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A building of unlimited height

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Abstract

We consider the overall buckling under own weight of a thin-walled column of circular cross-section and a radius that is a hyperbolic sine function of distance from the top of the column. The maximum stress is limited to a given value, but there is no limit to the height of the column. The wall thickness is determined by consideration of local buckling. It can be made to represent a building by adjusting the own weight of the column to include the weight of the floors, finishes, cladding and imposed load.

Keywords: Tallest building, buckling under own weight, optimization.

1 Introduction

Greenhill [1] was the first to obtain formulae for the buckling due to self-weight of columns of varying vertical profiles, including prisms and cones. But it was not untill 1966 that Keller & Niordson [2] found the optimum profile for a column. Both these papers assumed a material of unlimited strength and Naicu and Williams [3] extended the method to include non-linear elasticity.

In this paper we again combine the effect of buckling with the fact that the vertical stress has to be limited, but take a simpler approach by specifying the column profile, rather than using optimization. We shall assume that the material is linear elastic, but that the stress is limited to a given value.

If the cross-sectional dimensions of a column increase exponentially as one progresses downwards from the top, then the compressive stress due to self-weight can be made to approach a constant value. The same applies to the compressive stress in the columns and walls of a building, where one has to include the weight of floors, finishes and cladding, and imposed loads. Thus, there is no limit to the height of a building based upon stress, if one is prepared to sufficiently increase the footprint of the structure.

If the cross-sectional dimensions of a column (or the vertical structure of a building) increase exponentially downwards, then they must decrease exponentially upwards, which clearly cannot be possible because the structure will become too slender and buckle. Thus in this paper we consider the buckling of a column or building whose profile is described by the hyperbolic sine (sinh) function whose value starts at zero and then approaches the exponential. We will consider a column or idealized building which consists of a thin-walled circular tube whose radius is given by the sinh function and whose wall thickness is controlled by stress and local buckling of the wall.

2 Buckling of a column under own weight

Let us imagine a column made of a material of constant density ρ, whose cross-sectional area $A(x)$ varies with the distance x below the top of the column. In the case of a building we would need to adjust ρ artificially to include the weight of floors, finishes and so on. The volume of material above a cross-section at x is

$$V = \int_{u=0}^{x} A(u)\,du \tag{1}$$

and the axial stress in the column is

$$\sigma = \frac{\rho g V}{A} \tag{2}$$

where g is the acceleration due to gravity.
Now let us imagine that the column buckles and $y(x)$ is the sideways displacement of the column at x. Then the moment at x due to the weight of the column above is

$$M = \int_{u=0}^{x} (y(u) - y(x))\,\rho g A(u)\,du = \int_{u=0}^{x} y(u)\,\rho g A(u)\,du - y(x)\int_{u=0}^{x} \rho g A(u)\,du. \tag{3}$$

In this equation $\rho g A(u)\,du$ is the weight of an element of height du and $(y(u) - y(x))$ is the lever arm of that weight about the section at x. If we differentiate (3) with respect to x,

$$\frac{dM}{dx} = y(x)\,\rho g A(x) - \frac{dy}{dx}\int_{u=0}^{x} \rho g A(u)\,du - y(x)\,\rho g A(x) = -\frac{dy}{dx}\rho g V \tag{4}$$

in which $\frac{dy}{dx}$ is negative so that $\frac{dM}{dx}$ is positive. If we make the usual assumption in the linear theory of buckling of columns that $\frac{dy}{dx}$ is small, then the curvature is equal to $\frac{d^2y}{dx^2}$ and, for a linear elastic material, $M = EI\frac{d^2y}{dx^2}$ in which E is the Young's modulus of the material, which is assumed to be constant, and I is the second moment of area of the cross-section. Thus from (4),

$$\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) + \rho g V\frac{dy}{dx} = 0 \tag{5}$$

which is equivalent to equation IV(1) of Greenhill [1] and can be solved for y if we know the value of the constants E, ρ and g as well as V and I as functions of x and the boundary conditions.

3 Non-dimensional parameters

We will find it convenient to introduce the non-dimensional parameter,

$$\theta = \frac{x}{h} \tag{6}$$

in which

$$h = \frac{\sigma_{\max}}{\rho g} \tag{7}$$

and σ_{max} is the maximum stress that the material can carry. The maximum value of θ for a prismatic column is 1.0 since the stress at the base will then be σ_{max}. However in our case we shall see that θ can be increased without limit.
We shall use Greek letters to denote non-dimensional quantities, with the exception of σ which is conventionally used for stress.
If the material is assumed to be linear elastic up to σ_{max} we can write the yield strain,

$$\varepsilon_{max} = \frac{\sigma_{max}}{E}. \tag{8}$$

so that

$$\frac{E}{\rho g} = \frac{h}{\varepsilon_{max}}. \tag{9}$$

The rotation of the column due to buckling is

$$\phi = -\frac{dy}{dx} = -\frac{1}{h}\frac{dy}{d\theta} \tag{10}$$

in which the minus sign is there because y decreases as θ increases, so that (4) can now be written

$$\frac{1}{\varepsilon_{max} h}\frac{d}{d\theta}\left(I\frac{d\phi}{d\theta}\right) + V\phi = 0. \tag{11}$$

We now introduce another non-dimensional quantity,

$$\lambda = -\frac{\left(\frac{d\phi}{d\theta}\right)}{\phi} = -\frac{d}{d\theta}(\log\phi) \tag{12}$$

in which the minus sign is there because ϕ decreases as θ increases.
Then $\frac{1}{\varepsilon_{max} h}\frac{d}{d\theta}\left(I\frac{d\phi}{d\theta}\right) = -\frac{1}{\varepsilon_{max} h}\frac{d}{d\theta}(I\lambda\phi) = -\frac{1}{\varepsilon_{max} h}\left(\frac{d}{d\theta}(I\lambda) - I\lambda^2\right)\phi$ so that

$$I\frac{d\lambda}{d\theta} + \lambda\frac{dI}{d\theta} - I\lambda^2 - h\varepsilon_{max} V = 0. \tag{13}$$

In this equation only λ is unknown and we can integrate starting at the top of the column where $\theta = 0$, provided that we know the starting value of λ.
Note that this technique can be used for all column buckling problems to replace a second order differential equation with a first order, but usually there is little point in so doing because the second order equation is linear whereas the first order is not. However in our case they are both non-linear.

4 Local buckling of the column wall

If $R(x)$ and $T(x)$ are the radius and wall thickness of a thin walled circular tube, the cross-sectional area and second moment of area are

$$A = 2\pi RT \quad \text{and} \quad I = \pi R^3 T. \tag{14}$$

In article 11.6 of *Theory of Elastic Stability* Timoshenko and Gere [4] show that the stress necessary to cause local buckling of the wall of a circular tube is equal to

$$\sigma = \alpha E\frac{T}{R} \tag{15}$$

in which the constant $\alpha = \dfrac{\mu}{\sqrt{3(1-\nu^2)}}$ and E is again Young's modulus, ν is Poisson's ratio and $\mu \approx 0.2$ is a factor to take into account approximations in the theory and inaccuracies of manufacture. We shall treat α as known and hence we can calculate T if we know R and E and make the assumption that T is a small as possible consistent with avoiding local buckling of the wall. When the stress is equal to $\sigma_{\max}$, $\dfrac{T}{R} = \dfrac{\varepsilon_{\max}}{\alpha}$, so that if we take $\alpha \approx 0.1$ and $\varepsilon_{\max} \approx \dfrac{1}{1000}$ then $\dfrac{T}{R} \approx \dfrac{1}{100}$.
From (2) and (15) we have $V = \dfrac{2\pi\alpha h}{\varepsilon_{\max}} T^2$ and therefore $2\pi RT = A = \dfrac{dV}{dx} = \dfrac{4\pi\alpha}{\varepsilon_{\max}} T \dfrac{dT}{d\theta}$ so that

$$\frac{dT}{d\theta} = \frac{\varepsilon_{\max}}{2\alpha} R. \tag{16}$$

Now let us stipulate that the column radius is given by the hyperbolic sine function as stated in section1,

$$R = \beta h \sinh(\chi\theta) \tag{17}$$

where χ and β are non-dimensional constants that are to be determined. Then $\dfrac{dT}{d\theta} = \dfrac{\varepsilon_{\max}\beta h}{2\alpha} \sinh(\chi\theta)$ which has the solution

$$T = \frac{\varepsilon_{\max}\beta h}{2\alpha\chi} (\cosh(\chi\theta) - 1) \tag{18}$$

to give $T = 0$ when $R = 0$. Thus

$$\sigma = \frac{E\varepsilon_{\max}(\cosh(\chi\theta) - 1)}{2\chi \sinh(\chi\theta)} = \frac{\sigma_{\max}}{2\chi} \tanh\left(\frac{\chi\theta}{2}\right). \tag{19}$$

But we also have to ensure that σ is less than or equal to $\sigma_{\max}$ and therefore $\chi = \dfrac{1}{2}$ so that finally, writing

$$\xi = \frac{\theta}{2} \tag{20}$$

we have

$$R = \beta h \sinh\xi \tag{21}$$

$$T = \frac{\varepsilon_{\max}\beta h}{\alpha} (\cosh\xi - 1) \tag{22}$$

$$\sigma = \sigma_{\max} \tanh\left(\frac{\xi}{2}\right) \tag{23}$$

$$A = \frac{2\pi\varepsilon_{\max}\beta^2 h^2}{\alpha} \sinh\xi (\cosh\xi - 1) \tag{24}$$

$$V = \frac{2\pi\varepsilon_{\max}\beta^2 h^3}{\alpha} (\cosh\xi - 1)^2 \tag{25}$$

$$I = \frac{\pi\varepsilon_{\max}\beta^4 h^4}{\alpha} \sinh^3\xi (\cosh\xi - 1) = \frac{\pi\varepsilon_{\max}\beta^4 h^4}{\alpha} \sinh\xi (\cosh\xi + 1)(\cosh\xi - 1)^2. \tag{26}$$

We shall also need

$$\frac{dI}{d\theta} = \frac{\pi\varepsilon_{\max}\beta^4 h^4}{2\alpha} \sinh^2\xi \left(3\cosh^2\xi + \sinh^2\xi - 3\cosh\xi\right) = \frac{\pi\varepsilon_{\max}\beta^4 h^4}{2\alpha} (4\cosh\xi + 1)(\cosh\xi + 1)(\cosh\xi - 1)^2.$$

5 Integration of the differential equation

We can now substitute the results obtained at the end of section 4 into (13), which produces

$$(\cosh\xi+1)\left(\sinh\xi\left(\frac{d\lambda}{d\theta}-\lambda^2\right)+\frac{1}{2}(4\cosh\xi+1)\lambda\right)-\frac{2\varepsilon_{max}}{\beta^2}=0. \tag{27}$$

If we assume λ and $\frac{d\lambda}{d\theta}$ are both finite as $\theta \to 0$ then the value of λ at $\theta = 0$ is $\lambda_0 = \frac{2\varepsilon_{max}}{5\beta^2}$ so that

$$\beta=\sqrt{\frac{2\varepsilon_{max}}{5\lambda_0}}. \tag{28}$$

Thus

$$\frac{d\lambda}{d\theta}=\frac{1}{2\sinh\xi}\left(\frac{10\lambda_0}{\cosh\xi+1}-(4\cosh\xi+1)\lambda\right)+\lambda^2 \tag{29}$$

which almost certainly has to be integrated numerically. We cannot use (29) to calculate $\frac{d\lambda}{d\theta}$ at very small values of θ. Instead we have $\frac{d\lambda}{d\theta}=-5\frac{d\lambda}{d\theta}+\lambda_0^2$ so that $\frac{d\lambda}{d\theta}=\frac{\lambda_0^2}{6}$.

```
lambda[0] = 1.401381149d;
for (int i = 1; i <= n; i ++){
  double thetaMidPoint = ((double) i - 0.5d) * delta_theta;
  lambda[i] = lambda[i - 1];
  for (int run = 0; run <= 10; run ++){
    double lambdaMidPoint = (lambda[i - 1] + lambda[i]) / 2.0d;
    double hypcos = Math.cosh(thetaMidPoint / 2.0d);
    double hypsin = Math.sinh(thetaMidPoint / 2.0d);
    double lambdaDash;
    if (run == 0 && i == 1)lambdaDash = lambda[0] * lambda[0] / 6.0d;
    else lambdaDash = (10.0d * lambda[0] / (hypcos + 1.0d)
      - (4.0d * hypcos + 1.0d) * lambdaMidPoint) / (2.0d * hypsin)
      + lambdaMidPoint * lambdaMidPoint;
    lambda[i] = lambda[i - 1] + lambdaDash * delta_theta;}}
gamma[0] = 0.0;gamma[1] = gamma[0] - arbitraryValue * delta_theta;
for (int i = 1; i <= n - 1; i ++){
  double temporary = lambda[i] * delta_u / 2.0d;
  gamma[i + 1] = (2.0d * gamma[i] - gamma[i - 1] * (1.0d - temporary))
  / (1.0d + temporary);}
for (int i = 0; i <= n; i ++)gamma[i] = (gamma[n] - gamma[i]) / gamma[n];
```

Listing 1: Numerical Integration

5.1 Boundary condition at the bottom of the column

If we assume that the bottom of a column of finite height is fully encastré so that the rotation, $\phi = 0$, then $\lambda \to \infty$ in (12). Therefore let us introduce $\psi = \frac{1}{\lambda}$ so that

$$\frac{d\psi}{d\theta}=-\frac{1}{2\sinh\xi}\left(\frac{4\varepsilon_{max}}{\beta^2}\frac{\cosh\xi}{\cosh\xi+1}\psi^2-(4\cosh\xi+1)\psi\right)-1 \tag{30}$$

and $\psi \to 0$ at the bottom. However we want to imagine that a given column is actually only part of a larger column which *could* be infinitely tall. If θ is large in (30), $\frac{d\psi}{d\theta} = 2\psi - 1$ which has the solution

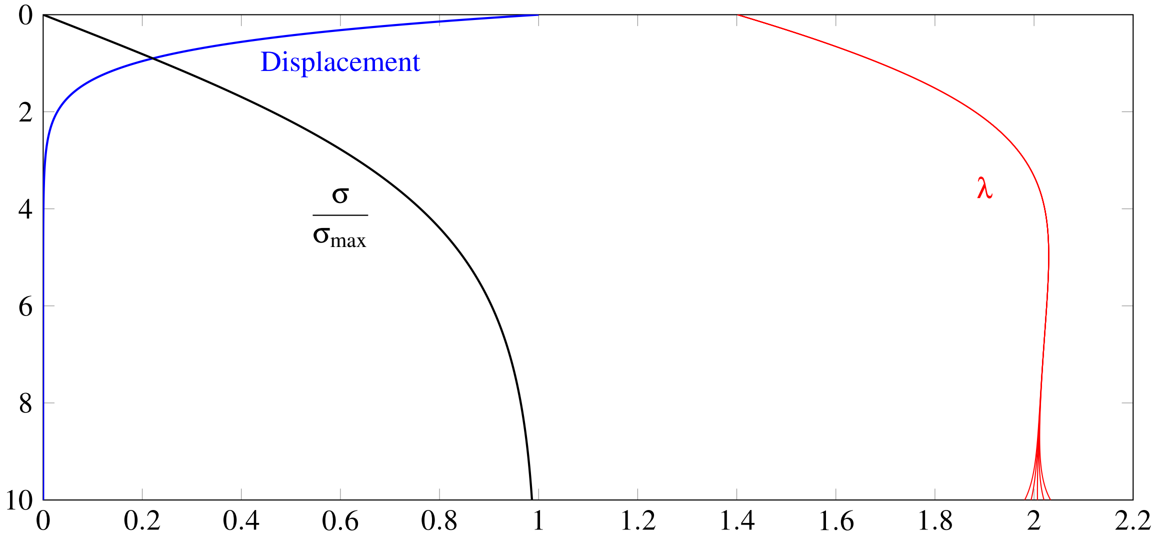

Figure 1: Lateral displacement in blue, $\frac{\sigma}{\sigma_{max}}$ in black and λ in red for different values of λ_0 plotted against non-dimensional distance from top of column $= \theta$ on the vertical axis.

Material	ρ kg/m^3	E GPa	σ_{max} MPa	$\varepsilon_{max} = \frac{\sigma_{max}}{E}$	$h = \frac{\sigma_{max}}{\rho g}$km	$\beta = \sqrt{\frac{2\varepsilon_{max}}{5\lambda_0}}$
Mild steel	7850	200	400	2×10^{-3}	5.0	0.0239
Concrete	2400	20	40	2×10^{-3}	1.7	0.0239
Timber	400	8	30	3.75×10^{-3}	7.6	0.033

Table 1: Approximate Material Properties, $g = 9.81\text{m/s}^2$ and the timber is Sitka spruce [5]

$\psi = \frac{1+\eta e^{2\theta}}{2}$ where η is a constant. From this we have to conclude that $\eta = 0$ and so $\psi \to \frac{1}{2}$ and $\lambda \to 2$ as $\theta \to \infty$.

5.2 Numerical integration

We have to integrate (29) numerically starting from λ_0. We can do this repeatedly adjusting the value of λ_0 to satisfy $\lambda \to 2$ for large θ. The core of the code is given in listing1 and the reason for the loop containing the counter 'run' is because we need the value of λ midway between λ_{i-1} and λ_i. y is found by solving the differential equation $\frac{d^2y}{d\theta^2} = -\lambda\frac{dy}{d\theta}$ using $y_{i+1} = \frac{2y_i - y_{i-1}\left(1-\lambda_i\frac{\delta\theta}{2}\right)}{1+\lambda_i\frac{\delta\theta}{2}}$. The code automatically sets the maximum displacement to an arbitrary value of 1.0.

The results of the numerical integration are given in figure 1. The values of λ_0 are in the range 1.401381147 to 1.401381151 and the column was split into 25000 segments. There was no change to the results if there were 10 times fewer or 10 times more segments. Thus we can take $\lambda_0 = 1.40$ with sufficient accuracy for practical purposes.

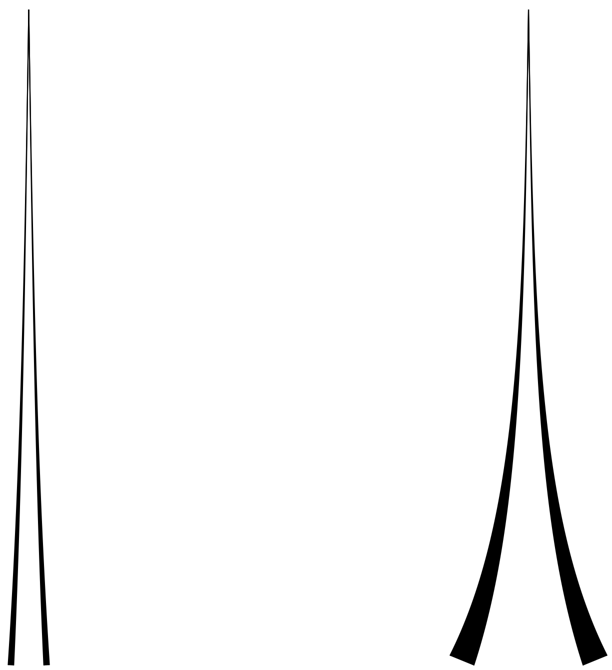

Figure 2: Column profiles with $\theta_{base} = 4$ on the left and $\theta_{base} = 8$ on the right. $\beta = 0.03$ for both.

6 Physical examples, wind and conclusions

Table 1 shows some typical material properties and figure 2 shows plots of columns with different values of the value of θ at the base. The wall thicknesses are exaggerated so that they can be seen. Both columns could be extended infinitely downwards and the column on the left is the top half of the column on the right drawn to a different scale. The value θ at the base has to be multiplied by the value of h in table 1 to give the height of the column. Thus the height of the column on the right in figure 2 is 60km tall.

Clearly this figure is at variance with what one might expect, since the maximum height of existing buildings is of the order of 1km. But we have to include safety factors and also all the weight of floors, finishes, cladding and imposed load which will effectively multiply ρ by some factor, hence reducing h.

We have not considered wind load. If we imagine a column with a conical profile and constant ratio of wall thickness to radius, then the section modulus at any section is proportional to the distance from the top cubed. If we assume that the wind load per unit area is constant, the moment due to wind is also proportional to the distance from the top cubed. Thus the bending stress is the same at all heights. This in turn means that for a column with a hyperbolic sine profile, the wind stresses are greatest at the top. However we have assumed that the ratio of wall thickness to radius tends to zero at the top, although it tends to a constant lower down. Therefore modification of the wall thickness will be required near the top.

Thus we can see that there is no limit to the height of a building, although the amount of material required grows exponentially with height. However tall a building we build, somebody else can come along and build a taller one.

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