Abstract—In this paper, we show that acoustoelasticity in hyperelastic materials can be understood using the framework of nonlinear wave mixing, which, when coupled with an induced static stress, leads to a change in the phase velocity of the propagating wave with no change in frequency. By performing Floquet wave eigenvalue analysis, we also show that band gaps for periodic composites, acting as 1-D phononic crystals, can be tuned using this static stress. In the presence of second-order elastic nonlinearities, the phase velocity of propagating waves in the phononic structure changes, leading to observable shifts in the band gaps. Finally, we present numerical examples as evidence that the band gaps are tuned by both the direction of the stress and its magnitude.

Index Terms—Floquet waves, hyperelasticity, nonlinear ultrasound, phononic crystals.

I. INTRODUCTION

MULTILAYERED periodic composites can be represented as 1-D phononic crystals [1], [2], comprising heterogeneous arrays of materials with different elastic properties. Such composites are increasingly being used in the automotive, marine, and aerospace industries as load bearing structures. For example, metal-polymer layered structures are now used in the fuselage of aircraft [3]. Similarly in the semiconductor industry, multilayered microstructures with different lattice spacings or with different thermal expansion coefficients that will induce differential stresses either during manufacture or in service. The lifetime and performance of such structures will depend upon the applied and/or the residual static stresses, induced either during their manufacture [4] and/or during their operation [5]. The detection of unwanted static stress states is key in determining the risk of failure in many of these safety-critical structures [6] as this may lead to the propagation of defects [7], which will ultimately affect their structural or functional performance.

Composite structures can be represented as 1-D phononic crystals, which can contain measurable band gaps [8] for ultrasonic waves. The spectral features of the ultrasound after interacting with the phononic structure will represent characteristics of its inherent periodic structure. Thus, when composite structures experience an applied or residual static stress, the ultrasonic wave propagation velocities will change within the different layers, as the characteristic of the band gap changes.

In this paper, starting from the nonlinear equations of motion [9], we deduce the dispersion relationships in such 1-D phononic structures in the presence of the static stress, which can be described as a zero-frequency wave [10]. These equations allow us to interpret the phenomena of acoustoelasticity in terms of a nonlinear wave mixing process [9] between a propagating wave and a zero-frequency wave. During this mixing, the output frequency does not shift, but the phase velocity of the propagating wave changes. The effect is enhanced by the presence of periodicity within the composite (which acts as a 1-D phononic crystal). As a consequence of the interplay between the phononic crystal and the ultrasonic nonlinear response to the static stress, the effect becomes observable, and indeed, provides a method of characterizing changes in the composite.

Using this principle, we now develop a new model to enable us to analyze how the composite, represented as a band gap in a 1-D phononic crystal, depends upon the applied static stress. Our analysis is underpinned using a nonlinear wave mixing [9] together with a recently revised acoustoelasticity theory in biaxially stressed, hyperelastic plate-like structures [11] and Floquet (Floquet–Bloch) wave theory for an infinite periodic medium [12].

Using this understanding of band gaps in composites potentially introduces an important new method for structural health monitoring. The fact that the band gaps can be controlled actively by the application of a static stress may also, in the future, lead to development of stress sensitive metamaterials [13]. In order to implement this concept, we used a recursive stiffness matrix method [14] to calculate the Floquet wavenumbers and reflection coefficients from a periodic semi-space. Despite the acoustoelastic effect being small [15], numerical analysis shows that the band gaps were very sensitive both to the static stress direction and to its magnitude—thus showing the potential of this method to detect defects in periodic composites. The model was also validated using
data from the previously reported works [11], [16] and these results are presented in Appendix A.

II. Method

The nonlinear equation of motion for an ideal isotropic solid is given in [9]

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mu \frac{\partial^2 \mathbf{v}_i}{\partial x_j \partial x_j} - (\lambda + 2\mu) \frac{\partial^2 \mathbf{v}_i}{\partial x_i \partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} \]

(1)

where \( \rho \) is the density of the undeformed medium, \( \mathbf{v} \) is the displacement vector in the solid, \( x_1 = x, x_2 = y, x_3 = z \), and \( t \) is the space and time coordinates, and \( \lambda \) and \( \mu \) are the Lamé constants. We use here the summation over repeated indices convention. The left-hand side corresponds to the linear wave equation and the right-hand side to the divergence of the nonlinear stress tensor given as

\[ \sigma_{ij}(\mathbf{v}) = (\mu + A/4) \left( \frac{\partial \mathbf{v}_i}{\partial x_j} \frac{\partial \mathbf{v}_i}{\partial x_j} + \frac{\partial \mathbf{v}_j}{\partial x_i} \frac{\partial \mathbf{v}_j}{\partial x_i} \right) + \frac{A}{4} \left( \frac{\partial ^2 \mathbf{v}_i}{\partial x_j \partial x_i} + \frac{\partial ^2 \mathbf{v}_j}{\partial x_i \partial x_j} \right) + B + \lambda \left( \frac{\partial \mathbf{v}_i}{\partial x_i} \frac{\partial \mathbf{v}_j}{\partial x_j} \delta_{ij} + 2 \frac{\partial \mathbf{v}_i}{\partial x_i} \frac{\partial \mathbf{v}_j}{\partial x_j} \right) \]

(2)

where \( A, B, \) and \( C \) are the third-order elastic constants in Landau and Lifshitz notation [17].

In order to model the nonlinear interaction between the static strain and an ultrasonic wave, we consider

\[ \mathbf{v} = \mathbf{e} \cdot \mathbf{r} + \mathbf{u} e^{-i\omega t - k \mathbf{r}} \]

(3)

where \( \mathbf{e} \) is a diagonal static strain tensor with the terms \( (e_{11}, e_{22}, \) and \( e_{33}) \) on the diagonal. Using (1) and (2), we define a modified dispersion relationship linking the static strain to the phase velocity for a wave propagating in the \( x_1 \)-direction (detailed calculations presented in these results are presented in Appendix B).

\[ \frac{\rho c_L^2}{c_L^2} = 2A/e_{11} + (2B + \lambda)(3e_{11} + e_{22} + e_{33}) + 2C(e_{11} + e_{22} + e_{33}) + 6\mu \rho c_{0L}^2 \]

(4)

\[ \frac{\rho c_S^2}{c_S^2} = (A/2 + 2\mu)(e_{11} + e_{33}) + (B + \lambda)(e_{11} + e_{22} + e_{33}) + \rho c_{0S}^2 \]

(5)

where the subscript \( S \) corresponding to the shear wave is either 22 or 33, and \( c_{0L} \) and \( c_{0S} \) are the longitudinal and shear wave velocities, respectively, in the unstrained medium. We show that nonlinear wave mixing and the induction of static stress lead to an effective change in the phase velocity of the propagating wave with a zero-frequency shift [10]. We note that propagation along different directions or nondiagonal stress tensors can also be deduced in a similar manner; however, the dispersion relations, (4) and (5), will contain more terms and the stressed media will show anisotropic properties.

To analyze the elastic wave propagation in these hyperelastic 1-D phononic crystals, we implement a recursive stiffness matrix method. The Floquet wave equation is given as [14]

\[ A_3 \cos(3k_z F H) + A_2 \cos(2k_z F H) + A_1 \cos(k_z F H) + A_0 = 0 \]

(6)

where \( k_z F \) is the vertical Floquet wavenumber and \( H \) is the unit cell thickness (Fig. 1). The following are the Floquet wave coefficients, \( A_i \), (these have been modified and updated from those originally reported [18]) in terms of the unit cell stiffness matrix

\[ A_3 = |K_{c1}^{21}| \]

(7)

\[ A_2 = 1/2(|M + K_{c1}^{21} + |M - K_{c1}^{21}|) - |M| \]

(8)

\[ A_1 = 1/2(|M + K_{c1}^{21} - |M + K_{c1}^{12}| + |K_{c1}^{21} - K_{c1}^{12}|) - 2|K_{c1}^{21}| \]

(9)

\[ A_0 = 1/4(|M + K_{c1}^{12} - K_{c1}^{21}| + |M - K_{c1}^{12} + K_{c1}^{21}|) - A_2 \]

(10)

where \( M = K_{c1}^{22} - K_{c1}^{11} \), \( |M| \) is the determinant of matrix \( M \) and \( K_{c1} \) is the whole stiffness matrix of the unit periodic cell [14].

The amplitude reflection coefficient from a submerged semi-space in terms of the Floquet wave equation parameters [19] can thus be written in the following form:

\[ R_{as} = \frac{\delta_{c}^{31} - \cos \theta / (i \omega p_{F} c_{c} F)}{\delta_{c}^{31} + \cos \theta / (i \omega p_{F} c_{c} F)} \]

(11)

where \( \theta \) is the wave incidence angle (Fig. 1), \( \omega = 2\pi f \), \( f \) is the wave frequency, \( p_{F} \) is the volumetric mass density, and \( c_{c} \) is the wave speed. \( \delta_{c}^{31} \) is the (3, 1) element in the 3 × 3 surface compliance matrix for a homogeneous or layered anisotropic semispace.

As an example of a 1-D phononic crystal (see Fig. 1), we now show results for a two-layered unit cell of the metal polymer periodic medium containing an aluminum and a polyvinylchloride (PVC) layer of 0.1 mm thickness. The following second-order elastic material properties were used in the eigenvalue analysis of the Floquet waves in the 1-D hyperelastic phononic structure: \( \lambda_1 = 54.307 \) GPa, \( \mu_1 = 27.174 \) GPa, and \( \rho_1 = 2704 \) kg/m\(^3\); whilst \( \lambda_2 = 3.8745 \) GPa, \( \mu_2 = 1.6335 \) GPa, and \( \rho_2 = 1350 \) kg/m\(^3\) for the aluminum and PVC layers, respectively [9], [11]. The corresponding
third-order elastic constants are in Murnaghans notation [20]:

\[ l_1 = -281.5 \text{ GPa}, m_1 = -339 \text{ GPa}, \text{ and } n_1 = -416 \text{ GPa} \]

\[ l_2 = -33.43 \text{ GPa}, m_2 = -20.88 \text{ GPa}, \text{ and } n_2 = -15.86 \text{ GPa} \]

The relationships between the third-order elastic constants in Landau and Lifshitz and Murnaghan’s notations are given in Appendix C. The above semispace fluid properties are

\[ c_F = 1480 \text{ m/s} \text{ and } \rho_F = 1000 \text{ kg/m}^3. \]

III. RESULTS

In our analysis three scenarios were investigated, namely:

1) using an applied static stress \( \sigma_{22} \), which is constant and equal to 200 MPa (tensile case), but where we vary both the angle \( \phi \) (see Fig. 1) in the range \( 0^\circ - 90^\circ \) with \( 1^\circ \) increments and the incident wave frequency \( f \) in the range \( 0-20.0 \text{ MHz} \) with 1 kHz increments (this scenario corresponds to \( \phi - f \) plane); 2) using an angle \( \phi \) which is constant and equal to 90\(^\circ\), but where we vary both the stress \( \sigma_{22} \) in the range \(-200-200 \text{ MPa} \) with 4.35 MPa increments and the incident wave frequency varies in the range \( 0-20.0 \text{ MHz} \), with 1 kHz increments (this scenario corresponds to \( \sigma - f \) plane); and 3) the scenario where the frequency of the incidence wave is kept constant and equal to 20 MHz, and where we vary both the angle \( \phi \) in the range \( 0^\circ-90^\circ \) and the stress \( \sigma_{22} \) in the range \(-200-200 \text{ MPa} \) (the last scenario corresponds to \( \sigma - \phi \) plane). In all three scenarios, the wave incidence angle was \( \theta = 35^\circ \) (to provide a representative illustration of the influence of the acoustoelastic effect on band gaps in the periodic semispace).

Fig. 2 shows the subsequent 3-D plot of the energy reflection coefficient from the fluid loaded stressed periodic semispace. We used a static stress limit of \( \pm 200 \text{ MPa} \), which can be either an applied or residual stress within the structure [21].

For band gaps which correspond to zones where \( R = 1 \), passbands are formed where \( R < 1 \). The results also show that the band gaps have a strong dependence (band gap becomes 2.5 times narrower due to the stress) on the static stress and the wave propagation direction \( \phi \) (Fig. 1) in the periodic semispace \( \phi - f \) plane. The band gaps depend significantly on the stress value when \( \phi = 90^\circ \), see plane \( \sigma - f \) in Fig. 2.

When \( \sigma = 0 \text{ MPa} \), the energy reflection coefficient corresponds to the unstressed semispace case. When the incidence wave frequency \( f \) is equal to 20.0 MHz (plane \( \sigma - \phi \)), the results also show that a band gap is formed in a wide range of the parameters \( \sigma \) and \( \phi \). The results are presented in more detail for the second scenario, which demonstrates the corresponding response for compression and tensile stress case in \( \phi - f \) plane.

Furthermore, we represent the ultrasonic wave response of the periodic structure in terms of Floquet wavenumbers, where the unit cell thickness product \( k_F = k_i \times n \) and energy reflection coefficients \( R \) from a periodic semispace loaded by a fluid when the angle \( \phi = 90^\circ, 45^\circ, 0^\circ \), see Fig. 1. Two separate cases are considered in the analysis with respect to the angle \( \phi \), namely, when the shear horizontal wave motion is not coupled to the sagittal wave motion [Fig. 3(a)–(c) and (g)–(i)], and second, when coupling occurs between the shear horizontal and sagittal wave motions [Fig. 3(d)–(f)]. In the reference case, where \( \sigma_{22} = 0 \text{ MPa} \), and \( m_{0,i} \) and \( m_{1,j} \) denote the out-of-plane shear wave and in-plane shear and longitudinal wave modes, respectively. Modes in both compressive and tensile cases are denoted as \( n_{0,i} \) and \( n_{1,j} \). For the reference case, the 1-D phononic crystal contains three well-defined zones [see Fig. 3(a)] comprising an effective homogeneous medium from 0 to 3.7 MHz, [see Fig. 3(b), mode \( m_{1,i} \)]; the first band gap in the frequency range 3.7–9.1 MHz, and finally, the main passband in the frequency range 10.0–17.1 MHz.

When the static stress direction is coincident with the wave propagation direction \( \phi = 90^\circ \), Fig. 3(a)–(c), significant changes occur in the band gap zones of the phononic crystal, despite the shear wave motion being decoupled from the sagittal wave motion. In this case, however, the first band gap is almost unaffected in the compression stress. However, in the tensile case, this band gap reduces from 3.7–9.1 MHz to 3.8–5.9 MHz, becoming 2.5 times narrower. This is due to a change in the mode dispersion of the structure as shown in Fig. 3(b) of the stressed mode \( n_{1,2} \) with respect to the same reference mode \( m_{1,2} \). The tensile static stress [Fig. 3(c)] also shifts down the second band gap which occurs in the frequency range 12.4–13.9 MHz, see the modes \( m_{1,2,3} \) and \( m_{1,4,3} \). The compression stress [Fig. 3(a)] causes a significant band-gap shift in the frequency range 10–13.3 MHz, this is shown in Fig. 3(b) where the reference mode \( m_{1,2} \) is shifted up, but the dispersion remains unchanged, whereas \( m_{1,2} \) mode does not shift significantly but its dispersion is changed. The out-of-plane modes \( m_{0,i} \) are relatively insensitive to the applied stress in comparison with \( m_{1,j} \) modes.

In the case of the \( m_{1,j} \) modes, the results show that the smallest difference in ultrasonic response from the stressed periodic structures is observed when the wave propagation and stress directions are orthogonal (\( \phi = 0^\circ \)) [see Fig. 3(g)–(i)]. In this case, the \( n_{0,i} \) modes are sensitive to the applied stress and these modes are consequently significantly shifted (see the modes \( n_{0,2} \) and \( n_{0,3} \)).

Finally, when the angle \( \phi = 0^\circ \) or \( \phi = 90^\circ \), the modes present in the structure are pure. However, when the angle \( \phi \) deviates from these values, the modes no longer remain pure and the shear horizontal wave motion couples into the
Fig. 3. Ultrasonic response from a periodic semispace loaded by fluid when the incident beam angle $\theta = 35^\circ$ and angle $\phi$ is (a)–(c) $\phi = 90^\circ$, (d)–(f) $\phi = 45^\circ$, and (g)–(i) $\phi = 0^\circ$. The energy reflection coefficients for the compression case [(a), (d), and (g)] and tensile stress case [(c), (f), and (i)] are shown. The corresponding Floquet wavenumber and unit cell thickness product $\Re(k_{_F} \times H)$ is depicted in (b), (e), and (h). Blue curves: the reference case when $\sigma_{22} = 0$ MPa. Green curves: the response when $\sigma_{22} = -200$ MPa. Red curves: the response $\sigma_{22} = 200$ MPa.

is caused by a change in the $m_{12}$ reference mode dispersion, see $n_{12}$. The main passband is also narrower as is seen in the frequency ranges 12.0–17.4 MHz and 10.9–15.0 MHz for the compression and tensile stress cases, respectively. These latter changes occur due to $m_{13}$ mode shifting up in the case of compression stress, and shifting down in the case of tensile stress, see mode $n_{13}$.

IV. CONCLUSION

In conclusion, we show that static stress together with second-order material nonlinearities have a significant influence on the band gaps in 1-D phononic crystals. This acoustoelastic effect can be understood as the nonlinear wave mixing between a zero-frequency wave and a propagating wave. Our results extend recently reported work [13], where analysis was conducted using small amplitude motions in a normal direction, while we now consider finite amplitude elastic waves having oblique angle propagation.

We show that the band gaps are highly tunable with respect to the direction and the magnitude of static stress. We also show that the effect is enhanced both by the resonances in the 1-D phononic crystals and by the coupling between the shear horizontal and sagittal wave. In the future, our study has the potential to enable a number applications in industries using periodic composites structures, including the integrity of large-scale composite structures used in the aerospace industry or of stresses caused by thermal mismatches in microstructures created within the semiconductor industry.

APPENDIX A

VALIDATION OF THE IMPLEMENTATION OF THE STIFFNESS MATRIX METHOD FOR GUIDED AND FLOQUET WAVES IN LAYERED STRUCTURES

Fig. 4 shows the Lamb wave phase velocity dependence for three modes, two of which are symmetric, $s_0$ and $s_1$, and one that is antisymmetric $a_1$, with respect to the angle sagittal wave motion. For example, this coupling is shown for the case when $\phi = 45^\circ$ in the reflection coefficient, for the frequency range 3.8–5.3 MHz [Fig. 3(f), mode $n_{11}$] in the tensile stress case (where weaker coupling is observed in the compression stress case). The modes $n_{12}$ and $n_{13}$ have a higher coupling to the sagittal wave motion. In the compression stress case [Fig. 3(d)], the first band gap is less affected than in the tensile stress case. The results show that in the tensile stress case, the first band gap is formed in the frequency range 5.3–7.5 MHz, and it is 2.5 narrower compared with the band gap when $\sigma_{22} = 0$ [see Fig. 3(f)]. The narrowing
\[ \sigma_{ij} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho \frac{\partial^2 u_j}{\partial t^2} - (\lambda + 2 \mu) \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) \]

where the polarization amplitude \( \mathbf{u} \) is parallel to the wave vector \( \mathbf{k} \). Similarly, we can determine the nonlinear stress tensor for the superposition (3)

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = -\mu_0 \rho e^{-i(\omega - \mathbf{k} \cdot \mathbf{r})} k^2 (2C(e_{11} + e_{22} + e_{33})2Ae_{11} + (2B + \lambda)(3e_{11} + e_{22} + e_{33}) + 6e_{11} \mu) . \]

Equating (12) and (13) allows us to introduce a new static strain dependent wave velocity, \( c_{0L} \), as defined by (4). The wave part of the superposition is then a solution of the linear wave equation taking this modified velocity into account.

In the same way, we proceed to evaluate the effect of the static strain on shear waves. In this case, the linear part of the wave equation evaluates to

\[ \frac{\partial^2 v_i}{\partial t^2} - \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \rho \frac{\partial^2 v_j}{\partial t^2} = u_1 e^{-i(\omega - \mathbf{k} \cdot \mathbf{r})} k^2 (\lambda + \mu) \]

where the polarization amplitude \( \mathbf{u} \) is perpendicular to the wave vector \( \mathbf{k} \). The nonlinear stress tensor for the superposition (3) in this case is equal to

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = -\mu_0 \rho e^{-i(\omega - \mathbf{k} \cdot \mathbf{r})} k^2 (A/2 + 2\mu)(e_{11} + e_S) + (B + \lambda)(e_{11} + e_{22} + e_{33}) \]

where the subscript \( S \) corresponds to the shear wave direction with indices equal to either 22 or 33. Similar to the longitudinal case, equating (14) and (15) allows us to introduce a new static strain dependent wave velocity, \( c_{0S} \), defined by (5). The wave part of the superposition is then a solution of the linear wave equation taking this modified velocity into account.

APPENDIX C

RELATIONSHIP BETWEEN THIRD-ORDER ELASTIC CONSTANTS FOR ISOTROPIC SOLIDS

The relationships between the third-order elastic constants in Landau and Lifshitz (\( A, \ B, \) and \( C \)) and Murnaghan’s notations are given as

\[ l = B + C, \ m = A/2 + B, \ n = A \]

where \( l, m, \) and \( n \) are the third-order elastic constants in Murnaghan’s notations.

ACKNOWLEDGMENT

The authors would like to thank Dr. J. Michaels from Georgia Tech and Dr. E. L. Tan from Nanyang Technological University for providing data used in Figs. 4 and 5, and also thank Dr. M. Tassieri for his helpful discussions. All data created during this research are openly available from the University of Glasgow at http://dx.doi.org/10.5525/gla.researchdata.600.

REFERENCES

Andriejus Demčenko received the B.Sc. degree in telecommunications engineering, the M.Sc. and Ph.D. degrees in measurement engineering from the Kaunas University of Technology, Kaunas, Lithuania, in 2001, 2003, and 2008, respectively, and the Ph.D. degree in mechanical engineering from the University of Twente, Enschede, The Netherlands, in 2014.

He is currently a Research Associate with the Division of Biomedical Engineering, School of Engineering, University of Glasgow, Glasgow, Scotland. His current research interests include physical linear and nonlinear ultrasonics in solid media.

Michael Mazilu was born in Bucharest, Romania, in 1969. He received the Magister degree in materials with a focus on electronic and mechanic properties and the Postgraduate Diploma degree in condensed matter and material physics from Louis Pasteur University, Strasbourg, France, and the Ph.D. degree in optoelectronics from the University of St. Andrews, St. Andrews, Scotland.

He is currently a Senior Lecturer with the School of Physics and Astronomy, University of St. Andrews. His research interests include fundamental and applied photonics and phononics.

Rab Wilson received the B.Sc. degree (Hons.) in mathematics with theoretical physics from the University of Aberdeen, Aberdeen, Scotland, in 1990, and the M.Sc. degree in advanced materials and the Ph.D. degree with a focus on ferroelectric thin films from Cranfield University, Bedfordshire, U.K., in 1998 and 2002, respectively.

He is currently a Research Associate with the University of Glasgow, Glasgow, Scotland, working on phononic crystal waveguide structures and devices.

Arno W. F. Volker received the M.Sc. degree in applied physics and the Ph.D. degree (Hons.) in geophysics from the Delft University of Technology, Delft, The Netherlands, in 1997 and 2002, respectively. In 1996, he joined TNO, Den Haag, The Netherlands, as a Student and a Staff Member in 1997. He has been working on ultrasonic measurement solutions for industrial applications for more than 20 years. He is currently a Senior Scientist with the Acoustics and Sonar Department, TNO.

Dr. Volker is a member of Society of Exploration Geophysicists and European Association of GeoScientists and Engineers.

Jonathan M. Cooper holds The Wolfson Chair in Biomedical Engineering. He is an EPSRC Research Fellow and holds a European Research Council Advanced Programme Grant. His major research interests are in ultrasonics, fluidics, and medical diagnostics.

Prof. Cooper was elected as a Fellow of the Royal Academy of Engineering (U.K.’s national academy of engineering) as well as a Fellow of the Royal Society of Edinburgh (Scotland’s national academy of arts, humanities and sciences).