On the Mechanics of Electrical Cables in Subsea Control Umbilicals

By

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Abstract

This thesis is concerned with the mechanics of electrical cables in subsea umbilicals. Subsea umbilicals are used in the offshore hydrocarbon industry to control and operate remote seabed equipment. They are characterised by an armoured and sheathed bundle of hydraulic hoses and electrical cables. Experience to date has shown that the electrical cables in subsea umbilicals are prone to mechanical failure. One of their most common modes of failure is the formation of kinks in the insulated copper conductors leading to rapid fatigue damage. The key physical phenomenon involved in this buckling type of failure is the non-linear material properties of the copper conductors which results in the build up of compressive forces under strain controlled cyclic loading.

Two main analytical models are developed in this thesis to investigate the mechanical loads in the electrical cables in subsea umbilicals. One model is concerned with the structural response of the umbilical for axi-symmetric loads and the other is concerned with the structural response for flexural loads. These models use an energy formulation and are based on the differential geometry of deformed helices. The models incorporate an original sub-structuring approach to take into account the compliance of the core. The models also include new features to take into account the non-linear material properties of the conductors, finite friction coefficients and a methodology to predict the equivalent material properties of the composite core.

The above analytical models are verified with full-scale tests and with other experimental data published in the public domain. A good level of agreement is achieved giving confidence in the application of this work to the kinking and fatigue analysis of the electrical cables in subsea umbilicals. The kinking analysis uses the analogy of the insulated copper conductors to beams on elastic
foundations to predict the critical buckling stresses. The critical stresses calculated from this model are expressed as a function of the global load limits. The fatigue analysis is carried out using the as-built material properties of the conductors which are determined experimentally. The thesis concludes by proposing a design methodology that can be used to optimise the mechanical performance of electrical cables in subsea umbilicals.
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Nomenclature

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* The general convention defines the lay angle as the angle between the longitudinal axis and the helix. The helix angle according to this convention defines the angle between the axis of the cross section and the helix. This convention has not been used in this thesis and all references to the helix angle define the lay angle.

\[ \alpha \] Helix angle or lay angle
\[ A \] Cross-sectional area
\[ \beta \] Ratio of helix radius to bending radius or foundation stiffness
\[ \mathbf{b} \] Unit binormal vector
\[ b \] Width or fatigue strength exponent
\[ c \] Hardening exponent or fatigue ductility exponent
\[ \Delta \] Slip distance
\[ d \] Outer diameter of hose
\[ D_u \] Ductility coefficient
\[ E \] Elastic modulus
\[ \phi \] Polar cylindrical co-ordinate
\[ F_z \] Umbilical global axial force
\[ F_\phi \] Umbilical global torsional moment
\[ F_r \] Umbilical global external pressure
\[ H \] Torsional moment
\[ I_n \] Second moment of area in the normal direction
\[ I_b \] Second moment of area in the binormal direction
\[ \gamma, \tau \] Twist
\[ G \] Shear modulus
\[ G_n \] Bending moment in the normal direction
\[ G_b \] Bending moment in the binormal direction
\[ J \] Polar moment of inertia
\[ \kappa \] Curvature
\( \kappa_n \): Curvature in the normal direction
\( \kappa_b \): Curvature in the binormal direction
\( \kappa_u \): Umbilical curvature
\( k \): Spring stiffness
\( K \): Strength coefficient
\( K_n \): External moment in the normal direction
\( K_b \): External moment in the binormal direction
\( \lambda \): Equivalent radial modulus of core or non-dimensional axial load
\( \ell \): Buckle half wavelength
\( L \): Umbilical length
\( \mu \): Friction coefficient or perturbation parameter
\( m \): Number of wires in armour layer
\( \nu \): Poisson’s ratio
\( \nu_c \): Dilatation coefficient of core
\( n \): Unit normal vector
\( n \): Number of wires in a conductor
\( N_n \): Shear force in the normal direction
\( N_b \): Shear force in the binormal direction
\( N_f \): Number of cycles to failure
\( p \): Internal pressure in a hose
\( \rho \): Bending radius and radius of toroid
\( \Theta \): External torsional moment
\( r \): Position vector
\( RA \): Reduction in area at necking
\( R \): Radial cylindrical co-ordinate
\( r \): Radial cylindrical co-ordinate of local reference cylinder
\( R_s \): Mean radius of sheath layer
\( R_c \): Outer radius of umbilical core
\(\sigma\) Stress
\(t\) Unit tangent vector
\(t\) Thickness of insulation and filler layers
\(t_s\) Thickness of sheath layer
\(T\) Axial force
\(s\) Arc length
\(u_z\) Longitudinal displacement
\(u_\phi\) Rotational displacement
\(u_R\) Radial displacement
\(u_s\) Displacement along the arc length
\(U\) Internal strain energy
\(\omega\) Toroid parametric angle
\(w\) Lateral displacement
\(W\) Work of external forces
\(\varepsilon\) Strain
\(X\) External force in the normal direction
\(y\) Non-dimensional lateral displacement
\(Y\) External force in the binormal direction
\(z\) Longitudinal cylindrical co-ordinate
\(Z\) External force in the tangent direction
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1 Introduction

Subsea systems are at the heart of new innovative technology that allows marginal offshore hydrocarbon reservoirs in deep water and hostile environments to be exploited economically. These systems consist primarily of the seabed equipment needed for drilling and production and the different modules required for their control. A generic layout of a subsea system is shown in Figure 1.1. The different seabed equipment includes items like blow out preventors for drilling, production trees and pumps. The different modules required to control this remote and often inaccessible equipment include surface control stations, subsea control pods and subsea umbilicals. The umbilicals span the water column between the surface and subsea control modules. These umbilicals also provide the chemical injection functions. A typical cross section of an umbilical is shown Figure 1.2. These umbilicals are mainly an armoured bundle of hydraulic hoses and electrical cables that are used to transmit chemical fluids, power and communication signals required for the safe control and operation of the remote seabed equipment.

The aim of this thesis is to investigate the mechanics of electrical cables in subsea umbilicals. The electrical cables were traditionally used to monitor the temperature, pressure and other physical properties of the produced hydrocarbon fluids. Since many of the sensors used to monitor these properties are intrusive to the hydrocarbon flow and would eventually erode, the reliability and the service life of the electrical cables used for these non-critical functions were not significant issues. In recent years the electrical cables have seen a rapid expansion in their duties. They are now required to power and control an increasing number of subsea electronics responsible for a wider range of critical functions that include valve operation. As a result, the reliability of subsea systems is increasingly dependent on the electrical cables within subsea umbilicals. However, experience to date has shown that the electrical cables in subsea umbilicals are very prone to mechanical failure
(Knight 1990 and Sasanow 1998). Mechanical failure, which is defined by failure of one or more of the insulated electrical conductor cores, is largely due to excessive mechanical loads or fatigue damage. Knight (1990) surveyed a number of umbilicals with reported failures and he showed that a large proportion of the reported umbilical failures is due to mechanical failure of the electrical cables. The reported umbilical failures due to failure of the electrical cables are 48% during manufacture, 58% during service and 25% during installation. Sasanow (1998) stated that for one major oil company a large percentage of its maintenance budget was spent on umbilical repairs related also to the mechanical failure of electrical cables.

A significant factor in the high failure rate of electrical cables in subsea umbilicals is the limited knowledge of their mechanical behaviour. This is the case given the lack of analytical methods for predicting the mechanical loads in the conductors. The bulk of published research on umbilicals is focused on predicting the global mechanical properties of an umbilical with the bundle of hydraulic hoses and electrical cables taken as a homogenous layer. The global mechanical properties, such as the axial, torsional and flexural stiffnesses, are dominated to a certain extent by the steel armour layers and research to date has been focused on these components of the umbilical. Other aspects of the published works on umbilicals that limit their capability of predicting the mechanical loads in the conductors of an electrical cable are the assumptions of linear material properties. This is a reasonable assumption for the steel armour wires, but the conductors are made predominantly from annealed copper that exhibits non-linear material properties. These non-linear properties of the conductors, which have not been considered in any of the earlier research, is taken into account in this thesis. These aspects account for the most common failure mode of the cores known as kinking which is also considered here.
Kinking is a buckling type of failure which leads to the formation of kinks in the insulated copper cores. A kinked conductor core is illustrated in Figure 1.3. Kinks subsequently result in rapid fatigue failure of the conductors. A number of different loading regimes can generate compressive loads, but most experimental and field data show that kinks usually occur during installation or during cyclic loading. These two loading regimes correlate strongly with the non-linear material properties of the copper conductors as shown in this thesis. To show how kinking failure during installation correlates with the non-linear material properties of copper, the mechanical loads acting on the cores as the umbilical is laid are investigated. As the umbilical is laid, a point on the umbilical will travel from the surface where the umbilical is subject to high tensile loads down towards the seabed where the umbilical is under very low tensile loads. The path of the point and the level of tensile loads as the umbilical is laid are shown in Figure 1.4. Near the surface the umbilical is under high tensile loads and as a result the cores are likely to experience plastic deformation. The plastic deformation will translate to residual extension near the seabed as the tensile loads on the umbilical are relaxed. However, the surrounding armour and polymer layers restrain the cores and their residual extension can only be accommodated if the cores are compressed. The forces required to compress the cores can reach critical values which causes kinking.

To summarise, the main objective of this thesis is to develop analytical models that lead to reliable predictions of the mechanical loads acting on the electrical cables in subsea umbilicals. These analytical models incorporate the non-linear material properties of the electrical conductors and kinking analysis. Both of these are original aspects not covered in the published literature. The analytical models are verified with full-scale tests and with other related experimental work published in the public domain. The analytical models incorporating these original aspects help identify design parameters that
optimise the performance and increase the reliability of these critical components in the offshore environment.

1.1 Historical Background

Although the earliest offshore activities date back to the end of the nineteenth century, the offshore industry, as defined by oil and gas exploration and production out at sea, traces its origin to some fifty years ago when mobile drilling platforms and drill ships were first introduced. These mobile platforms and drill ships replaced piers and fixed platforms that are uneconomical for exploration far from the shore. As a result, the early offshore activities were confined to a handful of geographical sites where offshore hydrocarbons gave away their presence by bubbling to the surface.

Mobile drilling platforms and drill ships were introduced in the early 1950s. As shown in Figure 1.5, mobile drilling platforms like submersibles and jack-ups evolved to allow greater water depths to be explored but their operational depth is still limited to about 50 and 100 metres respectively. Drill ships in turn evolved to allow much greater depths and hostile waters to be explored (Figure 1.5). Unlike piers and fixed platforms, which were built to accommodate drilling and production equipment in anticipation that a well drilled will produce oil or gas, mobile drilling platforms and drill ships were assigned the role of exploration drilling. If an explored location was to be subsequently exploited a fixed platform would be installed and a number of wells drilled and produced from this platform. Variants to fixed production platforms used include steel jacket platforms and concrete gravity platforms which are shown in Figure 1.6. The first steel jacket and concrete gravity platforms were used in 1948 and 1976 respectively.
A common feature of submersibles, jack-ups and fixed platforms including piers is a surface facility which, supported by the seabed, experiences minimal wave induced motion. The constraints to minimise the wave induced motion include the requirement to minimise the risk of damage to a conductor extended through the water column. The conductor consists of a wide casing anchored in the well bore and cantilevered to the surface platform. This conductor allows drilling and production equipment to be installed on the surface platform. The conductor is equipped with surface installed drilling and production equipment that provides the means to contain the combustible and toxic hydrocarbon fluids of the well being drilled or produced. Any damage to the conductor through the water column would render operating this equipment futile. The consequences would be severe and the resulting blow out may lead to the loss of human life. Thus, the use of drill ships and floating platforms dictated some modifications to drilling and production techniques to eliminate the requirement of a conductor. This could be achieved by locating the drilling and production equipment that contain the well on the seabed. This seabed equipment and the different modules required for its control constitute the subsea system.

The earliest subsea systems were used when floating drill ships were introduced in the early 1950s. These systems included a stack of five or six valves, a surface control station and an umbilical. With a handful of valves to operate, the umbilicals of these systems consisted of a few high pressure hoses of large diameter. Each hose was dedicated to a valve actuator as shown in the schematic of Figure 1.7 and the umbilical included redundant hoses to ensure an extended service life of the umbilical. For a number of reasons the use of subsea production systems lagged considerably behind the use of subsea drilling systems. One reason is the different political and legislative measures of the 1950s to increase reserves but to conserve production. This contributed to an increase in exploration activities using drill ships in water depths exceeding the operational capability of mobile drilling platforms. The use of drill ships consequently resulted in an increase in the use of subsea
drilling systems. However, production was confined to the extensive shallow continental shelves that until only relatively recently submersibles and jack-ups allowed their exploration. Another reason why the use of subsea production systems lagged behind is the higher level of complexity involved in controlling multiple wells with different duties (production, water injection and gas injection) as compared to the level of complexity of controlling one individual well being drilled. To control each individual valve of a production well with a dedicated hose, as was the case in controlling subsea drilling systems, would entail either a great number of umbilicals or umbilicals of relatively large diameter. A great number of umbilicals or umbilicals of relatively large diameter are impractical solutions.

In anticipation that hydraulic control methods used for subsea drilling systems would not accommodate the higher level of complexity of subsea production systems, subsea production systems of the early 1960s were designed to be operated by robots or were housed in pressurised chambers for direct intervention by technicians. These control methods proved to be inadequate and subsea production systems used in later years were limited to small fields or for tying back the production of one or more remote wells to an existing fixed platform. The Argyll field of the North Sea is a typical example. The Argyll field is the first oil field to be developed in the U.K. sector of the North Sea in 1975. For this field of eight wells with wet trees, the use of subsea production systems was dictated by the need to employ a semi-submersible production platform. The semi-submersible was expected to be used for other fields once the recoverable reserves of Argyll were exploited. With eight wells to produce in a water depth of 80 metres, the subsea production systems were controlled hydraulically using the same direct hydraulic control used for controlling subsea drilling systems.

Other variants to hydraulic control methods introduced in later years for more complex subsea systems include piloted and stepped hydraulic control. These
variants, which lead to an umbilical of smaller diameter, incurred additional subsea components which affected reliability and limited the operational capability of the subsea system. Other drawbacks of hydraulic control methods include the high attenuation in deep water and larger distances. These drawbacks led to the introduction of electro-hydraulic subsea systems. These systems which gained acceptance in the offshore oil and gas industry in the late 1980s employ the same principles as pilot hydraulic subsea systems which include a number of pilot valves. Pilot hydraulic subsea systems involve pilot valves actuated hydraulically from the surface control station through hoses of small diameter integrated in the umbilical. These valves direct pressurised fluids from a subsea accumulator to the actuator of the valve to be operated. Electro-hydraulic subsea systems employ the same principle but solenoid valves replace pilot hydraulic valves and two electrical cables replace the large number of small diameter hoses in the umbilical. One of these two electrical cables transmits electrical power to an electronic subsea control pod and the other cable transmits digital data to control and monitor the subsea system. Many modern control systems are multiplexed systems which superimpose the electrical signals and power in one electrical cable. Although these control methods were introduced in the late 1980s their use has increased dramatically in recent years. The increase in their use results from the increase in the use of floating production platforms which are increasingly deployed in deep water.

In summary, the early use of subsea systems was confined to drilling and for producing marginal fields that did not require complex control. The majority of these early systems used hydraulic control methods to execute the critical functions that included operating the subsea valves. In later years hydraulic control methods evolved to allow more complex subsea production systems to be used, but these hydraulic control methods affected the reliability and operational capability of the subsea systems. In deep water and over large distances the drawbacks of hydraulic control methods increase. Multiplexed electro-hydraulic control methods resolved some of these technical issues but
dual redundant control systems were used to ensure adequate reliability. The high level of redundancy needed to ensure reliability make multiplexed electro-hydraulic subsea systems less attractive in economical terms. In recent years and with subsea control experience gained over two decades, the redundancy of multiplexed subsea systems was reduced but most systems still used dual redundant components in the umbilical. The drawbacks of using dual redundant components include a larger outer diameter of the umbilical and the corresponding installation and handling difficulties. The use of dual redundant components is due to the limited knowledge on the structural behaviour of subsea umbilicals and the lack of reliable analytical means to assess the mechanical loads in its constituent elements.

1.2 Deep Water Applications

The definition of deep water for today’s offshore oil and gas applications is water depths in excess of 300 metres. In the Gulf of Mexico and offshore Brazil, the development of oil and gas fields in these water depths is driven by the maturing shallow water reserves and by the need to sustain the flow through the existing seabed infrastructure constructed during the boom days of offshore hydrocarbon production. The same argument applies to other geographical locations like the Atlantic Frontier of Northwest Europe. Here in addition to an increase in water depths, the environmental conditions are hostile with extreme significant wave heights in excess of 15 metres.

In deep water applications conventional fixed platforms are not economically viable and the general trend has been an increase in the use of floating production platforms. Variants to floating production platforms include semi-submersibles, ship shaped floating production storage and off-loading platforms (FPSO) and tension leg platforms. A layout of an offshore field development using an FPSO is shown in Figure 1.8. Tension leg platforms, tethered to the seabed, experience minimal wave induced motions allowing
the wells to be drilled and completed at the surface. Semi-submersibles and FPSOs are freely floating and are used in conjunction with subsea systems. The increase in the use of FPSOs is shown in Figure 1.9. This figure also shows the increase in the use of subsea well completions. As shown in Figure 1.9, the use of FPSOs has increased fourfolds in the last decade and there is a strong correlation between the increase in use of these platforms and the increase in subsea completions.

The use of FPSOs presents technical challenges for the different components of the subsea system. In particular, the use of these platforms presents technical challenges for the subsea umbilicals. These umbilicals, spanning the water column between a surface control station and a subsea control pod, are subject to severe dynamic loads instigated by the wave induced motion of the surface platform and the hydrodynamic forces on the umbilical. The effects of these loads in deep water and hostile environments, combined with the high tensile loads required to support the suspended weight of the umbilical, lead to considerable mechanical loads on the various constituent components of the umbilical.

Other technical challenges of applying subsea umbilicals in deep water and hostile environments include the requirement of high reliability. This high reliability minimises expensive intervention operations. In deep water, where floating production platforms are used, intervention requires a vessel that can hover above the wellhead to retrieve the different components requiring repair or maintenance. These vessels are of short supply and with a typical daily rate in excess of 100,000 pounds, intervention in terms of lost production and hire expenses is a costly exercise. High reliability is currently achieved by using dual umbilicals or by using dual redundant components within the umbilical. In either case, this approach incurs additional cost and renders ventures in deep and hostile waters uneconomical. The high reliability should
be achieved by better methods of analysing the mechanical loads acting on the different constituent elements of the umbilical.

The most critical constituent elements of modern and future umbilicals are the electrical cables. These cables are responsible for the critical functions of operating the solenoid valves that direct pressurised fluids to valve actuators. In future applications the cables will be responsible for a wider range of critical functions. These critical functions will include directing electric power to electrically rather than hydraulically actuated valves. These critical functions will also include powering and controlling new equipment such as pumps, power distribution networks and intelligent well instrumentation. These and other issues, which will be highlighted in this thesis, emphasise the significance of research on the mechanics of electrical cables in subsea umbilicals.

1.3 Thesis Overview

The aim of this thesis is to investigate the mechanics of electrical cables in subsea umbilicals. This will be achieved by developing analytical models that allow more accurate assessment of the mechanical loads in these critical elements for different loading regimes on the whole umbilical assembly.

The thesis, apart from this chapter, is made of six chapters detailing the different aspects considered to achieve the objectives of this research. Previous literature on subsea umbilicals and similar structures made of helical and cylindrical layers is reviewed in chapter two. This chapter also details an investigation into the mechanics of the helical armour wires of subsea umbilicals that constitute the dominant components contributing to the mechanical properties of umbilicals. In investigating the mechanics of the helical armour wires, axi-symmetric and flexural loads are considered. The axi-symmetric loads include the axial tension, the torsional moments and the
radial forces applied to the umbilical assembly. For axi-symmetric loads the analysis is carried out using the principles of differential geometry allowing the axial strain, the twist and curvatures of the helical armour wires to be calculated as a function of the global deformation variables of the umbilical. Flexural loads are also analysed using the principles of differential geometry. Under flexural loads the helical wires develop non-uniform axial and torsional loads along the length of the wire. For equilibrium, the non-uniform loads should be supported by frictional forces at the contact surfaces of the wires and the adjacent layers. These frictional forces can exceed a certain threshold value resulting in wire slip. Different slip mechanisms are investigated and these include slip along the loxodromic path and slip along the geodesic path. The slip along the loxodromic path is carried out for finite friction coefficients. The results are in agreement with experimental results reported in the public domain. For geodesic slip a new formulation is employed and the results reveal additional terms which have been ignored by other authors. The results also reveal some errors associated with ignoring the integration limits.

Chapter three details an analytical approach to modelling the structural response of subsea umbilicals for axi-symmetric and flexural loads. The difficulty of predicting the structural response of subsea umbilicals for axi-symmetric loads is due to the coupling in the axial, torsional and radial loads. Other difficulties arise due to separation of the layers. This leads to the introduction of additional variables into the equilibrium equations of the umbilical. Often iterative numerical techniques are used to predict the structural response of subsea umbilicals and in some cases the radial compliance of the umbilical core is taken into account by introducing an apparent Poisson’s ratio. Both methods have their limitations. A novel analytical approach to predict the structural response of umbilicals is presented in chapter three. This approach is based on variational energy methods. The approach is also based on the sub-structuring technique. The principles of variational energy are applied to the different layers of each sub-structure and this leads to expressions relating the applied loads to
deformation variables of the umbilical. These expressions constitute the stiffness matrix of a sub-structure. The stiffness matrix of the sub-structure is then partitioned and this approach eliminates the radial displacement from the equilibrium equations of the sub-structure. The stiffness matrix of the umbilicals can then be assembled from the stiffness matrix of the sub-structures.

Other aspects relating to the structural response of subsea umbilicals for axi-symmetric loads are also discussed in chapter three. These include a new methodology to predict the equivalent material properties of the umbilical core. In discussing the equivalent material properties of the core, it is shown quantitatively that the dominant material properties influencing the structural response under axi-symmetric loads are the equivalent radial stiffness and the dilatation coefficient. Analytical models are presented to calculate these mechanical properties.

In addition to modelling the structural response for axi-symmetric loads, chapter three presents an analytical model to predict the structural response for flexural loads. The model takes into account the structural response for the different slip mechanisms discussed in earlier chapters. For loxodromic slip and finite friction coefficients, the model is developed to take into account the spread of the slip region. The model is also based on the principles of variational energy and an incremental approach is used to take into account the non-linear response.

In chapter four, the models developed to predict the structural response for axi-symmetric and flexural loads are verified against full-scale experimental data. This chapter also details experimental tests that were carried out to model the equivalent mechanical properties of the core. Umbilicals with different constructional designs were examined and the differences between the experimental and theoretical results were found to be within acceptable limits. The theoretical and experimental investigation of umbilicals with
different constructional designs allows an assessment of the influence of the
design parameters on the mechanical properties of umbilicals. These design
parameters relate mainly to the number of armour layers, the lay angle of
these layers and the construction of the umbilical core.

In chapter four, the experimental procedure adopted to measure the bending
stiffness of the umbilical is described. Some of the limitations of this
experimental procedure, which is widely used in the industry, are examined.
These limitations include errors in measuring the bending stiffness for small
curvature increments. These errors are comparable in magnitude to the errors
used in analysing the experimental data. Due to this limitation, the
experimental results are limited to measuring the full slip bending stiffness of
the umbilical. The experimental results are compared with the theoretical
bending stiffness assuming different slip mechanisms. It is shown that the
bending stiffness for loxodromic slip is more representative of the
experimentally measured bending stiffness. Chapter four concludes with a
non-dimensional analysis on the influence of the main constructional design
parameters.

The mechanics of electrical cables is investigated in chapter five. This chapter
describes an analytical approach to take into account the double helix
construction of the electrical conductors. The analytical approach leads to
expressions relating the axial strain, the twist and curvatures of the conductors
to the global deformation variables of the umbilical. The influence of the
residual contact forces on the slip mechanism of the conductors is also
examined. The residual contact forces are the forces generated during the
manufacturing process. Chapter five also describes a general solution which
allows the axial strain in the conductors to be calculated directly from the
predicted structural response of the umbilical. This general solution takes into
account the double helix configuration of the cores within the umbilical
assembly. A case study is also examined in this chapter and the mechanical
loads in the conductors are calculated for the maximum installation loads.
The case study examined relates to an umbilical with a relatively low axial stiffness.

Chapter five also details the load regime in the electrical conductors as the umbilical is subjected to cyclic loading. The contribution of the conductors to the mechanical properties of the umbilical is small as compared to the contribution of the armour wires. Consequently, the conductors in an umbilical assembly are under strain controlled loading. This loading regime is unique to the conductors of subsea umbilicals. The influence of this loading regime is examined taking into account the non-linear material properties of the conductors. It is also shown in this chapter that while the wires of the electrical conductors are specified to be in an annealed condition, the manufacturing process imposes some work hardening. The level of work hardening is determined experimentally for two of the three main umbilical manufacturers. The level of work hardening in the conductors for a commercially available electrical cable is also determined experimentally to allow an assessment of the minimum work hardening expected in the conductors of subsea umbilicals.

The kinking failure of the copper cores is examined in chapter six. The cores of an electrical cable are embedded in low modulus polymeric insulation and filler layers. This allows an analogy to be drawn between the kinking of the conductors and the buckling of supported beams. The influence of the compressive mechanical properties of the polymeric insulation and filler layers on the kinking of the cores is also analysed in this chapter. It is shown that the compressive mechanical properties of these polymeric layers leads to a non-linear differential equation for the kinking of the cores. Perturbation techniques are used to solve this non-linear equation and the post-buckling curve is analysed.

Chapter six also presents an assessment of the influence of the construction of the conductor and the choice of insulation and filler layers on the kinking
resistance. Analytical expressions that relate the critical kinking stress to these parameters are derived. These expressions take into account the inelastic kinking of the cores. To take into account the inelastic kinking, the Ramberg-Osgood equation is used to model the non-linear stress-strain relationship. A comparative study is carried out to take into account different coefficients for the analytical stress-strain relationship.

Chapter six also examines the influence of the work hardening on the fatigue properties of the copper cores. Using the universal slopes equation the fatigue curves for copper in the annealed condition are compared with the fatigue curves in the as-built condition. It is shown that the influence of the work hardening leads to superior fatigue properties. The superior fatigue properties are well pronounced for the high cycle fatigue regime. In the final part of chapter six, a design application is presented. This design application allows loading and unloading limits to be specified so that the risk of kinking in the cores is minimised. The application is applied for a case study examined in earlier chapters.

The discussion and conclusions are presented in chapter seven. The discussion reflects on all the aspects considered in this thesis. The discussion also highlights the dominant parameters that should be considered in analysing the structural response of subsea umbilicals and the mechanics of the electrical cables. Conclusions, recommendations and scope for future work are also presented in this final chapter.
2 Structural Analysis of Umbilicals

During the different stages of their life cycle, subsea umbilicals experience complex loading regimes that can be divided into two main classes which are axi-symmetric and axi-asymmetric loads. This chapter is divided into two main sections to deal with these two types of loads. One section details the structural analysis of the different constituent elements when the umbilical is subject to axial, torsional and radial loads and the other section covers the structural analysis of these elements when the umbilical is subject to flexural loads. Before proceeding to these sections, the previous related work by other authors is first described.

2.1 Previous Work

The structural analysis of subsea umbilicals has its origins in the earlier work on ropes and overhead transmission lines. The main similarity between subsea umbilicals and these structures is the helical construction of the different layers. However, differences exist in the construction details and material selection. Ropes and transmission lines are made of steel and aluminium strands while umbilicals are made of a wide range of materials including steel, copper and a number of different polymers. The presence of all these different materials has a considerable effect on the assumptions that can be used regarding the compressibility of the umbilical core and the extent of linear material behaviour.

Hruska (1951) presented one of the earliest works on wire ropes. Hruska presented a linear solution for the axial stress in each wire of a rope when the rope is subjected to axial loads. The resultant twisting moments and radial forces due to these stresses were derived in subsequent publications by Hruska
Lutchansky (1969) discussed the axial stresses in the armour wires of submarine cables when the cable is subjected to flexural loads. Lutchansky obtained a general solution for the displacement of the armour wires and demonstrated experimentally that, for a certain bending curvature, the armour wires slip at the neutral axis of bending. Machida and Durelli (1973) took account of the change in geometry of the armour wires due to elongation and rotation of the wire rope. Using differential geometry they obtained expressions for the bending and twisting strains in the wires.

While the above researchers adopted an approach based on geometrical considerations of the helical wires, Phillips and Costello (1973) used Kirchoff's equilibrium equations for thin rods (Love, 1944) to analyse the stresses in the helical wires of a rope under axial and torsional loading. In other publications, Costello and Phillips (1976) and Costello and Sinha (1977) derived expressions for the effective axial and torsional stiffnesses of wire ropes. Costello (1990) also discussed stresses in wire ropes due to bending and assumed that the frictional forces between the different wires could be ignored. Each wire in the rope can then be considered as a free helical spring. Other issues related to wire ropes and armoured cables were discussed by Raoof and Hobbs (1984), Hobbs (1986) and Nabijou and Hobbs (1995). Raoof and Hobbs (1984) discussed the stresses in the wires of armoured cables close to terminations. Hobbs and Raoof (1986) presented analytical methods to predict the interwire slippage and fatigue in the stranded tethers of tension leg platforms. Nabijou and Hobbs (1995) presented an experimental study on the frictional performance of wire and fibre ropes bent over sheaves of different groove profiles.

Most of the models discussed above were developed primarily for wire ropes and strands where the assumption of no radial deformation leads to reasonably accurate results. In the presence of polymeric layers and other low modulus materials within the core of the umbilical, the above assumption of no radial deformation does not hold. Knapp (1979) took account of the radial deformation of submarine power cables and derived expressions for the
stresses in the different component layers of these structures assuming the core can be approximated to being either rigid or incompressible. Knapp also investigated the stresses due to bending (1986) and external pressure loading (1988). Knapp's analysis of wire stresses due to bending was carried out for the two extreme cases of infinite and zero frictional forces.

While many of the above papers on the modelling of wire ropes and cables address in depth the response to axi-symmetric axial-torsional loading, research into the response under bending is scarce and often two limiting cases are considered. These limiting cases are the full slip and the no slip conditions. These assumptions are justified for predicting the global stiffness of the umbilical but are questionable for predicting the stresses in the umbilical components. Hale (1984) discussed the effect of internal friction on the change in tension when a cable is passed over small rollers. Hale derived an expression relating the length of the slip region of a helical wire to the curvature of the cable. The derived expression was based on the assumption that slip occurs when the shear stress between the wire and the underlying cylinder exceeds the product of the friction coefficient and the contact pressure. Another publication on the shear stresses and length of slip regions is due to Spillers et al (1983). Spillers used Kirchoff's equilibrium equations and computed the variation of strains, contact forces and in-plane forces of helical rectangular wires. While the work of Hale (1984) and Spillers et al (1983) is based on the assumption that wire slip occurs in the tangential direction, other authors have argued that the helical wires have the tendency to follow a geodesic path. Feret and Bournazel (1987) and Out (1997) adopted this assumption and derived expressions for the tangential and transverse slip towards the geodesic. Other literature dealing with bending and the influence of slip on the structural response of helically constructed structures is due to LeClair and Costello (1986) and Waloen et al (1992). LeClair and Costello (1986) discussed the stresses in the wires of a single lay strand for two extreme cases of full slip and no slip. LeClair and Costello showed that for the no slip condition, the strand should be under relatively high axial loads. In this case the stresses due to practical bending curvatures
are small in comparison with the stresses due to the imposed axial loads. As a result of this, LeClair and Costello concluded that the axial stresses due to bending, in the wires of a single lay strand, could be ignored. Waloen et al (1991) developed analytical models to predict the slip of the armour wires of subsea umbilicals when the umbilical is bent onto a sheave.

Before proceeding to discuss the theory behind the structural analysis of subsea umbilicals a few more publications on umbilical modelling will be discussed here. These publications deal with combined axial, torsional and flexural loading. Lanteigne (1985) used an energy approach to estimate the stiffness matrix and discussed the influence of internal radial forces on the flexural rigidity. Lanteigne emphasised the experimentally observed fact that under bending and for high axial loads only the outermost helical wires slip relative to the underlying core. Lanteigne’s results showed that the bending stiffness is not only a non-linear function of the curvature but is also dependent on the magnitude of the axial force applied to the cable. Witz and Tan (1992a, 1992b) also used an energy approach to model umbilicals but they took account of the change in core dimensions, a parameter which was ignored by Lanteigne. Their numerical model can thus be viewed as most applicable for subsea umbilicals that are characterised by relatively soft cores in comparison with wire ropes and power cables. Witz and Tan did not introduce any assumptions regarding the compressibility of the core but they numerically computed the change in diameter as a function of the radial forces and internal pressures of the different components.

2.2 Co-ordinates systems

The different co-ordinate systems used in this work are a global Cartesian co-ordinate system \((X, Y, Z)\), a local co-ordinate system \((n, b, t)\) and a cylindrical co-ordinate system \((R, \phi, z)\). The global Cartesian co-ordinate system \((X, Y, Z)\)
is defined with the origin at the centre of the cross section at one end of the umbilical and the $Z$-axis directed along the longitudinal axis of the umbilical. The $X$-axis and the $Y$-axis are chosen arbitrarily to form a right hand Cartesian system. The local co-ordinate system $(n,b,t)$ is defined with the origin at the centre of the wire cross section. The axes $n$, $b$ and $t$ represent the normal, binormal and tangent axes of the helical wire cross section. The cylindrical co-ordinate system $(R,\phi,z)$ is defined with the origin at the centre of the cross section at one end of the umbilical, $R$ is the radius of the cylinder, $\phi$ is the polar co-ordinate on the circumference of the cylinder and $z$ is the longitudinal co-ordinate along the centre line of the cylinder. These distinct co-ordinate systems are shown in Figure 2.1.

2.3 Axi-symmetric Loading

The axi-symmetric loads of interest are the axial loads along the longitudinal axis of the umbilical, the twisting moments about this axis and the external hydrostatic pressure. The structural analysis of the different constituent elements of the umbilical for these loads is presented in the following sections.

2.3.1 Geometry of Stretched and Twisted Helices

To study the geometry of stretched and twisted helices, a reference cylindrical surface is introduced. The position vector of any arbitrary curve on this cylinder with reference to the global Cartesian co-ordinate system $(X,Y,Z)$ is given by:

$$\mathbf{r}(\phi,z) = \{R\cos\phi, R\sin\phi, z(\phi)\}$$

(2.1)
where $R$ is the radius of the cylinder, $\phi$ is the polar co-ordinate along the circumference of the cylinder and $z(\phi)$ is the co-ordinate along the longitudinal axis of the cylinder. The differential length of this arbitrary curve is then given by:

$$ds(\phi) = \sqrt{dz^2 + R^2 d\phi^2} \quad (2.2)$$

If the reference cylinder is subject to axi-symmetric deformations, the deformed position vector and the differential length of the arbitrary curve are given by:

$$\overline{r}(\phi) = \{\overline{R}\cos(\phi), \overline{R}\sin(\phi), \overline{z}\} \quad (2.3)$$

$$d\overline{s}(\phi) = \sqrt{d\overline{z}^2 + \overline{R}^2 d\phi^2} \quad (2.4)$$

where an over-bar denote the deformed state. The axi-symmetric deformations considered here are an axial displacement $u_z$ along the centre line of the reference cylinder, a radial displacement $u_R$ and a twist rotation $u_\phi$. These deformations are illustrated in Figure 2.2. The deformed co-ordinates are then given by:

$$\begin{align*}
\overline{z} &= z + u_z \\
\overline{\phi} &= \phi + u_\phi \\
\overline{R} &= R + u_R
\end{align*} \quad (2.5)$$

Substituting equation (2.5) into equation (2.4), the differential length of the arbitrary curve in the deformed state is given by:

$$d\overline{s} = \sqrt{dz^2 \left(1 + \frac{du_z}{dz}\right)^2 + R^2 d\phi^2 \left(1 + \frac{u_R}{R}\right)^2 \left(1 + \frac{du_\phi}{d\phi}\right)^2} \quad (2.6)$$

Equation (2.6) which is derived for any arbitrary curve on the reference cylinder can be applied to calculate the differential length of a helix. The helix is defined by a linear parametric representation $z(\phi)$ given by:

$$z(\phi) = \frac{R}{\tan \alpha} \phi \quad (2.7)$$

where $\alpha$ is the lay angle of the helix, $R$ is the radius of the helix and $\phi$ is the polar co-ordinate of the helix. The helix can be right handed or left handed depending on the rotation produced on following the path of the wire. If the rotation produced on following the path of the wire away from the origin of the
global co-ordinate system is anticlockwise, the helix is right handed; otherwise
the helix is left handed as shown in Figure 2.3. Substituting equation (2.7) into
equations (2.2) and (2.6), the differential lengths of the helix in the initial and
deformed states are given by:

\[ ds = \frac{dz}{\cos \alpha} \]  

(2.8)

\[ d\bar{s} = dz \sqrt{\left(1 + \frac{du_z}{dz}\right)^2 + \tan^2 \alpha \left(1 + \frac{u_R}{R}\right)^2 \left(1 + \frac{R \frac{du_\phi}{dz}}{\tan \alpha \frac{du_z}{dz}}\right)^2} \]  

(2.9)

The axial strain in the helix is then given by:

\[ \frac{du_z}{ds} = -1 + \left[ \left(1 + \frac{du_z}{dz}\right)^2 \cos^2 \alpha + \left(1 + \frac{R \frac{du_\phi}{dz}}{\tan \alpha \frac{du_z}{dz}}\right)^2 \left(1 + \frac{u_R}{R}\right)^2 \right] \sin^2 \alpha \]

where \( u_z \) is the displacement along the centre line of the helix. Using a
binomial expansion and ignoring second order terms, the above equation can
be linearised to:

\[ \frac{du_z}{ds} = \cos^2 \alpha \frac{du_z}{dz} + \frac{\sin \alpha \cos \alpha du_\phi}{dz} \frac{du_\phi}{dz} + \frac{u_R}{R} \sin^2 \alpha \]  

(2.10)

Equation (2.10) relates the axial strain in the helix to the axial strain, the radial
strain and the twist of the reference cylinder. This equation is similar to that
obtained by Hruska (1951) but with an additional allowance for the twist and
the radial strain. Lanteigne (1985), Knapp (1979), and Witz and Tan (1992a)
have also derived the same result but with different definitions for the helix
angle. In this work the helix angle is taken as the angle the helical wire makes
with the longitudinal axis of the reference cylinder. The helical angle is positive
if the helix is a right handed, otherwise the helical angle is negative.

In addition to the change in the differential length of the helix due to the
deformations of the reference cylinder, the helix twist and curvature will also
change due to these deformations. The change in twist and curvature will
result in twisting and bending strains. The derivations for the change in twist
and curvature are also carried out for any arbitrary curve on the reference
cylinder. The change in curvature and twist result from a change in the
directions of the tangent and binormal vectors respectively. The unit tangent, unit normal and unit binormal vectors of any arbitrary curve are given by:

\[ \mathbf{t} = \frac{\mathbf{r}'}{\sqrt{\mathbf{r}' \cdot \mathbf{r}'}}, \]

\[ \mathbf{n} = \frac{\mathbf{r}''(\mathbf{r}' \cdot \mathbf{r}') - \mathbf{r}'(\mathbf{r}'' \cdot \mathbf{r}')}{\sqrt{(\mathbf{r}' \cdot \mathbf{r}'')^2 - (\mathbf{r}'' \cdot \mathbf{r}')^2}}, \]

\[ \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{\sqrt{(\mathbf{r}' \cdot \mathbf{r}'')^2 - (\mathbf{r}'' \cdot \mathbf{r}')^2}} \]

where a hyphen indicates differentiation with respect to \( \phi \). The curvature and the twist are given by:

\[ \kappa = \frac{\sqrt{(\mathbf{r}' \cdot \mathbf{r}'')(\mathbf{r}' \cdot \mathbf{r}') - (\mathbf{r}' \cdot \mathbf{r}''')^2}}{(\mathbf{r}' \cdot \mathbf{r}')^{3/2}} \]

\[ \gamma = \frac{\mathbf{r}' \cdot (\mathbf{r}' \times \mathbf{r}'')}{(\mathbf{r}' \cdot \mathbf{r}')(\mathbf{r}' \cdot \mathbf{r}'') - (\mathbf{r}' \cdot \mathbf{r}'')^2} \]

and the rotation vector along the path of the curve is given by:

\[ \omega = \kappa \mathbf{b} + \gamma \mathbf{t} \]

The projections of this rotation vector along the unit tangent, unit normal and unit binormal vectors are given by:

\[ \tau = \mathbf{t} \cdot (\kappa \mathbf{b} + \gamma \mathbf{t}) = \gamma \]

\[ \kappa_n = \mathbf{n} \cdot (\kappa \mathbf{b} + \tau \mathbf{t}) = 0 \]

\[ \kappa_b = \mathbf{b} \cdot (\kappa \mathbf{b} + \tau \mathbf{t}) = \kappa \]

where \( \tau \) is the twist of the helix, \( \kappa_n \) is the curvature in the normal direction and \( \kappa_b \) is the curvature in the binormal direction. Using the general linear parametric representation \( z(\phi) \) to express a helix on the surface of the reference cylinder, the twist and curvature in the binormal direction are given by:

\[ \tau = \frac{dz/d\phi}{R^2 + (dz/d\phi)^2} \]

\[ \kappa_b = \frac{R}{R^2 + (dz/d\phi)^2} \]

Similarly, the twist and curvature in the binormal direction of the deformed helix are given by:
\[ \tau = \frac{d\bar{z}}{d\phi} \frac{d\bar{z}}{R^2 + (d\bar{z}/d\phi)^2} \] (2.22)

\[ \kappa_b = \frac{R}{R^2 + (d\bar{z}/d\phi)^2} \] (2.23)

where an over-bar denotes the deformed state. Substituting for the deformed co-ordinates \( R, \bar{z} \) and \( \bar{\phi} \) given in equation (2.5), the change in twist and curvature of the helix are given by:

\[
\Delta\tau = \frac{\sin \alpha \cos \alpha}{R} \left\{ (1 - 2 \cos^2 \alpha) \frac{du_z}{dz} - 2 \sin^2 \alpha \frac{u_R}{R} + \frac{R}{\tan \alpha} (1 - 2 \sin^2 \alpha) \frac{du_\phi}{dz} \right\} 
\] (2.24)

\[
\Delta\kappa_b = \frac{\sin^2 \alpha}{R} \left\{ \left[ 1 - 2 \sin^2 \alpha \right] \frac{u_R}{R} - 2 \cos^2 \alpha \frac{du_z}{dz} \right\} 
\] (2.25)

The expressions for the change in twist and curvature as expressed by the above equations are a function of the global deformation variables of the reference cylinder. Hence, from geometry, the axial strain, the change in twist and the change in curvature of a helix are calculated subject to the assumption that the deformations of the reference cylinder are small. This formulation is used to evaluate the mechanical strains in the helical wires of an umbilical.

The difficulty of employing the above formulation to calculate the mechanical strains in the helical elements of a subsea umbilical is the fact that the axisymmetric deformations of the umbilical are, in general, not known. These have to be calculated for a given loading on the umbilical. Moreover, the radial deformation of the helical elements is dependent on the contact pressures generated by the axial forces in these elements. Hruska (1953) showed that the contact line pressure \( X \) generated by a helical wire can be estimated from the formula \( X = -\kappa_b T \), where \( T \) and \( \kappa_b \) are the tension and binormal curvature of the wire. The contact pressure is thus a function of the axial forces in the wire.
which are in turn a function of the radial strain of the layer. An iterative technique is usually used for solving the above equations where the contact pressure generated by each helical layer is computed and the resulting radial strain is estimated prior to any subsequent iteration. An alternative analytical approach eliminating the need for this iterative technique is presented in chapter three of this thesis.

2.3.2 Equilibrium Equations

The equations of equilibrium for thin rods were first developed by Kirchoff and subsequently presented by Love (1944). Love (1944) used these equations to examine the equilibrium of free helical springs under tension and torsion. The six equations derived by Kirchoff, assuming the rod is inextensible and the curvatures are small, can be expressed in matrix form as:

\[
\begin{bmatrix}
N_n \\ N_b \\ T \\ G_n \\ G_b \\ H
\end{bmatrix}
\frac{d}{ds}
\begin{bmatrix}
0 & -\bar{\tau} & \bar{K}_b & 0 & 0 & 0 \\ -\bar{K}_b & 0 & -\bar{K}_n & 0 & 0 & 0 \\ -1 & 0 & 0 & -\bar{\tau} & \bar{K}_b & 0 \\ 1 & 0 & 0 & -\bar{K}_n & 0 & -\bar{K}_b \\ 0 & 0 & 0 & -\bar{K}_b & 0 & \bar{K}_n \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
N_n \\ N_b \\ T \\ G_n \\ G_b \\ H
\end{bmatrix}
+ \begin{bmatrix}
X \\ Y \\ Z \\ K_n \\ K_b \\ \Theta
\end{bmatrix} = 0 \quad (2.26)
\]

where \(N_n\) and \(N_b\) are the resultant shear forces in the normal and binormal direction, \(T\) is the axial force, \(X, Y\) and \(Z\) are the externally applied forces in the normal, binormal and tangential directions, \(G_n\) and \(G_b\) are the resultant bending moments in the normal and binormal directions, \(H\) is the torque, \(K_n, K_b\) and \(\Theta\) are the externally applied moments in the normal, binormal and tangential directions, \(\bar{K}_n\) and \(\bar{K}_b\) are the curvatures in the normal and binormal directions and \(\bar{\tau}\) is the twist of the deformed rod. Figure 2.4 clarifies these definitions.
Earlier, expressions for the change in curvatures and twist of a helical wire were derived from differential geometry. Assuming a linear stress-strain relationship, the resultant torque $H$ and moments $G_n$ and $G_b$ can be expressed as:

$$H = GJ\Delta \tau$$
$$G_n = EI_n \Delta \kappa_n$$
$$G_b = EI_b \Delta \kappa_b$$

where $E$ is the modulus of the material, $I_n$ and $I_b$ are the second moments of area in the normal and binormal directions, $G$ is the shear modulus of the material, $J$ is the polar moment of the area, $\tau$ is the twist in the helical wire and $\kappa_b$ and $\kappa_n$ are the curvatures in the normal and binormal directions. For a helical wire in an umbilical assembly it is assumed that under axial-torsional loading the resultant forces are constant along the helix and the externally applied moments are zero. Given these assumptions, equation (2.26) reduces to:

$$-\overline{\tau} N_b + \overline{\kappa}_b T + X = 0$$
$$-N_b - \overline{\tau} G_b + \overline{\kappa}_b H = 0$$

Substituting for $H$ and $G_b$ from equations (2.27) and (2.29) respectively, the above system of equations leads to the following expression for the radial force $X$ in terms of the initial and final curvatures:

$$X = \left\{GJ\Delta \tau \overline{\kappa}_b - EI_b \Delta \kappa_b \overline{\tau}\right\} - T \overline{\kappa}_b$$

Equation (2.32) is very similar to the expression derived by Hruska (1953) but with additional terms to account for the internal shear force in the binormal direction.
2.4 Flexural Loading

Structural analysis of umbilicals under flexural loading is much more complex compared with that for axi-symmetric loading. Witz (1994) highlighted this complexity where he discussed a case study in the cross-sectional analysis of a flexible riser. The study showed a considerable scatter in the results for flexural loading while the results for axi-symmetric loading were more consistent. A sample of the results reported in this case study is shown in Table 2.1. It is seen from this table that the scatter in the results is more pronounced for the bending stiffness with a normalised standard deviation of 1.1. In comparison, the normalised standard deviations for the axial and right hand (RH) torsional stiffnesses are 0.16 and 0.18 respectively.

The difficulties in analysing umbilicals under flexural loading can be attributed to a number of factors. Umbilicals are bent to high curvatures such that small deflection theory can not be fully justified. Other difficulties arise from interaction of the helical and cylindrical layers when the umbilical is bent. Under flexural loading a shear force at the contact surface between a helical wire and an underlying layer develops due to non-uniform strains along the helix. This shear force cannot exceed a certain value otherwise a slip mechanism develops. This slip mechanism results in a drop in the magnitude of the bending stiffness. Two different slip mechanisms are often considered. The first is slip along the loxodromic path and the second is slip towards the geodesic path. The loxodromic path is defined as the path where the angle between the tangent of the helical wire and the bent centre line of the toroid is constant. The geodesic path is defined as the shortest path between any two points on the surface of the toroid. For the helical wire to slip towards the geodesic path a transverse displacement is required. These slip mechanisms are discussed in this section.
Table 2.1: A case study in the cross-sectional analysis of a flexible riser

<table>
<thead>
<tr>
<th>Participant number</th>
<th>Axial stiffness (MN)</th>
<th>RH Torsional Stiffness (Nm²)</th>
<th>Bending Stiffness (Nm²)</th>
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<td>167</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
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<td>235</td>
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<td>Std. Deviation</td>
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<td>31</td>
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</table>

2.4.1 Geometry of Bent Helices

A helical wire wrapped around a cylinder is shown in Figure 2.5. The figure shows the helical wire in the initial state and in the deformed state when the cylinder is bent to a toroid. The position vector of the helix in the initial state is given by equation (2.1) presented earlier. When the cylinder is bent to a toroid the position vector of the deformed helix is given by:

\[ \mathbf{r} = \{ R \cos \phi, \rho - (\rho - R \sin \phi) \cos \omega, (\rho - R \sin \phi) \sin \omega \} \] (2.33)

where \( \rho \) is the radius of the toroid, \( R \) is the radius of the cylinder, \( \phi \) is the polar co-ordinate along the circumference of the cylinder and \( \omega \) is the parametric angle denoting the position vector with reference to the centre of the toroid as shown in Figure 2.5. The differential lengths of the wire in the initial and deformed states respectively are thus given by:

\[ ds = \frac{R}{\sin \alpha} \, d\phi \] (2.34)

\[ d\bar{s} = \sqrt{R^2 + (\rho - R \sin \phi)^2 \left( \frac{d\omega}{d\phi} \right)^2} \, d\phi \] (2.35)
where an over-bar denotes the deformed state. The twist and curvature of the helix in the initial state are given in equation (2.20) and equation (2.21) respectively. The change in the twist and curvature of the helix in the deformed state can be calculated from the general expressions presented in equation (2.14) and equation (2.15) for any given \( \omega(\phi) \).

### 2.4.2 Loxodromic Slip

The loxodromic slip is defined here as the slip along the loxodromic curve. The loxodromic curve is defined by the curve where the angle between the tangent to the curve and the axis of the toroid is equal to the helical angle of the undeformed helix. In this case the parametric angle \( \omega \) and the angle \( \phi \) are related by:

\[
\omega = \frac{R}{\rho \tan \alpha} \phi + c \tag{2.36}
\]

where \( c \) is a constant and \( \alpha \) is the helical angle. Substitution of equation (2.36) into equation (2.35) and using a Taylor’s series expansion up to the second order term, the differential length of the bent helix is thus given by:

\[
d\tilde{s} = \frac{R}{\sin \alpha} \left( 1 - \frac{R}{\rho} \cos^2 \alpha \sin \phi + \frac{R^2}{2\rho^2} \sin^2 \alpha \cos^2 \alpha \sin^2 \phi \right) d\phi \tag{2.37}
\]

The total length of the deformed helix over one pitch length can be obtained by integrating the above equation. The total length over one pitch length is thus given by:

\[
\bar{s} = \frac{2R\pi}{\sin \alpha} \left( 1 + \frac{1}{4} \frac{R^2}{\rho^2} \sin^2 \alpha \cos^2 \alpha \right) \tag{2.38}
\]

where \( \bar{s} \) is the deformed length of the wire over one pitch length of the helix. The second term within the brackets in equation (2.38) represents a residual extension in the length of the helix. This is not physically feasible since under pure bending the lengths of the helix in the initial and deformed states should
be identical. To employ this physical contraint the second order term must be ignored. The axial strain due to bending is then given by:

\[
\frac{du_s}{ds} = \frac{d\bar{s} - ds}{ds} = -\frac{R}{\rho} \sin \phi \cos^2 \alpha \tag{2.39}
\]

The above expression is of reasonable accuracy for typical bending radii encountered in umbilical applications. The axial strain along the helical wire, as given in the above equation, varies sinusoidally around the circumference of the bent cylinder with maximum absolute values at the inner and outer sides of the toroid. For this strain state to hold, the helical wire must be subject to external forces. These external forces are the shear forces at the contact surfaces between the wire and the adjacent layers. Figure 2.6 shows a differential element of a helical wire in contact with an arbitrary layer. For equilibrium the shear force per unit length \( Z \) at the contact surface of the wire and the layer is given by:

\[
Z = \frac{dT}{ds} = EA \frac{d}{ds} \left( \frac{du_s}{ds} \right) \tag{2.40}
\]

where \( E \) is the modulus of the wire, \( A \) is the cross-sectional area of the wire, \( T \) is the axial force in the wire and \( du_s/ds \) is given by equation (2.39). The shear force \( Z \) can then be expressed as:

\[
Z = -EA \cos^2 \alpha \frac{\cos \phi}{\rho} \sin \alpha \tag{2.41}
\]

The shear force \( Z \) as expressed in the above equation is inversely proportional to the radius of the toroid \( \rho \). To illustrate the evolution of the slip mechanism it will be assumed that the toroid is progressively bent to a tighter bend radius. As the bend radius decreases the shear force as expressed in equation (2.41) increases. Should this shear force exceed the maximum shear force \( Z_{\text{max}} \) that can be sustained by the contact surface, slip would occur first at the neutral axis of bending \( (\phi = 0, \pi) \) where the shear force is of greatest magnitude. With any further decrease of the bend radius the slip region will spread and the distribution of the shear and axial forces along one pitch length of the helix will
be as shown in Figure 2.7. Assuming that the slip region is bounded by ±φₜ, the axial force distribution can be expressed mathematically by:

\[
T = \begin{cases} 
-\frac{EA}{\rho} R \sin \phi \cos^2 \alpha & \phi_t < |\phi| < \frac{\pi}{2} \\
Z_{\text{max}} \frac{R}{\sin \alpha} & |\phi| < \phi_t
\end{cases}
\] (2.42)

The distribution of the axial force along the helix should be continuous at ±φₜ, otherwise a concentrated force will be required. For continuity at ±φₜ:

\[
EA \frac{R}{\rho} \sin \phi \cos^2 \alpha = -Z_{\text{max}} \frac{R}{\sin \alpha} \phi_t
\] (2.43)

If it is assumed that the maximum shear force is given by Coulomb's friction law, that is the maximum shear force is the product of the friction coefficient μ of the contact surface and the contact line pressure X, then the relationship between the curvature and the slip region can be expressed as:

\[
\rho = \frac{-EA \sin \alpha \cos^2 \alpha \sin \phi_t}{\mu X \phi_t}
\] (2.44)

A similar expression to equation (2.44) was derived by Witz and Tan (1992b) for the critical curvature at the inception of slip. At the inception of slip, φₜ is small and the critical bending curvature ρₜ can be written as:

\[
\rho_{\text{crit}} = \frac{-EA \sin \alpha \cos^2 \alpha}{\mu X}
\] (2.45)

Equation (2.44) shows that slip is a function of the wire axial stiffness, the contact line pressure and the friction coefficient of the contact surface. Slip is also a function of the lay angle. Figure 2.8 shows a plot of the critical bending radius for different lay angles. Figure 2.8 shows that the critical bending radius is greatest for lay angles of ±37 degrees.

To explain further the slip mechanism, a typical circular steel armour wire of an umbilical assembly will be examined. The wire is 5 mm in diameter and is laid at a helical angle of 30 degrees and a helical radius of 50 mm. The axial stiffness EA of the wire is then 4.2 MN and the curvature K is then 0.005 /mm. Assuming that the stress in the wire due to the imposed axial and torsional
loads is 70 MPa, which is equivalent to one third of the yield stress of mild steel, the axial force $T$ in the wire is then 1300 N. The contact line pressure $X$ generated by this armour wire which is given by $X = -\kappa T$ is then -7 N/mm. Assuming the friction coefficient at the contact surface of the wire and the underlying layer is 0.2, the critical bending radius is greater than 1000 metres. It is thus evident that for all flexural loads of practical significance the armour wires can reasonably be assumed to be in a full slip condition. However, this is not true for other helical elements within the core of the umbilical. This is best illustrated by considering a reinforced thermoplastic hose 20 mm in diameter and with an equivalent axial stiffness of 320 kN. Assuming the polymeric sheath layers transmit pressure, the pressure within the core of the umbilical is given by $X \cos \alpha / d$ where $X$ is the contact line pressure generated by the armour wire, $\alpha$ is the lay angle of the wire and $d$ is the diameter of the wire. The pressure within the core is then -1.2 MPa. Assuming the hose is laid at a helical angle of 10 degrees, the equivalent line pressure for the hose is thus -24 N/mm and the critical bending radius is 11 metres. The effective unit width of the hose is given by $d / \cos \alpha$, where $d$ and $\alpha$ are the outer diameter and helical angle of the hose respectively. Thus, while the critical bending radius of the armour layers is in excess of 1000 metres, the critical bending radius of the reinforced thermoplastic hose is only 11 metres. It is thus evident that slip is a strong function of the axial stiffness of the helical element and its lay angle. It is then possible that helical elements of low axial stiffness and small helical angle, such as the electrical cables and the hydraulic hoses, will undergo considerable straining before slip occurs.

Once the helical wire slips, the strain along the centre line of the wire is given by:

$$\frac{du_{slip}}{ds} = -\mu X \frac{R}{EA \sin \alpha} \phi$$

(2.46)

where $u_{slip}$ is the slip displacement along the loxodromic path. The magnitude of slip along the loxodromic path is thus given by:
\[ \Delta_t = \int_{-\pi/2}^{\phi} \left( du_{\text{slip}} - du_s \right) \]

\[ = \int_{-\pi/2}^{\phi} \mu X \frac{R^2}{EA \sin^2 \alpha} \phi d\phi - \int_{-\pi/2}^{\phi} \frac{R^2}{\rho \sin \phi} \frac{\cos^2 \alpha}{\sin \alpha} d\phi \]

where \( \Delta_t \) is the slip distance along the loxodromic path. The substitution of \( ds = (R/\sin \alpha) d\phi \) from equation (2.34) was used to obtain the above equation.

Carrying out the integrals, the magnitude of slip is given by:

\[ \Delta_t = \frac{\mu X R^2}{EA \sin^2 \alpha} \left( \frac{\pi^2}{8} - \frac{\phi^2}{2} \right) - \frac{R^2 \cos^2 \alpha}{\rho \sin \alpha} \cos \phi \]

The above expression defines the slip along the loxodromic path taking into account the frictional forces. Feret and Bournazel (1987) and Sævik (1992) obtained similar expressions for the case where the frictional forces are zero. In this case the slip is given by only the last term of equation (2.48). Equation (2.48) shows that the maximum slip magnitude occurs at the neutral axis of bending (\( \phi = 0, \pi \)) and the slip is zero at the inner and outer sides of the toroid (\( \phi = \pm \pi / 2 \)). The above equation also shows that as the friction coefficient decreases the slip magnitude increases (the contact line pressure \( X \) is negative).

To study the influence of the frictional forces the expression \( X = -\kappa_b T \) is substituted into equation (2.48). This substitution allows the slip distances in the outer armour wires of an umbilical to be analysed. The maximum slip distance at the neutral axis of bending is thus given by:

\[ \Delta_t = R \mu \frac{du_s}{ds} \left( \frac{\pi^2}{8} \right) - \frac{R^2 \cos^2 \alpha}{\rho \sin \alpha} \]

where \( du_s/ds \) is the axial strain in the armour wire. The above equation shows that the slip at the neutral axis of bending, for any given lay angle, is greater for small values of axial strain and for small friction coefficients. In this respect equation (2.48) confirms the experimental results of Benjaminsen et al (1992) who observed experimentally that the slip movement is greater in lubricated
umbilicals. The lubrication of an umbilical results in a low friction coefficient $\mu$. Equation (2.48) also confirms typical experimental results of cyclic bending over sheaves as those published by Ricketts and Kipling (1995). Ricketts and Kipling tested a number of umbilicals for different imposed axial loads on the umbilical assembly. They observed that for low axial loads, which results in low frictional forces, the fatigue life of the umbilical is greater than the fatigue life under high axial loads. This is confirmed by equation (2.48) since low axial loads, resulting in lower contact pressures within the umbilical, lead to greater slip movement. A greater slip movement in turn results in lower stress amplitudes.

The axial slip along the loxodromic path occurs when the shear forces along the helical wire exceed the frictional forces at the contact surface. In a similar manner the wires will experience a rotational twist. To discuss the twist rotation of a helical wire, the last row of the equilibrium matrix of a thin rod presented earlier in equation (2.26) is expressed as:

$$\frac{dH}{ds} - \bar{K}_bG_n + \bar{K}_nG_b + \Theta = 0 \quad (2.50)$$

where $H$ is the internal twist moment, $G_n$ and $G_b$ are the resultant bending moments in the normal and binormal directions, $\bar{K}_n$ and $\bar{K}_b$ are the curvatures in the normal and binormal curvatures in the deformed state, $\bar{\tau}$ is the twist in the deformed state and $\Theta$ is the externally applied twist moment. The curvatures in the normal and binormal directions and the twist in the wire once the wire slips along the loxodromic path are given by the projections of the imposed curvature on the umbilical. As shown in Figure 2.9, the curvatures in the normal and binormal directions, $\bar{K}_n$ and $\bar{K}_b$ and the twist $\bar{\tau}$ are given by:

$$\bar{K}_n = \frac{1}{\rho} \sin \phi \cos \alpha \quad (2.51)$$

$$\bar{K}_b = \frac{1}{\rho} \cos \phi \quad (2.52)$$

$$\bar{\tau} = \frac{1}{\rho} \sin \phi \sin \alpha \quad (2.53)$$
Substituting for the resultant bending and twisting moments, given in equations (2.27), (2.28) and (2.29), equation (2.50) can be simplified to:

\[ \frac{dH}{ds} - \kappa_n EI_b \kappa_b + \Theta = 0 \]  

(2.54)

In this case the external applied twisting moment \( \Theta \) required to hold the wire in equilibrium is given by:

\[ \Theta = \frac{\sin^2 \alpha}{R \rho} \left( EI_b \sin \phi \cos \alpha - GJ \cos \phi \right) \]  

(2.55)

This twisting moment is provided by the frictional force at the contact surface of the wire. In a similar manner to the axial slip, this frictional force cannot exceed a certain value otherwise a slip mechanism develops, but in this case the rotational twisting slip occurs at the inner and outer sides of the toroid. This is the case because the external twisting moment is greatest in these regions.

### 2.4.3 Geodesic Slip

The geodesic slip is defined here as the slip towards the geodesic curve on the surface of the toroid. The geodesic curve on any given surface is defined by the curve of shortest length on the surface. The geodesic curve is also defined by the curve where the unit tangent and the unit normal vectors of the curve are in the same plane as the unit normal vector of the surface. In this respect a helix is a geodesic since the normal to the curve coincides with the normal to the surface.

The arc length between any two given points on the surface of the toroid can be obtained by integrating \( d\bar{s} \) in equation (2.35). Using the following substitution:

\[ L(\phi) = \sqrt{R^2 + (\rho - R \sin \phi)^2 \left( \frac{d\omega}{d\phi} \right)^2} \]  

(2.56)

the arc length can be calculated from:
\[ s = \int_{\phi_1}^{\phi_2} L(\phi) d\phi \]  

(2.57)

where \( s \) is the arc length and \( \phi_1 \) and \( \phi_2 \) are the two integration limits defining the position of the points on the surface of the toriod. The geodesic curve is then obtained by solving for \( \omega(\phi) \) that minimises the arc length defined by the above integral. The solution is given by Euler-Lagrange equation (Kreysig, 1991):

\[
\frac{\partial L}{\partial \omega} - \frac{d}{d\phi} \left( \frac{\partial L}{\partial \omega'} \right) = 0
\]

(2.58)

where a hyphen indicates differentiation with respect to \( \phi \). The first term of the above equation is zero since \( L \) is not explicitly dependent on \( \omega \). Integrating equation (2.58) leads to:

\[
\left( \frac{\partial L}{\partial \omega'} \right) = b
\]

(2.59)

where \( b \) is a constant. Substituting for \( L \) from equation (2.56), the solution for the Euler-Lagrange equation can be expressed as:

\[
\frac{d\omega}{d\phi} = \frac{bR}{(\rho - R\sin\phi)\sqrt{(\rho - R\sin\phi)^2 - b^2}}
\]

(2.60)

Substituting \( c = b / \rho \) and \( \beta = R / \rho \), the above equation can be written as:

\[
\frac{d\omega}{d\phi} = \frac{c\beta}{(1 - \beta\sin\phi)\sqrt{(1 - \beta\sin\phi)^2 - c^2}}
\]

(2.61)

An approximate analytical solution for the above equation can be obtained by using a Taylor series expansion with respect to \( \beta \). Retaining only terms of second order in \( \beta \):

\[
\frac{d\omega}{d\phi} = \frac{c\beta}{\sqrt{1 - c^2}} \left[ 1 + \frac{\beta\sin\phi}{1 - c^2} + \beta\sin\phi \right]
\]

(2.62)

where the constant \( c \) can be obtained from the equation:

\[
\lim_{\beta \to 0} \frac{d\omega}{d\phi} = \frac{\beta}{\tan\alpha}
\]

(2.63)
The solution of the above equation leads to $c = \cos \alpha$. Substituting for $c$ in equation (2.62), the solution for the geodesic is given by:

$$\omega = \frac{\beta}{\tan \alpha} \left( \phi - \beta \cos \phi \left( \frac{1}{\sin^2 \alpha} + 1 \right) \right) + d \tag{2.64}$$

where $d$ is an integration constant. Equation (2.64) can also be written as:

$$\omega = \frac{\beta}{\tan \alpha} \left( \phi - 2\beta \cos \phi - \frac{\beta \cos \phi}{\tan^2 \alpha} \right) + d \tag{2.65}$$

The above expression is similar to that derived by Sævik (1992) except for the last term within the brackets. Sævik (1992) adopted a different approach based on the Lagrangian multiplier technique. Substituting the above expression for $\omega$ into equation (2.35), the differential length along the geodesic path is then given by:

$$d\bar{s} = \frac{R}{\sin \alpha} \left( 1 + \frac{\beta \sin \phi}{\tan^2 \alpha} \right) \tag{2.66}$$

Integrating the above expression over one pitch of the helix, the total arc length of the wire is given by:

$$\bar{s} = \frac{2R\pi}{\sin \alpha} \tag{2.67}$$

The total length of the geodesic path as expressed in the above equation is identical to the length along the loxodromic path. The reason for this discrepancy is the fact that higher order terms in $\beta$ were ignored in the approximate method used above. The differential length of the loxodromic and geodesic curves can then be assumed to be identical. This assumption can be used to calculate the slip magnitude. The slip magnitude can be calculated by integrating the differential lengths of the initial helix and the geodesic expressed in equations (2.34) and (2.66) respectively. This assumption leads to:

$$\int_{\phi_i}^{\phi_i^*} d\bar{s} = \int_{\phi_1}^{\phi_1^*} ds \tag{2.68}$$

where $d\bar{s}$ and $ds$ are the differential lengths of the wire in the deformed and initial configurations, $\phi_1^*$ and $\phi_1$ are the initial integration points along the
geodesic and the original helix, and \( \phi^* \) and \( \phi \) are the end integration points along the geodesic and initial helix. The above equation leads to:

\[
\phi^* - \frac{\beta \cos \phi^*}{\tan^2 \alpha} = \phi - \phi_1 + \left( \phi_1^* - \frac{\beta \cos \phi^*}{\tan^2 \alpha} \right)
\]  

(2.69)

Substituting for \( \phi^* = \phi + \Delta \phi \) and assuming \( \Delta \phi = \left( \phi^* - \phi \right) \ll 1 \), the above equation can be written as:

\[
\phi^* = \phi + \frac{\beta (\cos \phi - \cos \phi_1)}{\beta \sin \phi + \tan^2 \alpha} - \frac{\tan^2 \alpha (\phi_1^* - \phi_1^*)}{\beta \sin \phi + \tan^2 \alpha}
\]  

(2.70)

It is seen from the above equation that the assumption of equal differential lengths is not sufficient to define the position of the geodesic on the surface of the toroid. While the differential lengths are the same their position in space cannot be determined. Moreover, if a mechanical constraint is introduced by imposing the boundary condition \( \phi_1^* = \phi_1 \), the assumption of free slip does not hold and the above formulation would not be applicable. Out (1997) used such constraint to study the slip mechanism for different helical wires anchored at various locations around the circumference of the toroid. In the absence of any mechanical constraint the above equation can only determine the position \( \phi^* \) with respect to \( \phi \) if the the initial integration points are chosen at the points of intersection of the two curves. At these points \( \phi_1^* = \phi_1 \) and equation (2.70) can be expressed as:

\[
\phi^* = \phi + \frac{\beta (\cos \phi - \cos \phi_1)}{\beta \sin \phi + \tan^2 \alpha}
\]  

(2.71)

From equation (2.33) defining the position vector of any arbitrary curve on the bent cylinder, the intersection points are given by \( \omega^I = \omega^g \) where \( \omega^I \) and \( \omega^g \) are the parametric representations for the loxodromic and the geodesic paths respectively. The parametric representation for the loxodromic path is given in equation (2.36) presented earlier. The parametric representation for the geodesic path is given in equation (2.64). Equations (2.36) and (2.64) lead to:

\[
2\beta \cos \phi_1 + \frac{\beta \cos \phi_1}{\tan^2 \alpha} = 0
\]  

(2.72)
with solutions of $\phi_{1,n} = (n + 1)\pi/2$ where $\phi_{1,n}$ are the initial integration points. These points refer to points on the inner and outer sides of the toroid. The slip distance in this case would be:

$$\Delta = R \sqrt{\left(\phi^* - \phi\right)^2 + \left(\omega^e - \omega^l\right)^2}$$

(2.73)

where $\Delta$ is the slip distance along the surface of the toroid. Substituting for $\left(\phi^* - \phi\right)$ and $\left(\omega^e - \omega^l\right)$, the slip in the helical wire is given by:

$$\Delta = R\beta \frac{\cos \phi}{\tan \alpha} \sqrt{\frac{\tan^2 \alpha}{\left(\tan^2 \alpha + \beta \sin \phi\right)^2} + \left(1 + \frac{1}{\sin^2 \alpha}\right)^2}$$

(2.74)

The above formulation is applicable for small helical angles typical of the armour wires and the electrical cables in subsea umbilicals. Feret and Bournazel (1987), and based on their work Saevik (1992), derived a similar expression which is given by:

$$\Delta = R\beta \frac{\cos \phi}{\tan \alpha} \sqrt{\frac{1}{\tan^2 \alpha} + (2)^2}$$

(2.75)

To estimate the error involved in using the above equation to estimate the slip at the neutral axis of bending, the ratio $\chi$ given by:

$$\chi = \sqrt{\frac{1 + \left(\frac{\tan \alpha}{\sin \alpha \cos \alpha} - 1\right)}{1 + 4 \tan^2 \alpha}}$$

(2.76)

where $\chi$ is the ratio of the slip distance as calculated in this work to the ratio of the slip distance calculated by Feret and Bournazel (1987). Figure 2.10 shows a plot of this ratio for different lay angles. As shown this figure this ratio 2 for a lay angle of 30 degrees. This indicates that the slip magnitude calculated using the formulation presented by Feret and Bournazel (1987) would be half the slip magnitude calculated using the formulation developed in this work. The results are compared with the experimental data published by Benjaminsen et al (1992). Experimental measurements were only reported for the outer core of an electrical cable that is shown in Figure 2.11. From the dimensions and the lay configuration given it can be shown that the helix angle is 9.3 degrees and the
helix radius is 7.7 mm. The specimen was bent onto a sheave with a diameter of 330 mm. The experimentally measured slip distance when the cable is on the sheave was found to be dependent on the amplitude of the axial load. For the range of loads considered the slip varied from 2 mm to 10 mm. The small slip distances correspond to high frictional forces. The greater slip distances correspond to low frictional forces. The slip distance for the parameters of the helical core as calculated from the expressions derived in this work is 8.7 mm. The slip distance as calculated from the expressions derived by Feret and Bournazal (1987) is 1.4 mm. It is seen that the slip, as calculated from the expressions derived in this work, agrees well with the experimentally measured slip distances for low frictional forces. On the other hand the slip calculated from the Feret and Bournazal expression under-predicts the slip distance for even for high frictional forces. Bearing in mind that influence of the frictional forces was ignored in the theoretical expressions, the calculated slip distances should exceed the measured values. In this respect the expressions derived in this work are more representative of the slip distances likely to occur in the helical components of a subsea umbilical.

2.5 Concluding Summary

The structural analysis of subsea umbilicals was presented in this chapter for axi-symmetric and flexural loads. For axi-symmetric loads expressions for the axial strains in a helical wire, the change in the normal and binormal curvatures and the change in twist of the wire were derived from differential geometry. These expressions are expressed in terms of the axi-symmetric global deformation variables of the umbilical. These are the axial displacement, the radial displacement and the twist rotation deformation variables. The equations of equilibrium for thin rods were used to find the contact line pressures generated by the mechanical loads within the helical wire. This contact pressure is a function of the tension and shear forces in the wire. The
analysis presented using differential geometry is in agreement with the reviewed literature for axi-symmetric loading.

For flexural loads the structural analysis was presented for different slip mechanisms. These are slip along the loxodromic path and slip along the geodesic path. The slip along the loxodromic path is the slip along the tangential axis of the helical wire. This slip mechanism first occurs at the neutral axis of bending. The loxodromic slip mechanism was analysed for finite friction coefficients. The results are in agreement with experimental results that show that the axial slip distances are greater in lubricated umbilicals due to the smaller friction coefficient in a lubricated umbilical as compared with an un lubricated umbilical. The slip along the loxodromic path also involves rotational twist slip. Using the Kirchoff's equilibrium equations for thin rods it was shown that for a circular wire the rotational twist occurs first at the inner and outer sides of the bent umbilical. The geodesic slip, which is defined as the slip towards the path of minimum arc length, was also discussed in this chapter. A different formulation to that presented by Feret and Bournazel (1987), and based on their work Saevik (1992), was used and the results were shown to be in agreement only for large helix angles. For small helix angles which are typical of the helix angles of the armour wires in subsea umbilicals, it was shown that the slip distance would be greater than those predicted by the two authors mentioned above. From experimentally measured slip distances it was shown that the expressions for the slip derived in this chapter are more representative than the expressions derived by the previously mentioned authors.

It was also shown that for all bending radii of practical significance, the armour wires of subsea umbilicals can be assumed in a full slip condition. A numerical example was given to illustrate this point. The main reason why this is the case is the fact that the contact pressures generated by the helical wires are small and these elements are not subject to external applied pressure. Umbilicals are operated in a fully flooded condition and in this case the net external pressure is
zero. However, it was shown that for other constituent elements of low axial stiffness the above assumption does not hold. The other constituent elements like the electrical cables and the hydraulic hoses are thus likely to experience considerable mechanical strains before such a slip mechanism develops.
3 Structural Response of Umbilicals

A subsea umbilical is mainly an armoured and sheathed bundle of electrical cables and hydraulic hoses. A typical cross section is shown in Figure 3.1 and the different components are identified as the functional bundle, the armour layers and the inner and outer sheath layers. The functional bundle consists of the electrical cables and the hydraulic hoses that are usually laid in a helical configuration. Another lay configuration often used for the components of the functional bundle is the S-Z configuration which is shown in Figure 3.2. In this lay configuration a component is wound in one direction for a certain length and then the lay direction is reversed.

Umbilical manufacturers strive to position the electrical cables of the umbilical as near to the centre of the cross section as possible. This design requirement is recommended by the American Petroleum Institute (API 17E, 1994) and is believed to minimise the flexural loads in these components. The hydraulic hoses are positioned in the annulus between the electrical cables and the inner sheath layer and filler rods are used to fill the interstices. The cross section of the umbilical could be asymmetric and this would be dictated by the requirement to construct as compact an umbilical cross section as is possible. An asymmetric umbilical cross section is shown in Figure 3.3. The armour layers, which are made of steel wires that provide the mechanical support of the sheathed functional bundle, are also laid in a helical configuration. The general layout described above allows subsea umbilicals to be idealised as composite structures made of a cylindrical core, a number of armour layers and an outer sheath layer. The cylindrical core is considered as a uniform cylindrical rod and the outer sheath and is considered as a thin cylindrical shell. This idealised structural model is shown in Figure 3.4.
The structural response of subsea umbilicals idealised as a composite structure made of an inner core, a number of armour layers and an external sheath layer is discussed in this chapter. The structural response is analysed for axi-symmetrical loads and for flexural loads. For axi-symmetrical loads a variational energy approach is used and an analytical solution for the axi-symmetrical structural stiffness matrix is derived taking into account the radial compliance of the core. This analytical solution is new and it eliminates the requirement of iterative numerical techniques often used to predict the structural stiffness matrix of subsea umbilicals. Other new features of the analytical solution are a methodology to predict the mechanical properties of the core and a sub-structuring technique to take into account the separation of the layers. The methodology to predict the mechanical properties of the core, which include the equivalent radial modulus and dilation coefficient, allows an accurate representation of the influence of the core compliance on the structural response of umbilicals. In a similar manner, the sub-structuring technique to take into account the separation of the layers allows for more accurate representation of the complex behaviour of subsea umbilicals. For flexural loads, a variational energy approach is also used and the structural response is analysed for limiting cases of infinite and zero frictional coefficients. The cases of infinite and zero frictional coefficients represent respectively upper and lower bounds for the flexural stiffness of the umbilical.

The analytical results for the structural response under axi-symmetrical and flexural loads are compared with experimental data. For axi-symmetrical loads the experimental data compared include the axial and torsional stiffnesses of four different types of umbilicals. These umbilicals represent a range of different construction designs and allow the influence of different design parameters to be analysed. Such design parameters relate to the number of armour layers, the helix angle of the armour wires and the layout of the core. The experimental data also includes some results to predict the equivalent mechanical properties of the core. These experiments relate to the radial
stiffness of the hydraulic hoses for loads representative of the in-situ contact pressure generated by the constriction of the armour wires.

3.1 Axi-symmetric Loads

Umbilicals are installed and operated under specified top end tensile forces. These tensile forces are required to support the weight of the umbilical and the hydrodynamic forces induced by waves and currents. To assess the influence of these top end tensile forces and other axi-symmetric loads on the mechanics of the different components, a relationship between these loads and the deformation of the umbilical needs to be established. This relationship constitutes the structural response of subsea umbilicals.

The relationship between the axi-symmetric loads and the deformation of the umbilical is usually expressed in matrix form which represents the structural stiffness matrix. Due to the helical configuration of the armour wires and other components in the umbilical assembly, this stiffness matrix contains non-diagonal terms due to the coupling between the different deformation variables.

The variational energy approach is used in this section to model the structural response of subsea umbilicals for axi-symmetric loads. The variational energy approach is based on imposing an admissible virtual displacement on the structure and finding the corresponding external forces required for equilibrium. Assuming the umbilical is made of helical armour layers, cylindrical sheath layers and a uniform core, the total strain energy stored in the umbilical is given by:

\[ U = U_h + U_s + U_c \]  
\( (3.1) \)
where the subscripts \( h \), \( s \) and \( c \) are used to designate the helical armour layers, the polymeric outer sheath layer and the core respectively. The work of the external body and surface forces is given by:

\[
W = F_z u_z + F_\phi u_\phi + 2\pi R L F_r u_r
\]  
(3.2)

where \( F_z \), \( F_\phi \) and \( F_r \) are the axial force, torsional moment and external hydrostatic pressure, \( u_z \), \( u_\phi \) and \( u_r \) are the axial displacement, twist rotation and radial displacement, and \( R \) and \( L \) are the outer radius and the length of the umbilical. The total potential energy \( \Pi \) is thus given by:

\[
\Pi = U - W
\]  
(3.3)

where \( U \) is the internal strain energy and \( W \) is the work done by the external body and surface forces. The variation of the total potential is then given by:

\[
\delta \Pi = \delta U - \delta W = 0
\]  
(3.4)

The variation of the total potential energy leads to the equilibrium equations and the boundary conditions.

3.1.1 Strain Energy of Helical Layers

The axial strain, change in twist and curvature of a helical wire as a function of the deformation the umbilical were derived in chapter two. The axial strain in a helical wire \( \varepsilon \) (\( \varepsilon = du_s/ds \)) is given by:

\[
\frac{du_s}{ds} = \cos^2 \alpha \frac{du_z}{dz} + R \sin \alpha \cos \alpha \frac{du_\phi}{dz} + \frac{u_R}{R} \sin^2 \alpha
\]  
(3.5)

where \( u_s \) is the axial displacement along the centre line of the wire, \( \alpha \) and \( R \) are the helix angle and radius respectively, and \( u_z \), \( u_\phi \) and \( u_R \) are the axial displacement, twist rotation and radial displacement. The changes in twist and binormal curvature of the helical wire are given by:
Δτ = \frac{\sin \alpha \cos \alpha}{R} \left\{ \left(1 - 2 \cos^2 \alpha\right) \frac{du_z}{dz} - 2 \sin^2 \alpha \frac{u_R}{R} \right\} + \frac{R}{\tan \alpha} \left(1 - 2 \sin^2 \alpha\right) \frac{du_\phi}{dz} \right\} \tag{3.6}

\Delta \kappa_b = \frac{\sin^2 \alpha}{R} \left\{ \left(1 - 2 \sin^2 \alpha\right) \frac{u_R}{R} - 2 \cos^2 \alpha \frac{du_z}{dz} \right\} - 2R \sin \alpha \cos \alpha \frac{du_\phi}{dz} \right\} \tag{3.7}

where Δτ is the change in twist of the helical wire and Δκ_b is the change in the binormal curvature. Assuming that the umbilical is made of i helical layers and each helical layer in turn is made of m wires, the strain energy stored in the helical layers is then given by:

\[ U_h = \sum_i \frac{m_i}{2 \cos \alpha_i} \int_L \left( EA_i (\varepsilon_i)^2 + EI_{b,i} (\Delta \kappa_i)^2 + GJ_i (\Delta \tau_i)^2 \right) dz \tag{3.8} \]

where EA, EI_b and GJ are the axial, binormal bending and torsional stiffnesses of the wire respectively. The contribution of the bending and twisting terms will be of the order \( r_i^2 / R_i^2 \) smaller than the contribution of the axial strain terms where \( r_i \) is the radius of the armour wire and \( R_i \) is the helix radius of the armour layer. For typical helical angles and wire sizes of the armour layers in subsea umbilicals these terms are of second order and can be ignored. The expression for the internal strain energy can then be simplified to:

\[ U_h = \sum_i \frac{m_i}{2 \cos \alpha_i} \int_L EA_i (\varepsilon_i)^2 dz \tag{3.9} \]

Substituting for the axial strain in the helical wire, the variation of the strain energy in the helical layers is given by:

\[ \delta U_h = \sum_i \frac{m_i EA_i}{\cos \alpha_i} \left( \cos^2 \alpha_i \delta u_z + R_i \sin \alpha_i \cos \alpha_i \delta u_\phi + L \frac{\sin^2 \alpha_i}{R_i} \delta u_r \right) \varepsilon_i \tag{3.10} \]
The above general expression for the variation of the internal strain energy in the helical layers uses the assumption that the radial displacement is the same for all the helical layers. This assumption is valid provided all the helical layers remain in contact. If this is not the case then a sub-structuring approach can be employed as discussed later in this chapter. The sub-structuring approach entails that layers in contact are considered a sub-structure. Tan (1992) used this approach to model numerically the structural response of generic flexible structures.

### 3.1.2 Strain Energy of the Sheath Layer

The sheath layer is modelled as a thin walled cylindrical shell. The stress-strain relationships for thin walled shells under axi-symmetric loading are given by:

\[
\sigma_z = \frac{E}{(1-\nu^2)} \left[ \frac{du_z}{dz} + \frac{u_R}{R} \right] \tag{3.11}
\]

\[
\sigma_\theta = \frac{E}{(1-\nu^2)} \left[ \frac{u_R}{R} + \nu \frac{du_z}{dz} \right] \tag{3.12}
\]

\[
\sigma_{z\theta} = GR \frac{du_\phi}{dz} \tag{3.13}
\]

where \( \sigma_z \), \( \sigma_\theta \) and \( \sigma_{z\theta} \) are the stress in the longitudinal direction of the shell, the hoop stress and shear stress, \( E \) and \( G \) are the Young’s and shear moduli, \( \nu \) is Poisson’s ratio and \( R \) is the mean radius of the sheath layer. The internal strain energy of the sheath layer is then given by:

\[
U_s = \frac{\pi R_s t_s E_s}{(1-\nu_s^2)} \int \left( \frac{du_z}{dz} + \nu_s \frac{u_R}{R_s} \right) \frac{du_z}{dz} dz + \pi R_s^3 t_s G_s \int \left( \frac{du_\phi}{dz} \right)^2 dz + \frac{\pi R_s t_s E_s}{(1-\nu_s^2)} \int \left( \frac{u_R}{R_s} + \nu_s \frac{du_z}{dz} \right) \frac{u_R}{R_s} dz 
\]  

\tag{3.14}
where the subscript $s$ refers to the sheath layer and $R_s$ and $t_s$ are the radius and the thickness of the layer. The variation of the strain energy is thus given by:

$$
\delta U_s = \frac{2\pi R_s t_s E_s}{(1 - \nu_s^2)} \left( \frac{du_z}{dz} + \nu_s \frac{u_R}{R_s} \right) \delta u_z + 2\pi R_s^2 t_s G_s \frac{du_\phi}{dz} \delta u_\phi \\
+ \frac{2\pi t_s E_s}{(1 - \nu_s^2)} \left( \frac{u_R}{R_s} + \nu_s \frac{du_z}{dz} \right) \delta u_R
$$

(3.15)

It should be pointed that in the above formulation, no distinction is made between the radial displacement of the sheath layer and the radial displacement of the helical layers. As shown later, such a distinction will only influence the structural response when the global loads acting on the umbilical result in separation of the layers. In this case the sub-structuring approach and the radial displacement of the helical layers in a sub-structure will be either equal to the radial displacement of the sheath layer or the radial displacement of the core. In the case where the layers do remain in contact the radial displacement of all the layers, including the outer sheath layer, will be equal to the radial displacement of the core. This approximation is based on ignoring second order terms related to the change in the thickness of the armour wires.

### 3.1.3 Strain Energy of the Core

The core consists of the sheathed bundle of hydraulic hoses, electrical cables and filler rods. Assuming the core is a uniform cylinder with a given equivalent elastic modulus and dilation coefficient, then the stress-strain equations are given by:

$$
\sigma_z = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \left[ \frac{du_z}{dz} + \frac{2\nu}{1 - \nu} \frac{u_r}{r} \right]
$$

(3.16)
\[ \sigma_\theta = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{u_r}{r} + \nu \frac{du_z}{dz} \right] \]  
\[ \sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{u_r}{r} + \nu \frac{du_z}{dz} \right] \]  
\[ \sigma_{r\theta} = Gr \frac{du_\phi}{dz} \]

where \( r \) is the radial distance, \( E \) and \( \nu \) are the equivalent modulus and dialation coefficient of the core and \( G \) is the equivalent torsional modulus. The dialation coefficient is analogous to Poisson's ratio and represents the ratio of lateral strain associated with a given axial strain. The variation of the internal strain energy of the core is thus given by:

\[ \delta U_c = \frac{2\pi R_c LE_c}{(1 + \nu_c)(1 - 2\nu_c)} \left( \frac{u_R}{R_c} + \nu_c \frac{du_z}{dz} \right) \delta u_R \]

\[ + \frac{\pi R_c^2 E_c (1 - \nu_c)}{(1 + \nu_c)(1 - 2\nu_c)} \left( \frac{du_z}{dz} + \frac{2\nu_c}{1 - \nu_c} \frac{u_R}{R_c} \right) \delta u_z \]

\[ + \frac{1}{2} \frac{\pi R_c^4 G_c}{dz} \delta u_\phi \]

where the subscript \( c \) refers to the core and \( R_c \) is the outer radius of the core. The above expressions for the strain energy in the different constituent layers of the umbilical are used to derive the equilibrium equations of the umbilical for different boundary conditions. The core in the above model was assumed to be a uniform core with given material properties. It is shown later that this assumption does not lead to erroneous results given that the only dominant material properties of the core that influence the structural response of the umbilical are the equivalent radial modulus and the corresponding dialation coefficient.
3.1.4 Equilibrium Equations

The variational energy formulation discussed above can be used to derive the equilibrium equations of the umbilical. Substituting equations (3.10), (3.15) and (3.20) into equation (3.4), the equilibrium equations are given by:

\[
F_z = \sum_i m_i E A_i \cos \alpha_i \varepsilon_i + \left( 2\nu_s \pi t_s E_s \div (1 - \nu_s^2) + \frac{2\nu_c \pi R_c E_c}{(1 + \nu_c)(1 - 2\nu_c)} \right) u_R
\]

\[
+ \left( \frac{2\pi R_s t_s E_s}{(1 - \nu_s^2)} + \frac{\pi R_c^2 E_c}{(1 + \nu_c)(1 - 2\nu_c)} \right) du_z \div dz
\]

\[
F_\phi = \sum_i m_i E A_i R_i \sin \alpha_i \varepsilon_i + \left( 2\pi R_s^3 t_s G_s + \frac{1}{2} \pi R_c^4 G_c \right) \frac{du_\phi}{dz}
\]

\[
2\pi R F_r = \sum_i \frac{m_i E A_i \sin^2 \alpha_i \varepsilon_i}{R_i \cos \alpha_i} + \left( \frac{2\pi t_s E_s}{(1 - \nu_s)} + \frac{2\pi E_c}{(1 + \nu_c)(1 - 2\nu_c)} \right) u_R
\]

\[
+ \left( \frac{2\nu_s \pi R_s t_s E_s}{(1 - \nu_s^2)} + \frac{2\nu_c \pi R_c E_c}{(1 + \nu_c)(1 - 2\nu_c)} \right) du_z \div dz
\]

where the strain in the helical wire \( \varepsilon_i \) is given by equation (3.5). It should be emphasised that the above equations were derived assuming the different layers remain in contact. If this is not the case then the above equations should be modified to take into account the separation of layers by using a sub-structuring approach. The sub-structuring approach entails assuming the umbilical to be made of a number of sub-structures. The above equations are then used for each of the sub-structures and the stiffness matrix of the umbilical is assembled taking into account the fact that the axial strain and the twist deformation of these sub-structures is identical to the axial strain and the twist of the umbilical. The approach of sub-structuring was used by Witz and Tan (1992a) to study the axi-symmetric structural response of generic flexible structures.
To illustrate the sub-structuring approach the case of an umbilical subject to torsional moments is considered. The umbilical considered is made of an external sheath layer, two armour layers and a core as shown in Figure 3.4. The umbilical is assumed to be in a flooded condition and in this case the net external pressure load $F_r$ is zero. This is the case since in a flooded condition the pressure on the inner and outer surfaces of the sheath layer, which is modelled as a thin cylindrical shell, are of equal magnitude. For this loading condition it is noted that the selection of the sub-structures depends on the direction of the torsional moments but for either direction of the torque the umbilical consists of two sub-structures. For this illustrative example it is assumed that the outer armour layer is a right handed helix and the inner armour layer is a left handed helix. Then for an anti-clockwise torque the outer armour layer will unwind while the inner armour layer will tighten. As the outer armour layer unwinds, the helix angle decreases and as a result the path of the centre line of the armour wire will stretch. The helix angle in the deformed geometry $\bar{\alpha}$ is given by:

$$\tan \bar{\alpha} = \frac{1 + \frac{u_R}{R} + \frac{\tan \alpha}{R} \frac{du_\phi}{dz} \tan \alpha}{1 + \frac{du_z}{dz}}$$

Assuming that the ends of the umbilical are restrained against axial elongation, compressive forces develop in the wires and as a result the resultant contact radial pressure will act outwards. While the stresses in the outer armour wires will be compressive, the stresses in the inner armour wires will be tensile. These tensile stresses arise since anti-clockwise torque will tend to tighten this layer and as a result the helix angle will increase. As the inner armour layer tightens the forces in the wires will be tensile and the resultant radial contact pressure will act inwards.

Due to the opposing directions of the radial contact forces, the two armour layers separate. As a result, the outer sheath layer and the outer armour layer
form one sub-structure and the core and the inner armour layer form the other sub-structure. The equilibrium equations for the first sub-structure made of the outer armour layer and the sheath layer can then be written as:

\[
F_z^{(1)} = m_{out} EA_{out} \cos \alpha_{out} \varepsilon_i + \frac{2\nu \pi t_i E_s}{(1 - \nu_s^2)} u_R^{(1)}
\]

(3.25)

\[
F_\phi^{(1)} = m_{out} EA_{out} R_{out} \sin \alpha_{out} \varepsilon_{out} + 2\pi R_s^3 G_s \frac{du_\phi}{dz}
\]

(3.26)

\[
2\pi R_s F_{r,s} = \frac{2\pi t_s R_s}{(1 - \nu_s^2)} \left( u_R^{(1)} + \nu_s R_s \frac{du_z}{dz} \right) + \frac{m_{out} EA_{out} \sin^2 \alpha_{out}}{R_{out} \cos \alpha_{out}} \varepsilon_{out}
\]

(3.27)

where \(out\) denotes the outer armour layer and (1) denotes this first sub-structure. The equilibrium equations of the second sub-structure made of the core and the inner armour layer can be written as:

\[
F_z^{(2)} = m_{in} EA_{in} \cos \alpha_{in} \varepsilon_{in}
\]

\[
+ \frac{\pi R_c^2 E_c (1 - \nu_c)}{(1 + \nu_c)(1 - 2\nu_c)} \frac{du_z}{dz} + \frac{2\nu_c \pi R_c E_c}{(1 + \nu_c)(1 - 2\nu_c)} u_R^{(2)}
\]

(3.28)

\[
F_\phi^{(2)} = m_{in} EA_{in} R_{in} \sin \alpha_{in} \varepsilon_{in} + \frac{1}{2} \frac{\pi R_c^4 G_c}{(1 + \nu_c)(1 - 2\nu_c)} \frac{du_\phi}{dz}
\]

(3.29)

\[
2\pi R_c F_{r,c} = \frac{2\pi E_c}{(1 + \nu_c)(1 - 2\nu_c)} u_R^{(2)} + \nu_c R_c \frac{du_z}{dz} + \frac{m_{in} EA_{in} \sin^2 \alpha_{in}}{R_{in} \cos \alpha_{in}} \varepsilon_{in}
\]

(3.30)

where \(in\) denotes the inner armour layer and (2) denotes this second sub-structure. Given that the total external axial forces and torsional moments are the sum of the external forces and torsional moments in the two sub-structures, the above equilibrium equations can be solved to assess the structural response of the umbilical under axi-symmetric loads.

For the other case where the direction of torque is clockwise the stresses in the outer armour wires are tensile and the the resultant contact radial forces are inwards. The resultant contact radial forces of the inner armour layer are outwards. Assuming that the magnitude of the inward contact radial forces
generated by the outer armour layer is greater than the outward contact radial forces generated by the inner armour layer, the umbilical can be divided into two sub-structures. The first sub-structure is made of the outer sheath layer and the second sub-structure is made of the two armour layers and the core. The equilibrium equations for the first sub-structure are then given by:

\[
F_z^{(1)} = \frac{2\pi R_s t_s E_s}{(1 - v_s^2)} \frac{du_z}{dz} + \frac{2\nu_s \pi t_s E_s}{(1 - v_s^2)} u_R^{(1)}
\]

(3.31)

\[
F_\phi^{(1)} = 2\pi R_s^3 t_s G_s \frac{d\phi}{dz}
\]

(3.32)

\[
2\pi R_s F_{r,s} = \frac{2\pi t_s E_s}{(1 - v_s^2)} \left( u_R^{(1)} + \nu_s R_s \frac{du_z}{dz} \right)
\]

(3.33)

Similarly the equilibrium equations of the second sub-structure made of the core and the two armour layers can be written as:

\[
F_z^{(2)} = \left\{ \sum_i m_i E_A \cos^3 \alpha_i + \frac{\pi R_c^2 E_c (1 - v_c)}{(1 + v_c)(1 - 2v_c)} \right\} \frac{du_z}{dz} + \sum_i m_i E_A \cos \alpha_i \sin \alpha_i \frac{du_z}{dz} + \sum_i m_i E_A \sin^3 \alpha_i u_R^{(2)}
\]

(3.34)

\[
F_\phi^{(2)} = \sum_i m_i E_A R_i \cos \alpha_i \sin \alpha_i \frac{du_z}{dz} + \sum_i m_i E_A \sin \alpha_i u_R^{(2)}
\]

(3.35)

\[
2\pi R_c F_{r,c} = \sum_i \frac{m_i E_A \sin^2 \alpha_i}{R_i \cos \alpha_i} e_i + \frac{2\pi E_c}{(1 + v_c)(1 - 2v_c)} u_R^{(2)} + \nu_c R_c \frac{du_z}{dz}
\]

(3.36)
Given the differences in the equilibrium equations for the case of clockwise and anti-clockwise torque, the structural response will be a non-linear function and it will be influenced by the direction of twist. It should be noted that equation (3.36) for the case when the umbilical is in a flooded condition can be written as:

\[
\frac{1}{2\pi R_c} \sum_i m_i E A_i \sin^2 \alpha_i \cos \alpha_i \xi_i = \frac{E_c}{(1 + \nu_c)(1 - 2\nu_c)} \left( \frac{\mu_R^{(2)}}{R_c} + \nu_c \frac{du_z}{dz} \right) 
\]

The right hand side of above equation is simply the radial pressure on the core which is also given in equation (3.18). Thus the above expression shows that the radial pressure on the core is the sum of the line contact pressure generated by all the helical wires divided by the circumference of the core. The above expression also shows that the line contact pressure generated by a helical armour wire is the product of the tension in the wire and the initial curvature of the wire. The tension in a wire is given by \( E A \varepsilon \), the curvature is given by \( \sin^2 \alpha / R \) and the circumference is given by \( 2\pi R_c \). In this respect, the more complex expression for the line contact pressure derived in chapter two and given by:

\[
X = \left\{ G J \Delta \tau \kappa_b - E I_b \Delta \kappa_b \right\} \bar{\tau} - T \kappa_b 
\]

is not within the accuracy employed in developing this model. This is due to the fact that the second order terms related to bending and twist of the helical wire were ignored in the expression for the strain energy presented in equation (3.9). This assumption is valid since in a subsea umbilical, unlike the case in a wire strand, the radius of the wire is small compared to the radius of the helix. In a wire strand the radius of the wire is half the radius of the helix. In a subsea umbilical, this ratio is at least an order of magnitude smaller than a half.

For the above analytical approach the sub-structures have to be identified. These sub-structures can be identified for single loading conditions like axial
tension, torsion or uniform external pressure. For applied torque it was explained earlier that the umbilical can be divided into two sub-structures. For tensile loads the umbilical can also be divided into two sub-structures. In this case the outer sheath layer forms one sub-structure and the other layers of the umbilical form the other sub-structure. This is because the radial contact pressure generated by the two armour layers will be inwards and as a result these radial forces will constrict the core.

For combined loads predicting the separation of layers is not as straightforward. This is illustrated by considering the case of an umbilical under combined axial and torsional loads. To simplify the illustration of this phenomenon the umbilical is assumed to be made of an external sheath layer, a single armour layer and a core. Assuming the umbilical is initially tensiled and then twisted as to unwind the helical layer, the contact pressure on the core will progressively be decreased. During this stage the umbilical torque-twist relation will follow a linear relation. At one stage when the stresses in the armour wires are compressive, the armour wires will start to exert an outward contact pressure and will distend the outer sheath. This would result in a marked difference in the torque-twist relationship as the radial stiffness of the sheath layer will not be equivalent to the radial stiffness of the inner core. To assess the consequences of the opposing loading condition where the umbilical is initially twisted and then loaded axially the same argument can be used. Here, it is seen that as the umbilical is tensiled the internal contact pressure on the outer sheath layer is gradually reduced and at a certain tensile load this pressure would be zero. With a further increase in the axial tension, the stresses in the armour wires become tensile and the armour wires exert a radial contact pressure on the core. This point will again mark a difference in the tension-extension relationship.
3.2 Analytical Stiffness Matrix

The equilibrium equations for subsea umbilicals and other similar flexible structures were derived in equations (3.21), (3.22) and (3.23). The equilibrium equations, assuming all the layers remain in contact, represent a system of equations that can be solved to calculate the structural response of the umbilical for given loading conditions. However, the difficulty in analysing these structures is the fact that the structural response of the umbilical is dependent on the loading conditions and the separation of layers. The separation of the layers result from the constriction of the core due to the radial contact pressures generated by the helical armour layers. The separation of the layers introduces additional unknowns into the equilibrium equations and as a result the equilibrium equations become indeterminate. To solve this indeterminate set of equilibrium equations an iterative technique is often employed. This iterative technique involves the imposition of a certain displacement on the umbilical assembly and, as a first step in the iteration, calculating the axial stresses in the helical and cylindrical elements of the umbilical ignoring the radial constriction of the core. The iterations then proceed by calculating the radial constriction of the core and adjusting the computed stresses in the elements. Knapp (1979), Feld (1992) and Witz and Tan (1992a) have used such an iterative technique. Knapp (1979) and Feld (1992) studied the structural response of high voltage power cables that were idealised as made of a core and a number of armour layers. They ignored the influence of separation of the layers as high voltage cables are often operated under high tensile forces and minimal twist deformation. Such loading conditions ensure the layers of the cable remain in contact. Witz and Tan (1992a) took account of the separation of layers and the sub-structuring approach discussed earlier was used.

In this chapter an analytical approach to solve the indeterminate equilibrium equations is developed. The analytical model takes into account the
separation of layers by using a sub-structuring approach. This approach allows the umbilical to be modelled as made of a number of sub-structures. The equilibrium equations can be solved for each sub-structure and the structural response of the umbilical can be calculated using the constraint that the axial strain and twist rotation of these sub-structures is equal to the axial strain and twist of the umbilical. The structural response will differ depending on the separation of layers. The underlying mathematical formulation of the sub-structuring technique is writing the equilibrium equation for each sub-structure \( n \) as:

\[
\begin{bmatrix}
F_z \\
F_\phi \\
F_r 
\end{bmatrix}^n =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}^n
\begin{bmatrix}
\ddot{u}_z \\
\ddot{u}_\phi \\
u^n_R
\end{bmatrix}
\]

(3.39)

where \( a_{ij} \) are the coefficients of the stiffness matrix of the sub-structure, dot indicates differentiation with respect to \( z \) and \( n \) denotes the number of the sub-structure. The coefficients of the stiffness matrix can be obtained from equations (3.21), (3.22) and (3.23) which were derived assuming that all the layers of the umbilical remain in contact. When all the layers remain in contact the umbilical is considered to be made of one sub-structure. If this is not the case and the layers of the umbilical separate, the expressions for these coefficients depend on whether the armour layers constrict the core or distend the outer sheath layer. In the former case the coefficients are given by:

\[
a_{11} = \sum_i m_i E A_i \cos^3 \alpha_i + \frac{\pi R_c^2 E_c (1 - \nu_c)}{(1 + \nu_c)(1 - 2\nu_c)}
\]

(3.40)

\[
a_{12} = a_{21} = \sum_i m_i E A_i R_i \cos^2 \alpha_i \sin \alpha_i
\]

(3.41)

\[
a_{13} = 2\pi R_c a_{31} = \sum_i \frac{m_i E A_i}{R_i} \cos \alpha_i \sin^2 \alpha_i + \frac{2\nu_c \pi R_c E_c}{(1 + \nu_c)(1 - 2\nu_c)}
\]

(3.42)

\[
a_{22} = \sum_i m_i E A_i R_i^2 \sin^2 \alpha_i \cos \alpha_i + \frac{1}{4} \pi R_c^4 G_c
\]

(3.43)
\[ a_{23} = 2\pi R_c a_{32} = \sum_i m_i E A_i \sin^3 \alpha_i \]  
\[ a_{33} = \frac{1}{2\pi R_c} \left\{ \sum_i m_i E A_i \frac{\sin^4 \alpha_i}{R_i^2 \cos \alpha_i} + \frac{2\pi E_c}{(1 + \nu_c)(1 - 2\nu_c)} \right\} \] (3.44)

and in the later case the coefficients are given by:

\[ a_{11} = \sum_i m_i E A_i \cos^3 \alpha_i + \frac{2\pi R_s t_s E_s}{1 - \nu_s^2} \] (3.46)

\[ a_{12} = a_{21} = \sum_i m_i E A_i R_i \cos^2 \alpha_i \sin \alpha_i \] (3.47)

\[ a_{13} = 2\pi R_s a_{31} = \sum_i m_i E A_i \cos \alpha_i \sin^2 \alpha_i + \frac{2\pi \nu_s R_s E_s}{1 - \nu_s^2} \] (3.48)

\[ a_{22} = \sum_i m_i E A_i R_i^2 \sin^2 \alpha_i \cos \alpha_i + 2\pi R_s^3 t_s G_s \] (3.49)

\[ a_{23} = 2\pi R_s a_{32} = \sum_i m_i E A_i \sin^3 \alpha_i \] (3.50)

\[ a_{33} = \frac{1}{2\pi R_s} \left\{ \sum_i m_i E A_i \frac{\sin^4 \alpha_i}{R_i^2 \cos \alpha_i} + \frac{2\pi t_s E_s}{1 - \nu_s^2} \right\} \] (3.51)

Equation (3.39), using matrix partitioning, leads to:

\[
\begin{bmatrix}
F_z^n \\
F_\phi^n
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}^n
\begin{bmatrix}
\dot{u}_z \\
\dot{u}_\phi
\end{bmatrix} + \frac{1}{a_{33}^n} \begin{bmatrix}
a_{13} \\
a_{23}
\end{bmatrix}^n u_R^n
\] (3.52)

\[ u_R^n = \frac{1}{a_{33}^n} \left\{ F_r - \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}^n \begin{bmatrix} \dot{u}_z \\
\dot{u}_\phi
\end{bmatrix} \right\} \] (3.53)

Substituting for \( u_R^n \) into equation (3.52), the equilibrium equations of the substructure can be expressed as:

\[
\begin{bmatrix}
F_z^n \\
F_\phi^n
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}^n
\begin{bmatrix}
\dot{u}_z \\
\dot{u}_\phi
\end{bmatrix} + \frac{1}{a_{33}^n} \begin{bmatrix}
a_{13} \\
a_{23}
\end{bmatrix}^n \left\{ F_r - \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}^n \begin{bmatrix} \dot{u}_z \\
\dot{u}_\phi
\end{bmatrix} \right\} \] (3.54)
The above equation can be factorised and expressed as:

$$
\left[ \begin{array}{c}
F_z - \frac{a_{13}F_r}{a_{33}} \\
F_\phi - \frac{a_{23}F_r}{a_{33}}
\end{array} \right]^n = \left[ \begin{array}{ccc}
a_{11} - \frac{a_{31}a_{13}}{a_{33}} & a_{12} - & a_{13}a_{32} \\
a_{21} - \frac{a_{23}a_{31}}{a_{33}} & a_{22} - & a_{23}a_{31}
\end{array} \right]^n \begin{bmatrix}
\ddot{u}_z \\
\ddot{u}_\phi
\end{bmatrix}
$$

(3.55)

Using the constraints that the axial deformation and twist rotation of all the sub-structures are identical to the axial deformation and twist deformation of the umbilical, the structural response of the umbilical can then be evaluated from the following expression:

$$
F = \sum_n F^n
$$

(3.56)

where $F$ is the axial-torsional load matrix of the umbilical and $F^n$ is the axisymmetric load matrix of sub-structure $n$ and is as given in the following equation:

$$
F^n = K^n u
$$

(3.57)

where $K^n$ is the coefficient matrix of equation (3.55). The advantages of the above analytical solution include an efficient algorithm which avoid the inherent difficulties of non-convergence of numerical models. This is also a more versatile and adaptable approach to deal with different loading scenarios.

### 3.3 Core Material Properties

The above analytical model for the structural response of subsea umbilical was developed assuming the core is a uniform cylindrical rod. In reality, the core is a composite cylinder which consists of a cylindrical inner sheath layer, hydraulic hoses, electrical cables and filler rods. However, closer examination of the equilibrium equations shows that the equivalent radial modulus and corresponding dialation coefficient of the core are the only dominant
mechanical properties affecting the structural response of the umbilical. To illustrate this point, each coefficient of the stiffness matrix which is dependent on the material properties of the core is examined. These coefficients are:

\[ a_{11} = \sum_i m_i EA_i \cos^3 \alpha_i + \frac{\pi R_c^2 E_c (1 - v_c)}{(1 + v_c)(1 - 2v_c)} \]  (3.58)

\[ a_{13} = 2\pi R_c a_{31} = \sum_i m_i \frac{EA_i}{R_i} \cos \alpha_i \sin^2 \alpha_i + \frac{2v_c \pi R_c E_c}{(1 + v_c)(1 - 2v_c)} \]  (3.59)

\[ a_{22} = \sum_i m_i EA_i R_i^2 \sin^2 \alpha_i \cos \alpha_i + \frac{1}{4} \pi R_c^4 G_c \]  (3.60)

\[ a_{33} = \frac{1}{2\pi R_c} \left\{ \sum_i m_i EA_i \frac{\sin^4 \alpha_i}{R_i^2 \cos \alpha_i} + \frac{2\pi E_c}{(1 + v_c)(1 - 2v_c)} \right\} \]  (3.61)

For this illustration the umbilical core radius is assumed to be 50 mm, and the material modulus and dilatation coefficient are assumed to be 600 MPa and 0.42 respectively. The umbilical is also assumed to be made of one single helical layer. The diameter of the armour wire and the helix angle of the armour wire are assumed to be 5 mm and 30 degrees respectively.

Using the above parameters, the first term in equation (3.58) is equivalent to 144 MN while the second term is equivalent to 12 MN. The contribution of the core in this equation is thus one order of magnitude smaller than the contribution of the helical armour wires. The same argument hold for equation (3.59) and equation (3.60) where the contribution of the core is one order of magnitude smaller than the contribution of the helical armour wires. It should be pointed out that for the different coefficients of the stiffness matrix, the choice of the design parameters used in the above example were selected as to minimise the contribution of the helical armour wires and at the same time maximise the contribution of the core. Thus, for typical umbilicals the contribution of the armour wires in equations (3.58), (3.59) and (3.60) would be significantly greater than the contribution of the core and as a result
the contribution of the core can be ignored without introducing any significant errors.

However, for the design parameters used above the contribution of the core in equation (3.61) is comparable to the contribution of the helical armour wires. The contribution of the core is 16592 MPa and the contribution of the armour wires is 5800 MPa. Given the above, it can be concluded that the only material properties of the core that affect the structural response of the umbilical are the equivalent radial modulus and the dialation coefficient of the core. A methodology to evaluate these equivalent properties is discussed below.

For typical subsea umbilicals the centre of the core is occupied by the electrical bundle. The hoses and filler rods are laid helically around the electrical bundle and a sheathed layer is extruded over these hoses as shown in Figure 3.1 presented earlier. To evaluate the equivalent radial modulus and dialation coefficient, the core can thus be idealised as made of a rigid central element, a set of linear springs representing the hoses and filler rods and a sheath layer. This idealised model of the core is shown in Figure 3.5. The equations of equilibrium for the sheath layer of the idealised core can be written as:

\[
    P_{\text{out}} - P_{\text{in}} = \frac{E_s t_s}{R_s} \sigma_\theta = \frac{E_s t_s}{R_s(1 - \nu^2)} \left( \frac{u_R}{R_s} + \nu_s \frac{du_z}{dz} \right)
\]

where \( P_{\text{out}} \) and \( P_{\text{in}} \) are the external and internal pressures, \( t_s \) and \( R_s \) are the thickness and mean radius of the layer, \( \sigma_\theta \) is the hoop stress, \( E_s \) is the material modulus, \( \nu_s \) is the Poisson’s ratio and \( u_R \) is the radial displacement. Assuming the hoses and filler rods represent a series of springs, the equilibrium equation for these springs is given by:

\[
    P_{\text{in}} = \frac{u_R}{2\pi R_s} \sum k_i
\]
where $k_i$ is the stiffness of spring $i$. The above equation shows that the contact pressure on the springs is assumed to be the sum of the forces in the springs divided by the circumference of the sheath layer. Substituting equation (3.63) in equation (3.62), the equilibrium equation of the composite structure can be written as:

$$P_{out} = \left[ \frac{E_s t_s}{R_s (1 - v_s^2)} + \frac{1}{2\pi} \sum k_i \right] \frac{u_R}{R_s} + \frac{v_s E_s t_s}{R_s (1 - v_s^2)} \frac{du_z}{dz} \quad (3.64)$$

Assuming that the idealised core is equivalent to a homogeneous circular rod with an identical outer diameter, the pressure-radial displacement of this rod should be equal to the pressure-radial displacement of the idealised core. The equilibrium equations for a circular rod and a given external pressure $P_{out}$ is given by:

$$P_{out} = \frac{E_c}{(1 + v_c)(1 - 2v_c)} \frac{u_R}{R_s} + \frac{v_c E_c}{(1 + v_c)(1 - 2v_c)} \frac{du_z}{dz} \quad (3.65)$$

The equivalent radial modulus and dilatation coefficient of the core can thus be written as:

$$\lambda = \frac{E_c}{(1 + v_c)(1 - 2v_c)} = \left\{ \frac{E_s t_s}{R_s (1 - v_s^2)} + \frac{1}{2\pi} \sum k_i \right\} \quad (3.66)$$

$$\nu_c = \left(1 + \frac{R_s (1 - v_s^2)}{2\pi E_s t_s} \sum k_i \right)^{-1} v_s \quad (3.67)$$

The equivalent modulus of the core as expressed in the above equation represents an analytical approximation to predict the equivalent radial modulus and dilatation coefficient. The equivalent radial modulus of the core is a function of the physical and material properties of the sheath layer and is also a function of the stiffness of the hoses and filler rods. For the limiting case where the stiffness of the springs is zero, the equivalent radial stiffness and the dilatation coefficient of the core are equal to the radial stiffness and the Poisson’s ratio of the sheath layer.
It is pointed out the use of equivalent modulus properties to model the core is advantageous over other techniques often employed such as the use of the apparent Poisson’s ratio (Waloen et al, 1992). This apparent Poisson’s ratio is specified to be dependent on the design and manufacture, but means of calculating this apparent ratio has never been formulated and its value is often specified according to the similarity between one structure and another. The value for the apparent Poisson’s ratio range from zero for the case of a rigid core to more than 20 for the case of a soft core. The lack of analytical means to calculate the apparent Poisson’s ratio introduces subjectivity to the analysis methods. Moreover, the use of an apparent Poisson’s ratio as demonstrated later in this thesis does not does take into account the complex interaction of the different helical and cylindrical layers.

It is further pointed out that the methodology used to calculate the equivalent radial modulus and dilatation coefficients does not take into account any local effects due to radially stiff elements within the core. This approach would not lead to erroneous results as under the imposed radial load the core will always deform in a circular manner. In other words the core will not ovalise or experience local deformation. For such a deformation to occur the armour wires will have to undergo considerable shear distortion. Based on the principles of minimum energy, it can be shown that a more likely deformation to accommodate the influence of the stiff elements is unequal radial displacement in the different elements of the core. This is explained with reference to Figure 3.6. This figure shows a cylindrical shell element connected to a rigid central core with a number of springs of different stiffnesses. It is shown that if the shell is constrained to remain circular, the central rigid core will be displaced from its initial position. The new position of this element will be dictated by the requirement to minimise the internal strain energy stored in the springs. In this respect the above methodology can be developed to deal with asymmetric umbilical cross sections.
3.4 Flexural Loads

For axi-symmetric loads an assessment of the structural response is needed to establish a relationship between the loads and the deformation of the umbilical. This relationship is also needed to assess the static configuration and the global dynamic response. The same argument holds for flexural loads. However, while most experimental tests confirm a reasonably linear load-deflection relationship for moderate axi-symmetric loads, the load-deflection relationship for flexural loads is generally non-linear. Typical experimental moment-curvature plots of subsea umbilicals are shown in Figure 3.7. This figure shows that the moment-curvature curve is characterised by an initial region where for small curvatures the umbilical exhibits a high bending stiffness. This region represents the case where the armour wires of the umbilical are in a no slip condition. At a certain curvature limit, which represents the inception of slip at the critical bend radius discussed in chapter two, the bending stiffness decreases rapidly. When all the wires are in a full slip condition, the bending stiffness of the umbilical is some orders of magnitude smaller than the initial bending stiffness before the critical bend radius was reached. To account for the non-linear response under flexural loads a bilinear moment-curvature relationship is often employed. The initial high bending stiffness represents the limiting condition of an infinite friction coefficient and the low bending stiffness represents the limiting case of zero friction. The limiting cases of infinite and zero friction coefficients represent the cases of no slip and full slip respectively.

This section presents an analytical solution for the bending stiffness of subsea umbilicals taking into account finite friction coefficients. The analytical solution is based on the variational energy method presented earlier to analyse the structural response for axi-symmetric loads. The variational energy method leads to the equilibrium equations and is given by:

$$\delta \Pi = \delta U - \delta W = 0 \quad (3.68)$$
where $\Pi$ is the total potential, $U$ is the internal strain energy and $W$ is the work done by the external forces. The internal strain energy $U$ is given as the sum of the strain energies stored in the helical layers, the sheath layers and the core and is given by:

$$ U = U_h + U_s + U_c \quad (3.69) $$

where the subscript $h$, $s$ and $c$ are used to designate the helical armour layers, the sheath layers and the core.

The strain in the helical wires for a given bending radius was discussed earlier in chapter two. Assuming the slip region is bounded by $\pm \phi_t$, the strain $\varepsilon$ ($\varepsilon = du_x/ds$) in a helical wire is given by:

$$ \frac{du}{ds} = \begin{cases} 
- \frac{\mu X R \phi}{EA \sin \alpha} & 0 \leq |\phi| < \phi_t \\
- \frac{R \sin \phi \cos^2 \alpha}{\rho} & \phi_t \leq |\phi| < \frac{\pi}{2} 
\end{cases} \quad (3.70) $$

where $\rho$ is the bending radius of the umbilical, $\phi$ is the angular position of the wire, $R$ and $\alpha$ is the helix radius and angle, $\mu$ and $X$ are the friction coefficient and the contact pressure and $EA$ is the axial stiffness of the wire.

In addition to the axial strain in the helical wires, these elements experience flexural and torsional strains resulting the twist and curvatures in the normal and binormal curvatures. These can be obtained from the projections of the imposed curvature on the umbilical as shown in Figure 2.9 presented in chapter two. The twist and curvatures in the normal and binormal direction are thus given by:

$$ \tau = \frac{1}{\rho} \sin \phi \sin \alpha \quad (3.71) $$

$$ k = \frac{1}{\rho} \cos \phi \quad (3.72) $$
where $\tau$ is the twist, $\kappa_n$ is the curvature in the normal direction and $\kappa_b$ is the curvature in the binormal direction. The internal strain energy stored in a half pitch length of a helical wire is thus given by:

$$U_h = \frac{R}{2 \sin \alpha} \int_{-\pi/2}^{\pi/2} (E A e^2 + EI_n \kappa_n^2 + EI_b \kappa_b^2 + GJ \tau^2) d\phi$$

where $EI_n$, $EI_b$ and $GJ$ are the bending stiffness in the normal direction, the bending stiffness in the binormal direction and the torsional stiffness of the wire respectively.

As mentioned earlier, the structural response of subsea umbilicals under flexural loading exhibits a non-linear load-deformation relationship. The non-linear relationship arises as a result of the spread of the slip region which is given by:

$$\rho = \frac{E A \sin \alpha \cos^2 \alpha \sin \phi_t}{\mu X \Phi_t}$$

where $\phi_t$ is the slip region along the arc length of the wire. To take into account the spread of the slip region an incremental approach is employed. The incremental approach allows the variational energy method to be used assuming that over the small increment in displacement the total energy potential is path independent. Denoting the curvature of the umbilical by $\kappa_u = 1/\rho$ and substituting $\kappa_u = \kappa_u^i + \zeta \kappa_u^i$, where $\kappa_u^i$ is the instantaneous curvature and $\zeta \kappa_u^i$ is a small increment in the curvature, the expression for the strain energy given in equation (3.74) can be expressed as:
\[ U_h = \frac{R}{2 \sin \alpha} \int_0^{\phi_1} \left\{ EA \left( -\frac{\mu X R}{EA \sin \alpha} \right)^2 \phi^2 \right\} d\phi \]
\[ + \frac{R}{2 \sin \alpha} \int_{\phi_t}^{\pi/2} \cos \alpha \sin^2 \phi \left( \kappa_u^j + \xi \kappa_u^j \right)^2 d\phi \]
\[ + \frac{R}{2 \sin \alpha} \int_0^{\pi/2} E I_b \cos^2 \phi \left( \kappa_u^j + \xi \kappa_u^j \right)^2 d\phi \]
\[ + \frac{R}{2 \sin \alpha} \int_0^{\pi/2} \left( E I_n \cos^2 \alpha + G J \sin^2 \alpha \right) \sin^2 \phi \left( \kappa_u^j + \xi \kappa_u^j \right)^2 d\phi \]

Taking the variation of the strain energy and integrating with respect to \( \phi \), the above expression leads to:

\[ \delta U_h = \frac{R}{\sin \alpha} \frac{E A R^2 \cos^4 \alpha}{2} \left( \frac{\pi}{2} - \phi_t + \frac{1}{2} \sin 2\phi_t \right) \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]
\[ + \frac{\pi R}{2 \sin \alpha} \left\{ E I_b + E I_n \cos^2 \alpha + G J \sin^2 \alpha \right\} \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]

Assuming the umbilical is made of \( i \) helical layers, and each layer is made up of \( m \) wires, the variation of internal strain energy stored in the helical layers of the umbilical is thus given by:

\[ \delta U_h = \sum_i \frac{m_i}{\sin \alpha_i} \frac{E A_i R_i^3 \cos^4 \alpha_i}{2} \left( \frac{\pi}{2} - \phi_{t,i} + \frac{1}{2} \sin 2\phi_{t,i} \right) \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]
\[ + \sum_i \frac{\pi m_i R_i}{2 \sin \alpha_i} \left\{ E I_{b,i} + E I_{b,i} \cos^2 \alpha_i + G J_i \sin^2 \alpha_i \right\} \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]

Similarly, the variational of the internal strain energy stored in the sheath layer and the core are given by:

\[ \delta U_s = E I_s L \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]
\[ \delta U_c = E I_c L \left( \kappa_u^j + \xi \kappa_u^j \right) \delta \left( \xi \kappa_u^j \right) \]
and the variational of the external forces is given by:

$$\delta W = L (M^j + \zeta M^j) (\zeta \kappa_u^j)$$

(3.81)

where $EI_s$ and $EI_c$ are the bending stiffnesses of the sheath and the core, $L$ is the length of the umbilical, $M^j$ is the instantaneous bending moment and $\zeta M^j$ is the increment in the bending moment.

Substituting the expressions for the variation in the internal strain energy and the external body forces into the equation of equilibrium given in equation (3.68), the structural response for flexural loading is given by:

$$\zeta M^j = \frac{2}{\pi} \sum_i m_i E A_i R_i^2 \cos^3 \alpha_i \left\{ \frac{\pi}{2} - \phi_{t,i} + \frac{1}{2} \sin 2\phi_{t,i} \right\} \left( \kappa_u^j + \zeta \kappa_u^j \right)$$

$$+ \sum_i m_i \left\{ EI_{b,i} + (EI_{n,i} \cos^2 \alpha_i + GJ_i \sin^2 \alpha_i) \right\} \left( \kappa_u^j + \zeta \kappa_u^j \right)$$

$$EI_s (\kappa_u^j + \zeta \kappa_u^j) + EI_c (\kappa_u^j + \zeta \kappa_u^j) - M^j$$

(3.82)

The above equation allows an assessment of the bending stiffness taking into account the spread of the slip region $\phi_t$. Assuming the instantaneous moment $M^j$ and curvature $\kappa_u^j$ are in equilibrium, the equilibrium state for any increment $\zeta M^j$ in the applied bending moment is given by:

$$\zeta M^j = \frac{2}{\pi} \sum_i m_i E A_i R_i^2 \cos^3 \alpha_i \left\{ \frac{\pi}{2} - \phi_{t,i} + \frac{1}{2} \sin 2\phi_{t,i} \right\} (\zeta \kappa_u^j)$$

$$+ \sum_i m_i \left\{ EI_{b,i} + (EI_{n,i} \cos^2 \alpha_i + GJ_i \sin^2 \alpha_i) \right\} (\zeta \kappa_u^j)$$

$$+ EI_s (\zeta \kappa_u^j) + EI_c (\zeta \kappa_u^j)$$

(3.83)

For the limiting case of no slip the first term in the right hand side of the above equation provides the dominant contribution to the bending stiffness of the umbilical. Using the assumption that the diameter of the wire $r$ is small in comparison to the radius of the helix $R$, the first term in the right hand side of equation (3.83) for the case of no slip will be of the order of $R^2 / r^2$ greater
that the contribution of the terms including the bending and torsional stiffnesses of the wire. For this limiting case equation (3.83) can be written as:

\[ \zeta_{M}^{j} = \left\{ \sum_{i} m_{i}EA_{i}R_{i}^{2} \cos^{3} \alpha_{i} + EI_{s} + EI_{c} \right\} \zeta_{\kappa}^{j}_{u} \]  
(3.84)

For the case of full axial slip, equation (3.83) can be written as:

\[ \zeta_{M}^{j} = \left\{ \sum_{i} m_{i}(EI_{b,i} + EI_{n,i} \cos^{2} \alpha_{i} + GJ_{i} \sin^{2} \alpha_{i})d \right\} \zeta_{\kappa}^{j}_{u} \]  
(3.85)

\[ + \left( EI_{s} + EI_{c} \right) \zeta_{\kappa}^{j}_{u} \]

and for the case of full axial and rotational slip equation (3.83) can be written as:

\[ \zeta_{M}^{j} = \left\{ \sum_{i} m_{i}(EI_{b,i} + EI_{i} + EI_{c}) \right\} \zeta_{\kappa}^{j}_{u} \]  
(3.86)

The above formulation was presented for the case of slip along the loxodromic path. It was discussed in chapter two that for circular armour wires the loxodromic slip will also involve rotational slip. For rectangular wires this rotational slip will be restricted and the wires could slip towards the geodesic. The twist and curvature in the binormal direction when the wire slips towards the geodesic are given by (Sævik, 1992):

\[ \tau = -3 \cos^{2} \alpha \sin \phi \kappa_{u} \]  
(3.87)

\[ \kappa_{b} = -\sin \alpha \cos \alpha \left( \frac{1}{\sin^{2} \alpha} - 3 \right) \cos \phi \kappa_{u} \]  
(3.88)

and in this case the full slip equilibrium equation of the umbilical is given by:
In the case where the umbilical is made up of circular and rectangular wires, equation (3.86) and equation (3.68) can be combined to predict the structural response for flexural loads.

The above analysis was carried out assuming bending stiffness of the core $EI_c$ and the bending stiffness of the sheath layer $EI_s$ are constant. In this case, for both loxodromic and geodesic full slip, the equilibrium equations are linear. The no slip equilibrium equation is also linear. The non-linear flexural response occurs in the transitional region between the no slip and the full slip conditions. It is noted that the flexural contribution of the core and the sheath is taken as linear. In reality this is not strictly the case owing to the non-linear relationship between stress and strain for most polymers. This non-linear relationship can readily be incorporated in the above flexural model.

To illustrate the non-linear flexural response of subsea umbilicals, a helical wire 5mm in diameter is considered. The helix radius and angle of the wire are assumed to 50mm and 30 degrees respectively. The contact line pressure and the friction coefficient are taken as -50N/mm and 0.2 respectively. For these parameters the critical curvature, defined as the curvature at the inception of slip is 0.0065/m and the full slip curvature defined by the curvature where the slip region extends over the length of the wire is given by 0.009/m. The moment-curvature curve of this wire which is shown in Figure 3.8 is representative of the moment-curvature relationship of the umbilical. This figure shows that the moment-curvature curve exhibits non-linear
behaviour and the curve at the critical curvature is asymptotic to the bending stiffness for the no slip condition. For the full slip curvature the curve is asymptotic to the bending stiffness in the full slip condition.
4 Experimental Verification

In the previous chapters a theoretical analysis of the structural response of subsea umbilicals was presented and analytical models to predict this response were developed. These models were developed for axi-symmetric loads and for flexural loads and the models incorporate a methodology to predict the equivalent radial modulus and the dilatation coefficient of the composite umbilical core. The equivalent radial modulus and the dilatation coefficient were shown to be the dominant mechanical properties contributing to the structural response of the umbilical. In this chapter the analytical models are compared with full-scale experimental data relating to four umbilicals of different constructional designs. A comparative study on the influence of the different constructional designs is carried out to assess the influence of design parameters on the structural response. The full-scale experimental data include measurements of the axial, torsional and bending stiffnesses of these four umbilicals.

The full-scale experimental data to validate the analytical models for axi-symmetric loads was made available to the author by the umbilical manufacturers. The data relates to four different umbilical designs, schematics of which are shown in Figure 4.1. Two of these umbilicals were dynamic umbilicals and were used in deep water hostile environments. One of these is made of 4 armour layers laid at a helix angle of 10 degrees and the other is made of 5 armour layers laid at a helix angle of 30 degrees. The umbilical made of 4 armour layers is designated as Umbilical I and the umbilical made of 5 armour layers is designated as Umbilical II. The other two umbilicals were used offshore Brazil and both are made of 2 armour layers laid at helix angles between 14 and 18 degrees. Umbilical III and Umbilical IV designate these two umbilicals respectively. The measured axial and torsional stiffnesses of these different umbilicals are shown in Table 4.1.
The table shows the measured axial stiffnesses, the left hand (LH) torsional stiffnesses, and the right hand (RH) torsional stiffnesses for fixed and free end conditions.

The choice of the above four umbilicals allows a number of issues regarding the design parameters of subsea umbilicals to be addressed. These design parameters relate mainly to the number of armour layers, the lay angle of the wires and the construction of the core.

The full-scale experiments to validate the flexural response were carried out by the author. The aim of these experiments was to confirm the analytical solution showing that for flexural loads of practical significance the bending stiffness is the stiffness for the full slip condition as discussed in chapter two. In chapter two a non-linear expression for the bending stiffness was derived and it was argued that except for very small curvature increments the armour wires can be assumed to be in a full slip condition. Two different slip mechanisms were analysed. The first is a slip along the loxodromic path and the second is a slip towards the geodesic path. For circular wires slip along the loxodromic path involves axial and rotational slip. For rectangular wires rotational slip is restricted due to the surrounding layers. The slip towards the geodesic path is the slip towards the path of shortest length. The experimental flexural tests are restricted to validating the assumption that for curvatures of practical significance the armour wires are in a full slip condition due to the limitations of the available flexural experimental methods. These experimental methods which are widely used in the umbilical industry lack the required accuracy to measure with confidence very small curvature increments. The experimental results will also allow an assessment of the slip mechanism most representative of the armour wires in subsea umbilicals.
Experimental measurements to determine the bending stiffness of an umbilical are based on applying an axial load on a bent umbilical. The lateral displacement is measured at a number of equally spaced reference points near the mid-section of the umbilical. A polynomial curve is then fitted to the measured lateral displacement and the curvature at the midpoint of the test specimen is calculated from this curve. The moment is calculated as the product of the applied axial load and the amplitude of the displacement. This experimental method cannot be applied to measure very small curvature increments as explained later in this chapter.

Table 4.1: Measured axial and torsional stiffnesses of umbilicals I, II, III and IV

<table>
<thead>
<tr>
<th>Umbilical</th>
<th>Hose pressure (MPa)</th>
<th>Axial stiffness (MN)</th>
<th>LH Torsional stiffness (Nm²)</th>
<th>RH Torsional stiffness (Nm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Free ends</td>
<td>Fixed ends</td>
<td>Free ends</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>1400</td>
<td>1400</td>
<td>--</td>
</tr>
<tr>
<td>II</td>
<td>0</td>
<td>--</td>
<td>142</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>--</td>
<td>178</td>
<td>--</td>
</tr>
<tr>
<td>III</td>
<td>14</td>
<td>--</td>
<td>220</td>
<td>--</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>125</td>
<td>120</td>
<td>43100</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>67</td>
<td>64</td>
<td>23200</td>
</tr>
</tbody>
</table>

4.1 Moduli of Hoses and Filler Rods

The structural response of subsea umbilicals is dependent on the compliance of the core under the influence of the radial contact pressures generated by the constriction of the helical armour wires. A methodology was introduced to evaluate the equivalent radial modulus and dilatation coefficient and these two parameters were found to depend on the stiffness of the hoses and filler rods. In an umbilical assembly the hoses and filler rods are subject to local contact forces. For these complex loading conditions a conservative approach was adopted and it was assumed that the hoses and filler rods are spring
To evaluate the theoretical structural response of the above umbilicals, experimental tests were carried out to assess the equivalent spring stiffness of the hoses and filler rods. The experimental tests were carried out for different internal pressures and the test arrangement included squashing the pressurised hose between two flat plates. The predicted spring stiffness using the above test arrangement should lead to conservative results since the hoses in-situ would be subject to contact loads distributed over a number of contact points. The experimental results showing the deflection of the plates with the hoses pressurised to 7 MPa and 34 MPa is presented in Figure 4.2. The experimental tests show that the equivalent spring stiffness of the hoses for an internal pressure of 7 MPa and 34 MPa is 16 MPa and 42 MPa respectively. The same experimental procedure was used to determine the equivalent spring stiffness of medium density polyethylene (MDPE) filler rods and this was found to be 360 MPa.

Benjaminsen and Waloen (1995) carried out similar experimental tests on a range of hose sizes of different construction material. Based on these experimental tests, they developed an empirical expression relating the equivalent spring stiffness of the hose to the internal pressure. This expression is given by:

\[ k = 20 + 0.87p \]  

where \( k \) is the equivalent spring stiffness and \( p \) is the internal pressure of the hose. Thus for an internal pressure of 7 MPa and 34 MPa, the equivalent spring stiffness of a hose as predicted using the above expression is 26 MPa and 49 MPa respectively. Comparing the experimental results presented earlier with these empirical results it can be said that the difference in the measured and predicted values for an internal pressure of 7 MPa and 34 MPa is 38% and 14% respectively. However, given that a range of hose sizes and construction material are often used in an umbilical assembly, it reasonable to assume that the empirical expression derived by Benjaminsen and Waloen can be used in analysing the equivalent radial stiffness of the hoses. It is worth
mentioning that Benjaminsen and Waloen carried out experimental tests where the hoses were squashed between four circular rods as shown in Figure 4.3. Using this experimental procedure, which was considered to be more representative of the contact forces seen by the hoses in an umbilical assembly, the results did not show any significant difference in the measured equivalent stiffness as compared to experimental tests where the hose was loaded between two flat plates. In other words, for both squashing the hoses between two flat plates and squashing the hoses between four circular rods, the equivalent stiffness of a hose does not vary by a significant factor.

The above experimental methods used to determine the equivalent radial stiffnesses of the hoses and the filler rods were carried out for one manufacturer who supplied the experimental data for the axial stiffnesses of the umbilical II. The main advantages of these experimental tests is their simplicity and the fact they are representative of the localised contact loads between the different components of the functional bundle. It should be pointed out that this simple approach to determine the spring stiffness of these components leads to conservative results as shown later in this chapter.

4.2 Umbilical I

A schematic of Umbilical I is shown in Figure 4.1 and the material properties and dimensions of the different layers are shown in Table 4.2. Positive and negative helix angles denote right handed and left handed helices respectively. The core of the umbilical is made of a central electrical unit surrounded by two layers of hoses and filler rods. These two layers are modelled as springs in series. The sheath layer of the core is made of a polymer with an elastic modulus of 240 MPa. The thickness and the mean radius of this sheath layer are 4 mm and 64.2 mm respectively.
The inner layer of hoses and filler rods is made of 6 hoses and three MDPE filler rods. The outer layer is made of eight hoses and four MDPE filler rods. The calculated stiffness of the inner layer is estimated to be 1248 MPa. The stiffness of the outer layer is estimated to be 1632 MPa. The expressions for the equivalent radial modulus and the dilatation coefficient were derived in chapter two and are given by:

\[
\lambda = \left\{ \frac{E_s t_s}{R_s (1 - \nu_s^2)} + \frac{1}{2\pi} \sum k_i \right\} 
\]

\[
\nu_c = \left( 1 + \frac{R_s (1 - \nu_s^2)}{2\pi E_s t_s} \sum k_i \right)^{-1} \nu_s
\]

where \( \lambda \) and \( \nu_c \) are the equivalent radial modulus and dilatation coefficient of the core, \( R_s \) and \( t_s \) are the mean radius and wall thickness of the sheath layer, \( E_s \) and \( \nu_s \) are the modulus and Poisson’s ratio of the sheath layer and \( k \) is the spring stiffness of the hoses and filler rods. The limiting values for the dilatation coefficient are zero and a half. The limiting value of zero is for the case when the stiffness of the hoses is large. The limiting case of a half is for the case when the stiffness of the hoses is small. From equation (4.2) the contribution of the two layers of hoses and filler rods to the radial modulus of the core for zero hose internal pressure is 112 MPa. The contribution of the sheath layer is 23 MPa. Therefore, the equivalent radial modulus of the core is 135 MPa and the dilatation coefficient is 0.06.

Using the design parameters shown in Table 4.2 and the equivalent radial modulus and dilatation coefficient of the core calculated above, the analytical axial stiffness for fixed and free end conditions is predicted to be 1250 MN. The corresponding experimental stiffness is 1400 MN as shown in Table 4.3. Comparing the two sets of values, the analytical model underpredicts the axial stiffness by about 7%. This good level of agreement between the analytical and experimental axial stiffnesses of the umbilical allows other mechanical properties of the umbilical to be assessed. These mechanical properties refer
to the torsional balance, the twist of the umbilical and the radial compression and contact pressure on the core. These properties are also shown in Table 4.3. For illustration of the use of these mechanical properties, the contact pressure is calculated for the maximum installation load of 400 kN. Given the contact pressure per unit load as shown in Table 4.3 is 1.03 Pa/N, the contact pressure for the maximum installation load is 0.412 MPa. The same approach can be used to calculate the end twist or the end torque for any particular water depth and applied top tension.

Table 4.2: Constructional details of umbilical I

<table>
<thead>
<tr>
<th>Inner sheath</th>
<th>Mean radius (mm)</th>
<th>64.2</th>
<th>Modulus (MPa)</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Thickness (mm)</td>
<td>4</td>
<td>Poisson's ratio</td>
<td>0.42</td>
</tr>
<tr>
<td>Armour 1</td>
<td>Mean radius (mm)</td>
<td>69.5</td>
<td>Lay angle (degrees)</td>
<td>-10</td>
</tr>
<tr>
<td></td>
<td>No. of Wires</td>
<td>59</td>
<td>Wire size/diameter (mm)</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Modulus (GPa)</td>
<td>210</td>
<td>Poisson's ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Armour 2</td>
<td>Mean radius (mm)</td>
<td>76.5</td>
<td>Lay angle (degrees)</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>No. of Wires</td>
<td>65</td>
<td>Wire size/diameter (mm)</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Modulus (GPa)</td>
<td>210</td>
<td>Poisson's ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Armour 3</td>
<td>Mean radius (mm)</td>
<td>81.25</td>
<td>Lay angle (degrees)</td>
<td>-10</td>
</tr>
<tr>
<td></td>
<td>No. of Wires</td>
<td>56</td>
<td>Wire size/diameter (mm)</td>
<td>2.5x8.5</td>
</tr>
<tr>
<td></td>
<td>Modulus (GPa)</td>
<td>210</td>
<td>Poisson's ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Armour 4</td>
<td>Mean radius (mm)</td>
<td>83.75</td>
<td>Lay angle (degrees)</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>No. of Wires</td>
<td>59</td>
<td>Wire size/diameter (mm)</td>
<td>2.5x8.5</td>
</tr>
<tr>
<td></td>
<td>Modulus (GPa)</td>
<td>210</td>
<td>Poisson's ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Outer sheath</td>
<td>Mean radius (mm)</td>
<td>89.5</td>
<td>Modulus (MPa)</td>
<td>240</td>
</tr>
<tr>
<td></td>
<td>Thickness (mm)</td>
<td>7</td>
<td>Poisson's ratio</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 4.3: Analytical and experimental mechanical properties of umbilical I

<table>
<thead>
<tr>
<th>Mechanical property</th>
<th>$F_{x\infty}$ end conditions $F_{\infty\infty}$ end conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
</tr>
<tr>
<td>Axial stiffness (MN)</td>
<td>1250</td>
</tr>
<tr>
<td>End torque (Nm/N)</td>
<td>-1.12</td>
</tr>
<tr>
<td>End twist (rad/m/N)</td>
<td>--</td>
</tr>
<tr>
<td>Radial displacement (mm/N)</td>
<td>3.35 x 10^7</td>
</tr>
<tr>
<td>Contact pressure (Pa/N)</td>
<td>1.03</td>
</tr>
</tbody>
</table>
4.3 Umbilical II

A schematic of umbilical II is shown in Figure 4.1 and the material properties and dimensions of the different layers are shown in Table 4.4. The sheath layer of the core is made of a polymer with an elastic modulus of 350 MPa. The thickness and mean radius of this sheath layer are 5 mm and 85.9 mm respectively. The core of the umbilical is made of a central electrical unit surrounded by a layer of 9 hoses. Filler rods of low elastic modulus are used to fill the interstices within this layer. For this umbilical the experimental measurement of the axial stiffness was determined for different internal hose pressures. These measurements are shown in Table 4.1 presented earlier. Taking into account the spring stiffness of the hoses for different internal pressures, Table 4.5 show the equivalent radial modulus and dilatation coefficient of the core.

Table 4.6 show the analytical and experimental measurements of the axial stiffnesses of this umbilical for different internal hose pressures. The table shows that for fixed end conditions and with the hoses empty, the axial stiffness of the umbilical is 108 MN. With the hoses pressurised to 7 MPa and 34 MPa the axial stiffnesses are 112 MN and 182 MN respectively. Comparing the experimental and theoretical values presented in this table, it can be said that the difference between the analytical and experimental values is less than 25%. The analytical model, as was the case with umbilical I, underpredicts the axial stiffness and the results are hence conservative. It will be shown later that the main reason the analytical model underpredicts the axial stiffness by 25% is the sensitivity of the structural response of this particular umbilical to the predicted modulus of the core. This is the case due to the large helix angle of the armour wires which results in greater radial contact pressure on the core. The radial contact pressure for a wire of a helical angle of 30 degrees is more than eight times the contact pressure of an identical wire laid with the same helix radius but a with a helical angle of 10 degrees. The factor of eight
can be calculated from the linear expression that equates that the contact pressure to the product of the wire curvature and the wire axial force (Hruska, 1952). Given this considerable increase in the magnitude of the contact pressure it will be shown that the structural response of this umbilical is very dependent on the equivalent material properties of the core.

Table 4.4: Constructional details of umbilical II

<table>
<thead>
<tr>
<th>Inner sheath</th>
<th>Mean radius (mm)</th>
<th>85.9</th>
<th>Modulus (MPa)</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness (mm)</td>
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<td>Poisson’s ratio</td>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>Armour 1</td>
<td>Mean radius (mm)</td>
<td>91.9</td>
<td>Lay angle (degrees)</td>
<td>-30</td>
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<tr>
<td>No. of Wires</td>
<td>79</td>
<td>Wire size/diameter (mm)</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Modulus (GPa)</td>
<td>210</td>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Armour 2</td>
<td>Mean radius (mm)</td>
<td>97.6</td>
<td>Lay angle (degrees)</td>
<td>30</td>
</tr>
<tr>
<td>No. of Wires</td>
<td>84</td>
<td>Wire size/diameter (mm)</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Modulus (MPa)</td>
<td>210</td>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Armour 3</td>
<td>Mean radius (mm)</td>
<td>103.8</td>
<td>Lay angle (degrees)</td>
<td>-30</td>
</tr>
<tr>
<td>No. of Wires</td>
<td>90</td>
<td>Wire size/diameter (mm)</td>
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<td></td>
</tr>
<tr>
<td>Modulus (MPa)</td>
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<td>Poisson’s ratio</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Armour 4</td>
<td>Mean radius (mm)</td>
<td>110.0</td>
<td>Lay angle (degrees)</td>
<td>-30</td>
</tr>
<tr>
<td>No. of Wires</td>
<td>94</td>
<td>Wire size/diameter (mm)</td>
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<td>Modulus (MPa)</td>
<td>210</td>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Armour 5</td>
<td>Mean radius (mm)</td>
<td>116.4</td>
<td>Lay angle (degrees)</td>
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<td>No. of Wires</td>
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<td>Wire size/diameter (mm)</td>
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<td></td>
</tr>
<tr>
<td>Modulus (MPa)</td>
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<td>Poisson’s ratio</td>
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</tr>
<tr>
<td>Outer sheath</td>
<td>Mean radius (mm)</td>
<td>122.9</td>
<td>Modulus (MPa)</td>
<td>350</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>7</td>
<td>Poisson’s ratio</td>
<td>0.42</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.5: Equivalent radial modulus and dilatation coefficient of the core of umbilical II

<table>
<thead>
<tr>
<th>Internal pressure (MPa)</th>
<th>Empty</th>
<th>7</th>
<th>34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial stiffness of hose (MPa)</td>
<td>7</td>
<td>14</td>
<td>38</td>
</tr>
<tr>
<td>Equivalent stiffness of hose layer (MPa)</td>
<td>10</td>
<td>20</td>
<td>54</td>
</tr>
<tr>
<td>Equivalent stiffness of sheath layer (MPa)</td>
<td>24.7</td>
<td>24.7</td>
<td>24.7</td>
</tr>
<tr>
<td>Equivalent modulus of core (MPa)</td>
<td>34.7</td>
<td>44.7</td>
<td>78.7</td>
</tr>
<tr>
<td>Dilatation coefficient</td>
<td>0.3</td>
<td>0.23</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Table 4.6: Analytical and experimental axial stiffnesses of umbilical II

<table>
<thead>
<tr>
<th>Hose pressure (MPa)</th>
<th>Axial stiffness for fixed end conditions (MN)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
<td>Percentage difference</td>
</tr>
<tr>
<td>Empty</td>
<td>108</td>
<td>142</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>142</td>
<td>178</td>
<td>19</td>
</tr>
<tr>
<td>34</td>
<td>182</td>
<td>220</td>
<td>17</td>
</tr>
</tbody>
</table>

4.4 Umbilical III

A schematic of umbilical III is shown in Figure 4.1 and the material properties and dimensions of the different layers are shown in Table 4.7. This umbilical was the subject of experimental tests to determine its axial and torsional stiffnesses for different boundary conditions. In this section, the same approach adopted for umbilicals I and II is used to verify the analytical model developed. However, due to the availability of additional data the verification of the analytical model can be extended to compare the torsional stiffnesses of the umbilical and the coupled axial-torsional response.

The core of the umbilical is made of a central electrical unit and 9 hydraulic hoses. The sheath layer is made of MDPE with a mean radius of 41 mm and a thickness of 6 mm. The equivalent radial modulus and the dilatation coefficient of the core are calculated as 125 MPa and 0.33 respectively. Using the analytical model, the theoretical axial stiffness for fixed and free end conditions is 110 MN. The theoretical axial stiffnesses being equal for the fixed and free end conditions indicate that the umbilical is torsionally balanced. The experimental axial stiffnesses for fixed and free end conditions are 120 MN and 125 MN respectively. Table 4.8 show these theoretical and experimental values. Comparing the theoretical and experimental axial
stiffness for free end conditions the difference is within 12%. Comparing the theoretical and experimental axial stiffness for fixed end conditions the difference is within 9%. For both end conditions, the analytical model under predicts the axial stiffnesses and this is consistent with the case for umbilicals I and II. It is noted that the accuracy of the analytical model for these two umbilicals is comparable to the accuracy of the predicted results calculated for umbilical I.

For the three umbilicals discussed so far, a good level of agreement was obtained for the theoretical and experimentally axial stiffnesses for free and fixed end conditions. For umbilicals I and II verification of the analytical model was restricted to these parameters due to the lack of experimentally measured data on the torsional stiffnesses. For umbilical III, torsional stiffnesses for fixed and free boundary conditions were measured experimentally and these measurements are shown in Table 4.1 presented earlier. Table 4.9 show these experimental values together with the theoretical values and the differences between these values. For free end conditions, it is seen from Table 4.9 that the difference between the predicted and measured torsional stiffnesses is 33% and 3.7% respectively. For fixed end conditions the difference for left hand and right hand torsional stiffnesses is 38%. While these differences are higher than those associated with the predicted axial stiffnesses, these errors are still within reasonable limits. The reasons for the higher level of discrepancy as compared to the theoretical axial stiffnesses could not be clearly identified. However, it should be pointed out that the experimental equipment used to measure the torsional stiffness had an accuracy of ±1 degrees and the maximum level of twist that was achieved was 1.5 degrees/m. Given the umbilical was 6.3m the total end rotation would be 9.5 degrees. As a result the experimentally measured torsional stiffnesses have a 10% error. Other reasons for the discrepancy between the theoretical and experimental values could be due to end effects. Experience with torsional experiments using typical umbilical end terminations shows that end
effects have a significant influence on the measured results. This is mainly due to localised high twist near the end termination resulting in non-uniform twist along the length of the test sample. Bearing in mind that the analytical model was developed assuming that twist is uniform along the umbilical, some level of discrepancy between theoretically and analytical results is expected. The compounded effect of the errors resulting from the inaccuracy of the measurements taken and the errors due to non-uniform twist along the test specimen justifies that the errors of the analytical model are within reasonable limits.

Table 4.7: Constructional details of umbilical III

<table>
<thead>
<tr>
<th></th>
<th>Mean radius (mm)</th>
<th>Thickness (mm)</th>
<th>Modulus (MPa)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner sheath</td>
<td>41</td>
<td>6</td>
<td>650</td>
<td>0.42</td>
</tr>
<tr>
<td>Armour 1</td>
<td>43.5</td>
<td>-18</td>
<td>5.5</td>
<td>0.3</td>
</tr>
<tr>
<td>Armour 2</td>
<td>50.5</td>
<td>15</td>
<td>6</td>
<td>0.3</td>
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<tr>
<td>Outer sheath</td>
<td>55</td>
<td>6</td>
<td>650</td>
<td>0.42</td>
</tr>
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Table 4.8: Analytical and experimental axial stiffnesses of umbilical III

<table>
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<tr>
<th></th>
<th>Axial stiffness for free ends</th>
<th>Axial stiffness for fixed ends</th>
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</thead>
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<tr>
<td>Theory (MN)</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>Experiment (MN)</td>
<td>125</td>
<td>120</td>
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<tr>
<td>Percentage error</td>
<td>12</td>
<td>9</td>
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Table 4.9: Analytical and experimental torsional stiffnesses of umbilical III

<table>
<thead>
<tr>
<th></th>
<th>Torsional stiffness for free ends</th>
<th>Torsional stiffness for fixed ends</th>
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</thead>
<tbody>
<tr>
<td>Left hand</td>
<td>Right hand</td>
<td>Left hand</td>
</tr>
<tr>
<td>Theory (Nm²)</td>
<td>57600</td>
<td>34200</td>
</tr>
<tr>
<td>Experiment (Nm²)</td>
<td>43100</td>
<td>35900</td>
</tr>
<tr>
<td>Percentage error</td>
<td>33</td>
<td>4.7</td>
</tr>
</tbody>
</table>
4.5 Umbilical IV

A schematic of umbilical IV is shown in Figure 4.1 and the material properties and dimensions of the different layers are shown in Table 4.10. The core of this umbilical is made of a central hose surrounded by a thick filler layer of low elastic modulus. Nine hoses are used in the annulus between the central hose and the sheath layer of the core. This sheath layer of the core is made of a polymer with an elastic modulus of 600 MPa. The thickness and mean radius of the sheath layer are 6 mm and 35 mm respectively. The equivalent radial modulus and the dilatation coefficient of the core are calculated as 72 MPa and 0.35 respectively.

As shown in Table 4.1 presented earlier, the experimentally measured axial stiffnesses for fixed and free end conditions are 67 MN and 64 MN respectively. The corresponding theoretical stiffnesses are 68 MPa and 64 MN respectively. Comparing these experimental and theoretical values, which are shown in Table 4.11, the differences in the axial stiffness for fixed and free end conditions are less than 1%. For this umbilical the level of accuracy achieved is higher than the umbilicals analysed earlier.

Umbilical IV was also subjected to experimental tests to determine its torsional stiffnesses. The availability of this data allows the theoretically predicted torsional stiffnesses to be verified. As shown in Table 4.12, the experimental left hand and right hand torsional stiffnesses of this umbilical for free end conditions are 15900 Nm² and 12000 Nm² respectively. Comparing these values with the experimental values of 23200 Nm² and 11900 Nm², the differences are 31% and 1% respectively. For fixed end conditions the differences between the predicted and experimentally measured left hand and right hand torsional stiffness are 28% and 12% respectively. Thus, it can be said that the error in predicting the torsional stiffness of the umbilical is comparable to the error observed in analysing umbilical III. As was the case
for umbilical III, the experimental equipment used to determine the torsional stiffness had an accuracy of ±1 degrees. Taking this into account and the possibility that uniform twist along the test specimen was not achieved, it can be argued that the difference between the experimental and theoretical torsional stiffnesses is within the accuracy of the test equipment used. The accuracy of the test equipment, taking into account that one end of the umbilical was rotated 10 degrees, is about 10%.

Table 4.10: Constructional details of umbilical IV

<table>
<thead>
<tr>
<th></th>
<th>Mean radius (mm)</th>
<th>Modulus (MPa)</th>
<th>Thickness (mm)</th>
<th>Poisson's ratio</th>
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<tbody>
<tr>
<td>Inner sheath</td>
<td>31</td>
<td>600</td>
<td>6</td>
<td>0.42</td>
</tr>
<tr>
<td>Armour 1</td>
<td>36.4</td>
<td>210</td>
<td>-17</td>
<td>0.3</td>
</tr>
<tr>
<td>No. of Wires</td>
<td>50</td>
<td>3.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modulus (GPa)</td>
<td>3.15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Armour 2</td>
<td>40.4</td>
<td>210</td>
<td>14</td>
<td>0.3</td>
</tr>
<tr>
<td>No. of Wires</td>
<td>56</td>
<td>3.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modulus (MPa)</td>
<td>210</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Outer sheath</td>
<td>44</td>
<td>600</td>
<td>6</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 4.11: Analytical and experimental axial stiffnesses of umbilical IV

<table>
<thead>
<tr>
<th></th>
<th>Axial stiffness for free ends</th>
<th>Axial stiffness for fixed ends</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory (MN)</td>
<td>68</td>
<td>64</td>
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<tr>
<td>Experiment (MN)</td>
<td>67</td>
<td>64</td>
</tr>
<tr>
<td>Percentage error</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.12: Analytical and experimental torsional stiffnesses for umbilical IV

<table>
<thead>
<tr>
<th></th>
<th>Torsional stiffness for free ends</th>
<th>Torsional stiffness for fixed ends</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left hand</td>
<td>Right hand</td>
</tr>
<tr>
<td>Theory (Nm²)</td>
<td>15900</td>
<td>12000</td>
</tr>
<tr>
<td>Experiment (Nm²)</td>
<td>23200</td>
<td>11900</td>
</tr>
<tr>
<td>Percentage error</td>
<td>31</td>
<td>1</td>
</tr>
</tbody>
</table>
4.6 Summary of Axial-Torsional Results

In summary, the analytical models predicted accurate results for the axial stiffness of umbilicals I, III and IV but this accuracy was not achieved for umbilical II. Given the limited experimental data made available, it is not possible to ascertain the full reasons for this discrepancy. However, previous experience has shown that manufacturing defects, in particular the presence of gaps between the different layers of an umbilical, could lead to a reduction in the axial stiffness of the umbilical. This could be one of many reasons for this particular case. Other reasons for the discrepancy is the sensitivity of the predicted axial stiffness of this umbilical to the equivalent radial modulus of the core. This is discussed in further detail later in this chapter and it is shown that the large helical angle of the armour wires for this particular umbilical has a dominant role in reducing the axial and torsional stiffnesses of this umbilical. It is noted that the measured axial stiffness of umbilical II which is made of 5 armour layers is an order of magnitude smaller than the axial stiffness of umbilical I which is made of 4 armour layers. The measured axial stiffness of umbilical I was 1400 MN while the maximum axial stiffness for umbilical II was 220 MN. The analytical model, given the accuracy of the experimental equipment used, also predicted accurate results for the torsional stiffnesses of umbilicals III and IV. The accuracy of the theoretical torsional stiffnesses was less than 40%. Given the complexity of analysing umbilicals, it can be said that the above novel analytical approach to predict the axial-torsional response of subsea umbilicals leads to relatively accurate results.

4.7 Flexural Stiffness

Experimental measurements of the bending stiffness of umbilical I was carried out by the author. As shown in Figure 4.4, the test specimen that was 6 m long was placed on a relatively smooth, clean floor surface with the two ends connected together via a block and tackle. A load cell was incorporated in this
arrangement to measure the eccentrically applied axial end load. The specimen was bent by applying a small increment to the axial load. Due to the eccentricity of the applied load, this increment in the axial load generates an increment in the bending moment along the length of the test specimen. Before taking any measurements the specimen was raised by a small distance from the floor surface and then lowered when the loading process was stopped. The objective of this procedure was to minimise the influence of the frictional forces between the specimen and the floor. The lateral displacement of three equally spaced points of the umbilical was measured from a horizontal reference axis. The bending moment at the midpoint of the umbilical was determined as the product of the eccentricity and the magnitude of the end axial force. The curvature at this point was determined from a parabolic curve which was fitted to the experimentally measured lateral displacements. This experimental procedure has its limitations but its simplicity is one of its main advantages. Other experimental methods often used to determine the bending stiffness of umbilicals is the four point bending method. For this type of experiment the test specimen is usually about 1 m long. The relatively high diameter to length ratio of this test arrangement raises some questions to the accuracy of this procedure as shear distortion is likely to influence the results.

It was discussed earlier that for subsea umbilicals the helical wires are in a full slip condition for very small curvatures. The aim of this experiment is to demonstrate this analytically observed phenomenon. This was achieved by comparing the experimentally measured bending stiffness with the full slip analytical bending stiffness. To maximise the accuracy of the experimental procedure used, ten measurements of the bending stiffness were collected. The mean of these measurements was calculated to be 11.7 kNm². The analytical bending stiffness of the umbilical was calculated to take into account different slip mechanisms for the rectangular armour wires. The circular armour wires as discussed in chapter three will slip towards the
lo xo d ro m ic path and this slip mechanism will also include rotational slip. For the rectangular armour wires it was argued that two different slip mechanisms can take place, one along the loxodromic and one towards the geodesic. For rectangular wires rotational slip will be constrained by the surrounding layers. Table 4.13 shows the predicted values for loxodromic slip and geodesic slip. The predicted bending stiffness for loxodromic slip is 13.0 kNm\(^2\). The predicted bending stiffness for geodesic slip is 22.6 kNm\(^2\). The contribution of the core to the bending stiffness was ignored in calculating the theoretical bending stiffness. This is not likely to introduce significant errors as the core is made of a number of small diameter hoses and electrical cables. The bending stiffness of these components is small as compared with the bending stiffness of the sheaths and armour layers. Figure 4.4 shows that the theoretical bending stiffnesses for loxodromic slip and the experimental bending stiffness are in good agreement with a difference of 11%. This is not the case for geodesic slip where the difference exceeds 90%. Thus, it can be argued that the case of loxodromic slip is more representative of the slip mechanism in subsea umbilicals. This is the case since geodesic slip is the path of shortest length but not necessarily the path of minimum energy.

The good agreement between the theoretical bending stiffness for loxodromic slip and the experimental results confirms the analytical model which shows that the helical armour wires are in a full slip condition. At the same time the experimental results confirm that the loxodromic slip is more representative of the slip mechanism in subsea umbilical. It is worth noting that Tan (1992) who examined different types of flexible structures also argued that for these structures the bending stiffness can be predicted analytically assuming that the helical wires are in a full slip condition. Tan did not discuss the different slip mechanisms that are often assumed but his analysis was confined to the case of loxodromic slip.
It was pointed out earlier that the experimental procedure described above is not applicable for measuring small increments in the curvature where the condition of no slip can be investigated. To explain this further, it is assumed that the lateral deflection of the umbilical is a circular function satisfying the geometric and kinematic boundary conditions. Assuming the co-ordinate system is as shown in Figure 4.4, the displacement is then given by:

\[ y = A \cos \frac{\pi x}{L} \quad (4.4) \]

where \( y \) is the lateral displacement, \( A \) is the displacement amplitude and \( L \) is the length of the umbilical specimen. The change in curvature at the midpoint is given by:

\[ \Delta \kappa_u = \frac{\pi^2}{L^2} (\Delta A) \quad (4.5) \]

Assuming the change in the amplitude of displacement is measured in steps of 5 mm, the corresponding change in curvature for a specimen 6 metres in length is 0.001 /m. This curvature exceeds the critical curvature at which slip is initiated. If the amplitude of displacement is carried out in steps smaller than 5 mm then the desired accuracy in measuring the curvature would not achieved given the errors inherent to curve fitting algorithms.

| Table 4.13: Analytical and experimental bending stiffnesses of umbilical I |
|------------------|--------|----------------|
|                  | Magnitude | Percentage difference |
| Theory (kNm²)    | 11.7    | ---             |
| Loxodromic slip (kNm²) | 13.0 | 11              |
| Geodesic slip (kNm²)  | 22.6   | 93              |
4.8 Discussion and Conclusions

Comparing the axial stiffnesses of umbilical I and umbilical II, the axial stiffness of umbilical I is one order of magnitude greater than the axial stiffness of the umbilical II. The main reason for this is the difference in the helix angles of the armour wires of the two umbilicals. For umbilical I the helix angle of the armour wires is about 10 degrees and for umbilical II the helix angle is 30 degrees. Other reasons for this difference are the equivalent material properties of the cores. A plot of the axial stiffnesses of umbilical I and umbilical II for different assumed values for the modulus of the core is shown in Figure 4.5. The figure shows curves where the axial stiffness and the core modulus have been normalised. The axial stiffness is normalised by dividing the predicted value by the experimentally measured value. The core modulus is normalised by dividing the assumed value by the modulus of the sheath layer. The assumed value of the core modulus was taken for two extreme cases referred to in this section as the case of a compliant core and the case of a rigid core. The case of compliant core is taken as the case where the radial modulus of the hoses and filler rods is zero. In this case the only contribution to the radial modulus of the core is the radial rigidity of the sheath layer. The other extreme case of a rigid core is equivalent to assuming the core is a solid rod with identical material properties to the sheath layer. Figure 4.5 shows that the axial stiffness of umbilical I varies by a factor of 1.11 over the range of assumed core moduli while the axial stiffness of umbilical II varies by a factor of six over the same range. This shows that the higher axial stiffness of umbilical I is mainly due to the small helical angle of the armour layers. Moreover, Figure 4.5 shows that for small helical angles the stiffness of the umbilical is less sensitive to the radial modulus of the core.

To verify the influence of the helical angle on the axial stiffness of umbilical I, the helical angle was increased while maintaining the same coverage for the helical areas. The coverage is given as the percentage ratio of the projected
sum of the wire diameters divided by the circumference of the helical layer. The coverage of the helical layer is given by the expression:

$$C = \frac{nd}{2\pi R \cos \alpha}$$  \hspace{1cm} (4.6)

where $C$ is the coverage, $n$ is the number of wires, $d$ is the diameter of the wire, $R$ is the mean radius of the helical layer and $\alpha$ is the helix angle. Figure 4.6 shows a plot of the predicted axial stiffness of the umbilical for different helix angles and for different core moduli. Over the range of assumed core moduli, the axial stiffness of the umbilical for helical angles of 10 degrees and 25 degrees varies by a factor of 1.11 and 4 respectively. Also for the same armour coverage, the axial stiffness of the umbilical decreases by one order of magnitude for the case of a compliant core. This significant reduction in the axial stiffness of the umbilical results in considerable mechanical loads in the electrical cables of subsea umbilicals for typical installation and operational loads.

It is also noted that either incorporating solid filler rods in the umbilical core or increasing the thickness and modulus of the inner sheath layer can increase the equivalent radial modulus of the core. This is particularly significant in cases where the manufacturing and operational constraints dictate large helical angles for the armour wires. Examples of these manufacturing constraints include the capability of the machinery involved in assembling the umbilical to handle a fixed number of armour wires. Should the number of wires for a small helical angle exceed the capability of the machinery, umbilical manufacturers tend to increase the helical angles which allows less wires to be used.

The effect of increasing the modulus of the core for different helix angles is shown in Figure 4.6 presented earlier. There is a common need to increase the helix angles of the armour wires in deep water applications where the
umbilical is required to incorporate a greater number of large diameter hydraulic hoses because of increased flow needs. This dictates an increase in the diameter of the functional bundle and as a result a greater number of armour wires is required. The requirement of a greater number of armour wires often exceeds the capability of the manufacturer and the alternative of increasing the helix angle in used. This is the main manufacturing constraint which requires an increase in the helix angles of the armour wires. Other operational constraints also require the helix angle of the armour wires to be increased in order to provide radial strength to the umbilical. These operational constraints relate mainly to deep water and to geographic locations where the umbilical is likely to experience accidental impact loads. In deep water, the radial strength is required to support the radial forces imposed through the tensioners.

In conclusion, the analytical models discussed earlier in this thesis were verified against full-scale experimental data. It was shown that the differences between the theoretical and experimental results are within acceptable limits and the analytical models lead to conservative results. The differences between the experimental and theoretical results are small for umbilicals where the armour wires are laid with a small angle. For umbilicals made with large helical angles the differences are slightly greater owing to the sensitivity of these structures to the predicted equivalent material properties of the umbilical core. It was shown that the constriction of the armour wires laid with a large angle leads to greater contact forces. Due to this greater contact forces the equivalent mechanical properties of the core have a dominant influence of the structural response of the umbilical.
5 Mechanics of Electrical Cables

So far the analytical models developed in this thesis have been concerned with the global structural response of subsea umbilicals. The analysis is carried forward in this chapter to investigate the mechanics of the electrical cables within these. The aim of this investigation is to allow the different failure mechanisms of the electrical cables and the global loads leading to these failures to be evaluated accurately. This investigation also allows an assessment of the influence of the constructional design parameters on the mechanical loads in the copper cores.

Typical cross sections of signal and power electrical cables are shown in Figure 5.1. The conductors are required to be made of plain or tinned annealed copper wires. The conductors are insulated with a polymeric layer to form a core. The cores are then bundled together in a helical configuration and this bundle is embedded in a low modulus polymeric filler to form a circular cross-sectional area. For signal cables, a metallic signal shield layer is applied over the filler and the cable is then sheathed with a polymer of a relatively high elastic modulus. The metallic screen layer is usually made of annealed copper tape and the sheath is usually made of a thermoplastic polymer. Table 5.1 presents a list of the different polymer types used in the construction of electrical cables. The choice of the material types to be used depends on their electrical properties and their suitability for operating in sea water. For additional mechanical protection, a layer of steel armour wires can be added to the generic cross section of an electrical cable. In addition to providing mechanical strength, the armour layer also helps to promote slip of the electrical cable by two main mechanisms. First, an armoured electrical cable is of a much higher stiffness than an unarmoured cable. Consequently, the shear forces that drive the slip mechanism will be of greater magnitude for any given bending curvature as explained in chapter two. Second, the steel wires
provide a contact surface with a low friction coefficient which also helps in promoting slip within the electrical cable.

A number of electrical cables are often incorporated in an umbilical assembly and these cables are laid in either a helical or S-Z configuration. The helical and S-Z lays, which are also known as planetary and oscillatory lay configurations, are shown in Figure 3.2 presented earlier. In the planetary configuration, the centre line of the cable forms a constant angle with the centre line of the umbilical. This lay configuration generates twist of the cable that is often minimised by back twist of the bobbins feeding the assembly line. In the oscillatory configuration, a certain length of the cable is laid at a constant helical angle and then the lay direction is reversed. This lay configuration eliminates the requirement of a back twist. It is often seen that when the electrical cables are bundled together in a central electrical unit, these cables are laid in a planetary configuration. On the other hand, when the electrical cables are bundled with the hydraulic hoses these cables are laid with an oscillatory configuration. The main reason for this approach is the difficulty of introducing back twist to the large reels required to carry the hydraulic hoses.

One of the main failure mechanisms of interest to the subject of this thesis is the formation of kinks in the electrical conductors. The formation of kinks results in rapid fatigue and in ultimate failure of the cores. Kinks are a buckling type of failure and their formation is evidently due to compressive forces in the conductors. These compressive forces can arise due to twist and bending. Compressive forces can also arise under global tensile loads in cases where the radial constriction of the electrical cables is large or the cables are laid with a large helical angle. This can be deduced from the expression relating the axial strain in a helical element to the global deformation variables of the umbilical presented earlier in equation (3.5). However, the majority of kinking failures reported in the field and in experimental tests show that the
formation of kinks arise after installation and under cyclic loading. These kinking failures can be attributed to the non-linear material properties of annealed copper. The non-linear material properties are investigated in this chapter and experimental tests are carried out to determine the amount of work hardening that the conductors experience due to the manufacturing process. While the wires of the conductors are required to be in an annealed condition, some work hardening is expected to occur during the manufacturing process. The level of work hardening is assessed by comparing the stress-strain curves of copper wires in the as-built and annealed conditions. The cores were extracted from the cables of two main umbilical manufacturers. Experimental tests were also carried out to assess the work hardening in the conductors of a commercially available electrical cable.

<table>
<thead>
<tr>
<th>Polymer Type</th>
<th>Abbreviated Name</th>
<th>Elastic Modulus (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Density Polyethylene</td>
<td>HDPE</td>
<td>690-1800</td>
</tr>
<tr>
<td>Medium Density Polyethylene</td>
<td>MDPE</td>
<td>600-850</td>
</tr>
<tr>
<td>Low Density Polyethylene</td>
<td>LDPE</td>
<td>120-240</td>
</tr>
<tr>
<td>Ethylene Propylene Rubber</td>
<td>EPR/EPM</td>
<td>2-6</td>
</tr>
<tr>
<td>Cross Linked Polyethylene</td>
<td>XLPE</td>
<td>600</td>
</tr>
<tr>
<td>Ethylene Propylene Co-polymer</td>
<td>EPC</td>
<td>600</td>
</tr>
<tr>
<td>Ethylene Propylene Diene Rubber</td>
<td>EPDM</td>
<td>2-6</td>
</tr>
<tr>
<td>Plasticised Polyvinyl Chloride</td>
<td>PPVC</td>
<td>2-6</td>
</tr>
</tbody>
</table>

5.1 Deformed Geometry of a Double Helix

It was mentioned earlier that the cores within the electrical cables are laid in a helical configuration and the cables are in turn laid in a helical or S-Z configuration within the umbilical assembly. Emphasis is given here to the helical lay of the electrical cable since this lay configuration constitutes the
majority of lay configurations used in assembling subsea umbilicals. This lay configuration results in a double helix lay for the copper cores.

To analyse the influence of the double helical lay of the copper cores a recursive approach is used. This recursive approach is similar to the approach used in analysing wire ropes. For wire ropes the mechanical properties of each individual strand of a rope are determined and these mechanical properties are used to evaluate the mechanical properties of the rope. This approach takes into account the lay angle of the wires within the strand and the lay angle of the strands within the rope. A similar approach can be used to analyse the mechanical loads on the cores of an electrical cable.

The deformation of a helical curve as a function of the deformation of its reference cylinder was discussed in chapter two. The equations for the axial strain, the twist and the binormal curvature of the curve for axi-symmetric deformation of this global reference cylinder are given as:

\[ \varepsilon_1 = \cos^2 \alpha_1 \frac{du_z}{dz} + R_1 \sin \alpha_1 \cos \alpha_1 \frac{du_\phi}{dz} + \frac{\sin^2 \alpha_1}{R_1} u_{R,1} \]  
\[ (5.1) \]

\[ \tau_1 = \frac{\sin \alpha_1 \cos \alpha_1}{R_1} \left\{ (1 - 2 \cos^2 \alpha_1) \frac{du_z}{dz} - 2 \frac{\sin^2 \alpha_1}{R_1} u_{R,1} \right\} + \frac{R_1}{\tan \alpha_1} (1 - 2 \sin^2 \alpha_1) \frac{du_\phi}{dz} \]  
\[ (5.2) \]

\[ \kappa_{1,b} = \frac{\sin^2 \alpha_1}{R_1} \left\{ -2 \cos^2 \alpha_1 \frac{du_z}{dz} + (1 - 2 \sin^2 \alpha_1) \frac{u_{R,1}}{R_1} \right\} - 2R_1 \sin \alpha_1 \cos \alpha_1 \frac{du_z}{dz} \]  
\[ (5.3) \]

where \( \varepsilon_1 \) is the axial strain in the helical curve, \( \tau_1 \) and \( \kappa_{1,b} \) are the twist and the change in the binormal curvature of the curve, \( R_1 \) and \( \alpha_1 \) are the helix radius and angle, \( u_z \) and \( u_R \) are the axial and radial displacement of the global
reference cylinder and $u_\phi$ is the twist rotation of this cylinder. The subscript 1 is used to denote the first helical configuration. If the reference cylinder is also subjected to flexural loads, additional terms are introduced to the deformations of the helical curve presented above. These additional terms can be expressed as:

$$\varepsilon_1 = -R_1 \kappa_u \sin \phi_1 \cos^2 \alpha_1$$  \hspace{1cm} (5.4)

$$\tau_1 = -\kappa_u \sin \phi_1 \sin \alpha_1$$  \hspace{1cm} (5.5)

$$\kappa_{1,b} = -\kappa_u \sin \phi_1 \cos \alpha_1$$  \hspace{1cm} (5.6)

$$\kappa_{1,n} = \kappa_u \cos \phi_1$$  \hspace{1cm} (5.7)

where $\kappa_u$ is the curvature of the global reference cylinder and $\phi_1$ is the polar angle along the circumference of this cylinder. The above equations were derived for a helical curve assuming the longitudinal axis of the reference cylinder is a straight line. It is also assumed that the helical curve is not in a slip condition. In other words it is assumed that the helical curve is effectively bonded to the surface of its reference cylinder. If this helical curve is in turn the longitudinal axis of a reference cylinder, the deformation of any helical curve on this local reference cylinder can be determined by a recursive approach. The co-ordinate systems used are a global and a local polar co-ordinate systems $(R_1, \phi_1)$ and $(r_2, \phi_2)$ which are shown Figure 5.2. In these co-ordinate systems $R_1$ and $r_2$ are the radii of the global and the local reference cylinders and $\phi_1$ and $\phi_2$ are the polar angles along the circumferences of these cylinders.

### 5.1.1 Axial Strain

Using the recursive approach explained above and using the subscript 2 to define the double helical curve, the axial strain of this double helix is given by:
\[
\varepsilon_2 = \cos^2 \alpha_2 \varepsilon_1 + r_2 \sin \alpha_2 \cos \alpha_2 \tau_1 + \frac{\sin^2 \alpha_2}{r_2} u_{r,2} \\
- r_2 \sin \phi_2 \cos^2 \alpha_2 \kappa_{1,n} - r_2 \cos \phi_2 \cos^2 \alpha_2 \kappa_{1,b}
\]

(5.8)

where \( \varepsilon_1 \) is the sum of the expressions given in equations (5.1) and (5.4), \( \tau_1 \) is the sum of the expressions given in equations (5.2) and (5.5), \( \kappa_{1,n} \) is given in equation (5.7) and \( \kappa_{1,b} \) is the sum of the expressions given in equations (5.3) and (5.6). Substituting for \( \varepsilon_1, \tau_1, \kappa_{1,b} \) and \( \kappa_{1,n} \) and ignoring second order terms, the axial strain in the double helix is given by:

\[
\varepsilon_2 = \cos^2 \alpha_2 \left( \cos^2 \alpha_1 \frac{du_z}{dz} + R_1 \sin \alpha_1 \cos \alpha_1 \frac{du_\phi}{dz} + \frac{\sin^2 \alpha_1}{R_1} u_{R,1} \right) \\
- r_2 \kappa_u \sin \phi_1 \sin \alpha_1 \sin \alpha_2 \cos \alpha_2 - r_2 \kappa_u \sin \phi_2 \cos \phi_1 \cos^2 \alpha_2 \\
+ r_2 \kappa_u \cos \phi_2 \sin \phi_1 \cos^2 \alpha_2 \cos \alpha_1 + \frac{\sin^2 \alpha_2}{r_2} u_{r,2} \\
- R_1 \kappa_u \sin \phi_1 \cos^2 \alpha_1 \cos^2 \alpha_2
\]

(5.9)

The axial strain in the double helix as given in the above equation varies sinusoidally along the circumference of the local and global reference cylinders. The above equation was derived assuming the local reference cylinder is in a no slip condition. This no slip condition as discussed in chapter two can arise if the axial stiffness of the helix is small. If this local reference cylinder is in a full slip condition, the last term in equation (5.9) reduces to zero.

If the equivalent axial stiffness of the helix is small, it was shown that the external shear forces that develop at the contact surfaces could not be sufficient to overcome the frictional forces for a given contact pressure. However, it can be shown that irrespective of whether the helix slips or not, the slip of the double helix will depend on its axial stiffness. To illustrate this point the substitution:
\[ d\phi_2 = \frac{\sin \alpha_2}{r_2} ds_2 \]  
(5.10)

\[ d\phi_1 = \frac{\sin \alpha_1 \cos \alpha_2}{R_1} ds_2 \]  
(5.11)

is used to evaluate the shear force \( Z \) along the double helix which is given by:

\[ Z = EA \frac{d\varepsilon_2}{ds_2} = EA \left( \frac{\partial \varepsilon_2}{\partial \phi_2} \frac{d\phi_2}{ds_2} + \frac{\partial \varepsilon_2}{\partial \phi_1} \frac{d\phi_1}{ds_2} \right) \]  
(5.12)

where \( EA \) is the axial stiffness of the double helix. The above equation leads to the following expression for the shear force:

\[ Z = EA \sin \phi_1 \sin \phi_2 \sin \alpha_2 \cos^2 \alpha_2 \cos \alpha_1 \kappa_u \]
\[ + EA \cos \phi_1 \cos \phi_2 \sin \alpha_2 \cos^2 \alpha_2 \kappa_u \]
\[ + EA \cos \phi_1 \sin \alpha_1 \cos^3 \alpha_2 \cos^2 \alpha_1 \kappa_u \]  
(5.13)

As seen in the above equation the shear force is a function of the axial stiffness of the double helix. If the axial stiffness of the double helix is relatively large, the double helix will slip irrespective of whether its local reference cylinder slips or not.

### 5.1.2 Change in Twist and Curvature

The twist, normal curvature and binormal curvature of a helix as the function of the bending curvature of its reference cylinder were given in equations (5.5), (5.6) and (5.7) respectively. These equations are applicable for bending in one direction. If the reference cylinder is bent in more than one direction, these equations are modified and the twist, binormal and normal curvatures are the sum of the components generated by these two bending directions.

The above concept is applied to the double helix considered here. In this case, the twist of the double helix is the sum of the twist generated by the imposed
normal and binormal bending curvatures and the twist of the local reference cylinder. The normal and binormal curvatures of the double helix are also the sum of their respective components generated by the imposed normal and binormal bending of the local reference cylinder. The twist and curvatures in the normal and binormal directions are then given by:

\[
\tau_2 = (\kappa_{1,n} \sin \phi_2 - \kappa_{1,b} \cos \phi_2) \sin \alpha_2 + \tau_1 \quad (5.14)
\]

\[
\kappa_{2,b} = (\kappa_{1,n} \sin \phi_2 - \kappa_{1,b} \cos \phi_2) \cos \alpha_2 \quad (5.15)
\]

\[
\kappa_{2,n} = \kappa_{1,n} \cos \phi_2 + \kappa_{1,b} \sin \phi_2 \quad (5.16)
\]

where \( \tau_1 \) is the twist of the local reference cylinder and \( \kappa_{1,n} \) and \( \kappa_{1,b} \) are bending curvatures in the normal and binormal directions respectively. Substituting for the twist and curvatures from equation (5.5), equation (5.6) and equation (5.7), the twist and curvatures in the double helix can be expressed as:

\[
\tau_2 = (\cos \phi_1 \sin \phi_2 + \sin \phi_1 \cos \phi_2 \cos \alpha_1) \kappa_\mu \sin \alpha_2 \quad (5.17)
\]

\[
\kappa_{2,b} = (\cos \phi_1 \sin \phi_2 + \sin \phi_1 \cos \phi_2 \cos \alpha_1) \kappa_\mu \cos \alpha_2 \quad (5.18)
\]

\[
\kappa_{2,n} = (\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \alpha_1) \kappa_\mu \quad (5.19)
\]

Considering one half pitch length of the electrical cable between \( \phi_1 = -\pi / 2 \) and \( \phi_1 = \pi / 2 \), the maximum twist and the maximum binormal curvature occur at \( \phi_1 = \phi_2 = \pm \pi / 4 \). The maximum normal curvature occurs at the neutral axis of bending \( (\phi_1 = \phi_2 = 0) \). At this location the curvature is the curvature of the global reference cylinder and the twist and binormal curvature are zero.

5.2 Mechanical Loads in Electrical Cores

The electrical cables are laid in a helical configuration within the umbilical assembly and the electrical cores are in turn laid in a helical configuration
within the electrical cable. The cores can thus be modelled as a double helix. The electrical cable represents the local reference cylinder of this double helix and the umbilical represents the global reference cylinder. The axial strain in an electrical core is thus given by:

\[
\varepsilon_2 = \cos^2\alpha_2 \left( \cos^2\alpha_1 \frac{du_z}{dz} + R_1 \sin\alpha_1 \cos\alpha_1 \frac{d\phi}{dz} + \frac{\sin^2\alpha_1}{R_1} u_{R,1} \right) - r_2 \kappa_u \sin\phi_1 \sin\alpha_1 \sin\alpha_2 \cos\alpha_2 - r_2 \kappa_u \sin\phi_2 \cos\phi_1 \cos^2\alpha_2
\]

\[
+ r_2 \kappa_u \cos\phi_2 \sin\phi_1 \cos^2\alpha_2 \cos\alpha_1 + \frac{\sin^2\alpha_2}{r_2} u_{r,2}
\]

\[-R_1 \kappa_u \sin\phi_1 \cos^2\alpha_1 \cos^2\alpha_2 \]

where \( R_1 \) and \( \alpha_1 \) are the radius and lay angle of the electrical cable, \( r_2 \) and \( \alpha_2 \) are the radius and lay angle of the core, \( u_z \) and \( u_\phi \) are the axial extension and twist of the umbilical, \( u_{R,1} \) and \( u_{r,2} \) are the radial displacement of the electrical cable and the core and \( \kappa_u \) is the curvature of the umbilical. The above equation was derived assuming the electrical cables are under contact pressures sufficient to prevent their slip. The axial strain in the cores is seen to vary sinusoidally along their arc length. For this strain state to hold, the core should be subjected to an external shear force. This shear force was derived in equation (5.13) presented earlier. This shear force can exceed the maximum frictional force of the contact surface and the core can slip in a similar manner to the slip of the armour wires discussed in chapter two.

For slip of the armour wires it was shown that for typical wires the critical curvature occurs at a bending radius exceeding 1000 metres. The armour wires analysed were 5 mm in diameter laid at an angle of 30 degrees and a helix radius of 50 mm. The frictional coefficient was assumed to be 0.2. The contact line pressure, assuming the stress in the armour wires is equivalent to one third the yield stress of steel, was calculated to be 7 N/mm. For the simplest case where the lay angle of the cores is zero, the critical bending radius of the core is given by:
\[
\rho_{\text{crit}} = \frac{-EA\sin\alpha_1 \cos^2\alpha_1}{\mu X}
\]  

(5.21)

where \(\mu\) is the friction coefficient and \(X\) is the line contact pressure. Assuming the cores are under a contact line pressure of equal magnitude to the contact pressure of the armour wires and the frictional coefficient is also 0.2, the critical bending radius for a 2.5 mm\(^2\) signal core is 155 metres. This critical bend radius is small compared to the critical bend radius of an armour wire. However, for typical helical radii of electrical cables the corresponding mechanical strain for this critical bending radius is small. Assuming the electrical cable is laid at a helical radius of 50 mm and a helical angle of 30 degrees, the corresponding strain is less than 0.0001\%. This small level of strain can be ignored. Ignoring this level of strain is equivalent to assuming that the copper cores are in a full slip condition. In this case the only mechanical strain in a copper core due to global flexural loads on the umbilical is the strain resulting from bending about the neutral axis. However, given the critical function of these components other aspects of the slip mechanism needs to be considered. These for example include the contact forces generated during the manufacturing process. The contact forces generated by the manufacturing process could be of sufficient magnitude to reduce the critical bending radius. As a result the strains in the copper cores at the inception of slip would be of significant magnitude.

### 5.2.1 Residual Contact Pressure

Above, the slip mechanism of the copper cores was analysed and it was shown that for a 2.5 mm\(^2\) signal core the critical bending radius at the inception of slip is 155 metres. The contact pressure was evaluated as the pressure generated by the constriction of the helical armour wires. This is not strictly the case as these components will be subject to residual contact forces generated during the manufacturing process and the hydrostatic pressure as the electrical cables
are designed to be water-tight. An example of the contact forces generated during the manufacturing process is the shrinkage of the sheath layer of the electrical cable. These contact forces can be of considerable significance depending on the material type and dimensions of the sheath. To evaluate the magnitude of the contact forces generated due to the shrinkage of the sheath it is assumed that the core is rigid and the temperature is uniform throughout the thickness. In this case, the contact pressure at the inner surface of the sheath layer \( \sigma_r \) is given by:

\[
\sigma_r = \frac{(1 - v)E}{(1 + \nu)(1 - 2\nu)} \left( \varepsilon_z + \frac{\nu}{1 - \nu} \varepsilon_\theta \right)
\]  

(5.22)

where \( \nu \) and \( E \) are the Poisson’s ratio and the elastic modulus of the sheath layer, \( \varepsilon_z \) and \( \varepsilon_\theta \) are the axial and hoop strain resulting from shrinkage and are given by:

\[
\varepsilon_z = \varepsilon_\theta = -\alpha(T_1 - T_2)
\]  

(5.23)

where \( \alpha \) is the coefficient of thermal expansion, and \( T_1 \) and \( T_2 \) are the initial and final temperatures. Over the temperature range considered, the elastic modulus of the sheath layer varies from zero to 150 MPa assuming the layer is made of low density polyethylene (LDPE). To take this into account, an incremental approach is used and the contact pressure on the inner side of the sheath is then given by:

\[
\sigma_r = \frac{\alpha}{(1 + \nu)(1 - 2\nu)} \int_{T_1}^{T_2} EdT
\]  

(5.24)

It is assumed that the elastic modulus of the polymer varies linearly for the temperature range between zero and the maximum service temperature. The maximum service temperature of LDPE is 82 °C (Harper, 1975). At higher temperatures the polymer is assumed to be in a fully plastic state.
For LDPE the coefficient of thermal expansion is 0.0001 and the Poisson’s ratio is 0.42. Assuming the electrical cable is cooled at 5 °C, the contact pressure on the inner side of the sheath layer is 3.2 MPa. In calculating this contact pressure it was assumed that the electrical cables are cooled rapidly after the sheath layer is extruded and in this case stress relaxation can be ignored. In assembling the electrical cables cooling tanks are used to ensure the thermoplastic sheath layer sets rapidly before the cable is reeled. This also helps ensure the circularity of the sheath. In this respect ignoring the stress relaxation would not lead to erroneous results for the contact pressure shortly after the cable is assembled. After a certain length of time, the visco-elastic properties of LDPE will result in creep and stress relaxation and consequently the contact pressure will decrease. However, some residual of the initial contact pressure will still be present after the umbilical is installed and in operation. This has been confirmed from observations on dissected electrical cables. In addition to the contact forces generated by the shrinkage of the sheath, the cores within the electrical cable will be subjected to contact forces generated by the hydrostatic pressure acting on the sheath layer of the electrical cable. This is the case given the electrical cables are designed to be water-tight. The cores will also be subjected to contact pressures generated during the other stages of the manufacturing process such as armouring the cables or applying the signal shield layer. The cumulative effect of these contact pressures could increase the frictional resistance at the contact surfaces and consequently slip would occur at a considerably lower bending radius. Given the critical function of the electrical cables it is thus reasonable to assume that the cores of these cable will be in a no slip condition. This case leads to conservative results and is designated by the partial slip condition.

It is pointed out that evidence for the high residual forces generated by shrinkage of the sheath layer has been reported in a survey by Knight (1990). Knight reported that in one case the presence of kinks has been reported after the sheath layer was extruded over the bundled cores. This was attributed to
the relatively high modulus and thickness of the sheath layer. On shrinkage of this layer, high axial forces where imposed on the cores and the magnitude of these forces were sufficient to buckle the cores. To assess the magnitude of the axial force generated due to cooling of the sheath layer discussed above, it is assumed that during cooling the radial strain is linear through the thickness of the sheath. The axial stress in the sheath layer is then given by:

\[ \sigma_z = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} \left( \varepsilon_z + \frac{\nu}{1 - \nu} (\varepsilon_\theta + \varepsilon_r) \right) \]  

(5.25)

where:

\[ \varepsilon_z = \varepsilon_\theta = \varepsilon_r = -\alpha (T_1 - T_2) \]  

(5.26)

The axial stress is then given by:

\[ \sigma_z = -\frac{\alpha}{(1 - 2\nu)} \int_{T_1}^{T_2} EdT \]  

(5.27)

Using the values for the different coefficients and assuming a linear modulus-temperature the axial stress is 4.5 MPa. The axial force in the electrical cable is thus equal to:

\[ F = -9\pi r_{\text{mean}} t \]  

(5.28)

where \( r_{\text{mean}} \) and \( t \) are the mean radius and thickness of the sheath. As seen in the above equation the resultant axial force is directly related to the dimensions of the sheath. If the thickness is relatively large then the resultant axial force on the electrical cable can cause kinking of the cores. In a similar manner to the build up of radial contact stresses in the cores, the visco-elastic properties will lead to a degree of stress relaxation. Nonetheless, it is expected that some residual of the axial compressive force will be present in the electrical cable after the umbilical has been installed.
5.2.2 General Solution

In the previous section it was argued that the electrical cores are subject to residual contact forces generated during the manufacturing process and contact forces due to the hydrostatic pressure acting on the outer sheath of the electrical cable. It can be shown that these contact forces could be of a considerable magnitude as to reduce the critical bending radius defined as the bending radius at the inception of slip. Consequently, the mechanical loads on the copper cores can be of a significant magnitude. A conservative partial slip condition was proposed to take into account these contact forces that can not be quantified for a generic design model. These contact forces depend on the manufacturing process and the choice of material used in assembling the electrical cables. This conservative slip condition can be justified given the critical function of the cores and the experience that has shown the vulnerability of the cores to kinking and fatigue failures. The maximum axial strain in the copper cores can then be expressed as:

\[ \varepsilon_2 = \cos^2 \alpha_2 \varepsilon_1 + \frac{\sin^2 \alpha_2}{r_2} u_{r,2} + r_2 \kappa_u \cos^2 \alpha_2 \]  

(5.29)

where \( \varepsilon_1 \) is the axial strain in the electrical cables and is given by:

\[ \varepsilon_1 = \cos^2 \alpha_1 \frac{du_z}{dz} + R_1 \sin \alpha_1 \cos \alpha_1 \frac{du_\phi}{dz} + \sin^2 \alpha_1 \frac{u_{R,1}}{R_1} \]  

(5.30)

It is seen from the above equations that the strain in the cores is dependent on the helical angles of both the cores and the electrical cables. The optimal lay angle for the cores to minimise the strain induced by the axi-symmetric and flexural deformation of the umbilical is 90 degrees. The optimal lay angle to maximise these loads is zero. However, the choice of the lay angle is usually constrained by geometry and by the requirement to minimise the signal attenuation. The maximum lay angle of a helical element is given by:

\[ \alpha = \cos^{-1}(nr/\pi R) \]  

(5.31)
where \( r \) is the radius of the helical element and \( R \) is the helix radius. Thus, if 4 electrical cores are to be bundled together in a helical configuration, the lay angle of these cables can not exceed 25 degrees. For this lay angle the length of the electrical cable per unit length of the umbilical would be \( \cos \alpha^{-1} = 1.10 \). The 10% increase in the length of the electrical cable results, in addition to greater cost, in higher attenuation and inductance of the signals being transmitted. As a result, for both the electrical cables and the cores of these cables, manufacturers tend to lay these components as straight as the manufacturing process allows. Assuming this is the case, the expression for the axial strain in the copper cores can be simplified to:

\[
\varepsilon_2 = \cos^2 \alpha_2 \left( \cos^2 \alpha_1 \frac{du_z}{dz} + R_1 \sin \alpha_1 \cos \alpha_1 \frac{du_\phi}{dz} \right) + r_2 K_u \cos^2 \alpha_2 \tag{5.32}
\]

The axial strain in the copper cores as given in the above equation is a function of the axi-symmetric and flexural deformation of the umbilical. The expression which ignores the small radial displacement resulting from the small helical angles leads to conservative results. This is the case since the radial displacement tends to reduce the axial strain in these components.

Given that umbilicals are installed and operated at specified axi-symmetric loads, a relationship need to be established to relate the strain in the electrical cables to the corresponding loads. The relationship between the applied axi-symmetric loads and the deformation of the umbilical was discussed in chapter three. This relationship using a matrix notation can be expressed as:

\[
\begin{bmatrix}
F_z - a_{13}F_r \\
F_\phi - a_{23}F_r \\
\end{bmatrix} = \begin{bmatrix}
a_{11} - a_{31}a_{13} & a_{12} - a_{31}a_{32} \\
a_{21} - a_{23}a_{31} & a_{22} - a_{23}a_{33} \\
\end{bmatrix} \begin{bmatrix}
\frac{du_z}{dz} \\
\frac{du_\phi}{dz} \\
\end{bmatrix} \tag{5.33}
\]

where \( F_z \) is the axial force on the umbilical, \( F_\phi \) is the torsional load, \( F_r \) is the radial force and \( a_{ij} \) are the coefficients of the structural stiffness matrix. The inverse of the above equation can be written as:
Rewriting the axial strain in the copper cores as:

\[
\varepsilon_2 = \begin{bmatrix} \cos^2 \alpha_2 \cos^2 \alpha_1 & R_1 \cos^2 \alpha_2 \sin \alpha_1 \cos \alpha_1 \end{bmatrix} \left[ \begin{bmatrix} \frac{du_z}{dz} \\ \frac{du_{\phi}}{dz} \end{bmatrix} \right]^T + r_2 \kappa \cos^2 \alpha_2
\]

(5.35)

the above equation can be expressed as:

\[
\varepsilon_2 = \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} F_z - \frac{a_{13} F_r}{a_{33}} \\ F_{\phi} - \frac{a_{23} F_r}{a_{33}} \end{bmatrix} + r_2 \kappa \cos^2 \alpha_2
\]

(5.36)

where the coefficients \(b_{11}\) and \(b_{12}\) are given by:

\[
b_{11} = \cos^2 \alpha_2 \cos^2 \alpha_1
\]

(5.37)

\[
b_{12} = R_1 \sin \alpha_1 \cos \alpha_1 \cos^2 \alpha_2
\]

(5.38)

and the coefficient matrix \(C\) is given by:

\[
C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} - \frac{a_{31} a_{13}}{a_{33}} & a_{12} - \frac{a_{13} a_{32}}{a_{33}} \\ a_{21} - \frac{a_{23} a_{31}}{a_{33}} & a_{22} - \frac{a_{23} a_{31}}{a_{33}} \end{bmatrix}^{-1}
\]

(5.39)

The above equations constitute an analytical approach where the structural response of the umbilical can be used to evaluate the mechanical strain in the copper cores.

5.3 Case Study

The above analytical approach to assess the mechanical loads in the copper cores of a subsea umbilical is implemented in this section for umbilical II.
discussed earlier in chapter four. This umbilical is made of 5 armour layers
and is of a relatively low axial stiffness as compared to the other umbilicals
analysed. The axial stiffness of this umbilical with the hoses empty was
estimated to be 108 MN. The electrical cables of this umbilical are laid with a
helical angle of 10 degrees. Other data required for this case study is
presented in Table 5.2. This table shows the dimensions and lay configuration
of the different layers.

In this case study, the mechanical strains in the different constituent elements
are investigated for the maximum installation loads. The maximum
installation loads considered are the laying tension, the tensioners crushing
load and the curvature of the laying wheel or reel overboard shute. The
maximum laying tension and crushing load are a function of the submerged
weight of the umbilical and the water depth. The curvature of the laying
wheel is specific to the type of lay vessel and for this particular umbilical the
curvature of the wheel is 0.17/m. The maximum laying tension and crushing
load are given by the following empirical expressions:

\[ F_z = c_1 \times w \times h \]  \hspace{1cm} (5.40)

\[ F_r = c_2 \frac{F_z}{A_c} \]  \hspace{1cm} (5.41)

where \( F_z \) and \( F_r \) are the laying tension and crushing load respectively, \( w \) and
\( h \) are the submerged weight of the umbilical and the water depth, \( c_1 \) and \( c_2 \)
are constants which take into account the catenary configuration, dynamic
amplification and friction at the contact surface of the tensioners, and \( A_c \) is the
contact length of the tensioners. For this umbilical the submerged weight is
given as 98.3 kg/m and the water depth is 350 metres. The constants \( c_1 \) and
\( c_2 \) are 1.5 and 0.025 respectively. The tensioners are made of 4 tracks each 3.68
metres in length resulting in a contact length of 14.72 m. The laying tension
and the crushing load are then 507 kN and 0.915 MPa.
Table 5.2: Constructional details of umbilical II.

<table>
<thead>
<tr>
<th>Inner sheath</th>
<th>Mean radius (mm)</th>
<th>Modulus (MPa)</th>
<th>Thickness (mm)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>85.9</td>
<td>350</td>
<td>5</td>
<td>0.42</td>
</tr>
<tr>
<td>Armour 1</td>
<td></td>
<td></td>
<td>79</td>
<td>-30</td>
</tr>
<tr>
<td></td>
<td>91.9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Armour 2</td>
<td></td>
<td></td>
<td>84</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>97.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Armour 3</td>
<td></td>
<td></td>
<td>84</td>
<td>-30</td>
</tr>
<tr>
<td></td>
<td>103.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Armour 4</td>
<td></td>
<td></td>
<td>94</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>110.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Armour 5</td>
<td></td>
<td></td>
<td>101</td>
<td>-30</td>
</tr>
<tr>
<td></td>
<td>116.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For this case study, the axial strain in the cores is calculated for the maximum installation loads. Different loading regimes are examined to take into account the variations of the applied loads along the section of the umbilical near the proposed laying assembly. The proposed laying assembly that is shown in Figure 5.3 consists of a large diameter spool, tensioners and a profiled laying wheel. As the umbilical is reeled off the spool, the load regime can be idealised as free bending. As the spooled length enters the tensioners, the umbilical is under external pressure with minimum axial load and zero curvature. Near the other extremity of the tensioners the umbilical is under combined axial load and external pressure. The most critical loading regime occurs near the tip of the laying wheel where the umbilical is under combined axial load, flexural load and asymmetric external pressure. This asymmetric pressure results from contact with the sheave. It is noted that the contact line pressure can be calculated as the product of the axial tension and the curvature of the laying wheel. This contact line pressure is calculated as 84.5 N/mm. Assuming the effective width of the contact surface is equal to the diameter of the umbilical, the effective contact pressure is 0.2 MPa.
Table 5.3 shows the axial strain in the electrical cores for these three different loading regimes assuming the armour wires are in a full slip condition and the copper cores are in a partial slip condition. It is seen from this table that the axial strain in the electrical cores resulting from the maximum installation tension is one order of magnitude greater than the axial strain in the most critical armour layer. Moreover, the axial strain in the inner armour layer is compressive. This is due to the excessive radial deformation of the core. It can also be said that the armour layers of this umbilical do not share equally the imposed loads and the compressive stresses in the inner armour layer result in a reduction in the axial stiffness of the umbilical. As for the axial strain in the different components of the umbilical due to the imposed crushing load, it is seen that the strain in the copper cores are tensile while the strain in the armour wires are compressive. The magnitudes of these compressive strains are smaller that the magnitude of the strains due to the maximum laying tension.

It is noted from Table 5.3 that the strains in the armour wires for the maximum installation loads are within the elastic regime while the strains in the copper cores exceed their elastic limits. If the copper cores are in an annealed condition, the plastic strain in the cores for the maximum installation tension is about 0.35%. This is calculated assuming the yield stress of annealed copper is 70 MPa. The plastic deformation of the cores results in a residual extension which translates to compressive forces once the global loads on the umbilical are relaxed. This is largely due to the strain controlled loading conditions of the copper cores as explained in the following section.
Table 5.3: Strain in different components of umbilical for maximum installation loads

<table>
<thead>
<tr>
<th>Component</th>
<th>Strain (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maximum laying tension=507 kN</td>
</tr>
<tr>
<td>Electrical core</td>
<td>0.409</td>
</tr>
<tr>
<td>Armour 1</td>
<td>-0.019</td>
</tr>
<tr>
<td>Armour 2</td>
<td>0.004</td>
</tr>
<tr>
<td>Armour 3</td>
<td>0.024</td>
</tr>
<tr>
<td>Armour 4</td>
<td>0.042</td>
</tr>
<tr>
<td>Armour 5</td>
<td>0.058</td>
</tr>
</tbody>
</table>

5.3.1 Strain Controlled Loading

The wires in the conductors of electrical cables are required by the International Electrotechnical Commission to be made of plain or tin-coated annealed copper (IEC 228, 1978). This requirement is incorporated into the specification of the American Petroleum Institute (API 17E, 1994). If this is the case and the conductors within the umbilical assembly are in a fully annealed condition, the cores upon loading will develop a high level of plastic strain. This plastic strain upon unloading will translate to compressive loads. This is explained with reference to Figure 5.4. This figure shows a system of two springs connected in parallel and loaded as to extend the two springs by $\Delta x$. One of these springs which is denoted by the subscript 1 is an elastic spring and the other spring denoted by the subscript 2 is an elasto-plastic spring with a yield point $\Delta x_y < \Delta x$. For the given extension, the elasto-plastic spring will experience a plastic strain that is given by $\Delta x_p = \Delta x - \Delta x_y$. The equilibrium equation of the loaded system is given by:

$$F = K_1 \Delta x + K_2 (\Delta x - \Delta x_p)$$  \hspace{1cm} (5.42)
Due to the plastic deformation of the elasto-plastic spring, the unloaded system will be under a residual extension. This residual extension which ensures the equilibrium of the unloaded system is given by:

\[ \Delta x = \frac{\Delta x_p}{1 + K_1/K_2} \]  

(5.43)

As seen in the above equation, the residual extension is dependent on the ratio of the spring stiffnesses. If this ratio is small as compared to one, the residual extension is equal to the plastic extension of the elasto-plastic spring. As this ratio is increased the residual extension approaches zero.

Subsea umbilicals can be idealised using the spring system discussed above. The strains in the armour layers are often well within the elastic limits. Also, the stiffness of the armour layers is considerably greater than the axial stiffness of the electrical cores. Thus, it can be concluded that the umbilical will not develop any significant residual strain. Given the above, the loading-unloading path on the cores will be under strain controlled loading where the maximum strain is reached near the water surface. As the umbilical is installed the strain level at any specific point will reduce and on the seabed the strain in the cores would be zero. Assuming the cores are made of annealed copper, the loading-unloading path will result in compressive forces as illustrated in Figure 5.5. As the umbilical is loaded, the strain in the cores will follow the loading path labelled A-B and the cores will experience plastic strain. As the umbilical is unloaded the strain in the cores will follow the linear unloading path labelled B-C and the slope of this path is equal to the equivalent modulus of the core. Given the umbilical does not suffer any residual strain, the unloading path will continue along the non-linear path labelled C-D and the cores will be subject to compressive forces.

In the above discussion it was assumed that the cores will be in an annealed condition. However, the copper cores are likely to work harden as the
electrical cables and the umbilical are assembled. In this case the cores will not be in an annealed condition and an assessment of the in-situ mechanical properties is discussed in the following section.

5.3.2 As-Built Work Hardened Condition

The amount of work hardening in the cores depends on the manufacturing process and is expected to vary for different umbilicals and for different manufacturers. This is the case because the loads used in the manufacturing process are often not specified but selected to meet the requirements of the assembly process.

As assessment of this work hardening for the copper conductors of subsea umbilicals was carried out experimentally. The experimental procedure to assess the work hardening involves comparing the stress-strain curve in the as-built condition with the stress-strain curve in the annealed condition. The cores were dissected from an umbilical assembly and samples of the core were annealed to determine their stress-strain curve in the annealed condition. This was carried out for conductors dissected from umbilicals of two different manufacturers and conductors dissected from a commercially available electrical cable. A measure of the work hardening is achieved by offsetting the origin of the stress-curves as explained later.

For one of the manufacturers designated by manufacturer I, the conductors tested were made of 7 wires with a diameter of 0.65 mm. A sample of the wires was annealed to allow a comparison between the as-built mechanical properties and the annealed mechanical properties. A plot of the stress-strain curves for annealed and as-built conditions are shown Figure 5.6. From this figure it can be concluded that during the manufacturing process the wires of
the conductors were pre-strained by 17% and the corresponding yield stress is about 160 MPa.

The experimental tests to determine the pre-strain of the cores were carried out using a tensometer which is shown in Figure 5.7. This type of test machine measures the force applied on the test specimen by measuring the deflection of a load beam. The axial extension is measured from the deflection at one end of the test specimen. To obtain the optimal magnification, a 300 N beam was used. Typical experimental results using this tensometer are shown in Figure 5.8. The results show an irregularity at the start of the test. This irregularity is expected as any slack in the test specimen is taken out. Subsequent to this irregularity and as the specimen is tensiled some errors are expected in measuring the strains in the elastic region. The main contributory factor for these errors is the deflection of the beam which is of a significant proportion as compared to the elastic strains to be measured. Another contributor factor is the movement of the clamps. The type of clamps used consist of an eccentrically connected wheel as shown in Figure 5.9. As the load is increased the wheel rotates and the eccentric centre of rotation results in a clamping pressure on the test specimen. At low load levels the clamping pressure would not be sufficient to fully restrain the movement of the specimen within the clamp. Such distortions in measuring the elastic strains using tensile testing machines can not be avoided and such errors are evident in the experimental stress-strain plots presented by Benham (1961) and Morrow (1965). This was confirmed by using an alternative test machine which is shown in Figure 5.10. This test machine measures the load using a load cell and the strains in the specimen can be measured using an extensometer. Typical stress-strain curves obtained from these experiments are shown in Figure 5.11. The errors observed are similar to the errors seen using a tensometer. The source of error in this case is the alignment of the extensometer arms. As the jaws of the extensometer arms are closed the arms will be subject to forces which can result in their bending. As the load is
increased the arms will not only separate but they will also be straightened out. Due to this, measurements of the strains in the elastic region will also be distorted. Ideally measurements of strain in the elastic region should be carried out using strain gauges. However, the smallness of the test specimen prevents the application of this test procedure.

Nonetheless, it is emphasised that the error in the experimentally determined stress-strain curves is only significant at strains within the elastic region. This error decreases at higher strain levels as the material extension will be large and the load increments will be small. In this case, the material deflection is the dominant contribution compared with the deflection of the test equipment for the corresponding small load increment. This allows the procedure of comparing the stress-strain curves of as-built and annealed cores to be used with a good level of confidence. This is explained with reference to Figure 5.12 that show a stress-strain curve of specimens in the as-built condition and the stress-strain curve of specimens pre-strained by 17%. The pre-strained specimens were annealed and then loaded so that the strain reached was 17%. The good level of agreement between the two curves shows that the results obtained earlier are of a good accuracy. In other words, the method of offsetting the stress-strain curve to determine the amount of pre-strain can be used irrespective of the errors seen in measuring the characteristics of the curve in the elastic region. The two curves shown in Figure 5.12 are shown without any correction to take into account the deflection of the beam or the movement within the clamps. If any correction is to be applied, the correction will be identical for both curves and the good level of agreement will be preserved.

The same experimental procedure was carried out for the conductors of another manufacturer designated by manufacturer II. The conductors were made of 7 wires with a diameter of 0.85 mm. The stress-strain curves for the as-built and annealed conditions are shown in Figure 5.13. It is seen from this
figure that the pre-strain level due to the manufacturing process is 18% and the corresponding yield stress is 160 MPa. This pre-strain level is equivalent to the pre-strain level in the conductors of manufacturer I.

The experimental results showed that the pre-strain level of the conductors is of equivalent magnitude for both major umbilical manufacturers. The main reason the pre-strains are of equal magnitude is the fact that the dominant work hardening of the conductors occurs as the wires of the conductors are wound together. During this stage of the manufacturing process, axial and flexural loads are applied to the wires to wind them as tightly as possible. These loads are supported only by the wires that are in an annealed condition as specified by IEC 28. This is illustrated with reference to Figure 5.14 which shows experimental stress-strain curves of wires stripped from a standard commercially available electrical cable. This figure shows the strain-curves in the as-built condition and in the annealed condition. The figure shows that the level of pre-strain is equivalent to 15%. This level of pre-strain is comparable with the level of pre-strain seen in the conductors of manufacturers I and II. It can thus be concluded that a level of pre-strain between 15% and 18% is generic to the copper conductors of the size considered here. These are namely 2.5 mm² and 4 mm² conductors which constitute the majority of signal and power conductors used in subsea umbilicals.

The pre-strain of the copper cores during the manufacturing process influences both the cyclic material properties and the fatigue characteristics of the cores. The influence on the fatigue characteristics is discussed in chapter six. The influence on the cyclic material properties was discussed by Klesnil and Lukas (1980). Klesnil and Lucas showed that annealed copper exhibits cyclic hardening while a pre-strain exceeding 20% results in a cyclic softening behaviour as shown in Figure 5.15. Cyclic hardening is a material property where under cyclic loads, the stress required to strain a specimen between two
defined limits increases with each cycle. Cyclic softening is the reverse phenomenon where the required stress decreases. After a number of cycles, the cyclic stress-strain curve attains a stable state. Cyclic softening and hardening is illustrated in Figure 5.16. From the experimental results of Klesnil and Lucas it can be assumed that the cores will have the tendency to work soften. However, the differences between the cyclic and monotonic stress-strain will be too small to be of any practical significance.

5.4 Concluding Summary

The mechanics of electrical cables were discussed in this chapter taking into account the double helical configuration of the copper cores. The effect of the double helical configuration was analysed using a recursive approach which is often used in analysing the mechanics of wire ropes. The results showed that the cores are subject to contact forces resulting from the constriction of the armour wires, the manufacturing process and hydrostatic pressure. It was shown that the contact forces due to shrinkage of the sheath are of considerable magnitude. This is in agreement with a survey carried out by Knight (1992). Knight pointed out that one of the problems reported by a manufacturer was the presence of kinks during the manufacturing process. Kinking as explained earlier is a buckling type of failure resulting from compressive axial forces on the conductors. The presence of these kinks was attributed to the application of an MDPE sheath. The shrinkage of the sheath led to axial and radial forces that were of significant magnitude to buckle the cores. The grade of the sheath was changed and its thickness was reduced. These measures eliminated the formation of kinks in this case.

The residual contact forces resulting from the manufacturing process are difficult to quantify and will depend on a number of parameters such as cooling rate, stress relaxation and the creep properties under triaxial loading
conditions. As a conservative approach it was argued that the cores of a subsea umbilical should be considered to be in a partial slip condition. This is the slip condition where the electrical cables within the umbilical slip but the cores within the electrical cable do not slip. In other words, the electrical cables are assumed to behave as a uniform helical rod. This conservative assumption is justified given the critical function of the electrical cables and the high frequency of their failures.

An analytical model was developed to assess the axial strains in the copper cores from the predicted structural response of the umbilical. This model was implemented in a case study for an umbilical of relatively low axial stiffness. It was shown that for the maximum installation loads the axial strains in the copper cores can exceed the axial strains in the armour wires. This is the case given the low helix angles of the electrical cables and the cores as compared with the helical angle of the armour wires. For the case study analysed, the strains in the copper cores exceeded the elastic limit while the strains in the armour wires were well within their elastic limits. Given the dominant contribution of the armour wires to the axial stiffness of the umbilical, it was argued that the cores will be under strain controlled loading. As a result, the cores will be compressed once the umbilical is unloaded. The experimental tests carried out showed that the yield stress and the corresponding yield strain of the cores in the as-built condition are 160 MPa and 0.1% respectively. Thus, it can be said that for any cyclic loads on the umbilical resulting in strains in the cores exceeding these limits, the cores will develop compressive forces. The compressive forces are beneficial as far as fatigue is concerned but in some cases, these compressive forces can reach critical limits and cause buckling of the conductors.
6 Kinking and Fatigue of Copper Conductors

In the earlier chapters analytical models were developed to predict the global structural response of umbilicals and the corresponding mechanical loads in the copper conductors of these umbilicals. These models allow the failure modes of the electrical cables to be addressed and two fundamental modes are considered in this chapter. The first is the buckling failure of the copper conductors due to compressive stress and the second is the fatigue failure due to cyclic stress. The buckling failure of the copper conductors is also known as kinking.

A kinked conductor is shown in Figure 1.3 presented earlier. This figure shows a kink in a 7 wire signal conductor. The figure shows that the wavelength of the kink is approximately 9 mm which is equivalent to 12 times the diameter of the wire in the conductor. Kinking is largely associated with the non-linear material properties of the conductors and their cyclic strain controlled loading within the umbilical assembly. Under cyclic strain controlled loading and for loads resulting in plastic deformation the conductors develop compressive forces. These compressive forces can reach critical limits resulting in the formation of kinks with the subsequent failure of the electrical cable. Failure of electrical cables due to the formation of kinks is one of the most common failure modes of subsea umbilicals. These failures are often reported subsequent to the installation of the umbilical and also during manufacturing.

In many respects, the kinking failure of the copper conductors is analogous to the buckling of beams on elastic foundation. This is explained with reference to the model shown in Figure 6.1. This figure shows a conductor whose longitudinal axis coincides with the longitudinal axis of the electrical cable. The annulus between the conductor and the sheath of the electrical cable is
occupied by the insulating polymer layers. These layers restrain the lateral displacement generated by kinking. In a similar manner, the foundation restrains the buckling displacement of supported beams. The restraint can exhibit a softening or a hardening behaviour depending on the boundary conditions and the material properties. For kinking of the conductors, it is shown that the insulation and filler layers exhibit a hardening resistance.

Fatigue failure of the copper conductors is the second failure mode addressed in this chapter. Fatigue failure is an important failure mode in dynamic subsea umbilicals. These umbilicals spanning the water column between the surface production vessel and the seabed experience a spectrum of loads induced by currents, waves and motions of the production vessel. Their specified design life is between 20 and 30 years. Over this design life it is likely that the conductors will fail due to excessive fatigue damage unless the loads in these conductors are kept within certain limits. In assessing their fatigue damage, it is often assumed that the copper conductors are in a fully annealed condition. However, the conductors are not truly in an annealed condition due to the pre-strain induced during the manufacturing process. The level of pre-strain was determined in chapter five to be between 15% and 18%. The influence of pre-strain on the fatigue properties of the copper cores is analysed and representative fatigue design curves for the cores are proposed in this chapter.

6.1 Buckling and Stability

Prior to examining kinking of the electrical conductors, the general issues associated with buckling and stability need to be reviewed. Buckling and stability are two closely related engineering problems. In its simplest form, buckling analysis is a study of the critical loads at which more than one
equilibrium state can be found. Stability analysis, on the other hand, is the study of these equilibrium states. This is illustrated with reference to the axially compressed links shown in Figure 6.2. In this figure two different links are shown, one link is supported by a rotational spring and the other is supported by a lateral spring. Both links will buckle at certain load magnitudes but the equilibrium of the buckled state is stable for the link with a rotational spring and unstable for the link with a lateral spring.

Buckling analysis of the link models presented above is carried out by imposing a small perturbation to the initial equilibrium state. Using a linear formulation, the equilibrium equations are then solved and the load magnitude at which a non-trivial solution exists is determined. This load magnitude is defined as the critical load. It is then assumed that at the critical load any small perturbation will cause the structure to deviate from the initial equilibrium state to the buckled equilibrium state. Stability analysis is the study of this buckled equilibrium state. The buckled equilibrium state is stable if an increase in the load magnitude is required to increase the buckle displacement otherwise this equilibrium state is unstable.

While a buckling analysis can be carried out using a linear formulation, stability analysis can only be carried out using a non-linear formulation. This is explained here for the two link models shown in Figure 6.2. For the link model with a rotational spring of stiffness $C$ the non-linear equilibrium equation of the buckled state is given by:

\[ P = \frac{C\theta}{L\sin\theta} \quad (6.1) \]

where $P$ is the load magnitude and $L$ is the length of the link. For the link model with a lateral spring of stiffness $K$ the non-linear equilibrium equation of the buckled state is given by:

\[ P = KL\cos\theta \quad (6.2) \]
Assuming small rotations the above equations can be linearised and the buckling loads for the two link models can be derived. However, while for the link with a rotational spring the equilibrium of the buckled state is stable, this is not the case for the other link model. This can be inferred from studying the post-buckling curves shown also in Figure 6.2. The post-buckling curve of the model with a rotational spring shows that when the model is perturbed from the straight configuration, an increase in the load magnitude is required to deflect the link beyond this perturbed state. As for the link model with a lateral spring, the link deviates further away from the perturbed state even if the magnitude of the load is decreased. Thus, it can be said that at the buckling load the link with a rotational spring is stable, while the other link is unstable.

The above discussion serves to illustrate the significance of a non-linear formulation in studying stability. While incorporating non-linear terms in the above link models was simple and straightforward, incorporating such non-linear terms for continuous structures such as the copper conductors is not that simple. The equilibrium equations of continuous structures are expressed in the form of differential equations and introducing any non-linearity will considerably complicate the solution of these equations. Nonetheless, a non-linear formulation is necessary in order to examine the post-buckling path. If the post-buckling path is unstable, then any imperfection would result in catastrophic failures at much lower loads than those predicted from linear theory. If the post-buckling path is stable, then any imperfection would eliminate the presence of a bifurcation point but the curve would still follow a stable path. Thus, it can be argued that as far as calculating a safe load limit is concerned, the influence of small imperfections is less pronounced in stable post-buckling states than in unstable post-buckling states.
6.2 Kinking of Copper Conductors

To highlight the main factors involved in analysing kinking of the copper conductors of electrical cables a simplified model is considered first. This model is shown in Figure 6.1 presented earlier and it is assumed that the longitudinal axis of the conductor is aligned with the longitudinal axis of the electrical cable. The annulus between the conductor and the outer sheath layer is occupied by the insulation and filler polymeric layers. The equilibrium equation of the conductor can be thus be written as:

\[ \frac{d^2 M}{dx^2} + T \frac{d^2 w}{dx^2} - X = 0 \]  

(6.3)

where \( M \) is the bending moment, \( w \) is the displacement, \( T \) is the compressive axial load, \( X \) is the contact line force distributed along the conductor and \( x \) is the distance along the longitudinal axis of the conductor. Substituting for \( M \) as the product of the bending stiffness \( EI \) and the curvature \( \frac{d^2 w}{dx^2} \) and assuming the lateral distributed force is a function of the stiffness of the foundation \( \beta \) and the displacement \( w \), the above equation can be written as:

\[ \frac{d^4 w}{dx^4} + T \frac{d^2 w}{dx^2} + f(\beta, w) = 0 \]  

(6.4)

In the above equation, it is assumed that the displacement is small and the influence of any imperfections is ignored. The foundation in this model is the polymeric layers in the annulus between the core and the outer sheath.

6.2.1 Stiffness of Polymeric Layers

The insulation and filler layers are mainly made of elastomeric material with stress-strain characteristics similar to the stress-strain characteristics of rubber. The relationship between the stress and the extension of rubber is given by the empirical relationship:
\[ \sigma = \frac{E}{3}(e - e^{-2}) \]  
(6.5)

where \( \sigma \) is the nominal stress, \( E \) is modulus of elasticity and \( e \) is the extension ratio which is related to the strain by the following expression:

\[ e = 1 + \varepsilon \]  
(6.6)

A comparison of the empirical stress-extension relationship with experimental results is shown in Figure 6.3. The figure shows a very good agreement between the theoretical and experimental curves. Substituting for the extension ratio and using a binomial expansion up to second order terms, the stress-strain relationship of an elastomer can be expressed as:

\[ \sigma = E(\varepsilon - \varepsilon^2) \]  
(6.7)

If the insulating polymer is to be modelled as a foundation of a beam, the above stress-strain relationship should be modified to ensure its symmetry. This is the case given that whether the beam deflection is positive or negative, such deflection will compress the polymer. Assuming the thickness of the polymer is \( t \), the strain can be expressed as:

\[ \varepsilon = \begin{cases} \frac{w}{t} & w < 0 \\ -\frac{w}{t} & w > 0 \end{cases} \]  
(6.8)

where \( t \) is the thickness of the insulation and \( w \) is the deflection of the beam. The load-deflection relationship of the beam can then be expressed as:

\[ X = \begin{cases} Eb\left(\frac{w^2}{t^2} - \frac{w}{t}\right) & w < 0 \\ Eb\left(\frac{w^2}{t^2} + \frac{w}{t}\right) & w > 0 \end{cases} \]  
(6.9)

where \( b \) is the width of the beam. Analytical solutions using the above expression would be difficult in the context of the problem in hand and an alternative expression for the resistance of the foundation stiffness can be written as:
The above equation, as shown in Figure 6.4, compares well with the analytical expression derived by Treloar (1958) for compressive strains less than 70%. In this respect the analytical expression given in equation (6.10) can be used with reasonable accuracy. The equilibrium equation of the core shown in Figure 6.1, taking into account the non-linear foundation can thus be written as:

\[
E I \frac{d^4 w}{dx^4} + T \frac{d^2 w}{dx^2} + \beta_1 w + \beta_2 w^3 = 0
\] (6.11)

where \( \beta_1 \) and \( \beta_2 \) are given by:

\[
\beta_1 = \frac{E b}{t}
\] (6.12)

\[
\beta_2 = 7.2 \frac{E b}{t^3}
\] (6.13)

Equation (6.11) was derived assuming that the longitudinal axis of the conductor coincides with the longitudinal axis of the electrical cable. In this case the insulating layer in the annulus between the conductor and the outer sheath is of uniform thickness. As a result the stiffness of the foundation, which is a function of the insulation thickness, will be symmetric about the centre line of the conductor. If the longitudinal axes of the core is at an offset from the longitudinal axis of the electrical cable, the stiffness of the foundation will not be symmetric.

6.2.2 Elastic Buckling of Conductors

The equilibrium equation of a conductor taking into account the non-linear resistance of the foundation is given in equation (6.11). This fourth order non-linear differential equation is solved using a perturbation technique. First, the
The equilibrium equation is expressed in a non-dimensional form using the substitutions:

\[ w = \left( \frac{\beta_1}{\beta_2} \right)^{1/2} y \]  
(6.14)

\[ x = \left( \frac{EI}{\beta_1} \right)^{1/4} z \]  
(6.15)

\[ T = 2\lambda(EI\beta_1)^{1/2} \]  
(6.16)

The differential equation is then given by:

\[ y^{'''} + 2\lambda y'' + y + y^3 = 0 \]  
(6.17)

where a hyphen indicates differentiation with respect to \( z \). The above non-linear equation is solved using the Lindstedt-Poincare perturbation method. For this perturbation method, the non-dimensional load parameter \( \lambda \) and the displacement \( y \) expressed as functions of a small parameter \( \mu \) are given by:

\[ y = \mu^i y_i \]  
(6.18)

\[ \lambda = \lambda_0 + \mu^i \lambda_i \]  
(6.19)

The perturbation parameter \( \mu \) is set to be equal to the displacement amplitude. Substituting the above two expressions into the equilibrium equation and equating the coefficients of \( \mu \) to zero, a system of differential equations can be obtained. The first three of these differential equations are:

\[ L(y_1) = 0 \]  
(6.20)

\[ L(y_2) = -2\lambda_1 y_1'' \]  
(6.21)

\[ L(y_3) = -2\lambda_1 y_2'' - 2\lambda_2 y_1'' + y_1^3 \]  
(6.22)

where \( L(y_i) = y_i^{'''} + 2\lambda y_i'' + y_i \). The solution of the above equations leads to the following solution for the load-lateral deflection curve:

\[ \lambda = 1 + \frac{3}{8} \mu^2 \]  
(6.23)
It is seen from the above equation that the buckling load increases as the displacement increases and it can be concluded that the buckled conductor exhibits a stable post-buckling path. This stable path differs from the case of rail track and subsea pipelines due to the hardening foundation of the conductor as compared with the softening foundation of these other structures. The buckling deformation, the critical buckling load and the buckle half wavelength as calculated from the above perturbation method are given by:

\[ y = \frac{65}{64} \cos z - \frac{1}{64} \cos^3 z \]  \hspace{1cm} (6.24)

\[ T_{crit} = 2\sqrt{EI\beta_1} \]  \hspace{1cm} (6.25)

\[ \ell = \pi \sqrt{\frac{EI}{\beta_1}} \]  \hspace{1cm} (6.26)

The above equations demonstrate that in the absence of any imperfections the conductor will buckle once the load reaches the critical limit given in equation (6.25). Due to the stable post-buckling path, the buckle amplitude increases as the load is increased. Thus, it can be said that the conductors can sustain loads greater than this critical load. This is the case as long as the critical load leads to stresses within the elastic limits of the conductors. If the critical load lead to stresses exceeding the elastic limits, buckling of the conductors should be analysed using an inelastic formulation.

The solution presented above does not truly represent the localised kinking seen in the conductors. For localised kinking the above perturbation method is not feasible and an alternative linear formulation presented by Hetenyi (1946) can be used. The essence of the linear formulation presented by Hetenyi, where the higher order terms in the stiffness of the foundation are ignored, is the introduction of a small lateral imperfection generated by a concentrated point load \( P \). The solution of the equilibrium equation
satisfying the kinematic and geometric boundary conditions assuming the displacement is small is then given by:

\[ w = \frac{P}{4\sqrt{\beta_1 EI}} \frac{e^{-bx}}{ab} (a \cos ax + b \sin ax) \]  

(6.27)

where \( a \) and \( b \) are given by:

\[ a = \sqrt{\frac{\beta_1}{4EI} + \frac{T}{4EI}} \]  

(6.28)

\[ b = \sqrt{\frac{\beta_1}{4EI} - \frac{T}{4EI}} \]  

(6.29)

Using the symmetry at the origin of the beam and assuming the point load \( P \) is a small parameter, the only non-trivial solution is given by \( ab = 0 \). The solution of this equation leads to the same critical buckling load calculated using the perturbation method and given in equation (6.25). However, using this approach the buckling displacement is modulated and this leads to localised buckling patterns representative of the kinking displacement of the core. This localised buckling occurs at a load exceeding the critical buckling load just as predicted in the non-linear formulation using perturbation methods.

As a conservative assumption it is assumed that the critical buckling load is the load at which kinks are formed. This is only the case when the resulting critical stress is below the yield stress of the conductor. Assuming the conductor is made of \( n \) wires, the critical stress is given by:

\[ \sigma_c = \left( \frac{\beta_1 E}{n\pi} \right)^{1/2} \]  

(6.30)

In the above equation it is assumed that the conductor is made of circular wires and the bending stiffness is the sum of the bending stiffnesses of the wires. It is seen from the above equation that the critical stress increases with an increase in the linear stiffness of the foundation and decreases with an
increase in the number of the wires. To assess the influence of increasing the number of wires, the ratio of the critical stress of a conductor made of $n$ wires to the critical stress of a conductor made of one wire can be expressed as:

\[ \chi = n^{-1/2} \]  

(6.31)

The ratio $\chi$ represents a normalised buckling stress and a plot of this parameter versus the number of wires $n$ is shown in Figure 6.5. It is seen from this figure that this ratio decays exponentially and approaches zero as the number of wires increases. This is in agreement with experimental observations which show that conductors made of a large number of wires tend to kink more frequently. For the conductor made of $n$ wires the ratio of the bucklewave length to the wire diameter is given by:

\[ \frac{a}{d} \approx 3 \left( \frac{nE}{\beta_1} \right)^{1/4} \]  

(6.32)

For typical values for the stiffness of the insulation, it is seen that this ratio is always greater than 10 which indicates that shear distortion of the conductor can be ignored.

A signal core is considered as an example for calculating the critical buckling stress. Assuming that the linear resistance of the insulation is 30 MPa, the buckling stress of the core as calculated from equation (6.30) is given by:

\[ \sigma_c = \frac{1061 \text{ MPa}}{\sqrt{n}} \]  

(6.33)

To take into account the different types of conductors that are used, two cases are examined. These are 7 wire and 48 wire constructions which have a buckling stress of 401 MPa and 153 MPa respectively. For the core made of 48 wires it is seen that the buckling stress is nearer to the yield stress of the cores which was determined experimentally in chapter five to be 160 MPa. For the core made of 7 wires, the buckling stress exceeds the yield stress and in this case the core buckles inelastically.
6.2.3 Inelastic Kinking

To assess the influence of non-linear material properties on the buckling loads of beams two different methods are often employed. One method is known as the tangent modulus theory and the other is known as the reduced modulus theory. These two different methods are discussed in detail by Shanley (1947). Using the tangent modulus theory the modulus of the material is assumed to be the tangent modulus at the stress level. This is one of the earliest theories presented to deal with the inelastic buckling of beams. This theory ignores the fact that on bending the stresses on the convex side of the beam will be unloaded elastically. The reduced modulus theory was proposed to take this into account. The reduced modulus theory treats the beam as made of two materials of different elastic modulii and the equivalent modulus, which is known as the reduced modulus, is assumed to be the modulus of the beam. Shanley discussed these two theories and showed experimentally that the tangent modulus leads to more accurate results. A comparison of these two different methods with experimental data is shown in Figure 6.6. The main reason the tangent modulus theory leads to more accurate results is the fact that to increase the bending curvature of the beam, the axial loads on the beam should be increased. The net effect of an increase in these two parameters is an increase in the compressive stresses on the convex side of the beam. Thus, the hypothesis that the material on the convex side of the beam is unloaded does not hold. This was also confirmed experimentally by Shanley.

Using the tangent modulus theory, the critical buckling load of a conductor is assessed in this section. Assuming the stress-strain relationship of the cores is given by the Ramberg-Osgood equation, this relationship can be written as:

\[
\varepsilon = \frac{\sigma}{E} + \left( \frac{\sigma - \sigma_y}{K} \right)^c
\]  

(6.34)
where $K$ and $c$ are the strength coefficient and hardening exponent and $\sigma_y$ is the yield stress. The tangent modulus can then be expressed as:

$$E_t = \frac{E}{1 + cEK^{-c}(\sigma - \sigma_y)^{c-1}}$$  \hspace{1cm} (6.35)

The critical buckling stress is then calculated by substituting the tangent modulus given by the above expression into equation (6.30). The critical buckling stress is then given by:

$$\sigma_c = \left(\frac{\beta_1 E_t}{n\pi}\right)^{1/2}$$  \hspace{1cm} (6.36)

and the ratio of the wavelength of the buckle to the wire diameter is given by:

$$\frac{a}{d} \approx 3 \left(\frac{nE_t}{\beta_1}\right)^{1/4}$$  \hspace{1cm} (6.37)

For this illustrative example the core is assumed to be made of 7 wires. The modulus of the copper and the yield stress are assumed to be 118 GPa and 160 MPa respectively. Different cases are examined to take into account different hardening exponents. For all cases the strength coefficient is taken as 900 MPa and the linear stiffness of the insulation is taken as 30 MPa. Stress-strain curves using the above parameters are shown in Figure 6.7. It is seen from this figure that as the strain hardening exponent increases, the tangent modulus beyond the yield point increases.

Table 6.1 shows the ratio of the buckling stress to yield stress and the ratio of the buckle wavelength to the wire diameter for different values of strain hardening exponent. It is seen that as the strain hardening exponent increases, the ratio of the critical buckling stress to the yield stress increases. For low strain hardening exponents this ratio is very close to one. In other words for material with a marked yield point the critical buckling stress is approximately equal to the yield stress. This is in agreement with
Timoshenko’s (1936) remarks that for a given material with a marked yield point the critical buckling stress can be assumed to be the yield stress. It is also seen that as the strain hardening exponent increases the buckle wavelength decreases. As a result assuming the cores fail at a stress exceeding their yield stress, the ratio of the buckle wavelength to the wire diameter will be about 16. For a typical signal core made of 7 wires where the diameter of each wire is 0.67 mm, the buckle wavelength would be 10.8 mm. This is in agreement with the kink shown in Figure 1.3 presented earlier. Assuming the cores fail within the elastic region the buckle half wavelength for the same wire diameter is 28 mm. For a signal core made of 48 wires where the diameter of each wire is 0.2 mm the buckle wavelength for elastic and inelastic buckling is 7.2 mm and 3.8 mm respectively.

<table>
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<th>$\sigma_c/\sigma_y$</th>
<th>$E_t$ (GPa)</th>
<th>$a/d$</th>
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<tbody>
<tr>
<td>1.5</td>
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<td>9</td>
<td>15.0</td>
</tr>
<tr>
<td>2.0</td>
<td>1.15</td>
<td>11</td>
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</tr>
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<td>17.0</td>
</tr>
<tr>
<td>3.0</td>
<td>1.50</td>
<td>19</td>
<td>19.0</td>
</tr>
<tr>
<td>3.5</td>
<td>1.66</td>
<td>24</td>
<td>19.2</td>
</tr>
<tr>
<td>Large</td>
<td>3.66</td>
<td>118</td>
<td>28.6</td>
</tr>
</tbody>
</table>

The kinking analysis shows the dominant parameters involved in assessing the kinking failure of the cores. It is shown that kinking of the cores is mainly related to the number of wires, the thickness of the insulation and filler layers, and the elastic modulus of these layers. It is also shown that kinking exhibits a stable post-buckling path. In this respect and as a conservative assumption for elastic buckling the failure load can be defined as the critical load as calculated in equation (6.25). In a similar manner for inelastic buckling the failure load is
defined as the load resulting in compressive yield of the conductor. These conservative assumptions are justified given the critical function of the conductors and the requirement to minimise the risk of kinking failure. The compressive buckling stress in the cores should then satisfy the following two conditions:

\[ \sigma_c \leq \left( \frac{\beta_1 E}{n \pi} \right)^{1/2} \]  
\[ \sigma_c \leq \sigma_y \]

where \( \sigma_c \) is the compressive stress and \( \sigma_y \) is the yield stress. The above equations constitute a design requirement to minimise the risk of kinking failure. It is pointed out that this design requirement was derived ignoring the helix angle of the conductors within the electrical cable. This leads to a simplified solution owing to the symmetry in the stiffness of the conductor foundation.

### 6.3 Fatigue Design Curves

The design life of dynamic subsea umbilicals is often specified to be around 25 years. During this life cycle dynamic subsea umbilicals are subjected to a wide range of cyclic loads induced by waves, currents and motions of the surface vessel. As with all other structures subjected to cyclic loads, the cumulative fatigue damage can lead to fatigue failure before the desired life is achieved. To minimise the risk of such a failure, adequate fatigue analysis tools should be used. Fatigue curves are an essential part of these analysis tools. These curves relate the number of cycles to failure for any given stress range. Failure is denoted by a 100% fatigue damage.

The most common fatigue curves are empirical curves based on the work Basquin (1910), Coffin (1954) and Manson (1953). Basquin observed that for
stresses below the yield stress of any given material, the number of cycles to failure is related to the stress range by the following relationship:

\[
\Delta \varepsilon_e = 2 \frac{\sigma'_f}{E} (2N_f)^b
\]  

(6.40)

where \(\Delta \varepsilon_e\) is the elastic strain range, \(\sigma'_f\) is the fatigue strength coefficient, \(E\) is the elastic modulus of the material, \(N_f\) is the number of cycles to failure and \(b\) is the fatigue strength exponent. For stresses exceeding the yield stress of the material, Coffin (1954) and Manson (1953) working independently showed that fatigue damage is related to plastic deformation by the following relationship:

\[
\Delta \varepsilon_p = 2 \varepsilon'_f (2N_f)^c
\]  

(6.41)

where \(\Delta \varepsilon_p\) is the plastic strain range, \(\varepsilon'_f\) is the fatigue ductility coefficient, and \(c\) is the fatigue ductility exponent. The fatigue coefficients and exponents are material properties that also depend on environmental conditions. These relationships lead to the well-known empirical fatigue curve given by:

\[
\Delta \varepsilon = 2 \frac{\sigma'_f}{E} (2N_f)^b + 2 \varepsilon'_f (2N_f)^c
\]  

(6.42)

where \(\Delta \varepsilon\) is the total strain range. This relationship as discussed by Lemaitre and Chaboche (1994) can be written as:

\[
\Delta \varepsilon = 3.5 \frac{\sigma_u}{E} (N_f)^{-0.12} + D_u^{0.6} (N_f)^{-0.6}
\]  

(6.43)

where \(\sigma_u\) is the ultimate tensile strength and \(D_u\) is the ductility coefficient which is expressed as a function of the reduction in area at necking \(RA\) by:

\[
D_u = -\ln(1 - RA)
\]  

(6.44)

The above fatigue curve has been verified against a large number of experimental tests for different materials (Lemaitre and Chaboche, 1994). This fatigue curve is known as the universal slopes equation and it is widely accepted for design purposes. It is pointed out that fatigue curves based on
the above approach are crack initiation fatigue curves. Once a crack develops, the material can still sustain a number of cycles before the crack propagates through the specimen. The number of cycles after a crack has been initiated depends on the size of the test specimen. For the wire sizes considered here the number of these cycles is not expected to be large.

In assessing the fatigue damage in the constituent elements of a subsea umbilical, emphasis is often given to fatigue damage in the armour wires and the fatigue damage in the copper cores. These represent the constituent elements whose service lives are primarily dictated by fatigue as a result of applied mechanical loads. Failure of the other components such as the optical fibres, the hydraulic hoses and the polymeric sheath layers is governed by parameters other than cyclic mechanical loads such as material degradation and static fatigue. For the armour wires, an established fatigue curve is used by most manufacturers and designers. This is the Department of Energy Class C curve (1990) which was developed for welded plates and takes into account the sea environment and stress concentration factors resulting from defects in the welds. This curve is in general agreement with the limited data on the fatigue of armour wires in subsea umbilicals and flexible pipes. The main reason for this agreement is due to the fact that the use of the Class C curve allows defects resulting from fretting to be taken into account.

For the copper cores empirical fatigue curves based on the universal slopes method are used assuming the copper cores are in an annealed condition. The universal slopes method is often modified to take into account mean stress. This approach leads to very conservative results and is not realistic given that under strain controlled loading annealed material exhibits mean stress relaxation (Bannantine, 1990). In chapter five it was determined experimentally that the copper cores during the manufacturing process experience a pre-strain between 15% and 18%. This level of pre-strain results in a permanent reduction in the cross-sectional area by 17%. It is assumed that
the pre-strain is equivalent to the reduction in cross-sectional area by cold working. In this case the mechanical properties of the cores in the as-built condition as compared with the mechanical properties in the annealed condition are shown in Table 6.2. The data shown in this table is derived from the experimental results published by the Copper Development Association (1959) on copper wires 2 mm in diameter. The fatigue curve for annealed copper wires is given by:

$$\Delta \varepsilon = 0.0044\left(N_f\right)^{-0.12} + 2.016\left(N_f\right)^{-0.6}$$

and the fatigue curve for copper in the as-built condition is given by:

$$\Delta \varepsilon = 0.0063\left(N_f\right)^{-0.12} + 1.694\left(N_f\right)^{-0.6}$$

These two fatigue curves are shown in Figure 6.8. It is seen from this figure that while the two curves are in agreement for low cycle fatigue, the curves are not in agreement for high cycle fatigue. In the high cycle region the fatigue life is one order of magnitude greater for the as-built conductors as compared with annealed conductors.

Table 6.2: Mechanical properties of copper wires in the annealed and as-built conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>Annealed</th>
<th>As-built</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile strength (MPa)</td>
<td>147</td>
<td>214</td>
</tr>
<tr>
<td>Percentage reduction in area at necking (RA)</td>
<td>96</td>
<td>91</td>
</tr>
</tbody>
</table>
6.4 Design Applications

In an earlier section (6.2) a design requirement was specified to minimise the risk of kinking in the copper conductors of subsea umbilicals. This requirement is given by:

\[
\sigma_c \leq \left( \frac{\beta_1 E}{n\pi} \right)^{1/2} \quad (6.47)
\]

\[
\sigma_c \leq \sigma_y \quad (6.48)
\]

where \(\sigma_c\) is the compressive stress and \(\sigma_y\) is the yield stress. The above two equations should be satisfied simultaneously. It was also pointed out that the main mechanism resulting in compressive stresses in the cores is the global loading and unloading of the umbilical. Therefore, the above equations should be expressed in terms of the global load limits. The global axial load limits are considered first. The strain in the copper cores due to these global axial loads can be expressed as:

\[
\varepsilon = C\frac{F_z}{K} \quad (6.49)
\]

where \(C\) is a constant, \(F_z\) is the axial force on the umbilical and \(K\) is the axial stiffness of the umbilical. It is assumed that the axial loads on the umbilical oscillate between two limits which are given by \(F_{z,1}\) and \(F_{z,2}\). In this case, the axial strain limits are given by:

\[
\varepsilon_1 = C\frac{F_{z,1}}{K} \quad (6.50)
\]

\[
\varepsilon_2 = C\frac{F_{z,2}}{K} \quad (6.51)
\]

It is pointed out that the above strain limits are always tensile given that umbilicals are often operated with tensile axial forces. In this case, and as shown in Figure 6.9, the compressive strain in the copper cores is given by:
\[ \sigma_c = E(\varepsilon_1 - \varepsilon_2) - \sigma_y \]
\[ = E \frac{C}{K} (F_{z,1} - F_{z,2}) - \sigma_y \quad (6.52) \]

The above equation is only applicable if the cores experience plastic deformation. If this is not the case the compressive stress is zero. Modifying the above equation to take into account this criterion:

\[ \sigma_c = \begin{cases} 
E \frac{C}{K} (F_{z,1} - F_{z,2}) - \sigma_y & \varepsilon_1 > \varepsilon_y \\
0 & \varepsilon_1 < \varepsilon_y 
\end{cases} \quad (6.53) \]

where \( \varepsilon_y \) is the yield strain. Substituting for the design requirements given in equation (6.47) and equation (6.48), the load limits on the umbilical for \( \varepsilon_1 > \varepsilon_y \) should be specified as:

\[ \frac{C}{K} (F_{z,1} - F_{z,2}) \leq \left( \frac{\beta_1}{n \pi E} \right)^{1/2} + \frac{\sigma_y}{E} \quad (6.54) \]

\[ \frac{C}{K} (F_{z,1} - F_{z,2}) \leq 2 \frac{\sigma_y}{E} \quad (6.55) \]

The above equations are illustrated with reference to the case study analysed in chapter five. In this case study the maximum installation load was given as 507 kN and the axial stiffness of the umbilical was given as 108 MN. Using these parameters the axial strain in the copper cores was calculated as 0.409%. The constant \( C \) is then 0.87. The conductors in this case study are made of 7 wires and the insulation and filler layers are made of ethylene propylene rubber (EPR) and plasticised polyvinyl chloride (PPVC) respectively. The diameter of the wire is 2 mm and the thickness of both the insulation and filler layer is 4 mm. The elastic modulus of EPR and PPVC is about 60 MPa. The linear stiffness of the foundation \( \beta_1 \) is then 30 MPa. Using these parameters, the axial load upon unloading should be given by \( F_{z,2} \geq 171 \) kN. In other words, if the load upon unloading is smaller than 171 kN, the cores will kink. In comparison it is noted that if the axial stiffness of the umbilical was significantly greater than 108 MN, say 1080 MN, the tensile strain in the
copper cores upon loading will be within the elastic limits. In this case the copper cores will not be subjected to compressive stress on subsequent unloading. This highlights the advantages of designing the umbilical to be axially stiff as far as the reliability of the cores is concerned. The higher axial stiffness can be achieved by laying the armour wires with a small helical angle. On the other hand laying the armour layers at a small helix angle results in a lower radial stiffness of the umbilical. This would pose difficulties during handling and installation due to the imposed radial forces at the tensioners. This also would pose difficulties during manufacturing since to achieve the desired coverage the wires would have to be larger or their numbers would have to be increased. These difficulties are more pronounced in deep water. In deep water the suspended weight of the umbilical increases and so would the required radial forces at the tensioners. The umbilicals would also be larger in deep water due to increased flow requirements.

The same design criterion can be applied for bending but here the compressive stress results from the direct application of flexure. The compressive stress due to bending is given by:

$$\sigma = E\cos^2\alpha_2 \frac{r_2}{\rho_u}$$  \hspace{1cm} (6.56)

where $E$ is the modulus of copper, $r_2$ and $\alpha_2$ are the helix radius and angle of the conductor and $\rho_u$ is the bending radius of the umbilical. For a 7 wire conductor it was shown that typical critical stress values at which kinking occurs is about 160 MPa. A typical signal conductor with a lay angle of 10 degrees and helix radius of 2 mm is considered here. For this stress limit to be reached the critical bending radius of the umbilical is about 1.4 m. If the umbilical is subject to combined axial and bending loads the critical bend radius calculated above would be smaller owing to the fact that the tensile axial loads reduce the direct compressive stress generated due to bending.
For fatigue considerations the stresses within the copper conductors can be minimised by the careful choice of some of the key parameters of the umbilical. However, some of these parameters could be constrained by other requirements of the umbilical. The choice of the global axial stiffness is one example explained earlier. Decreasing the lay angle of the armour wires leads to a higher axial stiffness of the umbilical resulting in lower stresses in the conductors for a given axial load. Yet, decreasing the lay angle of the armour wires leads to an umbilical with lower radial strength posing difficulties during installation. The choice the weight to diameter ratio is another example. The weight to diameter ratio can be selected to optimise the dynamic response of the umbilical but this parameter is restricted to be of equal magnitude to the weight to diameter ratio of the production risers.

Effective and simple measures to minimise fatigue in the conductors are lubricating and tin coating the wires of the conductors. These measures help to promote slip and reduce the frictional stresses. Other measures to reduce fatigue have also been suggested by Legallais and Stratfold (1992) and O’Hear (1992). Legallais and Stratfold argued that armouring the electrical cables improve the fatigue performance of electrical cables. This as mentioned earlier in chapter five is due to the relatively high axial stiffness of armoured electrical cables as compared with unarmoured cables leading to greater slip within the cable assembly. O’Hear suggested the use of copper alloys such as copper cadmium to improve the fatigue properties of the conductors. In the cold worked condition this alloy is of superior mechanical properties to the copper types often used in umbilicals. However, this alloy is of inferior electrical properties and its conductivity is 20% lower than copper. Owing to this promoting slip by lubricating and armouring the cables is a more favourable solution to minimise fatigue damage.
7 Conclusion

The objectives of this work have been to develop analytical tools to predict the mechanical loads and failure modes of the electrical conductors in subsea umbilicals. Evidence for the significance of this work is found in the reported vulnerability of these components to mechanical damage during the different stages of the umbilical life cycle. The approach used to achieve these objectives, the conclusions and recommendations for further work are presented in this final chapter.

7.1 Overview

Subsea umbilicals are composite structures made of helical and cylindrical layers. The previous work related to umbilicals and similar generic flexible structures is reviewed in chapter two. This review illustrates the fact that the mechanics of the electrical cables in subsea umbilicals has not been researched prior to this work. The axial and rotational strains of the helical components are also analysed in chapter two. These strains are expressed as a function of the global axi-symmetric and flexural deformation of the umbilical. For axi-symmetric deformation the expressions derived are similar to the expressions derived by other researchers. For flexural deformation, the axial and rotational strains are discussed for the loxodromic and geodesic slip mechanisms. Loxodromic slip is analysed for finite friction coefficients. Geodesic slip is analysed using a new formulation and it is shown that the loxodromic slip is more representative of the slip mechanism in the armour wires of flexible structures.
The analytical models to predict the structural response of subsea umbilicals are presented in chapter three. For axi-symmetric loads the model uses a novel sub-structuring approach to take into account the interaction of the different helical and cylindrical layers. This approach eliminates the numerical iterative techniques often used to model these structures. The model for axi-symmetric loads also incorporates a new methodology to predict the equivalent material properties of the composite core. For flexural loads the structural response is analysed taking into account the spread of the slip region. This leads to a non-linear flexural response which is characterised by initial and final regions of high and low bending stiffness respectively. The structural response for flexural loads is also analysed to take into account the different slip mechanisms.

The analytical models are verified with full-scale experimental data in chapter four. Umbilical manufacturers provided the experimental data for axi-symmetric loads. The differences between the theoretical and experimental results are within acceptable limits. For flexural loads the response was determined using an experimental method incorporating an eccentrically applied axial load. The experimental results were compared with the full slip bending stiffnesses calculated using the two different slip mechanisms. It is shown that the experimental results compare well with the theoretical results assuming loxodromic slip. For geodesic slip the theoretical results over predict the bending stiffness.

Chapter five presents an assessment of the mechanical loads acting on the copper conductors of subsea umbilical taking into account their double helical configuration. Expressions for axial and rotational strains in the conductors are derived using a recursive approach. It is shown that these components can slip within the umbilical assembly irrespective of whether the electrical cables slip or not. However, owing to the possibility of high contact forces these cores should be considered in a partial slip condition. The partial slip
condition is defined as the case where the electrical cables slip but the cores within the cables do not slip. Chapter five also presents an experimental assessment of the pre-strain of the copper conductors in the as-built condition. It is shown that although the conductors are specified to be in an annealed condition, the as-built conductors are in a pre-strained condition. This pre-strain was determined experimentally to be approximately 15%.

The influence of the pre-strain in the as-built condition has been taken into account to analyse the kinking and fatigue failure of the copper conductors. This analysis is presented in chapter six. In analysing the kinking failure, the conductors are modelled as beams supported by a non-linear elastic foundation. The non-linear effects arise due to the compressive characteristics of the polymeric insulation layers. The kinking failure was analysed for elastic and inelastic material properties and it is shown that the susceptibility to kinking failure is higher for conductors made of a large number of wires. The influence of the pre-strain was also taken into account to define the coefficients of the universal slopes fatigue curve. The universal slopes fatigue curve is a widely accepted design curve which is dependent on the tensile strength and the ductility of the material. It is shown the pre-strain improves the fatigue characteristics of the conductors.

7.2 Discussion

The analytical models developed to predict the axi-symmetric structural response of subsea umbilicals use an original sub-structuring approach to take into account the interaction of the different helical and cylindrical layers. The model also include an new methodology to predict the core’s equivalent material properties which are of significance to the structural response for axi-symmetric loads. This methodology is based on a quantitative assessment of
the dominant equivalent mechanical properties contributing to the structural response of the umbilical. This quantitative assessment shows that the contribution of the axial and torsional stiffnesses of the core is one order of magnitude smaller than the contribution of the axial and torsional stiffnesses of the armour layers. On the other hand the contribution of the radial stiffness of the core is of the same order of magnitude as the contribution of the radial stiffness of the armour layers. This assessment allows the core to be modelled as a composite rod with a given equivalent radial modulus and a corresponding dilation coefficient. The axial and torsional moduli are ignored. This is not likely to introduce any significant errors in the analysis bearing in mind the other uncertainties in the structural analysis of umbilicals. These uncertainties relate to manufacturing defects such as the presence of gaps between the different layers, the residual stresses in the different components and differences between the design and as-built products. All these factors contribute to errors which are comparable to the errors introduced in ignoring the second order terms resulting from the axial and torsional contribution of the core.

The model for the axi-symmetric structural response is validated with full-scale experimental data relating to different umbilical designs. These designs include an umbilical made of four armour layers laid at 10 degrees, an umbilical made of five armour layers laid at 30 degrees and another two umbilicals made of two armour layers. The armour layers of these other two umbilicals are laid with angles between 14 degrees and 18 degrees. The differences between the experimental and theoretical results for the axial stiffnesses of these umbilicals are less than 23%. The differences are greatest for the umbilical made of five armour layers laid at 30 degrees. The main reason for this is the sensitivity of the axial stiffness of this structure to the calculated equivalent radial modulus of the core. It is shown that when the armour layers are laid with a large helix angle, the contact pressure generated due to the constriction of these layers is of a relatively greater magnitude resulting in higher radial displacement. As a result the predicted axial
stiffnesses of the umbilical are sensitive to the estimated equivalent radial modulus of the core. Nonetheless, the results are still within acceptable limits.

The methodology used to predict the equivalent material properties of the core is an alternative approach to specifying an apparent Poisson’s ratio. It is evident that this alternative methodology eliminates the subjectivity in the analysis. Specifying an apparent Poisson’s ratio is an approach that is often used in analysing subsea umbilicals and similar composite structures made of helical and cylindrical layers. No methodology to calculate this parameter has been developed prior to this work and the apparent Poisson’s ratio is often specified according to the similarity between one umbilical structure and another. This is likely to introduce subjectivity to the analysis methods. Moreover, the use of an apparent Poisson’s ratio would not take into account the complex interaction of the different helical and cylindrical layers. It is shown that the interaction of the layers is dependent on a number of parameters relating to the structural design of the umbilical. The use on an apparent Poisson’s ratio would not allow the effect of these design parameters to be taken into account. It is clear that the methodology to predict the equivalent material properties of the core overcomes the substantial disadvantages of specifying an apparent Poisson’s ratio.

The differences between the experimental and theoretical results for the torsional stiffnesses of the four umbilicals considered are less than 40%. These are higher differences that those predicted for the axial stiffnesses. The reasons for these higher differences have been attributed to the errors expected from the experimental procedure. The experimental procedure used to determine the torsional stiffnesses is likely to induce non-uniform twist along the test specimen. This is largely the case since the end terminations are not often designed to minimise the end effects. However, although the differences between the experimental results and the theoretical results are greater than what would be desirable, these errors are still within acceptable limits. Umbilicals are the products of a difficult assembly process with the
manufacturing process sometimes lasting several days. It is unlikely that the manufacturing conditions remain constant over this period and as a result differences between the umbilical design and the as-built product are normally encountered. Such differences cannot be quantified at the design stage and the theoretical results predicted using the models developed in this work still serve as adequate tools.

The loxodromic and geodesic slip of the armour wires and its influence on the flexural response have also been examined in this work. The loxodromic slip was analysed for finite friction coefficients and it is shown that for flexural loads of practical significance the armour wires could be assumed to be in a full slip condition. This is the case given that for a typical armour wire the critical bending radius, defined as the bending radius at the inception of slip, is 1000 m. The magnitude of this bending radius is too great to be of practical significance. Typical bending radii during installation and service can be as low as 5 m. The maximum flexural stress in the wire for this minimum bending radius is 105 MPa. The maximum flexural and shear stresses in the wire for the critical bending radius is 0.5 MPa and 10.5 MPa respectively. The maximum shear stress occurs on the convex side of the bent umbilical. As the bending radius is decreased the slip region extends at the neutral axis but this maximum stress does not increase in magnitude. In this respect the frictional stress in a helical wire can be assumed to be the maximum axial stress at the inception of slip.

The influence of the different slip mechanisms on the flexural response is also examined. It is shown that slip results in a non-linear flexural response characterised by initial and final regions of high and low bending stiffnesses respectively. The initial region of high bending stiffness corresponds to the case prior to the inception of slip. The final region of low bending stiffness corresponds to the case when the armour wires are in a full slip condition. The bending stiffnesses for different slip mechanisms are compared with experimental results. It is shown that the bending stiffness calculated
assuming loxodromic slip compares well with the experimental results. On the other hand the bending stiffness calculated assuming geodesic slip over-predicts the bending stiffness by a factor of two. It is argued that this can be attributed to fact that while the geodesic slip is a slip towards the path of minimum length, it is not necessarily the path of minimum potential energy.

The analytical models developed to predict the structural response are of significance to predicting accurately the mechanical loads in the electrical conductors of subsea umbilicals. The conductors offer little contribution to the mechanical strength of the umbilical and as a result they experience strain controlled loading. To assess the strain limits the structural response of the umbilical needs to be determined accurately. This is illustrated with reference to a system of two springs connected in parallel. One of these springs is of a relatively greater stiffness as compared to the stiffness of the other spring. The response of this system is dominated by the response of the spring of higher stiffness. If this spring exhibits linear response for cyclic loads between two specified load limits, the weak spring will be under strain controlled loading. The response of the armour wires and the electrical conductors is similar to the response of the above system. The armour wires are equivalent to the stiff spring and the conductors are equivalent to the weak spring and will be under strain controlled loading. If the conductors experience plastic deformation, they will be subjected to compressive forces. Compressive forces are generally favourable for fatigue considerations providing the conductors do not kink.

The mechanical loads in the conductors are assessed using a recursive approach to take into account the double helix configuration of the conductors. It is shown that for umbilical designs of low axial stiffnesses the conductors can experience axial strains exceeding the axial strains in the armour wires. This is illustrated in a case study presented in chapter five. For this case study the axial strain in the conductors is 0.41% while the axial strain in the outermost armour layer is 0.06%. The axial strain in the innermost
The armour layer is approximately 0.02%. The axial strain in the armour layers is of a smaller absolute magnitude owing to the fact that, due to their helical configuration, these layers are allowed to move radially. On the other hand, the radial displacement of the conductors is more restricted due to their central position within the umbilical assembly. For this case it is evident that the conductors experience plastic deformation. Once the global loads on the umbilical are relaxed the armour wires revert to their initial state while the conductors will be subjected to compressive forces.

For flexural loads it is argued that the possibility of high residual contact forces arising during the manufacturing process and the hydrostatic pressure acting on the outer sheath of the electrical cable can restrict the slip of the conductors within the electrical cable. An assessment of the contact forces generated due to shrinkage of the cable sheath layer is carried out. It is shown that these contact forces are of a significant magnitude. As a conservative assumption it is proposed that the conductors should be considered in a partial slip condition. This assumption leads to higher estimates of the mechanical loads in the conductors. However, this assumption is justified given the critical function of the conductors and their vulnerability to mechanical damage as reported in the published literature.

It is also shown in this work that while the wires of the conductors are specified to be in an annealed condition, these wires experience some work hardening during the manufacturing process. An assessment of the as-built work hardened condition was carried out experimentally. This experimental procedure involves comparing the stress-strain curves of conductors dissected from an umbilical with their stress-strain curves after annealing. The results of these experiments show the conductors during manufacturing experience a pre-strain of approximately 15%. These results were calculated for specimens dissected from umbilicals made by two of the main manufacturers. An equivalent level of pre-strain was also found in the conductors of a commercially available electrical cable. This indicates that the most likely
Origin of work hardening is at the manufacturing stage where the wires of the conductors are wound. Here, the manufacturing loads should be sufficient to ensure a compact cross section of the conductor. In this respect, a level of pre-strain of 15% can be assumed to be generic to the electrical conductors of subsea umbilicals.

The influence of the mechanical properties in the as-built condition has been taken into account to analyse the kinking and fatigue failure of the copper conductors. A design methodology was also devised to assess global load limits that can be imposed on the umbilical without causing excessive compressive stress in the conductors.

In analysing the kinking failure, the conductors were modelled as beams supported by a non-linear elastic foundation. The non-linear effects arise due to the compressive characteristics of the polymeric insulation and filler layers. It is shown that the kinking failure exhibits a stable post-buckling behaviour indicating that the cores could sustain greater compressive stresses than the bifurcation buckling stress. It is also shown that an increase in the number of wires of the conductors results in a reduction in the buckling stress. For a conductor made of 48 wires the buckling stress is a factor of two lower than the buckling stress of a conductor made of 7 wires. For the majority of modern umbilicals where the conductors are made of 7 wires and the usual polymers are used in the insulation and filler layers the buckling stress is greater than the yield stress. For this case the kinking failure is analysed taking into account the non-linear material properties of the conductors. The non-linear material properties were modelled using the Ramberg-Osgood equation. It is shown that for the conductors the inelastic kinking stress is equivalent to their compressive yield stress due to the marked yield point of the conductors. In other words, when the compressive stress is equal to the yield stress of the conductors the conductors are assumed to fail by kinking.
The influence of the as-built mechanical properties on the fatigue life was also considered and it was shown that the pre-strain arising during the manufacturing process improves the fatigue characteristics of the conductors. This improvement is significant in the high cycle fatigue region. The fatigue curves presented are based on the universal slopes equation which is an established and widely accepted fatigue design tool.

In concluding this section it can be said that umbilicals are complex structures and their response should be analysed carefully taking into account the radial displacement of their relatively soft cores. The traditional method of introducing an apparent Poisson's ratio is subjective and is likely to lead to erroneous results. This would have direct consequences on predicting accurately the mechanical loads in the copper cores. The methodology to obtain the equivalent mechanical properties of the core, which is derived in this work and verified with full-scale experimental data, offers a more consistent analysis tool. Care should also be used in using the assumption of geodesic slip. While this assumption leads to a mathematically stimulating discussion, its relevance to the slip mechanism of armour wires is still to be fully examined. Evidence presented in this work suggests that the loxodromic slip is more representative of the slip mechanism of the armour wires of umbilicals. It can also be concluded that for umbilicals of relatively low axial stiffness, there are advantages in positioning the electrical cables away from the centre line of the umbilical. This would allow these components to accommodate larger axial displacement without undergoing large mechanical strains. Away from the centre line, the electrical cables can move radially and consequently experience lower mechanical loads for a given axial displacement of the umbilical. For axially stiff umbilicals, these advantages could be offset by increased signal attenuation. It can also be concluded from this work that conductors made of 7 wires offer superior resistance to kinking. Nonetheless, the kinking resistance would be reduced significantly if the compressive yield stress is exceeded. The design methodology presented
allows the specification of load limits that ensure such a criterion is not reached.

### 7.3 Further Work

It is recommended that further experimental work is carried out to determine more representative fatigue curves of the copper conductors in the as-built condition. It is proposed that this experimental work should be dedicated to testing of the electrical cables rather than the umbilical assembly. A number of experimental research programmes have been carried out in the past but these programmes have been concerned with testing the whole umbilical assembly. The prohibitive cost of these tests limit the data to few thousand cycles for a limited number of load parameters. In comparison, the tests proposed here not only can be carried out for a greater number of cycles but can also examine the influence of different constructional designs. For example the tests can be carried out to examine the influence of different conductor constructions such as solid and stranded conductors. The tests can also be carried out to test the influence of armouring the electrical cables and the influence of different insulation and filler materials. While these proposed experimental tests are not strictly representative of the loads in the conductors, their simplicity and low cost should at least give an indication of the most optimal constructional design.

It is also proposed that further work is carried out on umbilicals which incorporate high voltage power conductors. Only a hand full of these umbilicals have been installed to date, but their use is set to increase as more equipment are deployed on the sea bed. This equipment includes amongst others electrically powered valves, subsea pumps and power distribution networks. The proposed further work should examine issues related to
mechanical and fatigue damage taking into account the thermal and electrical stresses in these high voltage power conductors.

7.4 Concluding Statement

This thesis has developed new models and methodologies for application in the design of subsea umbilicals incorporating electrical cables. The application of this work will result in an increased understanding of the mechanics of these structures and improve their reliability in deep water applications.
Acknowledgments

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Figure 1.1: Generic layout of subsea systems. The different modules are a surface control station, a subsea control pod and a subsea umbilical.
Figure 1.2: Typical cross section of an umbilical. The cross section is made of an armoured and sheathed bundle of electrical cables and hydraulic hoses. The electrical cables are used to transmit the communication signals and the electrical power. The hoses are used to transmit the hydraulic fluids and chemicals required for flow assurance.

Figure 1.3: A kinked conductor and subsequent failure due to excessive fatigue.
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Initial state: Asymmetric core with springs of different stiffnesses

Deformed state: Central rigid element displaced to minimise the internal strain energy stored in system

Figure 3.6: Idealised asymmetric core model. The outer cylindrical layer constrained to remain circular, the central radially rigid core is displaced to minimise the strain energy stored in the system.
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\[ F_1 = K_1 \Delta x \]

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stress (MPa)

![Graph of stress-strain curves](image)

Figure 5.15: Cyclic and monotonic stress-strain curves of pre-strained copper. In the annealed condition the curves show a hardening behaviour. For pre-strain of 20, 30 and 40% the curves show a softening behaviour (Klesnil and Lucas, 1980).

![Graph of hysteresis curves](image)

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