ON LINEAR AND NONLINEAR INSTABILITY IN BOUNDARY LAYERS WITH CROSSFLOW

by

Dominic Andrew Robert Davis

Supervisor:
Professor F.T. Smith, D.Phil, F.R.S.
ABSTRACT

The instability and transition of an incompressible boundary layer along a flat surface are considered theoretically and computationally in various linear and nonlinear regimes, with crossflow being the major new concern. The characteristic Reynolds number is taken as a large parameter throughout. The majority of the work describes so-called vortex/wave interactions which involve short-scale/long-scale interaction phenomena and divide roughly into two categories, namely for weakly nonlinear and strongly nonlinear theory. In Chapters Two to Five, the former theory is applied to two low-amplitude three-dimensional (3D) Tollmien-Schlichting waves present in a 3D boundary layer. The nonlinear interaction, near to the lower branch of the neutral curve, is controlled by a partial-differential system for the vortex flow together with an ordinary-differential equation for each wave pressure. Three distinct possibilities emerge for the nonlinear behaviour of the flow solution downstream - an algebraic finite-distance singularity, far-downstream exponential wave-decay or irregular oscillations - depending on the input amplitudes upstream, the wave angles, the size of the perturbed basic-flow wall-shear and the size of the crossflow. In Chapter Six, the interactive 3D
boundary-layer equations from lower-branch theory are examined in the double limit of small streamwise distances and high frequencies, whereupon the equations are converted into a quasi-two-dimensional form. A nonlinear periodic solution for the displacement decrement is subsequently obtained. Chapters Seven and Eight describe strongly nonlinear Rayleigh-wave/vortex interactions for a slightly 3D boundary layer. Here, as the crossflow is increased, the more important part of nonlinear interaction is concentrated in a relatively thin buffer layer in the midst of the boundary layer. The problem reduces to solving the generalised Rayleigh-wave pressure equation subject to a third-derivative wave-pressure discontinuity across the buffer layer, and a solution is obtained near the streamwise input station.
CONTENTS

Page

Abstract 2

Acknowledgements 6

Chapter 1 7
  Introduction

Chapter 2 27
  Weakly Nonlinear Vortex/Wave Interaction

Chapter 3 45
  Weakly Nonlinear Interactions for \( \omega_0 = O(\nu^{6/5}) \)

Chapter 4 85
  TS Stability Properties and Further Small Crossflow Analysis

Chapter 5 125
  Limiting Analyses for \( O(1) \) Crossflow
Chapter 6  163
Nonlinear Disturbances in 3D Boundary Layers

Chapter 7  185
Strongly Nonlinear Rayleigh-Wave/Vortex Interactions in Incompressible Boundary Layers for Increasing Crossflow

Chapter 8  217
Initial Streamwise Developments

Chapter 9  244
Summary

References  250
ACKNOWLEDGEMENTS

It is a great pleasure to thank my supervisor Professor Frank T. Smith for his invaluable guidance and encouragement throughout the preparation of this thesis.

I am also grateful to the Mathematics department at University College London for their generous accommodation and provision of facilities, and to S.E.R.C. for supplying financial support.

Thanks are also due to Mrs. J. Lucas for typing.

D.A.R.D.
CHAPTER 1 : INTRODUCTION

This thesis is devoted to the theoretical and computational study of linear and nonlinear instability in an incompressible boundary layer along a flat surface, with crossflow effects being the major new concern. The majority of the work addresses so-called vortex/wave interactions. These involve short-scale waves self-interacting to induce long-scale vortex flow, and roughly divide into two categories according to weakly nonlinear and strongly nonlinear theory.

Very early experiments by Reynolds (1883) showed that a well-ordered laminar motion, if subjected to a sufficiently large disturbance, could eventually destabilise into a chaotic (turbulent) state. Thereafter, many attempts were made to understand the nature of a flow's transition to turbulence, these attempts starting with linear (infinitesimal) disturbance theory.

The earliest ideas in linear instability theory revolved around the classical Orr-Sommerfeld equation, and assumed that a parallel-flow approximation is valid. In particular, Tollmien (1929) and Schlichting (1933)
obtained approximate theoretical solutions for two-dimensional boundary-layer flows. Subsequent experiments by Schubauer and Skramstad (1948) for the Blasius boundary layer on a flat plate yielded considerable agreement with Tollmien's and Schlichting's predictions. The experimental findings were further verified by numerical solutions of the Orr-Sommerfeld equation, e.g. Shen (1954), Jordinson (1970), Gaster (1974) and many others more recently.

There is a fundamental problem with the Orr-Sommerfeld approach however, in that the parallel-flow approximation is inaccurate for non-large values of the Reynolds number. Attempts were made to include very slight non-parallel-flow effects by Bouthier (1973) and Gaster (1974), by perturbing the Orr-Sommerfeld problem. Their results show a slightly improved agreement with the experiments, but still discrepancies remain near the critical Reynolds number. A more systematic method was used by Smith (1979) who examined the non-parallel-flow stability of a Blasius boundary layer, for asymptotically large values of the Reynolds number, via triple-deck theory.
Although important during the early stages of a transitional flow, or prior to that, linear instability theory is incapable of describing the flow's state completely during transition, where the disturbances are fully nonlinear in nature. Hence, the ultimate goal is to obtain a theoretical understanding of nonlinear instability properties.

The first significant progress was made by Stuart (1960) and Watson (1960) in their studies of plane Poiseuille flow in a channel. The nonlinearity involved is 'weak', in the sense that the basic flow is not altered by an $O(1)$ amount, but has a significant influence on the slow amplitude-growth of the disturbances near the neutral-stability curve of the linear theory. The theory was later developed by Stewartson and Stuart (1971) for the weakly nonlinear amplitude response of wave systems, near the critical Reynolds number of plane Poiseuille flow, and numerous other weakly-nonlinear theories followed.

With the substantial advance of computer resources in the 1980s, it became possible for the first time to perform large direct computational simulations on the
full Navier-Stokes equations (notably Wray and Hussaini (1984), Spalart and Yang (1987)), and the results have provided much insight into transition.

Our main concern here is ultimately with 'strong' (i.e. full) nonlinearity in boundary-layer flows. All our work is conducted in the high-Reynolds-number framework, which has some natural advantages. Firstly, it allows the consistent use of the parallel-flow approximation, which greatly simplifies the governing equations. Secondly, there is more chance of gaining a rational understanding of the effects of nonlinearity. Thirdly, we are able to capture, with considerable precision, the flow structures and fine scales that are involved. The high-Reynolds-number approach has helped to generate three current theories on strong nonlinearity, namely vortex/wave interactions, triple-deck interactions and Euler-type interactions, for successively larger disturbance sizes. All three are known to be connected in the limits of their disturbance sizes, e.g. as described in Smith and Burggraf (1985) for large disturbance frequencies in the context of pressure/displacement interactions.
Concerning weakly nonlinear effects first, in recent years a strong belief has developed that weakly nonlinear vortex/wave interactions can play a key role as an early stage in transition to turbulence of boundary layers and duct flows. In the boundary-layer context, the evidence is mainly from experimental observations (Aihara et al. (1965, 1969, 1981, 1985), Tani and Sakagami (1962), Bippes and Gortler (1972)), but also from supporting computational work (Wray and Hussaini (1984), Spalart and Yang (1987)).

This has provided the stimulus for recent theoretical studies in this field notably for boundary layers (Hall and Smith (1989)) and channel flows (Hall and Smith (1987), Blennerhassett and Smith (1991), the latter including a correction of Hall and Smith (1989)).

The weakly nonlinear interactions arising between two three-dimensional (3D) Tollmien-Schlichting (TS) waves and their induced streamwise vortices have been studied theoretically for a two-dimensional (2D) boundary layer over a flat surface by Hall and Smith (1989). The work in Chapters Two to Five below is an extension of their paper, considering the same problem but for a
3D boundary layer. Thus we are concerned primarily with the effects of crossflow, and in particular the role it plays, if any, in determining the ultimate nonlinear behaviour of the flow solution downstream from the initial station.

We address disturbances sufficiently close to the first, lower-branch, neutral station, where the TS waves are governed mainly by the triple-deck structure (Smith 1979). If the coordinate scales controlling the vortex are taken first to be comparable with those for the wave, this structure additionally incorporates the induced-vortex motion, as we see below. The nonlinear evolution process is principally contained within the lower deck of the triple-deck structure wherein the velocities $\mathbf{u}_\infty(u, v, w)$, the Cartesian coordinates $l_\infty(x, y, z)$, the pressure $p_\infty l_\infty^2 p$ and the time $l_\infty u_\infty^{-1} t$ are scaled in the form

$$\left[ u, v, w, \rho, x-x_0, y, z-z_0, t \right] = \left[ Re.^{-1/8} \tilde{u}, Re.^{-3/8} \tilde{v}, Re.^{-1/8} \tilde{w}, Re.^{-1/14} \tilde{\rho}, Re.^{-3/8} \tilde{x}, Re.^{-5/8} \tilde{y}, Re.^{-3/8} \tilde{z}, Re.^{-1/14} \tilde{t} \right]$$

near a typical $O(1)$ station $x = x_0, z = z_0$. Here $l_\infty, u_\infty, p_\infty, v_\infty$ represent, in turn, the typical
length of the plate (such as an airfoil chord), the flow speed in the outer stream (in the \( \text{x} \)-direction), the fluid density and the kinematic fluid viscosity. The Reynolds number \( Re = \frac{u_\infty l_\infty \nu_\infty^{-1}}{\nu} \) is taken to be a large parameter.

From the substitution into the Navier-Stokes equations, the scaled variables are then governed by the unsteady, interactive boundary layer equations

\[
\begin{align*}
\ddot{u}_T + \ddot{u} \dddot{u}_x + \ddot{u} \dddot{u}_y + \dddot{u} \dddot{u}_z &= -\dddot{\rho}_x + \dddot{u}_y, \quad (1.1a) \\
\ddot{w}_T + \ddot{w} \dddot{w}_x + \ddot{w} \dddot{w}_y + \dddot{w} \dddot{w}_z &= -\dddot{\rho}_z + \dddot{w}_y, \quad (1.1b) \\
\ddot{u}_x + \ddot{u}_y + \dddot{w}_z &= 0, \quad (1.1c)
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\ddot{u} = \ddot{w} = \dddot{u} = \dddot{w} &= 0 \quad \text{at} \quad Y = 0, \quad (1.1d) \\
\ddot{u} \sim Y + A, \quad \ddot{w} \sim Y^{-1} \quad \text{as} \quad Y \to \infty. \quad (1.1e)
\end{align*}
\]

Here the pressure \( \dddot{\rho} \) and the displacement decrement \( A \) are unknown functions of \( X, Z \) and \( T \) linked via the expression

\[
\dddot{\rho}(X, Z, T) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(ue \partial \partial Z + we \partial \partial Z)^2 A(\eta, Z, T) d\eta d\eta}{\left[(X - \eta)^2 + (Z - \eta)^2\right]^{1/2}}, \quad (1.1f)
\]
arising from the main and upper deck analyses, where 
\((U_e, O, \omega_e)\) denotes the basic flow outside the boundary layer, and the bars on the integral signs denote the principal value or the finite part.

Next, the oblique TS waves are represented by

\[ E_{1,2} \equiv \exp \left[ i (\xi + \beta_{1,2} z/2 - \omega T) \right] , \quad (1.2a,b) \]

where \( \xi, \beta_1, \beta_2 \) and \( \omega \) are real constants. The waves interact nonlinearly to induce streamwise-vortex flow in the manner \( E_1 E_2^{-1} = E_3 \), where

\[ E_3 \equiv \exp \left[ i (\beta_1 - \beta_2) z/2 \right] . \quad (1.3) \]

Equally we note the properties \( E_1 E_3^{-1} = E_2 \) and \( E_2 E_3 = E_1 \), which correspond to the vortex combining with the first wave to provoke the second wave, and combining with the second wave to provoke the first wave, respectively.

The scales and the flow structure for the TS/vortex interaction in a full 3D boundary layer are examined in Sections 2.1 to 2.5 below, for which a partial-differential system for the vortex flow and an ordinary
differential equation for each wave pressure are derived. These interaction equations are addressed in Section 2.6 and special attention is paid to the case of zero-incident vortex flow. Here it is possible to deduce an integral form for the vortex-streamwise skin-friction factor. Finally, in Section 2.7, we modify the present weakly-nonlinear formulation to include nonparallelism. In fact, the subsequent alterations to the interaction equations are minimal.

In Chapter Three, we consider the balance where the crossflow order of magnitude (relatively small) is such that it plays a direct part in controlling the pressure variations over the vortex-streamwise scale. The interaction equations (presented in Section 3.1) are addressed numerically (Section 3.2) and analytically (Section 3.3) and comparisons are noted in Section 3.4. Certain of the nonlinear properties found tend to agree with the corrected version of the Hall and Smith (1989) results (see Blennerhassett and Smith (1991)), in particular in the case where the flow solution breaks up after a finite distance downstream. In this case the crossflow effects become minor. Other nonlinear properties are vastly different from those described by Blennerhassett and Smith, and this is mainly where the
crossflow plays a significant part. These include both the feature of exponential wave decay far downstream and a curious phenomenon of repeated 'cycles' where the flow solution undergoes irregular oscillations. All the nonlinear features occurring downstream are believed to be entirely dependent on the input conditions, the spanwise wavenumbers, the perturbed basic-shear and the crossflow.

The effect of crossflow on the neutral stability of a relatively small TS wave is captured, in essence, by the associated linear eigenrelation

\[ \alpha^{5/3} = (\alpha^2 + \beta^2/\lambda)^{1/2} (\alpha + \beta w_0/2)^2, \quad (1.4) \]

where \( \alpha \) has the approximate value of 1 (see Smith (1986), Stewart and Smith (1987), Section 2.5 below).

In Chapter Four, we generate by a simple iterative scheme on (1.4), the stability curves that correspond to selected values of the crossflow, including the example of zero crossflow. Some interesting features arise from these.
Firstly, amongst the $\text{We} \neq 0$ curves, it is found that the curves for smaller crossflow tend to admit a finite streamwise-wavenumber interval wherein for any fixed value, four distinct roots are obtained for the associated spanwise wavenumber. This is explained rationally in Section 4.2, where a threshold value of 0.204 (approximately) is derived for the crossflow, above which the aforesaid feature is not observed. This is highly significant in the weakly-nonlinear situation of Chapter Two because, for fixed $\text{Re}$ and $\text{La}$, it is possible to have six different interactions (corresponding to the six distinct 'pairs' of TS waves) whenever this threshold value is not exceeded. (In particular, in the regime of very small crossflow (Section 4.4), each type of interaction possesses its own unique characteristics, and this is confirmed by the vastly different downstream behaviours.) Furthermore, it may be possible to generalise the weakly-nonlinear formulation to three or four TS waves, as discussed in Section 4.5.

Of the four modes, two of these are the simple counterparts of the Hall and Smith zero-crossflow modes. They are also, incidentally, the modes described in Chapter Three (in the appropriate small-crossflow
regime). The remaining two modes however, are
crossflow generated and have no direct connection with
2D boundary-layer instabilities. Again for small
crossflow, they are epitomised by the asymptotic
expansions

\[ \beta \sim (-2\omega) \omega_{e}^{-1} \pm (\epsilon \lambda^{1/3}) \omega_{e}^{-1/2} + O(\lambda^{-1/3}), \quad (1.5a,b) \]

where \( \epsilon \) has the approximate value of two. Notably,
the wave motion becomes increasingly aligned with the
negative spanwise direction as the crossflow dies out,
with singularities emerging in the spanwise wavenumbers.

Secondly, all the curves for \( \omega_{e} \neq 0 \) share some
striking differences with the zero-crossflow curve.
The main interest here concerns the maximum value of
\( \lambda \); for \( \omega_{e} = 0 \) this occurs for the 2D wave \( \beta = 0 \) and
has the approximate value of \( 1 \), but for any \( \omega_{e} \neq 0 \)
it is removed, in effect, to \( \beta = -\infty \), where it becomes
infinite. More precisely, for \( O(1) \) crossflow similar
asymptotic expansions to (1.5a,b) hold for large \( \lambda \),
except now \( \epsilon \) has the value \( 2 \lambda^{1/2} (1 + \omega_{e}^{2})^{-1/2} \). This
points to the existence of important short-scale
instabilities, (for example in the weakly-nonlinear
context as described in Section 5.1), where the
characteristic streamwise scale $\lambda$ is small. The interactive equations (1.1a-f) continue to hold until we reach the balance $\lambda = O(Re^{-3/56})$ (i.e. $\lambda = O(Re^{-3/17})$ globally). Then significant vertical disturbance-acceleration enters the boundary layer and provokes a jump in disturbance-pressure across the boundary layer invalidating (1f) above (see Section 5.2). In particular, it is found that linear neutral disturbances are admissible only up to a finite cut-off value for the related streamwise wavenumber. Hence, our current vortex/wave interaction ideas, which depend entirely on dominant wave-neutrality, cannot be applied beyond this value. After a brief examination of the weakly nonlinear interactions involving small-spanwise waves (Section 5.3), the remainder of the chapter (Section 5.4) is designated to the study of high frequencies in linear TS disturbance theory.

The high-frequency approach has been utilised with much success in recent years as a means of gaining rational insight in both linear (Stewart and Smith (1987)) and nonlinear (Smith (1985), Smith and Burggraf (1985)) lower-branch theories. In the nonlinear theories, the high-frequency method has been applied to 2D systems in the main, where it is possible to obtain solutions of the governing equations.
For 3D systems, the equations remain insoluble in general, in the high-frequency regime, although we can compensate for this difficulty by considering relatively short-scale instabilities, whereupon (1.1a-f) are restructured into a quasi-2D form (see Chapter Six). Subsequent high-frequency analysis reduces the controlling equations to a condition on the displacement decrement, which has an analogous form to the classical Benjamin-Ono equation (Benjamin (1967), Ono (1975)). Notably, the Benjamin-Ono equation was obtained in the fully-nonlinear 'Stage 2' theory of Smith and Burggraf (1985) for 2D boundary layers, and is known to submit both a nonlinear periodic solution and a solitary wave solution. Motivated by this, equivalent solutions are sought (and acquired) in our problem.

Concerning strongly nonlinear effects, the remainder of the research in the thesis is in boundary-layer transition where again nonlinear crossflow effects are the main interest; specifically the strongly nonlinear interactions occurring between a relatively small Rayleigh wave and its induced vortex are considered.
The chosen mathematical model is one of numerous possibilities in the rapidly developing theories on strongly nonlinear vortex/wave interactions. (See, for example, Hall and Smith (1991), Bennett, Hall and Smith (1991), Smith and Walton (1989).)


The inviscid Rayleigh wave is a well-known phenomenon in disturbance theory and in the high Reynolds-number framework is traditionally described by a two-tiered
structure. This consists of an active inviscid layer (of a thickness comparable with the boundary layer) atop a thin viscous Stokes layer which is usually passive. Moreover, the inviscid layer (or loosely speaking, the boundary layer) accommodates a narrow viscous critical layer about some fixed location where the mean-flow velocity equals the wavespeed. In the midst of the boundary layer, the linear Rayleigh wave has the typical scalings

\[ [u, v, w, \rho] = h [\hat{u}, \hat{v}, \hat{w}, \hat{\rho}] + \ldots \]  

(1.6a-d)

for the nondimensionalised velocity and pressure, where the only criterion on \( h \) is that it must be \( o(1) \). The associated nondimensional coordinate scalings are

\[ [x - x_0, y, z - z_0, t] = \varepsilon^{-1/2} [X, Y, Z, T] \]  

(1.6e-h)

near the station \( x = x_0, z = z_0 \). Here, the method of the nondimensionalisation (and of the subsequent definition of the Reynolds number) is exactly that used for the weakly-nonlinear problem above.

The nonlinearity involved in our interaction problem is 'strong' in the sense that the mean flow
throughout the boundary layer is completely altered from its initial undisturbed state. The theory as it stands, for zero crossflow, has been rigorously developed in Hall and Smith (1991); there it is shown that the nonlinear interaction between the wave and the vortex is effectively localised in the thin critical layer, because although the wave is affected by the vortex (via streamwise forcing) throughout the boundary layer, the wave's feedback on the vortex (via inertial forcing) is confined to the critical layer.

The Rayleigh wave has streamwise and temporal periodicity defined in the manner

$$E \equiv \exp \left[ i (\xi K - \eta T) \right], \quad (1.7)$$

where \(\xi\) and \(\eta\) are real constants, but has more general dependence on \(\eta\) and \(\zeta\). The induced vortex has a longer streamwise scale than the wave (typically \(O(1)\)) but crucially varies over the fast spanwise scale \(\xi\), enabling an \(O(1)\) distortion of the incident mean-flow profile to occur when \(h\) is comparable with \(\left[ \text{Re}^{-1} \ln \text{Re} \right]^{1/2}\).
In the zero-crossflow interactions (i.e. Hall and Smith (1991)), the problem reduces to solving a nonlinear, viscous partial-differential system for the vortex (which is driven by a wave-induced spanwise-shear discontinuity at the critical-layer boundary) coupled with a mean-flow forced partial-differential system for the wave pressure. These equations require a numerical treatment in general, but the task at hand is quite considerable. As yet, Hall and Smith have obtained no such solutions.

The Hall and Smith theory is summarised in Section 7.1, because it forms the basis for the subsequent analysis in Section 7.2, i.e. the effects of enhanced crossflow giving the boundary layer slight three-dimensionality. It is found that the majority of the boundary layer is now inviscid except for a thin buffer layer of viscous vortex motion surrounding the even thinner critical layer. It is in the buffer layer that the more important effects of the nonlinear vortex/wave interaction are observed. Firstly, the vortex motion therein is driven by wave-forcing at the critical-layer/buffer-layer interface. Secondly, the vortex motion provokes a fourth-order discontinuity in wave-pressure across the buffer layer and this feeds
into the inviscid layer outside as a third-derivative wave-pressure jump. Essentially now the problem reduces to solving the governing wave-pressure system in the inviscid layer in conjunction with the aforesaid jump condition. (This determines a spanwise evolution equation for the critical-layer wave-pressure, in principle.) Again the equations pose considerable numerical difficulties, and the author has obtained no solutions.

In an attempt to gain some rational insight regarding the flow solution, the nonlinear interaction is examined close to the upstream input station in Chapter Eight. Firstly in Section 8.1, the structural alterations, rescalings and governing interaction equations are derived analytically. It is shown that some rational headway is achieved as anticipated in this limiting regime, but no exact solutions are deduced due to the still complex nature of the wave-pressure system outside the buffer layer. However, for the special case of a broken-line mean-flow profile (Section 8.2) and allowing the principal wave-pressure to be spanwise periodic, a solution for the associated wave-pressure amplitude is obtained.
The above theory has many applications to other flows of interest such as channel flows, pipe flows and plane Couette flow. The only criterion in fact, is that the basic flow admits comparatively short-scale waves, and so other motions including free shear layers, separations and flow over surface obstacles can be accommodated by this theory.

A summary highlighting the key results in the thesis and suggesting further developments, is presented in Chapter Nine.
2.1 - Determination of the Scales

Linear or nonlinear Tollmien-Schlichting waves are governed by the interactive boundary-layer equations, (1.1a-f), associated with the triple-deck structure. If the amplitude of the waves is small, say of order \( \hat{h} \) compared with their fully nonlinear size, and strictly such that \( \hat{h} \gg R e^{i \omega} \) (for any \( m > 0 \)), then a vortex flow of \( O(\hat{h}^2) \) is induced via nonlinear coupling of the waves. Then, in particular, it is found that the spanwise inertial effects decay slowly (like \( \gamma^{-2} \)) far from the plate surface, causing the spanwise vortex velocity, \( \omega_v \), to grow logarithmically. This singular response is eventually damped out in a buffer deck, lying between the main and lower decks, where the boundary-layer shear-inertial operator, \( \gamma \partial_{\hat{x}_v} \), becomes of the same order as the diffusion operator \( \delta_{\gamma} \), where \( \hat{x}_v \) is the modulated streamwise length scale associated with the vortex. If the buffer deck relative thickness is \( \delta (\gg 1) \), then \( |\partial_{\hat{x}_v}| \sim \delta^{-3} \). There \( \omega_v \hat{h}^2 \ln \delta \), and furthermore from continuity, \( \omega_v \hat{h}^2 \delta^3 \ln \delta \), where \( \omega_v \) is the relative streamwise velocity of the vortex. In Hall and Smith (1989) it was assumed (and subsequently justified) that sensitive
nonlinear interactions would happen if the wave-variations over the longer scale $\bar{X}_v$ were controlled to some extent by the vortex shear from the buffer deck. So, proceeding in a similar vein, we seek the balance $|\partial \bar{X}_v/\partial \delta(\sim \hat{h}^2 \delta^2 \ln \delta)$ which, upon combining with first $\delta - \bar{X}_v$ balance, indicates the required regime to be

$$\delta \sim h^{-2/5}, \quad \bar{X}_v \sim h^{-6/5} \quad \text{where} \quad \hat{h} = h[-2(\ln h / 5)]^{1/2}$$

Unlike the analogous work in Hall and Smith (1989), we have not included nonparallelism here, although it is relatively easy to incorporate these additional effects as described at the end of the chapter.

2.2 - The Lower Deck

In the lower deck, where $y = Re^{-5/8} Y$, viscous forces play a prominent role; this is readily observed from the unsteady boundary-layer equations which hold here, namely

$$\tilde{u}_T + \tilde{u} \tilde{u}_x + \tilde{v} \tilde{u}_y + \tilde{w} \tilde{u}_z = -\tilde{\rho}_x + \tilde{u} \gamma \gamma, \quad (2.1a)$$

$$\tilde{w}_T + \tilde{u} \tilde{w}_x + \tilde{v} \tilde{w}_y + \tilde{w} \tilde{w}_z = -\tilde{\rho}_z + \tilde{w} \gamma \gamma, \quad (2.1b)$$

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0, \quad (2.1c)$$

with $\tilde{\rho}$ independent of $y, \quad (2.1d)$

and the no-slip condition: $\tilde{u} = \tilde{v} = \tilde{w} = 0$ at $y = 0$. \quad (2.1e)
We refrain from applying the outer constraint (1.1e) here because, in effect, the original lower deck has split into two separate decks, characterised by the current regime and the buffer deck which is thicker; it is in the latter region where application of (1.1e) takes place. Here we express the flow solution in the form

$$\bar{u} = \lambda Y + h L \bar{u}^{(1)} + h^{6/5} \lambda_3(Z) Y + h^2 L \bar{u}^{(3)} + h^{11/5} \bar{u}^{(1)} + \ldots, \quad (2.2a)$$

$$\bar{w} = h L \bar{w}^{(1)} + h^2 L \bar{w}^{(3)} + h^{11/5} \bar{w}^{(1)} + \ldots, \quad (2.2b)$$

$$\bar{w} = h L \bar{w}^{(1)} + h^2 L \bar{w}^{(3)} + h^{11/5} \bar{w}^{(1)} + \ldots, \quad (2.2c)$$

$$\bar{\rho} = h L \bar{\rho}^{(1)} + h^{16/5} \bar{\rho}^{(1)} + \ldots + h^{16/5} \bar{\rho}^{(3)} + \ldots, \quad (2.2d)$$

where $L \equiv [-2 \ln h/5]^{-1/2}$ and $\lambda Y$ is the basic-flow shear arising from the main deck. The terms superscripted (1) are TS contributions, with the single- and double-tilde quantities denoting leading order and second order effects, in turn. The induced-vortex contributions have the superscript (3) whilst the quantity $\lambda_3(Z) Y$ (where $\lambda_3 \equiv \lambda_{33} E_3 + \text{c.c.}$) is the vortex-streamwise shear that arises through feedback from the buffer deck. We note that two streamwise coordinate scales are active: the triple-deck scale $X$, and the modulated scale $\chi = h^{-6/5} \lambda X$ say, responsible for wave growth and vortex variations. Hence $\partial_x \rightarrow \partial_x + h^{6/5} \partial_\chi$ effectively. Finally, we expand $\lambda$ as $1 + h^{6/5} \lambda_1 + \ldots$, where $\lambda_1$ is real; this is possible provided the lower deck quantities adjust to accommodate the local variations of the skin-friction. (Smith and Burggraf (1985)).
We now substitute the above expansions into (2.1a-e) to obtain

\[
\begin{align*}
\tilde{u}_{n}^{(1)} + \gamma \tilde{u}_{x}^{(1)} + \tilde{u}_{x}^{(1)} &= -\tilde{\rho}_{x}^{(1)} + \tilde{u}_{n}^{(1)}, \\
\tilde{u}_{n}^{(1)} + \gamma \tilde{w}_{x}^{(1)} &= -\tilde{\rho}_{x}^{(1)} + \tilde{w}_{n}^{(1)}, \\
\tilde{u}_{x}^{(1)} + \tilde{u}_{y}^{(1)} + \tilde{w}_{z}^{(1)} &= 0, \\
\tilde{u}_{n}^{(1)} + \gamma (\tilde{u}_{x}^{(1)} + \tilde{u}_{x}^{(1)}) + (\lambda_{3} + \lambda_{1}) \gamma \tilde{u}_{x}^{(1)} &+ \tilde{u}_{z}^{(1)} + \tilde{w}_{z}^{(1)} \lambda_{3} \gamma \tilde{u}_{x}^{(1)} = -\tilde{\rho}_{x}^{(1)} + \tilde{u}_{n}^{(1)}, \\
\tilde{w}_{n}^{(1)} + \gamma (\tilde{w}_{x}^{(1)} + \tilde{w}_{x}^{(1)}) + (\lambda_{3} + \lambda_{1}) \gamma \tilde{w}_{x}^{(1)} &= -\tilde{\rho}_{x}^{(1)} + \tilde{w}_{n}^{(1)}, \\
\tilde{u}_{x}^{(1)} + \tilde{u}_{x}^{(1)} + \tilde{u}_{y}^{(1)} + \tilde{w}_{z}^{(1)} &= 0, \\
\tilde{u}_{n}^{(1)} + \tilde{u}_{x}^{(1)} + \tilde{u}_{y}^{(1)} + \tilde{w}_{z}^{(1)} + \tilde{w}_{z}^{(1)} &= \tilde{u}_{n}^{(1)}, \\
\tilde{w}_{n}^{(1)} + \tilde{w}_{n}^{(1)} + \tilde{w}_{n}^{(1)} + \tilde{w}_{z}^{(1)} &= \tilde{w}_{n}^{(1)}, \\
\tilde{u}_{y}^{(1)} + \tilde{w}_{z}^{(1)} &= 0,
\end{align*}
\]

for the main TS, forced TS and vortex flow, respectively, where the associated pressures \( \tilde{\rho}_{1}^{(1)} \), \( \tilde{\rho}_{1}^{(1)} \), \( \tilde{\rho}_{1}^{(1)} \) are independent of \( y \), and zero speed is observed at the wall.

The TS- and vortex- properties described in Chapter 1 are now introduced; that is, we write

\[
\begin{align*}
\tilde{u}_{n}^{(1)} &= \tilde{u}_{n}^{(1)}(x, y) E_{1} + \tilde{u}_{n}^{(1)}(x, y) E_{2} + c.c., \\
\tilde{u}_{n}^{(1)} &= \tilde{u}_{n}^{(1)}(x, y) E_{1} + \tilde{u}_{n}^{(1)}(x, y) E_{2} + c.c., \\
\tilde{u}_{n}^{(1)} &= \tilde{u}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{u}_{2}^{(1)} &= \tilde{u}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{u}_{n}^{(1)} &= \tilde{u}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{w}_{n}^{(1)} &= \tilde{w}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{w}_{n}^{(1)} &= \tilde{w}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{w}_{n}^{(1)} &= \tilde{w}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\tilde{w}_{n}^{(1)} &= \tilde{w}_{n}^{(1)}(x, y) E_{3} + c.c., \\
\end{align*}
\]

(where c.c. denotes complex conjugate).
etc., and equate the coefficients of $E_1, E_2 \propto E_3$
as appropriate, in (2.3a-c), (2.4a-c), (2.5a-c). We
find, successively, the governing equations

\begin{align}
&i (dY - \Omega) \tilde{u}_{in} + \tilde{u}_{in} = - i 2 \tilde{p}_{in} + \tilde{u}_{inYY}, \quad (2.7a) \\
&i (dY - \Omega) \tilde{w}_{in} = - i \beta_n \tilde{p}_{in}/2 + \tilde{w}_{inYY}, \quad (2.7b) \\
&i d \tilde{u}_{in} + \tilde{u}_{inY} + i \beta_n \tilde{w}_{in}/2 = 0, \quad (2.7c)
\end{align}

\begin{align}
&i (dY - \Omega) \tilde{u}_{in} + Y \tilde{u}_{inX} + i d \lambda_1 Y \tilde{u}_{in} + i d \left\{ \frac{\lambda_{23} \tilde{u}_{12}}{-\lambda_{23} \tilde{u}_{11}} \right\} \\
&+ \tilde{u}_{in} + \tilde{u}_{in} \lambda_1 + \left\{ \frac{\lambda_{33} \tilde{u}_{12}}{-\lambda_{33} \tilde{u}_{11}} \right\} + \frac{i}{2} (\beta_1 - \beta_2) Y \left\{ \frac{\lambda_{33} \tilde{u}_{12}}{-\lambda_{33} \tilde{u}_{11}} \right\} \\
&= - i d \tilde{p}_{in} - \tilde{p}_{inX} + \tilde{u}_{inYY}, \quad (2.8a) \\
&i (dY - \Omega) \tilde{w}_{in} + Y \tilde{w}_{inX} + i d \lambda_2 Y \tilde{w}_{in} + i d \left\{ \frac{\lambda_{33} \tilde{w}_{12}}{-\lambda_{33} \tilde{w}_{11}} \right\} \\
&+ \tilde{w}_{in} + \tilde{w}_{in} \lambda_1 + \left\{ \frac{\lambda_{33} \tilde{w}_{12}}{-\lambda_{33} \tilde{w}_{11}} \right\} \\
&= - i \beta_n \tilde{p}_{in}/2 + \tilde{w}_{inYY}, \quad (2.8b) \\
&i d \tilde{u}_{in} + \tilde{u}_{inX} + \tilde{u}_{inY} + i \beta_n \tilde{w}_{in}/2 = 0, \quad (2.8c)
\end{align}

(with $^*$ denoting complex conjugate)

\begin{align}
&\tilde{u}_{11} \tilde{u}_{12}^* + \tilde{u}_{12}^* \tilde{u}_{11}^* + \tilde{u}_{33} + i (\beta, \tilde{u}_{11} \tilde{w}_{12}^* - \beta_2 \tilde{u}_{12}^* \tilde{w}_{11})/2 = \tilde{u}_{33YY}, \quad (2.9a) \\
&i d (\tilde{u}_{12}^* \tilde{w}_{11} - \tilde{u}_{11} \tilde{w}_{12}^*) + \tilde{u}_{12}^* \tilde{w}_{11Y} + \tilde{u}_{12} \tilde{w}_{11}^* + i (\beta_1 - \beta_2) \tilde{w}_{11} \tilde{w}_{12}^*/2 \\
&\tilde{w}_{33} + i (\beta_1 - \beta_2) \tilde{w}_{33}^*/2 = 0, \quad (2.9b)
\end{align}

where $\tilde{p}_{in}, \tilde{p}_{in}, \tilde{p}_{33}$ are $Y$-independent, and

\begin{align}
&\tilde{u}_{in} = \tilde{w}_{in} = \tilde{w}_{in} = 0 \text{ at } Y = 0 \text{ (likewise for } (\tilde{u}_{in}, \tilde{w}_{in}, \tilde{w}_{in}) \text{ and } (\tilde{u}_{33}, \tilde{u}_{33}, \tilde{w}_{33}), \text{ for } n = 1, 2 \text{ respectively.}
\end{align}

These systems of equations will be addressed in detail later,
once we have established some necessary results from
the buffer, main and upper decks. For now, we observe
the asymptotic features of the two TS waves and the vortex in the outer extremes of the lower deck.

These are

\[ u_{1n} \sim 1, \quad \bar{u}_{1n} \sim Y, \quad \bar{w}_{1n} \sim Y^{-1}, \quad \bar{p}_{1n} \sim 1; \quad (2.10a-d) \]
\[ u_{x3} \sim 1, \quad \bar{u}_{x3} \sim Y, \quad \bar{w}_{x3} \sim Y^{-1}, \quad \bar{p}_{x3} \sim 1; \quad (2.11a-d) \]
\[ u_{33} \sim Y^3 \ln Y, \quad \bar{u}_{33} \sim Y \ln Y, \quad \bar{w}_{33} \sim \ln Y, \quad \bar{p}_{33} \sim 1; \quad (2.12a-d) \]

as \( Y \to \infty \).

Whilst (2.10a-d), (2.11a-d) follow easily from inspection of (2.7a-c), (2.8a-c), the results (2.12a-d) rely on the waves being neutral and on imposing a derivative condition at the wall, namely

\[ \bar{w}_{33} (0) = \int_0^\infty F(Y) dY, \quad (2.13) \]

where \( F \) is the wave-forcing on the left-hand side of (2.9b). Otherwise, we would find that \( \bar{w}_{33} \sim Y \ln Y \), for large \( Y \).

### 2.3 - The Buffer Deck

Here \( y = \kappa^{-5/3} h^{2/5} \hat{y} \), where \( \hat{y} \) is \( O(1) \).

This region is needed to adjust the vortex flow to the outer constraint in (1.1) above. We write the velocity and pressure as

\[ u = h^{-2/5} \hat{y} + h^{6/5} (\hat{\gg} + \hat{\gg}) + \ldots + h^{9/5} \hat{\gg} + \ldots + h^{11/5} \hat{\gg} + \ldots, \quad (2.14a) \]
\[ \bar{u} = h^{3/5} \hat{\gg}^{(1)} + \ldots + h^{8/5} \hat{\gg}^{(3)} + \ldots + h^{9/5} \hat{\gg}^{(1)} + \ldots, \quad (2.14b) \]
\[ \bar{w} = h^{7/5} \hat{\gg}^{(1)} + \ldots + h^2 \hat{\gg}^{(3)} + \ldots + h^{13/5} \hat{\gg}^{(1)} + \ldots, \quad (2.14c) \]
\[ \bar{p} = h^{12/5} \hat{\gg}^{(1)} + \ldots + h^{11/5} \hat{\gg}^{(3)} + \ldots + h^{16/5} \hat{\gg}^{(3)} + \ldots, \quad (2.14d) \]
Thus we generate the sets of equations

\[
\hat{y} \hat{U}_x^{(1)} + \hat{U}_x^{(1)} = 0, \quad (2.15a)
\]
\[
\hat{y} \hat{\omega}_x^{(1)} = -\hat{\rho}_z^{(1)}, \quad (2.15b)
\]
\[
\hat{U}_x^{(1)} + \hat{u}_y^{(1)} = 0, \quad (2.15c)
\]

\[
\hat{y} \left( \hat{U}_x^{(1)} + \hat{U}_x^{(1)} \right) + (\hat{U}_x^{(2)} + \lambda, \hat{y}) \hat{U}_x^{(1)} + \hat{U}_x^{(1)}(\hat{U}_y^{(2)} + \lambda_1) + \hat{U}_x^{(1)} = 0, \quad (2.16a)
\]
\[
\hat{y} \left( \hat{U}_x^{(2)} + \hat{U}_x^{(1)} \right) + (\hat{U}_x^{(3)} + \lambda, \hat{y}) \hat{U}_x^{(2)} = -\hat{\rho}_z^{(2)}, \quad (2.16b)
\]
\[
\hat{U}_x^{(1)} + \hat{U}_x^{(2)} + \hat{U}_y^{(2)} = 0, \quad (2.16c)
\]

\[
\hat{y} \hat{U}_x^{(3)} + \hat{U}_x^{(3)} = \hat{U}_x^{(3)}, \quad (2.17a)
\]
\[
\hat{y} \hat{\omega}_x^{(3)} = \hat{\omega}_x^{(3)}, \quad (2.17b)
\]
\[
\hat{U}_x^{(3)} + \hat{u}_y^{(3)} + \hat{U}_z^{(3)} = 0, \quad (2.17c)
\]

for the main TS, forced TS and vortex flows, in turn, where again the corresponding pressures are independent of the normal scale. Each set of equations is to be solved subject to the external displacement condition (1.1e) where, for each of the above systems, this reduces to

\[
\left( \hat{U}_x^{(1)}, \hat{U}_x^{(2)}, \hat{U}_x^{(3)} \right) \to \left( \hat{\lambda}_x^{(1)}, \hat{\lambda}_x^{(2)}, \hat{\lambda}_x^{(3)} \right) \}
\]
\[
\left( \hat{\omega}_x^{(1)}, \hat{\omega}_x^{(2)}, \hat{\omega}_x^{(3)} \right) \sim \left( \hat{\gamma}_x^{-1}, \hat{\gamma}_x^{-1}, \hat{\gamma}_x^{-1} \right) \}
\]
\[
(2.18a-f)
\]

Here the displacement decrement A has the expansion

\[
A = h^{1/5} \hat{A}^{(2)} + h L \hat{A}^{(1)} + \ldots + h^{11/5} \hat{A}^{(1)} + \ldots. \quad (2.19)
\]
For the main TS part we find simple slip-effect solutions hold:

\[
\begin{align*}
\hat{\mathbf{u}}^{(1)} &= \mathbf{A}^{(1)}, \\
\hat{\mathbf{c}}^{(1)} &= \hat{\gamma} \mathbf{A}^{(1)}, \\
\hat{\mathbf{a}}^{(1)} &= \left\{ \frac{-\beta_1 \tilde{p}_{11}}{2 \lambda \tilde{\gamma}} \mathbf{E}_1 - \frac{\beta_2 \tilde{p}_{12}}{2 \lambda \tilde{\gamma}} \mathbf{E}_2 \right\} + \text{c.c.}, \\
\hat{\rho}^{(1)} &= \hat{\rho}^{(1)},
\end{align*}
\]

where \( \mathbf{A}^{(0)} = \mathbf{A}_{11} \mathbf{E}_1 + \mathbf{A}_{12} \mathbf{E}_2 + \text{c.c.} \) \( \quad (2.21) \)

Consistent lower-deck matching is required as \( \hat{\gamma} \to 0^+ \), suggesting that

\[
\hat{\mathbf{u}}^{(1)} \to \mathbf{A}^{(1)} \quad \text{(and hence} \quad \hat{u}_{1n} \to \mathbf{A}_{1n}, \text{for} \quad n = 1, 2 \text{ as} \quad \gamma \to \infty \quad \text{in the lower deck}. \quad (2.22)
\]

Next, \((2.16a-c)\), solved in conjunction with \((2.18b)\), yield in particular

\[
\begin{align*}
\hat{\mathbf{u}}^{(1)} &= \hat{\mathbf{u}}^{(3)} \mathbf{A}^{(1)} + \mathbf{A}^{(1)}, \\
\hat{\mathbf{c}}^{(1)} &= -\hat{\gamma} \left( \mathbf{A}^{(1)}_{x} + \mathbf{A}^{(1)}_{x} \right) - \hat{\mathbf{u}}^{(3)} \mathbf{A}^{(1)},
\end{align*}
\]

where \( \mathbf{A}^{(1)} = \mathbf{A}_{11} \mathbf{E}_1 + \mathbf{A}_{12} \mathbf{E}_2 + \text{c.c.} \)

Compatible matching with the lower deck solution will be guaranteed so long as

\[
\hat{\mathbf{u}}^{(1)} \to \lambda_3 \mathbf{A}^{(1)} + \mathbf{A}^{(1)} \quad \text{as} \quad \gamma \to \infty \quad (2.24)
\]

in the lower deck.
The equations for the vortex, (2.17a-c), are partial-differential, being dependent on \( \tilde{\mathbf{y}} \) and \( \mathbf{X} \). They are insoluble at present since they depend on the unknown wave-pressure terms \( \rho_{1}, \rho_{2} \) via a slip-condition on \( \tilde{W}_{\mathbf{f}}(\mathbf{z}) \) at the buffer-deck wall. Therefore, we must determine the dominant wave variations over the longer scale \( \mathbf{X} \), and this requires us to solve the pressure-displacement law (1.1f) for both the leading and forced TS waves, and subsequently coupling the results with each corresponding problem in the lower deck. Since the vortex pressure is mainly passive here, we need not concern ourselves with the vortex motion beyond this layer.

### 2.4 - Pressure-Displacement Relations

An alternative to solving the Cauchy-Hilbert integral (1.1f) directly is to consider the Laplacian equation and associated boundary conditions, from which the integral was derived, namely

\[
\left( \partial_{x}^{2} + \partial_{y}^{2} + \partial_{z}^{2} \right) \rho' = 0,
\]  
(2.25a)

with \( \rho' \to 0 \) as \( y' \to \infty \),

\[
\rho' \to \tilde{\rho}, \quad \rho'_y \to (\omega_{e} \partial_x + \omega_{e} \partial_z)^2 \mathbf{A}, \text{ as } y' \to 0.
\]  
(2.25b,c,d)

Here \( y' (\equiv \Re^{-3/8} y) \) is the scaled upper deck transverse coordinate, and the pressure \( \rho' \) has the splitting

\[
\rho' = h \rho'^{(0)} + \ldots + h^{1/5} \rho'^{(1)} + \ldots,
\]  
(2.26)

where

\[
\rho'^{(0)} = \rho''(\mathbf{x}, y') \mathbf{E}_1 + \rho'^{z}(\mathbf{x}, y') \mathbf{E}_2 + \text{c.c.},
\]

\[
\rho'^{(1)} = \rho^{'''}(\mathbf{x}, y') \mathbf{E}_1 + \rho^{''z}(\mathbf{x}, y') \mathbf{E}_2 + \text{c.c.}.
\]
Insertion of (2.26) into (2.25a-d) gives

\[ \frac{\partial^2 \rho_n'}{\partial y^2} - \left( \ell^2 + \beta_n^2 / 4 \right) \rho_n' = 0, \quad (2.27a) \]

with \( \rho_n' (\bar{x}, \infty) = 0, \)

\( \rho_n' (\bar{x}, 0) = \bar{\rho}_n, \quad \rho_n'' (\bar{x}, 0) = -\left( \ell u_e + \beta_n \omega_e / 2 \right)^2 A_n (\bar{x}), \quad (2.27b, c, d) \)

(for \( n = 1, 2 \)), to leading order. This yields

\[ \left( \ell^2 + \beta_n^2 / 4 \right)^{1/2} \bar{\rho}_n (\bar{x}) = \left( \ell + \beta_n \omega_e / 2 \right)^2 A_n (\bar{x}) \quad (2.28a, b) \]

(for \( n = 1, 2 \)) where, without loss of generality, \( u_e = 1. \)

The second level equations are

\[ \frac{\partial^2 \rho_n''}{\partial y^2} - \left( \ell^2 + \beta_n^2 / 4 \right) \rho_n'' = -2 i \ell \rho_n' (\bar{x}), \quad (2.29a) \]

with \( \rho_n'' (\bar{x}, \infty) = 0, \)

\( \rho_n'' (\bar{x}, 0) = \bar{\rho}_n, \)

\( \rho_n'' (\bar{x}, 0) = -\left( \ell + \beta_n \omega_e / 2 \right)^2 A_n (\bar{x}) + 2 i \ell \left( \ell + \beta_n \omega_e / 2 \right) A_n (\bar{x}), \quad (2.29c, d) \)

(for \( n = 1, 2 \))

which we note is linear, due to the absence of any vortex forcing. In fact, only the slow streamwise modulation serves to drive the secondary waves.

The solution of (2.29a-d) is

\[ \left( \ell^2 + \beta_n^2 / 4 \right)^{1/2} \bar{\rho}_n (\bar{x}) = \left( \ell + \beta_n \omega_e / 2 \right)^2 A_n (\bar{x}) \]

\[ -2 i \left( \ell + \beta_n \omega_e / 2 \right) A_n (\bar{x}) \]

\[ + i \ell \left( \ell + \beta_n \omega_e / 2 \right)^2 A_n (\bar{x}) / \left( \ell^2 + \beta_n^2 / 4 \right), \quad (2.30a, b) \]

for \( n = 1, 2. \)
2.5 - Neutral Eigenrelations and Pressure-Amplitude Equations

Having established the necessary upper-deck pressure-displacement laws, we now return to the lower deck and solve the leading TS and forced TS systems of equations therein, together with (2.28a,b), (2.29a,b) above.

So, for the leading waves we have

\[ i(\lambda - \omega)\tilde{u}_m + \tilde{v}_m = -i\lambda \tilde{a}_m + \tilde{w}_m \gamma, \]  
\[ (\lambda^2 + \beta_n^2/4)^{1/2} \tilde{a}_m = (\lambda + \beta_n \omega_e/2)^2 \tilde{A}_m, \]

for \( n = 1, 2 \). From this we can deduce the dispersion relations

\[ \lambda^2 A_i'(\gamma_0)/\kappa = (i\lambda)^{1/3} (\lambda^2 + \beta_n^2/4)^{1/2} (\lambda + \beta_n \omega_e/2)^2, \]

for \( n = 1, 2 \) where \( \gamma_0 = -i^{1/3} \kappa /\omega^{1/3} \), \( \kappa \equiv \int_{\gamma_0}^{i^{1/3}\infty} A_i(s) ds \), and \( A_i \) is the Airy function. The neutrality of the waves implies the values

\[ \gamma_0 = -\lambda, A_i'(\gamma_0)/\kappa = (\lambda, \lambda dz)^{1/3}, \]  

for \( n \approx 2.3, d_z \approx 2.3 \) (Lin (1955), Smith (1979), Drazin and Reid (1981)), so
that (2.32a,b) become

\[ \mathcal{L}^{5/3} \left( \frac{d_1}{d_2} \right) = \left( \mathcal{L}^2 + \beta^2 \right)^{1/4} \left( \mathcal{L} + \beta \omega e / 2 \right)^2, \quad (2.33a,b) \]

for \( n = 1,2 \). Equations (2.33a,b) encapsulate the neutral stability of the two 3D disturbances. Many interesting features about linear and nonlinear stability can be made for extreme values of the parameters involved, e.g. \( 0 < \omega e << 1, \Omega >> 1, \mathcal{L} >> 1 \), etc., and these are studied in Chapters 4,5 below.

For the forced TS waves the governing equations are

\[
i (\mathcal{L} Y - \mathcal{A}) \bar{u}_{in} + Y \bar{u}_{in} \bar{\kappa} + i \mathcal{L} \lambda_1 \bar{u}_{in} + i \mathcal{L} \left\{ \lambda_{33} \bar{u}_{12} \begin{pmatrix} \lambda_{33} \bar{u}_{12} \\ -\lambda_{33}^* \bar{u}_{11} \end{pmatrix} \right. \]

\[ + \bar{u}_{in} + \bar{u}_{in} \lambda_1 + \left\{ \lambda_{33} \bar{u}_{12} \begin{pmatrix} \lambda_{33} \bar{u}_{12} \\ -\lambda_{33}^* \bar{u}_{11} \end{pmatrix} \right. \]

\[ = -i \mathcal{L} \tilde{p}_{in} - \tilde{p}_{in} + \bar{u}_{inYY}, \quad (2.34a) \]

\[
i (\mathcal{L} Y - \mathcal{A}) \bar{w}_{in} + Y \bar{w}_{in} \bar{\kappa} + i \mathcal{L} \lambda_1 \bar{w}_{in} + i \mathcal{L} \left\{ \lambda_{33} \bar{w}_{12} \begin{pmatrix} \lambda_{33} \bar{w}_{12} \\ -\lambda_{33}^* \bar{w}_{11} \end{pmatrix} \right. \]

\[ = -i \mathcal{L} \tilde{p}_{in} - \tilde{p}_{in} + \bar{w}_{inYY}, \quad (2.34b) \]

\[
i \mathcal{L} \tilde{u}_{in} + \tilde{u}_{in} \lambda_1 + \tilde{u}_{in} \lambda_1 + i \beta_1 \tilde{w}_{in} = 0, \quad (2.34c) \]

\[ \bar{u}_{in} \sim \left\{ \lambda_{33} \bar{A}_{12} \begin{pmatrix} \lambda_{33} \bar{A}_{12} \\ -\lambda_{33}^* \bar{A}_{11} \end{pmatrix} \right. \]

\[ \bar{w}_{in} \sim Y^{-1} \quad \text{as } Y \to \infty, \quad (2.34d) \]

and

\[
i (\mathcal{L}^2 + \beta^2 / 4)^{1/2} \tilde{p}_{in} = (\mathcal{L} + \beta \omega e / 2)^2 \bar{A}_{in} \]

\[ - 2i (\mathcal{L} + \beta \omega e / 2) \bar{A}_{in} \bar{k} + i \mathcal{L} (\mathcal{L} + \beta \omega e / 2) \bar{A}_{in} \bar{k} \mathcal{L} / (\mathcal{L}^2 + \beta^2 / 4), \quad (2.34e) \]

for \( n = 1,2 \). After much working, it is found that

\[ a_1 \frac{d \tilde{p}_{in}}{d \mathcal{L}} + b_1 \lambda_1 \tilde{p}_{in} + c_1 \lambda_{33} \tilde{p}_{12} = 0, \quad (2.35a) \]

\[ a_2 \frac{d \tilde{p}_{12}}{d \mathcal{L}} + b_2 \lambda_1 \tilde{p}_{12} + c_2 \lambda_{33} \tilde{p}_{11} = 0, \quad (2.35b) \]
hold, where

\[ a_n' = 2B_n \tilde{\eta}_0 \gamma_i 0 \tau_n^2 / \left( 3\omega^3 \Delta^{2/3} \right) \]
\[ - i B_n^{-1/2} \left( -2 \lambda^2 + 2 \lambda^2 + \lambda^2 \right) \left( -5 \lambda^2 + 2 \lambda^2 \right), \]  
(2.36a)

\[ b_n' = -2B_n \tilde{\eta}_0 \gamma_i 0 \tau_n^2 / \left( 3\omega^2 \Delta^{5/3} \right) - 5B_n^{1/2} / \left( 2 \Delta \right), \]  
(2.36b)

\[ c_n' = i \Delta \tilde{\eta}_0 \Delta \left( 2 \lambda^2 \lambda^2 \right) \tau_n^2 / \left( \beta \Delta^2 \left( \lambda, \tau_2 \right)^2 \right) \]
\[ - \Gamma_2 \beta \lambda^2 \Delta \tau_n^2 / \left( 8 \Delta^2 \right) - 2^{-1} B_n^{-1/2} \left( 5 \lambda^2 \lambda^2 \right) \tau_n^2 / \left( \beta \Delta^2 \lambda^2 \right) \]
\[ + 3 \beta \lambda^2 \xi \Delta^2 / \beta \]  
(2.36c)

for \( n = 1, 2 \). Here \( B_n = (\omega^2 + \beta_n^2 / 4) \), \( \tau_n = (\omega + \beta_n \omega / 2) \), \( \delta_n = \left( 1 - \beta_1 / \beta_n \right) \), \( \Delta \equiv i \omega \), \( \Gamma_1 \equiv A_i \left( \tilde{\eta}_0 \right) / A_i' \left( \tilde{\eta}_0 \right) \), \( \Gamma_2 \equiv H / A_i' \left( \tilde{\eta}_0 \right) \) and \( \tilde{\eta}_0, H \) are as defined previously.

In the above, (2.35a,b) constitute the pressure-amplitude equations, and show that the growth rates of each wave are affected by the perturbed basic-flow skin-friction \( (\mathcal{A}_1) \) and the buffer-deck vortex-streamwise shear \( (\lambda_{33}) \), as hypothesised in Section 2.1. We are prevented from solving the amplitude equations as they stand, because the vortex (and hence \( \lambda_{33} \)) has implicit dependence on the waves, as argued in Section 2.3 above. Instead we must solve the wave- and vortex-equations interactively, although firstly we need to ascertain the matching conditions for the vortex at the buffer-deck wall.
Now, in the lower deck, wave-inertial forcing provokes logarithmic growth in the vortex-spanwise velocity component. Substituting the asymptotic properties

\begin{align}
\tilde{u}_{ln} &= \tilde{A}_{ln} + (\beta_n^2 \tilde{p}_{ln} / (4d^2)) Y^{-1} + \ldots, \\
\tilde{U}_{ln} &= -i(\omega \tilde{A}_{ln}) Y + (i \omega \tilde{A}_{ln} - i \omega (1 + \beta_n^2 / (4d^2)) \tilde{p}_{ln}) + \ldots, \\
\tilde{u}_l &= -(\beta_n^2 \tilde{p}_{ln} / (2d)) Y^{-1} + \ldots, \\
\end{align} (2.37a)

for the leading waves in the lower deck, as \( Y \to \infty \), into (2.9b) above, and implementing the main pressure-displacement laws (2.28a,b), culminates in the vortex condition

\begin{equation}
\tilde{w}_{33} \sim i K \tilde{p}_{11} \tilde{p}_{22}^* / Y^2 \quad \text{(or } \tilde{w}_{33} \sim -i K \tilde{p}_{11} \tilde{p}_{22}^* \ln Y \text{)} , \quad \text{for } Y \gg 1 ,
\end{equation}

where

\begin{equation}
K = (\beta_1 - \beta_2) (1 + \beta_1 \beta_2 / (4d^2)) / 2 .
\end{equation} (2.38)

This logarithmic growth in \( \tilde{w}_{33} \) is known to induce the other properties \( \tilde{u}_{33} \sim Y^3 \ln Y, \tilde{U}_{33} \sim Y \ln Y \) as \( Y \to \infty \), in the lower deck. Therefore, the inner constraints for the vortex in the buffer deck are

\begin{equation}
\begin{cases}
\tilde{u}_{33} \to 0, \\
\tilde{U}_{33} \to 0, \\
\tilde{u}_{33} \to -i K \tilde{p}_{11} \tilde{p}_{22}^* \\
\end{cases}
\quad \text{as} \quad \tilde{y} \to \tilde{O}^+ . \quad (2.39)
\end{equation}

2.6 - The Interaction Equations

In summary, the nonlinear vortex/TS interaction is embodied in the equations
\[
\begin{align*}
\hat{U}_{x} + V &= U \hat{g}, \quad \text{(2.40a)} \\
\hat{W}_{x} &= W \hat{g}, \quad \text{(2.40b)} \\
U_{x} + V_{y} + i(\beta_{1} - \beta_{2})W/2 &= 0, \quad \text{(2.40c)}
\end{align*}
\]

with

\[
\begin{align*}
\hat{U}(\infty, \infty) &= A, \quad W(\infty, \infty) = 0, \quad \text{(2.40d)} \\
\hat{U}(\infty, 0) &= 0, \quad W(0, 0) = -iK \rho_{1} \rho_{12}^*, \quad \text{(2.40e)}
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dx} \rho_{11} + b_{1} \lambda_{1} \rho_{11} + C_{1} \lambda_{33} \rho_{12} &= 0, \quad \text{(2.40f)} \\
\frac{d}{dx} \rho_{22} + b_{2} \lambda_{2} \rho_{22} + C_{2} \lambda_{33} \rho_{21} &= 0, \quad \text{(2.40g)}
\end{align*}
\]

where \( \hat{U}_{33}, \hat{V}_{33}, \hat{W}_{33}, \hat{A}_{33}, \hat{P}_{1}, \hat{P}_{2} \) have, in turn, been replaced by \( U, V, W, A, P_{1}, P_{2} \) and \( b_{n} = b_{n}^{'n}/a_{n}, c_{n} = c_{n}^{'n}/a_{n}^{'n} \) (\( n = 1, 2 \)).

Defining \( \zeta = U \hat{g} \) (so that \( \zeta(g=0)=\lambda_{33} \)), \( (2.40a-c) \)

simplify to

\[
\begin{align*}
\hat{g} \hat{g} - \hat{g} \hat{g}_{x} &= -i(\beta_{1} - \beta_{2})W/2, \quad \text{(2.41a)} \\
W \hat{g} \hat{g} - \hat{g} \hat{W}_{x} &= 0, \quad \text{(2.41b)} \\
\zeta &\hat{g}_{\infty} = W(\infty, \infty) = 0, \quad \text{(2.41c)} \\
\hat{g}_{\infty} &= 0, \quad W(\infty, 0) = -iK \rho_{1} \rho_{12}^*, \quad \text{(2.41d)}
\end{align*}
\]

In principle, we may solve \( (2.41a-d) \) collectively with \( (2.40f,g) \), given some prescribed input conditions in \( \hat{X} \); however, no analytical solutions seem to exist.

Analogous equations to \( (2.40a-g) \), \( (2.41a-d) \) were obtained by Blennerhassett and Smith (1991) for zero crossflow, where the authors corrected the original zero-crossflow "interaction equations" in Hall and Smith (1989). In both papers, a partial-differential finite-difference scheme was applied directly to the vortex-wave equations.
It is possible to reduce the $\tau-W$ system above to an integral equation for $\lambda_{33}$ if we consider the special case of zero-input vortex flow, i.e. $W=\tau=0$ at $\chi=\chi_0$ (say), where $\chi_0$ is some specified initial value upstream. Defining the origin shift variable $\chi_1=\chi-\chi_0$ (so that $\chi_1=0$ is our initial station), we apply the Laplace transform in $\chi$, to (2.41a-d) so that

$$\tau_{gg'}-s\hat{\tau} = -i(\beta_1-\beta_2)\hat{W}/2, \quad (2.42a)$$

$$\hat{W}_{gg'} - s\hat{W} = 0, \quad (2.42b)$$

subject to

$$\tau(s,\infty)=0, \hat{W}(s,\infty)=0, \quad (2.42c)$$

$$\tau_g(s,0)=0, \hat{W}(s,0)=-iK\hat{\rho}. \quad (2.42d)$$

Here $\hat{\tau} = \int_0^\infty \tau(\chi_1,\hat{\gamma})e^{-s\chi_1}d\chi_1$, $\hat{W} = \int_0^\infty W(\chi_1,\hat{\gamma})e^{-s\chi_1}d\chi_1$, $\hat{\rho} = \int_0^\infty P_{11}(\chi_1)P_{12}(\chi_1)e^{-s\chi_1}d\chi_1$ are the Laplace transforms of $\tau, W$ and $P_{11}P_{12}^*$ respectively. Solving (2.42a-d) gives

$$\hat{W} = -iK\hat{\rho}A_i'(s^{1/3}\hat{\gamma})/A_i(0), \quad (2.43a)$$

and $\hat{\tau} = (\beta_1-\beta_2)K\hat{\rho}s^{-2/3}A_i'(s^{1/3}\hat{\gamma})[2A_i(0)]$. \quad (2.43b)

Evaluating (2.43b) at $\hat{\gamma}=0$ gives

$$\hat{\lambda}_{33} = (\beta_1-\beta_2)KA_i'(0)s^{-2/3}\hat{\rho}/[2A_i(0)] \quad (2.44)$$

where $\hat{\lambda}_{33} = \int_0^\infty \lambda_{33}(\chi_1)e^{-s\chi_1}d\chi_1$ is the Laplace transform of $\lambda_{33}$. Using the result $\mathcal{L}(x_{-1/3}) = s^{-2/3}\int(2/3)$ where

$$\int(2/3) = \int_0^\infty t^{-1/3}e^{-t}dt, \quad \text{and } \mathcal{L} \text{ is the Laplace transform},$$

we employ the Convolution theorem to invert (2.44).

Therefore

$$\lambda_{33}(\chi) = -\frac{(\beta_1-\beta_2)KA_i'(0)}{2\Lambda_i(0)\Gamma(2/3)} \int_\chi^\infty \frac{\hat{\lambda}_{33}(\chi_1)\hat{\rho}_{11}(\chi_1)\hat{\rho}_{12}(\chi_1)(\chi_1-\chi)^{-1/3}d\chi_1}{\chi_0}. \quad (2.45)$$
Then unifying (2.45) and (2.40f,g), and prescribing values for \( P_{i,j} \) and \( \bar{\mathcal{K}} \) at \( \bar{\mathcal{K}} = \bar{\mathcal{K}}_0 \), we can determine the flow solution for \( \bar{\mathcal{K}} > \bar{\mathcal{K}}_0 \). Numerically, our task is much easier, since we have eliminated one variable \( \hat{\gamma} \) entirely and we do not therefore need to resort to the potentially difficult and computationally expensive two-variable finite-difference schemes. On the other hand, for non-zero starting-vortex flow the same Laplace transform scheme for (2.41a-d) would yield additional terms, generally triple integrals, on the right-hand side of (2.45), and in this case the finite-difference scheme would possibly be the better choice.

For all the ensuing weakly nonlinear analysis, we concentrate on the effect of zero-starting vortices. As mentioned in the Introduction, no computations have yet been performed for a full 3D boundary layer, i.e. \( \mathcal{W}_e = O(1) \). However for the case \( \mathcal{W}_e = O(\varepsilon^{6/5}) \), where crossflow has a direct effect on the pressure-amplitude equations, computational results have been obtained by the author and the full details of the numerical scheme involved as well as comparisons with certain analytical features are presented in Chapter 3.

2.7 - Effects of Nonparallelism

The global basic-flow velocity \( (\mathcal{U}_j, \mathcal{L}_e^{-1/2} \mathcal{U}, \mathcal{W}) \) has the expansion
\[ u = \bar{u}(\bar{y}) + (x - x_0) \bar{u}_b(\bar{y}) + \ldots \quad (2.46a) \]
\[ v = \bar{v}_b(\bar{y}) + \ldots \quad (2.46b) \]
\[ w = \bar{w}_b(\bar{y}) + (x - x_0) \bar{w}_b(\bar{y}) + \ldots \quad (2.46c) \]

as \( x \rightarrow x_0^+ \), in the main deck.

Nonparallelism becomes significant when the second terms in \( u, w \) and the first in \( v \), start to drive the vortex-induced waves over the longer scale \( \bar{x} \).

Hence, we seek the balance

\[ \Re^{-3/8} h^{-6/5} \sim h^{6/5}, \text{ i.e. } h \sim \Re^{-5/32}. \]

The subsequent alteration to the interaction equations is the insertion of additional terms \( (b_1 \lambda_b \bar{x}) \rho_1 \) and \( (b_2 \lambda_b \bar{x}) \rho_2 \) in (2.40f,g) respectively, where \( \lambda_b = \bar{u}_b'(0) \).

This may possibly lead to a substantial change in the development of the flow solution, especially if the nonlinear structure avoids breaking up after a finite distance downstream, but on the other hand nonparallelism may be relatively passive. Much depends on the wave angles and the input conditions and this will become clearer in later chapters.

We observe that for \( h \gg \Re^{-5/32} \), nonparallelism is negligible. Thus, with hindsight, the analysis preceding this section is seen to be valid in the regime \( \Re^{-5/32} \ll h \ll 1 \).
CHAPTER 3: WEAKLY NONLINEAR INTERACTIONS FOR $\mathcal{O}(\epsilon^{6/5})$

3.1 Analysis

Here we address the problem of weakly nonlinear vortex/TS interaction for small crossflow; specifically we are interested in the balance $\omega_{e} \sim \epsilon^{6/5}$ where crossflow directly influences the long-range stability of both waves.

Letting $\omega_{e} = \epsilon^{6/5} \mathcal{W}_{e}$ where $\mathcal{W}_{e}$ is $O(1)$, we wish to retain all the scalings and parameters laid out in Chapter 2, except that we expand $\beta_{\omega}$ as

$$\beta_{\omega} = \beta_{0\omega} + \epsilon^{6/5} \beta_{1\omega} + O(\epsilon^{12/5})$$

for $\omega = 1, 2$.

The equations governing the principal motion of the two TS waves are then (2.31a-f), except that $\beta_{0\omega}$ and $0$ replace $\beta_{\omega}$ and $\omega_{e}$ respectively. Hence, we have

$$(d_{1}/d_{2}) = \lambda^{1/2}(\lambda^{2} + \beta_{0\omega}^{2}/4)^{1/2},$$

as the neutral eigenrelations for each wave, in turn. Here (3.2a,b) submit equal and opposite roots for $\beta_{01}$, $\beta_{02}$, i.e. $\beta_{01} = -\beta_{02} = \beta_{0}$ say,

where $\beta_{0}$ is taken as non-negative without loss of generality. The neutral eigenrelations above are
precisely those obtained in Hall and Smith (1989) for zero crossflow, and this verifies the diminished crossflow influence on the main TS motion.

The principal vortex system in the buffer deck becomes

\[ \tau y - y \tau = -i \beta_0 W, \quad (3.3a) \]
\[ W y - y W = 0, \quad (3.3b) \]

with

\[ \tau (\infty, \infty) = W(\infty, \infty) = 0, \quad (3.3c) \]
\[ \tau_y (\infty, 0) = 0, \quad W_y (\infty, 0) = -i K_0 \beta_{11} \beta_{12}, \quad (3.3d) \]

where

\[ K_0 = \beta_0 (1 - \beta_0^2 / 4 \alpha^2). \quad (3.4) \]

Then for zero-starting vortex flow we deduce a similar integral form for \( \lambda_{33} \) as in Chapter 2; this time

\[ \lambda_{33}(\infty) = \frac{-\beta_0 K_0 A_i'(0)}{A_i(0) \Gamma(2/3)} \int_{\infty}^{\infty} \frac{\rho_{11}(u) \rho_{12}(u) (\infty - u)^{1/3}}{d u}. \quad (3.5) \]

Moving on to the secondary TS waves, we now find that they are directly influenced by both crossflow \( (W_e) \) and the spanwise perturbations \( (\beta_{11}) \), in addition to the driving effects of Chapter 2, i.e. the vortex shear \( (\lambda_{33}) \), the basic-flow modified shear \( (\lambda_1) \) and the slow streamwise modulation \( (\alpha \sim \partial / \partial x) \). The resulting wave-pressure equations are

\[ \frac{d P_{11}}{d x} + b_1 \lambda_1 P_{11} + d_1 \left( \frac{\beta_{11} d}{4 \alpha} + W_e \right) P_{11} + c_1 \lambda_{33} P_{12} = 0, \quad (3.6a) \]
\[ \frac{d P_{12}}{d x} + b_1 \lambda_1 P_{12} - d_1 \left( \frac{\beta_{12} d}{4 \alpha} + W_e \right) P_{12} - c_1 \lambda_{33}^* P_{11} = 0, \quad (3.6b) \]

where

\[ b_1 = b / \alpha, \quad c_1 = c / \alpha, \quad d_1 = d / \alpha \] with
\[ a = 2B_0 \gamma_0 \Omega 1/3 \Delta^{2/3} - i B_0^{-1/2} (4/3 + \beta_o^2/12 \lambda^2), \quad (3.7a) \]
\[ b = -2B_0 \gamma_0 \Omega B_0 D/13 \Delta^{5/3} - 5 B_0^{-1/2} / 3 \lambda, \quad (3.7b) \]
\[ c = i B_0 \Delta^{2/3} (2B_0 \gamma_0 / 3 + \beta_o^2 / 4) \]
\[ - \lambda^{-1} B_0^{-1/2} (5 \lambda^2 / 3 - \beta_o^2 / 3), \quad (3.7c) \]
\[ d = \beta_o B_0^{-1} / \lambda^2, \quad (3.7d) \]

and \( B_0 = (\lambda^2 + \beta_o^2 / 4). \) All other notation is as defined in Chapter 2. We note that the interaction coefficients \( a, b, c \) are equal in value to the corresponding terms in Hall and Smith (1989), whilst \( d \) is introduced by the spanwise perturbations and the crossflow.

A natural question here would be whether the analyses in Chapters 2 and 3 are directly connected, that is, whether we have a consistent solution match for \( We \ll 1 \) and \( We \gg 1 \) respectively. Clearly, since the size of \( \lambda \) is maintained in each chapter, and both \( \lambda_1 \) and \( \lambda_{33} \) have invariant orders of magnitude, also, we would require \( |\lambda_1| \sim |\lambda_{33}| \sim |\lambda| \sim 1 \) as \( We \to \infty. \) Then, from (3.6a,b), the only possible option is
\[ \beta_{1n} \sim -4B_0 We / \lambda^2 + o(1), \text{ as } We \to \infty, \quad (3.8a,b) \]
for \( n=1,2. \) This is consistent with the analysis in the regime \( h^{6/5} \ll \omega_0 \ll 1, \) as shown in Chapter 4.
If \( \beta_{in} + 4 \beta_0 \omega \omega \approx 0(1) \), then the possibilities for large \( \omega \) are plentiful according to its asymptotic behaviour. For the case where \( \beta_{in} + 4 \beta_0 \omega \omega \sim \omega \) as \( \omega \to \infty \), it follows that \( |d_x| \sim \omega \) and, to retain nonlinear interaction, \( \lambda_{33} \sim \omega \), using (3.6a,b). The latter balance suggests that \( \rho_{1} \sim \rho_{2} \sim \omega^{s/6} \) using (3.5), so that as \( \omega \to h^{-6/5} \) (restoring 0(1) crossflow), 
\( \rho_{1} \sim \rho_{2} \sim h^{-1} \), \( \lambda_{33} \sim h^{-6/5} \), \( \lambda \sim h^{-6/5} \) and we find that the interactions now become fully nonlinear, in the sense that the disturbance size is as large as the basic-flow shear in the lower deck. Aspects of this full nonlinear problem are considered later in Chapter 6.

Although the more relevant problem (in terms of its associations with Chapter 2) is where (3.8a,b) is observed, we instead consider the case where \( \beta_{in} \equiv 0 \) (for \( n = 1, 2 \)), and \( \lambda_{i}, \omega \) are non-negative, since it possesses many interesting nonlinear features. Hence, our interaction equations are

\[
\frac{d \rho_{11}}{d x} + (b_{i} \lambda_{i} + d_{i} \omega) \rho_{11} + c_{i} \lambda_{33} \rho_{2} = 0, \quad (3.9a)
\]
\[
\frac{d \rho_{2}}{d x} + (b_{i} \lambda_{i} - d_{i} \omega) \rho_{2} + c_{i} \lambda_{33}^{*} \rho_{11} = 0, \quad (3.9b)
\]
and

\[
\lambda_{33}(x) = - \frac{\beta_{0} k_{0} a_{i}(0)}{A_{i}(0) \Gamma(2/3)} \int \frac{x^{1/2}}{X_{x}} \rho_{11}(u) \rho_{2}^{*}(u)(X - u)^{-1/3} du, \quad (3.9c)
\]

where

\[
(d_{1}/d_{2}) = \lambda^{1/3} (\alpha^{2} + \beta^{2}/4)^{1/2}, \quad (3.10)
\]

We now devise a numerical scheme for (3.9a-c), enabling computational results to be obtained.
3.2 - Numerical Work

Rewriting the pressure equations in the form

\[
\frac{dP_1}{d\chi} = -(b, \lambda_1 + d, \Psi_\omega) P_{11} - c_1 \lambda_{33} P_{12}, \quad (3.11a)
\]
\[
\frac{dP_2}{d\chi} = -(b, \lambda_1 - d, \Psi_\omega) P_{12} - c_1 \lambda_{33}^* P_{11}, \quad (3.11b)
\]

we applied a predictor-corrector scheme of second order accuracy, to advance (3.11a,b) in distance as follows.

In the predictor step, provisional quantities

\[
P_{11p}^{(n+1)}, P_{12p}^{(n+1)}
\]

are calculated using the equations

\[
P_{11p}^{(n+1)} = P_{11}^{(n)} + (\Delta \chi) \left[-(b, \lambda_1 + d, \Psi_\omega) P_{11}^{(n)} - c_1 \lambda_{33}^{(n)} P_{12}^{(n)} \right], \quad (3.12a)
\]
\[
P_{12p}^{(n+1)} = P_{12}^{(n)} + (\Delta \chi) \left[-(b, \lambda_1 - d, \Psi_\omega) P_{12}^{(n)} - c_1 \lambda_{33}^{(n)} P_{11}^{(n)} \right], \quad (3.12b)
\]

given the solution \((P_{11}^{(n)}, P_{12}^{(n)}, \lambda_{33}^{(n)})\) at the \(n^{th}\) station, \(\chi_n = \chi_0 + n(\Delta \chi)\), where \(\chi_0\) is the starting point and \(\Delta \chi\) is some suitably small step-length. In the corrector step, we determine the quantities

\[
P_{11}^{(n+1)}, P_{12}^{(n+1)}
\]

by

\[
P_{11}^{(n+1)} = \left[ P_{11}^{(n)} + P_{11p}^{(n+1)} \right] / 2
\]
\[
+ (\Delta \chi) \left[ -(b, \lambda_1 + d, \Psi_\omega) P_{11p}^{(n+1)} - c_1 \lambda_{33}^{(n+1)} P_{12p}^{(n+1)} \right] / 2, \quad (3.13a)
\]
\[
P_{12}^{(n+1)} = \left[ P_{12}^{(n)} + P_{12p}^{(n+1)} \right] / 2
\]
\[
+ (\Delta \chi) \left[ -(b, \lambda_1 - d, \Psi_\omega) P_{12p}^{(n+1)} - c_1 \lambda_{33}^{(n+1)} P_{11p}^{(n+1)} \right] / 2, \quad (3.13b)
\]

where \(\lambda_{33}^{(n+1)}\) is obtained using a suitable numerical scheme on (3.9c). The scheme we use is a simple trapezoidal rule, and this is also used to determine
the 'corrected' \((n+1)^{th}\) iterate \(\lambda_{33}^{(n+1)}\). We have
\[
\lambda_{33}^{(n)}(\bar{x}) = \int_{\bar{x}_0}^{\bar{x}} p(u) f(\bar{x} - u) \, du,
\]  
where
\[
p(u) = p_i(u) p_i^*(u),
\]  
and
\[
f(\bar{x} - u) = \left[\frac{\beta_0 K_0 A_i'(0)}{A_i(0)[2/3]}\right](\bar{x} - u)^{-1/3}. \tag{3.16}
\]
So, \(\lambda_{33}^{(n)}(\bar{x}_n) = \int_{\bar{x}_0}^{\bar{x}_n} p(u) f(\bar{x}_n - u) \, du\). Now for \(n = 0\), we have \(\lambda_{33}^{(0)}(\bar{x}_0) = 0\), the starting value.

For \(n \geq 1\),
\[
\lambda_{33}^{(n)}(\bar{x}_n) = S_n + Q_n,
\]
where
\[
S_n = \begin{cases} 
\sum_{j=1}^{n-1} \int_{\bar{x}_j}^{\bar{x}_{j+1}} f(\bar{x}_n - u) \, du, & n \geq 2, \\
0, & n = 1;
\end{cases} \tag{3.17}
\]
and
\[
Q_n = \int_{\bar{x}_{n-1}}^{\bar{x}_n} p(u) f(\bar{x}_n - u) \, du. \tag{3.18}
\]

Let us firstly consider \(Q_n\): since \(0 \leq \bar{x}_n - u \leq \bar{x}_n - \bar{x}_{n-1} = \Delta \bar{x}\) and \(\Delta \bar{x}\) is sufficiently small, it is clear from (3.16) that \(f\) will be large over the range \([\bar{x}_{n-1}, \bar{x}_n]\) becoming infinite as \(u \rightarrow \bar{x}_n\). Thus, we approximate \(Q_n\) by
\[
Q_n \approx \frac{1}{2} \left[ p(\bar{x}_{n-1}) + p(\bar{x}_n) \right] \int_{\bar{x}_{n-1}}^{\bar{x}_n} f(\bar{x}_n - u) \, du, \text{ i.e.}
\]
\[
Q_n \approx \beta_0 K_0 \delta(\Delta \bar{x})^{2/3} \left[ p(\bar{x}_{n-1}) + p(\bar{x}_n) \right] / 2, \tag{3.19}
\]
where
\[
\delta = -A_i'(0)/[A_i(0)[2/3]] \approx 0.54. \tag{3.20}
\]

For \(S_n\), a trapezoidal rule is employed for \(n \geq 2\), and we deduce the approximate values
\[
S_2 \approx (\Delta \bar{x}) \left[ p(\bar{x}_0) f(\bar{x}_2 - \bar{x}_0) + p(\bar{x}_1) f(\bar{x}_2 - \bar{x}_1) \right] / 2, \tag{3.21a}
\]
\[
S_n \approx (\Delta \bar{x}) \left[ p(\bar{x}_0) f(\bar{x}_n - \bar{x}_0) + 2 \sum_{j=1}^{n-3} p(\bar{x}_j) f(\bar{x}_n - \bar{x}_j) \right.
\]
\[
+ p(\bar{x}_{n-1}) f(\bar{x}_n - \bar{x}_{n-1}) \big] / 2, \tag{3.21b}
\]
for \(n \geq 3\).
We conclude, therefore, that

\[
\lambda_{33r}^{(1)} = \delta(\Delta X)^{2/3} \rho_0 K_0 \left[ \frac{\rho_p^{(o)} + \rho_p^{(1)}}{2} \right]; \quad (3.22a)
\]

\[
\lambda_{33r}^{(2)} = \delta(\Delta X)^{2/3} \rho_0 K_0 \left[ 2^{-1/3} \rho_p^{(o)} + 2 \rho_p^{(1)} + \rho_p^{(2)} \right]/2; \quad (3.22b)
\]

\[
\lambda_{33r}^{(n+1)} = \delta(\Delta X)^{2/3} \rho_0 K_0 \left[ n^{-1/3} \rho_p^{(o)} + 2 \rho_p^{(n)} \right]/2, \quad (3.22c)
\]

for \( n \gg 3 \). Here

\[
\rho_p^{(n)} \equiv \rho_p^{(n)}(\rho_p^{(o)})^*. \quad (3.23)
\]

Similar equations hold for \( \lambda_{33r}^{(n)} \), the corrected iterative values, except that \( \rho_p^{(n)} \) is replaced by \( \rho^{(n)} \), where

\[
\rho^{(n)} \equiv \rho_{11}^{(n)}(\rho_{12}^{(n)})^*. \quad (3.24)
\]

The above procedure proved to be stable and accurate for suitably small step-lengths. Interaction results were obtained for sample values of \( \beta_0, \lambda, W_e \) and all starting at \( \delta v_0 = -1 \) upstream of the neutral TS point. The input value for each wave-pressure was fixed at 0.1. A selection of these results is presented at the end of this chapter, with explanations accompanying each figure in turn.

The next stage is to try to pin down various analytical criteria which decide the eventual nature of the nonlinear interactions. Comparisons with the numerical results will be made afterwards.
3.3 - Ultimate Flow Behaviour

There appear to be several options open to us concerning the ultimate behaviour of the nonlinear interactive flows; we now consider each of these in turn.

(A) Algebraic Singularity at a Finite Distance Downstream

Here we suppose that the flow solution breaks up at some finite position, say as \( x \to x_s \), where the proposed local response has the form

\[
\rho_{11} \sim \tilde{\gamma}^{-m_1} e^{i\Phi_1(\tilde{\gamma})}\tilde{P}_{11} + \ldots,
\]

\[
\rho_{12} \sim \tilde{\gamma}^{-m_2} e^{i\Phi_2(\tilde{\gamma})}\tilde{P}_{12} + \ldots, \tag{3.25a}
\]

and

\[
\tilde{\alpha}_{33} \sim \tilde{\gamma}^{-m_3} e^{i\Phi_3(\tilde{\gamma})}\tilde{\alpha}_{33} + \ldots, \tag{3.25c}
\]

where \( \tilde{\gamma} = (x_s - x) \) and \( 0 < \tilde{\gamma} \ll 1 \). Here \( m_1, m_2, m_3 \) (all positive) are, as yet, unknown, whilst \( \Phi_1, \Phi_2, \Phi_3 \) are real-valued \( \tilde{\gamma} \)-dependent functions, also unknown.

Now, the dominant balances in (3.9a,b) consist of modulation- and vortex-forcing, and requires \( m_1 = m_2 = m \) (say), \( m_3 = 1 \) and \( \Phi_3 = (\Phi_1 - \Phi_2) \). Equating real coefficients of \( e^{i\Phi_1} \) in (3.9a,b) respectively, yields at leading order

\[
m |\tilde{P}_{11}| + c_{1r} \tilde{\alpha}_{33} |\tilde{P}_{12}| = 0, \tag{3.26a}
\]

\[
m |\tilde{P}_{12}| + c_{1r} \tilde{\alpha}_{33} |\tilde{P}_{11}| = 0, \tag{3.26b}
\]

(where \( c_{1r} = P_2, [c_1] \)), from which it follows that

\[
|\tilde{P}_{11}| = |\tilde{P}_{12}|, \tag{3.27}
\]

and

\[
m = -\tilde{\alpha}_{33} c_{1r}. \tag{3.28}
\]
To avoid inconsistency among the imaginary components of \( e^{i\varphi_1}, e^{i\varphi_2} \) in (3.9a,b), we must expand \( \varphi_1, \varphi_2 \) as

\[
\begin{align*}
\varphi_1 &\sim \varphi_{10} \ln \gamma + \varphi_{11} + \varphi_{12} \gamma + \ldots, \\
\varphi_2 &\sim \varphi_{20} \ln \gamma + \varphi_{21} + \varphi_{22} \gamma + \ldots,
\end{align*}
\]

(3.29a, b)

where \( \varphi_{10}, \varphi_{11}, \ldots, \varphi_{20}, \varphi_{21}, \ldots \) are unknown constants. Then, at the first level we find

\[
\varphi_{10} = \varphi_{20} = -c_{ic} \tilde{\alpha}_{33},
\]

(3.30a, b)

where \( c_{ic} = \text{Im} \{ c_i \} \). Therefore,

\[
\varphi_3 = (\varphi_{11} - \varphi_{21}) + O(\gamma),
\]

(3.31)

from the above definition, so that (3.9c) gives the principal relation

\[
\tilde{\alpha}_{33} = \beta_0 K_0 \delta |\tilde{\alpha}_{11}| |\tilde{\alpha}_{22}|,
\]

(3.32)

(where \( \delta \) is given by (3.20)), and the evaluation

\[
m = 5/6.
\]

(3.33)

Finally, we combine (3.28), (3.32) and (3.33) to get

\[
5/6 = -\beta_0 \delta |\tilde{\alpha}_{11}| |\tilde{\alpha}_{22}| (K_0 c_{ir}),
\]

which implies the solvability condition

\[
K_0 c_{ir} < 0.
\]

(3.34)

This may be interpreted as a constraint on the positive angle \( \Theta \), made by one of the waves with the \( x \)-direction (the other wave angle being \( -\Theta \)), where

\[
\Theta \equiv \tan^{-1}(\beta_0/2\delta).
\]

(3.35)
Computations give the approximate range of validity as

$$32.2^\circ < \theta < 45^\circ.$$ (Blennerhassett and Smith (1991)).

Significantly, this option was found to occur in Hall and Smith (1989), for the 2D boundary-layer case, leading to the same compatibility relation (3.34). In retrospect, this is understandable, since the difference between the two theories in question, lies in the types of linear forcing present, i.e. the perturbed basic-flow shear and the crossflow here compared with the nonparallelism in Hall and Smith (1989), all of which become negligible within the breaking-up regime above. This is, however, the only option common to both theories.

(B) Exponential Wave-Decay at Downstream Infinity.

(i) $\rho_1 \neq 0$ and/or $W_2 \neq 0$.

Here we suppose that

$$\rho_{11} \sim e^{-S_1 \bar{X}} |\mathcal{P}_{11}| + \ldots,$$
$$\rho_{12} \sim e^{-S_2 \bar{X}} |\mathcal{P}_{12}| + \ldots,$$

as $\bar{X} \to \infty$, where $S_1, S_2$ are generally complex, with positive real parts. Then the skin-friction integral suggests that

$$\lambda_{33} \sim \bar{X}^{-1/3} e^{i\phi_3(\bar{X})} |\hat{\mathcal{X}}_{33}| + \ldots,$$

where the main contribution is the cumulative sum of
the integrand across the entire range of integration

\[ \int_{x_0}^{x} \] for \( x \gg 1 \). Here, \( \hat{\phi}_3 \) is real but unknown.

Hence, linear effects will dominate in the pressure-amplitude equations, producing the conditions

\[ -S_1 + (b_1 \lambda_1 + d_1 \omega_0) = 0, \] (3.38a)
\[ -S_2 + (b_1 \lambda_1 - d_1 \omega_0) = 0. \] (3.38b)

Taking real parts of these gives

\[ S_{1r} = b_{1r} \lambda_{1r} + d_{1r} \omega_0, \] (3.39a)
\[ S_{2r} = b_{1r} \lambda_{1r} - d_{1r} \omega_0, \] (3.39b)

where \( b_{1r} = R_\omega (b_1), \) \( d_{1r} = R_\omega (d_1). \) Calculations have shown that \( b_{1r} > 0 \) and \( d_{1r} < 0 \), for all wave angles, and thus (3.39a,b) are valid provided

\[ \lambda_1 > \left( -\frac{d_{1r}}{b_{1r}} \right) \omega_0. \] (3.40)

Critically, there are no restrictions on the wave angles, here.

(ii) \( \lambda_1 \omega_0 = 0 \).

In this case the pressure equations reduce to

\[ \frac{dP_1}{dx} + c_1 \lambda_{33} P_2 = 0, \] (3.41a)
\[ \frac{dP_2}{dx} + c_1 \lambda_{33}^* P_1 = 0, \] (3.41b)

which, upon combining with the initial conditions in \( \bar{X} \), and (3.9c), implies that \( P_{1i} = P_{2i} = P \) (say), and
that $\mathcal{A}_{33}$ is real. Furthermore, real parts of (3.41a,b) both give
\[
\frac{d}{d\kappa}|\rho| + c_{1r} \lambda_{33} |\rho| = 0,
\] (3.42)
while (3.9c) reveals that
\[
\lambda_{33} = \beta_0 K_0 \delta \int_{X_0}^{X} |\rho|^2 (X-u)^{-1/3} \, du.
\] (3.43)

Now we seek the balance
\[
|\rho| \sim e^{-s} X^m |\rho| + \ldots, \text{ for } X \gg 1,
\] (3.44)
where $s(>0)$ and $m$ are unknown constants. Substitution of (3.44) into (3.43), for large $X$, leads to
\[
\lambda_{33} \sim (\beta_0 K_0 \delta I) X^{-1/3} + \ldots, \quad (3.45)
\]
where $I$ is a positive, undetermined quantity related to the shear-integral. Therefore, a balance in (3.42) is achieved if
\[
m = 2/3, \quad (3.46)
\]
whereupon the principal equation is
\[
-\frac{2}{3} s + (\beta_0 \delta I) K_0 c_{1r} = 0. \quad (3.47)
\]
With $\beta_0 \delta I > 0$, it follows that
\[
K_0 c_{1r} > 0 \quad (3.48)
\]
must necessarily hold, for this option to work. This
condition is the opposite to that derived in (A) above. Clearly then
\[ \theta < 32.2^\circ \text{ or } \theta > 45^\circ \text{ is required.} \]

\[(C) \quad K_0 = 0.\]

In the special case of zero interaction, \(K_0 = 0\), the two waves have angles \(\pm 45^\circ\), so that they are mutually perpendicular. Evidently (3.9c) implies

\[ A_{33} \equiv 0, \quad \text{for} \quad X \gg X_0, \quad (3.49) \]

whereas (3.9a,b) imply

\[ |P_1| = |P_{11}(X_0)| e^{S_1r(X-X_0)}, \quad (3.50a) \]
\[ |P_2| = |P_{12}(X_0)| e^{S_2r(X-X_0)}, \quad (3.50b) \]

where \(S_{1r}, S_{2r}\) are defined in (3.39a,b) above. Four distinct cases are found to occur:

(i) \(0 \leq \alpha_1 < (-d_{1r}/b_{1r})W_0\): in this instance \(|P_{11}|\)
and \(|P_{12}|\) grow and decay exponentially, in turn;

(ii) \(\alpha_1 > (-d_{1r}/b_{1r})W_0\): both waves decay exponentially;

(iii) \(\alpha_1 = W_0 = \alpha_0\): \(|P_{11}| = |P_{11}(X_0)| = \text{const.}, |P_{12}| = |P_{22}(X_0)| = \text{const.};\)

(iv) \(\alpha_1 = (-d_{1r}/b_{1r})W_0\): \(|P_{11}| = |P_{11}(X_0)| = \text{const.},\)
and \(|P_{12}|\) decays exponentially.
This special case has corresponding \( \theta = \Theta_1 \approx 32.2^\circ \)

Again (3.50a,b) hold, but now (3.9c) implies that

\[
\lambda_{33} \sim (\beta_0 K_0 \delta \hat{I}) \chi^{-1/3} + \ldots, \tag{3.51}
\]

as \( \chi \to \infty \), where \( \hat{I} \) is an unknown positive constant.

Hence, the vortex shear will decay algebraically far downstream. Options (i)-(iv) again apply for the wave-pressures.

(E) **Cyclic Behaviour**

This option, relevant for \( K_0 \neq 0, C_{ir} \neq 0 \), is an interesting phenomenon occurring in the range

\[
0 \leq \alpha_1 < \left( C_{ir}/L_{ir} \right) W_e.
\]

At the input station we have zero vortex flow, suggesting that the dominant terms in the pressure equations near \( \chi^+ \) are linear. Hence, we may expect

\[
|P_{11}| \sim |P_{11}(\chi_0)| e^{S_{ir}(\chi - \chi_0)}, \tag{3.52a}
\]

\[
|P_{22}| \sim |P_{22}(\chi_0)| e^{S_{ir}(\chi - \chi_0)}, \tag{3.52b}
\]

for \( 0 < (\chi - \chi_0) < 1 \), where \( S_{ir} > 0 \) and \( S_{ir} < 0 \) are as in (3.39a,b). This means that \( |P_{11}| \) and \( |P_{22}| \) grow and decay, respectively; meanwhile in this local regime, the vortex shear has \( 0(1) \) oscillatory motion of period \( \pi/(d_{1r} W_e) \).
Outside this regime, the vortex shear couples with the strong wave \(|P_{11}|\) to revitalise \(|P_{12}|\), and this corresponds to (3.9b) becoming nonlinear in form. It is believed that \(|P_{22}|\) and \(|\lambda_{33}|\) will subsequently grow faster than \(|P_{11}|\) until eventually a stage is reached where nonlinearity enters (3.9a) and alters the hitherto linear growth of \(|P_{11}|\). (Although there is no general analytical proof available here to verify the above claims, the author has found that in the limiting case of large \(W_c\), the indicated growths of \(|P_{12}|\) and \(|\lambda_{33}|\) are doubly exponential in nature. In addition, there is strong circumstantial evidence (eg, figures 4, 5, 6 below) to ratify the proposed behaviour for \(O(1)\) values of \(\lambda_i\) and \(W_c\).) This stage corresponds to the completion of one 'cycle' after which there appear to be two possible outcomes:

\(i\) **Algebraic Finite-Distance Blow-Up**

The analysis follows in a similar vein to that for option (A) above, so that once again the necessary criterion for this behaviour is,

\[ \theta_i < \theta < 45^\circ, \text{ where } \theta_i \approx 32.2^\circ. \]

In fact, the blow-up may be delayed until several further cycles have been performed, depending on how strong the linear effects are.
(ii) **Continual Cyclic Behaviour**

This is the most likely option for wave angles outside the $[\theta_i, 45^\circ]$ range.

We now summarise these options in diagrammatical form, for the three cases:

- $\lambda_i = W_e = 0$, $0 \leq \lambda_i < \left( \frac{d_{ir}}{b_{ir}} \right) W_e$
- $\lambda_i > \left( \frac{d_{ir}}{b_{ir}} \right) W_e$. 


EXPONENTIAL WAVE-DECAY AT INFINITY.

ALGEBRAIC FINITE-DISTANCE BREAK UP.

EXPONENTIAL WAVE-DECAY AT INFINITY.

$\theta_1 (~32.2^\circ)$

A1: CONSTANT WAVE-PRESSURES, ZERO VORTEX-SHEAR.

A2: CONSTANT WAVE-PRESSURES, FAR-DOWNSTREAM
ALGEBRAIC DECAY FOR VORTEX-SHEAR.
81: Exponential growth (decay) for \( p_{11}(p_{22}) \),
zero vortex-shear.

82: Exponential growth (decay) for \( p_{11}(p_{22}) \),
far-downstream algebraic decay for vortex-shear.
C1: Exponential decay at infinity for wave-pressures, zero vortex-shear.

C2: Exponential decay at infinity for wave-pressures, far-downstream algebraic decay for vortex-shear.
3.4 - Comparing Analysis and Computations

Firstly, we consider the case where \( \lambda_1 = \omega_e = 0 \). Figures 1 and 2, involving angles outside the range \( \left[ \theta_1, 45^\circ \right] \) both exhibit features attributable to option (B)(ii) above, that is, fast (exponential) decay for each wave-pressure, but relatively slower (algebraic) decay for the vortex shear. Although not readily obvious from the graphs, further computations by the author over a greater range show that \( \lambda_{33} \) does indeed tend to zero in both examples. On the other hand, for \( \phi \in (\theta_1, 45^\circ) \) the predicted finite-distance break-up of option (A) above is seen to occur in figure 3. It is interesting that there is a long period of dormancy before the flow solution blows up in an 'explosive' manner; a similar observation was reported in Hall and Smith (1989), when they discussed certain numerical results which possessed the algebraic blow-up behaviour.

Secondly, for \( \lambda_1 \) in the interval \( \left[ 0, \left( -\alpha_r / b_{ir} \right) \omega_e \right] \) figure 4 (where \( \phi \notin \left[ \theta_1, 45^\circ \right] \)) reflects contrasting properties with figure 5 (where \( \phi \in (\theta_1, 45^\circ) \)). In the latter case, we observe an ultimately singular response, as anticipated in (E)(i) above; in the former case however, a curious sequence of cycles (as postulated
in (E)(ii) above) illustrate the flow solution. The first few cycles indicate rapid changes in the flow solution over a small region but further on downstream, the fluid motion becomes more benign with the cycles seeming to develop a more uniform pattern.

Finally, for \( \lambda_i > \left( \frac{\alpha_{1r}}{\beta_{1r}} \right) W_e \), the pressure-exponential-decay/vortex-shear-algebraic-decay option, (B)(i) above, is seen to take effect in figures 6, 7 and 8. On a cautious note, however, the alternative possibility of finite-distance break-up for \( \theta \in (\theta, 45^\circ) \) will almost certainly occur at some unknown critical value of \( \lambda_i \) (given \( \theta \) and \( W_e \)), when the linear presence is sufficiently subdued.

Thus, overall, the agreement between analysis and computations is felt to be considerable.
FIGURE CAPTIONS

Figure 1  Nonlinear-interaction computed results for 
\( \beta = 1 \), wave angle \( \theta = 28.95^\circ \), \( \Lambda_1 = 0 \), 
\( \Omega_e = 0 \). (a) TS pressure moduli \( |P_{11}, P_{12}| \) \( \Lambda x \), (b) Vortex shear modulus 
\( |\Lambda_{33}| \) \( \Lambda x \). Initial TS pressure moduli 
both \( 0.1 \), zero-initial vortex field.
Grid size \( \Delta \Lambda x = 0.05 \).

Figure 2  Nonlinear-interaction computed results for 
\( \beta = 1.6 \), wave angle \( \theta = 46.70^\circ \), \( \Lambda_1 = 0 \), 
\( \Omega_e = 0 \). (a) \( |P_{11}, P_{12}| \) \( \Lambda x \), 
(b) \( |\Lambda_{33}| \) \( \Lambda x \). Start 
\( |P_{11}, P_{12}| \) both \( 0.1 \), \( |\Lambda_{33}| = 0 \).
Grid \( \Delta \Lambda x = 0.05 \).

Figure 3  Computational solutions of the nonlinear 
interaction for \( \beta = 1.3 \), wave angle \( \theta = 
37.80^\circ \), \( \Lambda_1 = 0 \), \( \Omega_e = 0 \). (a) \( |P_{11}| \) \( \Lambda x \), 
\( |P_{12}| \) \( \Lambda x \), (b) \( |\Lambda_{33}| \) \( \Lambda x \). Start 
\( |P_{11}, P_{12}| \) both \( 0.1 \), \( |\Lambda_{33}| = 0 \).
Grid \( \Delta \Lambda x = 0.025 \).
Figure 4: Computed results of the nonlinear interaction for $\beta=1$, wave angle $\theta=28.95^\circ$, $\lambda_1=0$, $W_e=1$. (a) $|P_{11}|, |P_{12}|$ vs. $\bar{X}$, (b) $|P_{21}|, |P_{22}|$ vs. $\bar{X}$, (c) $|\lambda_{33}|$ vs. $\bar{X}$. Start $|P_{11}|, |P_{12}|$ both 0.1, $|\lambda_{33}|=0$. Grid $\Delta \bar{X}=0.05$.

Figure 5: Results for $\beta=1.3$, wave angle $\theta=37.80^\circ$, $\lambda_1=0$, $W_e=1$. (a) $|P_{11}|, |P_{12}|$ vs. $\bar{X}$, (b) $|\lambda_{33}|$ vs. $\bar{X}$. Start $|P_{11}|, |P_{12}|$ both 0.1, $|\lambda_{33}|=0$. Grid $\Delta \bar{X}=0.004$.

Figure 6: Results for $\beta=1$, wave angle $\theta=28.95^\circ$, $\lambda_1=1$, $W_e=0$. (a) $|P_{11}|, |P_{12}|$ vs. $\bar{X}$, (b) $|\lambda_{33}|$ vs. $\bar{X}$. Start $|P_{11}|, |P_{12}|$ both 0.1, $|\lambda_{33}|=0$. Grid $\Delta \bar{X}=0.005$.

Figure 7: Results for $\beta=1.6$, wave angle $\theta=46.70^\circ$, $\lambda_1=5$, $W_e=1$. (a) $|P_{11}|, |P_{12}|$ vs. $\bar{X}$, (b) $|\lambda_{33}|$ vs. $\bar{X}$. Start $|P_{11}|, |P_{12}|$ both 0.1, $|\lambda_{33}|=0$. Grid $\Delta \bar{X}=0.0025$.

Figure 8: Results for $\beta=1.3$, wave angle $\theta=37.80^\circ$, $\lambda_1=5$, $W_e=1$. (a) $|P_{11}|, |P_{12}|$ vs. $\bar{X}$, (b) $|\lambda_{33}|$ vs. $\bar{X}$. Start $|P_{11}|, |P_{12}|$ both 0.1, $|\lambda_{33}|=0$. Grid $\Delta \bar{X}=0.005$. 
CROSSFLOW ANALYSIS

In this chapter we attempt to uncover properties regarding 3D boundary-layer stability associated with neutral 3D TS waves of the type encountered in Chapters 2 and 3. The equation describing such neutral stability is

\[ \xi^{5/3} \left( \frac{d_1}{d_z} \right) = (\xi^2 + \beta^2/4)^{1/2} \left( \xi + \beta \xi_c / 2 \xi \right)^2 \]  \hspace{1cm} (4.1)

for waves proportional to \( \exp \left[ i (\alpha X + \beta Z / 2 - \nu \xi T) \right] \) where \( \xi, \beta, \nu \) are real, and \( \lambda_1 \approx 2.3, \lambda_2 \approx 2.3 \).

4.1 - Neutral Curves

It is possible to generate the wavenumber coordinates \( (\xi, \beta) \), for fixed \( \xi_c \), without having to resort to numerical analysis. The method used is the following. Dividing each side of (4.1) by \( \xi^3 \) gives

\[ \xi^{-4/3} \left( \frac{d_1}{d_z} \right) = (1 + \beta^2/4 \xi^2)^{1/2} \left( 1 + \beta \xi_c / 2 \xi \right)^2, \]

or

\[ \xi = (\xi_1 / \xi_2)^{3/4} \left( 1 + c^2 \right)^{-3/8} \left( 1 + c \xi_c \xi_2 \right)^{-3/2}, \]  \hspace{1cm} (4.2)

where

\[ c = \beta / 2 \xi. \]  \hspace{1cm} (4.3)

Geometrically, \( c \) represents the tangent of the angle that the TS wave makes with the streamwise direction.
By choosing any value of $c$, we use (4.2) to determine the associated value of $\lambda$ and then deduce $\beta$ from (4.3). Doing this continuously, for sufficiently close values of $c$ over a wide enough range, enables us to produce the corresponding $\lambda-\beta$ curve.

The following results are for $w_e = 0, 0.01, 0.1, 2$ and 5, in turn. The approximation $d_1/d_2 = 1$ has been used.
FIGURE CAPTIONS

**Figure 1**  Plot of the $L-\beta$ neutral curve for zero crossflow.

**Figure 2**  Plot of the $L-\beta$ neutral curve for $\omega_e = 0.01$.

**Figure 3**  Plot of the $L-\beta$ neutral curve for $\omega_e = 0.1$.

**Figure 4**  Plot of the $L-\beta$ neutral curve for $\omega_e = 2$.

**Figure 5**  Plot of the $L-\beta$ neutral curve for $\omega_e = 5$. 

$w_x = 2$

Fig. 4
All the above curves for \( \omega_c > 0 \) share some striking differences with the \( \omega_c = 0 \) curve. Firstly, we note that the latter curve is symmetric about the \( \lambda \)-axis and extends across the entire \( \beta \)-range. This is not so for \( \omega_c > 0 \) however, where the curves are asymmetric and have finite cut-off values for \( \beta \), say \( \beta_c (> 0) \), above which no neutral solutions exist. Also, as \( \omega_c \) increases from zero upwards, \( \beta_c \) decreases, tending to zero in the limit \( \omega_c \to \infty \). Secondly, for zero crossflow each \( \beta \) value has one and only one corresponding \( \lambda \) value, unlike the case of non-zero crossflow where two such values exist for \( \beta < \beta_c \), one for \( \beta = \beta_c \), and none for \( \beta > \beta_c \). Finally, there is a vast difference between the respective maximum \( \lambda \) values; when \( \omega_c = 0 \) this value is \( (a_1/d_2)^{3/4} (\sim 1) \) and occurs when \( \beta = 0 \), i.e., for 2D TS waves; for \( \omega_c > 0 \) though, it is removed, in effect, to \( \beta = -\infty \), and becomes infinite. Much of the remaining work in this chapter, and some of Chapter 5, will be devoted to the \( (-\beta) >> 1 \) limit, since it points to potentially radical developments in 3D boundary-layer theory. Further distinctions between the graphical results occur amongst the \( \omega_c > 0 \) curves. Upon close inspection, we see that for \( \omega_c = 0.01 \) and \( \omega_c = 0.1 \), there exists an \( \lambda \)-interval in which, for each \( \lambda \), there are four corresponding \( \beta \) values. No such interval is found for \( \omega_c = 2 \) and \( \omega_c = 5 \) where, by comparison, the curves are 'tilted' well away from the
negative $\beta$-axis. This strongly suggests that there is some critical value of $\omega_e$, say $\omega_{ec}$, above which the said interval vanishes, and in the next section this is calculated by analytical means. The result is very significant when applied to the weakly nonlinear analysis of Chapter 2, since it shows that six pairs of interactions can possibly be set up, for fixed $\lambda$ in the critical interval, given $0 < \omega_e < \omega_{ec}$. (This problem shall be addressed for asymptotically small value of $\omega_e$ in Section 4.3.) Furthermore, it allows us to speculate on weakly nonlinear interactions involving three or four waves, each with common values of $\lambda$ and $\beta$. We briefly summarise this at the end of the chapter.

4.2 - Critical Crossflow

Consider the stability diagram for non-zero crossflow, and suppose that the 'four-mode' property holds:

![Stability diagram with four modes and critical crossflow range](image-url)
In the diagram we see that having the indicated range necessitates the existence of one maximum and one minimum stationary point on the upper branch. So let us now differentiate (4.1) implicitly with respect to $\beta$.

We find that

$$\frac{5}{3} \left( \frac{d}{d\epsilon} \right) \epsilon^{2/3} \left( d\omega / d\beta \right) = (\epsilon^2 + \beta^2/4)^{-1/2} \left( \alpha + \beta \omega_e/2 \right) \left[ \frac{1}{2} \left( \epsilon^2 + \beta^2/4 \right) \left( d\epsilon / d\beta \right) \right] + \omega_e/2 \right) + \frac{1}{2} \left( \alpha + \beta \omega_e/2 \right) \left( 2d\epsilon / d\beta + \beta / 2 \right) ] .$$

To locate stationary values, we set $d\epsilon / d\beta = 0$, so that

$$(\epsilon^2 + \beta^2/4) \omega_e + \beta \left( \alpha + \beta \omega_e/2 \right) / 4 = 0 , \text{ (ignoring the (0,0) stationary value).}$$

Hence, 

$$\left( \frac{3}{8} \omega_e \right) \beta^2 + \left( \frac{1}{4} \alpha \right) \beta + \epsilon^2 \omega_e = 0 , \quad (4.4)$$

which has roots

$$\beta = \frac{- \alpha/4 \pm \sqrt{\alpha^2/16 - 3\omega_e^2 \epsilon^2/2}}{(3\omega_e/4)} , \quad (4.5a,b)$$

using the quadratic formula. To guarantee that $\beta$ is real, and hence the existence of upper branch stationary values, we require $\alpha^2/16 - 3\omega_e^2 \epsilon^2/2 \geq 0$, i.e. $\omega_e \leq \frac{1}{\sqrt{24}}$. Therefore, our critical value of $\omega_e$ is

$$\omega_e^c = \frac{1}{\sqrt{24}} . \quad (4.6)$$

We observe, in passing, that if $\omega_e < \omega_e^c$, then two distinct stationary values on the upper branch are present (given by $S_1$ and $S_2$ on the above diagram) so that as $\omega_e \to \omega_e^c^-$, these coordinates coalesce at $(\epsilon^*, \beta^*)$. 
to form a point of inflexion, where \( \lambda^* = \left( \frac{36\sqrt{3}}{25\sqrt{5}} \frac{(d_1)}{(d_2)} \right)^{3/4} \)

\[ \text{and} \quad \beta^* = \frac{-\sqrt{24}}{3} \left[ \frac{36\sqrt{3}}{25\sqrt{5}} \frac{(d_1)}{(d_2)} \right]^{3/4} \]

Before considering small crossflow analysis, we briefly summarise the downstream options in the general \( \omega_* = O(1) \) context. These are no different from those laid down in Chapter 3 for \( \omega_* = O(h^{6/5}) \) apart from slight modifications to the solvability conditions in each option.

4.3 - Ultimate Flow Behaviour

(A) Algebraic Singularity at a Finite Point Downstream

The criteria necessary here are

\[ (\beta_1 - \beta_2)Kc_{1r} < 0 \quad \text{and} \quad (\beta_1 - \beta_2)Kc_{2r} < 0. \quad (4.7a, b) \]

If, in addition, \( b_{1r} > 0 \) and \( b_{2r} < 0 \) (or vice versa), then the singularity is delayed until after a non-zero number of cycles.

(B) Exponential Decay at Infinity

The required conditions are

\[ b_{1r} > 0 \quad \text{and} \quad b_{2r} > 0. \quad (4.8a, b) \]

However, we note that this condition is not sufficient, and that option (A) is still a possibility if (4.7a, b)
also holds. In this case, the outcome will generally depend on the value of $\beta_1$, (not determined analytically), given $\beta_1$ and $\beta_2$.

(C) Continuous Cycles

This option seems most likely to occur when $b_{1r} > 0$ and $b_{2r} < 0$ (or vice versa), and (4.7a,b) is not satisfied.

4.4 - Small Crossflow

Many limiting features can be derived from (4.1), which are of direct relevance to linear and weakly nonlinear analysis, and useful in suggesting fully nonlinear scales and structures. For the remainder of the chapter, we will consider the effects of small crossflow in the regime $h^{6/5} \ll \omega_e \ll 1$. Other limiting analyses will be addressed in Chapter 5.

So, here the 'four-mode' criterion, $\omega_e < \omega_{ec}$, will be satisfied and the neutral curve is illustrated below.
We will focus our attention on nine specific areas $(\mathbf{A} - \mathbf{I})$ on the curve which typify regions of interest to us. Firstly, $\mathbf{C}$ corresponds to the position on the negative upper branch where the small crossflow effects become highly significant, causing a deviation from the $\omega_e=0$ curve. The equivalent spanwise location on the lower branch is denoted by $\mathbf{D}$. For positive $\beta$, $\mathbf{I}$ represents the counterpart to $\mathbf{C}$. Next, $\mathbf{A}$ and $\mathbf{B}$ mark the points on the upper and lower branches, in turn, where crossflow effects are prominent and $\lambda$ is $O(1)$. Lastly, regions $\mathbf{E} - \mathbf{H}$ are where $\beta = O(1)$. Notably, the path joining $\mathbf{C}$ to $\mathbf{I}$ on the upper branch is arbitrarily close to the $\omega_e=0$ curve, and this is in marked contrast to the path linking $\mathbf{D}$ and $\mathbf{I}$ via the lower branch, which is non-vanishing as $\omega_e \to 0$. (This branch would ultimately 'disappear' though, once crossflow started to balance with inverse powers of the Reynolds number).

\[ (A) \quad (-\beta) \sim \omega_e^{-1}. \]

In the regime pertaining to $\mathbf{A}$ and $\mathbf{B}$, we seek the balance $\lambda \sim 1$ and expect significant crossflow, i.e. $\beta \omega_e/2 \neq o(\lambda)$ using (4.1). Thus, $(-\beta) \gg \lambda$ ($= O(1)$). Suppose that $\lambda + \beta \omega_e/2 = O(1); \omega_e)$; then for consistency in (4.1) we would require $|\beta| \omega_e \sim |\beta|^{-1/2} \ll 1$, so that $\beta \omega_e \ll \lambda$ contradicting the hypothesis. Hence,
our only option is to have \( \lambda + \beta \omega_e / 2 = 0(\epsilon) \), say, where \( \epsilon = o(1/\omega_e) \). Thus \( (-\beta) \sim 2\lambda \omega_e^{-1} \) and, via (4.1), \( \epsilon \sim \omega_e^{1/2} \). Therefore, summarising we have

\[
(-\beta) = (2\lambda)\omega_e^{-1} + \tilde{\beta}_1 \omega_e^{-1/2} + O(1). \tag{4.9}
\]

The constant \( \tilde{\beta}_1 \) is determined by inserting (4.5) into (4.1), and equating the second order balances. Two solutions are found, namely

\[
\tilde{\beta}_n = \pm 2(d_1/d_z)^{1/2} \lambda^{1/3}, \tag{4.10a,b}
\]

for \( n = 1,2 \), and these, we conclude, relate to the upper branch and lower branch, respectively. We observe that (4.10a,b) denote the principal interactive relationships connecting the upper and lower decks. The primary balance \( \beta \sim (-2\lambda)\omega_e^{-1} \) is merely an upper deck consistency law, independent of lower deck behaviour.

We are now in a position to consider related weakly nonlinear analysis of the type described in Chapter 2. It is found that, provided \( 0 < \lambda < (d_1/d_z)^{3/4} \), two further modes on the upper branch exist and these are labelled, along with the crossflow modes, in the following diagram:

![Diagram](image-url)
The modes $\beta_1$ and $\beta_2$ are effectively those of Chapter 3, at leading order; they lie within the $|\beta| \ll \omega_e^{-1/4}$ regime on the upper branch, (which uniformly converges to the $\omega_e = 0$ curve, as $\omega_e \to 0$), and have the expansions

$$\beta_1 = \beta_0 + \beta_{11} \omega_e + O(\omega_e^2), \quad (4.11a)$$

$$\beta_2 = -\beta_0 + \beta_{12} \omega_e + O(\omega_e^2), \quad (4.11b)$$

where $\beta_0 (> 0)$ satisfies

$$\alpha^{1/3}(d_1/d_2) = (\alpha^2 + \beta_0^2/4)^{1/2}, \quad (4.12)$$

and

$$\beta_{1n} = -4(\alpha^2 + \beta_0^2/4)/\alpha, \quad (4.13a,b)$$

for $n = 1, 2$.

In the limit $\omega_e \to O(\hbar^{1/5})$, matching with the $\beta$-expansion in Chapter 3 is obtained.

The other two modes have been considered above, and are known to have the form

$$\beta_3 = (-2\omega)\omega_e^{-1} + \beta_{13} \omega_e^{-1/2} + O(1), \quad (4.14a)$$

$$\beta_4 = (-2\omega)\omega_e^{-1} + \beta_{14} \omega_e^{-1/2} + O(1), \quad (4.14b)$$

where

$$\beta_{3n} = \pm 2(d_1/d_2)^{1/2} \alpha^{1/3}, \quad (4.15a,b)$$

for $n = 3, 4$.

We now review each of the six possible pairs of TS waves, in turn.

(i) $\beta_1$ and $\beta_2$.

The analysis describing this theory is essentially covered by Section 3.1 and we find that the interaction
equations are given exactly by (3.9a-c) above. Certain linear asymptotic properties are addressed in Section 4.4(D) below.

(ii) $\beta_1$ and $\beta_3$

The disparity in the modal sizes inevitably leads to different orders of magnitude for each wave, and compressions or enhancements of the forcing terms $\lambda_{33}$, $\partial / \partial x'$ and $\lambda_1$. This is gauged to some extent by the new forms for the interaction coefficients. One very interesting feature here is the absence of streamwise modulation, in all but the upper deck, on the wave having mode $\beta_3$.

In the lower deck, we find that the 'crossflow' wave (of mode $\beta_3$) has pressure of $O(\omega^{-7/4})$ relative to the 'regular' wave (of mode $\beta_1$), i.e. smaller, although its comparative flow speed of $O(\omega^{-1/4})$ is larger. The streamwise and normal coordinates for the vortex, in the buffer deck, are modified by amounts $O(\omega^{q/10}), O(\omega^{3/10})$ respectively, in order to preserve vortex-wave interaction. In the upper deck, the characteristic normal scale is reduced by a factor of $\omega_\infty$ in order to retain a consistent balance in the Laplacian operator, which is typically applied to each wave-pressure there.
The interaction equations are found to be

\[ \hat{\zeta} \frac{d}{d\hat{\eta}} \hat{\zeta} = -i \Delta \hat{\zeta}, \quad (4.16a) \]

subject to

\[ \hat{\zeta}(\hat{\eta}, \infty) = 0, \quad \hat{\zeta}(\hat{\eta}, 0) = 0, \quad (4.16c) \]

where

\[ \hat{K}_0 = -\beta_0/2, \quad (4.16e) \]

and

\[ \frac{d\hat{P}_{11}}{d\hat{\eta}} + \hat{b}_1 \hat{P}_{11} + \hat{c}_1 \hat{P}_{33} \hat{P}_2 = 0, \quad (4.16f) \]

\[ \frac{d\hat{P}_2}{d\hat{\eta}} + \hat{b}_2 \hat{P}_{12} + \hat{c}_2 \hat{P}_{33} \hat{P}_2 = 0, \quad (4.16g) \]

where

\[ \left[ \hat{X}, \hat{\eta}, \hat{\zeta}, \hat{\lambda}_{33}, \hat{P}_{11}, \hat{P}_{12}, \hat{A}_1 \right] \]

\[ = \left[ \omega e^{i\theta_0 / 2}, \omega e^{i\theta_0 / 3}, \omega e^{i\theta_0 / 4}, \omega e^{12/10}, \omega e^{-12/10}, \hat{A}_{33}, \hat{\lambda}_{11}, \omega e^{7/12} \hat{P}_2, \omega e^{-9/10} \hat{A}_1 \right]^+ \ldots \quad (4.17a-h) \]

Here \( \hat{b}_n = b_n/a_n \), \( \hat{c}_n = c_n/a_n \), \quad (4.18a-d)

for \( n = 1, 2 \), where

\[ a_1 = 2 \beta_0 \theta_0 \Gamma_i \Delta^{2/13} / \left[ 3 \delta \Delta^{2/13} - i \beta_0^{-1/2} \left( 4 + \beta_0^2 / 12 \Delta^2 \right) \right], \quad (4.19a) \]

\[ b_1 = -2 \beta_0 \theta_0 \Gamma_i \Delta / 3 \Delta^{2/13} - 5 \beta_0^{1/2} / 3 \Delta, \quad (4.19b) \]

\[ c_1 = i \Delta \theta_0 \Delta^{-2/13} \left( 2 \beta_0^{1/2} \Gamma_i \Delta^{7/3} / \left[ 3 \left( d_i/d_x \right) \right] + \Delta^2 / 2 \right) \]

\[ - \Delta^{-1} \beta_0^{-1/2} \left( 5 \beta_0^{1/2} \Delta^{7/3} / \left[ 3 \left( d_i/d_x \right) \right] - 3 \Delta^2 / 2 \right) \] \quad (4.19c)

\[ a_2 = -2 i / \left[ \Delta^{13/3} \left( d_i/d_x \right)^{1/2} \right] \] \quad (4.19d)

\[ b_2 = -2 \Delta \theta_0 \Gamma_i \Delta \left( d_i/d_x \right) \Delta^{2/3} / 3 \Delta^{2/3} - 5 / 3 \] \quad (4.19e)

\[ c_2 = \Gamma_i \Delta^{-2/13} \beta_0 \Gamma_i \left( d_i/d_x \right) / 4 \Delta^{13/3} + 3 \beta_0 / 4 \Delta \] \quad (4.19f)

are the new interaction coefficients, and \( \beta_0 \approx (\Delta^2 + \beta^2 / 14) \)

with all other notation as defined in Chapter 2 above.
The coefficients $a_1, b_1$ are simply those occurring for zero crossflow (c.f. Hall and Smith (1989), Chapter 3 above). This is understandable because the crossflow mode, $\beta_2$, has no influence on these linear terms. It does however affect $c$ (the nonlinear feature), as expected, but does not offset the distribution of its related forcing throughout the triple-deck structure, which remains in equal proportion. The streamwise modulation driving $\hat{A}_2$ is found to be concentrated in the upper deck only; this is verified by the simple form of $a_2$, which has no lower-deck related terms such as $\hat{A}_0, K$ and $A_i'$. The coefficients $b_2$ and $a_2$ also have a conveniently easy form. For zero-incident vortices, (4.16a-e) reduce to

$$\hat{A}_{23} = \frac{Z \beta \hat{A}_i'(0)}{2 A_i(0) \Gamma(2/3)} \int_{\hat{f}_0}^{\hat{f}_2} \hat{P}_i(u) \hat{P}_2(u)(\hat{f}_2 - u)^{1/3} du, \quad (4.20)$$

upon application of the Laplace transform in $\hat{f}_2$, where $\hat{f}_0$ is the upstream starting value.

The ultimate behaviour of the flow solution is governed by the options outlined in Section 4.3 above, where we note that

$$[(\beta_1 - \beta_2), K, b_{1r}, b_{2r}, c_{1r}, c_{2r}] \rightarrow [2 \omega, \hat{K}_0, \hat{b}_{1r}, \hat{b}_{2r}, \hat{c}_{1r}, \hat{c}_{2r}] \text{ in effect here.}$$

To determine the quantities $\hat{b}_{1r}, \hat{b}_{2r}, \hat{c}_{1r}, \hat{c}_{2r}$ we must utilise the neutral values appropriate to the triple-deck theory, i.e. $A_i'(\hat{f}_0)/K = (d_1/\beta_2) i^{1/3}, \hat{f}_0 = -i^{1/3} d_1$,.
where \( d, \approx 2.3, d, \approx 2.3 \). Furthermore, Professor F.T. Smith (personal communication 1991) supplied the approximate values

\[
\begin{align*}
  i^{-1/3} r_1 &= (-0.41, -0.55), \\
  i^{-1/3} r_2 &= (1.18, 0.88),
\end{align*}
\]

at the neutral TS point. Calculations then reveal that

\[
\begin{align*}
  \hat{b}_{1r} &= - (3 + \beta_0^2 / 2 d_2^2) 2 B_0^{1/2} d_1 (d_2 - 1) (i^{-1/3} r_1)_r / \left( \beta d^{2/3} |a_1|^2 \right), \\
  \hat{b}_{2r} &= - \alpha^{1/3} d_1 (d_2 - 1) (d_1 / d_2)^{3/2} (i^{-1/3} r_1)_r / 3,
\end{align*}
\]

where \((i^{-1/3} r_1)_r = \text{Re.} \left[ (i^{-1/3} r_1) \right]\). Hence, \(\hat{b}_{1r}\) and \(\hat{b}_{2r}\) are both positive, irrespective of \(\beta_0\). This, therefore, narrows the range of possibilities to just two: algebraic finite-distance break-up or exponential decay far downstream. The extra conditions for the first option are

\[
2 \mathcal{K}_0 \hat{c}_{1r} < 0 \quad \text{and} \quad 2 \mathcal{K}_0 \hat{c}_{2r} < 0,
\]

or \(\hat{c}_{1r} > 0, \hat{c}_{2r} > 0\), since \(\hat{K}_0\) is negative. Further analysis gives

\[
\hat{c}_{2r} = - d_1 (d_2 - 1) (d_1 / d_2)^{3/2} \beta_0 (i^{-1/3} r_2)_r / \left[ \beta d^{2/3} \right],
\]

where \((i^{-1/3} r_2)_r = \text{Re.} \left[ (i^{-1/3} r_2) \right]\).

Thus, \(\hat{c}_{2r} < 0\) for all values of \(\beta_0\) and, consequently, the waves will decay exponentially as \(\bar{x} \to \infty\).
(iii) $\beta_2$ and $\beta_3$.

The basic alterations to the flow structure are exactly the same as those described in (ii) above, only now our interaction equations have the form

\begin{align*}
\tilde{\zeta}_\tilde{\eta} - \tilde{\omega} \tilde{\zeta} &= -i \lambda \tilde{\omega}, \\
\tilde{\omega}_\tilde{\eta} - \tilde{\omega} \tilde{\zeta} &= 0, \\
\text{subject to} \\
\tilde{\zeta}(\tilde{\eta}, \infty) &= 0, \quad \tilde{\omega}(\tilde{\eta}, \infty) = 0, \\
\tilde{\zeta}_\tilde{\eta}(\tilde{\eta}, 0) &= 0, \quad \tilde{\omega}(\tilde{\eta}, 0) = -i \tilde{\omega}_0 \tilde{\eta} \tilde{\eta}^*,
\end{align*}

where $\tilde{\omega}_0 = \beta_0/2$, and

\begin{align*}
\frac{d\tilde{\eta}_1}{d\tilde{\eta}} + \tilde{b}_1 \tilde{\eta}_1 \tilde{\eta}_1 + \tilde{C}_1 \tilde{\eta}_3 \tilde{\eta}_3^* &= 0, \\
\frac{d\tilde{\eta}_2}{d\tilde{\eta}} + \tilde{b}_2 \tilde{\eta}_2 \tilde{\eta}_2 + \tilde{C}_2 \tilde{\eta}_3 \tilde{\eta}_3^* &= 0.
\end{align*}

The variables are scaled in a similar fashion to (4.17a-h), and

\begin{align*}
\tilde{\beta}_1 &= \hat{\beta}_1, \quad \tilde{\zeta}_1 = \hat{\zeta}_1, \quad \tilde{\beta}_2 = \hat{\beta}_2, \quad \tilde{\zeta}_2 = - \hat{\zeta}_2. 
\end{align*}

Simplifying (4.21a-e) leads to

\begin{align*}
\tilde{\eta}_3 &= -\frac{2 \beta_0 \text{Ai}'(0)}{2 \text{Ai}(0) \text{Ai}'(2/3)} \int_{\tilde{\gamma}_0}^{\tilde{\gamma}_0} \tilde{\eta}_1 (\tilde{\eta}_2 (\tilde{\eta}_2 - u)^{-1/3} du, \\
\text{where } \tilde{\gamma} &= \tilde{\eta}_0 \text{ is the input coordinate.}
\end{align*}

Using (4.22a-d) and analytical results above, we have

\begin{align*}
\tilde{b}_1 > 0, \quad \tilde{b}_2 > 0 \quad \text{and} \quad \tilde{C}_2 > 0.
\end{align*}
Then, reviewing the downstream options of Section 4.3, with 
\[ (\beta_1, \beta_2, K_J, L_1, L_2, C_1, C_2) \rightarrow (2J, K_0, L_1, L_2, C_1, C_2) \]
we again deduce that the waves must die out, in an exponential manner, far downstream.

(iv) \( \beta_1 \) and \( \beta_2 \).

In this case, the structural modifications of (ii) and (iii) apply and, as in both theories, the perturbation terms in the crossflow mode play a significant role in the interaction equations, which are

\[
\begin{align*}
\bar{c}(\eta, \infty) &= 0, \quad \bar{w}(\eta, \infty) = 0, \\
\bar{c}(\eta, 0) &= 0, \quad \bar{w}(\eta, 0) = -iK_0 \bar{P}_{11} \bar{P}_{12}^*,
\end{align*}
\]  
(4.24c)  
(4.24d)

where

\[
K_0 = -\beta_0/2
\]  
(4.24e)

and

\[
\begin{align*}
d\bar{P}_{11}/d\eta + \bar{b}_1 \bar{P}_{11} + \bar{c}_1 \bar{P}_{33} \bar{P}_{12} &= 0, \\
d\bar{P}_{12}/d\eta + \bar{b}_2 \bar{P}_{12} + \bar{c}_2 \bar{P}_{33} \bar{P}_{11} &= 0.
\end{align*}
\]  
(4.24f)  
(4.24g)

The scaled variables are essentially defined in (4.17a-h) above, with bars ( - ) replacing hats ( ^ ), and

\[
\bar{b}_1 = \hat{b}_1, \quad \bar{c}_1 = \hat{c}_1, \quad \bar{b}_2 = -\hat{b}_2, \quad \bar{c}_2 = -\hat{c}_2.
\]  
(4.25a-d)
Combining (4.25a-e) yields

\[ \overline{\mathbf{p}}_{23} = \frac{2 \beta_0 A_1'(0)}{2A_1(0) \Gamma(2/3)} \int_{\overline{\mathbf{p}}}^{\overline{\mathbf{p}}} \overline{\mathbf{p}}_1(\mathbf{u}) \overline{\mathbf{p}}_2(\mathbf{u})(\overline{\mathbf{p}}_i - \mathbf{u})^{-1/3} \, d\mathbf{u}, \]  

(4.26)

given \((\overline{\mathbf{p}}, \overline{\mathbf{e}}) = (0, 0)\) at \(\overline{\mathbf{p}} = \overline{\mathbf{p}}_0\).

Now, for the first time, the linear coefficients associated with \(\lambda\), have different signs, i.e. \(\overline{e}_{1r} > 0\), \(\overline{e}_{2r} < 0\), so that the cyclic option summed up in Section 4.3 is brought into play. Moreover, \(\overline{c}_{2r}\) is positive and \(\overline{c}_0\) is negative, whilst \((\beta_1 - \beta_2) / (2d) \omega e^{-1}\), so that (4.7b) will hold. Unfortunately, there appear to be no analytical means of determining \(\overline{c}_{1r}\) and, therefore, no definite claims regarding the exact nature of the flow downstream, except for the initial cycle.

(v) \(\beta_2\) and \(\beta_4\).

Again, the adaptations of (ii)-(iv) above, regarding the flow set-up, are found to be relevant, and the nonlinear equations become

\[ \frac{\dot{\mathbf{c}}_i}{\dot{\mathbf{c}}_j} - \overline{c}_i \overline{c}_j = -i \alpha \mathbf{w}, \quad (4.27a) \]

\[ \mathbf{w}_{\dot{\mathbf{c}}_{2r}} - \overline{c}_2 \mathbf{w}_{\dot{\mathbf{c}}_2} = 0, \quad (4.27b) \]

with

\[ \overline{c}_2(\overline{\mathbf{c}}_2, \infty) = 0, \overline{c}_2(\overline{\mathbf{c}}_2, 0) = 0, \quad (4.27c) \]

\[ \overline{c}_2(\overline{\mathbf{c}}_2, \infty) = 0, \overline{c}_2(\overline{\mathbf{c}}_2, 0) = -i \alpha_0 \overline{c}_{11} \overline{c}_{12}, \quad (4.27d) \]

where

\[ \kappa_0 = \beta_0 / 2, \quad (4.27e) \]

and

\[ \frac{d\dot{\mathbf{p}}_{11}}{d\gamma} + \dot{\mathbf{c}}_1 \dot{\mathbf{p}}_{11} + \mathbf{c}_1 \mathbf{p}_{33} \dot{\mathbf{p}}_{22} = 0, \quad (4.27f) \]

\[ \frac{d\dot{\mathbf{p}}_{12}}{d\gamma} + \dot{\mathbf{c}}_2 \dot{\mathbf{p}}_{12} + \mathbf{c}_2 \mathbf{p}_{33} \dot{\mathbf{p}}_{11} = 0. \quad (4.27g) \]
Scaled variables are as in \((4.17a-h)\), with \(\hat{v}\) replacing \(A\), and
\[
\begin{align*}
\hat{b}_1 &= \hat{b}_1, & \hat{c}_1 &= \hat{c}_1, & \hat{b}_2 &= -\hat{b}_2, & \hat{c}_2 &= \hat{c}_2. \\
(4.28a-d)
\end{align*}
\]

The vortex-shear integral is
\[
\frac{\hat{A}_{33}}{2A_{0}(0)} = \left[ \int_{0}^{\gamma} \hat{p}_1(u) \hat{p}_2(u) (\gamma - u)^{-1/3} \, du \right]^{\gamma}_{\gamma_{0}}.
(4.29)
\]

We see that \(\hat{b}_{1r} > 0\) and \(\hat{b}_{2r} < 0\) in a similar fashion to the counterpart values \(\overline{b}_{1r}, \overline{b}_{2r}\) of (iv) above. Also, the second condition necessary for eventual finite-distance break-up, \((2\lambda \hat{c}_{r} \hat{c}_{zr} < 0)\), is achieved for all \(\beta_{0}\) values, but lack of information regarding \(\hat{c}_{r}\) means that corroboration of the first condition \((2\lambda \hat{c}_{r} \hat{c}_{zr} < 0)\), is not possible analytically. So, again, only the limited conclusion of initial cyclic behaviour can be made.

(vi) \(\beta_{3}\) and \(\beta_{4}\).

This is possibly the most interesting of the six pairs of modes. One subtle feature arising from this particular interaction is the relative slowness of the vortex-spanwise variations compared to those for the waves, i.e. we have, in effect, \(|\partial/\partial z| \sim (\beta_{3}-\beta_{4}) = o(\beta_{n})\) \((n = 1, 2)\) for the vortex, using the mode expansions in \((4.14a, b)\). The orders of magnitude for each wave-pressure are taken as \(O(1)\) and this induces flow speed of \(O(\omega e^{-2})\) for both waves in the lower deck. In the
buffer deck, the streamwise and normal coordinates 
change by amounts $O(\omega e^{3/2})$, $O(\omega e^{1/2})$ respectively, and 
these are significantly smaller than the alterations 
made in (ii)-(v) above. Also here, the vortex-spanwise 
velocity is relatively strong compared to its streamwise 
and normal counterparts, and this is due to the 
enhancement of $|\partial/\partial z|^{-1}$. The influence of streamwise 
modulation on both waves is confined to the upper deck, 
where the transverse waves coordinate is rescaled by a factor $\omega e$.

The nonlinear variables now satisfy

$$\hat{\zeta}_q - \hat{\zeta}_q = -i (\beta_{q3} - \beta_{q4}) \hat{\omega} / 2,$$

$$\hat{\omega}_q - \hat{\omega}_q = 0,$$

subject to

$$\hat{\zeta}(\hat{q}, \infty) = 0, \hat{\omega}(\hat{q}, \infty) = 0,$$

$$\hat{\zeta}(\hat{q}, 0) = 0, \hat{\omega}(\hat{q}, 0) = -i \hat{K}_0 \hat{p}_{11} \hat{p}_{12},$$

where

$$\hat{K}_0 = (\beta_{q3} - \beta_{q4}) / 2,$$

and

$$\frac{d \hat{p}_{11}}{\frac{d \hat{q}}{\hat{q}}} + \hat{b}_1 \hat{\zeta}_1 \hat{p}_{11} + \hat{c}_1 \hat{A}_{33} \hat{p}_{12} = 0,$$

$$\frac{d \hat{p}_{12}}{\frac{d \hat{q}}{\hat{q}}} + \hat{b}_2 \hat{\zeta}_2 \hat{p}_{12} + \hat{c}_2 \hat{A}_{33} \hat{p}_{11} = 0.$$

Here

$$[\hat{x}, \hat{y}, \hat{w}, \hat{z}, \hat{A}_{33}, \hat{p}_{11}, \hat{p}_{12}, \hat{\lambda}] = \left[\omega e^{3/2} \hat{\omega}, \omega e^{1/2} \hat{\omega} \right],$$

$$\omega e^{-5/2} \hat{\omega}, \omega e^{-2} \hat{\omega}, \omega e^{-2} \hat{A}_{33}, \hat{p}_{11}, \hat{p}_{12}, \omega e^{-2} \hat{\lambda}],$$

and

$$\hat{b}_1 = \frac{b_1}{a_1}, \hat{c}_1 = \frac{c_1}{a_1}, \hat{b}_2 = -\hat{c}_1, \hat{c}_2 = -\hat{c}_1,$$

where

$$a_1'' = -2i \sqrt[3]{2^{1/3}(\alpha_1 \beta_2) \frac{1}{2^2}},$$

$$b_1'' = -2 \alpha_3 \gamma_1 \Delta e^{2/3} \sqrt[3]{\Delta^{5/3}} - 5/3,$$

$$c_1'' = 2i^{1/3} \Delta e^{3} \gamma_1 (\alpha_1 \beta_2) / 3 - 5/3.$$
Once more, the \(\mathcal{V-W}\) system simplifies, this time to
\[
\hat{\lambda}_{33} = \frac{(\beta_{13} - \beta_{14})^2}{4\text{Ai}'(0)} \int_{\text{Ai}(0)\Gamma(2/3)}^{\infty} \tilde{P}_1(u) \tilde{P}_2^*(u) (\tilde{u} - u)^{-1/3} du, \tag{4.34}
\]
where \(\beta_{13} = -\beta_{14} = 2(d_1/d_2)^{1/2} \mathcal{L}^{1/3}\), from above.

The criteria for algebraic finite-distance break-up,
\[(\beta_{13} - \beta_{14}) K_0 \hat{c}_{1r} < 0 \quad \text{and} \quad (\beta_{13} - \beta_{14}) K_0 \hat{c}_{2r} < 0\]
cannot be satisfied since \(\hat{c}_{1r} = -\hat{c}_{2r}\). Also, we observe that
\(\hat{c}_{1r} = -\hat{c}_{2r}\) ruling out exponential wave-decay at downstream infinity. Instead, we anticipate that the flow solution is continually cyclic.

Analogous theory to the current case is obtained for large values of \(\mathcal{L}\), with crossflow as \(O(1)\) (see Chapter 5).

The crossflow mode expansions (\((4.14a,b)\) above) remain valid for small \(\mathcal{L}\) provided the first term continues to dominate the second term, i.e. \(\mathcal{L} \omega_e^{-1} \gg \mathcal{L}^{1/3} \omega_e^{-1/2}\) or \(\mathcal{L} \gg \omega_e^{3/4}\). As \(\mathcal{L} \to O(\omega_e^{3/4})\) we have the onset of new analysis, which is considered next.

\[(B) \quad (-\beta) \sim \omega_e^{-1/4}\]

Here we concentrate on the regime associated with \(\mathcal{E}\) and \(\mathcal{D}\) where the crossflow effects first start to significantly affect the stability of the boundary layer.
Hence, as in (A) above, we anticipate that $|\beta| \omega_e \neq o(\omega)$ but, in contrast, we additionally seek the balances 

$(\alpha + \beta \omega_e/2) \alpha \sim \beta \omega_e$ so that consistency in (4.1) is achieved if $\alpha \sim \omega_e^{5/4}$ and $|\beta| \sim \omega_e^{-1/4}$. We note that $\beta$ can be positive or negative in sign, but for now we consider the former case. Then, re-addressing (4.1) with the formal expansions

$$\beta = \hat{\beta}_0 \omega_e^{-1/4} + o(\omega_e^{7/4}), \quad (4.35a)$$
$$\alpha = \hat{\alpha}_0 \omega_e^{5/4}, \quad (4.35b)$$

where $\hat{\beta}_0 < 0$, we find

$$(\alpha_1, \alpha_2) \hat{\alpha}_0^{5/3} = -\hat{\beta}_0 (\hat{\alpha}_0 + \hat{\beta}_0/2)^2/2, \quad (4.36)$$

at leading order. Three solutions will generally result from this cubic equation, for fixed $\hat{\alpha}_0$, although they may not all be real. It is believed that for some critical $\hat{\alpha}_0$ value, say $\hat{\alpha}_0$, three real solutions exist for $\hat{\alpha}_0 > \hat{\alpha}_0$, but only one for $\hat{\alpha}_0 < \hat{\alpha}_0$. This is verified by the neutral curve above, where, in the current regime, we have

Our interest lies with the case $\hat{\alpha}_0 > \hat{\alpha}_0$ where three real modes exist, two of which ($\hat{\beta}u_1$ and $\hat{\beta}u_2$) exist on
the upper branch whilst the other (\( \hat{\beta}_L \)) is on the lower branch. Before describing the nonlinear implications we firstly observe the linear asymptotic properties:

(i) \[
\hat{\beta}_0 = -2 \hat{\omega}_0 \pm 2 (d_1 / d_2)^{1/2} \hat{\omega}_0^{1/3} \ldots
\]
for \((-\hat{\beta}_0) \to \infty\); \((4.37a,b)\)

(ii) \[
\hat{\omega}_0 \gg 1 \quad \text{and} \quad \hat{\beta}_0 = -2 (d_1 / d_2)^{-1/3} \hat{\omega}_0^{-1/3} \ldots
\]
or \[
\hat{\omega}_0 \ll 1 \quad \text{and} \quad \hat{\beta}_0 = -2 (d_1 / d_2)^{1/3} \hat{\omega}_0^{5/9} \ldots
\]
for \((-\hat{\beta}_0) \to 0^+\). \((4.38a)\)

Clearly, \(\hat{\beta}_{U_2}\) and \(\hat{\beta}_L\) satisfy property (i) above, whereas \(\hat{\beta}_{U_1}\) and \(\hat{\beta}_L\) satisfy (4.38a,b).

In the following nonlinear interactions, the scaled pair of modes is labelled \((\hat{\xi}, \hat{\eta})\) where these are any two of \((\hat{\beta}_{U_1}, \hat{\beta}_{U_2}, \hat{\beta}_L)\). The required adjustments to the main variables are

\[
[\xi, \eta, \omega, \kappa, \lambda_{33}, \rho_{11}, \rho_{12}, \sigma_1] = \begin{bmatrix} \omega e^{21/20} \xi, \omega e^{7/10} \eta, \\
\omega e^{-\eta/15} \xi, \omega e^{-\eta/15} \eta, \rho_{11}, \rho_{12}, \omega e^{-\eta/15} \sigma_1 \end{bmatrix} + \ldots (4.39a-h)
\]

and the primary equations are

\[
\frac{\partial^2 \xi}{\partial \xi^2} - \frac{\partial \eta}{\partial \xi} = -i(\hat{\beta}_1 - \hat{\beta}_2) \xi/2,
\]
\[
\frac{\partial^2 \eta}{\partial \xi^2} - \frac{\partial \xi}{\partial \xi} = 0,
\]
with \(\xi(\frac{\xi}{\eta}, \infty) = 0, \eta(\frac{\xi}{\eta}, \infty) = 0\), \((4.40c)\)
\(\xi(\xi, 0) = 0, \eta(\xi, 0) = -i K_0 \rho_{11} \rho_{12}^*\), \((4.40d)\)
where \(K_0 = (\hat{\beta}_1 - \hat{\beta}_2) \hat{\omega}_0 \xi / 8 \hat{\omega}_0^2\) \((4.40e)\)
and
\[
\frac{d\rho_{11}}{d\xi} + \rho_{11} \rho_{11} + \xi \lambda_{33} \rho_{12} = 0,
\]
\[
\frac{d\rho_{12}}{d\xi} + \rho_{12} \rho_{11} + \xi \lambda_{33} \rho_{11} = 0.
\]
\((4.40f,g)\)
Here
\[ \eta_n = \frac{\eta}{\tilde{\eta}_n}, \quad \tilde{\eta}_n = \frac{\tilde{\eta}}{\tilde{\eta}_n}, \]
and
\[ \tilde{\eta}_n = \left| \tilde{\eta}_n \right| \tilde{\eta}_n D(\Delta_0/\Delta_0) \left[ \frac{32 \Delta_0^{2/3}}{\Delta_0^{1/3}} \right] - 2i \left| \tilde{\eta}_n \right|^2 \tilde{\eta}_n \left( \Delta_0/2 \tilde{\eta}_n - 5/12 \right) \Delta_0, \]

\[ \tilde{\eta}_n = - \left| \tilde{\eta}_n \right| \tilde{\eta}_n D(\Delta_0/\Delta_0) \Delta_0^{2/3} \left[ \frac{32 \Delta_0^{5/3}}{\Delta_0^{1/3}} \right] - 5 \left| \tilde{\eta}_n \right| / 6 \Delta_0, \]

\[ \tilde{\eta}_n = i D \tilde{\eta}_n \Delta_0^{2/3} \left( \left| \tilde{\eta}_n \right| \tilde{\eta}_n \tilde{\eta}_n / 6 \tilde{\eta}_n \left( \tilde{\eta}_n \right)^2 \right) - \left| \tilde{\eta}_n \right|^2 \tilde{\eta}_n / 8 \Delta_0^2, \]

\[ \tilde{\eta}_n = -2 \Delta_0 \left| \tilde{\eta}_n \right|^2 \left( 5 \left| \tilde{\eta}_n \right| \tilde{\eta}_n / 12 \tilde{\eta}_n \left( \tilde{\eta}_n \right)^2 \right) + 3 \left| \tilde{\eta}_n \right| \tilde{\eta}_n \Delta_0 / 8, \]

for \( n = 1, 2 \) where \( \tilde{\eta}_n = (\Delta_0 + \tilde{\eta}_n / 12), \Delta_0 \equiv i \Delta_0 \)

and all the other notation is as in Chapter 2. Once more, the zero-incident vortex conditions yield an integral form for \( \tilde{\eta}_n \):

\[ \tilde{\eta}_n = -\frac{\tilde{\eta}_n \left( \tilde{\eta}_n - \tilde{\eta}_n \right)^2 A_i'(0)}{16 \Delta_0^2 A_i(0) \Delta_0^{1/3}} \int \frac{\tilde{\eta}_n}{\Delta_0} \tilde{\eta}_n \tilde{\eta}_n (\tilde{\eta} - \tilde{\eta})^{-1/3} du. \]

In the present case, the lower deck normal scale is stretched by a factor \( \Delta_0^{-1} \equiv O(w_e^{-1/2}) \) to allow for inertial-viscous balances. Also, the streamwise modulation forcing in the lower deck has been restored (c.f. Sections 4.4(A)(ii)-(vi) above).
As detailed above (Sections 3.3, 4.3), the decisive factors determining the ultimate nature of the flow are the signs of the real parts of the interaction coefficients, the signs of \( \hat{\varepsilon}_0 \) and \( \hat{\beta}_1 - \hat{\beta}_2 \) (or more appropriately \( \hat{\varepsilon}_0 (\hat{\beta}_1 - \hat{\beta}_2) \)) which, with \( \hat{\beta}_n < 0 \), will be positive here for any \( \hat{\beta}_1, \hat{\beta}_2 \) satisfying (4.35) and possibly the size of \( \hat{X}_i \). The crucial terms affecting the signs of \( \Re(\hat{\varepsilon}_n) \) and \( \Re(\hat{\varepsilon}_n) \) are \( \hat{\varepsilon}_n \) and \( \hat{\sigma}_n \) and whilst no extensive progress has been made on a numerical front, we can at least deduce a few preliminary results. From the diagram it is immediately obvious that

\[
\hat{\beta}_L < \hat{\beta}_U < \hat{\beta}_U < 0, \tag{4.44}
\]

for any \( \hat{\omega} > \hat{\omega}_c \).

Rewriting (4.35) as

\[
\hat{\varepsilon}_0^2 (\hat{\varepsilon}_0 - \hat{\omega}_0) = \left(\frac{d_1}{d_2}\right) \hat{\omega}_0^{5/3}
\]

where \( \hat{\varepsilon}_0 = (\hat{\varepsilon}_0 + \hat{\beta}_0/\varepsilon) \),

we seek real roots. Elementary polynomial theory states that the roots of this equation, say \( \hat{\varepsilon}_1, \hat{\varepsilon}_2 \) and \( \hat{\varepsilon}_3 \) are constrained by

\[
\hat{\varepsilon}_1 \hat{\varepsilon}_2 \hat{\varepsilon}_3 = \left(\frac{d_1}{d_2}\right) \hat{\omega}_0^{5/3}, \tag{4.46a}
\]

\[
\hat{\varepsilon}_1 \hat{\varepsilon}_2 + \hat{\varepsilon}_1 \hat{\varepsilon}_3 + \hat{\varepsilon}_2 \hat{\varepsilon}_3 = 0, \tag{4.46b}
\]

\[
\hat{\varepsilon}_1 + \hat{\varepsilon}_2 + \hat{\varepsilon}_3 = \hat{\omega}_0. \tag{4.46c}
\]

Now (4.46a) implies that either one or all three of the roots are negative. In the latter case we would
have $\hat{\xi}_1 \hat{\xi}_2 + \hat{\xi}_1 \hat{\xi}_3 + \hat{\xi}_2 \hat{\xi}_3 > 0$ contradicting (4.46b). Hence, exactly one root ($\hat{\xi}_1$, without loss of generality) is negative whilst the other two are positive (where we impose $\hat{\xi}_3 > \hat{\xi}_2$). Clearly then

$$\hat{\xi}_1 = (\hat{\lambda}_0 + \hat{\beta}_L / 2), \hat{\xi}_2 = (\hat{\lambda}_0 + \hat{\beta}_U / 2) \text{ and } \hat{\xi}_3 = (\hat{\lambda}_0 + \hat{\beta}_U / 2). \quad (4.47a-c)$$

This leads to a very significant result concerning $\hat{b}_n$:

standard analysis reveals that

$$\hat{b}_n = -\left[\hat{\beta}^2 (d_1 (d_2 - 1) (d_1 d_2) (i^{-\frac{1}{2}}n_1)_r / 3 \lambda_0^2 \right] \hat{\xi}_n^{-1}, \quad (4.48)$$

so that

$$\hat{b}_n \begin{cases} < 0, \text{ if } \hat{\beta} = \hat{\beta}_L, \\ > 0, \text{ if } \hat{\beta} = \hat{\beta}_U, \text{ or } \hat{\beta}_U. \end{cases}$$

Therefore, if the two modes under consideration are $\hat{\beta}_U$ and $\hat{\beta}_U$, then the ensuing interactions either generate exponential wave decay far downstream or else an algebraic finite-distance singularity is provoked, depending on the signs of $\hat{\xi}_1$ and $\hat{\xi}_2$, for which no rational deductions have been made. On the other hand, if $\hat{\beta}_L$ is one of the modes with the other being $\hat{\beta}_U$ or $\hat{\beta}_U$, then the nonlinear interactions would involve the initial-cycle feature in the flow solution. This cyclic behaviour either continues indefinitely or else the solution blows up at some finite point downstream, again according to the signs of $\hat{\xi}_1$ and $\hat{\xi}_2$. 
Here we consider the regime associated with $\beta$, where the crossflow effects first cause deviations from the zero-crossflow curve, for positive values of $\beta$. The order of magnitude arguments follow in a similar manner to that described in (B) above, and likewise we find $\lambda \sim \omega_e^{3/4}$ and $\beta \sim \omega_e^{-1/4}$. Then, expanding $\beta$ and $\lambda$ in the form

$$\beta = \beta_0 \omega_e^{-1/4} + O(\omega_e^{3/4}),$$  \hspace{1cm} (4.49a)  

$$\lambda = \lambda_0 \omega_e^{3/4},$$  \hspace{1cm} (4.49b)  

where $\beta_0 > 0$, we equate leading order terms in (4.1) to obtain

$$(d_1/d_z) \lambda_0^{5/12} = \beta_0 (\lambda_0 + \beta_0/2)^2/2.$$  \hspace{1cm} (4.50)  

Alternatively, we have

$$\bar{c}_0^2 (\bar{c}_0 - \lambda_0) = (d_1/d_z) \lambda_0^{5/12},$$  \hspace{1cm} (4.51)  

where $\bar{c}_0 = (\lambda_0 + \beta_0/2)$. Again, from polynomial theory

$$\bar{c}_1 \bar{c}_2 \bar{c}_3 = (d_1/d_z) \lambda_0^{5/12},$$  \hspace{1cm} (4.52a)  

$$\bar{c}_1 \bar{c}_2 + \bar{c}_1 \bar{c}_3 + \bar{c}_2 \bar{c}_3 = 0,$$  \hspace{1cm} (4.52b)  

$$\bar{c}_1 + \bar{c}_2 + \bar{c}_3 = \lambda_0,$$  \hspace{1cm} (4.52c)  

where $\bar{c}_1, \bar{c}_2, \bar{c}_3$ are the (possibly complex) roots of (4.50). Suppose all these roots are real: then,
since $\beta_0 > 0$, it follows that $\zeta_i > 0$ (for $i=1,2,3$). Thus, $\zeta_1 \zeta_2 + \zeta_1 \zeta_3 + \zeta_2 \zeta_3 > 0$, violating (4.52b). Therefore, we conclude that only one real root may exist, with the other two being a complex conjugate pair. This makes sense, in hindsight, since the neutral curve exhibits just one possible mode in the regime under discussion, which, in fact, only exists provided some unknown 'cut-off' value, $\beta_c$, has not been exceeded.

So, with only one available mode (for $\beta_0 < \beta_c$), we are forced to seek nonlinear interactions with one of $(\hat{\beta}_{u_1}, \hat{\beta}_{u_2}, \hat{\beta}_L)$ as the complementary mode. The alterations to the flow structure are the same as those described in (B) above, and the interaction equations become

\begin{align*}
\hat{\zeta}_1 + \hat{\zeta}_2 &= -i (\beta_0 - \beta_o) \hat{W} / 2, \quad (4.53a) \\
\hat{W}_1 + \hat{W}_2 &= 0, \quad (4.53b) \\
\end{align*}

with

\begin{align*}
\hat{\zeta}_1 (\hat{\eta}, \infty) &= 0, \quad \hat{W}(\hat{\eta}, \infty) = 0, \quad (4.53c) \\
\hat{\zeta}_2 (\hat{\eta}, 0) &= 0, \quad \hat{W}(\hat{\eta}, 0) = -i K_0 \hat{\pi}_{11} \hat{\pi}_{22}^*, \quad (4.53d)
\end{align*}

where

\begin{align*}
K_0 &= (\beta_0 - \beta_o) \beta_o / 8 \Delta_0^2, \quad (4.53e)
\end{align*}

and

\begin{align*}
\frac{d \hat{\pi}_{11}}{d \hat{\eta}} + b_1 \hat{\lambda}_1 \hat{\pi}_{11} + c_1 \hat{\lambda}_{31} \hat{\pi}_{12} &= 0, \quad (4.53f) \\
\frac{d \hat{\pi}_{12}}{d \hat{\eta}} + c_2 \hat{\lambda}_1 \hat{\pi}_{12} + c_2 \hat{\lambda}_{32}^* \hat{\pi}_{11} &= 0. \quad (4.53g)
\end{align*}
Here the variables are defined in a similar vein to (4.39a-h) above, and \( \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2 \) are effectively given by (4.41a-d) above with \((\hat{\beta}_1, \hat{\beta}_2) \rightarrow (\hat{\beta}_0, \hat{\beta}_0)\).

We note that \( \hat{\beta}_0 \) can represent either of three roots \( \hat{\beta}_1, \hat{\beta}_2 \) or \( \hat{\beta}_L \). The zero-starting vortex-skin-friction integral becomes

\[
\hat{\lambda}_{33} = -\frac{\bar{\beta}_0 \hat{\beta}_0 (\bar{\beta}_o - \hat{\beta}_o)^2 A_i'(0)}{16 \int_0^\infty \bar{A}_i(\eta) \overline{\lambda}(2\lambda^2) A_i(\eta) \overline{A}_i(\eta)(\eta - u)^{-1/3} \ du. \tag{4.54}
\]

Equation (4.48) above holds for \( \hat{\lambda}_{1r} \) with \((\hat{\beta}^\wedge_{1r}, \hat{\beta}^\wedge_{1r}) \rightarrow (\bar{\beta}_0, \bar{\beta}_0)\) in effect, and since \( \overline{\lambda}_0 = (\bar{\lambda}_0 + \bar{\lambda}_1)/2 > \lambda_0 \), where \( \lambda_0 \) is positive, we surmise that \( \hat{\lambda}_{1r} > 0 \). Now if \( \hat{\beta}_0 = \hat{\beta}_u \) or \( \hat{\beta}_U \) then, from above, \( \hat{\lambda}_{2r} > 0 \); this suggests two possibilities in the ultimate flow behaviour, namely an algebraic finite-distance singularity or exponential wave-decay far downstream, depending once more on the unknown signs of \( \hat{\lambda}_{1r} \) and \( \hat{\lambda}_{2r} \), and the unknown magnitude of \( \hat{\lambda}_{1r} \). For the alternative case, \( \hat{\beta}_0 = \hat{\beta}_L \), we know that \( \hat{\lambda}_{2r} < 0 \). If, in addition, \( \hat{\lambda}_{1r} > 0 \) and \( \hat{\lambda}_{2r} > 0 \), then algebraic break-up occurs after a non-zero quantity of cycles; otherwise we have continual cyclic behaviour.

Before moving on to the next section we note the asymptotic properties implied by (4.50):-
(i) for $\bar{b}_0 \ll 1$ there are two possibilities:

$$I_0 \gg 1 \quad \text{and}$$

$$\bar{b}_0 = 2 \left( \frac{d_1}{d_2} \right) I_0^{-1/3} + \ldots \quad (4.55a)$$

or $I_0 \ll 1 \quad \text{and}$

$$\bar{b}_0 = 2 \left( \frac{d_1}{d_2} \right)^{1/3} I_0^{5/9} + \ldots \quad (4.55b)$$

corresponding, in turn, to the 'routes' taken along the upper and lower branches;

(ii) for $\bar{b}_0 \gg 1$, there are no possible neutral solutions (verifying the cut-off feature on the neutral curve).

(D) $|\beta| \sim |\delta| : \text{Upper Branch}$

The linear and nonlinear analyses associated with regions $\mathcal{E}$ and $\mathcal{F}$ are covered in Chapter 3 for the special case of $\text{We} = 0$. Here we merely recall the neutral eigenrelation

$$(\frac{d_1}{d_2}) = \lambda^{1/3} \left( \lambda^2 + \beta^2/4 \right)^{1/2}, \quad (4.56)$$

which yields equal and opposite solutions of $\beta$, given $\lambda$.

Important limiting properties are:

(i) $\lambda \ll 1$ and

$$\beta \sim \pm 2 \left( \frac{d_1}{d_2} \right) \lambda^{-1/3} \text{ as } |\beta| \to \infty. \quad (4.57a,b)$$

(ii) $\lambda \to \left( \frac{d_1}{d_2} \right)^{3/4} \text{ as } \beta \to 0^\pm. \quad (4.57c,d)$

We notice that as $\lambda \to \text{We}^{3/4} \lambda_0$ in (i), a consistent wavenumber match is obtained with the $|\beta| \sim \text{We}^{-1/4}$ theory
above (i.e. (4.38a) and (4.55a) for positive and negative $\beta$, in turn).

(E) \( |\beta| \sim 1: \text{Lower Branch} \)

The analytical results here are merely those obtained in the small-streamwise wavenumber limit of the \( |\beta| \sim \omega e^{-1/4} \) theories in (B), (C) above, involving the two lower branch modes. Hence, no new physical features are brought into play at this stage. This regime (depicted by \[G\] and \[H\] in the main diagram above) is, however, a convenient base to lay down the dominant interaction equations, which are found to yield unusual properties. The wavenumber expansions are expressed as

\[
\beta_{1,2} = \pm \tilde{\beta}_0 + O(\omega e^2),
\]

where, by (4.1),

\[
\tilde{\beta}_0 = 2 (d_1/d_2)^{1/3} \omega e^{1/4} \]

As \( \omega \to e^{-9/20} \omega_0 \) (so that \( \omega \to e^{1/14} \omega_0 \)), (4.58(a), (4.59) imply that

\[
\beta_{1,2} \to \left[ \pm 2 (d_1/d_2)^{1/3} \omega e^{1/14} \right] \omega e^{-1/4},
\]

matching with the \( |\beta| \sim \omega e^{-1/4} \) analyses for small \( \tilde{\omega}_0, \omega_0 \) in turn (i.e. (4.38b) and (4.55b)).

Turning now to the nonlinear analysis, we label the two scaled modes \( \tilde{\beta}_0^1 \) and \( \tilde{\beta}_0^2 \) where \( \tilde{\beta}_0^\pm = \pm \tilde{\beta}_0 \), for
respectively. Consequently, the interaction equations become

\[
\frac{\tilde{\zeta}}{\tilde{\zeta}} \tilde{\xi} - \tilde{\xi} \frac{\tilde{\zeta}}{\tilde{\zeta}} = -2i \left[ \frac{(d_1/d_2)^{1/2}}{d_0^{5/4}} \right] \tilde{\omega}, \quad (4.61a)
\]

\[
\tilde{\omega} \frac{\tilde{\zeta}}{\tilde{\zeta}} - \tilde{\xi} \frac{\tilde{\omega}}{\tilde{\xi}} = 0, \quad (4.61b)
\]

with

\[
\tilde{\zeta} (\zeta, \infty) = 0, \quad \tilde{\omega} (\zeta, \infty) = 0, \quad (4.61c)
\]

\[
\tilde{\zeta} (\zeta, 0) = 0, \quad \tilde{\omega} (\zeta, 0) = -i \tilde{\kappa}_0 \tilde{\rho}_1 \tilde{\rho}^*_2, \quad (4.61d)
\]

where

\[
\tilde{\kappa}_0 = -2 (d_1/d_2) \frac{d_0}{\zeta}^{-1/3}, \quad (4.61e)
\]

and

\[
\frac{d \tilde{\rho}_1}{d \zeta} + \tilde{\zeta} \frac{\tilde{\rho}_1}{\zeta} + \tilde{\zeta} \tilde{\lambda}_{33} \tilde{\rho}_2 = 0, \quad (4.61f)
\]

\[
\frac{d \tilde{\lambda}_2}{d \zeta} + \tilde{\zeta}_2 \tilde{\rho}_2 + \tilde{\zeta}_2 \tilde{\lambda}_{33} \tilde{\rho}^*_1 = 0. \quad (4.61g)
\]

Here

\[
\tilde{b}_n = \tilde{b}_{1n} / \tilde{b}_{2n}, \quad \tilde{c}_n = \tilde{c}_{1n} / \tilde{c}_{2n}, \quad (for \ n = 1, 2), \quad (4.62a-d)
\]

where

\[
\tilde{b}_{1n} = \left[ 2i \left( \frac{1}{2} \right)^{1/3} \frac{\tilde{\kappa}_0}{\zeta} D (d_1/d_2)^{4/3} / 3 + 5i (d_1/d_2)^{1/3} / 3 \right] \tilde{\omega}_0^{-4/3} \tilde{\zeta}_n, \quad (4.63a,b)
\]

\[
\tilde{b}_{2n} = -\left[ 2i \left( \frac{1}{2} \right)^{1/3} \frac{\tilde{\kappa}_0}{\zeta} D (d_1/d_2)^{4/3} / 3 + 5 (d_1/d_2)^{1/3} / 3 \right] \tilde{\omega}_0^{-4/3} \tilde{\zeta}_n, \quad (4.63c,d)
\]

\[
\tilde{c}_{1n} = \left[ \frac{1}{13} \frac{D \tilde{\omega}_0}{\zeta} (2 \zeta_1 + \zeta_2) / 3 \right] (d_1/d_2)^{4/3}
\]

\[
- 4 (d_1/d_2)^{1/3} / 3 \right] \tilde{\omega}_0^{-4/3}, \quad (4.63e,f)
\]

and all the other symbols are as defined in Chapter 2.

Also, the main variables here have the underlying form

\[
\begin{bmatrix}
\tilde{\chi}, \tilde{g}, \tilde{w}, \tilde{\zeta}, \tilde{\lambda}_{33}, \tilde{\rho}_1, \tilde{\rho}_2, \tilde{\lambda}_1
\end{bmatrix} = \begin{bmatrix}
\tilde{\omega}_e^{18/25} \tilde{\chi}, \tilde{\omega}_e^{6/25} \tilde{g},
\tilde{\omega}_e^{12/25} \tilde{w}, \tilde{\omega}_e^{-48/25} \tilde{\zeta}, \tilde{\omega}_e^{-48/25} \tilde{\lambda}_{33}, \tilde{\rho}_1,
\tilde{\rho}_2, \tilde{\omega}_e^{-48/25} \tilde{\lambda}_1
\end{bmatrix} + \ldots. \quad (4.64a-h)
In the lower deck, the transverse scale undergoes a stretch of relative order $We^{-2/5}$ to retain convective-diffusive balances. The dominant equations governing the waves' motions are found to be three-dimensional in form. This is surprising in some ways, because intuitively we might expect a principal cross-stream motion due to the relative largeness of $\beta$ to $\lambda$. However, the main basic-flow component driving the waves is purely streamwise, i.e. $(Y,0,0)$, and this necessitates the 3D structure of the waves' equations. In the buffer deck the re-scaling factors associated with the streamwise-vortex and normal coordinates are given by (4.64a,b) above, whilst in the main and upper deck, where 2D cross-stream motion is mainly observed for the two waves, the coordinate scales remain unaltered.

Returning to the problem in hand, we now investigate the 'ultimate behaviour' possibilities. Firstly, we seek the sign of $\tilde{c}_{\alpha_r}$; as it turns out, no extensive analysis is required here, since (4.63b,e) reveal the interesting properties

$$\tilde{c}_{\alpha_n} = i\lambda_0 \tilde{c}_{\alpha_n},$$

and hence $\tilde{c}_{\alpha_n} = O_3$ (for $n = 1,2$). \hfill (4.65a,b)

For the nonlinear interaction coefficients, calculations yield

$$\tilde{c}_{\alpha_r} = d_1(d_{z-1})(d_1/d_2)^{5/3}(8Q + 15R)/36 - d_1^2(d_{z-1})^2(d_1/d_2)^{5/3}S_6, \quad (4.66a,b)$$
for \( n = 1, 2 \), where \( Q \approx -0.41, R \approx 1.18, S \approx -0.96. \)

Therefore \( \tilde{\mathbf{c}}_{n r} > 0 \) for \( n = 1, 2 \), which means that the necessary condition for an algebraic finite-distance singularity is satisfied here. In fact, with \( \tilde{\mathbf{c}}_{n r} = 0 \) eliminating the other possibilities outlined in Section 4.3, we suspect that this option will occur throughout this regime.

### 4.5. - Weakly Nonlinear Interactions Involving Four Waves

The above findings in Section 4.2 show that it is possible to have four distinct waves, with common values of \( \mathbf{d} \) and \( \mathbf{\alpha} \), provided \( |\omega_0| < \frac{k}{\sqrt{2}} \) and \( \mathbf{d} \) lies within a certain (crossflow-dependent) interval. If these are introduced into a boundary layer simultaneously, six vortices will be generated via wave-inertia effects, and the buffer deck will be brought into play as before. Subsequently, these vortices will provide feedback effects to each of the four waves, and a fully interactive system will be in operation. Tentatively, we suggest that the governing equations have the form

\[
\begin{align*}
(\mathcal{C}_{jk})_{y} - \mathcal{C}_{jk}x &= -i(\beta_j - \beta_k)W_{jk}/2, \quad (4.67a-f) \\
(W_{jk})_{y} - \mathcal{C}_{jk}x &= 0, \quad (4.67g-l)
\end{align*}
\]

subject to

\[
\begin{align*}
\mathcal{C}_{jk}(x, \infty) &= 0, \quad W_{jk}(x, \infty) = 0, \quad (4.67m-r) \\
(\mathcal{C}_{jk})_{y}(x, 0) &= 0, \quad W_{jk}(x, 0) = -iK_{jk}P_{j}P_{k}^{*}, \quad (4.67s-x)
\end{align*}
\]

and

\[
\frac{dP_{j}}{dx} + b_j \gamma_j P_j + \sum_{k=1 \atop k \neq j}^{+} c_{jk} \lambda_{jk} P_k = 0, \quad (4.68a-d)
\]
where \( j, k \in \{1, 2, 3, 4\} \) and \( j < k \), without loss of generality. Also, \( b_j = b_j (\beta_j, W_e) \), \( c_{jk} = c_{jk} (\beta_j, \beta_k, W_e) \) and
\[
K_{jk} = (\beta_j - \beta_k) (1 + \beta_j \beta_k / 4d^2) / 2,
\]
for \( j, k \in \{1, 2, 3, 4\}, j < k \). Physically, \((W_{jk}, \chi_{jk})\) represents the vortex induced by nonlinear coupling of the \( j \)th and \( k \)th waves, and if these have zero starting values we deduce that
\[
A_{jk} = \frac{-(\beta_j - \beta_k) K_{jk} A_i'(0)}{2 A_i(0) I(2/3)} \int_{\chi_0}^{\chi} \rho_j(u) \rho_k^*(u) (\chi - u)^{-1/2} du,
\]
for \( j, k \in \{1, 2, 3, 4\}, j < k \).

Then, given starting values for the wave-pressures, we can in principle solve numerically using a method similar to that described in Section 3.2, but it would possibly present a sizeable task in practice. No solutions of the above have been attempted yet.
Here we examine the effects of $O(1)$ crossflow and consider other limiting analyses of the general theory outlined in Chapter 2. Firstly, we address the problem of large streamwise (and subsequently large spanwise) wavenumbers.

5.1 - $\alpha \gg 1$ Theory

We recall that the neutral stability curve relating real wavenumbers $\alpha, \beta$ associated with a linear perturbation of a 3D boundary layer with outer velocity $(1,0,\omega_z)$, is as sketched below.

Here $\alpha, \beta$ and $\omega_z$ are constrained by (4.1) above. The regimes $(-\beta) \gg 1$ and $|\beta| \ll 1$ have been labelled on
the diagram, and we shall briefly consider the latter in Section 5.3, before discussing some high-frequency analysis in Section 5.4. For the present case, we note that as \( \lambda \) increases in order of magnitude, the neutral eigenrelation only remains consistent if

\[
\beta \sim (2 \omega_e^{-1}) \lambda + \varepsilon \lambda^{1/3} + O(\lambda^{-1/3}), \quad (5.1)
\]

where \( \varepsilon \) is double-valued,

\[\varepsilon_1, \varepsilon_2 = \pm 2(a_1/a_2)^{1/2} \omega_e^{-1/2} \left(1 + \omega_e^2\right)^{-1/4}. \quad (5.2a,b)\]

There are clear analogies here with the work described in Section 4.3(A)(vi) above, concerning the two crossflow modes for \( \omega_e \ll 1 \) and \( \lambda = O(1) \) and in a like manner, the values \( \varepsilon_1, \varepsilon_2 \) represent the upper and lower branches, in turn. Unlike the last-mentioned theory, however, only two modes (given by (5.1), (5.2a,b)) instead of four, exist for fixed streamwise wavenumbers. These two modes are brought into being solely through the influence of nonzero crossflow. In other words, the linear and nonlinear analyses associated with the current regime have no connection whatsoever with 2D boundary layers. Consequently, the presence of crossflow is crucial here.
We label the 'upper' and lower branch modes as \( \beta_1, \beta_2 \) so that

\[
\begin{align*}
\beta_1 &= -(2\omega_0^{-1})\lambda + e_1 \lambda^{1/3} + O(\lambda^{-1/3}), \\
\beta_2 &= -(2\omega_0^{-1})\lambda + e_2 \lambda^{1/3} + O(\lambda^{-1/3}),
\end{align*}
\]

(5.3a)

(5.3b)

where \( e_1, e_2 \) are as defined above. To preserve neutrality we adjust the dominant wave-frequency according to the \( \lambda-\Re \), lower branch condition \( \Omega/\lambda^{2/3} \approx \lambda_1 \), where \( \lambda_1 \approx 2.3 \). Therefore

\[
\Omega \equiv \lambda_1 \lambda^{2/3} \quad (5.4)
\]

for \( \lambda > 1 \). This maintains temporal-inertial-viscous balances for the two waves' motions in the lower deck, so long as the normal scale contracts by an amount \( O(\lambda^{-1/3}) \).

In the buffer deck, the \( \bar{x}, \bar{y} \) scales alter by factors of \( O(\lambda^{-3/5}), O(\lambda^{-1/5}) \) respectively, in order to keep nonlinearity in the vortex- and wave-interaction equations. The normal scale again changes, this time by an amount \( O(\lambda^{-1}) \) in the upper deck to preserve a balance in Laplace's equation for the wave-pressures (i.e. (2.25a) above). Finally, the relative spanwise scale for the vortex \( (\sim |\beta_1 - \beta_2|^{-1}) \) is reduced to \( O(\lambda^{-1/3}) \) compared with the \( O(\lambda^{-1}) \) reduction for the waves.
The governing nonlinear equations are

\[ \overline{\mathcal{P}}_t = -i (\varepsilon_1 - \varepsilon_2) \overline{\mathcal{W}} / 2, \quad (5.5a) \]

\[ \overline{\mathcal{W}}_t = -i \overline{\mathcal{P}}_t = 0, \quad (5.5b) \]

subject to

\[ \overline{\mathcal{P}}(\overline{\tau}, \infty) = 0, \overline{\mathcal{W}}(\overline{\tau}, \infty) = 0, \quad (5.5c) \]

\[ \overline{\mathcal{P}}(\overline{\tau}, 0) = 0, \overline{\mathcal{W}}(\overline{\tau}, 0) = -i K_0 \overline{\mathcal{P}}_1 \overline{\mathcal{P}}_2^*, \quad (5.5d) \]

where

\[ K_0 = \omega e^{-2 (1 + \omega^2)} (\varepsilon_1 - \varepsilon_2) / 2, \quad (5.5e) \]

and

\[ \frac{d\overline{\mathcal{P}}_1}{d\overline{\tau}} + b_1 \overline{\mathcal{P}}_1 + c_1 \overline{\mathcal{P}}_3 \overline{\mathcal{P}}_2 = 0, \quad (5.5f) \]

\[ \frac{d\overline{\mathcal{P}}_2}{d\overline{\tau}} + b_2 \overline{\mathcal{P}}_2 + c_2 \overline{\mathcal{P}}_3^* \overline{\mathcal{P}}_1 = 0, \quad (5.5g) \]

upon substitution of

\[ \begin{bmatrix} \overline{\mathcal{P}}, \overline{\mathcal{W}}, \overline{\mathcal{C}}, \overline{\mathcal{A}}_{33}, \overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2, \overline{\mathcal{A}}_1 \end{bmatrix} = \begin{bmatrix} \omega^{-315} \overline{\mathcal{P}}, \omega^{-315} \overline{\mathcal{W}}, \\
\omega^{115} \overline{\mathcal{W}}, \omega^{415} \overline{\mathcal{C}}, \omega^{415} \overline{\mathcal{A}}_{33}, \overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2, \omega^{415} \overline{\mathcal{A}}_1 \end{bmatrix} + \ldots, \quad (5.6a-b) \]

into (2.33a-d), (2.32f,g) from above. The interaction coefficients

\[ \overline{b}_n = \overline{b}_{1n} / \overline{a}_{1n}, \quad \overline{c}_n = \overline{c}_{1n} / \overline{a}_{1n} \quad (f o r \ n = 1, 2), \quad (5.7a-d) \]

are such that

\[ \overline{a}_{1n} = -4i (1 + \omega e^2)^{112} /[\omega \varepsilon_n \varepsilon_{n}], \quad (5.8a,b) \]

\[ \overline{b}_n = i^{1/3} (1 + \omega e^2)^{3/2} \overline{\delta}_0 \varepsilon_n \varepsilon_{n} / 6 \]

\[ -5 (1 + \omega e^2)^{112} /[3 \omega \varepsilon_n], \quad (5.8c,d) \]

\[ \overline{c}_n = i^{1/3} (1 + \omega e^2)^{3/2} \overline{\delta}_0 \varepsilon_n \varepsilon_{n} / 6 \]

\[ -5 (1 + \omega^2)^{112} /[3 \omega \varepsilon_n], \quad (5.8e,f) \]
where \((a^\prime_n, b^\prime_n, c^\prime_n) \sim (\alpha^{-1/3} \overline{a}_{1n}, \overline{b}_{1n}, \overline{c}_{1n}) + \ldots \) \((5.9a-f)\)

for \(n=1,2\) respectively, and \(a^\prime_n, b^\prime_n, c^\prime_n\) are the global interaction coefficients. In the above equations, we observe that there is no explicit dependence on \(\alpha\) since this parameter has been scaled out using \((5.3a,b), (5.4)\).

An important linear feature is the nature of the streamwise forcing on the waves which is concentrated entirely in the upper deck. (This again bears out the close analogy with Section 4(A)(vi) where this same feature is observed).

Here the zero-input vortex-flow condition implies

the integral expression

\[
\overline{A}_{33} = -\omega_0 \varepsilon^2 (1+\omega_0^2) (\varepsilon_1 - \varepsilon_2) \overline{A}_i'(0) \int_{\gamma_0}^{\gamma} \overline{P}_{11}(u) \overline{P}_{12}(u) (\overline{\gamma} - u)^{-1/3} du.
\]

\((5.10)\)

We can easily deduce the likely eventual behaviour of the flow because clearly \(\overline{b}_1 = -\overline{b}_2\) and \(\overline{c}_1 = -\overline{c}_2\), violating the necessary conditions for finite-distance blow-up and far-downstream decay. In accordance with the related work in Section 4(A)(vi), the suspected outcome is that of the flow solution undergoing repetitious cycles.
Before discussing the implications from the current analysis for $\mathcal{L}$ stepping up in orders of magnitude comparable with powers of the Reynolds number, we show that the present work matches for $\text{We} \to 0^+$ with the above analysis in Section 4(A)(vi) for its limiting case $\mathcal{L} \gg 1$.

From (5.2a, b) it is easily seen that

$$E_n \sim \pm \left[2(\lambda, l d_2)^{11/2}\right] \text{We}^{-1/2} + O(\text{We}^{1/2}) , \quad (5.11a, b)$$

for $\text{We} \ll 1$. Hence (5.8a-f) yield

$$\overline{a}_n \sim \mp \left[2i (\lambda, l d_2)^{-11/2}\right] \text{We}^{-3/2} + O(\text{We}^{-1/2}) , \quad (5.12a, b)$$

$$\overline{b}_n \sim \left[2i^{11/3} \lambda_0 r_0 (\lambda, l d_2)^{3} \right] \text{We}^{-1} + O(1) , \quad (5.12c, d)$$

$$\overline{c}_n \sim \left[2i^{11/3} \lambda_0 r_0 (\lambda, l d_2)^{3} \right] \text{We}^{-1} + O(1) \quad (5.12e, f)$$

for $\text{We} \ll 1$. Meanwhile, for $\mathcal{L} \gg 1$ in Section 4(A)(vi), equations (4.33a-c) suggest

$$a_n^{''} \sim \mp \left[2i (\lambda, l d_2)^{-11/2}\right] \mathcal{L}^{-11/3} , \quad (5.13a, b)$$
$$b_n^{''} \sim 2i^{11/3} \lambda_0 r_0 (\lambda, l d_2)^{3} \mathcal{L}^{-5/3} , \quad (5.13c, d)$$
$$c_n^{''} \sim 2i^{11/3} \lambda_0 r_0 (\lambda, l d_2)^{3} \mathcal{L}^{-5/3} , \quad (5.13e, f)$$

where globally,

$$(a_n', b_n', c_n') \sim (\text{We}^{-3/2} a_n'', \text{We}^{-1} b_n'', \text{We}^{-1} c_n'') + \ldots . \quad (5.14a-f)$$
Thus, the dominant sets of interaction coefficients in each case are equal in the matching regime, with the orders of magnitude being $O\left(\lambda^{-1/3} \omega_e^{-3/2}\right)$, $O(\omega_e^{-1})$, $O(\omega_e^{-1})$, in turn. Also in the $\omega_e \ll 1$, $\lambda = O(1)$ theory we have

$$\beta_n \sim -(2\lambda) \omega_e^{-1} + \beta_n \omega_e^{-1/2} + O(1),$$

where $\beta_n = \pm 2(\lambda, d_z) \lambda^{1/2} \lambda^{1/2}$, and again agreement is plainly obtained in the connecting regime.

We now approach the question of what happens as $\lambda$ becomes so large that the triple-deck structure starts to be significantly distorted. The first new physical effect to come into play is wave acceleration, in the normal direction, in the main deck (typically $\lambda \partial u / \partial x$), which becomes of the same order as the normal pressure gradient $(\partial \rho / \partial y)$. The latter quantity had hitherto been solely dominant inducing zero wave-pressure change across the main deck. (Here we note that $(\bar{u}, \bar{v}, \bar{w}, \rho)$, $(\bar{u}, \bar{v}, \bar{w}, \rho)$, $(x, y, z)$ denote the basic flow, the disturbance (relatively small) and the Cartesian coordinates in the main deck.) It is well-known from linear TS theory (Smith 1979) that the disturbance flow for $Re \gg 1$ has the underlying form

$$(\bar{u}, \bar{v}, \bar{w}, \bar{p}) \sim \left(Re^{-1/3} \hat{u}, Re^{-11/14} \hat{v}, Re^{-11/18} \hat{w}, Re^{-11/14} \hat{p}\right) + \ldots,$$

in the main deck, where

$$\hat{u} = \bar{u} \gamma A, \hat{v} = -i \left(2 \bar{u} + \beta \bar{w}/2\right)A,$$  \hspace{1cm} (5.15a-d)

$$\hat{w} = \bar{w} \gamma A, \hat{p} = \rho,$$  \hspace{1cm} (5.16a-b,c,d)
and \( \mathcal{L}, \beta, \rho, \alpha \) are defined in the Introduction. For \( \mathcal{L} \gg 1 \), (1.1a-f) suggest that

\[
A \sim \mathcal{L}^{1/3}, \quad \rho \sim 1
\]

and therefore

\[
\hat{\omega} \sim \mathcal{L}^{4/3}, \quad \hat{\beta} \sim 1.
\]

Finally, since we have the global properties \( x \sim \text{Re}^{-3/8} \mathcal{L}^{-1} \), \( y \sim \text{Re}^{-1/8} \), we see that our new regime becomes activated when

\[
\mathcal{L}^{7/13} \text{Re}^{3/8} \sim \text{Re}^{11/12}, \quad \text{i.e.} \quad \mathcal{L} \sim \text{Re}^{3/56}.
\]

This implies that the new streamwise and spanwise scales for the perturbations are \( \mathcal{O}(\text{Re}^{-3/17}) \). Also, since the upper and lower deck widths respond like \( \mathcal{L}^{-1}, \mathcal{L}^{-1/3} \) respectively, for \( \mathcal{L} \gg 1 \), the new decks have overall sizes \( \mathcal{O}(\text{Re}^{-3/17}), \mathcal{O}(\text{Re}^{-9/14}) \) in turn. Formal analysis of the linear theory, based on the above findings, is addressed next.
5.2. \( \mathbf{x} = O\left(Re^{-3/17}\right) \) Theory

We firstly derive the governing equations in the fully nonlinear case, and then linearise these by considering a small wave-like perturbation. In the lower layer the solution has the expansion

\[
\begin{align*}
U &= Re^{-117} U + \ldots, \\
V &= Re^{-514} V + \ldots, \\
W &= Re^{-117} W + \ldots, \\
\rho &= Re^{-217} \rho + \ldots,
\end{align*}
\]  

\text{(Diagram of the flow structure in the present regime.)}
where \( U, V, W, \rho \) are principally dependent on the variables
\[
(\hat{X}, \hat{Y}, \hat{Z}, \hat{T}) = (Re^{3/17} x, Re^{-1/14} y, Re^{3/17} z, Re^{2/17} t).
\] 

The leading order equations of motion are
\[
\begin{align*}
U_{\hat{T}} + UV_{\hat{X}} + VW_{\hat{Y}} + WU_{\hat{Z}} &= -\rho_{\hat{X}} + U\rho_{\hat{Y}}, \\
W_{\hat{T}} + UW_{\hat{X}} + VV_{\hat{Y}} + VU_{\hat{Z}} &= -\rho_2 + W\rho_{\hat{Y}}, \\
U_{\hat{Z}} + V_{\hat{Y}} + W_{\hat{Z}} &= 0,
\end{align*}
\]
subject to the boundary conditions
\[
U = V = W = 0 \text{ at } \hat{Y} = 0, \quad U \sim \hat{Y} + A, \quad W \sim \hat{Y}^{-1} \text{ as } \hat{Y} \to \infty,
\]
where \( \rho(\hat{X}, \hat{Z}, \hat{T}) \) and \( A = A(\hat{X}, \hat{Z}, \hat{T}) \) is the scaled boundary-layer displacement decrement. This is precisely the system of equations holding in the lower deck of triple-deck theory (c.f. (1.1a-e) above).

In the main layer, the velocity and pressure expand in the manner
\[
\begin{align*}
u &= \vec{u} + Re^{-3/17} u_0(\hat{X}, \hat{Y}, \hat{Z}, \hat{T}) + \ldots, \\
\omega &= Re^{-3/17} \omega_0(\hat{X}, \hat{Y}, \hat{Z}, \hat{T}) + \ldots, \\
\rho &= Re^{-2/17} \rho_0(\hat{X}, \hat{Y}, \hat{Z}, \hat{T}) + \ldots,
\end{align*}
\]
where
\[
y = Re^{-1/2} \bar{y}.
\]
Here the basic-flow profile \((\bar{u}, \bar{O}, \bar{w}, \bar{O})\), is independent of the fast scales \((\hat{x}, \hat{z}, \hat{T})\) in contrast to the main disturbance flow \((u_o, v_o, w_o, \rho_o)\). The equations of motion governing the disturbance are

\[
\begin{align*}
\bar{u} \frac{\partial u_o}{\partial x} + u_o \frac{\partial \bar{u}}{\partial x} + \bar{w} \frac{\partial u_o}{\partial z} &= 0, \\
\bar{u} \frac{\partial u_o}{\partial x} + \bar{w} \frac{\partial u_o}{\partial z} &= -\rho_o g, \\
\bar{u} \frac{\partial \bar{w}}{\partial x} + u_o \frac{\partial \bar{w}}{\partial z} + \bar{w} \frac{\partial \bar{w}}{\partial z} &= 0, \\
\frac{\partial u_o}{\partial x} + u_o \frac{\partial w_o}{\partial x} + \frac{\partial w_o}{\partial z} &= 0,
\end{align*}
\]

which yields the solution

\[
\begin{align*}
u_o &= \bar{u} \bar{g} A, \\
v_o &= -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) A, \\
w_o &= \bar{w} \bar{g} A, \\
\rho_o &= \int_{0}^{\bar{g}} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)^2 A d\bar{y}, + \rho,
\end{align*}
\]

where the lower deck matching condition \(\rho_o(0) = \rho\) has been applied. We note that whilst the velocity solutions are unchanged from the main deck results in triple-deck theory, the pressure is no longer independent of \(\bar{g}\) and gives rise to a pressure jump

\[
I = \int_{0}^{\infty} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)^2 A d\bar{y},
\]

across the main layer.

In the upper layer, we find that the slower streamwise scales

\[
(\tilde{x}, \tilde{z}) = (Re^{3/28}, Re^{3/28})
\]

(5.26a,b)
play an active role. This is because we require the momentum operator

$$J \equiv (a \partial a^x + \omega_e \partial a^z) \quad (5.27)$$

to have zero effect on the disturbance motion.

Plainly, this is analogous to the $L \gg 1$ analysis of the previous section, where the consistency relation

$$(L + \beta \omega_e / 2)$$

holds. (Indeed, this equation is obtained in the linearised case, where $\partial a^x \to iL, \partial a^z \to i\beta / 2$
in effect.) The flow solution has the expanded form

$$u = 1 + Re^{-114} \tilde{u}_0 (\hat{x}, \hat{y}, \hat{z}, \hat{T}, \hat{X}, \hat{Z}) + \ldots, \quad (5.28a)$$

$$v = Re^{-114} \tilde{v}_0 (\hat{x}, \hat{y}, \hat{z}, \hat{T}, \hat{X}, \hat{Z}) + \ldots, \quad (5.28b)$$

$$w = \omega_e + Re^{-114} \tilde{w}_0 (\hat{x}, \hat{y}, \hat{z}, \hat{T}, \hat{X}, \hat{Z}) + \ldots, \quad (5.28c)$$

$$\rho = Re^{-217} \tilde{\rho}_0 (\hat{x}, \hat{y}, \hat{z}, \hat{T}, \hat{X}, \hat{Z}) + \ldots, \quad (5.28d)$$

where

$$y = Re^{-317} \tilde{y} \quad (5.28e)$$

and substitution into the Navier-Stokes equations yields

$$J_t \tilde{u}_0 = -\tilde{\rho}_0 \hat{x}, \quad (5.29a)$$

$$J_t \tilde{v}_0 = -\tilde{\rho}_0 \hat{y}, \quad (5.29b)$$

$$J_t \tilde{w}_0 = -\tilde{\rho}_0 \hat{z}, \quad (5.29c)$$

$$\tilde{u}_0 \hat{x} + \tilde{v}_0 \hat{y} + \tilde{w}_0 \hat{z} = 0, \quad (5.29d)$$

for the main perturbation. Here $(1, 0, \omega_e)$ is the constant external basic flow, $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0, \tilde{\rho}_0)$ is the perturbation velocity and pressure, and

$$J_t \equiv (a \partial a^x + \omega_e \partial a^z). \quad (5.30)$$
Simplification of (5.29a-d) gives

$$\nabla^2 \tilde{p}_0 = 0,$$  \hspace{1cm} (5.31)

where

$$\nabla^2 \equiv (\partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2).$$

The boundary conditions complementing (5.31) are

$$\tilde{p}_0 \to 0 \quad \text{as} \quad \tilde{y} \to \infty,$$ \hspace{1cm} (5.32a)

$$\tilde{p}_0 \to 1 + P, \quad \tilde{p}_0 \to J_1^2 A \quad \text{as} \quad \tilde{y} \to 0,$$ \hspace{1cm} (5.32b,c)

whereupon we deduce the pressure-displacement law

$$P = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left(\frac{\partial}{\partial x} \tilde{z} + \omega_\epsilon \frac{\partial}{\partial \tilde{y}} \tilde{z} \right)^2 A}{\left[(\tilde{x} - \tilde{y})^2 + (\tilde{z} - \tilde{z}_0)^2 \right]^{\frac{1}{2}}} \, d\tilde{y} \, d\tilde{z} - I. \hspace{1cm} (5.33)$$

The deviation from the triple-deck theory is essentially captured by the Coanda integral $I$, in (5.33), which in turn arose through the significant emergence of fluid acceleration in the normal direction. The nonlinear stability of the flow is determined by unifying the lower layer equations, (5.21a-e), and (5.33). We observe that the boundary layer growth is still governed by viscous-inviscid interaction and, again as in triple-deck theory, the critical layer is contained within the viscous wall layer.
Having established the controlling disturbance equations in scaled nonlinear form, we are in a position to linearise these by considering the perturbation to be relatively small, say of \( O(\epsilon) \), (where \( 0 < \epsilon < 1 \)). Hence, \((U, V, W, P, A)\) are expressed as

\[
(U, V, W, P, A) = \epsilon (\bar{U}, \bar{V}, \bar{W}, \bar{P}, \bar{A}) + \ldots . \tag{5.34a-e}
\]

where

\[
\bar{U} = \bar{u}, (\bar{x}, \bar{z}, \bar{y}) \text{ E + c.c.} \tag{5.35}
\]

(and likewise for \(\bar{V}, \bar{W}, \bar{P}, \bar{A}\) and

\[
\bar{E} = e^{\exp \left\{ i (\bar{x} \bar{\omega} \bar{\zeta} + \bar{\beta}_0 \bar{Z}/2 - \bar{\gamma}_0 \bar{T}) \right\}} . \tag{5.36}
\]

Now the operators \(\partial/\partial \bar{x}, \partial/\partial \bar{z}, \partial/\partial \bar{T}\) have effective 'values' \(i\bar{\omega}_0, i\bar{\beta}_0/2\) and \(-i\bar{\gamma}_0\) respectively, where \(\bar{\omega}_0, \bar{\beta}_0, \bar{\gamma}_0\) are all real, so that the upper layer constraint implies the condition

\[
\bar{\beta}_0 = -2\bar{\omega}_0 \omega^{-1}. \tag{5.37}
\]

Inserting (5.34d,e) into (5.33) gives the main result

\[
\bar{\rho} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\omega^2 \bar{\omega} \bar{Z} E + \text{c.c.})}{[(\bar{x} - \bar{\eta})^2 + (\bar{z} - \bar{\eta})^2]^\frac{3}{2}} d\bar{\eta} d\bar{\zeta} - \bar{I}, \tag{5.38}
\]

where

\[
\bar{I} = \epsilon \bar{I} + \ldots , \tag{5.39}
\]

and we have chosen to have the disturbance independent of \(\bar{X}\).
Upon use of standard complex analysis, (5.38) yields

$$\bar{P}_1 = \omega_0^2 \tilde{A}_1 \tilde{z} \tilde{z} / \bar{V}_o - \tilde{I}_1, \quad (5.40)$$

where

$$\tilde{z} = Z_0 \omega_0^{-1} (1 + \omega_0^2)^{1/2}, \quad (5.41)$$

and

$$\tilde{I}_1 = -\tilde{A}_1 \int_0^\infty (\alpha_0 \bar{u} + \beta_0 \bar{w} / 2)^2 d\bar{y}. \quad (5.42)$$

Using (5.37), we reduce (5.42) to

$$\tilde{I}_1 = -Z_0^2 \tilde{A}_1 I^*, \quad (5.43)$$

where

$$I^* = \int_0^\infty (\bar{u} - \bar{w} / \omega_0)^2 d\bar{y}. \quad (5.43)$$

Thus,

$$\bar{P}_1 = \omega_0^2 \tilde{A}_1 \tilde{z} \tilde{z} / \bar{V}_o + Z_0^2 \tilde{A}_1 I^*. \quad (5.44)$$

Solving the linear problem in the lower layer, we find the standard small-perturbation result

$$\left( A_i' (\varphi_0) / K \right) \tilde{A}_1 = (i \bar{Z}_0)^{1/3} (1 + \hat{\beta}_0^2 / 4 \bar{Z}_0^2) \bar{P}_1, \quad (5.45)$$

holds which, in conjunction with (5.44), implies

$$(i \bar{Z}_0)^{1/3} \hat{\delta}_0 \omega_0^2 \tilde{A}_1 \tilde{z} \tilde{z} = Z_0^2 \left[ A_i'(\varphi_0) / K - \left( i \bar{Z}_0 \right)^{1/3} \hat{\delta}_0 I^* \right] \tilde{A}_1. \quad (5.46)$$
The neutral lower branch results

\[ A_i'(\xi_0) / H = (d_1/d_2)i^{1/3} \xi_0 = -d_i^{1/3} \text{ (where } d_i \approx 2.3, d_2 \approx 2.3) \]

apply here, and consequently the neutral equation is

\[ \ddot{A}_i = -S \dot{A}_i \quad \text{(5.47)} \]

where

\[ S = \omega_e^{-2} (1 + \omega_e^2)^{-1/2} \left[ (d_1/d_2) \xi_0^{-2/3} \omega_e^2 - \xi_0^{-2} (1 + \omega_e^2) I^* \right] \quad \text{(5.48)} \]

Equation (5.47) determines the boundary-layer stability over the slower scale \( \xi \), and will clearly depend heavily upon the sign of \( S \). Simple analysis reveals that \( S = 0 \) when \( \xi_0 = \xi_0^c \) (say) where

\[ \xi_0^c = \left\{ \frac{(d_1/d_2) \omega_e^2}{(1 + \omega_e^2) I^*} \right\}^{3/7} \quad \text{(5.49)} \]

Moreover, \( S \) is positive for \( 0 < \xi_0 < \xi_0^c \), but negative for \( \xi_0 > \xi_0^c \). Hence, in the former case we write \( S = \rho^2 \) (where \( \rho \) is real), and (5.47) yields \( \cos(\rho \xi) \) and \( \sin(\rho \xi) \) as two independent solutions. Therefore, for \( \xi_0 \) in this interval, all modes are neutrally stable. On the other hand, for \( \xi_0 > \xi_0^c \) we write \( S = -\mu^2 \) (where \( \mu \) is real) leading to the independent solutions \( e^{\mu \xi} \) and \( e^{-\mu \xi} \). Thus, instability generally occurs for \( \xi_0 \) values beyond \( \xi_0^c \).
Another zero of $S$ occurs at the origin, and employing Rolle's Theorem from elementary real analysis, we anticipate at least one maximum (positive) $S$-value in the range $(0, \tilde{z}_{OC})$. Exactly one is found at $\tilde{z}_0 = \tilde{z}_{OM}$ where

$$\tilde{z}_{OM} = \left[2(d_{11}ld_{12})\frac{w_e}{q(1+w_e^2)}\right]^{3/7} \quad (5.50)$$

and the corresponding value of $S$, say $S_M$, is given by

$$S_M = \frac{7(d_{11}ld_{12})}{9w_e(1+w_e^2)^{1/2}} \left[2(d_{11}ld_{12})\frac{w_e}{q(1+w_e^2)}\right]^{3/7} \quad (5.51)$$

The value of $\tilde{z}_{OM}$ is approximately 52.5% of $\tilde{z}_{OC}$, and this ratio is independent of crossflow. Diagrammatically, the relation between $S$ and $\tilde{z}_0$ is as shown below.
Returning to the case where $\mathcal{L}_0 < \mathcal{L}_{oc}$, the two spanwise modes are effectively given by

$$\mathcal{P}_{1,2} = \mathcal{P}_0 \pm \text{Re}^{-1/28} \left( \lambda/2 \right) + O\left( \text{Re}^{-1/14} \right),$$

(5.52a,b)

where $\lambda$ is real, so that

$$\left( \mathcal{P}_1 - \mathcal{P}_2 \right) \approx \text{Re}^{-1/28} \lambda + O\left( \text{Re}^{-1/14} \right).$$

(5.53)

As $\mathcal{L}_0 \to \mathcal{L}_{oc}$, $\lambda \to 0$, so that the modes converge. For $\mathcal{L}_0 > \mathcal{L}_{oc}$, $\lambda$ is replaced by $-\mu i$, where $\mu$ is real, in (5.52a,b), (5.53), so that no neutral solutions exist.

The upshot of this is to strongly suggest the occurrence of loop 'closure' for the global $\lambda-\beta$ neutral curve, i.e.

The shaded region inside the loop represents stability whilst outside we have unstable boundary-layer growth.

The smaller regimes ($O(\text{Re}^{-3/17})$) correspond to the triple-deck theory, studied in Chapters 2, 3, 4 and Section 5.1 above.
Some asymptotic properties are now presented.

Firstly, for $0 < \tilde{z}_0 < 1$, 

$$ S \sim \left[ (\alpha_1/d_z) / (\omega_e (1 + \omega_e^2)^{1/2}) \right] \tilde{z}_0^{2/3}, \quad (5.54) $$

so that

$$ \lambda \sim \pm \left[ 2 (\alpha_1/d_z)^{1/2} / (\omega_e^{1/2} (1 + \omega_e^2)^{1/4}) \right] \tilde{z}_0^{1/3}. \quad (5.55a,b) $$

Matching is obtained here with the $\lambda \gg 1$ analysis of Section 5.1, since as $\tilde{z}_0 \rightarrow \text{Re}^{-3/56} \lambda$,

$$ \tilde{\beta}_{1,2} \sim \text{Re}^{-3/56} \left[ - (2 \omega_e^{-1}) \lambda \right. $$

$$ + \left. \pm \left( 2 (\alpha_1/d_z)^{1/2} / (\omega_e^{1/2} (1 + \omega_e^2)^{1/4}) \right) \tilde{z}_0^{1/3} + \ldots \right], \quad (5.56a,b) $$

using (5.52a,b) and (5.37), which ties in with the expansion (5.1) above. As expected here the transverse pressure gradient effects (proportional to $I^e$) become negligible. This is not the case, however, in the limit $\tilde{z}_0 \rightarrow \infty$, where

$$ S \sim - \left[ (1 + \omega_e^2)^{1/2} I^e / \omega_e^3 \right] \tilde{z}_0^3. \quad (5.57) $$

and

$$ \mu \sim \pm 2 \left[ (1 + \omega_e^2)^{1/4} (I^e)^{1/2} / \omega_e^{3/2} \right] \tilde{z}_0^{3/2}. \quad (5.58a,b) $$

In this case, the viscous lower layer contributions have been deferred to higher order, and the instability is no longer governed by pressure/displacement interactions.
Next, for $\omega_e \gg 1$, \[ S \sim \left[ \left( \frac{d_1}{d_2} \right) \tilde{L}_0^{2/3} - \tilde{L}_0^3 \int_0^\infty \tilde{u}^2 \, d\tilde{y} \right] \omega_e^{-2} + \ldots \] (5.59) whilst the critical value $\tilde{\omega}_c$ has the limiting behaviour \[ \tilde{\omega}_c \to \left[ \left( \frac{d_1}{d_2} \right) \left( \int_0^\infty \tilde{u}^2 \, d\tilde{y} \right)^{-1} \right]^{3/7} \] (5.60)

Generally, the wave angle, measured relative to the \( \xi \)-direction, is given by \[ \Theta = \tan^{-1} \left( \frac{\tilde{\alpha}_c}{2 \tilde{L}_0} \right). \] (5.61) Further use of (5.37) gives \[ \Theta = \tan^{-1} \left( -\omega_e^{-1} \right). \] (5.62)

In the present case \[ \Theta \sim -\omega_e^{-1} + O(\omega_e^{-3}), \] (5.63) which means that the wave travels nearly parallel to the \( \xi \)-direction.

Finally, for $0 \ll \omega_e \ll 1$, \[ S \sim -\left[ \tilde{L}_0^3 \int_0^\infty \tilde{u}^2 \, d\tilde{y} \right] \omega_e^{-5} + \ldots \] (5.64) (provided $\tilde{L}_0 \gg \omega_e^{12/17}$), and \[ \tilde{\omega}_c \sim \left[ \left( \frac{d_1}{d_2} \right) \left( \int_0^\infty \tilde{u}^2 \, d\tilde{y} \right)^{-1} \right]^{3/17} \omega_e^{12/17} + \ldots. \] (5.65)
The critical value for $\mathcal{Z}_0$ has become arbitrarily close to zero, and hence, over the majority of the streamwise range, we have instability. The wave-angle expansion, this time, is

$$\theta \sim (-\pi/2) + O(\omega^2), \quad (5.66)$$

so that the wave travels nearly normal to the $x$-direction, and in an opposite direction to the external crossflow.

5.3 - Small $|\beta|$ Analysis

The diagram in Section 5.1 illustrates two distinct cases occurring for $|\beta| << 1$, corresponding to the upper and lower branches. We consider the weakly nonlinear analysis in each case separately.

(A) Lower Branch

It is discovered here that the $O(1)$ crossflow effects have no influence on the main flow solution, because the extreme smallness of $|\beta|$ forces the contribution $\beta w_x$ to become negligible. Consequently, (4.61a-g), (4.62a-d), (4.63a-f) above, which describe small-crossflow interactions on the lower branch, hold here. Therefore, ultimately, we anticipate an algebraic finite-distance singularity in the flow solution.
In this instance, as $|\beta| \to 0$, it is found that

$$\lambda \sim (d_1/d_2)^{3/4} + O(|\beta|)$$  \hfill (5.67)

using the general eigenrelation (4.1). Then consideration of (2.40f,g), (2.41a-d) in the limit of $|\beta| \ll 1$, yields the controlling vortex-wave interaction equations

$$\hat{\mu} \hat{\gamma} - \hat{\varphi} \hat{\nu} = -i \hat{\omega}$$  \hfill (5.68a)

$$\hat{\omega} \hat{\nu} - \hat{\varphi} \hat{\mu} = 0$$  \hfill (5.68b)

with

$$\hat{\mu} (\hat{\gamma}, \infty) = 0, \hat{\nu} (\hat{\gamma}, \infty) = 0$$  \hfill (5.68c)

$$\hat{\mu} (\hat{\gamma}, 0) = 0, \hat{\nu} (\hat{\gamma}, 0) = -i k_0 \hat{c}_1 \hat{c}_3$$  \hfill (5.68d)

where $k = 1$  \hfill (5.68e)

and

$$\frac{d}{d\hat{\gamma}} \hat{c}_1 \hat{c}_3 + e \hat{c}_1 \hat{c}_3 \hat{c}_2 \hat{c}_1 = 0$$  \hfill (5.68f)

$$\frac{d}{d\hat{\gamma}} \hat{c}_2 \hat{c}_3 + e \hat{c}_2 \hat{c}_3 \hat{c}_1 \hat{c}_2 = 0$$  \hfill (5.68g)

Here

$$[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \alpha_{33}, \hat{p}_{11}, \hat{p}_{22}, \hat{\lambda}_1] = \left[|\beta|^{-6/5} \frac{\hat{\gamma}}{\gamma}, \right.$$  \hfill (5.69a-h)

$$|\beta|^{-2/5} \frac{\hat{\gamma}}{\gamma}, \left. |\beta|^{-1/5} \frac{\hat{\gamma}}{\gamma}, |\beta|^{-1/5} \hat{\lambda}_{33}, \hat{p}_{11}, \hat{p}_{22}, |\beta|^{-1/5} \hat{\lambda}_1 \right] + \ldots$$

and

$$\hat{b}_n = b/\alpha, \hat{c}_n = c/\alpha, \text{(for } n = 1,2\text{)},$$  \hfill (5.70a-d)

where

$$\alpha = 2 \gamma_0 \hat{\gamma} D i^{-2/3}(d_1/d_2)^{11/4}/3 - 4 i (d_1/d_2)^{-3/4}/3$$  \hfill (5.71a)

$$b = 2 i^{1/3} \gamma_0 \hat{\gamma} D (d_1/d_2)/3 - 5/3$$  \hfill (5.71b)

$$c = 2 i^{1/3} \gamma_0 \hat{\gamma} D (d_1/d_2)/3 - 5/3$$  \hfill (5.71c)
The equations (5.68a-g) reflect the interactions involving two almost 2D waves and their induced vortex. Once again we note the passiveness of $\omega_\infty$, despite being $O(1)$, due to the decreased size of $|\beta|$.

Regarding the interaction coefficients (5.70a-d), we find

$$ \hat{b}_{\alpha r} = \hat{c}_{\alpha r} = -2 d_i (d_z - 1) (d_i / d_z)^{1/4} (i^{-1/3} r_i)_r / |a_{\alpha r}| > 0, $$

for $\alpha = 1, 2$, since $(i^{-1/3} r_i)_r (\equiv Re \cdot [i^{-1/3} r_i]) < 0$, and $d_i \approx 2.3, d_z \approx 2.3$.

Thus, because $\hat{K} > 0$, finite-distance break-up is eliminated, and with $\hat{b}_{\alpha r} > 0$ for $\alpha = 1, 2$, we deduce that the waves will decay exponentially, far downstream.

### 5.4 - $\sqrt{\lambda} \gg 1$ Analysis

So far in this thesis we have concentrated entirely on weakly nonlinear analysis problems that occur on the lower branch of the neutral $\lambda - Re$ curve, for $Re \gg 1$. There, the evaluations $\mathcal{J} = d_i \lambda^{1/3}, A_i' (\eta_0) / K = (d_i / d_z) i^{1/3} (d_i \approx 2.3, d_z \approx 2.3)$ hold, so that in particular, $\mathcal{J} \sim \lambda^{1/3}$ in the limit $\lambda \to \infty$. Physically, this means that a temporal-inertial balance is maintained in the viscous wall layer.
Here we are concerned with the linear development of dominantly neutral wave disturbances, for large values of the frequency \( \mathcal{L} \), which are not governed by the lower branch of the neutral \( \mathcal{L}-\mathcal{L}_{\mathcal{L}} \) curve. The linearised triple-deck structure is our starting point, where the dispersive eigenrelation

\[
\mathcal{L}^2 \frac{A_i'(\mathcal{L})}{H} = (i \mathcal{L})^{1/3} \left( \mathcal{L}^2 + \beta^2 \right)^{1/2} (\alpha + \beta \omega e/2)^2, \tag{5.72}
\]

\[
( \mathcal{L}_0 = -i^{1/3} \mathcal{L} / \mathcal{L}^{2/3}, H = \int_{\mathcal{L}_0}^{i^{1/3} \infty} A_i(q) dq )
\]

describes the boundary-layer instability due to a relatively small wave perturbation. For \( \mathcal{L} \to \infty \), we assume that \( \mathcal{L}_0 \to \infty \) (and hence \( \mathcal{L} \gg \mathcal{L}^{2/3} \)). The asymptotic nature of \( A_i'(\mathcal{L}_0)/H \) is deduced as follows. For any complex variable \( s \) satisfying the condition \( \arg(s) = \pi/6 \), we have

\[
A_i'(s)/H(s) = -\left( \int_{s}^{i^{1/3} \infty} H(s_i) ds_i \right)/H(s) - S, \tag{5.73}
\]

using properties of Airy functions.

Then for \( |s| \gg 1 \),

\[
A_i'(s)/H(s) \sim -s - A_i(s)/A_i'(s) + \ldots, \tag{5.74}
\]

using \( \text{l'Hopital's Rule} \) twice on the first term on the right-hand side of (5.73). Employing the standard results on Airy functions

\[
A_i(s) \sim |s|^{-1/4} e^{(2/3) \gamma s^{3/2}}, \quad A_i'(s) \sim -|s|^{-1/4} s^{1/2} e^{(2/3) \gamma s^{3/2}} \tag{5.75a,b}
\]
for \( |s| \to \infty \), we deduce that in our case

\[ A_i'(\zeta_0)/\zeta \sim -\zeta_0 + \zeta_0^{1/2} + \ldots \]  

(5.76)

for \( |\zeta_0| >> 1 \). Then, (5.72) infers the \( \alpha, \beta \)- expansions

\[ (\alpha, \beta) = \mathcal{N}^{1/2}(\alpha_0, \beta_0) + \mathcal{N}^{-1/2}(\alpha_1, \beta_1) + O(\mathcal{N}^{-3/2}) \]  

(5.77)

The lower deck, which we address next, is found to split into two basic regions: a viscous wall layer of the Stokes variety, and a thicker inviscid outer layer which recaptures the temporal-inertial balance.

Firstly, in the Stokes layer, where \( \gamma = \mathcal{N}^{-1/2} \gamma \), the disturbance velocity and pressure expand as

\[ \tilde{U} = \tilde{U}_0 + \mathcal{N}^{-1} \tilde{U}_1 + \ldots \]  

(5.78a)

\[ \tilde{V} = \tilde{V}_0 + \mathcal{N}^{-1} \tilde{V}_1 + \ldots \]  

(5.78b)

\[ \tilde{W} = \tilde{W}_0 + \mathcal{N}^{-1} \tilde{W}_1 + \ldots \]  

(5.78c)

\[ \tilde{P} = \mathcal{N}^{1/2} \tilde{P}_0 + \mathcal{N}^{-1/2} \tilde{P}_1 + \ldots \]  

(5.78d)

Substituting (5.78a-d) into the linearised form of (1.1a-d) yields

\[ -i \tilde{U}_{01} = -i \alpha_0 \tilde{P}_0 + \tilde{U}_0 \gamma \gamma \]  

(5.79a)

\[ -i \tilde{W}_{01} = -i \beta_0 \tilde{P}_0 / 2 + \tilde{W}_0 \gamma \gamma \]  

(5.79b)

\[ i \alpha_0 \tilde{V}_{01} + \tilde{V}_0 \gamma + i \beta_0 \tilde{W}_0 / 2 = 0 \]  

(5.79c)

subject to

\[ \tilde{U}_{01} = \tilde{V}_{01} = \tilde{W}_{01} = 0 \text{ at } \gamma = 0, \]  

(5.79d)

where

\[ \tilde{u}_0 = \tilde{u}_0 e + c.c., \]  

(5.80)
and so on, and

\[ E \equiv \exp \left[ i (\mathbf{x} \cdot \mathbf{x} + \mathbf{r} \cdot \mathbf{z} / 2 - \mathbf{r} \cdot \mathbf{r}) \right]. \]  

Here

\[ (\mathbf{x}, \mathbf{z}, \mathbf{r}) \equiv (\mathbf{r}^{1/2} \chi, \mathbf{r}^{1/2} \mathbf{z}, \mathbf{r} \mathbf{T}), \]  

and

\[ (\mathbf{r}, \mathbf{p}) = (\mathbf{r}_0, \mathbf{p}_0) + \mathbf{r}^{-1} (\mathbf{r}_1, \mathbf{p}_1) + O(\mathbf{r}^{-2}). \]  

Solving (5.79a-d) gives

\[ \tilde{U}_{01} = \mathbf{r}_0 \tilde{P}_{01} \left( 1 - e^{-i \tilde{r}^2} \right), \]  

\[ \tilde{V}_{01} = -i \mathbf{r}_0^2 \tilde{P}_{01} (\tilde{r} - r^{-1} e^{i \tilde{r}^2} + r^{-1}), \]  

\[ \tilde{W}_{01} = \beta_0 \tilde{P}_{01} \left( 1 - e^{-i \tilde{r}^2} \right) / 2, \]  

where \( \tilde{r} \equiv e^{3 \pi i / 4}, \mathbf{r}_0^2 \equiv (\mathbf{r}_0^2 + \beta_0^2 / 4) \) and \( \tilde{P}_{01} \) is \( \tilde{r} \)-independent.

Secondly, in the outer inviscid (intermediate) layer, where \( \mathbf{r} = \mathbf{r}^{1/2} \tilde{\mathbf{r}} \), we express \((\tilde{U}, \tilde{V}, \tilde{W}, \tilde{P})\) as

\[ \tilde{U} = \tilde{U}_0 + \mathbf{r}^{-1} \tilde{U}_1 + \ldots, \]  

\[ \tilde{V} = \mathbf{r} \tilde{V}_0 + \tilde{V}_1 + \ldots, \]  

\[ \tilde{W} = \tilde{W}_0 + \mathbf{r}^{-1} \tilde{W}_1 + \ldots, \]  

\[ \tilde{P} = \mathbf{r}^{1/2} \tilde{P}_0 + \mathbf{r}^{-1/2} \tilde{P}_1 + \ldots. \]  

The controlling equations are found to be

\[ i(\mathbf{r}_0 \tilde{r} - 1) \hat{\mathbf{r}}_{01} + \hat{\mathbf{r}}_{01} = -i \mathbf{r}_0 \hat{\mathbf{r}}_{01}, \]  

\[ i(\mathbf{r}_0 \tilde{r} - 1) \hat{\mathbf{r}}_{01} = -i \beta_0 \hat{\mathbf{r}}_{01} / 2, \]  

\[ i \mathbf{r}_0 \hat{\mathbf{r}}_{01} + \hat{\mathbf{r}}_{01} \mathbf{r} + i \beta_0 \hat{\mathbf{r}}_{01} / 2 = 0. \]
subject to
\[ \hat{U}_{01} \rightarrow A_{01}, \hat{W}_{01} \sim \hat{y}^{-1} \text{ as } \hat{y} \rightarrow \infty, \quad (5.86d) \]
and Stokes layer matching:
\[ (5.86e) \]

\[ i(2 \hat{y} - 1) \hat{U}_{11} + i \hat{y} \hat{U}_{01} + \hat{V}_{11} = -i \hat{y}_0 \hat{P}_{11} - i \hat{y}_1 \hat{P}_{01}, \quad (5.87a) \]
\[ i(2 \hat{y} - 1) \hat{W}_{11} + i \hat{y} \hat{W}_{01} = -i \beta_0 \hat{P}_{11} / 2 - i \beta_1 \hat{P}_{01} / 2, \quad (5.87b) \]
\[ i \hat{y}_0 \hat{U}_{11} + i \hat{y}_1 \hat{U}_{01} + \hat{V}_{11} + i \beta_0 \hat{W}_{11} / 2 + i \beta_1 \hat{W}_{01} / 2 = 0, \quad (5.87c) \]

subject to
\[ \hat{U}_{11} \rightarrow A_{11}, \hat{W}_{11} \sim \hat{y}^{-1} \text{ as } \hat{y} \rightarrow \infty, \quad (5.87d) \]
and Stokes layer matching,
\[ (5.87e) \]
at first and second order, respectively. Here the negative boundary layer displacement is given by
\[ \bar{A} = A_0 + i \bar{A}_1 + \ldots, \quad (5.88) \]
and we are employing the splitting
\[ \hat{U}_0 = \hat{U}_{01} E + \text{c.c., } \hat{U}_1 = \hat{U}_{11} E + \text{c.c.}, \quad (5.89a,b) \]
etc. The solution of (5.86a-e) is
\[ \hat{U}_{01} = \left[ \beta_0^2 \left\{ \hat{y} \hat{y}_0 - (2 \hat{y}_0 \hat{y} - 1) \right\} \right] \hat{P}_{01} + A_{01}, \quad (5.90a) \]
\[ \hat{U}_{01} = -i \hat{y}_0 \hat{A}_{01}, \quad (5.90b) \]
\[ \hat{W}_{01} = -i \left[ \beta_0 \left\{ (2 \hat{y} - 1) \hat{y} - 1 \right\} \right] \hat{P}_{01}, \quad (5.90c) \]
with the associated compatibility equation
\[
(\lambda_0^2 + \beta_0^2/4) \tilde{p}_0 = \lambda_0 A_{01}, \tag{5.91}
\]

being necessary to ensure a successful match with the Stokes layer solutions. Next, (5.87a-e) bear the results
\[
\hat{u}_{11} = \left[ \frac{\beta_0^2}{4\lambda_0 (\lambda_0 \hat{Y} - 1)} \right] \tilde{p}_0 + \left[ \frac{\beta_0 (\beta_0 \lambda_0 - \beta_0 \lambda_1)/2 \lambda_0^2 + \beta_0 \gamma_{1/4}}{(\lambda_0 \hat{Y} - 1)} \right] \tilde{p}_0 + A_{11},
\]
\[
\hat{v}_{11} = -i (\lambda_0 A_1 + \lambda_1 A_0) \hat{Y} - i \gamma_{1/4} \tilde{p}_0,
\]
\[
\hat{w}_{11} = - (\beta_0 \tilde{p}_0 + \beta_1 \tilde{p}_0) / 2 (\lambda_0 \hat{Y} - 1) + \left[ \frac{\lambda_0 \beta_0 \hat{Y}}{2 (\lambda_0 \hat{Y} - 1)} \right] \tilde{p}_0, \tag{5.92c}
\]

where this time we require
\[
(\lambda_0^2 + \beta_0^2/4) \tilde{p}_0 = \lambda_0 A_{11} + \left[ \frac{(\lambda_0 \lambda_0^{-1} + \gamma_{1/4}) (\lambda_0^2 + \beta_0^2/4)}{2 \lambda_0^2 + \beta_0 \gamma_{1/4}} \right] \tilde{p}_0,
\]
\[
\tag{5.93}
\]

to hold, in order to secure a correct match with the Stokes layer solutions. Non-neutral effects are present at this order, and given, in effect, by the complex constant \( \gamma \), defined above. The disturbance velocities as given in (5.90a-c), (5.92a-c) above, are valid everywhere except at \( \lambda \hat{Y} = \lambda_0 \), where singularities occur. To resolve these, a viscous critical layer of relative thickness \( O(\lambda_0^{-1}) \) is introduced around \( \lambda \hat{Y} = \lambda_0 \), enabling the singular effects to be smoothed out. We do not give details of this layer here, since it has no impact on the linear stability analysis.
In the upper deck, where the normal coordinate is
\( y' = x^{1/2} y \), the wave-pressure \( p' \) has the expansion
\[
p' = \mathcal{N}^{1/2} p_0' + \mathcal{N}^{1/2} p_1' + \ldots 
\] (5.94)

Inserting (5.94) into the upper-deck equations (2.25a-d) of Chapter 2, yields the first and second level systems of equations
\[
\frac{\partial^2 p_{01}'}{\partial y''^2} - \left( \omega_0^2 + \beta_0^2/4 \right) p_{01}' = 0, \quad (5.95a)
\]
with \( p_{01}' \to 0 \) as \( y'' \to \infty \), \( (5.95b) \)
\[
p_{01}' \to \tilde{p}_{01}, \quad p_{01}' y'' \to -\left( \omega_0 + \beta_0 \omega_e/2 \right)^2 A_{01} \text{ as } y'' \to 0; \quad (5.95c,d)
\]
and
\[
\frac{\partial^2 p_{11}'}{\partial y''^2} - \left( \omega_0^2 + \beta_0^2/4 \right) p_{11}' = 2 \left( \omega_0 \omega_1 + \beta_0 \beta_1/4 \right) p_{01}', \quad (5.96a)
\]
with \( p_{11}' \to 0 \) as \( y'' \to \infty \), \( (5.96b) \)
\[
p_{11}' \to \tilde{p}_{11}, \quad p_{11}' y'' \to -\left( \omega_0 + \beta_0 \omega_e/2 \right)^2 A_{11}
-2 \left( \omega_0 + \beta_0 \omega_e/2 \right) \left( \omega_1 + \beta_1 \omega_e/2 \right) A_{01} \text{ as } y'' \to 0. \quad (5.96c,d)
\]
Here the wave-like expansions
\[
p_{01}' = p_{01}'(y'')E + \text{c.c.} \quad p_{11}' = p_{11}'(y'')E + \text{c.c.} \quad (5.97a,b)
\]
have been used. Equations (5.95a-d), (5.96a-d) submit the pressure-displacement laws
\[
\left( \omega_0^2 + \beta_0^2/4 \right)^{1/2} \tilde{p}_{01} = \left( \omega_0 + \beta_0 \omega_e/2 \right)^2 A_{01}, \quad (5.98)
\]
\[
\left( \omega_0^2 + \beta_0^2/4 \right)^{1/2} \tilde{p}_{11} + 2 \left[ \left( \omega_0 \omega_1 + \beta_0 \beta_1/4 \right) / \left( \omega_0^2 + \beta_0^2/4 \right) \right]^{1/2} \tilde{p}_{01}
= \left( \omega_0 + \beta_0 \omega_e/2 \right)^2 A_{11} + 2 \left( \omega_0 + \beta_0 \omega_e/2 \right) \left( \omega_1 + \beta_1 \omega_e/2 \right) A_{01}, \quad (5.99)
\]
respectively. Then, the dominant stability equation

\[ \omega_0 = \left( \alpha_0^2 + \beta_0^2 / 4 \right)^{1/2} \left( \omega_0 + \beta_0 \, \omega_e / 2 \right)^2, \quad (5.100) \]

is obtained by combining (5.91) and (5.98), and eliminating \( \tilde{P}_0, A_0 \).

We illustrate the neutral eigenrelation (5.100), in the form of an \( \omega-\beta \) curve, for fixed values of \( \omega_e \), as follows:
There are immediate similarities between this curve, and a typical one representing the lower branch of the neutral \( \lambda - \beta \) curve for \( \omega_e \neq 0 \) (see Chapter 4).

Firstly, there is the "cut-off" feature for positive \( \beta \), where for \( \beta > \beta_c \) (say), no neutral modes exist. Secondly, the curve develops two branches which in particular, for negative \( \beta \), lie above and below the dotted curve \( \beta = (-2 \omega e^{-1}) \lambda \). In the limit of \( \lambda \gg 1 \), it is found that

\[
\beta \sim -(2 \omega e^{-1}) \lambda \pm \tilde{\beta} + O(\lambda^{-1}),
\]

\[\text{(5.101a,b)}\]

where

\[
\tilde{\beta} = 2 \omega e^{-1/2}/(1+\omega^2)^{1/4},
\]

\[\text{(5.102)}\]

for the upper branch and lower branch, in turn.

Next, we combine (5.93) and (5.99) and eliminate \( \bar{P}_{\lambda \lambda}, A_{\lambda \lambda} \) to obtain

\[
2 (\omega_0^2 + \beta_0^2/4) (\omega_1 + \beta_1 \omega e/2)
+ (\beta_1/4 \omega_0) (\omega_0 \beta_1 - \beta_0 \omega_1) (\omega_0 + \beta_0 \omega e/2)
= \omega_0 r^{-1} (\omega_0^2 + \beta_0^2/4) (\omega_0 + \beta_0 \omega e/2).
\]

\[\text{(5.103)}\]

This equation has no solutions for which \( \omega_1 \) and \( \beta_1 \) are both real, because \( r \) is complex; therefore, it governs the long-range stability of the boundary layer.
Further use of (5.100) enables us to reduce (5.103) to

\[
(2/\omega_{0}^{2}S_{0}^4 - \beta_{0}^2S_{0}/4) \omega_{1} + \left( \omega_{0}/\omega_{0}^{2}S_{0}^4 + \beta_{0} \omega_{0} S_{0}/4 \right) \beta_{i} = (-1 - i) \left[ \frac{1}{2} \frac{\omega_{0}^{2}S_{0}^2}{2} \right] \] (5.104)

where

\[ S_{0} = (1 + \beta_{0} \omega_{0} / 2 \omega_{0}) \] (5.105)

and \( \Gamma \) has been evaluated. Writing \( \omega_{1} \) and \( \beta_{i} \) as

\[
\omega_{1} = \omega_{1r} + i \omega_{1i}, \quad (5.106a)
\]
\[
\beta_{i} = \beta_{ir} + i \beta_{ii}, \quad (5.106b)
\]

where \( \omega_{1r}, \omega_{1i}, \beta_{ir}, \beta_{ii} \) are unknown real constants,

the imaginary components of (5.104) equate in the manner

\[
(2/\omega_{0}^{2}S_{0}^4 - \beta_{0}^2S_{0}/4) \omega_{1i} + \left( \omega_{0}/\omega_{0}^{2}S_{0}^4 + \beta_{0} \omega_{0} S_{0}/4 \right) \beta_{ii} = -1 \left[ \frac{1}{2} \frac{\omega_{0}^{2}S_{0}^2}{2} \right] \] (5.107)

We note that the dominant streamwise- and spanwise-
growth factors are \( e^{-\omega_{1r} \hat{X}}, e^{-\beta_{ii} \hat{Z}/2} \), respectively,

occurring over the relatively long triple-deck scales

\[
(\hat{X}, \hat{Z}) = \mathcal{N}^{-1/2}(X, Z). \] (5.108a,b)

So clearly we have instability if either \( \omega_{1i} < 0 \) or
\( \beta_{ii} < 0 \). Otherwise, the flow is stable.
Two special cases are now considered.

(A) \( \beta_{ii} = 0 \).

Here (5.107) simplifies to

\[
R_o \mathcal{A}_{ii} = -1 / (2^{1/2} S_0^3), \quad (5.109)
\]

where

\[
R_o = \left(2 / \omega_0^2 S_0^4 - \beta_0^2 S_0 / 4 \right). \quad (5.110)
\]

Our aim is to determine how the sign of \( \mathcal{A}_{ii} \) varies
with the wave-angle \( \theta_0 \), where

\[
\theta_0 \leq \tan^{-1} \left( \beta_0 / 2 \omega_0 \right) \quad (5.111)
\]

Firstly, we note that \( S_0 = 0 \) if and only if \( \theta_0 =
\tan^{-1} (-\omega_0) = \theta_1 \), say. For \( \theta > \theta_1 \), \( S_0 \) is positive
whereas for \( \theta < \theta_1 \), \( S_0 \) is negative. At first sight,
there appear to be no easy analytical means of determining
the zero(es), if any exist, of \( R_o \). However, deeper
investigation reveals that

\[
R_o = 0 \quad \text{exactly when} \quad \frac{\partial R_o}{\partial \omega_0} = 0.
\]
(The term on the left-hand side of (5.104) is effectively obtained through streamwise perturbation of the leading order eigenrelation, say \( \beta_0 = \rho_0 (\omega_0) \), (apart from a multiplicative factor), i.e., we let \( \lambda_0 \rightarrow \lambda_0 + \sigma^{-1} \lambda_1 + \ldots \), and keep \( \beta_0 \) fixed, so that \( \beta_0 = \beta_0 (\lambda_0 + \sigma^{-1} \lambda_1 + \ldots) \).

At second order this yields the quantity \( \lambda_1 (\partial \beta_0 / \partial \lambda_0) \) (via a Taylor expansion). Hence the result). The above diagram indicates one such point, \( \beta = \beta_c \), the maximum \( \beta_0 \) value (with corresponding angle \( \theta_\circ \)) where \( \partial \beta_0 / \partial \lambda_0 = 0 \). For \( \theta_\circ > \theta_c \), \( \lambda_0 > 0 \) whereas for \( \theta_\circ < \theta_c \), \( \lambda_0 < 0 \).

Thus, we may represent the long-range stability in the following way for any \( \omega \in \mathbb{R} \),

\[ \begin{array}{c}
\text{Stable.} \\
\text{--- Unstable.}
\end{array} \]
where we observe that the disturbance is

\[
\begin{align*}
\text{unstable if } & \theta_1 < \theta_0 < \theta_c ; \\
\text{stable if } & \theta_0 < \theta_1 \text{ or } \theta_0 > \theta_c ; \\
\text{undefined at } & \theta_0 = \theta_1 \text{ and } \theta_0 = \theta_c . (|\text{d}L_1| \rightarrow \infty, \text{there}).
\end{align*}
\]

(B) \(L_1 = 0.\)

Now we find

\[Q_0 \beta_{1,i} = -1/(2^{1/2} S_0^3), \quad (5.112)\]

where

\[Q_0 = \left( \frac{\omega_e / R_0^2 S_0^4 + \beta_0 \lambda_0 S_0 / 4}{} \right). \quad (5.113)\]

Again, the sign of \( \beta_{1,i} \) is sought for variable \( \theta_0 \).

Following a similar argument to one described in (A) above, we have

\[Q_0 = 0 \quad \text{if and only if } \frac{d\lambda_0}{d\beta_0} = 0.\]

The \( \lambda_0 - \beta_0 \) curve drawn for sufficiently small crossflow illustrates two points on the negative portion of the upper branch where the gradient vanishes. In a similar vein to the 'critical crossflow' theory described in Section 4.2 above, there exists a threshold value of \( \omega_e \), say \( \omega_{ec} \), above which these points vanish.

Calculations reveal that \( \omega_{ec} = (\sqrt{24})^{-1} \). For \( \omega_e > \omega_{ec} \), \( Q_0 \)
is positive for all $\theta_0$; for $0 < \omega_c < \omega_{sc}$, $Q_{\theta}$ is negative where $\theta_2 < \theta < \theta_3$ (\(\theta_2, \theta_3\) being the positions where zero gradient occurs), but positive otherwise.

So now the long-range stability falls into two distinct categories:

(i) $0 < \omega_c < \omega_{sc}$.

Here the disturbance is

\[
\begin{cases}
\text{unstable if } \theta_1 < \theta_0 < \theta_2 \text{ and } \theta_0 > \theta_3; \\
\text{stable if } \theta_2 < \theta_0 < \theta_3 \text{ and } \theta_0 < \theta_1; \\
\text{undefined at } \theta_0 = \theta_1, \theta_0 = \theta_2 \text{ and } \theta_0 = \theta_3.
\end{cases}
\]
Here the disturbance is

\[
\begin{cases}
\text{unstable if } \Theta_0 > \Theta_1; \\
\text{stable if } \Theta_0 < \Theta_1; \\
\text{undefined at } \Theta_0 = \Theta_1.
\end{cases}
\]

The present high-frequency analysis would appear to remain valid for large \( L_0 \), until \( L_0 \sim Re^{1/24} \), where normal wave acceleration in the main part of the boundary layer becomes significant. At this stage,
the corresponding analysis has similarities to that in
Section 5.2 above, and in particular the feature of
'loop closure' is once again obtained. In the next
chapter we consider certain aspects of the 3D triple-
deck equations, and apply the results to large
disturbances in high-frequency regimes.
Here, in a brief diversion from vortex/wave interaction theory, we assess certain stability features of the 3D boundary layer associated with fully nonlinear disturbances. The disturbances considered are of a triple-deck nature, so that the normalised interactive boundary-layer equations

\begin{align}
U_T + U U_x + V U_y + W U_z &= -P_x + U Y Y, \\
W_T + U W_x + V W_y + W W_z &= -P_z + W Y Y, \\
U_x + V_y + W_z &= 0,
\end{align}

subject to

\begin{align}
U = V = W &= 0 \text{ at } Y = 0, \\
U \sim Y + A(x, z, T), W \sim Y^{-1} \text{ as } Y \to \infty,
\end{align}

and such that

\begin{equation}
\rho(x, z, T) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(u_0, \frac{\partial}{\partial \eta} + u_\eta, \frac{\partial}{\partial \eta})^2 A(\tau, z, T)}{[x - \xi]^2 + (z - \eta)^2} d\eta d\xi,
\end{equation}

hold yet again. Here \((U, V, W, \rho, A)\) denotes the lower-deck disturbance profile, \((u_0, \xi, u_\eta, w_\eta)\) denotes the upper-deck basic-flow velocity and \((\xi, \eta, \zeta, \tau)\) are the characteristic triple-deck scales.
No analytical methods of solution of the above equations are known to exist. The main difficulty is the presence of the nonlinear acceleration terms, i.e. $UU_x, UU_{xx}$ and so on, but further difficulties are posed by the complex nature of the pressure-displacement law (6.1f) and the three-dimensionality involved. In the regime where $|X| \ll 1$ (described next), it is possible to reduce the pressure-displacement law to a single-integral equation and to obtain a quasi-two-dimensional form for the interactive equations. (This proves to be highly useful in some related large-frequency analysis (discussed below) where ultimately a periodic, equilibrium solution is acquired for the displacement decrement).

In response to shortening the typical streamwise length scale, i.e. letting

$$X = \varepsilon \bar{X}$$  \hspace{1cm} (6.2)

where $\varepsilon$ is a small, positive constant and $\bar{X}$ is $O(1)$, the perturbation quantities expand in the manner

$$U = \varepsilon^{1/3} \bar{U} + \ldots, \hspace{1cm} (6.3a)$$
$$V = \varepsilon^{-1/3} \bar{V} + \ldots, \hspace{1cm} (6.3b)$$
$$W = \varepsilon^{1/3} \bar{W} + \ldots, \hspace{1cm} (6.3c)$$
$$\rho = \varepsilon^{2/3} \bar{\rho} + \ldots, \hspace{1cm} (6.3d)$$
$$A = \varepsilon^{1/3} \bar{A} + \ldots, \hspace{1cm} (6.3e)$$
and these depend primarily on
\[ \left[ \tilde{Y}, \tilde{Z}, \tilde{T} \right] \equiv \left[ e^{-1/3} Y, e^{-1} Z, e^{-2/3} T \right] \]  
(6.4 a-c)

and \( \bar{x} \). Substituting the above expansions into 
(6.1a-f) yields the principal equations

\[
\begin{align*}
\bar{U}_\tau + \bar{U}_\tau \bar{U}_x + \bar{V} \bar{U}_\tau + \bar{W} \bar{U}_\tau = & -\bar{p}_x + \bar{U}_{\tau\tau}, \quad (6.5a) \\
\bar{W}_\tau + \bar{U}_\tau \bar{W}_x + \bar{V} \bar{W}_\tau + \bar{W} \bar{W}_x = & -\bar{p}_z + \bar{W}_{\tau\tau}, \quad (6.5b) \\
\bar{U}_x + \bar{V} \bar{U}_x + \bar{W}_x = & 0, \quad (6.5c)
\end{align*}
\]

subject to

\[ \bar{U} = \bar{V} = \bar{W} = 0 \quad \Rightarrow \quad \bar{Y} = 0, \]  
(6.5d)

\[ \bar{U} \sim \bar{Y} + \mathcal{A}(\bar{x}, \bar{z}, \bar{T}, \hat{x}, \hat{z}, \ldots), \bar{W} \sim \bar{Y}^{-1} \quad \text{as} \quad \bar{Y} \to \infty, \]  
(6.5e)

with

\[
\bar{p} (\bar{x}, \bar{z}, \bar{T}, \hat{x}, \hat{z}, \ldots) = -\frac{1}{2\pi i} (w_e \partial / \partial \hat{x} + w_e \partial / \partial \hat{z})^2 \int \int \frac{\mathcal{A}(\xi, \eta, \bar{T}, \hat{x}, \hat{z}, \ldots)}{((\bar{x} - \eta)^2 + (\bar{z} - \eta)^2)^{1/2}} \partial \xi \partial \eta. \]
(6.5f)

Here the slower scales

\[ \left[ \hat{X}, \hat{Z} \right] = e^{-1/3} \left[ X, Z \right] \]  
(6.6a,b)

have been brought into play in (6.5f), because this equation is, in fact, the second-order balance of (6.1f)
above; the leading-order equation there is a solvability condition for $\tilde{A}$, namely

$$
\left( \frac{\partial A}{\partial x} + \frac{\partial w}{\partial z}\right) \tilde{A} = 0. 
$$

(6.7)

It is assumed that this property has the more general form

$$
(\frac{\partial A}{\partial x} + \frac{\partial w}{\partial z}) \tilde{Q} = 0, \quad (6.8)
$$

where $\tilde{Q}$ is any of $[\tilde{U}, \tilde{V}, \tilde{W}, \tilde{P}, \tilde{A}]$. (We note that in the corresponding linear theory, i.e. Section 5.1 above, the result $\beta = -2\tilde{w}\tilde{u}/w$ was obtained for the dominant wavenumbers $\tilde{z}, \tilde{p}$ (except $u_e = 1$ was imposed there). This is easily verified here by letting $\partial/\partial x \rightarrow i\tilde{x}, \partial/\partial z \rightarrow i\tilde{p}/2$ in the generalised equation (6.8)). Therefore, as a consequence of (6.8),

$$
\tilde{Q} = \tilde{Q} \left( \tilde{w} \tilde{x} - u_e \tilde{x}, \tilde{y}, \tilde{T}, \tilde{x}, \tilde{z}, \ldots \right). 
$$

(6.9)

This suggests that we introduce the skewed orthogonal coordinates
\[ m' = \left( w_e \bar{X} - u_e \bar{Z} \right) / q_e, \quad \bar{m} = \left( u_e \bar{X} + w_e \bar{Z} \right) / q_e, \quad (6.10a, b) \]

to replace the fast scales \( \bar{X}, \bar{Z} \) where

\[ q_e = \sqrt{u_e^2 + w_e^2} \quad (6.11) \]

is the magnitude of the basic flow in the outer stream.

Clearly then, \( \bar{Q} \) has no \( \bar{m} \)-variations. The diagram below illustrates the transformation.
From the above figure it is clear that the disturbance-velocity component in the $\vec{m}$-direction is given by

$$U \cos \left[ \tan^{-1} \left( \frac{u_e}{w_e} \right) \right] - \vec{w} \sin \left[ \tan^{-1} \left( \frac{u_e}{w_e} \right) \right] = \vec{u} \text{ (say)},$$

which reduces to

$$\vec{u} = \left( w_e \vec{u} - u_e \vec{w} \right) / q_e.$$ \hspace{1cm} (6.12)

Secondly, the disturbance-velocity component in the $\vec{n}$-direction, say $\vec{w}$, is

$$U \sin \left[ \tan^{-1} \left( \frac{u_e}{w_e} \right) \right] + \vec{w} \cos \left[ \tan^{-1} \left( \frac{u_e}{w_e} \right) \right],$$

so that

$$\vec{w} = \left( u_e \vec{u} + w_e \vec{w} \right) / q_e.$$ \hspace{1cm} (6.13)

In a similar vein to (6.10a,b) above, we apply a change of coordinates to the slower scales $\hat{x}, \hat{z}$, that is

$$\hat{u} = \left( w_e \hat{x} - u_e \hat{z} \right) / q_e \quad \hat{v} = \left( u_e \hat{x} + w_e \hat{z} \right) / q_e.$$ \hspace{1cm} (6.14a,b)
Then, in particular, we have the property

\[(u_e \, \partial / \partial \hat{x} + w_e \, \partial / \partial \hat{z}) \equiv q_e \, \partial / \partial \hat{\rho}. \tag{6.15}\]

We are now in a position to transform (6.5a-f) above into a skewed form that includes the quantities \(\tilde{u}, \tilde{w}\) and the coordinates \(\tilde{m}, \tilde{n}, \tilde{\tau}\) and \(\tilde{\tau}\). We find, after simple manipulation, that

\[
\begin{align*}
\tilde{u}_{\tau} + \tilde{u} \, \tilde{u}_{\tilde{m}} + \nabla \cdot \tilde{u}_{\tilde{\tau}} &= -\tilde{\rho}_{\tilde{m}} + \tilde{u}_{\tilde{\tau}}, \tag{6.16a} \\
\tilde{w}_{\tau} + \tilde{u} \, \tilde{w}_{\tilde{m}} + \nabla \cdot \tilde{w}_{\tilde{\tau}} &= \tilde{w}_{\tilde{\tau}}, \tag{6.16b} \\
\tilde{u}_{\tilde{m}} + \nabla_{\tilde{\tau}} &= 0, \tag{6.16c}
\end{align*}
\]

subject to

\[
\tilde{u} = \tilde{v} = \tilde{w} = 0 \text{ at } \tilde{\tau} = 0, \tag{6.16d}
\]

\[
\tilde{u} \sim \frac{w_e}{q_e} (\tilde{\tau} + \bar{\alpha}(\tilde{m}, \tilde{\tau}, \hat{m}, \hat{\tau}, \ldots)), \quad \tilde{w} \sim \frac{w_e}{q_e} (\tilde{\tau} + \bar{\alpha}(\tilde{m}, \tilde{\tau}, \hat{m}, \hat{\tau}, \ldots))
\] as \(\tilde{\tau} \to \infty, \tag{6.16e}\)

with

\[
\bar{P}(\tilde{m}, \tilde{\tau}, \hat{m}, \hat{\tau}, \ldots) = \frac{q_e^2 \, \delta}{\pi \, \partial \hat{\rho}^2} \left\{ \int_{-\infty}^{\infty} \left\{ \int_0^\infty \frac{\bar{\alpha}(s, \tilde{\tau}, \hat{m}, \hat{\tau}, \ldots) \, ds}{(\tilde{m} - \tilde{\tau})} \right\} \, d\tilde{\tau} \right\}. \tag{6.16f}
\]

where the lower limit, \(\delta\), on the inner integral in (6.16f) is problem-dependent in general.
Here we have used the derivative transformations

\[ \frac{\partial}{\partial \bar{x}} \rightarrow \left( \frac{\omega}{q_e} \right) \frac{\partial}{\partial \bar{m}} + \left( \frac{\omega}{q_e} \right) \frac{\partial}{\partial \bar{n}}, \quad (6.17a) \]

\[ \frac{\partial}{\partial \bar{z}} \rightarrow -\left( \frac{\omega}{q_e} \right) \frac{\partial}{\partial \bar{m}} + \left( \frac{\omega}{q_e} \right) \frac{\partial}{\partial \bar{n}}, \quad (6.17b) \]

and (6.15) above. The new system has a two-dimensional nature as expected, because of the flow quantities' independence of \( \bar{\rho} \). Despite this though, the equations are still generally insoluble.

So, in an attempt to gain some rational insight, we next examine the effects of large frequency. Mathematically speaking, we seek solutions for \( T \sim \mathcal{R}^{-1} \) where \( \mathcal{R} \) is a large parameter. As a result, preserving nonlinearity in (6.16a-f), we require

\[ \bar{U} = \mathcal{R}^{1/2} \bar{U}_0 (\bar{m}_0, \bar{\gamma}, \hat{\bar{m}}_0, \bar{\bar{n}}_0, \bar{T}_0, ...) + \ldots \quad (6.18a) \]

\[ \bar{V} = \mathcal{R}^{3/2} \bar{V}_0 (\bar{m}_0, \bar{\gamma}, \hat{\bar{m}}_0, \bar{\bar{n}}_0, \bar{T}_0, ...) + \ldots \quad (6.18b) \]

\[ \bar{W} = \mathcal{R}^{1/2} \bar{W}_0 (\bar{m}_0, \bar{\gamma}, \hat{\bar{m}}_0, \bar{\bar{n}}_0, \bar{T}_0, ...) + \ldots \quad (6.18c) \]

\[ \bar{P} = \mathcal{R} \bar{P}_0 (\bar{m}_0, \hat{\bar{m}}_0, \bar{\bar{n}}_0, \bar{T}_0, ...) + \ldots \quad (6.18d) \]

\[ \bar{A} = \mathcal{R}^{1/2} \bar{A}_0 (\bar{m}_0, \hat{\bar{m}}_0, \bar{\bar{n}}_0, \bar{T}_0, ...) + \ldots \quad (6.18e) \]
where \( \bar{\mu}, \bar{\nu}, \hat{\mu}, \hat{\nu} \) have been rescaled in the fashion
\[
(\bar{\mu}, \bar{\nu}, \hat{\mu}, \hat{\nu}) = \mathcal{R}^{-1/2}\left(\bar{\mu}_0, \bar{\nu}_0, \hat{\mu}_0, \hat{\nu}_0\right) + \ldots.
\] (6.19a-d)

while
\[
\bar{\gamma} = \mathcal{R}^{1/2} \hat{\gamma} \tag{6.20}
\]

and
\[
\bar{\tau} = \mathcal{R}^{-1} \bar{\tau}_0 + \ldots. \tag{6.21}
\]

Entering (6.18a-e) - (6.21) into (6.16a-f) and subsequently equating the leading order coefficients yields
\[
\bar{\omega}_0 \bar{\tau} + \bar{\omega}_0 \bar{\omega}_0 \hat{\nu}_0 + \bar{\nu}_0 \bar{\omega}_0 \hat{\gamma} = -\bar{\rho}_0 \bar{\omega}_0, \tag{6.22a}
\]
\[
\bar{\omega}_0 \bar{\tau} + \bar{\omega}_0 \bar{\omega}_0 \hat{\nu}_0 + \bar{\nu}_0 \bar{\omega}_0 \hat{\gamma} = 0, \tag{6.22b}
\]
\[
\bar{\omega}_0 \bar{\tau} + \bar{\nu}_0 \bar{\gamma} = 0, \tag{6.22c}
\]

subject to
\[
\bar{\nu}_0 = 0 \text{ at } \bar{\gamma} = 0, \tag{6.22d}
\]
\[
\bar{\omega}_0 \sim \frac{\omega}{\eta_0} (\bar{\gamma} + \bar{\alpha}_0), \quad \bar{\omega}_0 \sim \frac{\omega}{\eta_0} (\bar{\gamma} + \bar{\alpha}_0) \text{ as } \bar{\gamma} \rightarrow \infty, \tag{6.22e}
\]

with
\[
\bar{\rho}_0 = \frac{\eta_0^2}{\pi} \frac{\partial^2}{\partial \bar{\alpha}_0^2} \left\{ \int_{-\infty}^{\infty} \left[ \frac{1}{\mathcal{R}^2 \left(\bar{\mu}_0 - \bar{\gamma}\right)} \right] d\eta \right\}. \tag{6.22f}
\]
These equations resemble the Euler equations from classical inviscid theory, except of course that here the system is still interactive with the outer constraints being of the triple-deck kind. (Naturally, a viscous wall layer (of relative thickness $O(\alpha^{-1})$ here) lies beneath the current layer; therein, the nonlinear effects are still important but the basic-flow influence is negligible). It is the inviscidness of the above system which enables us to evaluate the negative displacement factor, $\bar{A}_0$, as the following work demonstrates.

We focus our attention on the 2D part of the above system of equations, i.e. we neglect the relatively unimportant equation for $\bar{W}_o$, (6.22b). To begin with we assume that the current layer, being inviscid, admits simple polynomial solutions for $\bar{U}_o$ and $\bar{V}_o$ which are regular near the wall. In view of the boundary conditions and the continuity equation above, it is proposed that

\[
\bar{U}_o = \frac{W_o}{q_e} (\bar{Y} + \bar{A}_o), \quad (6.23a)
\]

\[
\bar{V}_o = -\left(\frac{W_o}{q_e} \bar{A}_o \bar{a}_o\right) \bar{Y}, \quad (6.23b)
\]
are valid solutions. This would appear to be possible provided the solvability condition

\[
\frac{u_0}{q_e} (\vec{A}_0 \tau_0 + \frac{w_0}{q_e} \vec{A}_0 \vec{A}_0 \vec{m}_0) = -\vec{p}_0 \vec{m}_0,
\] (6.24)

arising from (6.22a), is satisfied. A second, independent expression for \( \vec{p}_0 \vec{m}_0 \) is

\[
\vec{p}_0 \vec{m}_0 = \frac{q_e^2 \partial^2}{\pi} \frac{\partial}{\partial \vec{m}_0^2} \left\{ \int_{-\infty}^{\infty} \frac{(\vec{A}_0 \delta_s)}{(\vec{m}_0 - \eta)} d\eta \right\}
\] (6.25)

by (6.22f) above, and this simplifies to

\[
\vec{p}_0 \vec{m}_0 = \frac{q_e^2 \partial^2}{\pi} \int_{-\infty}^{\infty} \frac{\vec{A}_0 (\eta, \tau_0, \vec{m}_0, \vec{n}_0, \ldots)}{(\vec{m}_0 - \eta)} d\eta
\] (6.26)

Hence, combining this with (6.24) above, we deduce that

\[
\frac{w_0}{q_e} \vec{A}_0 \tau_0 + \frac{w_0}{q_e} \vec{A}_0 \vec{A}_0 \vec{m}_0 = -\frac{q_e^2 \partial^2}{\pi} \int_{-\infty}^{\infty} \frac{\vec{A}_0}{(\vec{m}_0 - \eta)} d\eta.
\] (6.27)

This equation is analogous to the classical Benjamin-Ono equation (Benjamin (1967), Ono (1975)) for which
Benjamin obtained both a nonlinear periodic solution and a solitary wave solution. His results are only special cases however, and generalised forms for both types of solution have been obtained by Smith and Burggraf (1985). There, the authors essentially modified Benjamin's method of solution in each case to incorporate a hitherto unused constant. We proceed in a similar vein to this, and firstly seek a nonlinear periodic solution.

We start by defining the coordinate $\phi$ as

$$\phi = \frac{q_e^2}{\omega_e^2} \left( \bar{m}_o - \frac{C_i^* \omega_e}{q_e} \right),$$

where $C_i^*$ is some unknown, constant wave-speed factor, and endeavour to find solutions of the form

$$\bar{A}_o = \bar{A}_o(\phi, \bar{m}_o, \bar{n}_o, \ldots).$$

Thus

$$\frac{\partial}{\partial \bar{m}_o} \to -\frac{q_e^2 C_i^*}{\omega_e} \frac{\partial}{\partial \phi},$$

$$\frac{\partial}{\partial \bar{n}_o} \to \frac{q_e^2}{\omega_e} \frac{\partial}{\partial \phi},$$

so that (6.27) becomes

$$-C_i^* \bar{A}_o \phi + \bar{A}_o \bar{A}_o \phi = -\frac{q_e^2}{\pi} \frac{\partial}{\partial \phi} \left\{ \frac{q_e^2 \xi}{\omega_e^2} \int_{-\infty}^{\infty} \frac{\left( \int_{\tilde{\xi}}^\phi \tilde{A}_o \tilde{A}_o^* d\tilde{\eta} \right) d\eta}{(\bar{m}_o - \eta)} \right\}.$$
This we integrate with respect to $\phi$ to obtain

$$- C_0^k \bar{A}_0 + \frac{A_0^2}{2} + K = - \frac{q_e^2}{\pi \omega} \frac{\delta^2}{\delta \bar{A}_0^2} \int_{-\infty}^{\infty} \frac{(J_0^s \bar{A}_0 \, ds)}{(\bar{A}_0 - \eta)} \, d\eta, \quad (6.31)$$

where $K$ is an unknown constant.

Now, we express $\bar{A}_0$ as a Fourier series in $\phi$, such that

$$\bar{A}_0 = \sum_{n=-\infty}^{\infty} C_n e^{-i\pi n \phi / l}, \quad (6.32)$$

where

$$C_n = \frac{1}{2l} \int_{0}^{2l} \bar{A}_0(\phi, \ldots) e^{i\pi n \phi / l} \, d\phi \quad (6.33)$$

(for all integers $n$) is constant-valued, and $l$ depicts the unknown wavelength of $\bar{A}_0$. Now, upon entering (6.32) into (6.31) we find after some working that

$$C_1^k \sum_{n=-\infty}^{\infty} C_n e^{-i\pi n \phi / l} + \frac{1}{2} \sum_{n=-\infty}^{\infty} B_n e^{-i\pi n \phi / l} + K$$

$$= \frac{q_e^2 \ell}{\pi} \frac{\delta^2}{\delta \bar{A}_0^2} \sum_{n=0}^{\infty} \frac{C_n}{|n|} e^{-i\pi n \phi / l} \quad \sum_{n \neq 0}^{\infty} (6.34)$$
Here the standard result \[ \int_{-\infty}^{\infty} e^{i\pi x} \, dx = (\text{sgn} \cdot \pi) i\pi, \]
from complex analysis has been applied to the Cauchy integral in (6.31), and

\[ C_0 \hat{\alpha}_0 \hat{\alpha}_0 = 0 \quad (6.35) \]

must hold, in order that the aforesaid integral exists.
(Thus, \( C_0 = \hat{\alpha}_0 a_1(\hat{\alpha}_0) + a_2(\hat{\alpha}_0) \) in general, for arbitrary functions \( a_1, a_2 \).) Also here, we have expressed \( \hat{\alpha}_0^2 \)
as a Fourier series in the manner

\[ \hat{\alpha}_0^2 = \sum_{n=-\infty}^{\infty} \mathcal{B}_n e^{-i\pi n \phi/L} \quad (6.36) \]

where

\[ \mathcal{B}_n = \frac{1}{2L} \int_{-\phi}^{\phi} \hat{\alpha}_0^2(\phi, \ldots) e^{i\pi n \phi/L} \, d\phi, \quad (6.37) \]

for all integers \( n \). In fact, \( \mathcal{B}_n \) reduces to

\[ \mathcal{B}_n = \sum_{m=-\infty}^{\infty} C_m C_{n-m} \quad (6.38) \]

for each \( n \), using the Fourier-series definition for
\[ \bar{A}_1 \text{ in (6.37). Therefore, equating coefficients of } 
\ e^{-i\pi \eta / \ell} \text{ (for } n \neq 0 \text{) in (6.34) gives} 
\]
\[ \left[ C_1^* + \frac{q \lambda^2}{\pi n^2} \frac{\delta}{\delta \eta_{\omega}^2} \right] C_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} C_m C_{n-m}. \quad (6.39) \]

(We note that for \( n = 0 \), in (6.34), the corresponding coefficients merely produce a solution for the passive constant \( K \).)

The nonlinear expression on the right-hand side of (6.39) prevents us acquiring a direct solution for \( C_n \). To overcome this, we express \( C_n \) as

\[ C_n = \begin{cases} 
\Delta e^{-\rho|n|} e^{-i\pi \eta_{\omega}/(q \ell)} & \text{for } |n| > 0, \\
\Delta - \hat{C} & \text{for } n = 0, 
\end{cases} \quad (6.40) \]

and aim to solve the constants \( \Delta \) and \( \rho \) (the latter, we note, being positive to make \( C_n \) convergent as \( |n| \to \infty \)). The quantity \( \hat{C} \) is the crucial arbitrary constant omitted in Benjamin's original solution. Now (6.39) gives

\[ \left[ C_1^* - \frac{\pi |n|}{\ell} \right] \Delta e^{-\rho|n|} e^{-i\pi \eta_{\omega}/(q \ell)} 
\]
\[ = \frac{\Delta^2}{2} \left( S_n - 2\Delta^{-1} \hat{C} e^{-\rho|n|} \right) e^{-i\pi \eta_{\omega}/(q \ell)}, \quad (6.41) \]
where

\[ J_n = \sum_{m=-\infty}^{\infty} e^{-\rho |n-m|} e^{-\rho |n-m|} \quad (6.42) \]

For positive or negative \( \lambda \), in turn,

\[ J_n = \sum_{m=\lambda+1}^{\infty} e^{\pm \rho (n-m)} + \sum_{m=1}^{\infty} e^{\mp \rho n} \]

\[ + \sum_{m=0}^{\infty} e^{\mp \rho (n-2m)} \quad (6.43) \]

which reduces to

\[ J_n = \left\{ 1 + 2 \sum_{k=1}^{\infty} e^{-2k\rho} \right\} e^{-\ln 1 \rho} + \ln 1 e^{-\ln 1 \rho} \quad (6.44) \]

Moreover, the series contained in the braces is the standard representation of \( \text{coth} \rho \) and hence

\[ J_n = (|n| + \text{coth} \rho) e^{-|n| \rho} \quad (6.45) \]

Now equating coefficients of \( |n| e^{-|n| \rho} e^{-i\pi \lambda_0 / (2\rho) \epsilon} \) in (6.42) yields the solution

\[ \Delta = -2\pi / \nu \quad (6.46) \]
and likewise for \( e^{-in\lambda} \cdot e^{-\frac{n\pi \rho_0}{(q_x)}} \) we find that

\[
\rho = \coth^{-1}\left(-\frac{C^*}{\pi}\right),
\]

(6.47)

where

\[
C^* = c_i^* + \hat{C}
\]

(6.48)

must be negative to ensure that \( \rho > 0 \).

Having evaluated \( \Delta \) and \( \rho \) we insert (6.40) into (6.32) and simplify to find that

\[
\bar{A}_0 = \hat{C} + \frac{(g\pi^2/\hat{a}^2)}{1 - (1 - 16\pi^2/\hat{a}^2)^{1/2} - \cos \left[\frac{\pi}{2} \left(\hat{w}_0 - \frac{C^* w_0}{q_x} + \frac{\hat{n}_0}{q_x}\right) + \hat{n}_0/q_x\right]}
\]

(6.49)

for the boundary-layer displacement. Here \( \hat{a} \) is a positive arbitrary constant equal to \(-4C^*\), so that

\[
c_i^* = -\hat{C} - \hat{a}/4
\]

(6.50)

using (6.48) above. The value of \( \hat{C} \) will generally depend upon the nature of the problem under scrutiny. For example, if we have cosinusoidal behaviour far upstream (such as in the weakly-nonlinear "Stage 1" - type theory of Smith and Burggraf (1985)), then

\[
\hat{C} = -\left(\frac{g\pi^2}{\hat{a}^2}\right)
\]

is required. Another example is to have \( \hat{C} = 0 \), whereupon (6.49) reduces to the analogous
version of Benjamin's original solution. There appear
to be no further restrictions on \( \hat{C} \) here except that
(6.35) must be observed. Hence, we require

\[
\hat{C} \hat{n}_0 = 0 ,
\]  

(6.51)

since \( L \) (and thus \( \Delta \)) has been treated as an \( \hat{n}_0 \)-

independent quantity in the above analysis.

We mentioned above a second type of solution,
namely the solitary wave solution. Here \( \overline{A}_0 \) is written
as

\[
\overline{A}_0 = -\hat{\theta} + \int_{-\infty}^{\infty} C(x) e^{i\theta x} dx ,
\]  

(6.52)

where \( C \) is some suitably behaved function to be
determined and \( \hat{\theta} \) is an arbitrary constant. Rather
than solving by first principles however, we observe
that (6.52) is equivalent, in effect, to letting \( L \to \infty \)
whilst keeping \( \hat{\theta} \) fixed in the periodic case above.
Therefore, (6.49) reveals that

\[
-\overline{A}_0 = \hat{E} + \frac{\hat{\omega}}{\left[ 1 + \frac{\hat{\alpha}^2}{16} \left[ \frac{4 \hat{\zeta}}{\omega^2} \left( \hat{m}_0 - \frac{C_0 w e}{q_e} \right) + \hat{n}_0 / q_e \right] \right]^2}
\]  

is the required solution in this instance. Here \( \hat{C} \)
has been assumed to converge (to \( \hat{E} \)) as \( l \to \infty \), and \( c_*^* = -\hat{E} - \hat{\alpha}/4 \) is the relevant wavespeed.

Finally in this chapter, we investigate the solution properties of \( \overline{W}_o \), the velocity component in the \( \overline{\alpha}_o \)-direction, which is effectively decoupled from \( \overline{U}_o \) and \( \overline{V}_o \) in the skewed system of equations (6.22a-f) above. Firstly we rule out the possible solution

\[
\overline{W}_o = \frac{u_e}{q_e} (\gamma + \overline{\alpha}_o)
\]

(which satisfies the outer constraint (6.22e)). This approach was used for \( \overline{U}_o, \overline{V}_o \) but here it turns out that such an imposition would lead to a second independent constraint on \( \overline{\alpha}_o \), i.e.

\[
\frac{u_e}{q_e} \overline{\alpha}_o \bar{\omega}_o + \frac{u_e^2}{q_e^2} \overline{\alpha}_o \overline{\alpha}_o \bar{\omega}_o = 0,
\]

which then violates the first condition (6.27), for non-trivial solutions.

Instead we write \( \overline{W}_o \) as

\[
\overline{W}_o = \frac{u_e}{q_e} (\gamma + \overline{\alpha}_o) + \bar{\omega},
\]

(6.54)
where $\tilde{\omega}$ is an unknown function; in view of the $\phi$-dependent results above, we seek solutions of the form

$$\tilde{\omega} = \tilde{\omega}(\phi, \tilde{\gamma}, \ldots), \quad (6.55)$$

where $\phi$ is defined in (6.28) above. Substituting (6.54) into (6.22b), we generate the following partial-differential equation for $\tilde{\omega}$:

$$\left[\tilde{\gamma} + A_o - c_i^*\right] \tilde{\omega}_\phi - \tilde{\gamma} A_o \tilde{\omega}_\gamma = (u_e q_e / \omega_{e^2}) \tilde{p}_o^\prime, \quad (6.56)$$

where use has been made of (6.24) above.

In the far field, we require $\tilde{\omega} = o(1)$ to be satisfied automatically because of the given outer constraints. Further examination reveals that

$$\tilde{\omega} \sim \left(\frac{u_e q_e \tilde{p}_o}{\omega_{e^2}}\right) \tilde{\gamma}^{-1} + \left(\frac{u_e q_e (c_i^* - A_o)}{\omega_{e^2}} \tilde{p}_o\right) \tilde{\gamma}^{-2} + \ldots, \quad (6.57)$$

as $\tilde{\gamma} \to \infty$, which is consistent with our requirements.

Next, at the wall, we establish that

$$\tilde{\omega} \sim \frac{u_e q_e}{\omega_{e^2} (A_o - c_i^*)} \int \tilde{p}_o(\phi, \ldots) d\phi + O(\tilde{\gamma}), \quad (6.58)$$
as $\gamma \to 0^+$, from (6.56).

Applying the change of coordinates

$$\tilde{\gamma} = \frac{1}{2} \tilde{\gamma}^2 + \overline{A}_0 \tilde{\gamma}, \quad \tilde{\psi} = \varnothing,$$

where $\overline{A}_0 = \overline{A}_0 - c_1^*$,

so that

$$\frac{\partial}{\partial \tilde{\psi}} \rightarrow \frac{\partial}{\partial \tilde{\psi}} + \overline{A}_0 \tilde{\psi} \left[ (2 \tilde{\gamma} + \overline{A}_0)^{1/2} - \overline{A}_0 \right] \frac{\partial}{\partial \tilde{\gamma}}, \quad (6.60a)$$

$$\frac{\partial}{\partial \tilde{\gamma}} \rightarrow (2 \tilde{\gamma} + \overline{A}_0)^{1/2} \frac{\partial}{\partial \tilde{\gamma}}, \quad (6.60b)$$

we may determine from (6.56) that

$$\tilde{\omega} = \left( \frac{\omega \phi_0^*}{\omega \phi^*_0} \right) \int \tilde{\psi} \frac{\tilde{\phi}}{\overline{A}_0 \tilde{\phi}^* \left[ 2 \tilde{\gamma}^2 + \overline{A}_0^2 \left( \tilde{\phi} \right) \right]^{1/2}} d\tilde{\phi} + \tilde{\varrho} (\tilde{\gamma}), \quad (6.61)$$

where the value of $\tilde{\varrho}$ is dependent upon the input conditions upstream at $\tilde{\psi} = \tilde{\psi}_0$ (say), where $\tilde{\psi}_0$ may be negatively infinite.
Thus, overall, we conclude that

$$\bar{W}_o = \left( \frac{u_e}{a_e} \right) \left( 2 \tilde{\gamma} + \bar{A}_o^2 \right)^{1/2}$$

$$+ \frac{u_e a_e}{\omega_o} \int_{\hat{\psi}_o} \frac{P_{o \psi_1}}{\left[ 2 \tilde{\gamma} + \bar{A}_o^2 (\psi_1) \right]^{1/2}} d\hat{\psi}_1 + \bar{g} (\tilde{\gamma}) \quad (6.62)$$

is the component of velocity in the $\bar{A}_o$-direction.

In view of (6.22f), $\bar{W}_o$ is essentially a function of $\bar{A}_o$ only, for prescribed input conditions. Hence, for a given solution of $\bar{A}_o$ satisfying (6.30) above (e.g. the nonlinear periodic solution (6.49) or the solitary wave solution (6.53)), it is possible to evaluate $\bar{W}_o$, in principle.
Chapter 7 - Strongly Nonlinear Rayleigh-Wave/Vortex Interactions in Incompressible Boundary Layers for Increasing Crossflow

7.1 Preliminaries

Strongly nonlinear interactions occurring between a single linear Rayleigh wave and its induced vortex in a boundary layer, have been studied in the instance of small crossflow \( O(\ell_z^{-1/2}) \) by Hall and Smith (1989; subsequently corrected 1991) which we now summarise. There, the crucial feature enabling a relatively small wave to self-interact and distort the mean-flow profile is the fast scale

\[
Z = \Re^{1/2} (3 - 3_o), \tag{7.1a}
\]

near some spanwise station \( 3 = 3_o \). This governs not only the wave motion, but more significantly, the vortex motion. The streamwise and temporal coordinates controlling the wave are

\[
[X, T] = \Re^{1/2} [x - x_o, t], \tag{7.1b,c}
\]

close to the position \( x = x_o \) and the time \( t = 0 \); this greatly contrasts with the vortex flow which varies over the \( O(1) \) scales \( x, t \).
In the boundary layer, where \( y = \Re^{-1/2} \) with \( \Re \) of \( O(\cdot) \), a strong vortex flow induced by wave-inertial forcing is sought. There, the unknown velocity and pressure amplitude for the wave is \( \Pi \), and this is determined below. The traditional singular response of the Rayleigh-wave velocities, like \((\hat{y} - \hat{f})\) near some value \( \hat{y} = \hat{f}(\xi, \mathbf{Z}) \) (where the combined temporal-inertial force driving the wave vanishes), necessitates the existence of a viscous critical layer there of comparative thickness \( O(\Re^{-1/6}) \), wherein the singularities are damped out. The wave-forcing in this layer is relatively large, and we expect a provoked vortex motion to ensue provided

\[
\hat{\mathcal{W}}_{y} \sim \frac{\Pi}{\Re^{1/2}}/\mathbf{Y}^{2} \quad \text{as} \quad \hat{\mathbf{Y}} \to \pm \infty, \quad (7.2)
\]

where \( \hat{\mathcal{W}}_{y} \) is the unscaled spanwise velocity of the vortex and \( \hat{\mathbf{Y}} \equiv \Re^{1/4} (\hat{y} - \hat{f}) \) is the critical-layer normal coordinate. In particular,

\[
\hat{\mathcal{W}}_{y} \sim \frac{\Pi}{\Re^{1/2}} \left[ k_{+} \hat{\mathbf{Y}} + O(\mathcal{N}_{1}/\mathbf{Y}) \right] \quad \text{for} \quad |\hat{\mathbf{Y}}| \gg 1, \quad (7.3)
\]

where \( (k_{+} - k_{-}) \) is a measure of the wave-nonlinearity driving the vortex, which we evaluate below. Hence, a vortex-spanwise velocity of \( O(\Pi^{2} \Re^{2/3}) \) feeds into the main part of the boundary layer, and gives rise to
an $O(1)$ vortex-streamwise velocity in the regime

$$\Pi \sim Re^{-7/12},$$

(7.4)

by mass-conservation arguments, (i.e. balancing the terms $(u_v)_x$ and $(w_v)_3$ where $u_v, w_v$ denote the unscaled vortex-streamwise and spanwise velocities outside the critical layer).

The flow structure is represented diagrammatically below.
In the core region, where $y = Re y^2$ (but $y^2 < 1$), we express the flow solution in the form

$$u = u(x, y, z, t) + Re^{-1/2} \omega^{(1)}(x, y, z, T) + \ldots,$$  \hspace{1cm} (7.5a)

$$v = Re^{-1/2} \omega(x, y, z, t) + Re^{-1/2} \omega^{(1)}(x, y, z, T) + \ldots,$$  \hspace{1cm} (7.5b)

$$w = Re^{-1/2} \omega(x, y, z, t) + Re^{-1/2} \omega^{(1)}(x, y, z, T) + \ldots,$$  \hspace{1cm} (7.5c)

$$p = Re^{-1/2} \omega^{(1)}(x, y, z, T) + \ldots + Re^{-1} \omega^{(1)}(x, y, z, T) + \ldots.$$  \hspace{1cm} (7.5d)

Here the barred terms denote vortex quantities, whereas the wave is represented by the superscript $(1)$.

Insertion into the Navier-Stokes equations yields

\begin{align*}
\bar{u}\bar{u}_x + \bar{u}\bar{u}_y + \bar{u}\bar{u}_z &= \bar{u}\bar{y} + \bar{u}\bar{z}, \hspace{1cm} (7.6a) \\
\bar{u}\bar{u}_x + \bar{u}\bar{u}_y + \bar{u}\bar{u}_z &= \bar{p}_x + \bar{p}_y + \bar{p}_z, \hspace{1cm} (7.6b) \\
\bar{u}\bar{u}_x + \bar{u}\bar{u}_y + \bar{u}\bar{u}_z &= \bar{p}_x + \bar{p}_y + \bar{p}_z, \hspace{1cm} (7.6c) \\
\bar{u}_x + \bar{u}_y + \bar{u}_z &= 0, \hspace{1cm} (7.6d)
\end{align*}

subject to the boundary conditions

\begin{align*}
\bar{u} = \bar{v} = \bar{w} &= 0 \text{ at } y = 0, \hspace{1cm} (7.6e) \\
\bar{u} \to 1, \bar{w} \to w_0(x) \text{ and } \bar{u}_y \to 0 \text{ as } y \to \infty, \hspace{1cm} (7.6f)
\end{align*}

and
subject to the boundary conditions

\[ u^{(1)} = 0 \text{ at } \bar{y} = 0, \]  
\[ (u^{(1)}, u^{(1)}, w^{(1)}, \rho^{(1)}) \to 0 \text{ as } \bar{y} \to \infty, \]

for the dominant vortex and the dominant wave respectively. We note that spatial-vortex development only is being addressed here (accounting for the absence of \( \overline{u}, \overline{v}, \overline{w} \) in (7.6a-c)) and we have no external pressure gradient.

Now we decompose the wave in the following manner:

\[ u^{(1)} = \overline{u}(\bar{y}, Z) E + \text{c.c.}, \]  

and so on, where

\[ E \equiv \exp\left[i\lambda (x - cT)\right], \]

and both \( \lambda \) and \( c \) are real and \( x \)-dependent. Then,
(7.7a-f) imply

\[ \begin{align*}
    i\omega (\bar{u} - c) \bar{\nu} + \bar{\nu} \bar{u}_y &+ \bar{\nu} \bar{u}_z = -i\omega \bar{\rho}, \\[7.10a] 
    i\omega (\bar{u} - c) \bar{\nu} &= -\bar{\rho}_y, \quad [7.10b] 
    i\omega (\bar{u} - c) \bar{\nu} &= -\bar{\rho}_z, \quad [7.10c] 
    i\omega \bar{u} + \bar{\nu}_y + \bar{\nu}_z &= 0, \quad [7.10d]
\end{align*} \]

with

\[ \begin{align*}
    \bar{\nu} &= 0 \quad \text{at} \quad y = 0, \\[7.10e] 
    (\bar{u}, \bar{\nu}, \bar{\nu}, \bar{\rho}) &\to 0 \quad \text{as} \quad y \to \infty. \quad [7.10f]
\end{align*} \]

upon equating coefficients of \( E \). These equations reduce to the generalised Rayleigh-wave pressure equation

\[ \begin{align*}
    \bar{\rho}_{yy} + \bar{\rho}_{zz} &- \frac{2}{(\bar{u} - c)} \left[ \bar{u}_y \bar{\rho}_y + \bar{u}_z \bar{\rho}_z \right] - \omega^2 \bar{\rho} = 0, \quad [7.11a] \end{align*} \]

which has the boundary conditions

\[ \begin{align*}
    \bar{\rho}_y (0, z) &= 0, \quad \bar{\rho} (\infty, z) = 0. \quad [7.11b,c]
\end{align*} \]

Here we observe the direct influence exerted by the vortex-streamwise velocity on the wave motion throughout this region. This contrasts with the wave forcing acting on the vortex which is concentrated in the critical layer. To determine the aforementioned forcing, it is therefore necessary to analyse the flow
solution within this layer. Before this, however, we summarise some important asymptotic properties associated with (7.11a), namely that

\[ \tilde{\rho} \sim \tilde{\rho}_0 + (\tilde{\gamma} - \tilde{f}_0) \tilde{\rho}_1 + (\tilde{\gamma} - \tilde{f}_0)^2 \tilde{\rho}_2 + \ldots, \] (7.12)

as \( \tilde{\gamma} \to \tilde{f}_0^\pm \) where \( \tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2 \) are uniquely determined from

\[ \tilde{\rho}_1 \Delta = \tilde{\rho}_0 \tilde{f}_z, \] (7.13a)

\[ -2 \tilde{\rho}_2 \Delta - f_{zz} \tilde{\rho}_1 + 2(\tilde{\gamma} - \tilde{f}_0) \left( f_z \tilde{\rho}_1 - \tilde{\rho}_{oz} \right) \]

\[ + \tilde{\rho}_{oz} - \Delta^2 \tilde{\rho}_0 = 0, \] (7.13b)

\[ -2 \tilde{\rho}_2 \tilde{f}_z - 2 \tilde{\rho}_2 \left( f_{zz} + 3 \Delta \mu / \lambda - 2 \lambda \tilde{f}_z / \lambda \right) + \tilde{\rho}_{zz} \]

\[ + \left( 2 / \lambda \right) \left( \mu f_z - 2 \lambda \right) \tilde{\rho}_{zz} + \tilde{\rho}_1 \left[ - \Delta^2 + 3 \mu f_z / \lambda - \mu f_{zz} / \lambda \right] \]

\[ + \left[ \mu \left( \tilde{\rho}_{zz} - \Delta^2 \tilde{\rho}_0 \right) - 2 \mu \tilde{\rho}_{oz} \right] / \lambda = 0. \] (7.13c)

Here the basic flow expansion

\[ \tilde{\mathcal{U}} = c + (\tilde{\gamma} - \tilde{f}_0) \lambda + (\tilde{\gamma} - \tilde{f}_0)^2 \mu + \ldots, \] (7.14)

as \( \tilde{\gamma} \to \tilde{f}_0^\pm \), has been employed and

\[ \Delta \equiv (1 + \tilde{f}_0^2). \]

In the critical layer, where \( \tilde{\gamma} - \tilde{f}_0(x, z) = Re^{-1/6} \tilde{\gamma} \), and \( \tilde{\gamma} \) is of order unity, it is found that \( (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\rho}) \) have the form
Again, bars denote the vortex quantities, while the wave quantities are designated by the numbered superscripts.

Entering (7.15a-d) into the Navier-Stokes equations firstly gives

\[ U^{(0)} = f_z W^{(0)} , \]
\[ U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda = -\rho^{(0)} + \Delta U^{(0)} , \]
\[ U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda = -\rho^{(0)} + \Delta U^{(0)} , \]
\[ U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda = -\rho^{(0)} + \Delta U^{(0)} , \]

for the important wave motion, and secondly

\[ \langle U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda \rangle > + \cdots = \Delta U^{(0)} , \]
\[ \langle U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda \rangle > + \cdots = -\rho^{(0)} + \Delta U^{(0)} , \]
\[ \langle U^{(0)} + V^{(2)} + W^{(0)} f_z \lambda \rangle > + \cdots = f_z \lambda \Delta U^{(0)} , \]

where \( \lambda = \bar{f}_z \bar{W} \),

for the primary induced-vortex motion. Here \( < > \)
refers to the vortex components only in the enclosed terms. We may eliminate \( P_z \) by multiplying equation (7.17c) by \( f_z \) and adding the result to (7.17b) to obtain
\[
\langle u^{(1)} W_{x}^{(1)} + v^{(2)} W_{y}^{(1)} + w^{(1)} W_{z}^{(1)} - f_z W^{(2)} W_{y}^{(2)} + f_z f_{zz}(W^{(2)})^2/\Delta \rangle
\]
where (7.17d) has been put to use. This leads to the condition
\[
\mathcal{W}_{\gamma}(\infty) - \mathcal{W}_{\gamma}(-\infty) = \int_{-\infty}^{\infty} F(\gamma, Z) d\gamma,
\]
where
\[
F = \Delta^{-1} \langle u^{(1)} W_{x}^{(1)} + v^{(2)} W_{y}^{(1)} + w^{(1)} W_{z}^{(1)} - f_z W^{(2)} W_{y}^{(2)} + \Delta^{-1} f_z f_{zz}(W^{(2)})^2 \rangle.
\]
After extensive working, the right-hand side of (7.19) is found to be given by
\[
\int_{-\infty}^{\infty} F(\gamma, Z) d\gamma,
\]
where
\[
\int (l/3) = \int_{0}^{\infty} e^{-t} dt, \quad \alpha = \alpha \Delta^{-1}. \]
The above condition feeds into the core-flow solution, forcing
the constraint

$$\tilde{\omega}_y (f^\pm) = \Gamma^\pm$$  \hspace{1cm} (7.22)

there, (i.e. a discontinuity in vortex-spanwise shear across the critical layer), where

$$\Gamma^+ - \Gamma^- = \frac{2\pi (2/3)^{2/3} \Gamma(1/3)}{\Delta^5 \alpha^{6/3}} \left\{ \frac{\partial}{\partial Z} \left( |\alpha z|^2 \right) - \frac{1}{3} \left( \frac{5 \alpha z}{\Delta} + \frac{1}{\Delta} \right) |\alpha z|^2 \right\}$$  \hspace{1cm} (7.23)

This embodies the wave's feedback effect on the vortex motion in the core.

Thus, condensing the above results, our vortex-wave interaction system is, in essence, captured by the equations (7.6a-f), (7.11a-c) and (7.22) above. These may be solved numerically in principle, by marching downstream in $\alpha$, after prescribing a sufficient number of input conditions. No computations have yet been attempted.

In the next section, we build on the above findings, by considering the effects of raising the crossflow from its present order of magnitude.
Now our interest is in describing the nature of the vortex-wave interaction problem above in the regime of large crossflow. Ideally, we would like to be able to understand the corresponding set up at $O(1)$ values of the global crossflow (i.e. $\mathcal{W} = O(\ell^2)$ here). Then we would have a model describing possible transition processes in three-dimensional boundary layers which are of great importance in practice. However, the following work is believed by the author to be valid only if the global crossflow remains smaller than $O(\ell^{-1/6})$ and the reasons for this are given at the end of the chapter. Hence, we cannot deduce the analysis for $O(1)$ global crossflow, here).

We suppose that $\mathcal{W} = O(\sigma)$ where the parameter $\sigma$ is large, although smaller than any inverse power of the Reynolds number, and we retain the Rayleigh scales. Then, a non-trivial balance in (7.6d) above requires

$$\mathcal{U} \sim \sigma$$  \hspace{1cm} (7.24a)

and, because of wave-inertia in (7.6b,c), we further deduce that

$$\bar{P}_2 \sim \sigma^2.$$

\hspace{1cm} (7.24b)
Next, to ensure a significant flow development in the streamwise direction, we must balance the nonlinear operator components $\bar{u} \partial / \partial x$ and $\bar{w} \partial / \partial z$. Hence, upon preserving the finite order of $\bar{u}$, we require

$$x = \sigma^{-1} \hat{x}, \quad (7.24c)$$

say, where now $\hat{x}$ is $O(1)$. Finally, we note that the viscous terms are subdominant in the main vortex equations, across the majority of the boundary layer. Viscosity becomes active in a sublayer however, where the vortex flow adjusts to the no-slip constraint at the wall, and secondly, near the critical layer where the wave-coupling condition, (7.22), is applied; the zones have relatively thin widths of $O(\sigma^{-1/3})$ and $O(\sigma^{-1/2})$, respectively, from inertial-viscous balancing.

Diagrammatically, we summarise the emerging structural changes as follows.
We begin by examining the flow properties in the core, where $\bar{y}$ is $O(1)$. Firstly, for the vortex we have

$$\bar{u} = \bar{u}_0 + \sigma^{-1/2} \bar{u}_1 + \ldots$$  \hspace{1cm} (7.25a)
$$\bar{v} = \sigma^{-1/2} \bar{v}_0 + \sigma^{-1/2} \bar{v}_1 + \ldots$$  \hspace{1cm} (7.25b)
$$\bar{w} = \sigma^{-1/2} \bar{w}_0 + \sigma^{-1/2} \bar{w}_1 + \ldots$$  \hspace{1cm} (7.25c)
$$\bar{p}_2 = \sigma^{-3/2} \bar{p}_{20} + \sigma^{-3/2} \bar{p}_{21} + \ldots$$  \hspace{1cm} (7.25d)
where all the quantities depend solely on the variables $\tilde{x}, \tilde{y}, \tilde{z}$. Inserting (7.25a-d) into (7.6a-f) primarily yields

\[
\begin{align*}
\tilde{u}_0 \tilde{u}_{0x} + \tilde{v}_0 \tilde{v}_{0y} + \tilde{w}_0 \tilde{w}_{0z} &= 0, \\
\tilde{u}_0 \tilde{w}_{0x} + \tilde{v}_0 \tilde{w}_{0y} + \tilde{w}_0 \tilde{w}_{0z} &= -\tilde{p}_{20 y}, \\
\tilde{u}_0 \tilde{u}_{0x} + \tilde{v}_0 \tilde{u}_{0y} + \tilde{w}_0 \tilde{u}_{0z} &= -\tilde{p}_{20 z}, \\
\tilde{u}_0 \tilde{w}_{0x} + \tilde{v}_0 \tilde{w}_{0y} + \tilde{w}_0 \tilde{w}_{0z} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{v}_0 &= 0 \text{ at } \tilde{y} = 0, \\
\tilde{u}_0 \to 1, \tilde{w}_0 \to \tilde{w}_{0o}(x) \text{ and } \tilde{u}_0 \tilde{y} \to 0 \text{ as } \tilde{y} \to \infty.
\end{align*}
\]

Notably, this system is inviscid and consequently we may relax the no-slip constraint on both $\tilde{u}_0$ and $\tilde{w}_0$ at the wall.

Secondly, regarding wave motion in the core, we express $\tilde{\rho}$ as

\[
\tilde{\rho} = \sigma^{-1} \left[ \tilde{\rho}_0(y, Z) + \sigma^{-1/2} \tilde{\rho}_1(y, Z) + \ldots \right],
\]

where $r$ is, as yet, unknown. Subsequently, $\tilde{\rho}_0$ satisfies

\[
\begin{align*}
\tilde{\rho}_{0yy} + \tilde{\rho}_{0zz} - \frac{2}{(\tilde{u}_0 - c_o)} \left[ \tilde{u}_0 \tilde{g}_{0y} + \tilde{w}_0 \tilde{p}_{0z} \right] - \chi^2 \tilde{\rho}_0 &= 0, \\
\end{align*}
\]

with

\[
\begin{align*}
\tilde{\rho}_{0y}(0, Z) = 0, \quad \tilde{\rho}_0(\infty, Z) = 0,
\end{align*}
\]
upon substitution of (7.27) into (7.11a-c) above. Here we note that

\[ C = C_0 + \sigma^{-1/2} C_1 + \sigma^{-1} C_2 + \ldots, \]  
\[ \chi = \chi_0 + \sigma^{-1/2} \chi_1 + \ldots. \]  

(7.29a)  
(7.29b)

Near the buffer layer interface, \( \bar{y} = f_0(\hat{x}, \hat{z}) \), \( \bar{p}_0 \) is expanded in the form

\[ \bar{p}_0 = \pi_0 + \pi_1(\bar{y} - f_0) + \pi_2(\bar{y} - f_0)^2 \]

\[ + \pi_{3L} (\bar{y} - f_0)^3 \ln |\bar{y} - f_0| + \pi_{4L} (\bar{y} - f_0)^4 + \ldots, \]  

(7.30)

where \(|\bar{y} - f_0| \ll 1\) and

\[ f = f_0(\hat{x}, \hat{z}) + \sigma^{-1/2} f_1(\hat{x}, \hat{z}) + \ldots. \]  

(7.31)

We observe that logarithmic terms have been included; this is allowable here, because we are still outside the critical-layer boundary. Another striking feature is the coefficient of \((\bar{y} - f_0)^3\) which differs for \( \bar{y} < f_0 \), and implies a third-derivative 'jump' in the wave-pressure across the buffer layer. This is induced by wave-driven vortex forcing therein, as shown later.

The controlling equations for \( \Pi_0, \Pi_1, \Pi_L \) and \( \Pi_{3L} \) are

\[ \Delta_0 \pi_0 = f_{0z} p_{0z}, \]  

(7.32a)

\[ 2 \Delta_0 \pi_1 = \pi_{0zz} - \chi_0^2 \pi_0 + 2(\alpha_{oz}/\alpha_0)(f_{0z} \Pi_1 - \Pi_{0z}) - f_{0zz} \Pi_1, \]  

(7.32b)

\[ 3 \Delta_0 \pi_{3L} = 2 \pi_{2z} f_{0z} + 2 \pi_2(f_{0zz} + 3 \Delta_0 \Pi_0/\alpha_0 - 2 \alpha_{oz} f_{0z}/\alpha_0) \]

\[ - \pi_{1zz} - (2/\alpha_0)(\mu_0 f_{0z} - \alpha_{oz}) \pi_{1z} \]

\[ - \Pi_1(-\chi_0^2 + 2 \mu_{oz}/\alpha_0 - \mu_0 f_{0zz}/\alpha_0) - \lambda_0^{-1}[\mu_0(\pi_{0zz} - \alpha_0^2 \pi_0) - 2 \mu_{oz} \pi_{0z}], \]  

(7.32c)
where we have expanded $\lambda$ and $\mu$ as

$$\lambda = \lambda_0 + \sigma^{-1/2} \lambda_1 + \ldots, \quad (7.33a)$$

$$\mu = \mu_0 + \sigma^{-1/2} \mu_1 + \ldots, \quad (7.33b)$$

and $\Delta_0 \equiv 1 + f_0 z^2$.

No information regarding $\mathcal{P}_3^\pm$ can be gleaned from the current asymptotic treatment; instead we must address the global problem (7.29a-c), although no rational deductions are apparent in the general case.

Next, we address the buffer layer, where $\mathcal{Y} - f_0 = \sigma^{-1/2} \mathcal{Y}$ (with $\mathcal{Y} \sim 1$). In this comparatively narrow region, viscosity re-enters the dominant vortex equations and a match with the corresponding critical-layer solutions is possible at the common boundaries (i.e. $\mathcal{Y} \to f_1^\pm$). Thus, in particular, we apply the wave-induced constraint on vortex-spanwise shear (7.22) there.

We write the velocity and pressure of the vortex as

$$\mathcal{U} = c_0(\hat{x}) + \sigma^{-1/2} U_1(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \sigma^{-1} U_2(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \ldots, \quad (7.34a)$$

$$\hat{\mathcal{U}} = \sigma^{-1/2} \hat{U}_0(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \hat{U}_1(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \ldots, \quad (7.34b)$$

$$\mathcal{W} = \sigma^{-1/2} W_0(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \sigma^{-1/2} W_1(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \ldots, \quad (7.34c)$$

$$\tilde{\mathcal{W}} = \sigma^{-1/2} \tilde{W}_0(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \sigma^{-1/2} \tilde{W}_1(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \ldots, \quad (7.34d)$$

$$\tilde{p}_2 = \sigma^2 P_{20}(\hat{x}, \mathcal{Z}) + \sigma^{-3/2} P_{21}(\hat{x}, \mathcal{Y}, \mathcal{Z}) + \ldots, \quad (7.34e)$$
where
\[ \hat{v} = \hat{v}_0 - f_{0z} \hat{w}_0 - f_{0z} \hat{w} \]  \hfill (7.35)

is the "skewed normal" velocity, i.e. the normal component relative to the buffer layer surface
\[ \bar{y} = f_0(\hat{x}, z) . \]

Substituting (7.34 a-d) into (7.6a-f) generates the "zeroth-order" condition
\[ C_0 \hat{u}_z = 0 \]  \hfill (7.36)

and
\[ C_0 U_1 \hat{u}_z + \hat{v}_0 U_1 + W_0 U_1 + \Delta_0 U_{1y} + \Delta_0 W_{0y} \]  \hfill (7.37a)
\[ C_0 W_0 \hat{u}_z + \hat{v}_0 W_0 + W_0 W_0 + L_0(C_0 W_0) \]
\[ = - \Delta_0^{-1} P_2 \hat{u}_z + \Delta_0 W_{0y} \]  \hfill (7.37b)
\[ \hat{v}_y + W_0 = 0 , \]  \hfill (7.37c)
\[ P_{20y} = 0 \]  \hfill (7.37d)

\[ C_0 U_2 \hat{u}_z + U_1 U_1 \hat{u}_z + \hat{v}_0 U_2 + \hat{v}_1 U_1 \]
\[ + W_0 U_2 + W_1 U_1 = \Delta_0 U_{2y} + \]  \hfill (7.38a)
\[ [2f_{02} \partial / \partial z + f_{02z}] U_{1y} , \]
\[ C_0 W_1 \hat{u}_z + U_1 W_0 \hat{u}_z + \hat{v}_0 U_2 + \hat{v}_1 U_1 \]
\[ + W_0 W_1 + W_1 W_1 + L_1(C_0 \hat{v}_0, W_0, U_1, W_1) \]
\[ = - \Delta_0^{-1} P_2 \hat{u}_z + \Delta_0 W_{1y} - [2f_{0z} \partial / \partial z + f_{0zz}] W_{0y} , \]  \hfill (7.38b)
\[ U_1 \hat{u}_z + \hat{v}_1 \hat{u}_z + W_1 = 0 , \]  \hfill (7.38c)
\[ P_{21y} = (f_{0z} / \Delta_0) P_{20z} - L_0 / f_{0z} , \]  \hfill (7.38d)
to first and second order respectively, where

\[ L_0 \equiv \frac{f_{oz}}{\Delta_o} \left\{ C_0^2 f_{ozz} + 2C_0 W_0 f_{ozz} + W_0^2 f_{ozz} \right\}, \quad (7.39a) \]

\[ L_1 \equiv \frac{f_{oz}}{\Delta_o} \left\{ 2 \left[ C_0 U_1 f_{ozz} + (C_0 W_1 + U_1 W_0) f_{ozz} \right. \\
+ W_0 W_1 f_{ozz} \right] + f_{oz} \left[ C_0 U_1 f_{ozz} + W_0 U_1 z \right] \\
+ \left[ C_0 \hat{V}_o f_{ozz} + W_0 \hat{V}_o z \right] \left\} - \frac{f_{oz}}{\Delta_o} \left[ f_{oz} U_{yy} + \hat{V}_{0yy} \right] \\
+ 2 \frac{f_{oz}^2 f_{ozz}}{\Delta_o} W_{oy} \right\}, \quad (7.39b) \]

We may expect the critical-layer wave-feedback constraint to affect the principal vortex components \( U_1, \hat{V}_o, W_0 \), but the following analysis of the wave structure in the present layer, suggests that the forcing in question is deferred to the next level, i.e. involving \( U_2, \hat{V}_1, W_1 \).

The wave-pressure in the buffer layer expands as

\[ \tilde{\rho} = \sigma^{-1} \left( \hat{\rho}_0 + \sigma^{-1/2} \hat{\rho}_1 + \sigma^{-1} \hat{\rho}_2 \right. \\
+ \sigma^{-3/2} \left[ -\frac{1}{2} \ln \sigma \right] \hat{\rho}_{3L} + \sigma^{-3/2} \hat{\rho}_{3T} + \ldots \right), \quad (7.40) \]
where \( \hat{\rho}_0, \hat{\rho}_1, \ldots \), are dependent on the variables \( Y \) and \( Z \) only. Insertion into the Rayleigh-wave pressure equation (7.11a) yields, to leading order

\[
\Delta_0 \hat{\rho}_{0YY} - \frac{2 \Delta_0 U_{1Y}}{(U_1 - c_1)} \hat{\rho}_{0Y} = 0 . \quad (7.41)
\]

Consistent matching with the outer solution requires, for \( |Y| \gg 1 \),

\[
\hat{\rho}_0 \sim \pi_0 + O(Y^{-1}) , \quad (7.42)
\]

which is only possible here for the trivial case

\[
\hat{\rho}_0 \equiv \pi_0 . \quad (7.43)
\]

The equations at successively higher orders are

\[
\Delta_0 \hat{\rho}_{1YY} - \frac{2 \Delta_0 U_{1Y}}{(U_1 - c_1)} \hat{\rho}_{1Y} = -\frac{2 f_{oz} U_{1Y}}{(U_1 - c_1)} \pi_{oz} ; \quad (7.44)
\]

\[
\Delta_0 \hat{\rho}_{2YY} - \frac{2 \Delta_0 U_{1Y}}{(U_1 - c_1)} \hat{\rho}_{2Y} = \left[ f_{oz} \hat{\rho}_{1Y} + 2 f_{oz} \hat{\rho}_{2Y} \right] + \frac{2 \Delta_0 U_{2Y}}{(U_1 - c_1)} \hat{\rho}_{1Y} - \frac{2 f_{oz} (U_{1Y} \hat{\rho}_{2zY} + U_{1Z} \hat{\rho}_{1Y})}{(U_1 - c_1)}
\]

\[
- \frac{2 \Delta_0 (U_2 - c_2)}{(U_1 - c_1)^2} U_{1Y} \hat{\rho}_{1Y} + \frac{2 (U_{1Z} - f_{oz} U_{2Y}) \pi_{oz}}{(U_1 - c_1)}
\]

\[
- \pi_{oz} + \omega_0^2 \pi_0 ; \quad (7.45)
\]

\[
\Delta_0 \hat{\rho}_{3LYY} - \frac{2 \Delta_0 U_{1Y}}{(U_1 - c_1)} \hat{\rho}_{3LY} = 0 ; \quad (7.46)
\]
\[
\Delta_0 \hat{\rho}_{3YY} - \frac{2 \Delta_0 U_{1Y}}{(U_1 - c_1)} \hat{\rho}_{2Y} = \left[ f_{oz} \hat{\rho}_{2Y} + 2 f_{oz} \hat{\rho}_{2zY} \right] \\
+ \frac{2}{(U_1 - c_1)} \left\{ \Delta_0 U_{2Y} \hat{\rho}_{2Y} - f_{oz} (U_{1Y} \hat{\rho}_{2z} + U_{1z} \hat{\rho}_{2Y}) - \frac{\Delta_0 (U_2 - c_2)}{(U_1 - c_1)} U_{1Y} \hat{\rho}_{2Y} \right\} \\
+ \Delta_0 U_{3Y} \hat{\rho}_{1Y} - f_{oz} (U_{2Y} \hat{\rho}_{1z} + U_{2z} \hat{\rho}_{1Y}) - \frac{(U_2 - c_2)}{(U_1 - c_1)} U_{2Y} (\Delta_0 \hat{\rho}_{1Y} - f_{oz} \Pi_{oz}) \\
- \frac{(U_3 - c_3)^2}{(U_1 - c_1)} U_{1Y} (\Delta_0 \hat{\rho}_{1Y} - f_{oz} \Pi_{oz}) - f_{oz} U_{3Y} \Pi_{oz} + U_{3z} \hat{\rho}_{1z} + U_{3z} \Pi_{oz} \right\} \\
- \Pi_{1zz} + \alpha_0^2 \Pi_1 + 2 \alpha_0 \alpha_1 \Pi_0. 
\] (7.47)

In (7.44) - (7.47) above, the result (7.43) has been used in advance to simplify the equations. We address each of these in turn.

Firstly, (7.44) is divided through by \((U_1 - c_1)^2\) and integrated to reveal

\[
\Delta_0 \hat{\rho}_{1Y} = k_0(z) (U_1 - c_1)^2 + f_{oz} \Pi_{oz}, 
\] (7.48)

where \(k_0\) is unknown. This, in conjunction with the outer constraint

\[
\hat{\rho}_1 \sim \Pi_{1Y} + O(1) \quad \text{as} \quad |Y| \to \infty, 
\] (7.49)

implies that \(k_0 = 0\) and

\[
 \hat{\rho}_1 = f_{oz} \Pi_{oz} \Delta_0^{-1} Y + L_1(z). 
\] (7.50)

Hence, in particular, \(\Delta_0 \Pi_1 = f_{oz} \Pi_{oz}\), in agreement
with (7.32a). We note that the asymptotic properties

\[ U_1 \sim \lambda_0 Y + O(1), \quad (7.51a) \]
\[ U_2 \sim \mu_0 Y^2 + O(Y), \quad (7.51b) \]
as \(|Y| \to \infty\), are employed both here and below.

Next, the \( \hat{\rho}_2 \) equation is manipulated in a similar vein to (7.44), and reduces to

\[
\Delta \hat{\rho}_2 = H(Z)(U_1 - c_1) \int_\infty^Y \frac{l}{(U_1 - c_1)^2} \, dY
\]

\[
+ 2(\pi_{2z} - f_{oz} \pi_1)(U_1 - c_1)^2 \int_\infty^Y \frac{U_{iz}}{(U_1 - c_1)^3} \, dY
\]

\[
- 2f_{oz} (U_1 - c_1)^2 \int_\infty^Y \frac{\hat{\rho}_{iz} U_{iy}}{(U_1 - c_1)^3} \, dY + K_1(Z)(U_1 - c_1)^2, \quad (7.52)
\]

where \( H = f_{oz} \pi_1 + 2f_{oz} \pi_{1z} + \pi_{zz} \pi_1 - \pi_{zz} \), \( 7.53 \)
and \( K_1 \) is unknown. The bars on the integral signs in (7.52) denote the finite part or principal value.

The external condition for matching is

\[ \hat{\rho}_2 \sim \pi_2 Y^2 + O(Y) \text{ as } |Y| \to \infty, \quad (7.54) \]

and so, for consistency in (7.52) for large positive \( Y \), we require \( K_1 = 0 \). On the other hand, for \( Y \to -\infty \), (7.54) is violated unless we impose the compatibility
relation

\[ H I_1 + 2(\eta_0 z - f_0 z \eta_1) I_2 - 2 f_0 z I_3 = 0, \quad (7.55) \]

where

\[ I_1 = \int_{-\infty}^{\infty} \frac{1}{(U_1 - c_1)^2} \, dY, \quad (7.56) \]

\[ I_2 = \int_{-\infty}^{\infty} \frac{U_1 z}{(U_1 - c_1)^3} \, dY, \quad (7.57) \]

\[ I_3 = \int_{-\infty}^{\infty} \frac{\hat{\rho}_{1z} U_1 Y}{(U_1 - c_1)^3} \, dY, \quad (7.58) \]

This is essentially a solvability condition for \( U_1 \),

and appears to be too restrictive, unless the vortex

flow is simple to leading order, i.e. not affected by

wave-inertial forcing from the critical layer. Then,

(7.37a-d) admit the simple solutions

\[ U_1 = \mathcal{A}_0 (Y - f_1) + c_1, \quad (7.59a) \]

\[ W_0 = \mathcal{W}_0 (\hat{x}, Z), \quad (7.59b) \]

where \( \mathcal{W}_0 \) now satisfies

\[ c_0 \mathcal{W}_0 \hat{x} + \mathcal{W}_0 \mathcal{W}_0 \hat{z} + L_0 = -\Delta_0^{-1} \hat{P}_{20z}, \quad (7.60) \]

and (7.55) is trivially satisfied.
Next, (7.46) bears the straightforward solution

\[ \hat{\rho}_{3L} = \Pi_{3L}, \quad (7.61) \]

allowing a consistent match with the solution in the core.

Finally, we simplify (7.47) to

\[
\Delta \hat{\rho}_{3Y} = (U_1 - c_1)^2 \left\{ \right.

\int_{0}^{Y} \frac{G_1(Y)}{(U_1 - c_1)^2} \, dY_1
\]

\[
+ \frac{2}{(U_1 - c_1)^2} \left. \right|_{0}^{Y} \frac{G_2(Y)U_{12}(Y)}{(U_1 - c_1)^3} \, dY_1
\]

\[
+ \left. \right|_{0}^{Y} \frac{G_3(Y)}{(U_1 - c_1)^2} \, dY_1
\]

\[
+ \left. \right|_{0}^{Y} \frac{G_4(Y)}{(U_1 - c_1)^2} \frac{\partial}{\partial Y_1} \left[ \frac{(U_2 - c_2)}{(U_1 - c_1)} \right] \, dY_1
\]

\[
+ \left. \right|_{0}^{Y} \frac{G_5(Y)}{(U_1 - c_1)^2} \frac{\partial}{\partial Y_1} \left[ \frac{(U_2 - c_2)}{(U_1 - c_1)} \right] \, dY_1
\]

\[ + K_2(Z) (U_1 - c_1)^2, \quad (7.62) \]

after some working, where

\[ G_1 = (f_{o \bar{z} z} + 2f_{o\bar{z}} \partial / \partial z) \hat{\rho}_{2Y} + (\Delta_0^2 \hat{\rho}_i - \hat{\rho}_{izz}) \]

\[ + 2\partial_0 \partial_1 \Pi_0, \quad (7.63a) \]

\[ G_2 = \hat{\rho}_{i z} - f_{o \bar{z}} \hat{\rho}_{2Y} \quad \text{,} \quad (7.63b) \]

\[ G_3 = -f_{o \bar{z}} \hat{\rho}_{2z} \quad \text{,} \quad (7.63c) \]

\[ G_4 = (\Pi_{o \bar{z}} - f_{o \bar{z}} \Pi_1) \quad \text{,} \quad (7.63d) \]

\[ G_5 = \Delta_0^2 \hat{\rho}_{2Y} - f_{o \bar{z}} \hat{\rho}_{iz}, \quad (7.63e) \]
and $K_2$ is unknown. The associated external constraint is

$$\rho_3 \sim \pi_3 \pm 3Y^3 + O(Y^2) \quad \text{as} \quad Y \to \pm \infty, \quad (7.64a,b)$$

so that (7.62) yields the principal equations

$$3\pi_3^+ = K_2 \lambda_0^2, \quad (7.65a)$$

$$3\pi_3^- = K_2 \lambda_0^2 + \lambda_0^2 (J_1 + 2J_2 + 2J_3 + 2J_4 + 2J_5), \quad (7.65b)$$

for $Y \to \pm \infty$, respectively. Here

$$J_1 = \int_{-\infty}^{\infty} \frac{G_1(Y)}{(U_1 - c_1)^2} \, dY, \quad (7.66a)$$

$$J_2 = \int_{-\infty}^{\infty} \frac{G_2(Y) U_1 Z}{(U_1 - c_1)^3} \, dY, \quad (7.66b)$$

$$J_3 = \int_{-\infty}^{\infty} \frac{G_3(Y) U_1 Y}{(U_1 - c_1)^3} \, dY, \quad (7.66c)$$

$$J_4 = \frac{1}{(U_1 - c_1)^2} \left( \frac{d}{dZ} \left[ \frac{(U_2 - c_2)}{(U_1 - c_1)} \right] \right) \, dY, \quad (7.66d)$$

$$J_5 = \int_{-\infty}^{\infty} \frac{G_5(Y)}{(U_1 - c_1)^2} \left( \frac{2}{dY} \left[ \frac{(U_2 - c_2)}{(U_1 - c_1)} \right] \right) \, dY, \quad (7.66e)$$
and, of these, only the last two have non-zero values.

Hence, \((7.65a,b)\) combine to give

\[
\Pi^+_3 - \Pi^-_3 = \frac{2G}{3\Delta_0} \int_{-\infty}^{\infty} \frac{1}{(Y-f_1)^2} \frac{\partial}{\partial Y} \left[ \frac{(U_z-C_z)}{(Y-f_1)} \right] dY \\
+ \frac{2}{3\Delta_0} \int_{-\infty}^{\infty} G_5(Y) \frac{\partial}{\partial Y} \left[ \frac{(U_z-C_z)}{(Y-f_1)} \right] dY, \quad (7.67)
\]

where, in addition, we have used \((7.59a)\). The term on the right-hand side of \((7.67)\) represents the vortex forcing on the wave that induces the wave-pressure jump across the buffer layer. Hence, we expect the variable \(U_z\) (and subsequently \(\hat{V}_i, W_i\)) to be non-simple in the sense that it is affected by the dominant wave-pressure coupling condition from the critical layer. Consequently, \(\Gamma^\pm\) has the expansion

\[
\Gamma^\pm = \sigma \Gamma_0^\pm + \sigma^{1/2} \Gamma_1^\pm + \ldots, \quad (7.68)
\]

using \((7.22)\) and hence, by \((7.23)\),

\[
\widetilde{p} \sim \sigma^{1/2} \quad (7.69)
\]

which holds throughout the boundary layer. Thus, the leading order components in \((7.23)\) equate in the manner

\[
\Gamma_0^+ - \Gamma_0^- = \frac{2\pi (2/3)^{2/3} \Gamma (1/3)}{\Delta_0 \bar{a}_0^{5/3}} \left\{ \frac{\partial}{\partial Z} \left( |\Pi_{oz}|^2 \right) \right. \\
- \frac{1}{3} \left( \frac{5\lambda_{oz}}{\bar{a}_0} + \frac{\Pi_{oz} \dot{f}_{oz} \ddot{f}_{oz}}{\Delta_0} \right) |\Pi_{oz}|^2 \right\}, \quad (7.70)
\]

where

\[
\bar{a}_0 = \bar{a}_0 \lambda_0 \Delta_0^{-1}.
\]
We now reconsider the vortex equations (7.37a-d), (7.38a-d) given the above findings and, for convenience, introduce the Prandtl shift variable \( \mathcal{Y} \), where

\[
\mathcal{Y} = Y - f_1(\mathcal{X}, Z).
\]  

(7.71)

Thus, we have the derivative transformations

\[
\frac{\partial}{\partial Y} \rightarrow \frac{\partial}{\partial \mathcal{Y}}, \quad \frac{\partial}{\partial \mathcal{X}} \rightarrow \frac{\partial}{\partial \mathcal{X}} - f_1 \frac{\partial}{\partial \mathcal{Y}}, \quad \frac{\partial}{\partial Z} \rightarrow \frac{\partial}{\partial Z} - f_1 \frac{\partial}{\partial \mathcal{Y}}.
\]  

(7.72a-c)

The leading order vortex system (7.37a-d) becomes

\[
(C_0 \partial_0 \mathcal{Y} + C_0 c_1 \mathcal{X}) + V_0 \lambda_0 + W_0 \lambda_0 Z \mathcal{Y} = 0,
\]  

(7.73a)

\[
C_0 W_0 \mathcal{X} + W_0 W_0 Z + L_0 = -\Delta_0^{-1} \rho_{20 Z},
\]  

(7.73b)

\[
\hat{V}_0 \mathcal{Y} + W_0 Z = 0,
\]  

(7.73c)

\[
\rho_{20} = \rho_{20}(\mathcal{X}, Z),
\]  

(7.73d)

using (7.59a,b) above, where

\[
\hat{V}_0 \equiv \hat{V}_0 - C_0 f_1 \mathcal{X} - W_0 f_1 Z
\]  

(7.74)

is the leading-order normal velocity of the vortex relative to the critical layer surface \( Y = f_1(\mathcal{X}, Z) \).

We can partially solve (7.73a-d) to find
\[ W_0 = C_0 A_0 \int_0^\infty \frac{\alpha_0 \cdot \alpha_0}{\alpha_0^2} \, dZ_i + a_0(x), \quad (7.75a) \]

\[ \hat{V}_0 = -W_{0z} \bar{Y} - C_0 C_i z / A_0, \quad (7.75b) \]

where \( a_0 \) is unknown.

Next, the second order set of equations (7.38a-d) is now given by

\[ C_0 U_{z \bar{Y}} + (a_0 \bar{Y} + c_i)(a_0 \bar{Y} + c_i z) + \hat{V}_0 U_{z \bar{Y}} + \hat{V}_1 \alpha_0 \]

\[ + W_0 U_{z \bar{Z}} + W_1 \alpha_{0z \bar{Y}} = \Delta_0 U_{z \bar{Y}} - [2 f_{0z} \partial / \partial Z + f_{0z \bar{Z}}] \alpha_0, \quad (7.76a) \]

\[ C_0 W_{1 \bar{Y}} + (a_0 \bar{Y} + c_i)W_{0 \bar{Y}} + \hat{V}_0 W_{1 \bar{Y}} + W_0 W_{1z} + W_1 W_{0z} \]

\[ + L_1 = -\Delta_0^{-1} (p_{0z} - f_{0z} p_{1 \bar{Y}}) + \Delta_0 W_{1 \bar{Y}}, \quad (7.76b) \]

\[ (a_0 \bar{Y} + c_i) + \hat{V}_{1 \bar{Y}} + W_{1z} = 0, \quad (7.76c) \]

and the boundary conditions to be satisfied here are

\[ W_i \sim \nu_0(\bar{Y}, Z) \bar{Y}, \quad \hat{V}_i \sim -(a_0 \bar{Y}) \nu_{0z} \bar{Y}^2 / 2, U_z \sim \mu_0(\bar{Y}, Z) \bar{Y}^2 / 2, \quad (7.76d) \]

as \( \bar{Y} \to \infty \),

\[ W_{1 \bar{Y}}(0^+) = \Gamma_{0^+}, \quad (7.76e) \]

where \( (\Gamma_{0^+} - \Gamma_{0^-}) \) is defined, in (7.70) above. Here,
in a similar fashion to (7.74) above, we have used the convenient definition

\[ \hat{V}_1 \equiv V_1 - (\rho_0 \overline{Y} + \zeta_1) f_{1z} - \omega_1 f_{1z} \quad (7.77) \]

which signifies the second-order vortex-normal velocity relative to \( Y = f_1(\hat{x}, z) \).

In a similar vein, we apply the transformation (7.71) to the wave-pressure solutions in the vortex layer. Hence, firstly

\[
\hat{\rho}_0 = \rho_0, \quad (7.78)
\]

\[
\hat{\rho}_1 = \rho_1 \overline{Y} + \hat{\kappa}_0(Z), \quad (7.79)
\]

(where \( \hat{\kappa}_0 \) is unknown) are deduced for the two lowest-order pressures. Secondly, the next-order system reduces to

\[
\Delta_0 \hat{\rho}_{2\overline{Y}} = (2 \Delta_0 \overline{\eta}_2) \overline{Y} + \left[ (\overline{\eta}_2 - 2 \theta_2 \overline{\eta}_1) f_{1z} + f_{oz} \hat{\kappa}_{oz} \right], \quad (7.80)
\]

where full use of (7.59a) has been made. Thirdly, the pressure \( \hat{\rho}_3 \) satisfies
\[ \Delta_0 \hat{\rho}_{3Y} = Y \int_{-\infty}^{\infty} \frac{G_1(Y)}{Y_1^2} \, dY_1 
+ 2Y^2 \int_{-\infty}^{\infty} \frac{G_2(Y_1)(\partial_0 \bar{Y}_1 - f_1 \zeta_0)}{\partial_0 Y_1^3} \, dY_1 
+ 2Y^3 \int_{-\infty}^{\infty} \frac{G_3(Y_1)}{Y_1^3} \, dY_1 + 2G_4Y^2 \int_{-\infty}^{\infty} \frac{1}{Y_1^2} \left[ \frac{\partial}{\partial Z} - f_1 \frac{\partial}{\partial Y_1} \right] \left( \frac{U_z - C_z}{\partial_0 \bar{Y}_1} \right) \, dY_1 
+ 2Y^2 \int_{-\infty}^{\infty} \frac{G_5(Y_1)}{Y_1^2} \frac{\partial}{\partial Y_1} \left( \frac{U_z - C_z}{\partial_0 \bar{Y}_1} \right) \, dY_1 + \hat{k}_2(z) \partial_0 \hat{\rho}_{3Y}^2 \]

\[ \text{(7.81)} \]

where \( \hat{k}_2 \) is unknown. The associated jump takes the form

\[ \Pi_3^+ - \Pi_3^- = \frac{2G_4}{3\Delta_0} \int_{-\infty}^{\infty} \frac{1}{Y^2} \left[ \frac{\partial}{\partial Z} - f_1 \frac{\partial}{\partial Y} \right] \left( \frac{U_z - C_z}{\partial_0 \bar{Y}} \right) \, dY \n+ \frac{2G_5(Y)}{3\Delta_0 \partial_0 \bar{Y}} \int_{-\infty}^{\infty} \frac{\partial}{\partial Y} \left( \frac{U_z - C_z}{\partial_0 \bar{Y}} \right) \, dY \]

\[ \text{(7.82)} \]

under the Prandtl transformation. Hence, because the vortex is driven by an amplitude-squared wave forcing (i.e. (7.70) above) and both \( G_4 \) and \( G_5 \) are essentially proportional to \( |\Pi_0| \) (in terms of size), we conclude here that \( \Pi_3^+ - \Pi_3^- \) possesses a nonlinear
amplitude-cubed effect and, this is of a type similar to that encountered in the classical weakly nonlinear theory of the Stuart-Watson (Stuart (1960), Watson 1960) type. The corresponding jump evaluated from the core, i.e. using the Rayleigh-wave pressure equation (7.11a) along with its boundary constraints (7.11b,c), has general dependence on $\Pi_{02}, \Pi_{12}$ and $\Pi_0$.

Linking this with the inner expression leads to a spanwise evolution equation for $|\Pi_0|$, which is unknown here.

To complete the present $\sigma \gg 1$ analysis, we must guarantee the absence of logarithmic terms in $\hat{\rho}_0$, $\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3$, etc. at the critical-layer/buffer-layer boundary. Clearly, from (7.78) - (7.80) above, $\hat{\rho}_0, \hat{\rho}_1, \hat{\rho}_2$ exhibit no such behaviour, being smooth functions of $\bar{\gamma}$. Analysing the $\hat{\rho}_3$-equation, we firstly note the asymptotic properties

\begin{align*}
G_1 &= G_{10} + \left[ 2 \left( f_{0z} \Pi_{22} + 2 f_{0z} \partial \Pi_{2z} / \partial z \right) \Pi_z + \left( \omega_0^2 \Pi_1 - \Pi_{1z} \Pi_2 \right) \bar{\gamma} + \ldots \right] \\
G_2 &= G_{20} + \left( \Pi_{1z} - 2 f_{0z} \Pi_{2z} \right) \bar{\gamma} + \ldots \\
G_3 &= G_{30} + G_{31} \bar{\gamma} - f_{0z} \Pi_{2z} \bar{\gamma}^2 + \ldots \\
G_5 &= \left( f_{0z} \Pi_{1z} - f_{0z} \Pi_z \right) f_{1z} + \left( 2 \Delta_0 \Pi_z - f_{0z} \Pi_{1z} \right) \bar{\gamma} + \ldots \quad (7.83a)
\end{align*}
as $| \overline{y} | \rightarrow 0$, while $G_4$ is independent of $\overline{y}$ and given by (7.63d). Here $G_{10}, G_{20}, G_{30}, G_{31}$ are known values but, being passive in the subsequent analysis, are not evaluated here for the sake of brevity.

From (7.81) we conclude that

$$
\Delta_0 \tilde{\rho}_{3\overline{y}} = M_0(Z) + M_1(Z) \overline{y} + \left\{ 2 f_{oz} \Pi_{zz} + 2 f_{oz} \Pi_{z \overline{y}} + 2 \Pi_{zz} \right\} 
$$

$$
+ 2 \left( \frac{\Pi_{zz}}{\alpha_0} \right) \left( \Pi_{zz} - 2 f_{oz} \Pi_{z \overline{y}} - 2 f_{oz} \Pi_{z \overline{y}} \right) 
$$

$$
+ 2 \left( \alpha_0 + f_{oz} \Pi_{z \overline{y}} \right) \left( \Pi_{zz} - 2 f_{oz} \Pi_{z \overline{y}} \right) + \cdots \overline{y} \ln \overline{y} + O(\overline{y}^2), \quad (7.84)
$$

for $| \overline{y} | \ll 1$, where $\Pi_{zz} = \frac{1}{2} \Omega_{\overline{y}} (\overline{y} = 0)$ and $M_0, M_1$ are known values. The criterion for no logarithms is therefore satisfied if

$$
2 \left( \Pi_{zz} - 2 f_{oz} \Pi_{z \overline{y}} \right) \frac{\partial}{\partial \overline{y}} \left( \frac{\Pi_{zz}^*}{\alpha_0} \right) + 2 \left( 2 \alpha_0 + f_{oz} \Pi_{z \overline{y}} \right) \left( \Pi_{zz}^* \right) 
$$

$$
= -2 \frac{\Pi_{zz}}{\alpha_0} \left( \Pi_{zz} - 2 f_{oz} \Pi_{z \overline{y}} \right) - \left[ 2 f_{oz} \Pi_{zz} + 2 f_{oz} \Pi_{z \overline{y}} \Pi_{zz} \right] 
$$

$$
+ \left( \Pi_{zz} - 2 f_{oz} \Pi_{z \overline{y}} \right) \right]. \quad (7.85)
$$

We interpret this equation as a constraint on the buffer-layer location $\overline{y} = f_0$ in terms of the curvature of the vortex flow at the inner boundary $\overline{y} = f_1$, which is proportional to $\Pi_{zz}$. For higher order terms in
wave-pressure ($\hat{p}_4$, $\hat{p}_5$, etc.) we generate successive constraints on $\hat{f}_1$, $\hat{f}_2$, etc., when applying the no-logarithm condition.

In summary, therefore, the interactive flow solution is determined by the equations (7.11a-c), (7.76a-e), (7.82) and (7.85) above. The current analytical shortcomings can be overcome to some extent by considering the initial streamwise developments, i.e. the regime $0 < \hat{x} << 1$. This problem is addressed in the next chapter.

It is believed by the author that the above analysis remains valid until the buffer layer is 'absorbed' by the critical layer, i.e. when $\sigma \sim Re^{1/3}$. Then the dominant interaction between the wave and vortex appears to be concentrated entirely within the critical layer, and no rational advances beyond this stage (i.e. $\sigma >> Re^{1/3}$) are evident yet.
8.1 - General Analysis

The interaction model laid down in Section 7.2 above is now addressed near the streamwise input station; for convenience this is taken as $\hat{x} = 0$, so that the regime of interest to us now is $O(1) \ll \hat{x} \ll 1$. We retain all the relevant coordinate scales apart from a necessary $O(1^{1/2})$ compression of the buffer layer, through the convective-diffusive balances.

Firstly, we note that the crossflow at the outer edge of the boundary layer, $U_{\infty}$, has the regular Taylor expansion

$$U_{\infty}(x) = U_{\infty 0} + \hat{x} U_{\infty 1} + \ldots$$

for the main vortex motion in the inviscid core layer. Here we observe that the dominant streamwise velocity component, $U_{\infty}$, has no $Z$-dependence, since it represents the oncoming mean-flow profile which generally varies over a slower spanwise scale (of $O(1)$ in terms of the scalings based on the Reynolds number).
Substituting (8.2a-d) into (7.26a-f) above principally yields

\[ \bar{u}_{oo} \bar{u}_{01} + \bar{u}_{oo} \bar{u}_{00} g = 0, \quad (8.3a) \]
\[ 2 \bar{u}_{oo} \bar{v}_{oz} + \bar{u}_{oz}^2 + \bar{u}_{oo} \bar{v}_{01} g + \bar{u}_{oo} \bar{v}_{oo} g + \bar{w}_{oo} \bar{w}_{oo} z = 0, \quad (8.3b) \]
\[ \bar{u}_{oo} \bar{u}_{01} + \bar{u}_{oo} \bar{u}_{00} g + \bar{w}_{oo} \bar{w}_{oo} z = -\bar{p}_{oo} g, \quad (8.3c) \]
\[ \bar{u}_{oo} \bar{w}_{01} + \bar{u}_{oo} \bar{w}_{00} g + \bar{w}_{oo} \bar{w}_{oo} z = -\bar{p}_{oo} z, \quad (8.3d) \]
\[ \bar{u}_{01} + \bar{u}_{00} g + \bar{w}_{00} z = 0, \quad (8.3e) \]
\[ 2 \bar{u}_{oz} + \bar{u}_{01} g + \bar{w}_{01} z = 0, \quad (8.3f) \]

with

\[ \bar{u}_{oo} = 0 \text{ at } \bar{g} = 0, \quad (8.3g) \]
\[ \bar{u}_{oo} \to 1, \bar{\omega}_{oo} \to \bar{\omega}_{oo}, \bar{w}_{oo} g \to 0 \text{ as } \bar{g} \to \infty. \quad (8.3h) \]

The important variables in the subsequent analysis are \( \bar{u}_{oo}, \bar{u}_{01} \) which drive the main wave-pressure systems in the core; hence, we need only consider (8.3a,e,g,h) here which, given \( \bar{u}_{oo}, \bar{w}_{oo} \), may be solved for \( \bar{u}_{oo}, \bar{u}_{01} \), i.e.

\[ \bar{u}_{oo} = -\left[ A + \frac{d^2}{d \bar{z}} \right] \bar{u}_{oo}, \quad \bar{u}_{01} = \left[ A + \frac{d^2}{d \bar{z}} \right] \bar{u}_{00} g, \quad (8.4a,b) \]

where

\[ \beta = \int_0^\bar{g} \frac{\bar{u}_{oo}}{\bar{u}_{oo}} d\bar{g}, \quad (8.5) \]

and \( A \) is an unknown constant.

Next, the wave-pressure in the core is written in the form

\[ \bar{p}_0 = \bar{p}_{00}(\bar{g}, \bar{z}) + \hat{\omega} \bar{p}_{01}(\bar{g}, \bar{z}) + \ldots, \quad (8.6) \]

and substituted into (7.28a-c) above. We obtain
\[ \ddot{\rho}_{00} g + \ddot{\rho}_{00} z z - \frac{2}{(\bar{u}_{00} - c_0)} \bar{u}_{00} g \ddot{\rho}_{00} g - \alpha^2 \ddot{\rho}_{00} = 0, \quad (8.7a) \]

subject to
\[ \ddot{\rho}_{00} g (0, Z) = 0, \quad \ddot{\rho}_{00} (\infty, Z) = 0; \quad (8.7b, c) \]

\[ \ddot{\rho}_{01} g + \ddot{\rho}_{01} z z - \frac{2}{(\bar{u}_{00} - c_0)} \bar{u}_{00} g \ddot{\rho}_{01} g - \alpha^2 \ddot{\rho}_{01} \]

\[ = \frac{2}{(\bar{u}_{00} - c_0)} \left[ \bar{u}_{01} g \ddot{\rho}_{00} g + \bar{u}_{01} z \ddot{\rho}_{00} z - \frac{\bar{u}_{01}}{(\bar{u}_{00} - c_0)} \bar{u}_{00} g \ddot{\rho}_{00} g \right], \quad (8.8a) \]

subject to
\[ \ddot{\rho}_{01} g (0, Z) = 0, \quad \ddot{\rho}_{01} (\infty, Z) = 0, \quad (8.8b, c) \]

at first and second order respectively. We note that

\( c_0 \) and (because \( R_0 \) is fixed) \( \alpha_0 \) have no

\( \hat{\xi} \) -expansions due to (7.36) above.

It is convenient here to introduce the spanwise-mode representations

\[ \ddot{\rho}_{00} = \hat{\ddot{\rho}}_{00} (\bar{g}) \cos \beta Z, \quad (8.9a) \]

\[ \bar{u}_{01} = u_0 (\bar{g}) + \sum_{n=1}^{\infty} u_n (\bar{g}) \cos n \beta Z, \quad (8.9b) \]

\[ \ddot{\rho}_{01} = \hat{\ddot{\rho}}_{01} (\bar{g}) + \sum_{n=1}^{\infty} \hat{\ddot{\rho}}_{1n} (\bar{g}) \cos n \beta Z, \quad (8.9c) \]
where $\beta$ is a real, unknown constant. Then (8.7a-c) reduce to

$$\hat{\rho}_{oo}'' - \frac{2}{(\bar{u}_{oo} - c_0)} \bar{u}_{oo}' \hat{\rho}_{oo}' - \sigma_i^2 \hat{\rho}_{oo} = 0, \quad (8.10a)$$

with

$$\hat{\rho}_{oo}'(0) = 0, \quad \hat{\rho}_{oo}(\infty) = 0, \quad (8.10b, c)$$

where

$$\sigma_i^2 \equiv (\varepsilon_0^2 + \beta^2), \quad (8.11)$$

and $' \equiv d/d\bar{y}$. Equation (8.10a) is essentially Rayleigh's pressure equation for a linear 2D wave having effective wavenumber $\sigma_i$. Consequently, there exists an inflexion point in $\bar{u}_{oo}$ at the critical layer, i.e.

$$\bar{u}_{oo}'' = 0 \quad \text{at} \quad \bar{y} = f_{oo} \quad (8.12)$$

(Lin (1955), Drazin and Reid (1981)). Here we have expanded $f_0$ as

$$f_0 = f_{oo} + \hat{\lambda} f_{01}(z) + \ldots, \quad (8.13)$$

and we note that $f_{oo}$ is $z$-independent in order to satisfy the principal critical-layer flow-speed condition

$$\bar{u}_{oo}(\bar{y}) = c_0 (=\text{constant}), \quad \text{at} \quad \bar{y} = f_{oo}. \quad (8.14)$$
The inflexion-point condition, (8.12) forces the wave-induced vortex-streamwise velocity in the buffer layer, $U_{\omega}$, to be of an order higher in $\mathcal{X}$, than would otherwise be the case. This, in turn, reduces the wave-pressure jump (calculated in the buffer layer, i.e. (7.82) above) so that it affects $\tilde{p}_0$ rather than $\tilde{p}_{oo}$ in the core. Consequently, (8.10a-c) becomes a standard inviscid eigenvalue problem, which would typically yield a stability relation $C_0 = C_0(\omega_0, \beta)$.

So, much interest lies with the solution of (8.8a-c) where we find, upon substitution of (8.9a-c),

$$\hat{\rho}_n'' - \frac{2\tilde{U}_{\omega}}{U_{\omega} - C_0} \hat{\rho}_n' - \gamma_n^2 \hat{\rho}_n = h_n, \quad (8.15a)$$

subject to $\hat{\rho}_n'(0) = 0, \hat{\rho}_n(\infty) = 0 \quad (8.15b, c)$ for each co-efficient of $\cos n\beta x$. Here

$$h_0 = \frac{1}{(U_{\omega} - C_0)} \left[ \hat{p}_{oo} \left( u_1' - \frac{U_{\omega}'}{U_{\omega} - C_0} u_1 \right) + \beta^2 \hat{p}_{oo} u_1 \right], \quad (8.16)$$

$$h_1 = \frac{2}{(U_{\omega} - C_0)} \left[ \hat{p}_{oo}' \left( u_2' - \frac{U_{\omega}'}{U_{\omega} - C_0} u_2 \right) + \frac{\hat{p}_{oo}'}{2} \left( u_2' - \frac{U_{\omega}'}{U_{\omega} - C_0} u_2 \right) + \beta^2 \hat{p}_{oo} u_2 \right], \quad (8.17)$$

$$h_n = \frac{1}{(U_{\omega} - C_0)} \left[ \hat{p}_{oo}' \left( u_{n-1}' - \frac{U_{\omega}'}{U_{\omega} - C_0} u_{n-1} \right) \right.$$

$$\left. - (n-1) \beta^2 \hat{p}_{oo} u_{n-1} + \hat{p}_{oo}' \left( u_{n-1}' - \frac{U_{\omega}'}{U_{\omega} - C_0} u_{n-1} \right) \right)$$

$$\left. + (n+1) \beta^2 \hat{p}_{oo} u_{n-1} \right], \quad (8.18)$$

for $n > 2$.

$$\gamma_n^2 \equiv (\omega_0^2 + n^2 \beta^2). \quad (8.19)$$
For each \( \alpha \), we seek a general solution of the form

\[
\hat{\beta}_n = Q_n(\gamma) \hat{\rho}_n(\gamma),
\]  

(8.20)

where \( \hat{\rho}_n \) satisfies the complementary equation in (8.15a). Substituting (8.20) into (8.15a) yields

\[
Q''_n \hat{\rho}_n + 2Q_n' \hat{\rho}_n' - \frac{2\tilde{u}_0}{(\tilde{u}_0 - C_0)} Q_n \hat{\rho}_n = h_n,
\]

(8.21)

which bears the solution

\[
Q_n = \frac{(\tilde{u}_0 - C_0)^2}{\hat{\rho}_n^2} \left[ \int_{0, \infty} \frac{h_n \hat{\rho}_n}{(\tilde{u}_0 - C_0)^2} d\gamma \right],
\]

(8.22)

(provided \( \hat{\rho}_n \tilde{g}(0) = \hat{\rho}_n(\infty) \equiv 0 \), for \( \gamma < \tilde{f}_{\infty}, \gamma > \tilde{f}_{\infty} \), in turn. Here we observe that the integral path does not cross \( \gamma = \tilde{f}_{\infty} \), where a singularity would arise in the integrand. Using the properties

\[
\hat{\rho}_n = \Pi_n(\gamma) + O[(\gamma - \tilde{f}_{\infty})] ,
\]

(8.23a)

\[
\tilde{u}_0 = C_0 + \delta_{\infty} (\gamma - \tilde{f}_{\infty}) + O[(\gamma - \tilde{f}_{\infty})^2],
\]

(8.23b)

as \( \gamma \to \tilde{f}_{\infty}^\pm \), in (8.22), we obtain

\[
Q_n = Q_n^0 + Q_n^1 (\gamma - \tilde{f}_{\infty}) + Q_n^2 (\gamma - \tilde{f}_{\infty})^2
\]

\[
+ Q_n^3 (\gamma - \tilde{f}_{\infty})^3 \ln |\gamma - \tilde{f}_{\infty}|
\]

\[
+ (\gamma - \tilde{f}_{\infty})^3 \left[ \frac{\tilde{u}_0^2}{3\pi \hat{\rho}_n^2} \Pi^+ + Q_n^3 \right] + O[(\gamma - \tilde{f}_{\infty})^4],
\]

(8.24)
as \( \overline{y} \to f_{oo} \pm \), where

\[
I_i^\pm = \int_{f_{oo}}^{f_{oo}} \frac{h_n \beta_n}{(U_{oo} - c_o)^2} \, d\overline{y},
\]

\[ (8.25) \]

and \( Q_{n0}, Q_{n1}, Q_{n2}, Q_{n3}, Q_{n4}, Q_{n5} \) are unknown functions, independent of \( \overline{y} \) and continuous across \( \overline{y} = f_{oo} \).

Hence, if we expand \( \hat{p}_{in} \) as

\[
\hat{p}_{in} = \pi_{0in} + \ldots + \pi_{3in}^\pm (\overline{y} - f_{oo})^3 + \ldots ,
\]

\[ (8.26) \]

we may conclude from (8.20), (8.23a) and (8.24) that

\[
\pi_{3in}^+ - \pi_{3in}^- = -\frac{2}{3} \int_{0}^{f_{oo}} \frac{h_n (\beta_n / \beta_{n0})}{(U_{oo} - c_o)^2} \, d\overline{y}.
\]

\[ (8.27) \]

Thus we have determined to leading order in \( \hat{\kappa} \), an 'outer' expression for the jump of the third-derivative wave-pressure across the buffer layer. Our main interest here (as clarified later) is with the pressure coefficient of \( \cos \beta z \), which is effectively a perturbation to the leading order pressure coefficient \( \hat{p}_{oo}(\overline{y}) \). Consequently, the associated complementary function \( \hat{p}_{ic} \) satisfies (8.10a) suggesting that a valid candidate for \( \hat{p}_{ic} \) is simply \( \hat{p}_{oo} \). Thus
\[ \pi_{31}^+ - \pi_{31}^- = \frac{-a_0^2}{3} \int_0^\infty \frac{h_1(\rho_{oo}/\hat{\rho}_{oo})}{(\rho_{oo} - c_o)^2} \, d\gamma, \quad (8.28) \]

where \( \hat{\rho}_{oo}(r_oo) = \pi_{oo} \).

A second, independent representation of the jump \((\pi_{31}^+ - \pi_{31}^-)\) comes from the buffer-layer analysis, which we now examine. To begin with, we describe the vortex motion where

\[ \hat{V}_0 = \hat{\omega} \hat{V}_{oo}(\bar{\gamma},\bar{Z}) + \hat{\omega}^{3/2} \hat{V}_{01}(\bar{\gamma},\bar{Z}) + \ldots, \quad (8.29a) \]

\[ \hat{w}_0 = \hat{w}_{oo}(\bar{Z}) + \hat{\omega} \hat{w}_{01}(\bar{Z}) + \ldots, \quad (8.29b) \]

\[ \hat{\rho}_{oo} = \hat{\rho}_{oo}(\bar{Z}) + \hat{\omega} \hat{\rho}_{01}(\bar{Z}) + \ldots, \quad (8.29c) \]

\[ \hat{u}_2 = \hat{\omega}^2 \hat{u}_{20}(\bar{\gamma},\bar{Z}) + \hat{\omega}^3 \hat{u}_{21}(\bar{\gamma},\bar{Z}) + \ldots, \quad (8.29d) \]

\[ \hat{\nu}_1 = \hat{\omega} \hat{\nu}_{10}(\bar{\gamma},\bar{Z}) + \hat{\omega}^2 \hat{\nu}_{11}(\bar{\gamma},\bar{Z}) + \ldots, \quad (8.29e) \]

\[ \hat{w}_1 = \hat{\omega}^{1/2} \hat{w}_{10}(\bar{\gamma},\bar{Z}) + \hat{\omega}^{3/2} \hat{w}_{11}(\bar{\gamma},\bar{Z}) + \ldots, \quad (8.29f) \]

\[ \hat{\rho}_{z1} = \hat{\omega}^{1/2} \hat{\rho}_{z10}(\bar{\gamma},\bar{Z}) + \hat{\omega}^{3/2} \hat{\rho}_{z11}(\bar{\gamma},\bar{Z}) + \ldots, \quad (8.29g) \]

and \( \hat{V}_0, \ldots, \hat{\rho}_{z1} \) are defined in Chapter 7 above. Here

\[ \bar{\gamma} = \hat{\omega}^{-1/2} \gamma \quad (8.30) \]

is the rescaled normal coordinate.
Entering (8.29a-g) into (7.75a,b), (7.76a-e) above gives, successively

\[ \hat{V}_{00} = -W_{00} z_0 \hat{\tau} - \frac{C_0 C_{11}}{2 \lambda_0}, \quad (8.31a) \]

\[ W_{00} = \frac{C_0}{\lambda_0} \int z_0 \lambda_1(z_1) d Z_1 + a_{00}, \quad (8.31b) \]

and

\[ \lambda_0 \hat{\tau} + V_{10}^* + W_{10} z = 0, \quad (8.32a) \]

\[ C_0 (2 U_{20} - \hat{\tau} U_{20} / 2) + \lambda_{00} \lambda_0 \hat{\tau}^2 + V_{10}^* \lambda_{00} \]

\[ = U_{20} \hat{\tau}^2 - f_{012z} \lambda_{00}, \quad (8.32b) \]

\[ C_0 (W_{10} / 2 - \hat{\tau} W_{10} / 2) = W_{10} \hat{\tau}, \quad (8.32c) \]

subject to

\[ W_{10} \sim V_{00}(z) \hat{\tau}, \quad \hat{V}_{10} \sim -\left( \lambda_0 + V_{00} z \right) \hat{\tau} / 2, \quad U_{20} \sim \mu_{00} \hat{\tau} / 2 \quad (8.32d) \]

as \( \hat{\tau} \to \pm \infty, \)

\[ W_{10}(0^\pm) = \Gamma_{00}^\pm, \quad (8.32e) \]

where

\[ V_1^* = \hat{V}_{10} + 3 C_{11} \hat{\tau} / 2. \quad (8.33) \]

The various parameter expansions that have been applied here are
\[ f_1 = \hat{x}^{1/2} f_{10}(z) + \hat{x}^{3/2} f_{11}(z) + \ldots, \]  
\[ c_1 = \hat{x}^{3/2} c_{10} + \hat{x}^{5/2} c_{11} + \ldots, \]  
\[ c_2 = \hat{x}^{5/2} c_{20} + \hat{x}^3 c_{21} + \ldots, \]  
\[ \mu_0 = \hat{x} \mu_{00}(z) + \hat{x}^2 \mu_{01}(z) + \ldots, \]  
\[ \nu_0 = \nu_{00}(z) + \hat{x} \nu_{01}(z) + \ldots, \]  
\[ a_0 = a_{00} + \hat{x} a_{01} + \ldots, \]  
and  
\[ \Gamma_0 = \Gamma_{00} + \hat{x} \Gamma_{01} + \ldots, \]  
(8.34a)  
(8.34b)  
(8.34c)  
(8.34d)  
(8.34e)  
(8.34f)  
(8.34g)

where, equating dominant balances in (7.70) above, we have

\[ \Gamma_0^+ - \Gamma_0^- = \frac{2 \pi (2/3)^{2/3} \Gamma (1/3)}{(\alpha_0 \lambda_{00})^{5/3}} \frac{\partial}{\partial z} \left( |\pi_{00(z)}|^2 \right), \]  
(8.35)

given that

\[ \pi_0 = \pi_{00}(z) + \hat{x} \pi_{01}(z) + \ldots. \]  
(8.36)
The wave-affected vortex system, (8.32a–e), may be solved as follows. Differentiating (8.32c) with respect to \( \overline{\eta} \) gives

\[
-\frac{c_0}{2} \overline{W_{10},\overline{n}} = \overline{W_{10},\overline{n}^2} \cdot \overline{\eta}, \quad (8.37)
\]

or equivalently

\[
(\overline{W_{10},\overline{n}} e^{c_0 \overline{n}^2 / 4}) \overline{\eta} = 0. \quad (8.38)
\]

Then, after some manipulation and use of the boundary conditions (8.32d,e), we find that

\[
W_{10} = K \left[ \overline{\eta} J^\pm(\overline{n}) + 2c_0^* e^{-c_0 \overline{n}^2 / 4} \right] + \nu_0^* \overline{\eta}, \quad (8.39)
\]

where

\[
J^\pm(\overline{n}) = \int_{\pm \infty}^{\overline{n}} e^{-c_0 \overline{n}^2 / 4} \, d\overline{n}, \quad (8.40)
\]

\[
K = \frac{-1}{2M} \left( \Gamma_0^+ - \Gamma_0^- \right), \quad (8.41)
\]

and

\[
M = \sqrt{\frac{\pi}{c_0}}. \quad (8.42)
\]

Next, we integrate (8.32a) with respect to \( \overline{\eta} \) and solve with the appropriate boundary conditions to obtain

\[
W_{10}^* = -K_z \left[ \left( \frac{1}{2} \overline{n}^2 + c_0^{-1} \right) J^\pm(\overline{n}) + c_0^{-1} \overline{n} e^{-c_0 \overline{n}^2 / 4} \right]

- \left( \sigma_{10} + \nu_{00} \overline{n} \right) \overline{n}^2 / 2 - A_z^{\pm}(\overline{z}), \quad (8.43)
\]
where

\[ A_2^+ - A_2^- = 2c_0^{-1} K_2 M \quad (8.44) \]

in order to make \( V_{10}^* \) continuous across the critical layer. Finally, we differentiate (8.32b) four times with respect to \( \mathcal{I} \) to get

\[-c_0 \eta \frac{U_{20}^{(iv)}}{2} + V_{10}^* \eta^{(iv)} \lambda_{00} = U_{20}^{(vi)}, \quad (8.45)\]

so that

\[ (\frac{U_{20}^{(iv)}}{c_0 \eta^{2/4}}, \eta^{(iv)}) = \lambda_{00} V_{10}^* \eta^{(iv)} c_0 \eta^{2/4}. \quad (8.46) \]

The solution, subject to (8.32d,e) above, is

\[ U_{20} = \frac{\lambda_{00} K_2}{2c_0^2} \left[ \left( 12 - c_0^2 \eta^2 \right) J_0(\eta) + 2\eta \left( 2 - c_0^2 \eta^2 \right) e^{-c_0 \eta^2/4} \right] \]

\[ + \frac{\lambda_{00}}{c_0} \left( \gamma_{00} - \gamma_{01} \right) \eta^2 + \left( \frac{\lambda_{00}}{c_0^2} A_2^\pm - \frac{\gamma_{01} A_2^\pm}{c_0} \right), \quad (8.47) \]

where additionally, continuity of \( U_{20}, U_{20} \eta \) across the critical layer has been imposed. In particular, (8.47) implies

\[ U_{20} \eta^2 = -\frac{\lambda_{00} K_2}{c_0^2} \left[ c_0 \eta^2 J_0(\eta) + 2\eta e^{-c_0 \eta^2/4} \right] \]

\[ + \frac{\lambda_{00}}{c_0} \left( \gamma_{00} - \gamma_{01} \right), \quad (8.48) \]

and this will be used in later analysis.
Now we turn to the wave-pressure motion in the buffer layer and address the wave-pressure jump there, given globally by (7.82) above, in the small-$\hat{\xi}$ regime.

Firstly, we note that

\[
\Pi_1 = \Pi_{10} \hat{\xi} + \Pi_{11} \hat{\xi}^2 + \ldots, \tag{8.49a}
\]

\[
\Pi_2 = \Pi_{20} + \Pi_{21} \hat{\xi} + \ldots, \tag{8.49b}
\]

\[
\Pi_{3L} = \Pi_{3L0} \hat{\xi} + \Pi_{3L1} \hat{\xi}^2 + \ldots, \tag{8.49c}
\]

for $0<\hat{\xi}<<1$, by necessity, in order to satisfy (7.32a-c) above where, in particular

\[
\Pi_{20} = \Pi_{20zz} - \varphi^2 \Pi_{100}. \tag{8.50}
\]

Hence, (7.79) and (7.80) imply

\[
\hat{\beta}_1 = O(\hat{\xi}^{3/2}), \tag{8.51}
\]

\[
\hat{\beta}_2 = [2\Pi_{20} \hat{\xi} + \Pi_{100} \varphi \Pi_{00z}] \hat{\xi}^{1/2} + O(\hat{\xi}^{3/2}), \tag{8.52}
\]

respectively. Furthermore, (7.63d,e) reveal that

\[
G_4 = \Pi_{0zz} + O(\hat{\xi}), \tag{8.53}
\]

\[
G_5 = [2(\Pi_{00zz} - \varphi^2 \Pi_{1000}) \hat{\xi} + \Pi_{100} \varphi \Pi_{00z}] \hat{\xi}^{1/2} + O(\hat{\xi}^{3/2}), \tag{8.54}
\]

in turn, so that to leading order in (7.82) we have

\[
\Pi_{31}^+ - \Pi_{31}^- = \frac{2}{3\lambda_0} \left[ \Pi_{00z} \mathbb{I}_1 + (\Pi_{00zz} - \varphi^2 \Pi_{1000}) \mathbb{I}_2 \right], \tag{8.55}
\]
given that
\[ \pi_3^\pm = \pi_3^0 + \pi_3^\pm \xi + \ldots \] \hfill (8.56)

Here
\[ I_1 = \int_{-\infty}^{\infty} \frac{1}{\ell^3} \frac{3}{2} \left( U_{z_0} - c_{z_0} \right) \, d\vec{\eta}, \] \hfill (8.57)
\[ I_2 = \int_{-\infty}^{\infty} \frac{1}{\ell} \frac{3}{2} \ln \left( \frac{U_{z_0} - c_{z_0}}{\ell} \right) \, d\vec{\eta}, \] \hfill (8.58)

and these may be evaluated as follows.

Firstly, we consider \( I_2 \) where integration by parts suggests the alternative form
\[ I_2 = \int_{-\infty}^{\infty} \frac{1}{\ell} \left( U_{z_0} - c_{z_0} \right) \, d\vec{\eta}. \] \hfill (8.59)

Integrating by parts twice more yields
\[ I_2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\ell} U_{z_0} \, d\vec{\eta}. \] \hfill (8.60)

Now we substitute the vortex solution (8.48), so that
\[ I_2 = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{2}{c_0} \left( \nu_{y_0 z} - \mu_{y_0} \right), \frac{1}{\ell} \right. \]
\[ \bigg. - \frac{2}{c_0} \frac{K_{yz}}{2c_0} \left[ c_0 \ ln \left( \frac{\mu_{z_0}}{\ell} \right) + 2c_0 \frac{\mu z^2}{4} \right] \bigg\} \, d\vec{\eta}, \] \hfill (8.61)
which reduces to

\[ I_2 = -\frac{\lambda_{oo} K_z M}{2 \epsilon_0} \]  
\[ (8.62) \]

Moreover, by (8.41), this may be rewritten as

\[ I_2 = \frac{\lambda_{oo} L_z}{4 \epsilon_0} \]  
\[ (8.63) \]

where \( L \equiv I_{\infty}^+ - I_{\infty}^- \). \[ (8.64) \]

Secondly, we note that

\[ I_1 = \frac{d}{dz} \left\{ \int_{-\infty}^{\infty} \frac{1}{4\pi} (U_{c_0} - C_{e_0}) d\vec{n} \right\} \]  
\[ (8.65) \]

i.e.

\[ I_1 = \frac{dI_2}{dz} \]  
\[ (8.66) \]

using (8.59) above. Thus,

\[ I_1 = \frac{\lambda_{oo}}{4 \epsilon_0} L_{zz} \]  
\[ (8.67) \]

and therefore (8.55) simplifies to

\[ \Pi_{31}^+ - \Pi_{31}^- = \frac{1}{6 \epsilon_0} \left[ (\Pi_{oo} L_z)_{zz} - \alpha_o^2 \Pi_{oo} L_z \right] \]  
\[ (8.68) \]

in view of (8.63) and (8.67).
Representing $\Pi_{00}, \Pi_{31}^\pm$ in their spanwise-mode forms

$$\Pi_{00} = \hat{\Pi}_{00} \cos \beta Z,$$

$$\Pi_{31}^\pm = \hat{\Pi}_{310}^\pm + \hat{\Pi}_{311}^\pm \cos \beta Z + \hat{\Pi}_{312}^\pm \cos 2\beta Z$$

$$+ \hat{\Pi}_{313}^\pm \cos 3\beta Z + \ldots , \quad (8.69)$$

where $\hat{\Pi}_{00}, \hat{\Pi}_{31}^\pm (n \geq 0)$ are unknown constants.

It is found that

$$L \left( \Xi \hat{\Pi}_{00}^+ - \hat{\Pi}_{00}^- \right) = \frac{2\sigma (2/3)^{2/3} \Gamma (1/3)}{(\omega_0 \lambda_{00})^{5/3}} \hat{\Pi}_{00} \left| \hat{\Pi}_{00} \right|^2 \sin 2\beta Z, \quad (8.70)$$

from (8.35); then the coefficients of $\cos \eta \beta Z (n \geq 0)$ in (8.68) equate in the manner

$$\Pi_{311}^+ - \Pi_{311}^- = -\beta^4 (\omega^2 - \beta^2) Q \frac{\hat{\Pi}_{00} \left| \hat{\Pi}_{00} \right|^2}{3c_0 (\omega_0 \lambda_{00})^{5/3}}, \quad (8.71)$$

$$\Pi_{313}^+ - \Pi_{313}^- = -\beta^4 (\omega^2 + 3\beta^2) Q \frac{\hat{\Pi}_{00} \left| \hat{\Pi}_{00} \right|^2}{3c_0 (\omega_0 \lambda_{00})^{5/3}}, \quad (8.72)$$

$$\Pi_{31n}^+ - \Pi_{31n}^- = 0, \text{ for } n \neq 1, 3, \quad (8.73)$$

where $Q \equiv \sigma (2/3)^{2/3} \Gamma (1/3)$ is a positive constant.

Therefore, comparing (8.71) with (8.28) above yields the compatibility relation

$$\frac{\beta^4 (\omega^2 - \beta^2) Q}{c_0 (\omega_0 \lambda_{00})^{5/3}} \frac{\hat{\Pi}_{00} \left| \hat{\Pi}_{00} \right|^2}{\lambda_{00}^2} \int_0^\infty \frac{h_i \left( \hat{\rho}_{00}/\hat{\Pi}_{00} \right)}{(\omega_{00} - c_0)^2} \, dt \, , \quad (8.74)$$
which determines \(|\hat{\pi}_{ooh}|\) in theory, and is the key result of the current work. In the next section, we evaluate \(|\hat{\pi}_{ooh}|\) for the special case of an incident broken-line mean-flow profile.

Finally here, we deduce the conditions for no-logarithm behaviour in the wave-pressures. Using the above expansions for \(f_{oo}, \pi_{oo}, \pi_{1}, \pi_{2}\) and \(\lambda_{0}\) along with

\[
\mu_{2}^{*} = \hat{\mu}_{20}^{*}(z) + \ldots \quad (8.75)
\]

and inserting into (7.86), produces the main result

\[
2 \pi_{00z}(\nu_{0ozz} - \lambda_{01}z) + 2(\nu_{0zz} - \lambda_{0} z^{2} \nu_{00})(\nu_{0oz} - \lambda_{01})
\]

\[
= C_{0} \left[ f_{01zz} \pi_{0oz} + f_{01zz} (\nu_{0zz} + \lambda_{0} z^{2} \pi_{00}) \right] , \quad (8.76)
\]

where we have used

\[
\mu_{20}^{*} (\equiv \frac{U_{20} \pi_{1}(\pi = 0)}{C_{0}}) = \frac{\Delta \pi}{C_{0}} (\nu_{0oz} - \lambda_{01}) \quad (8.77)
\]

Equation (8.76) illustrates how \(f_{01}\), the main perturbation to the buffer layer location, is constrained by the vortex-streamwise curvature at the critical layer. Notably \(f_{oo}\), the principal buffer layer position, is arbitrary, albeit \(z\)-independent. Employing the spanwise splittings
\[ \lambda_0 = \lambda_{010} + \lambda_{011} \cos \beta z + \ldots \]  \hspace{1cm} (8.78a)

\[ \nu_0 = \nu_{000} + \nu_{001} \sin \beta z + \ldots \]  \hspace{1cm} (8.78b)

\[ f_0 = f_{010} + f_{011} \cos \beta z + \ldots \]  \hspace{1cm} (8.78c)

we find (from (8.71)) that

\[ f_{011} = \frac{2(\beta \nu_{001} - \lambda_{011})}{\beta^2 c_0} \]  \hspace{1cm} (8.79a)

\[ \lambda_{010} = 0 \]  \hspace{1cm} (8.79b)

and \( f_{010} \) is arbitrary.

8.2 - Special Case: Broken-Line Mean-Flow Profile.

Here we consider the special instance where the incident mean-flow profile has the form

\[ \tilde{u}_{00} = \begin{cases} 
\lambda_{00} \tilde{y}, & \text{for } 0 < \tilde{y} < \lambda_{00}^{-1}; \\
1, & \text{for } \tilde{y} > \lambda_{00}^{-1}.
\end{cases} \]  \hspace{1cm} (8.80)

Clearly then

\[ \lambda_{00} = c_0 \tilde{f}_{00}^{-1} \]  \hspace{1cm} (8.81)

in order to satisfy the conditions at the critical layer. Furthermore, this layer must be located in the normal range \((0, \lambda_{00}^{-1})\) so that a non-trivial shear,
crucial to the vortex-wave interaction mechanism, is retained. Thus, we require

\[ 0 < f_{oo} < \lambda_{oo}^{-1} \quad (8.82) \]

which, in conjunction with (8.81), means that

\[ 0 < \omega_0 < 1. \quad (8.83) \]

Later results show that this property holds automatically without restricting the flow parameters.

The given profile is clearly discontinuous in streamwise shear across the junction \( \gamma = \lambda_{oo}^{-1} \). Consequently, we cannot resolve the wave-pressure in the core immediately; instead we need to solve for \( \gamma < \lambda_{oo}^{-1} \) and \( \gamma > \lambda_{oo}^{-1} \) separately, and apply the interface conditions of kinematic constraint and pressure-continuity across \( \gamma = \lambda_{oo}^{-1} \) to connect the solutions. We begin by addressing the principal system of equations for wave-normal velocity

\[
\hat{\nu}_{oo}'' - \left[ \frac{\hat{u}_{oo}''}{(\hat{u}_{oo} - c_o)} + (\lambda_o^2 + \beta^2) \right] \hat{u}_{oo} = 0, \quad (8.84a)
\]

subject to

\[
\hat{u}_{oo}(0) = 0, \quad \hat{u}_{oo}(\infty) = 0. \quad (8.84b, c)
\]
which is directly linked to (8.10a-c) above via the relation

\[ \hat{\omega}_{00} = -\frac{\hat{\rho}_{00}'}{i\omega (\hat{\omega}_{00} - \omega_0)} \]  

(8.85)

Firstly, for \( \bar{\gamma} < \omega_0^{-1} \) we have

\[ \hat{\omega}_{00}'' - (\omega_0^2 + \beta^2) \hat{\omega}_{00} = 0 \], \hspace{1cm} (8.86a)

\[ \hat{\omega}_{00}(0) = 0 \], \hspace{1cm} (8.86b)

which submit the solution

\[ \hat{\omega}_{00} = K \sinh(\gamma_1 \bar{\gamma}) \]  

(8.87)

where \( \gamma_1^2 \equiv (\omega_0^2 + \beta^2) \) \hspace{1cm} (8.88)

and \( K \) is an unknown constant. Secondly

\[ \hat{\omega}_{00}'' - (\omega_0^2 + \beta^2) \hat{\omega}_{00} = 0 \], \hspace{1cm} (8.89a)

\[ \hat{\omega}_{00}(\infty) = 0 \], \hspace{1cm} (8.89b)

hold for \( \bar{\gamma} > \omega_0^{-1} \), and this has the solution

\[ \hat{\omega}_{00} = S e^{-\gamma_1 \bar{\gamma}} \]  

(8.90)

where \( S \) is an unknown constant. To complete the analysis here, we apply the above-mentioned boundary conditions at \( \bar{\gamma} = (\omega_0^{-1})^2 \). The first of these
requires $\hat{u}_0(u_0-c_0)^{-1}$ to be continuous there, so that

$$R \sinh(\xi, \lambda_0^{-1}) = Se^{-\xi, \lambda_0^{-1}}. \quad (8.91)$$

Next, we need to have

$$[(u_0 - c_0)u_0 - \hat{u}_0u_0]$$

continuous across the junction, i.e.

$$[\xi_1(1-c_0)\cosh(\xi, \lambda_0^{-1}) - \lambda_0 \sinh(\xi, \lambda_0^{-1})] R$$

$$+ [\xi_1(1-c_0)e^{-\xi, \lambda_0^{-1}}] S = 0. \quad (8.92)$$

Combining these suggests that for non-trivial solutions $(R \neq 0, S \neq 0)$, it is necessary to have

$$\det\begin{pmatrix}
\sinh(\xi, \lambda_0^{-1}) & -e^{-\xi, \lambda_0^{-1}} \\
\left[\xi_1(1-c_0)\cosh(\xi, \lambda_0^{-1}) - \lambda_0 \sinh(\xi, \lambda_0^{-1})\right] & \xi_1(1-c_0)e^{-\xi, \lambda_0^{-1}}
\end{pmatrix} = 0,$$

from which

$$C_0 = 1 - \frac{e^{-\xi, \lambda_0^{-1}} \sinh(\xi, \lambda_0^{-1})}{\xi_1 \lambda_0^{-1}}, \quad (8.94)$$

is obtained, and this is the neutral stability eigenrelation associated with the Rayleigh wave. We observe that the function $f(x) = e^{-x} \sinh x - x$ has only one maximum, at $x = 0$, where $f(x) = 0$. Therefore, for $x > 0$, we have $f(x) < 0$, which means that $C_0 > 0$ is plainly satisfied. The second criterion, $C_0 < 1$, holds immediately. Hence, (8.83) is observed.
Given (8.87) and (8.90), we determine the corresponding pressures using (8.85). Firstly, in the interval \((O, A_{oo}^{-1})\) we deduce that

\[
\hat{\rho}_{oo} = \frac{\hat{\pi}_{oo}}{\text{Sinh}(\xi, f_{oo})} \left[ \text{Sinh}(\delta, y) - \delta(x)Cosh(\delta, y) \right], \quad (8.95)
\]

where the zero-normal pressure-gradient condition at the wall has been implemented. Notably

\[
\hat{\pi}_{oo} = i\frac{\omega}{\gamma_1^2} \text{Sinh}(\xi, A_{oo}^{-1}) R, \quad (8.96)
\]

and \(\hat{\rho}_{oo}(f_{oo}) = \hat{\pi}_{oo}\) as required. Secondly, for \(\bar{y} > A_{oo}^{-1}\), (8.85) implies

\[
\hat{\rho}_{oo} = i\frac{\omega}{\gamma_1} (1 - c_\infty) S e^{-\gamma_1 \bar{y}}, \quad (8.97)
\]

where external pressure decay has been imposed, as necessary.

Next we consider the unknown integral on the right-hand side of (8.74) which is proportional to the external wave-pressure jump. There, the function has some dependence on the, as yet, unknown vortex-streamwise velocity perturbations, \(u_8\) and \(u_z\).
Hence, it is vital to address (8.4a,b), (8.5) above, where insertion of the expansions

\[ \bar{u}_{\infty} = u_0(y) + \ldots, \quad \tilde{\omega}_{\infty} = \omega_0(y) + \omega_1(y) \sin \beta z + \ldots, \]

along with (8.9b) above, yields in particular the 2D displacement solutions

\[ u_\alpha = A \bar{u}_{\infty}', \quad \sigma_\alpha = -A \tilde{\omega}_{\infty}, \]

where \( A \) is some unknown constant, equal to the discontinuity in \( u_\alpha \) across \( \bar{y} = \lambda_{\infty}^{-1} \). (This discontinuity is smoothed out via a thin 'adjustment' layer across the junction, but we omit details here since it has no active effect on the vortex-wave interaction.) Conveniently, we are free to set \( u_z = 0 \), without contravening the flow solution.

Using (8.99a) in (8.17) leads to the reduction

\[ h_1 = -2A (\bar{u}_{\infty}')^2 \frac{\rho_{\infty}'}{\bar{u}_{\infty}' - c_0}, \]

so that the right-hand side of (8.74) becomes

\[ -2 \lambda_{\infty}^2 A \int_0^{\infty} \frac{(\bar{u}_{\infty}')^2}{(\bar{u}_{\infty}' - c_0)^{+}} \rho_{\infty} \rho_{\infty}' \, d\bar{y} = \hat{I} \text{ (say)}. \]
After further simplification

\[ \hat{I} = \frac{A}{3\pi_0} \int_0^{\lambda_0^{-1}} \left[ \frac{1}{(y-f_0)^3} \left( \hat{\rho}_{oo}^2 \right)' \right]^2 d\bar{y}, \quad (8.102) \]

which we integrate by parts several times to get

\[ \hat{I} = \frac{A}{3\pi_0} \left[ \Phi - \frac{1}{2} \int_0^{\lambda_0^{-1}} \frac{1}{(y-f_0)}(\hat{\rho}_{oo}^2)^{(iv)} d\bar{y} \right], \quad (8.103) \]

where

\[ \Phi = \left[ \frac{1}{(y-f_oo)^3} \left( \hat{\rho}_{oo}^2 \right)' + \frac{1}{2(y-f_oo)^2} (\hat{\rho}_{oo}^2)'' + \frac{1}{2(y-f_oo)} (\hat{\rho}_{oo}^2)''' \right]_0^{\lambda_0^{-1}}. \quad (8.104) \]

Now,

\[ (\hat{\rho}_{oo}^2)' = 2\hat{\rho}_{oo} \hat{\rho}_{oo}', \quad (8.105) \]

\[ (\hat{\rho}_{oo}^2)'' = 2 \left[ \hat{\rho}_{oo}^2 + \frac{2}{(y-f_oo)} \hat{\rho}_{oo} \hat{\rho}_{oo}''' + \frac{\lambda_1^2}{(y-f_oo)} \hat{\rho}_{oo}^2 \right], \quad (8.106) \]

\[ (\hat{\rho}_{oo}^2)''' = 2 \left[ \frac{6}{(y-f_oo)} \hat{\rho}_{oo}^2 + \left( 4\lambda_1 + \frac{2}{(y-f_oo)^2} \right) \hat{\rho}_{oo} \hat{\rho}_{oo}' \right. \]
\[ \left. + \frac{2\lambda_1^2}{(y-f_oo)} \hat{\rho}_{oo}^2 \right], \quad (8.107) \]

\[ (\hat{\rho}_{oo}^2)^{(iv)} = 2 \left[ 4 \left( \lambda_1 + \frac{5}{(y-f_oo)^3} \right) \hat{\rho}_{oo}^2 + \frac{4\lambda_1^2}{(y-f_oo)} \hat{\rho}_{oo} \hat{\rho}_{oo}' \right. \]
\[ \left. + 4\lambda_1 + \hat{\rho}_{oo}^2 \right], \quad (8.108) \]

where repeated use of (8.10a) has been made.

Substitution of (8.105) – (8.107) into (8.104) gives
\[ \Phi = \left[ \frac{7}{(y-f_{oo})^2} \hat{\rho}_{oo}^{2} + \left( 4 \xi_1^2 + \frac{6}{(y-f_{oo})^2} \right) \hat{\rho}_{oo} \hat{\rho}_{oo}' \right. \\
\left. + \frac{3\xi_1^2}{(y-f_{oo})^2} \hat{\rho}_{oo}^2 \right] \frac{\hat{\rho}_{oo}^{-1}}{0} \] 
which further simplifies to

\[ \Phi = \frac{\hat{\pi}_{oo}^{2} \xi_1^4}{\text{Sinh}^2(\delta_c)} \left[ 0 - 2 \delta (1-c_0) \left( 2 + \frac{3}{\delta^2 (1-c_0)} \right) \right] \text{Sinh}^2 \delta \\
-3 \right] 
\] 
from (8.94) and (8.95), where

\[ \delta \equiv \delta_1 \pi_{oo}^{-1}. \] 

Next, we define

\[ J = \frac{1}{2} \int_0^{\hat{\rho}_{oo}^{-1}} \frac{1}{(y-f_{oo})} \left( \frac{\hat{\rho}_{oo}^2}{(iv)} \right) d\delta, \] 

and, by exercising (8.95), (8.108) above, we eventually find that

\[ J = \frac{2\hat{\pi}_{oo}^{2} \xi_1^4}{\text{Sinh}^2(\delta_c)} \left[ 5 - 2 \delta (1-c_0) \right] \text{Sinh}^2 \delta. \] 

Therefore, reviewing (8.103) we see that

\[ \hat{\pi} = \frac{-A \xi_1^4}{\text{Sinh}^2(\delta_c)} \left[ \frac{2 \text{Sinh}^2 \delta}{\delta (1-c_0)} + 1 \right] \hat{\pi}_{oo}, \] 
from (8.110) and (8.113).
We recall that defined the right-hand side of (8.74), and hence, substituting (8.114) implies the solution

\[
|\hat{\Pi}_{oo}| = \left[ -\text{sgn} \left( \lambda_0 - |\beta| \right) \cdot A \right]^{1/2} \frac{\gamma_0^2 C_0^{1/2} \left( \lambda_0 \lambda_{oo} \right)^{5/6}}{\beta^2 (\lambda_0^2 - \beta^2)} \cdot \alpha^{1/2} \cdot \sinh(\delta C_0) \times \left[ \frac{2}{\delta (1 - C_0)} \sinh^2 \delta + 1 \right]^{1/2} 
\]

for the dominant wave-pressure amplitude. Two distinct cases emerge here, namely \( \lambda_0 > |\beta| \) and \( \lambda_0 < |\beta| \).

The first of these suggests that \( A < 0 \) must be satisfied, in order that (8.115) is not violated. Then, correspondingly, \( \mu_0 < 0 \) for \( 0 < \overline{\mu} < \lambda_{oo}^{-1} \) (see (8.99a)) pointing to flow reversal in the streamwise-velocity mean-flow correction. In contrast, the second case implies that \( A > 0 \) is our consistency condition, suggesting forward-travelling flow in the mean part of the velocity perturbation.

We see that as \( \lambda_0 \to |\beta| \), a singularity in wave-pressure arises leading to a modified flow structure. We have not analysed the corresponding regimes where \( (\lambda_0 \pm \beta) = o(1) \).

To evaluate \( |\hat{\Pi}_{oo}| \) numerically, we must first specify \( f_{oo} \). Then, (8.81) is used to replace the shear \( \lambda_{oo} \) in the stability equation (8.94), so that
\( C_0 = C_0(\omega_0, \beta) \) there. For given frequency \( \omega_0 \), we have a second independent relation \( C_0 = \omega_0 \omega_0^{-1} \), and if \( \beta \) is specified, we may deduce \( C_0 \) and \( \omega_0 \) in principle. Finally, we prescribe a value for \( A \), negative or positive, according to whether \( \omega_0 > |\beta| \) or \( \omega_0 < |\beta| \) respectively.

A theory analogous to the above has been obtained, again for small distances, in the case where the crossflow is \( O(\epsilon^{-1/2}) \) globally (i.e. for the vortex-wave interaction system with which we started in Chapter 7). It is found that in such a case the main interaction equations (and hence solutions) are identical in form to their counterpart versions above, and subsequently a result similar to (8.74) is obtained as a solvability equation for the leading-order wave-pressure near the critical layer. The reason for the lack of change appears to be that the crossflow has no leading order effect on the wave. As previously, a narrow viscous buffer region (of thickness \( O(\epsilon^{-1/2} \times^{1/2}) \) here) surrounds the critical layer, and contains the more important part of the vortex-wave interaction.
CHAPTER NINE: SUMMARY

The principal aim of this thesis has been to help to determine the effects of crossflow on the instability and transition of boundary layers, mainly in the context of vortex/wave interactions.

The weakly nonlinear interactions arising between two relatively low-amplitude \( O(h) \) 3D TS waves and their induced vortex (of relative amplitude \( k^2 \)) in a fully 3D boundary layer, were examined in Chapter Two. In the special case of zero incident vortex flow it was shown to be possible to reduce the nonlinear-interaction equations to an ordinary differential equation for each wave pressure coupled with a convolution integral for the vortex-streamwise skin-friction factor. Numerical treatments are required in general to solve these equations, and this has yet to be attempted.

However, for the small-crossflow balance \( \omega_e \sim h^{6/5} \) addressed in Chapter Three, some computational solutions were obtained based on a second-order finite-difference predictor-corrector scheme for the wave-pressure equations together with a trapezoidal rule (again of second-order accuracy) for the skin-friction integral.
The results compare favourably with analytical predictions of the downstream behaviour of the flow solution. Firstly, for wave angles between 32.2° (approximately) and 45°, the flow solution breaks up algebraically at some finite position downstream, provided the linear forcings (due to perturbed basic-flow wall-shear and crossflow) in each wave-pressure equation are sufficiently small. The singular response is induced by nonlinear growth in the wave-pressure equations, which suggests that a new regime will come into play near the break-up position and there the vortex-wave interaction will be of a stronger nonlinear nature (possibly fully nonlinear, i.e. where the basic flow is distorted by an $O(1)$ amount). This is yet to be confirmed formally, however. For sufficiently large linear forcing, the break-up option is avoided, and instead the flow continues to far downstream where it eventually decays. Next, for wave angles outside (32.2°, 45°) the far-downstream decay option is again observed except when the linear forcing effects for each wave-pressure have opposite signs; then the flow variables experience oscillatory behaviour.

In Chapter Four, the linear 3D eigenrelations controlling the neutral stability of the two TS waves,


