximating dependent rare events. Bernoulli, 19(4):1243–1267.
- [21] Choi, D. S., Wolfe, P. J., and Airoldi, E. M. (2012). Stochastic blockmodels with a growing number of classes. *Biometrika*, 99(2):273–284.

- [22] Crane, H. and Dempsey, W. (2018). Edge exchangeable models for interaction networks. Journal of the American Statistical Association, 113(523):1311–1326.
- [23] Davis, C. and Kahan, W. M. (1970). The rotation of eigenvectors by a perturbation. iii. SIAM Journal on Numerical Analysis, 7(1):1–46.
- [24] Durrett, R. (2007). *Random graph dynamics*, volume 200. Cambridge university press Cambridge.
- [25] Fan, J. and Yao, Q. (2008). Nonlinear time series: nonparametric and parametric methods. Springer Science & Business Media.
- [26] Gao, C., Lu, Y., Zhou, H. H., et al. (2015). Rate-optimal graphon estimation. The Annals of Statistics, 43(6):2624–2652.
- [27] Gao, F. and van der Vaart, A. (2017). On the asymptotic normality of estimating the affine preferential attachment network models with random initial degrees. *Stochastic Processes and their Applications*, 127(11):3754–3775.
- [28] Gong, Y. (1995). Speech recognition in noisy environments: A survey. Speech communication, 16(3):261–291.
- [29] Ji, P., Jin, J., et al. (2016). Coauthorship and citation networks for statisticians. The Annals of Applied Statistics, 10(4):1779–1812.
- [30] Kallenberg, O. (2006). Probabilistic symmetries and invariance principles. Springer Science & Business Media.
- [31] Karrer, B. and Newman, M. E. (2011). Stochastic blockmodels and community structure in networks. *Physical review E*, 83(1):016107.
- [32] Klopp, O., Tsybakov, A. B., and Verzelen, N. (2017). Oracle inequalities for network models and sparse graphon estimation. *The Annals of Statistics*, 45(1):316–354.
- [33] Krivelevich, M. and Sudakov, B. (2013). The phase transition in random graphs: A simple proof. Random Structures & Algorithms, 43(2):131–138.
- [34] Li, T., Lei, L., Bhattacharyya, S., Sarkar, P., Bickel, P. J., and Levina, E. (2018). Hierarchical community detection by recursive partitioning. arXiv preprint arXiv:1810.01509.
- [35] Lovász, L. (2012). Large networks and graph limits, volume 60. American Mathematical Soc.
- [36] Newman, M. E. (2005). Power laws, pareto distributions and zipf's law. Contemporary physics, 46(5):323–351.
- [37] Olhede, S. C. and Wolfe, P. J. (2013). Degree-based network models. arXiv preprint

arXiv:1211.6537.

- [38] Olhede, S. C. and Wolfe, P. J. (2014). Network histograms and universality of blockmodel approximation. *Proceedings of the National Academy of Sciences*, 111(41):14722–14727.
- [39] Ouadah, S., Robin, S., and Latouche, P. (2015). Degree-based goodness-of-fit tests for heterogeneous random graph models: independent and exchangeable cases. arXiv preprint arXiv:1507.08140.
- [40] Pagani, G. A. and Aiello, M. (2013). The power grid as a complex network: a survey. *Physica A: Statistical Mechanics and its Applications*, 392(11):2688–2700.
- [41] Papadopoulos, F., Kitsak, M., Serrano, M. A., Boguná, M., and Krioukov, D. (2012). Popularity versus similarity in growing networks. *Nature*, 489(7417):537.
- [42] Qin, T. and Rohe, K. (2013). Regularized spectral clustering under the degree-corrected stochastic blockmodel. In Advances in Neural Information Processing Systems, pages 3120– 3128.
- [43] Rohe, K., Chatterjee, S., Yu, B., et al. (2011). Spectral clustering and the high-dimensional stochastic blockmodel. *The Annals of Statistics*, 39(4):1878–1915.
- [44] Saldana, D. F., Yu, Y., and Feng, Y. (2017). How many communities are there? Journal of Computational and Graphical Statistics, 26(1):171–181.
- [45] Sarkar, P., Chakrabarti, D., Jordan, M., et al. (2014). Nonparametric link prediction in large scale dynamic networks. *Electronic Journal of Statistics*, 8(2):2022–2065.
- [46] Sole, R. V. and Montoya, J. M. (2001). Complexity and fragility in ecological networks. Proceedings of the Royal Society of London. Series B: Biological Sciences, 268(1480):2039– 2045.
- [47] Truquet, L. (2019). Local stationarity and time-inhomogeneous markov chains. The Annals of Statistics, 47(4):2023–2050.
- [48] Tsybakov, A. B. (2008). Introduction to Nonparametric Estimation. Springer Publishing Company, Incorporated, 1st edition.
- [49] Varin, C., Reid, N., and Firth, D. (2011). An overview of composite likelihood methods. Statistica Sinica, pages 5–42.
- [50] Veitch, V. and Roy, D. M. (2015). The class of random graphs arising from exchangeable random measures. *arXiv preprint arXiv:1512.03099*.
- [51] Veitch, V., Roy, D. M., et al. (2019). Sampling and estimation for (sparse) exchangeable graphs. *The Annals of Statistics*, 47(6):3274–3299.

- [52] Von Luxburg, U. (2007). A tutorial on spectral clustering. *Statistics and computing*, 17(4):395–416.
- [53] Wolfe, P. J. and Olhede, S. C. (2013). Nonparametric graphon estimation. arXiv preprint arXiv:1309.5936.
- [54] Wu, W. and Zhou, Z. (2017). Nonparametric inference for time-varying coefficient quantile regression. Journal of Business & Economic Statistics, 35(1):98–109.
- [55] Wu, W. B. (2005). Nonlinear system theory: Another look at dependence. Proceedings of the National Academy of Sciences, 102(40):14150–14154.
- [56] Zhang, Y., Levina, E., and Zhu, J. (2014). Detecting overlapping communities in networks using spectral methods. *arXiv preprint arXiv:1412.3432*.
- [57] Zhao, Y., Levina, E., and Zhu, J. (2012). Consistency of community detection in networks under degree-corrected stochastic block models. *The Annals of Statistics*, 40(4):2266–2292.
- [58] Zhou, Z. (2013). Heteroscedasticity and autocorrelation robust structural change detection. Journal of the American Statistical Association, 108(502):726–740.
- [59] Zhou, Z. (2014). Inference of weighted v-statistics for nonstationary time series and its applications. The Annals of Statistics, 42(1):87–114.
- [60] Zhou, Z. and Wu, W. B. (2010). Simultaneous inference of linear models with time varying coefficients. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 72(4):513-531.