The lure of infinity

Jennie Golding

In reading this article, I hope you'll need at least a pen and paper, but maybe also a graphing package or calculator, some elementary programming, perhaps a spreadsheet…..

'Some mathematics that made an impression on me': well, as a paid-up 'maths ed junkie', I am of course spoiled for choice. My own area of mathematics research was measure theoretic aspects of functional analysis, but as I look back over my mathematical life I realise that the mathematics that has had greatest impact on me has been that which served to 'set alight' my students. My teaching career was largely with 5 to 18 year olds in 'all-ability' state schools, though my first job in a boys' grammar school served both to benchmark my classroom expectations, and to highlight to me what I value in mathematics classrooms, namely the active engagement, excitement, empowerment and risk-taking involved in genuine mathematical activity. I believe that is available to the range of our young people, and that it serves to expose them in hands-on ways to our cultural and historical heritages, support them in constructive ways of learning to struggle through 'being stuck', and learning to choose and use their accumulating tools with confidence and discrimination.

Such beliefs relate not only to the nature of mathematics and its epistemology, but to discussions I've had in policy and academic circles for many years now: about 'who' should have access to 'what' mathematics. In a world where such discussion now often focuses more on data and digital literacy, and computational thinking, rather than on mathematics, I believe we have a moral purpose that includes exposing all our learners (in my case, at least all young people and beginner and practising teachers) to experience working in a range of mathematical ways and with mathematical ideas - and finding for themselves how that empowers both individuals and society. Yes, that takes time, and deliberate balance, but my experience is that taking that time leads to greater effectiveness of learning in the whole of our teaching.

Some such mathematical experiences of course stick in the teacherly mind: in my case the time when a year 4 class were engaged in working with Eratosthenes' sieve and one declared 'seems to me prime numbers are pretty special'; the year 7 classes who went home from an encounter with Goldbach's conjecture fired up with the idea that there are mathematical questions they understand, but to which we do not yet know the answer, though if they exert themselves when they go home then maybe….(When I was that age, it was the four-colour conjecture, but….); the GCSE resit student who, in the wake of a lightbulb moment when equivalent fractions suddenly made sense, and confident she could now produce any number of a family of equivalent fractions on a spreadsheet, so putting them together was going to be no problem, declared 'I think maths is a bit like the Tardis'; the sixth former who, trying to make sense of implicit equations for the first time, tried inputting 'siny=cosx' on her calculator and was blown away when she not only saw 'an' answer, but was able, with prompting, to explain both what she saw, and what the technology was not showing her.

One piece of classical mathematics sticks in my mind as enriching for so many of my students – and to my delight, the beginner and practising teachers I've worked with. I don't remember when I first met the Koch snowflake, but as an undergraduate in the early 1970's I was increasingly excited by ideas in newly-emergent chaos theory and developments in the field of fractals. The Koch snowflake is of course a relatively simple and elegant example of the latter, originally presented as an example of an everywhere continuous and nowhere differentiable curve. For those new to it, you've missed a treat. The 'snowflake' is constructed as follows:

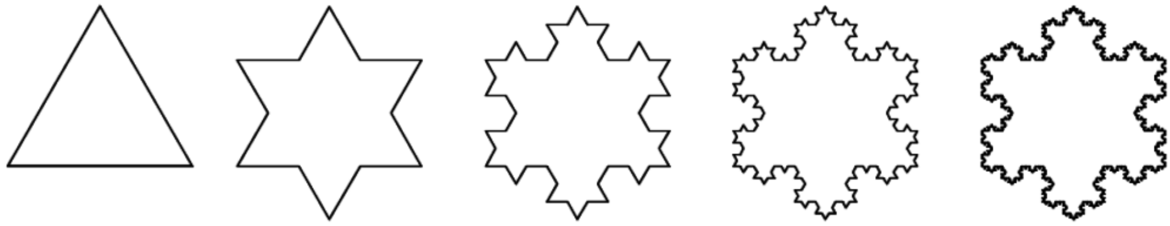

Figure 1: First five stages of the Koch snowflake

……

Start with an equilateral triangle. At each stage, divide each edge into thirds and build an equilateral triangle on the middle third, erasing the base of the equilateral triangle. Continue indefinitely. The snowflake is based on the Koch curve, which appeared in a 1904 paper entitled "On a Continuous Curve Without Tangents, Constructible from Elementary Geometry" by the Swedish mathematician Helge von Koch. There's an animation here that doesn't give away too much, but is suitable for prompting discussion, including of its intriguing fractal properties.

But what use is the Koch snowflake in a school classroom, and for whom? And what does engagement with it achieve, mathematically? Clearly, the everywhere continuous, nowhere differentiable nature of the snowflake can shake year 13s out of a complacency with calculus, beginning to extend their mathematical horizons, but there is more to the curve than that. For example, obvious questions to ask are, 'What happens to the number of edges, to the perimeter, and to the area, as the snowflake develops?' (Look away now if you want to avoid a spoiler….). It turns out that the perimeter increases indefinitely, whereas the area converges to 8/5 of the original area. Classical proofs of these results can be achieved through standard approaches to geometric series, but those are not everyday diets for the average teenager. So … what use have I made of this beautiful construction in the classroom, and why bother?

First, I think it's important young people, especially in a digital age, have the opportunity to experience the satisfaction of hands-on physical engagement with shapes, measure and geometry, from early years on. This might take the form of creating different structures of pattern, including those from a variety of cultures, exploring the construction and properties of regular polygons, designing and making boxes, creating Escher-type designs or seasonal decorations…. Fluency in the accurate measurement of lengths and angles is supported when there's also a moderately creative purpose in view, and lends an increasing meaning-making to related measures. Besides which, geometry is, globally, part of our human cultures. I have used construction of a Koch snowflake with year 9s upwards, but always building on previous experience, including with drawing a variety of shapes on plain, or variously-gridded paper. I also expect students to explore using Logo or Scratch or Geogebra for geometric constructions, but believe they should have experience of both digital and physical experiences and exploration.

I might introduce them to the snowflake as follows:

- Introduce a static slide or slides of the first 3 or 4 stages of the snowflake, or the first part of the above animation, ask students what they can see happening and what they predict might happen at the next stage(s).

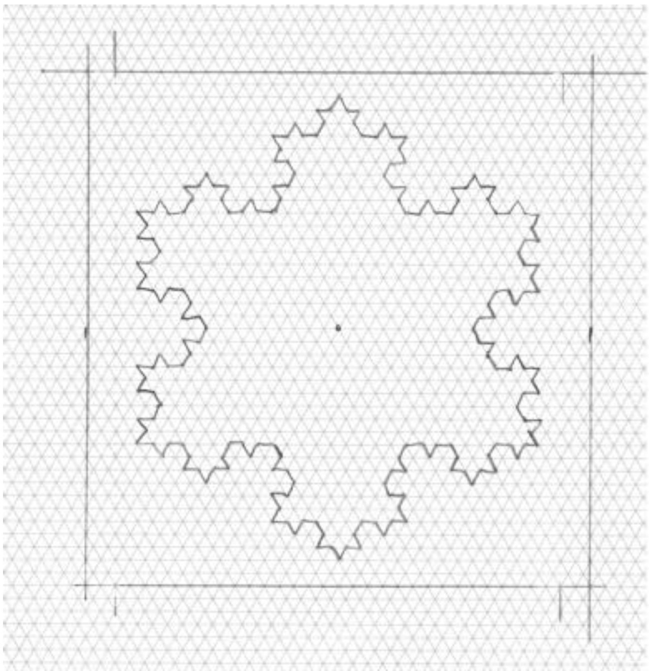
Figure 2: First four stages as drawn on isometric paper

- Indicate a bit of history (my experience is that the range of students love to feel connected with history, and with the mathematics being explored at given points in time). Here I might point to what else was going on in 1904 (though there was earlier work on fractals), e.g. early flight, definitely no electronics although there were other calculating devices…
- Ask for suggestions of what wouldbe a helpful way to set about constructing one, and finding out what happens next? Accept all suggestions but also allow students to compare and evaluate those made. What size triangle might we start with? (I want students to suggest probably isometric dotty paper, and an initial edge that's a multiple of 3, or better, a multiple of 9 units long, and to be able to explain why those starting points might be useful).
- We might talk about how many stages they'll be able to draw, and the contrasts between mathematical limitations (we could in principle continue indefinitely) and physical limitations.
- I'd ask them to see how far they could get, but while they were doing that, to think of some questions they'd like to ask about the snowflake.

From here, we have gone in a variety of directions, depending on the interests and resilience of the class. We'd usually talk about the recursive and infinite characteristics of the construction, and relate those to other situations we've worked with (most of my classes have wrestled with 1+1/2+1/4+1/8+1/16…. and a variety of other series at some stage). We'd pose some questions, and maybe share some ideas for how we might try to answer those. I'd always offer an opportunity to follow those up, but sometimes 'in a few weeks, when we've met some other situations and you can then choose which you want to work with.' I've found that offering a choice, including of group size (two to four can work), is very constructive for motivation and commitment - even if it means some students have more they want to explore than lesson time allows.

So where might students take the exploration? They will often want to know if they can programme a given shape using logo or Scratch or another language they're familiar with, how many edges there are at any stage, and the perimeter and area then. Some will assume both perimeter and area converge, but want to know what they converge to; others will assume both increase without limit. They will often choose calculators or spreadsheets as tools, and I encourage them to see for themselves the links between the recursive nature of the snowflake and replicative functions available on a spreadsheet. I have had GCSE resit students who have programmed the first 5 or 6 or 7 stages in logo, or who have represented the perimeter and area on a spreadsheet, the latter usually in terms of triangular units of area, gradually devising more efficient approaches. The successive patterns are important, and give rise to in-group conversations that are about reasoning and rigour, that throw up arguments about the relative merits of decimals and of fractions for such purposes, and about the accuracy and precision of what students see on a spreadsheet or diagram or calculator. Other students tire of me asking 'how do you know?' and develop greater rigour in their thinking and eventually begin to critique the limitations of their own argument, so that a few are ready to read about how other mathematicians have approached such situations ('shall I lend you an A Level book?' is wonderfully motivating to a 15-year-old). Others will then 'jump on board', very willing at that time to accept a neat little formula that seems to work, but impressed that such a formula exists.

Helpfully, much of what students might look up on the internet is unnecessarily complex, so they're typically thrown back on their own resources - 'the sets that are constructed this way form a Cauchy sequence in the Hausdorff metric'; 'if we start with a triangle with side length s, the area of the snowflake is $\frac{2\sqrt{3}s^2}{5}$ ', diagrams of the snowflake on a squared grid background - are three such examples. Fine for some teenagers, but not helpful for many. The important part, though, for most students is the idea of an infinite sequence of squiggles all crumpled up within a finite space: infinite perimeter, finite area. And their horizon can then be extended to know this infinite detail is characteristic of fractals; it's fascinating and mind-blowing. And more fractals can follow, even if dropped in as a slide background – they'll notice. Teenagers are very open to the curious, to the surprising, to the frankly beautiful. And typically, awed to be exposed to opportunities to be able to approach such phenomena, and pleased their teacher thinks they might want to – they'll tell you if they don't, but I've yet to meet that except where a student has brought significant other issues to class.

So which teenagers am I thinking about here? This particular piece of maths I haven't introduced before year 9, since I want to save it for when students can appreciate more mathematical ideas rather than fewer. I've often used it in the middle or towards the end of year 10, with groups working towards a GCSE grade 3 or more, who are beginning to build some mathematical stamina, and who have learnt that the purpose of what I introduce might not always be immediately obvious, but that it usually has benefits – in other words, they've learnt to trust me. I've had some wonderfully sophisticated conversations with students in 'top sets', but as above, more moderately-attaining students can also get drawn in to the geometry and mathematics here. The fact I'm asking them to engage with those is very affirming: this is clearly 'proper mathematics', and they can sometimes surprise themselves with the depth of their thinking.

What mathematical thinking would I want to pull out here? That again will vary with the students (teenagers or teachers!): I try to push them beyond their comfort zone, but not so far they drown…..

- The scale and units of analysis are 'up for grabs': mathematicians choose their own, and some are easier to work with than others ('mathematicians are lazy' is a commonly voiced aphorism in my classrooms: appealing to teenagers, but helpful because it makes thinking about the important things easier).
- Mathematicians ask questions about situations – and some questions turn out to be more interesting than others.
- They choose the tools they'll use. Some are good for exploration, others for going further than 'it looks like' to 'it has to be, because…'. Students come to appreciate the benefits of different digital tools – and their limitations for some purposes.
- Sometimes students can build on others' work, but if so, have to use that critically: explaining how it works to peers really helps one's grasp of new ideas.
- Thinking about infinite processes, the infinitely big and the infinitely small, is tricky, though the range of students finds that intriguing. It's sometimes surprising, and can be deeply satisfying.
- There are cultural and historical links: mathematics is a universal human activity, not only for utilitarian purposes, but because we are made pattern-makers and problem-solvers.
- And then there are specific mathematical learning points – importance of unit of analysis, scaling factors in length and in area, analysis of patterns, behaviour of infinite sequences and series…..

But here's the best bit: that's not all. It remains a source of great delight to me that, thirty-five years into my teaching career when I moved to Higher Education, I was still being taught about elementary mathematics by my students – and Koch snowflake work was no exception. I report here a conversation with a keen, but not obviously outstanding, year 10 student, David (now twenty-eight, and a systems analyst), that has stuck in my mind. He was working in a class that had a choice of two tasks, both giving rise to infinite geometric series. I urge you to enjoy my limitations as a teacher, but share my delight in the student's mathematical thinking. Then if the situation is not already familiar, take a pencil and discover the maths……

***David:** I got both of those out (the area and the perimeter) last night, using those nifty formulas in the A Level book with my numbers, and I think I followed those. They confirmed what the spreadsheet was saying, but I can see this is more convincing: it's more than 'it looks like it ought to be…'. And that's really satisfying.*

Me (scanning work, and hoping to build up good communication habits): *That looks great, and I do like the way you've set it out there, very nice and clear … some of my year 12s could learn from that.*

***David:** Me and Ghosh and Sophie, we all worked on this, but Ghosh and Sophie don't want me to tell them how I used it, they want to work it out for themselves.*

***Me:** Fair enough – they'll get much more satisfaction that way. Em …. I think most people are still wrestling with these problems, so tell me, what other questions do you think you might ask, that might be worth exploring?*

***David:** Well – I could ask what would happen if I started with a square instead of a triangle, and perhaps took out a quarter to build another square. No….. I like the symmetry here, so perhaps start with a square, and put another square on the middle third of each side. What would that do, do you think? **Me (mathematical heart overflowing as David homes in on valuing the symmetry):** I have no idea – really. There's only one way to find out! (David goes away, grinning).*

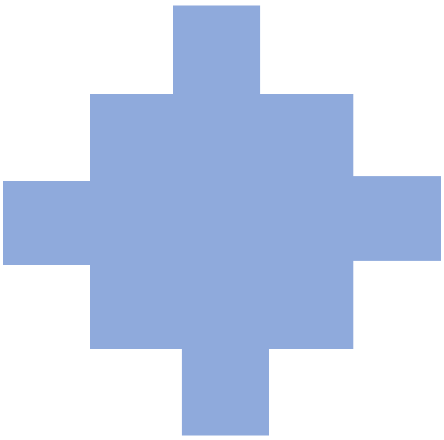

Figure 3: First two stages of David's construction

24 hours later……

***David:** I did it. I started with the drawing, so I could check the first couple of steps and get a feel for what was going on. I was worried about it overlapping, to begin with. It seemed to be working well, though, and the perimeter is heading off to infinity as you'd expect. But there's something about my area working that doesn't feel quite right. I went straight for the formula because it's neater, and I don't want to use a spreadsheet – it seems cheating almost, I want to be sure what I'm doing is right.*

***Me (mathematical heart again overflowing, as David talks about 'it not feeling right'):** OK, take me through what you've done.*

***(2 minutes later) David:** Oh no, that's not right, is it? Why couldn't I see that? It's like you say about talking to the cat. (One of my standard hints and tips: if you're stuck or think you might have gone wrong, try explaining the problem to the cat). OK, don't tell me, let me sort it out. … (David goes to his seat).*

***(5 minutes later) David:** It's working, it's working, but why is it that? …… (Ghosh: Pipe down David, the rest of us are trying to think).*

***(3 minutes after that) David:** I've got it! That is *so cool*, that's really neat.*

And it is, but I'd hate to spoil the satisfaction…..