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The second Weyl coefficient for a first order system

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Abstract

For a scalar elliptic self-adjoint operator on a compact manifold without boundary we have two-term asymptotics for the number of eigenvalues between 0 and λ when $\lambda \to \infty$, under an additional dynamical condition. (See [3, Theorem 3.5] for an early result in this direction.)

In the case of an elliptic system of first order, the existence of two-term asymptotics was also established quite early and as in the scalar case Fourier integral operators have been the crucial tool. The complete computation of the coefficient of the second term was obtained only in the 2013 paper [2]. In the present paper we simplify that calculation. The main observation is that with the existence of two-term asymptotics already established, it suffices to study the resolvent as a pseudodifferential operator in order to identify and compute the second coefficient.

Keywords: spectral theory, asymptotic distribution of eigenvalues.

MSC classes: primary 35P20; secondary 35J46, 35R01.

Contents

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1 Statement of the problem

Let A be a first order linear psedodifferential operator acting on m-columns of complex-valued half-densities over a connected closed (i.e. compact and without boundary) n-dimensional manifold M. Throughout this paper we assume that $m, n \geq 2$.

Let $A_1(x, \xi)$ and $A_{\mathrm{sub}}(x, \xi)$ be the principal and subprincipal symbols of A. Here $x = (x^1, \ldots, x^n)$ denotes local coordinates and $\xi = (\xi_1, \ldots, \xi_n)$ denotes the dual variable (momentum). The principal and subprincipal symbols are $m \times m$ matrix-functions on $T^*M \setminus \{\xi = 0\}$.

Recall that the concept of subprincipal symbol originates from the classical paper [4] of J. J. Duistermaat and L. Hörmander: see formula (5.2.8) in that paper. Unlike [4], we work with matrix-valued symbols, but this does not affect the formal definition of the subprincipal symbol.

We assume our operator A to be formally self-adjoint (symmetric) with respect to the standard inner product on m-columns of complex-valued half-densities, which implies that the principal and subprincipal symbols are Hermitian. We also assume that our operator A is elliptic:

$$\det A_1(x, \xi) \neq 0, \qquad \forall (x, \xi) \in T^*M \setminus \{0\}. \tag{1.1}$$

Let $h^{(j)}(x, \xi)$ be the eigenvalues of the matrix-function $A_1(x, \xi)$. Throughout this paper we assume that these are simple for all $(x, \xi) \in T^*M \setminus \{0\}$. The ellipticity condition (1.1) ensures that all our $h^{(j)}(x, \xi)$ are nonzero.

We enumerate the eigenvalues of the principal symbol $h^{(j)}(x, \xi)$ in increasing order, using a positive index $j = 1, \ldots, m^+$ for positive $h^{(j)}(x, \xi)$ and a negative index $j = -1, \ldots, -m^-$ for negative $h^{(j)}(x, \xi)$. Here m^+ is the number of positive eigenvalues of the principal symbol and m^- is the number of negative ones. Of course, $m^+ + m^- = m$.

Let λ_k and $v_k(x)$ be the eigenvalues and the orthonormal eigenfunctions of the operator A; the particular enumeration of these eigenvalues (accounting for multiplicities) is irrelevant for our purposes. Each $v_k(x)$ is, of course, an m-column of half-densities.

Let us define the two local counting functions

$$N_\pm(x, \lambda) := \begin{cases} 0 & \text{if} \quad \lambda \leq 0, \\ \sum_{0 < \pm\lambda_k < \lambda} \|v_k(x)\|^2 & \text{if} \quad \lambda > 0. \end{cases} \tag{1.2}$$

The function $N_+(x,\lambda)$ counts the eigenvalues λ_k between zero and λ, whereas the function $N_-(x,\lambda)$ counts the eigenvalues λ_k between $-\lambda$ and zero. In both cases counting eigenvalues involves assigning them weights $\|v_k(x)\|^2$. The quantities $\|v_k(x)\|^2$ are densities on M and so are the local counting functions $N_\pm(x,\lambda)$.

Let $\hat\rho:\mathbb{R}\to\mathbb{C}$ be a smooth function such that $\hat\rho(t)=1$ in some neighbourhood of 0 and the support of $\hat\rho$ is sufficiently small. Here 'sufficiently small' means that $\operatorname{supp}\hat\rho\subset(-\mathbf{T},\mathbf{T})$, where $\mathbf{T}$ is the infimum of the lengths of all possible loops. A loop is defined as follows. For a given j, let $(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))$ denote the Hamiltonian trajectory originating from the point (y,η), i.e. solution of the system of ordinary differential equations (the dot denotes differentiation in time t)

$$\dot x^{(j)}=h^{(j)}_\xi(x^{(j)},\xi^{(j)}),\qquad \dot\xi^{(j)}=-h^{(j)}_x(x^{(j)},\xi^{(j)})$$

subject to the initial condition $(x^{(j)},\xi^{(j)})\big|_{t=0}=(y,\eta)$. Suppose that we have a Hamiltonian trajectory $(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))$ and a real number $T>0$ such that $x^{(j)}(T;y,\eta)=y$. We say in this case that we have a loop of length T originating from the point $y\in M$.

We denote $\rho(\lambda):=\mathcal{F}^{-1}_{t\to\lambda}[\hat\rho(t)]$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform. See [2, Section 6] for details.

Further on we will deal with the mollified counting functions $(N_\pm*\rho)(x,\lambda)$ rather than the original discontinuous counting functions $N_\pm(x,\lambda)$. Here the star stands for convolution in the variable λ. More specifically, we will deal with the derivative, in the variable λ, of the mollified counting functions. The derivative will be indicated by a prime.

It is known [1, 2, 9, 10, 11, 12, 13, 15, 16] that the functions $(N'_\pm*\rho)(x,\lambda)$ admit asymptotic expansions in integer powers of λ:

$$(N'_\pm*\rho)(x,\lambda)=a^\pm_{n-1}(x)\,\lambda^{n-1}+a^\pm_{n-2}(x)\,\lambda^{n-2}+a^\pm_{n-3}(x)\,\lambda^{n-3}+\dots\quad\text{as}\quad\lambda\to+\infty.\tag{1.3}$$

Definition 1.1. We call the coefficients $a^\pm_k(x)$ appearing in formula (1.3) local Weyl coefficients.

Note that our definition of Weyl coefficients does not depend on the choice of mollifier ρ.

It is also known [1, 2, 9, 10, 11, 12, 13, 15, 16] that under appropriate geometric conditions we have

$$N_\pm(x,\lambda)=\frac{a^\pm_{n-1}(x)}{n}\lambda^n+\frac{a^\pm_{n-2}(x)}{n-1}\lambda^{n-1}+o(\lambda^{n-1})\quad\text{as}\quad\lambda\to+\infty.\tag{1.4}$$

Remark 1.2. Our Definition 1.1 is somewhat nonstandard. It is customary to call the coefficients appearing in the asymptotic expansion (1.4) Weyl coefficients rather than those in (1.3). However, for the purposes of this paper we will stick with Definition 1.1.

Further on we deal with the coefficients $a^+_k(x)$. It is sufficient to derive formulae for the coefficients $a^+_k(x)$ because one can get formulae for $a^-_k(x)$ by replacing the operator A by the operator $-A$.

If the principal symbol of our operator A is negative definite, then the operator has a finite number of positive eigenvalues and all the coefficients $a^+_k(x)$ vanish. So further on we assume that the principal symbol has at least one positive eigenvalue. In other words, we assume that $m^+\geq 1$.

The task at hand is to write down explicit formulae for the coefficients $a_{n-1}^+(x)$ and $a_{n-2}^+(x)$ in terms of the principal and subprincipal symbols of the operator A.

The explicit formula for the coefficient $a_{n-1}^+(x)$ has been known since at least 1980, see, for example, [9, 10, 11, 12, 13, 15, 16]. It reads

$$a_{n-1}^+(x)=\frac{n}{(2\pi)^n}\sum_{j=1}^{m^+}\ \int\limits_{h^{(j)}(x,\xi)<1} d\xi\,, \tag{1.5}$$

where $d\xi=d\xi_1\ldots d\xi_n$.

The explicit formula for the coefficient $a_{n-2}^+(x)$ was derived only in 2013, see [2, formula (1.24)]. This formula reads

$$\begin{aligned} a_{n-2}^+(x)=-\frac{n(n-1)}{(2\pi)^n}\sum_{j=1}^{m^+}\ \int\limits_{h^{(j)}(x,\xi)<1} & \Bigl([v^{(j)}]^*A_{\mathrm{sub}}v^{(j)} \\ & -\frac{i}{2}\{[v^{(j)}]^*,A_1-h^{(j)},v^{(j)}\}+\frac{i}{n-1}h^{(j)}\{[v^{(j)}]^*,v^{(j)}\}\Bigr)(x,\xi)\,d\xi\,. \end{aligned} \tag{1.6}$$

Here curly brackets denote the Poisson bracket on matrix-functions $\{P,R\}:=P_{x^\alpha}R_{\xi_\alpha}-P_{\xi_\alpha}R_{x^\alpha}$ and its further generalisation

$$\{F,G,H\}:=F_{x^\alpha}GH_{\xi_\alpha}-F_{\xi_\alpha}GH_{x^\alpha}\,, \tag{1.7}$$

where the subscripts x^α and ξ_α indicate partial derivatives and the repeated index α indicates summation over $\alpha=1,\ldots,n$.

Note that if $q(x,\xi)$ is a function on $T^*M\setminus\{0\}$ positively homogeneous in ξ of degree 0, then

$$\int\limits_{h^{(j)}(x,\xi)<1} q(x,\xi)\,d\xi$$

is a density on M. Hence, the quantities (1.5) and (1.6) are densities.

The problem with the derivation of formula (1.6) given in [2] was that it was very complicated. The aim of the current paper is to provide an alternative, much simpler, derivation of formula (1.6).

It may be that the approach outlined in the current paper would allow one, in the future, to calculate further coefficients in the asymptotic expansion (1.3). Note that for an operator that is not semibounded this is a nontrivial task.

2 Strategy for the evaluation of the second Weyl coefficient

Let $z\in\mathbb{C}$, $\operatorname{Im}z>0$. Our basic idea is to consider the resolvent $(A-zI)^{-1}$ and, by studying it, recover the second Weyl coefficient $a_{n-2}^+(x)$. Unfortunately, the operator $(A-zI)^{-1}$ is not of trace class, therefore one has to modify our basic idea so as to reduce our analysis to that of trace class operators.

Let us consider the self-adjoint operator

$$i\left[2(A-zI)^{1-n}-(A-2zI)^{1-n}-2(A-\bar zI)^{1-n}+(A-2\bar zI)^{1-n}\right]. \tag{2.1}$$

We claim that the operator (2.1) is of trace class. In order to justify this claim we calculate below, for fixed z, the principal symbol of the operator (2.1) and show that it has degree of homogeneity $-n-1$.

Let B be the parametrix (approximate inverse) of A, see [18, Section 5] for details. Then, modulo $L^{-\infty}(M)$ (integral operators with infinitely smooth integral kernels), we have

$$A - zI \equiv A - zAB = A(I - zB),$$
$$(A - zI)^{n-1} \equiv A^{n-1}(I - zB)^{n-1},$$
$$(A - zI)^{1-n} \equiv (I - zB)^{1-n}A^{1-n} \equiv (I - zB)^{1-n}B^{n-1}. \tag{2.2}$$

But

$$(I - zB)^{1-n} \equiv I + (n-1)zB - \frac{n(n-1)}{2}(zB)^2 + \dots, \tag{2.3}$$

where the expansion is understood as an asymptotic expansion in smoothness (each subsequent term is a pseudodifferential operator of lower order). Substituting (2.3) into (2.2), we get

$$(A - zI)^{1-n} \equiv B^{n-1} + (n-1)zB^n - \frac{n(n-1)}{2}z^2B^{n+1} + \dots. \tag{2.4}$$

Replacing z by $2z$, we get

$$(A - 2zI)^{1-n} \equiv B^{n-1} + 2(n-1)zB^n - 2n(n-1)z^2B^{n+1} + \dots. \tag{2.5}$$

Formulae (2.4) and (2.5) imply

$$2(A - zI)^{1-n} - (A - 2zI)^{1-n} \equiv B^{n-1} + n(n-1)z^2B^{n+1} + \dots. \tag{2.6}$$

Replacing z by $\bar{z}$, we get

$$2(A - \bar{z}I)^{1-n} - (A - 2\bar{z}I)^{1-n} \equiv B^{n-1} + n(n-1)\bar{z}^2B^{n+1} + \dots. \tag{2.7}$$

Formulae (2.6) and (2.7) imply that the operator (2.1) is a pseudodifferential operator of order $-n-1$ with principal symbol $-4n(n-1)(\operatorname{Re} z)(\operatorname{Im} z)A_1^{-n-1}$.

It might seem more natural to consider the operator

$$(A - zI)^{-n-1} \tag{2.8}$$

instead of (2.1). The operator (2.8) is also of order $-n-1$, hence, trace class. Unfortunately, the algorithm presented in the remainder of this section won't work for the operator (2.8). The reason is that if we start with (2.8), we end up with the integral

$$\int_0^{+\infty} \frac{\mu^{n-2}}{(\mu - z)^{n+1}}\, d\mu\,, \tag{2.9}$$

where the exponent in the numerator is lower that the exponent in the denominator by more than one. The integral (2.9) is a polynomial in $\frac{1}{z}$ (no logarithm!) and it does not experience a jump when z crosses the positive real axis. Starting with (2.8) one can recover $a_{n-2}^+ - (-1)^n a_{n-2}^-$, but it appears to be impossible to recover a_{n-2}^+ itself. We need a logarithm in order to separate contributions from positive and negative eigenvalues.

The operator (2.1) is a pseudodifferential operator of order $-n-1$, hence it has a continuous integral kernel. This observation allows us to introduce the following definition.

Definition 2.1. By $f(x,z)$ we denote the real-valued continuous density obtained by restricting the integral kernel of the operator (2.1) to the diagonal $x=y$ and taking the matrix trace tr.

The explicit formula for our density is

$$f(x,z)=i\sum_{\lambda_k}\left[\frac{2}{(\lambda_k-z)^{n-1}}-\frac{1}{(\lambda_k-2z)^{n-1}}-\frac{2}{(\lambda_k-\bar z)^{n-1}}+\frac{1}{(\lambda_k-2\bar z)^{n-1}}\right]\|v_k(x)\|^2\,. \tag{2.10}$$

This formula can be equivalently rewritten as

$$\begin{aligned} f(x,z)=i\int_0^{+\infty}&\left[\frac{2}{(\mu-z)^{n-1}}-\frac{1}{(\mu-2z)^{n-1}}-\frac{2}{(\mu-\bar z)^{n-1}}+\frac{1}{(\mu-2\bar z)^{n-1}}\right]N_+'(x,\mu)\,d\mu\\ &-(-1)^n\frac{2^n-1}{2^{n-1}}i\left[\frac{1}{z^{n-1}}-\frac{1}{\bar z^{n-1}}\right]\sum_{\lambda_k=0}\|v_k(x)\|^2\\ -(-1)^n\,i\int_0^{+\infty}&\left[\frac{2}{(\mu+z)^{n-1}}-\frac{1}{(\mu+2z)^{n-1}}-\frac{2}{(\mu+\bar z)^{n-1}}+\frac{1}{(\mu+2\bar z)^{n-1}}\right]N_-'(x,\mu)\,d\mu\,. \end{aligned} \tag{2.11}$$

The expression in the second line of (2.11) is the contribution from the kernel (eigenspace corresponding to the eigenvalue zero) of the operator A.

Let us also introduce another density

$$\begin{aligned} f^\rho(x,z):=i\int_0^{+\infty}&\left[\frac{2}{(\mu-z)^{n-1}}-\frac{1}{(\mu-2z)^{n-1}}-\frac{2}{(\mu-\bar z)^{n-1}}+\frac{1}{(\mu-2\bar z)^{n-1}}\right](N_+'{*}\rho)(x,\mu)\,d\mu\\ -(-1)^n\,i\int_0^{+\infty}&\left[\frac{2}{(\mu+z)^{n-1}}-\frac{1}{(\mu+2z)^{n-1}}-\frac{2}{(\mu+\bar z)^{n-1}}+\frac{1}{(\mu+2\bar z)^{n-1}}\right](N_-'{*}\rho)(x,\mu)\,d\mu\,. \end{aligned} \tag{2.12}$$

Put $z=\lambda e^{i\varphi}$, where $\lambda>0$ and $0<\varphi<\pi$. We will now fix the angle φ and examine what happens when $\lambda\to+\infty$.

Lemma 2.2. *The density $f^\rho(x,\lambda e^{i\varphi})-f(x,\lambda e^{i\varphi})$ tends to zero as $\lambda\to+\infty$.*

Proof See Appendix A. □

Lemma 2.3. *The density $f^\rho(x,\lambda e^{i\varphi})$ admits the asymptotic expansion*

$$f^\rho(x,\lambda e^{i\varphi})=b_1(x,\varphi)\lambda+b_0(x,\varphi)+o(1)\quad\text{as}\quad\lambda\to+\infty, \tag{2.13}$$

where

$$b_1(x,\varphi)=-4(\ln 2)(n-1)(\sin\varphi)\left[a_{n-1}^+(x)+(-1)^n\,a_{n-1}^-(x)\right], \tag{2.14}$$

$$b_0(x,\varphi)=-2\left[(\pi-\varphi)\,a_{n-2}^+(x)+(-1)^n\,\varphi\,a_{n-2}^-(x)\right]. \tag{2.15}$$

Proof See Appendices B and C. □

Lemmata 2.2 and 2.3 imply the following corollary.

Corollary 2.4. *The density $f(x,\lambda e^{i\varphi})$ admits the asymptotic expansion*

$$f(x,\lambda e^{i\varphi})=b_1(x,\varphi)\lambda+b_0(x,\varphi)+o(1)\quad\text{as}\quad\lambda\to+\infty, \tag{2.16}$$

where the coefficients $b_1(x,\varphi)$ and $b_0(x,\varphi)$ are given by formulae (2.14) *and* (2.15) *respectively.*

Suppose that we know the coefficient $b_0(x,\varphi)$ for all $\varphi \in (0,\pi)$. It is easy to see that formula (2.15) allows us to recover the second Weyl coefficient $a^+_{n-2}(x)$. Namely, if we take an arbitrary pair of distinct $\varphi_1, \varphi_2 \in (0,\pi)$ then

$$a^+_{n-2}(x) = \frac{\varphi_1\, b_0(x,\varphi_2) - \varphi_2\, b_0(x,\varphi_1)}{2\pi(\varphi_2 - \varphi_1)}\,. \tag{2.17}$$

Alternatively, the second Weyl coefficient $a^+_{n-2}(x)$ can be recovered by means of the identity

$$a^+_{n-2}(x) = -\frac{1}{2\pi} \lim_{\varphi\to 0^+} b_0(x,\varphi)\,. \tag{2.18}$$

Formulae (2.16)–(2.18) tell us that the problem of evaluating the second Weyl coefficient has been reduced to evaluating the second coefficient in the asymptotic expansion of the density $f(x,\lambda e^{i\varphi})$ as $\lambda \to +\infty$. Recall that the latter is defined in accordance with Definition 2.1.

3 The Weyl symbol of the resolvent

Let $z = \lambda e^{i\varphi}$, where $\lambda > 0$ and $0 < \varphi < \pi$. We formally assign to z a 'weight', as if it were positively homogeneous in ξ of degree 1. Our argument goes along the lines of [18, Section 9].

We performed formal calculations evaluating the symbol of the operator $(A - zI)^{-1}$ in local coordinates and then switched to the Weyl symbol. (One could have worked with Weyl symbols from the very start.) Further on we denote the Weyl symbol of the operator $(A - zI)^{-1}$ by $[(A - zI)^{-1}]_W$. We calculated $[(A - zI)^{-1}]_W$ in the two leading terms:

$$\begin{aligned}[(A - zI)^{-1}]_W &= (A_1 - zI)^{-1} - (A_1 - zI)^{-1} A_{\mathrm{sub}} (A_1 - zI)^{-1} \\ &+ \frac{i}{2}\{(A_1 - zI)^{-1}, A_1 - zI, (A_1 - zI)^{-1}\} + O[(1 + |\xi| + |z|)^{-2}(1 + |\xi|)^{-1}].\end{aligned} \tag{3.1}$$

Here the curly brackets denote the generalised Poisson bracket on matrix functions (1.7).

The concept of a Weyl symbol was initially introduced for pseudodifferential operators in $\mathbb{R}^n$, see [18, subsection 23.3]. In the case of pseudodifferential operators acting on half-densities over a manifold it turns out that the Weyl symbol depends on the choice of local coordinates. However, in the two leading terms the Weyl symbol does not depend on the choice of local coordinates, see Appendix D. Note that a consistent definition of the full Weyl symbol for a pseudodifferential operator acting on half-densities over a manifold requires the introduction of an affine connection, see [14]. In the current paper we do not assume that we have a connection.

See Appendix E for a discussion of symbol classes and an explanation of the origins of the particular structure of the remainder term in formula (3.1), as well as remainder term estimates in subsequent formulae. In (E.22) we obtain (3.1) in the appropriate symbol classes.

Note that the expression in the second line of (3.1) can be equivalently rewritten as

$$\{(A_1 - zI)^{-1}, A_1 - zI, (A_1 - zI)^{-1}\} = (A_1 - zI)^{-1}\{A_1, (A_1 - zI)^{-1}, A_1\}(A_1 - zI)^{-1}, \tag{3.2}$$

which is the representation used by V. Ivrii, see second displayed formula on page 226 of [11]. We mention (3.2) in order to put our analysis within the context of previous research in the subject.

Let us now express the principal symbol A_1 in terms of its eigenvalues $h^{(j)}$ and eigenprojections $P^{(j)}$:

$$A_1 = \sum_j h^{(j)} P^{(j)}. \tag{3.3}$$

In what follows we will be substituting (3.3) into our previous formulae. But before proceeding with the calculations let us discuss which expression, the one in the RHS of (3.2) or the one in the LHS of (3.2), is better suited for practical purposes. Substitution of (3.3) into the RHS of (3.2) gives a sum over five indices, whereas substitution of (3.3) into the LHS of (3.2) gives a sum over only three indices. Hence, we will stick with the representation from the LHS of (3.2).

Substituting (3.3) into (3.1) we get

$$\begin{aligned} [(A-zI)^{-1}]_W = & \sum_j \frac{P^{(j)}}{h^{(j)}-z} - \sum_{k,l} \frac{P^{(k)} A_{\mathrm{sub}} P^{(l)}}{(h^{(k)}-z)(h^{(l)}-z)} \\ & + \frac{i}{2} \sum_{j,k,l} (h^{(j)}-z) \left\{ \frac{P^{(k)}}{h^{(k)}-z}, P^{(j)}, \frac{P^{(l)}}{h^{(l)}-z} \right\} + O[(1+|\xi|+|z|)^{-2}(1+|\xi|)^{-1}]. \end{aligned} \tag{3.4}$$

Our eigenprojections satisfy the identity

$$P^{(k)} P^{(j)} = \delta^{kj} P^{(k)}. \tag{3.5}$$

The identity (3.5) allows us to rewrite formula (3.4) as

$$\begin{aligned} [(A-zI)^{-1}]_W = & \sum_j \frac{P^{(j)}}{h^{(j)}-z} - \sum_{k,l} \frac{P^{(k)} A_{\mathrm{sub}} P^{(l)}}{(h^{(k)}-z)(h^{(l)}-z)} \\ & + \frac{i}{2} \sum_{j,k,l} \frac{h^{(j)}-z}{(h^{(k)}-z)(h^{(l)}-z)} \{P^{(k)}, P^{(j)}, P^{(l)}\} \\ & - \frac{i}{2} \sum_{k,l} \frac{P^{(k)}\big(h^{(k)}_{x^\alpha} P^{(l)}_{\xi_\alpha} - h^{(k)}_{\xi_\alpha} P^{(l)}_{x^\alpha}\big) + \big(h^{(l)}_{\xi_\alpha} P^{(k)}_{x^\alpha} - h^{(l)}_{x^\alpha} P^{(k)}_{\xi_\alpha}\big) P^{(l)}}{(h^{(k)}-z)(h^{(l)}-z)} \\ & + O[(1+|\xi|+|z|)^{-2}(1+|\xi|)^{-1}]. \end{aligned} \tag{3.6}$$

4 The matrix trace of the resolvent

Let B be a matrix pseudodifferential operator acting on m-columns of half-densities, $v \mapsto Bv$. The action of such an operator can be written in more detailed form as

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \mapsto \begin{pmatrix} {B_1}^1 & {B_1}^2 & \dots & {B_1}^m \\ {B_2}^1 & {B_2}^2 & \dots & {B_2}^m \\ \vdots & \vdots & \ddots & \vdots \\ {B_m}^1 & {B_m}^2 & \dots & {B_m}^m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \tag{4.1}$$

where the ${B_j}^k$ are scalar pseudodifferential operators acting on half-densities.

Definition 4.1. The matrix trace of the operator (4.1) is the scalar operator

$$\operatorname{tr} B := {B_1}^1 + {B_2}^2 + \cdots + {B_m}^m. \tag{4.2}$$

Obviously, the Weyl symbol of the matrix trace of an operator is the matrix trace of the Weyl symbol of the operator. Hence, formula (3.6) implies

$$[\operatorname{tr}(A - zI)^{-1}]_W = \sum_j \frac{1}{h^{(j)} - z} - \sum_j \frac{\operatorname{tr}[A_{\mathrm{sub}} P^{(j)}]}{(h^{(j)} - z)^2}$$
$$+ \frac{i}{2} \sum_{j,k,l} \frac{h^{(j)} - z}{(h^{(k)} - z)(h^{(l)} - z)} \operatorname{tr}\{P^{(k)}, P^{(j)}, P^{(l)}\} + O[(1 + |\xi| + |z|)^{-2}(1 + |\xi|)^{-1}]. \tag{4.3}$$

Note that formula (4.3) does not contain terms with derivatives of the Hamiltonians $h^{(j)}$ because all such terms cancelled out after we took the matrix trace.

Formula (3.5) implies

$$\operatorname{tr}\{P^{(k)}, P^{(j)}, P^{(l)}\} = 2\delta^{kj}\delta^{jl} \operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\}$$
$$- \delta^{kj} \operatorname{tr}\{P^{(l)}, P^{(j)}, P^{(l)}\} - \delta^{jl} \operatorname{tr}\{P^{(k)}, P^{(j)}, P^{(k)}\} + \delta^{kl} \operatorname{tr}\{P^{(k)}, P^{(j)}, P^{(k)}\}. \tag{4.4}$$

Substituting (4.4) into (4.3) and using (3.3) we get

$$[\operatorname{tr}(A - zI)^{-1}]_W = \sum_j \frac{1}{h^{(j)} - z} - \sum_j \frac{\operatorname{tr}[A_{\mathrm{sub}} P^{(j)}]}{(h^{(j)} - z)^2}$$
$$+ \frac{i}{2} \sum_j \frac{\operatorname{tr}\{P^{(j)}, A_1 - h^{(j)} I, P^{(j)}\}}{(h^{(j)} - z)^2} + i \sum_j \frac{\operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\}}{h^{(j)} - z}$$
$$+ O[(1 + |\xi| + |z|)^{-2}(1 + |\xi|)^{-1}]. \tag{4.5}$$

Detailed calculations leading up to formulae (4.4) and (4.5) are presented in Appendix F.

Formula (4.5) provides a compact representation for the Weyl symbol of the matrix trace of the resolvent. Even though our intermediate calculations involved summation over several (up to three) indices, summation in our final formula (4.5) is carried out over a single index.

5 The matrix trace of a power of the resolvent

In order to implement the strategy outlined in Section 2 we need to write down the Weyl symbol of the operator $\operatorname{tr}(A - zI)^{1-n}$.

We have the operator identity

$$(A - zI)^{1-n} = \frac{1}{(n-2)!} \frac{d^{n-2}}{dz^{n-2}} (A - zI)^{-1}. \tag{5.1}$$

The operations of taking the matrix trace and differentiation with respect to a parameter commute, so formula (5.1) implies

$$\operatorname{tr}(A - zI)^{1-n} = \frac{1}{(n-2)!} \frac{d^{n-2}}{dz^{n-2}} \operatorname{tr}(A - zI)^{-1}. \tag{5.2}$$

The latter formula, in turn, implies

$$[\mathrm{tr}(A-zI)^{1-n}]_W=\frac{1}{(n-2)!}\,\frac{d^{n-2}}{dz^{n-2}}\,[\mathrm{tr}(A-zI)^{-1}]_W\,. \tag{5.3}$$

Substituting (4.5) into (5.3) we get

$$\begin{aligned}[\mathrm{tr}(A-zI)^{1-n}]_W=\sum_j\frac{1}{(h^{(j)}-z)^{n-1}}-(n-1)\sum_j\frac{\mathrm{tr}[A_{\mathrm{sub}}P^{(j)}]}{(h^{(j)}-z)^n}\\ +\frac{i}{2}(n-1)\sum_j\frac{\mathrm{tr}\{P^{(j)},A_1-h^{(j)}I,P^{(j)}\}}{(h^{(j)}-z)^n}+i\sum_j\frac{\mathrm{tr}\{P^{(j)},P^{(j)},P^{(j)}\}}{(h^{(j)}-z)^{n-1}}\\ +O[(1+|\xi|+|z|)^{-n}(1+|\xi|)^{-1}].\end{aligned} \tag{5.4}$$

We can view this as an explicit version of the result of applying $(d/dz)^{n-2}$ to the trace of (E.22) (cf. (E.39)).

6 Asymptotic expansion for the density f

We have previously defined the density $f(x,z)$, see Definition 2.1. In this section we shall derive the asymptotic expansion for the density $f(x,\lambda e^{i\varphi})$ as $\lambda\to+\infty$. The angle $0<\varphi<\pi$ will be assumed to be fixed.

Put

$$s^{(j)}_{1-n}(x,\xi,z):=\frac{1}{(h^{(j)}-z)^{n-1}}\,, \tag{6.1}$$

$$\begin{aligned}s^{(j)}_{-n}(x,\xi,z):=-(n-1)\frac{\mathrm{tr}[A_{\mathrm{sub}}P^{(j)}]}{(h^{(j)}-z)^n}+\frac{i}{2}(n-1)\frac{\mathrm{tr}\{P^{(j)},A_1-h^{(j)}I,P^{(j)}\}}{(h^{(j)}-z)^n}\\ +i\frac{\mathrm{tr}\{P^{(j)},P^{(j)},P^{(j)}\}}{(h^{(j)}-z)^{n-1}}\,,\end{aligned} \tag{6.2}$$

where the subscripts indicate the degree of homogeneity in ξ. Recall, yet again, that our convention is 'z and ξ are of the same order'. Comparing (5.4) with (6.1) and (6.2) we see that $\sum_j s^{(j)}_{1-n}$ is the leading (principal) component of the Weyl symbol of the operator $\mathrm{tr}(A-zI)^{1-n}$, whereas $\sum_j s^{(j)}_{-n}$ is the next (subprincipal) component.

The structure of formula (6.1) is very simple, whereas the structure of formula (6.2) is nontrivial. This warrants a discussion.

The first term in the RHS of (6.2) contains the expression $\mathrm{tr}[A_{\mathrm{sub}}P^{(j)}]$. It gives the 'obvious' contribution to the second Weyl coefficient. The expression $\mathrm{tr}[A_{\mathrm{sub}}P^{(j)}]$ appears in the early papers of V. Ivrii and G. V. Rozenblyum.

The second term in the RHS of (6.2) contains the expression $\mathrm{tr}\{P^{(j)},A_1-h^{(j)}I,P^{(j)}\}$. It gives a contribution to the second Weyl coefficient which is not so obvious. The expression $\mathrm{tr}\{P^{(j)},A_1-h^{(j)}I,P^{(j)}\}$ first appeared in [16].

Finally, the third term in the RHS of (6.2) contains the expression $\mathrm{tr}\{P^{(j)},P^{(j)},P^{(j)}\}$. It gives a U(1) curvature contribution to the second Weyl coefficient. This contribution to the second Weyl coefficient was identified in [2] and did not appear in previous publications.

The density $f(x,\lambda e^{i\varphi})$ is the value of the integral kernel of the operator

$$i\,\mathrm{tr}\left[2(A-zI)^{1-n}-(A-2zI)^{1-n}-2(A-\bar zI)^{1-n}+(A-2\bar zI)^{1-n}\right] \tag{6.3}$$

on the diagonal. We obtain the asymptotic expansion (2.16) for $f(x,\lambda e^{i\varphi})$ by replacing the operator (6.3) with its Weyl symbol and integrating in ξ. This gives the following formulae for the asymptotic coefficients:

$$b_1(x,\varphi)=\frac{1}{(2\pi)^n}\sum_j b_1^{(j)}(x,\varphi), \tag{6.4}$$

$$b_0(x,\varphi)=\frac{1}{(2\pi)^n}\sum_j b_0^{(j)}(x,\varphi), \tag{6.5}$$

where

$$b_1^{(j)}(x,\varphi)= i\int\left[2s_{1-n}^{(j)}(x,\xi,e^{i\varphi})-s_{1-n}^{(j)}(x,\xi,2e^{i\varphi})-2s_{1-n}^{(j)}(x,\xi,e^{-i\varphi})+s_{1-n}^{(j)}(x,\xi,2e^{-i\varphi})\right]d\xi, \tag{6.6}$$

$$b_0^{(j)}(x,\varphi)= i\int\left[2s_{-n}^{(j)}(x,\xi,e^{i\varphi})-s_{-n}^{(j)}(x,\xi,2e^{i\varphi})-2s_{-n}^{(j)}(x,\xi,e^{-i\varphi})+s_{-n}^{(j)}(x,\xi,2e^{-i\varphi})\right]d\xi. \tag{6.7}$$

The integrands in (6.6) and (6.7) decay as $|\xi|^{-n-1}$ as $|\xi|\to+\infty$, so these integrals converge.

Strictly speaking, we also have to consider the contributions from the terms $K^{(n)}$ in (E.35). However, it follows from the remark after (E.37) that they are $o(1)$ as $\lambda\to+\infty$.

7 The second Weyl coefficient

Let us us examine what happens to the integral (6.7) when $\varphi\to0^+$. It is easy to see that if j is such that $h^{(j)}<0$ then the integral (6.7) tends to zero as $\varphi\to0^+$: one can simply set $\varphi=0$ in the integrand. This means that only those j for which $h^{(j)}>0$ contribute to the limit of the expression (6.6) when $\varphi\to0^+$. Therefore, formulae (2.18) and (6.5) give us the following expression for the second Weyl coefficient:

$$a_{n-2}^+(x)=-\frac{1}{(2\pi)^{n+1}}\sum_{j=1}^{m^+}\lim_{\varphi\to0^+}b_0^{(j)}(x,\varphi)\,. \tag{7.1}$$

Here the enumeration of eigenvalues of the principal symbol A_1 is assumed to be chosen in such a way that $j=1,\dots,m^+$ correspond to positive eigenvalues $h^{(j)}$.

It remains only to evaluate $\lim_{\varphi\to0^+}b_0^{(j)}(x,\varphi)$ explicitly. Here $b_0^{(j)}(x,\varphi)$ is defined by formula (6.7), where the integrand is defined in accordance with (6.2).

Let us rewrite formula (6.2) as

$$s_{-n}^{(j)}(x,\xi,z)=s_{-n}^{(j;1)}(x,\xi,z)+s_{-n}^{(j;2)}(x,\xi,z), \tag{7.2}$$

where

$$s_{-n}^{(j;1)}(x,\xi,z) := -(n-1)\frac{\operatorname{tr}\left(A_{\mathrm{sub}}P^{(j)} - \frac{i}{2}\{P^{(j)}, A_1 - h^{(j)}I, P^{(j)}\}\right)}{(h^{(j)}-z)^n}, \tag{7.3}$$

$$s_{-n}^{(j;2)}(x,\xi,z) := i\frac{h^{(j)}\operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\}}{h^{(j)}(h^{(j)}-z)^{n-1}}. \tag{7.4}$$

Note that the numerators in (7.3) and (7.4) are positively homogeneous in ξ of degree zero.

Formula (6.7) now reads

$$b_0^{(j)}(x,\varphi) = b_0^{(j;1)}(x,\varphi) + b_0^{(j;2)}(x,\varphi), \tag{7.5}$$

where

$$b_0^{(j;k)}(x,\varphi) = \\ i\int\left[2s_{-n}^{(j;k)}(x,\xi,e^{i\varphi}) - s_{-n}^{(j;k)}(x,\xi,2e^{i\varphi}) - 2s_{-n}^{(j;k)}(x,\xi,e^{-i\varphi}) + s_{-n}^{(j;k)}(x,\xi,2e^{-i\varphi})\right]d\xi, \tag{7.6}$$

$k=1,2$.

Denote by $(S_x^*M)^{(j)}$ the $(n-1)$-dimensional unit cosphere in the cotangent fibre defined by the equation $h^{(j)}(x,\xi)=1$ and denote by $d(S_x^*M)^{(j)}$ the surface area element on $(S_x^*M)^{(j)}$ defined by the condition

$$\left[\frac{d}{d\mu}\int_{h^{(j)}(x,\xi)<\mu} g(\xi)\,d\xi\right]_{\mu=1} = \int_{(S_x^*M)^{(j)}} g(\xi)\,d(S_x^*M)^{(j)}, \tag{7.7}$$

where $g:\mathbb{R}^n\to\mathbb{R}$ is an arbitrary smooth function. This means that we introduce spherical coordinates in the cotangent fibre with the Hamiltonian $h^{(j)}$ playing the role of the radial coordinate, see also [17, subsection 1.1.10].

Switching to spherical coordinates, we see that each integral (7.6) is a product of two integrals, an $(n-1)$-dimensional surface integral over the unit cosphere and a 1-dimensional integral over the radial coordinate. Namely, we have

$$b_0^{(j;k)}(x,\varphi) = c^{(j;k)}(x)\,d^{(j;k)}(\varphi), \tag{7.8}$$

where

$$c^{(j;1)}(x) := -(n-1)\int_{(S_x^*M)^{(j)}}\operatorname{tr}\left(A_{\mathrm{sub}}P^{(j)} - \frac{i}{2}\{P^{(j)}, A_1 - h^{(j)}I, P^{(j)}\}\right)d(S_x^*M)^{(j)}, \tag{7.9}$$

$$c^{(j;2)}(x) := i\int_{(S_x^*M)^{(j)}} h^{(j)}\operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\}\,d(S_x^*M)^{(j)}, \tag{7.10}$$

$$d^{(j;1)}(\varphi) := i\int_0^{+\infty}\left[\frac{2}{(\mu-e^{i\varphi})^n} - \frac{1}{(\mu-2e^{i\varphi})^n} - \frac{2}{(\mu-e^{-i\varphi})^n} + \frac{1}{(\mu-2e^{-i\varphi})^n}\right]\mu^{n-1}\,d\mu, \tag{7.11}$$

$$d^{(j;2)}(\varphi) := \\ i\int_0^{+\infty}\left[\frac{2}{(\mu-e^{i\varphi})^{n-1}} - \frac{1}{(\mu-2e^{i\varphi})^{n-1}} - \frac{2}{(\mu-e^{-i\varphi})^{n-1}} + \frac{1}{(\mu-2e^{-i\varphi})^{n-1}}\right]\mu^{n-2}\,d\mu. \tag{7.12}$$

Integrating by parts we see that the integrals in the right–hand-sides of (7.11) and (7.12) have the same values, i.e. they do not depend on n. Hence, it is sufficient to evaluate the integral (7.12) for $n = 2$. We have

$$d^{(j;1)}(\varphi) = d^{(j;2)}(\varphi) = i \int_0^{+\infty} \left[\frac{2}{\mu - e^{i\varphi}} - \frac{1}{\mu - 2e^{i\varphi}} - \frac{2}{\mu - e^{-i\varphi}} + \frac{1}{\mu - 2e^{-i\varphi}} \right] d\mu \\ = -2(\pi - \varphi)\,, \quad (7.13)$$

so substituting (7.5), (7.8) and (7.13) into (7.1) we get

$$a^+_{n-2}(x) = \frac{1}{(2\pi)^n} \sum_{j=1}^{m^+} \left[c^{(j;1)}(x) + c^{(j;2)}(x) \right]. \quad (7.14)$$

Formulae (7.14), (7.9) and (7.10) give us the required explicit representation of the second Weyl coefficient. However, integrating over a unit cosphere is not very convenient, so we rewrite formulae (7.9) and (7.10) as

$$c^{(j;1)}(x) = -n(n-1) \int_{h^{(j)}(x,\xi)<1} \operatorname{tr} \left(A_{\mathrm{sub}} P^{(j)} - \frac{i}{2} \{P^{(j)}, A_1 - h^{(j)} I, P^{(j)}\} \right) (x,\xi)\, d\xi\,, \quad (7.15)$$

$$c^{(j;2)}(x) = n\, i \int_{h^{(j)}(x,\xi)<1} \left(h^{(j)} \operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\} \right) (x,\xi)\, d\xi\,. \quad (7.16)$$

Working with eigenprojections $P^{(j)}$ is also not very convenient, so we express them via the normalised eigenvectors $v^{(j)}$ of the principal symbol A_1 as

$$P^{(j)} = v^{(j)} [v^{(j)}]^*. \quad (7.17)$$

Substituting (7.17) into (7.15) and (7.16) we get

$$c^{(j;1)}(x) = -n(n-1) \int_{h^{(j)}(x,\xi)<1} \left([v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2} \{[v^{(j)}]^*, A_1 - h^{(j)} I, v^{(j)}\} \right) (x,\xi)\, d\xi\,, \quad (7.18)$$

$$c^{(j;2)}(x) = -n\, i \int_{h^{(j)}(x,\xi)<1} \left(h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \right) (x,\xi)\, d\xi\,. \quad (7.19)$$

The transition from (7.15) to (7.18) is quite straightforward, but the transition from (7.16) to (7.19) warrants an explanation. Here we have $\operatorname{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\} = -\operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) = -\{[v^{(j)}]^*, v^{(j)}\}$, where at the last step we made use of [2, formula (4.17)].

The advantage of formulae (7.18) and (7.19) is that they do not involve the matrix trace.

Combining formulae (7.14), (7.18) and (7.19) we arrive at (1.6).

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Appendix A Proof of Lemma 2.2

Let us introduce the functions

$$g_n(\mu,z):=\frac{2}{(\mu-z)^n}-\frac{1}{(\mu-2z)^n}-\text{c.c.},\quad n\in\mathbb{N},\quad \mu\in\mathbb{R},\quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{A.1}$$

Here and further on 'c.c.' stands for 'complex conjugate terms'.

The functions (A.1) possess the following properties:

$$\partial_1 g_n(\mu,z):=\partial_\mu g_n(\mu,z)=-n g_{n+1}(\mu,z), \tag{A.2}$$

$$|g_n(\mu,z)|\le\frac{4}{|\mu-z|^n}+\frac{2}{|\mu-2z|^n}\,. \tag{A.3}$$

Formula (2.12) can be rewritten as

$$\begin{aligned} f^\rho(x,z)=i\int_0^{+\infty} & g_{n-1}(\mu,z)\,(N'_+*\rho)(x,\mu)\,d\mu \\ -(-1)^n i\int_0^{+\infty} & g_{n-1}(\mu,-z)\,(N'_-*\rho)(x,\mu)\,d\mu, \end{aligned} \tag{A.4}$$

where

$$N'_\pm(x,\nu)=\sum_{\pm\lambda_k>0}\delta(\nu\mp\lambda_k)\|v_k(x)\|^2 \tag{A.5}$$

is a tempered distribution in ν supported on $\mathbb{R}_+$ and taking values in densities. The convolution

$$(N'_\pm*\rho)(x,\mu)=\int_0^{+\infty}N'_\pm(x,\nu)\,\rho(\mu-\nu)\,d\nu \tag{A.6}$$

is a continuous function of μ taking values in densities. It is known that

$$|(N'_\pm*\rho)(x,\mu)|\le c(x)(1+|\mu|^{n-1}),$$

where $c(x)$ is a fixed positive density. Arguing as in (2.2)–(2.7), it is easy to see that, for fixed z, the function $g_{n-1}(\mu,z)$ decays as $|\mu|^{-n-1}$ when $\mu\to\pm\infty$, so the integrals in (A.4) converge.

We have

$$\begin{aligned} &\int_0^{+\infty} g_{n-1}(\mu,z)\,(N'_\pm*\rho)(x,\mu)\,d\mu \\ &\qquad=\int_0^{+\infty} g_{n-1}(\mu,z)\left(\int_0^{+\infty}N'_\pm(x,\nu)\,\rho(\mu-\nu)\,d\nu\right)d\mu \\ &\qquad=\int_0^{+\infty} N'_\pm(x,\nu)\left(\int_0^{+\infty}g_{n-1}(\mu,z)\,\rho(\mu-\nu)\,d\mu\right)d\nu \\ &\qquad=\int_0^{+\infty} N'_\pm(x,\mu)\left(\int_0^{+\infty}g_{n-1}(\nu,z)\,\rho(\nu-\mu)\,d\nu\right)d\mu. \end{aligned} \tag{A.7}$$

In going from the second line of (A.7) to the third we changed the order of integration. This can be justified, for example, by replacing the infinite series (A.5) by a finite partial sum and going to the limit.

Substituting (A.7) into (A.4) and using formula (2.11), we find that

$$f^\rho(x,z) - f(x,z) = i\int_0^{+\infty} N'_+(x,\mu)\left(\int_0^{+\infty} g_{n-1}(\nu,z)\,\rho(\nu-\mu)\,d\nu - g_{n-1}(\mu,z)\right)d\mu$$
$$- (-1)^n i\int_0^{+\infty} N'_-(x,\mu)\left(\int_0^{+\infty} g_{n-1}(\nu,-z)\,\rho(\nu-\mu)\,d\nu - g_{n-1}(\mu,-z)\right)d\mu$$
$$+ (-1)^n \frac{2^n-1}{2^{n-1}} i\left[\frac{1}{z^{n-1}} - \frac{1}{\bar{z}^{n-1}}\right]\sum_{\lambda_k=0}\|v_k(x)\|^2.$$

Now, let $z = \lambda e^{i\varphi}$ with $\lambda > 0$ and fixed $\varphi \in (0,\pi)$. In view of the fact that $N_\pm(x,\lambda) = O(\lambda^n)$, in order to show that $f^\rho(x,\lambda e^{i\varphi}) - f(x,\lambda e^{i\varphi}) \to 0$ as $\lambda \to +\infty$ it is sufficient to prove that

$$\left|\int_0^{+\infty} g_{n-1}(\nu,\lambda e^{i\varphi})\,\rho(\nu-\mu)\,d\nu - g_{n-1}(\mu,\lambda e^{i\varphi})\right| \le \frac{\mathrm{const}_\varphi}{\lambda(1+\mu^{n+1})}, \quad \forall\lambda \ge 1, \quad \forall\mu \ge 0. \tag{A.8}$$

Recall that according to our definition of the mollifier ρ we have

$$|\rho(\nu)| \le \frac{c_p}{(1+|\nu|)^p}, \quad \forall p \in \mathbb{N}, \tag{A.9}$$

$$\int_{-\infty}^{+\infty} \rho(\nu)d\nu = 1, \quad \text{and} \quad \int_{-\infty}^{+\infty} \rho(\nu)\nu^m d\nu = 0, \quad \forall m \in \mathbb{N}. \tag{A.10}$$

Formula (A.10) implies that

$$\int_0^{+\infty} g_{n-1}(\nu,\lambda e^{i\varphi})\rho(\nu-\mu)d\nu - g_{n-1}(\mu,\lambda e^{i\varphi})$$
$$= \int_{-\infty}^{+\infty}\left[g_{n-1}(\nu,\lambda e^{i\varphi}) - g_{n-1}(\mu,\lambda e^{i\varphi})\right]\rho(\nu-\mu)d\nu - \int_{-\infty}^{0} g_{n-1}(\nu,\lambda e^{i\varphi})\rho(\nu-\mu)d\nu. \tag{A.11}$$

Using (A.3) and (A.9) with $p = n+3$ we get

$$\left|\int_{-\infty}^{0} g_{n-1}(\nu,\lambda e^{i\varphi})\rho(\nu-\mu)d\nu\right| \le \int_{-\infty}^{0} \frac{6}{\lambda^{n-1}|\sin\varphi|^{n-1}}\frac{c_{n+3}}{(1+|\nu|+\mu)^{n+3}}d\nu$$
$$\le \frac{6c_{n+3}}{\lambda^{n-1}|\sin\varphi|^{n-1}(1+\mu^{n+1})}\int_{-\infty}^{0}\frac{d\nu}{1+\nu^2} \le \frac{\mathrm{const}_\varphi}{\lambda(1+\mu^{n+1})}, \quad \forall\lambda \ge 1. \tag{A.12}$$

In order to estimate the first integral in the RHS of (A.11) let us perform a change of variable $\nu \mapsto \mu + \nu$,

$$\int_{-\infty}^{+\infty}\left[g_{n-1}(\nu,\lambda e^{i\varphi}) - g_{n-1}(\mu,\lambda e^{i\varphi})\right]\rho(\nu-\mu)\,d\nu$$
$$= \int_{-\infty}^{+\infty}\left[g_{n-1}(\mu+\nu,\lambda e^{i\varphi}) - g_{n-1}(\mu,\lambda e^{i\varphi})\right]\rho(\nu)\,d\nu. \tag{A.13}$$

Writing Taylor's formula with remainder in Lagrange's form and using (A.2), we get

$$g_{n-1}(\mu+\nu,\lambda e^{i\varphi}) - g_{n-1}(\mu,\lambda e^{i\varphi}) = -(n-1)g_n(\mu,\lambda e^{i\varphi})\,\nu$$
$$+ \frac{n(n-1)}{2}g_{n+1}(\mu,\lambda e^{i\varphi})\,\nu^2 - \frac{(n+1)n(n-1)}{6}R(\mu,\nu,\lambda,\varphi)\,\nu^3, \tag{A.14}$$

where

$$R(\mu,\nu,\lambda,\varphi)=g_{n+2}(\xi_{\mu,\mu+\nu},\lambda e^{i\varphi}) \tag{A.15}$$

and $\xi_{\mu,\mu+\nu}$ is some real number strictly between μ and $\mu+\nu$. From (A.10), (A.14) and (A.2) we obtain

$$\int_{-\infty}^{+\infty}\left[g_{n-1}(\mu+\nu,\lambda e^{i\varphi})-g_{n-1}(\mu,\lambda e^{i\varphi})\right]\rho(\nu)\,d\nu$$
$$=-\frac{(n+1)n(n-1)}{6}\int_{-\infty}^{+\infty}R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\,d\nu. \tag{A.16}$$

Comparing formula (A.8) with (A.11)–(A.13) and (A.16) we see that the proof of Lemma 2.2 has been reduced to proving that

$$\int_{-\infty}^{+\infty}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{\mathrm{const}_\varphi}{\lambda(1+\mu^{n+1})}\,,\quad\forall\lambda\ge1,\quad\forall\mu\ge0. \tag{A.17}$$

In order to prove (A.17) it is sufficient to prove the following two estimates:

$$\int_{-\infty}^{+\infty}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{\mathrm{const}_\varphi}{\lambda^{n+2}}\,,\quad\forall\lambda\ge1,\quad\forall\mu\in[0,\lambda], \tag{A.18}$$

$$\int_{-\infty}^{+\infty}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{\mathrm{const}_\varphi}{\lambda\mu^{n+1}}\,,\quad\forall\lambda\ge1,\quad\forall\mu\ge\lambda. \tag{A.19}$$

Observe that formulae (A.15) and (A.3) give us the rough estimate

$$|R(\mu,\nu,\lambda,\varphi)|\le\frac{6}{|\sin\varphi|^{n+2}\lambda^{n+2}}\,,\quad\forall\lambda>0,\quad\forall\mu,\nu\in\mathbb{R}. \tag{A.20}$$

Formulae (A.20) and (A.9) with $p=5$ imply (A.18).

Formulae (A.15) and (A.3) also tell us that

$$|R(\mu,\nu,\lambda,\varphi)|\le\frac{\mathrm{const}_\varphi}{\mu^{n+2}}\le\frac{\mathrm{const}_\varphi}{\lambda\mu^{n+1}}$$

uniformly over all $\mu\ge\lambda>0$ and $\nu\ge-\mu/2$. Using this estimate and formula (A.9) with $p=5$ we get

$$\int_{-\mu/2}^{+\infty}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{\mathrm{const}_\varphi}{\lambda\mu^{n+1}}\,,\quad\forall\lambda\ge1,\quad\forall\mu\ge\lambda. \tag{A.21}$$

Comparing formulae (A.21) and (A.19) we see that the proof of Lemma 2.2 has been reduced to proving that

$$\int_{-\infty}^{-\mu/2}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{\mathrm{const}_\varphi}{\lambda\mu^{n+1}}\,,\quad\forall\lambda\ge1,\quad\forall\mu\ge\lambda. \tag{A.22}$$

Using (A.20) and (A.9) with $p=n+5$ we get

$$\int_{-\infty}^{-\mu/2}\left|R(\mu,\nu,\lambda,\varphi)\,\nu^3\rho(\nu)\right|\,d\nu\le\frac{6c_{n+5}}{|\sin\varphi|^{n+2}\lambda^{n+2}}\int_{\mu/2}^{+\infty}\frac{d\nu}{\nu^{n+2}}$$
$$=\frac{6\cdot2^{n+1}c_{n+5}}{(n+1)|\sin\varphi|^{n+2}\lambda^{n+2}\mu^{n+1}}\,,\quad\forall\lambda\ge1,\quad\forall\mu\ge\lambda,$$

which implies (A.22). □

Appendix B Some integrals involving the functions g_n

In this appendix we evaluate some integrals involving the functions (A.1). These results will be used later in Appendix C.

Let us evaluate the following indefinite integral:

$$\int \frac{\mu^n d\mu}{(\mu - z)^n} = \int \left(1 + \frac{z}{\nu}\right)^n d\nu = \int \left[1 + \frac{nz}{\nu} + \sum_{k=2}^{n} \binom{n}{k} \frac{z^k}{\nu^k}\right] d\nu$$
$$= \nu + nz \log \nu + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{1-k} z^k \nu^{1-k}$$
$$= \mu + nz \log(\mu - z) + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{1-k} z^k (\mu - z)^{1-k}. \quad \text{(B.1)}$$

Here in performing intermediate calculations we used the change of variable $\nu = \mu - z$.

Similarly,

$$\int \frac{\mu^{n-1} d\mu}{(\mu - z)^n} = \int \left(1 + \frac{z}{\nu}\right)^{n-1} \frac{d\nu}{\nu} = \int \left[1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{z^k}{\nu^k}\right] \frac{d\nu}{\nu}$$
$$= \log \nu - \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{k} z^k \nu^{-k} = \log(\mu - z) - \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{k} z^k (\mu - z)^{-k}. \quad \text{(B.2)}$$

Formulae (A.1), (B.1) and (B.2) imply

$$\int g_n(\mu, z)\, \mu^n\, d\mu = 2nz \log(\mu - z) - 2nz \log(\mu - 2z)$$
$$+ 2 \sum_{k=2}^{n} \binom{n}{k} \frac{1}{1-k} z^k (\mu - z)^{1-k} - \sum_{k=2}^{n} \binom{n}{k} \frac{1}{1-k} 2^k z^k (\mu - 2z)^{1-k} - \text{c.c.}, \quad \text{(B.3)}$$

$$\int g_n(\mu, z)\, \mu^{n-1}\, d\mu = 2 \log(\mu - z) - \log(\mu - 2z)$$
$$- 2 \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{k} z^k (\mu - z)^{-k} + \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{k} 2^k z^k (\mu - 2z)^{-k} - \text{c.c.}. \quad \text{(B.4)}$$

Using (B.3) and (B.4) we can finally evaluate definite integrals:

$$\int_0^{+\infty} g_n(\mu, z)\, \mu^n\, d\mu = \left[2nz \log\left(\frac{\mu - z}{\mu - 2z}\right) - 2n\bar{z} \log\left(\frac{\mu - \bar{z}}{\mu - 2\bar{z}}\right) \right]\Bigg|_0^{+\infty}, \quad \text{(B.5)}$$

$$\int_0^{+\infty} g_n(\mu, z)\, \mu^{n-1}\, d\mu = \left[\log\left(\frac{\mu - z}{\mu - 2z}\right) + \log\left(\frac{\mu - z}{\mu - \bar{z}}\right) - \log\left(\frac{\mu - \bar{z}}{\mu - 2\bar{z}}\right) \right]\Bigg|_0^{+\infty}. \quad \text{(B.6)}$$

Here the complex logarithms log are continuous multivalued functions which have to be handled carefully.

Note that for any $z \in \mathbb{C} \setminus \mathbb{R}$ and any real positive μ we have

$$\operatorname{Im} \frac{\mu - z}{\mu - 2z} = \frac{\mu \operatorname{Im} z}{|\mu - 2z|^2} \neq 0,$$

$$\operatorname{Im}\frac{\mu - z}{\mu - \bar z} = \frac{2\operatorname{Im} z(\operatorname{Re} z - \mu)}{|\mu - z|^2} = 0 \quad \Rightarrow \quad \operatorname{Re}\frac{\mu - z}{\mu - \bar z} = \frac{(\operatorname{Re} z - \mu)^2 - (\operatorname{Im} z)^2}{|\mu - z|^2} < 0,$$

so neither of the two arguments of our $\log$ crosses the positive real axis $\mathbb{R}_+$. Hence, we are free to switch from $\log$ to the single-valued $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{R} + i[0, 2\pi)$ branch-cut along $\mathbb{R}_+$. Formulae (B.5) and (B.6) become

$$\int_0^{+\infty} g_n(\mu, z)\, \mu^n \, d\mu = \left[2nz \operatorname{Log}\left(\frac{\mu - z}{\mu - 2z}\right) - 2n\bar z \operatorname{Log}\left(\frac{\mu - \bar z}{\mu - 2\bar z}\right)\right]\Bigg|_0^{+\infty}$$
$$= 2n(z - \bar z)\ln 2 = 4ni(\ln 2)\operatorname{Im} z, \quad \text{(B.7)}$$

$$\int_0^{+\infty} g_n(\mu, z)\, \mu^{n-1} \, d\mu = \left[\operatorname{Log}\left(\frac{\mu - z}{\mu - 2z}\right) + \operatorname{Log}\left(\frac{\mu - z}{\mu - \bar z}\right) - \operatorname{Log}\left(\frac{\mu - \bar z}{\mu - 2\bar z}\right)\right]\Bigg|_0^{+\infty}$$
$$= \operatorname{Log}\left(\frac{\mu - z}{\mu - \bar z}\right)\Bigg|_0^{+\infty} = i\pi(1 + \operatorname{sgn}\operatorname{Im} z) - i \operatorname{Arg} z^2, \quad \text{(B.8)}$$

where $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to [0, 2\pi)$ is also branch-cut along $\mathbb{R}_+$.

Appendix C Proof of Lemma 2.3

Formula (1.3) tells us that

$$(N'_\pm * \rho)(x, \mu) = a^\pm_{n-1}(x)\mu^{n-1} + a^\pm_{n-2}(x)\mu^{n-2} + (1 + \mu)^{n-3} r^\pm(x, \mu), \quad \text{(C.1)}$$

where $r^\pm(x, \mu)$ is bounded uniformly in $\mu \ge 0$.

Let $g_n(\mu, z)$ be defined in accordance with (A.1). We have

$$g_n(\lambda\mu, \lambda z) = \lambda^{-n} g_n(\mu, z), \quad \forall \lambda > 0. \quad \text{(C.2)}$$

Using (C.2) we get

$$\int_0^{+\infty} g_{n-1}(\mu, \lambda e^{i\varphi})\, \mu^{n-1} \, d\mu = \lambda \int_0^{+\infty} g_{n-1}(\mu, e^{i\varphi})\, \mu^{n-1} \, d\mu\,, \quad \text{(C.3)}$$

$$\int_0^{+\infty} g_{n-1}(\mu, \lambda e^{i\varphi})\, \mu^{n-2} \, d\mu = \int_0^{+\infty} g_{n-1}(\mu, e^{i\varphi})\, \mu^{n-2} \, d\mu, \quad \text{(C.4)}$$

$$\int_0^{+\infty} g_{n-1}(\mu, \lambda e^{i\varphi})\, (1 + \mu)^{n-3}\, r^\pm(x, \mu) \, d\mu$$
$$= \frac{1}{\lambda} \int_0^{+\infty} g_{n-1}(\mu, e^{i\varphi}) \left(\frac{1}{\lambda} + \mu\right)^{n-3} r^\pm(x, \lambda\mu) \, d\mu = o(1) \quad \text{as} \quad \lambda \to +\infty. \quad \text{(C.5)}$$

Recall (see Appendix A) that the function $g_{n-1}(\mu, z)$ decays as μ^{-n-1} when $\mu \to +\infty$, so the integrals in (C.3)–(C.5) converge.

Substituting (C.3)–(C.5) into (A.4) we get

$$
\begin{aligned}
&f^\rho(x,\lambda e^{i\varphi}) \\
&= \lambda i \left[a^+_{n-1}(x) \int_0^{+\infty} g_{n-1}(\mu, e^{i\varphi})\, \mu^{n-1}\, d\mu \;-\; (-1)^n a^-_{n-1}(x) \int_0^{+\infty} g_{n-1}(\mu, e^{i(\varphi+\pi)})\, \mu^{n-1}\, d\mu \right] \\
&\quad + i \left[a^+_{n-2}(x) \int_0^{+\infty} g_{n-1}(\mu, e^{i\varphi})\, \mu^{n-2}\, d\mu \;-\; (-1)^n a^-_{n-2}(x) \int_0^{+\infty} g_{n-1}(\mu, e^{i(\varphi+\pi)})\, \mu^{n-2}\, d\mu \right] \\
&\qquad + o(1) \quad \text{as} \quad \lambda \to +\infty. \qquad (C.6)
\end{aligned}
$$

Formulae (B.7) and (B.8) give us the values of the integrals appearing in (C.6), so (C.6) becomes

$$
\begin{aligned}
f^\rho(x,\lambda e^{i\varphi}) &= -4(n-1)(\ln 2)(\sin\varphi) \left[a^+_{n-1}(x) + (-1)^n a^-_{n-1}(x) \right] \lambda \\
&\quad - 2 \left[a^+_{n+2}(x)(\pi - \varphi) + (-1)^n a^-_{n-1}(x)\varphi \right] + o(1) \quad \text{as} \quad \lambda \to +\infty,
\end{aligned}
$$

thus proving the lemma. □

Appendix D Weyl quantization on manifolds

[1] Let M be a compact manifold. A pseudodifferential operator of order $m \in \mathbb{R}$ is a continuous operator $A : C^\infty(M) \to C^\infty(M)$ which has a weakly continuous extension $\mathcal{D}'(M) \to \mathcal{D}'(M)$ such that, with K_A denoting the distribution kernel,

1) $\operatorname{sing\,supp} K_A \subset \operatorname{diag}(M \times M)$,

2) For every system of local coordinates $\gamma : \Omega \ni \rho \mapsto x \in \Omega' \subset \mathbb{R}^n$ where $\Omega \subset M$, $\Omega' \subset \mathbb{R}^n$ are open and γ a diffeomorphism, we have (identifying Ω and $\gamma(\Omega)$)

$$
Au(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\theta} a(x,\theta) u(y)\, dy\, d\theta + Ru, \quad u \in C_0^\infty(\Omega),\ x \in \Omega, \qquad (D.1)
$$

where R is smoothing ($K_R \in C^\infty(\Omega \times \Omega)$) and a is a symbol of order m; $a \in S^m(\Omega)$, which means that $a \in C^\infty(\Omega \times \mathbb{R}^n)$ and that for every $\widehat{K} \Subset \Omega$ and all $\alpha, \beta \in \mathbb{N}^n$, $\exists C = C_{\widehat{K},\alpha,\beta}$ such that

$$
|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C \langle\theta\rangle^{m-|\beta|},\ \forall\, (x,\theta) \in \widehat{K} \times \mathbb{R}^n,\ \text{where } \langle\theta\rangle = (1+|\theta|^2)^{1/2}. \qquad (D.2)
$$

If $\widetilde{\gamma} : \widetilde{\Omega} \ni \rho \mapsto \widetilde{x} \in \widetilde{\Omega}'$ is another local coordinate chart, then over the intersection $\Omega \cap \widetilde{\Omega}$ we can express $x = \kappa(\widetilde{x})$, where $\kappa = \gamma \circ \widetilde{\gamma}^{-1}$ and we have

$$
a(\kappa(\widetilde{x}), \theta) \equiv \widetilde{a}(\widetilde{x}, (\kappa'(\widetilde{x}))^{\mathrm{t}}\theta) \bmod S^{m-1}. \qquad (D.3)
$$

This allows us to define the symbol σ_A of A on T^*M up to symbols of order 1 lower. More precisely, we have a bijection

$$
L^m(M)/L^{m-1}(M) \ni A \mapsto \sigma_A \in S^m(T^*M)/S^{m-1}(T^*M), \qquad (D.4)
$$

[1] The content of this appendix can be found in a slightly more concentrated form in the appendix of [19]. The main ideas and related results appeared earlier in Appendix a.3 in [7]. We recovered these precise references only after completing the section and decided to keep it for the convenience of the reader. See also Section 18.5 in [8].

with the natural definition of the symbol classes $S^m(T^*M)$, and with $L^m(M)$ denoting the space of pseudodifferential operators on M of order m.

It is well known that we can replace $a(x,\theta)$ in (D.1) with $a((x+y)/2,\theta)$ and this leads to the same definition of σ_A in $S^m/S^{m-1}(T^*M)$. Thus, working with

$$Au(x) = \mathrm{Op}\,(a)u(x) + Ru, \quad a \in S^m(\Omega \times \mathbf{R}^n), \; K_R \in C^\infty, \tag{D.5}$$

leads to the same principal symbol map. Here we write[2]

$$\mathrm{Op}\,(a)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y) dy d\theta. \tag{D.6}$$

It seems to be a well-known result (though we did not find a precise reference) that if we fix a positive smooth density ω on M, restrict our attention to local coordinates for which $\omega = dx_1...dx_n$ and work with the Weyl quantization as in (D.5), (D.6), then (D.4) improves to a bijection

$$L^m/L^{m-2}(M) \ni A \mapsto \sigma_A \in S^m/S^{m-2}(T^*M). \tag{D.7}$$

A natural generalization of this is to consider pseudodifferential operators acting on 1/2-densities; $A : C^\infty(M;\Omega^{1/2}) \to C^\infty(M;\Omega^{1/2})$. When using the Weyl quantization, we get the local representation analogous to D.5:

$$A(u(y)dy^{1/2}) = (\mathrm{Op}\,(a)u)(x)dx^{1/2} + (Ru)dx^{1/2}, \tag{D.8}$$

where $dx = dx_1...dx_n$. Recall that Duistermaat and Hörmander [4] have defined invariantly the notion of subpincipal symbol of such operators when the symbols are sums of a leading positively homogeneous term of order m in ξ and a symbol of order $m-1$. This result, as well as the fixed density invariance mentioned above, follow from the next more or less well-known proposition (cf. the footnote on page 19).

Proposition D.1. *Let $L^m(M)$ denote the space of pseudodifferential operators on M of order m, acting on half densities. Then if $(x_1,...,x_n)$ and $(\widetilde{x}_1,...,\widetilde{x}_n)$ are two local coordinate charts and we use the representation (D.8), so that*

$$A(udx^{1/2}) \equiv (\mathrm{Op}\,(a)u)dx^{1/2} \equiv (\mathrm{Op}\,(\widetilde{a})\widetilde{u})d\widetilde{x}^{1/2},$$

modulo the action of smoothing operators, for $udx^{1/2} = \widetilde{u}d\widetilde{x}^{1/2}$ supported in the intersection of the two coordinate charts, then we have

$$a(\kappa(\widetilde{x}),\theta) \equiv \widetilde{a}(\widetilde{x}, \kappa'(\widetilde{x})^{\mathrm{t}}\theta) \bmod S^{m-2}, \tag{D.9}$$

implying that we have a natural bijective symbol map

$$L^m/L^{m-2}(M) \to S^m/S^{m-2}(T^*M). \tag{D.10}$$

Proof. We only verify (D.9) and omit the (even more) standard arguments for (D.10). Our proof will be a straightforward adaptation of the proof of the invariance of pseudodifferential operators under composition with diffeomorphisms by means of the Kuranishi trick (cf. [5]).

[2] Strictly speaking, when Ω is not convex we need here to insert a suitable smooth cutoff $\chi(x,y) \in C^\infty(\Omega\times\Omega)$ which is equal to one near the diagonal, the choice of which can affect the operator only by a smoothing one.

In the intersection of the two coordinate charts Ω and $\widetilde{\Omega}$, we have $u(y)dy^{1/2} = \widetilde{u}(\widetilde{y})d\widetilde{y}^{1/2}$. Here $y = \kappa(\widetilde{y})$, where κ is a diffeomorphism: $\widetilde{\gamma}(\Omega\cap\widetilde{\Omega}) \to \gamma(\Omega\cap\widetilde{\Omega})$, $\kappa = \gamma\circ\widetilde{\gamma}^{-1}$). Thus $u(y) = \widetilde{u}(\widetilde{y})(\det\kappa'(\widetilde{y}))^{-1/2}$, assuming that $\det\kappa' > 0$ for simplicity. Thus, modulo the action of smoothing operators

$$A(udy^{1/2}) \equiv (\mathrm{Op}\,(a)u)dx^{1/2} = (\det\kappa'(\widetilde{x}))^{1/2}(\mathrm{Op}\,(a)u)d\widetilde{x}^{1/2},$$

so up to a smoothing operator $\mathrm{Op}\,(\widetilde{a})$ coincides with

$$B:\ \widetilde{u} \mapsto (\det\kappa'(\widetilde{x}))^{1/2}\mathrm{Op}\,(a)u, \quad u(y) = \widetilde{u}(\widetilde{y})(\det\kappa'(\widetilde{y}))^{-1/2}.$$

We have

$$\begin{aligned}
B\widetilde{u}(\widetilde{x}) &= (\det\kappa'(\widetilde{x}))^{1/2}\iint e^{i(x-y)\cdot\theta}a\left(\frac{x+y}{2},\theta\right)u(y)dy\frac{d\theta}{(2\pi)^n}\\
&= (\det\kappa'(\widetilde{x}))^{1/2}\iint e^{i(\kappa(\widetilde{x})-y)\cdot\theta}a\left(\frac{\kappa(\widetilde{x})+y}{2},\theta\right)\widetilde{u}(\widetilde{y})(\det\kappa'(\widetilde{y}))^{-1/2}dy\frac{d\theta}{(2\pi)^n}\\
&= \iint e^{i(\kappa(\widetilde{x})-\kappa(\widetilde{y}))\cdot\theta}a\left(\frac{\kappa(\widetilde{x})+\kappa(\widetilde{y})}{2},\theta\right)\widetilde{u}(\widetilde{y})(\det\kappa'(\widetilde{x})\det\kappa'(\widetilde{y}))^{1/2}d\widetilde{y}\frac{d\theta}{(2\pi)^n}\,.
\end{aligned}$$

By Taylor's formula (and restricting to a suitably thin neighborhood of the diagonal by means of a smooth cutoff, equal to one near the diagonal), we get

$$\kappa(\widetilde{x}) - \kappa(\widetilde{y}) = K(\widetilde{x},\widetilde{y})(\widetilde{x}-\widetilde{y}),$$

where $\widetilde{K}(\widetilde{x},\widetilde{y})$ depends smoothly on $(\widetilde{x},\widetilde{y})$ and

$$K(\widetilde{x},\widetilde{y}) = \kappa'\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right) + O((\widetilde{x}-\widetilde{y})^2).$$

It follows that

$$\begin{aligned}
B\widetilde{u}(\widetilde{x}) &= \iint e^{i(\widetilde{x}-\widetilde{y})\cdot K^{\mathrm{t}}(\widetilde{x},\widetilde{y})\theta}a\left(\frac{\kappa(\widetilde{x})+\kappa(\widetilde{y})}{2},\theta\right)\widetilde{u}(\widetilde{y})(\det\kappa'(\widetilde{x})\det\kappa'(\widetilde{y}))^{1/2}d\widetilde{y}\frac{d\theta}{(2\pi)^n}\\
&= \iint e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\theta}}a\left(\frac{\kappa(\widetilde{x})+\kappa(\widetilde{y})}{2},K^{\mathrm{t}}(\widetilde{x},\widetilde{y})^{-1}\widetilde{\theta}\right)\widetilde{u}(\widetilde{y})\frac{(\det\kappa'(\widetilde{x})\det\kappa'(\widetilde{y}))^{1/2}}{\det K(\widetilde{x},\widetilde{y})}d\widetilde{y}\frac{d\widetilde{\theta}}{(2\pi)^n}.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{\kappa(\widetilde{x})+\kappa(\widetilde{y})}{2} &= \kappa\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right) + O((\widetilde{x}-\widetilde{y})^2),\\
K^{\mathrm{t}}(\widetilde{x},\widetilde{y})^{-1} &= \left((\kappa')^{\mathrm{t}}\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right)\right)^{-1} + O((\widetilde{x}-\widetilde{y})^2),\\
\det K(\widetilde{x},\widetilde{y}) &= \det\kappa'\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right) + O((\widetilde{x}-\widetilde{y})^2),\\
(\det\kappa'(\widetilde{x})\det\kappa'(\widetilde{y}))^{1/2} &= \det\kappa'\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right) + O((\widetilde{x}-\widetilde{y})^2).
\end{aligned}$$

Thus,

$$B\widetilde{u} = \mathrm{Op}\,(\widetilde{a})\widetilde{u} + \iint e^{i(\widetilde{x}-\widetilde{y})\cdot\widetilde{\theta}}b(\widetilde{x},\widetilde{y},\widetilde{\theta})u(\widetilde{y})d\widetilde{y}\frac{d\widetilde{\theta}}{(2\pi)^n},$$

where $\widetilde{a} \in S^m$ is related to a as in (D.9) and $b \in S^m(\widetilde{\gamma}(\Omega\cap\widetilde{\Omega})^2 \times \mathbb{R}^n)$ (in the sense that $\partial_{\widetilde{x}}^{\alpha}\partial_{\widetilde{y}}^{\beta}\partial_{\widetilde{\theta}}^{|\delta|}b = O(\langle\widetilde{\theta}\rangle^{m-\delta})$ uniformly in $\widetilde{\theta}$ and locally uniformly in $(\widetilde{x},\widetilde{y})$) and b vanishes to the second order on the diagonal, $\widetilde{x} = \widetilde{y}$. By standard arguments we have $B \equiv \mathrm{Op}(r)$, where $r \in S^{m-2}$ and the proposition follows. □

Appendix E The resolvent and its powers as pseudo-differential operators

Let $\gamma : M \supset \Omega \to \Omega' \subset \mathbb{R}^n$ be a chart of local coordinates and let us identify Ω' with Ω in the natural way. Let $a(x,\xi) \in S^1(\Omega \times \mathbb{R}^n)$ (defined modulo $S^{-\infty}(\Omega \times \mathbb{R}^n)$) be the Weyl symbol of

$$A_{|C_0^\infty(\Omega)} : C_0^\infty(\Omega) \to C^\infty(\Omega), \tag{E.1}$$

so that

$$Au(x) = \mathrm{Op}\,(a)u(x) + Ru(x),\ x \in \Omega \tag{E.2}$$

for every $u \in C_0^\infty(\Omega)$, where $R \in L^{-\infty}(\Omega)$ in the sense that $K_R \in C^\infty(\Omega \times \Omega)$. Here we identify 1/2 densities and scalar functions on Ω by means of the fixed factor $dx^{1/2}$. We first work in this fixed local coordinate chart and write simply A for the operator in (E.1). We notice that

$$a - z \in S(\Omega \times \mathbb{R}^n, \langle \xi, z\rangle) = S(\langle \xi, z\rangle), \tag{E.3}$$

in the sense that $a - z \in C^\infty(\Omega \times \mathbb{R}^n)$ and that for all $K \Subset \Omega$, $\alpha, \beta \in \mathbb{N}^n$,

$$|\partial_x^\alpha \partial_\xi^\beta (a - z)| \le C_{K,\alpha,\beta} \langle \xi, z\rangle \langle \xi\rangle^{-|\beta|}, \tag{E.4}$$

uniformly when $z \in \mathbb{C}$, $|z| > 1$, $x \in K$, $\xi \in \mathbb{R}^n$. Here, we write $\langle \xi\rangle = (1 + |\xi|^2)^{1/2}$, $\langle \xi, z\rangle = (1 + |z|^2 + |\xi|^2)^{1/2}$.

Similarly, if $\Gamma \subset \dot{\mathbb{C}}$ is a closed conic neighborhood of $\dot{\mathbb{R}}$ and until further notice we restrict our attention to $z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1))$, we have

$$(a - z)^{-1} \in S(\langle \xi, z\rangle^{-1}) \tag{E.5}$$

with the natural generalization of the definition (E.4).

Sometimes, we shall exploit the fact that $a - z$ and $(a - z)^{-1}$ belong to narrower symbol classes, used in [6]. We say that $b(x,\xi,z)$, defined for (x,ξ,z) as in (E.5), belongs to $S_1(\langle \xi, z\rangle^m)$, $m \in \mathbb{R}$, if

$$|\partial_x^\alpha \partial_\xi^\beta b(x,\xi,z)| \le C_{K,\alpha,\beta} \begin{cases} \langle \xi, z\rangle^m, \text{ when } \alpha = \beta = 0, \\ \langle \xi, z\rangle^m \frac{\langle \xi\rangle}{\langle \xi, z\rangle} \langle \xi\rangle^{-|\beta|}, \text{ when } (\alpha,\beta) \neq (0,0), \end{cases} \tag{E.6}$$

uniformly for $x \in K \Subset \Omega$, $\xi \in \mathbb{R}^n$, $z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1))$.

If $b_j \in S(\langle \xi, z\rangle^{m_j})$, $j = 1, 2$, the asymptotic Weyl composition

$$\begin{aligned} b_1 \# b_2 &= \left(e^{(i/2)\sigma(D_{x,\xi}; D_{y,\eta})} b_1(x,\xi) b_2(y,\eta)\right)_{\substack{y=x \\ \eta=\xi}} \\ &\sim \sum_{k=0}^\infty \frac{1}{k!} \left(\left(\frac{i}{2} \sigma(D_{x,\xi}; D_{y,\eta}) \right)^k b_1(x,\xi) b_2(y,\eta) \right)_{\substack{y=x \\ \eta=\xi}} \end{aligned} \tag{E.7}$$

is well defined in $S(\langle \xi, z\rangle^{m_1+m_2}) / S(\langle \xi, z\rangle^{m_1+m_2} \langle \xi\rangle^{-\infty})$, where

$$S(\langle \xi, z\rangle^{m_1+m_2} \langle \xi\rangle^{-\infty}) = \bigcap_{N \ge 0} S(\langle \xi, z\rangle^{m_1+m_2} \langle \xi\rangle^{-N})$$

and with the natural definition of the symbol spaces to the right. Here $\sigma(D_{x,\xi};D_{y,\eta}) = D_\xi \cdot D_y - D_x \cdot D_\eta$. Notice that

$$\frac{1}{k!}\left(\left(\frac{i}{2}\sigma(D_{x,\xi};D_{y,\eta})\right)^k b_1(x,\xi)b_2(y,\eta)\right)_{\substack{y=x\\ \eta=\xi}} \in S(\langle \xi, z\rangle^{m_1+m_2}\langle\xi\rangle^{-k}). \tag{E.8}$$

When $b_j \in S_1(m_j)$ this improves to

$$\frac{1}{k!}\left(\left(\frac{i}{2}\sigma(D_{x,\xi};D_{y,\eta})\right)^k b_1(x,\xi)b_2(y,\eta)\right)_{\substack{y=x\\ \eta=\xi}} \in \begin{cases} S_1(\langle \xi, z\rangle^{m_1+m_2}), \text{ when } k=0,\\ S(\langle \xi, z\rangle^{m_1+m_2-2}\langle\xi\rangle^{2-k}), \text{ when } k\geq 1.\end{cases} \tag{E.9}$$

In particular,

$$b_1\#b_2 \equiv b_1 b_2 \text{ mod } S(\langle \xi, z\rangle^{m_1+m_2-2}\langle\xi\rangle).$$

In the special case $b_1 = a-z$, $b_2 = (a-z)^{-1}$ we get

$$(a-z)\#(a-z)^{-1} = 1 + r, \tag{E.10}$$

$$\begin{aligned} r \sim & \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{i}{2}\sigma(D_{x,\xi};D_{y,\eta})\right)^k \left(a(x,\xi)(a(y,\eta)-z)^{-1}\right)_{\substack{y=x\\ \eta=\xi}} \\ & \in S(\langle \xi, z\rangle^{-2}\langle\xi\rangle)/S(\langle \xi, z\rangle^{-2}\langle\xi\rangle^{-\infty}), \\ r \equiv & \frac{i}{2}\sigma(D_{x,\xi};D_{y,\eta})\left(a(x,\xi)(a(y,\eta)-z)^{-1}\right)_{\substack{y=x\\ \eta=\xi}} \\ \equiv & \frac{i}{2}\{a,(a-z)^{-1}\} \text{ mod } S(\langle \xi, z\rangle^{-2}), \end{aligned} \tag{E.11}$$

with the Poisson bracket as defined in Section 1.

The symbolic inverse of $A-z$ is now

$$b(x,\xi,z) \sim (a-z)^{-1}\#(1 - r + r\#r ... \pm r^{\#k} + ...), \tag{E.12}$$

where

$$r^{\#k} = \underbrace{r\#r\#...\#r}_{k \text{ factors}} \in S\left((\langle\xi\rangle/\langle \xi, z\rangle^2)^k\right) \subset S\left(\langle \xi, z\rangle^{-k}\right).$$

We see that $b(x,\xi,z) \in S(\langle \xi, z\rangle^{-1})$ and that

$$b \equiv (a-z)^{-1} \text{ mod } S\left(\frac{\langle\xi\rangle}{\langle \xi, z\rangle^3}\right).$$

More precisely,

$$b \equiv (a-z)^{-1} - (a-z)^{-1}\#r \text{ mod } S\left(\frac{1}{\langle \xi, z\rangle^3}\right).$$

Here

$$\begin{aligned}(a-z)^{-1}\#r \sim (a-z)^{-1}r + \sum_{k\geq 1}\frac{1}{k!}\left(\left(\frac{i}{2}\sigma(D_{x,\xi};D_{y,\eta})\right)^k \left((a-z)^{-1}(x,\xi)r(y,\eta)\right)\right)_{\substack{y=x\\ \eta=\xi}} \\ \equiv (a-z)^{-1}r \text{ mod } S\left(\frac{1}{\langle \xi, z\rangle^3}\right),\end{aligned}$$

so

$$b(x,\xi,z) \equiv (a-z)^{-1} - (a-z)^{-1}r \\ \equiv (a-z)^{-1} - \frac{i}{2}(a-z)^{-1}\{a,(a-z)^{-1}\} \bmod S\left(\frac{1}{\langle \xi,z\rangle^3}\right), \tag{E.13}$$

where we also used (E.11).

If $b_j \in S(\langle \xi,z\rangle^m \langle \xi\rangle^{k-j})$ for $j=0,1,...$, we can apply a standard procedure to construct a symbol $b \in S((\langle \xi,z\rangle^m\langle \xi\rangle^k)$ such that

$$b - \sum_0^{N-1} b_j \in S(\langle \xi,z\rangle^m \langle \xi\rangle^{k-N})$$

for every $N \geq 1$ and we still write $b \sim \sum_0^\infty b_j$ where b is a concrete symbol (uniquely determined up to $S(\langle \xi,z\rangle^m\langle\xi\rangle^{-\infty})$). If b_j are holomorphic for $z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1))$, then the standard construction produces a symbol b which is also holomorphic.

If $b \in S(\langle \xi,z\rangle^m\langle\xi\rangle^k)$ is such a holomorphic symbol then by the Cauchy inequalities we get[3]

$$\partial_z^\ell b \in S(\langle \xi,z\rangle^m\langle\xi\rangle^k\langle z\rangle^{-\ell})$$

in the sense that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_z^\ell b| \leq C_{K,\alpha,\beta,\ell}\langle \xi,z\rangle^m\langle\xi\rangle^{k-|\beta|}\langle z\rangle^{-\ell}$$

for $x \in K \Subset \Omega$, $\xi \in \mathbb{R}^n$ and omitting the slight increase of $\Gamma \cup D(0,1)$, mentioned in the last footnote.

With the holomorphic z-dependence in mind we return to (E.11) and write

$$r \sim \sum_{k=1}^\infty r_k(x,\xi,z) \tag{E.14}$$

and get a concrete symbol $r \in S(\langle \xi,z\rangle^{-2}\langle\xi\rangle^{2-1})$ which is holomorphic in z, so that for every $N \geq 1$,

$$r - \sum_1^{N-1} r_k \in S(\langle \xi,z\rangle^{-2}\langle\xi\rangle^{2-N}) \tag{E.15}$$

and by the Cauchy inequalities

$$\partial_z^\ell\left(r - \sum_1^{N-1} r_k\right) \in S(\langle \xi,z\rangle^{-2}\langle\xi\rangle^{2-N}\langle z\rangle^{-\ell}). \tag{E.16}$$

From the explicit expression of the r_k (or from observing that they are defined for z in $(\dot{\mathbb{C}} \setminus \Gamma) \cup D(0,\langle\xi\rangle/C)$ when ξ is large), we see that

$$\partial_z^\ell r_k \in S(\langle \xi,z\rangle^{-2-\ell}\langle\xi\rangle^{2-k}), \tag{E.17}$$

$$\partial_z^\ell\left(\sum_1^{N-1} r_k\right) \in S(\langle \xi,z\rangle^{-2-\ell}\langle\xi\rangle^{2-1}). \tag{E.18}$$

[3] After replacing Γ with any closed conic set containing Γ in its interior and $D(0,1)$ with $D(0,1+\epsilon)$ for any $\epsilon > 0$

Choosing $N = \ell + 1$ in (E.16), (E.18), we get

$$\partial_z^\ell r \in S(\langle \xi, z\rangle^{-2-\ell}\langle \xi\rangle^1). \tag{E.19}$$

This argument shows that (E.14) is valid in the symbol space $\widetilde{S}(\langle \xi, z\rangle^{-2}\langle \xi\rangle^{2-1})$, where we say that $c \in \widetilde{S}(\langle \xi, z\rangle^m\langle \xi\rangle^k)$ if $c(x, \xi, z)$ is a smooth, holomorphic in z and

$$\partial_z^\ell c \in \widetilde{S}(\langle \xi, z\rangle^{m-\ell}\langle \xi\rangle^k), \text{ for all } \ell \geq 0.$$

In (E.12) we can choose $r^{\#k}$ and the asymptotic sums so that $b \in \widetilde{S}(\langle \xi, z\rangle^{-1})$ and so that (E.13) improves to

$$\begin{aligned} b(x, \xi, z) &\equiv (a - z)^{-1} - (a - z)^{-1}r \\ &\equiv (a - z)^{-1} - \frac{i}{2}(a - z)^{-1}\{a, (a - z)^{-1}\} \bmod \widetilde{S}\left(\frac{1}{\langle \xi, z\rangle^3}\right), \end{aligned} \tag{E.20}$$

where

$$r \in \widetilde{S}(\langle \xi, z\rangle^{-2}\langle \xi\rangle),\ (a - z)^{-1} \in \widetilde{S}(\langle \xi, z\rangle^{-1}). \tag{E.21}$$

In the main text we have

$$A = A_1 + A_0 + A_{-1},\ A_0 = A_{\mathrm{sub}},$$

where $A_j \in S(\langle \xi\rangle^j)$ and A_1, A_0 are positively homogeneous in ξ of degree 1 and 0 respectively, in the region $|\xi| \geq 1$. From the resolvent identity

$$\begin{aligned} (a - z)^{-1} = (A_1 - z)^{-1} &- (A_1 - z)^{-1}(a - A_1)(A_1 - z)^{-1} \\ &+ (A_1 - z)^{-1}(a - A_1)(a - z)^{-1}(a - A_1)(A_1 - z)^{-1} \end{aligned}$$

we infer that

$$(a - z)^{-1} \equiv (A_1 - z)^{-1} - (A_1 - z)^{-1}(A_0 + A_{-1})(A_1 - z)^{-1} \bmod \widetilde{S}(\langle \xi, z\rangle^{-3}),$$

hence,

$$(a - z)^{-1} \equiv (A_1 - z)^{-1} - (A_1 - z)^{-1}A_0(A_1 - z)^{-1} \bmod \widetilde{S}(\langle \xi\rangle^{-1}\langle \xi, z\rangle^{-2}).$$

In particular,

$$(a - z)^{-1} \equiv (A_1 - z)^{-1} \bmod \widetilde{S}(\langle \xi, z\rangle^{-2})$$

and from (E.20) we get

$$\begin{aligned} b \equiv (A_1 - z)^{-1} &- (A_1 - z)^{-1}A_0(A_1 - z)^{-1} \\ &- \frac{i}{2}(A_1 - z)^{-1}\{A_1, (A_1 - z)^{-1}\} \bmod \widetilde{S}\left(\frac{1}{\langle \xi\rangle\langle \xi, z\rangle^2}\right), \end{aligned} \tag{E.22}$$

which implies (3.1) (cf. (3.2)).

By construction, b is a realization of the symbolic inverse of $a - z$:

$$(a - z)\#b \equiv 1 \bmod \widetilde{S}(\langle \xi, z\rangle^{-2}\langle \xi\rangle^{-\infty}).$$

Let $B = \mathrm{Op}\,(b) : C_0^\infty(\Omega) \to C^\infty(\Omega)$ (where we also insert a suitable cutoff $\in C^\infty(\Omega\times\Omega)$, equal to 1 near $\mathrm{diag}\,(\Omega\times\Omega)$). Then

$$\partial_z^k B(z) = O(\langle z\rangle^{-k_1}) : H^s_{\mathrm{comp}}(\Omega) \to H^{s+k_2}_{\mathrm{loc}}(\Omega) \text{ uniformly for } z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1)), \quad \text{(E.23)}$$

when $1 + k = k_1 + k_2$, $k_j \geq 0$, $s \in \mathbb{R}$.

Let $\chi, \Phi \in C_0^\infty(\Omega)$, with $\Phi = 1$ near $\mathrm{supp}\,(\chi)$. Then,

$$(A - z)\Phi B \chi = \chi + R, \quad \text{(E.24)}$$

where $R = R(z)$ is a smoothing operator: $\mathcal{D}'(\Omega) \to C^\infty(\Omega)$, depending holomorphically on z, such that $Ru = 0$ when $\mathrm{supp}\,(u) \cap \mathrm{supp}\,(\chi) = \emptyset$ and

$$\partial_z^k R = O(\langle z\rangle^{-2-k}) : H^{-s}(\Omega) \to H^s_{\mathrm{loc}}(\Omega),\ z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1)), \quad \text{(E.25)}$$

for all $s \in \mathbb{R}$, $k \geq 0$. We omit the standard proof of this, based on the symbolic results above, starting with the identity

$$(A - z)\Phi B\chi = [A, \Phi]B\chi + \Phi(A - z)B\chi.$$

Let $M \subset \bigcup_1^N \Omega_j$ be a finite covering of M with coordinate charts as above. Recall that A is a globally defined pseudodifferential operator acting on 1/2 densities so we can now view $A - z$ as acting: $C_0^\infty(\Omega_j; \Omega^{1/2}) \to C^\infty(M; \Omega^{1/2})$ for each j. We have a corresponding operator B_j (as "B" above), now acting on 1/2-densities, so that

$$B_j(u dx^{1/2}) = (\mathrm{Op}\,(b_j)u)dx^{1/2},\ u \in C_0^\infty(\Omega_j) \quad \text{(E.26)}$$

where $dx^{1/2}$ is the canonical (and j-dependent) 1/2-density on Ω_j. Let $\chi_j \in C_0^\infty(\Omega_j)$ form a partition of unity on M. (E.24) becomes

$$(A - z)\Phi_j B_j \chi_j = \chi_j + R_j(z), \quad \text{(E.27)}$$

where R_j has the properties of "R" in (E.23), (E.25) except for the fact that R_j acts on 1/2-densities and that we can actually define R_j as an operator on M such that

$$\|\partial_z^k R_j\|_{\mathcal{L}(H^{-s}, H^s(M))} \leq C_s \langle z\rangle^{-2-k},\ z \in \dot{\mathbb{C}} \setminus (\Gamma \cup D(0,1)). \quad \text{(E.28)}$$

Here $H^s(M)$ denotes the Sobolev space of 1/2-densities of order $s \in \mathbb{R}$.

Let

$$B := \sum \Phi_j B_j \chi_j : \ C^\infty(M; \Omega^{1/2}) \to C^\infty(M; \Omega^{1/2}). \quad \text{(E.29)}$$

Then

$$(A - z)B(z) = 1 + R(z), \quad \text{(E.30)}$$

$$R(z) = \sum R_j(z), \quad \text{(E.31)}$$

$$\partial_z^k B(z) = O(\langle z\rangle^{-k_1}) : \ H^s \to H^{s+k_2},\ \text{when } k + 1 = k_1 + k_2,\ k_j \geq 0, \quad \text{(E.32)}$$

$$\partial_z^k R(z) = O_s(\langle z\rangle^{-2-k}) : \ H^{-s} \to H^s, \quad \text{(E.33)}$$

for all $s \in \mathbb{R}$.

On the other hand, by direct arguments, we know that $(A - z)^{-1}$ also enjoys the properties (E.32). Applying this operator to the left in (E.30), we get

$$(A - z)^{-1} = B(z) - K(z),\ K(z) = (A - z)^{-1}R(z). \quad \text{(E.34)}$$

Clearly, $K(z)$ also satisfies (E.33).

Using the operator identity (5.1) in (E.34), we get

$$(A - z)^{1-n} = B^{(n)}(z) - K^{(n)}(z), \tag{E.35}$$

$$B^{(n)} = \frac{1}{(n-2)!}\partial_z^{n-2} B(z), \tag{E.36}$$

$$K^{(n)} = \frac{1}{(n-2)!}\partial_z^{n-2} K(z) = O_s(\langle z\rangle^{-n}) : \ H^{-s} \to H^s. \tag{E.37}$$

From the last estimate it follows that $K^{(n)}$ is of trace class with a continuous distribution kernel which is uniformly $= O(\langle z\rangle^{-n})$.

Let x_0 be a point in a coordinate chart $\Omega = \Omega_j$ and assume for simplicity that $\chi = \chi_j$ is equal to 1 near that point. Then near (x_0, x_0) the distribution kernel of B (identified locally with an operator acting on scalar functions) coincides with that of $\mathrm{Op}\,(b)$, where b satisfies (E.20). Consequently,

$$B^{(n)} = \mathrm{Op}\,(b^{(n)}), \tag{E.38}$$

$$b^{(n)} \equiv (a-z)^{-n} - \frac{1}{(n-2)!}\partial_z^{n-2}\left((a-z)^{-1} r\right) \mod \widetilde{S}\left(\langle \xi, z\rangle^{-n-1}\right). \tag{E.39}$$

Appendix F Proof of formulae (4.4) and (4.5)

Formula (3.5) implies

$$(\partial P^{(k)})P^{(j)} + P^{(k)}\partial P^{(j)} = \delta^{kj}\partial P^{(k)}, \tag{F.1}$$

where ∂ is any partial derivative. We have

$$\begin{aligned}\mathrm{tr}\{P^{(k)}, P^{(j)}, P^{(l)}\} &= \mathrm{tr}\big[(\partial_{x^\alpha} P^{(k)})P^{(j)}\partial_{\xi_\alpha} P^{(l)} - (\partial_{\xi_\alpha} P^{(k)})P^{(j)}\partial_{x^\alpha} P^{(l)}\big] \\ &= \mathrm{tr}\big[\big((\partial_{x^\alpha} P^{(k)})P^{(j)}\big)\big(P^{(j)}\partial_{\xi_\alpha} P^{(l)}\big) - \big((\partial_{\xi_\alpha} P^{(k)})P^{(j)}\big)\big(P^{(j)}\partial_{x^\alpha} P^{(l)}\big)\big].\end{aligned}$$

Using (F.1), we can rewrite the above formula as

$$\begin{aligned}\mathrm{tr}\{P^{(k)}, P^{(j)}, P^{(l)}\} = \mathrm{tr}\big[&\big(\delta^{kj}\partial_{x^\alpha} P^{(j)} - P^{(k)}\partial_{x^\alpha} P^{(j)}\big)\big(\delta^{jl}\partial_{\xi_\alpha} P^{(j)} - (\partial_{\xi_\alpha} P^{(j)})P^{(l)}\big) \\ &- \big(\delta^{kj}\partial_{\xi_\alpha} P^{(j)} - P^{(k)}\partial_{\xi_\alpha} P^{(j)}\big)\big(\delta^{jl}\partial_{x^\alpha} P^{(j)} - (\partial_{x^\alpha} P^{(j)})P^{(l)}\big)\big].\end{aligned}$$

Expanding the parentheses in the above formula and rearranging terms, we get

$$\begin{aligned}\mathrm{tr}\{P^{(k)}, P^{(j)}, P^{(l)}\} = \delta^{kj}\mathrm{tr}\{P^{(j)}, P^{(l)}, P^{(j)}\} &+ \delta^{jl}\mathrm{tr}\{P^{(j)}, P^{(k)}, P^{(j)}\} \\ &- \delta^{kl}\mathrm{tr}\{P^{(j)}, P^{(k)}, P^{(j)}\}.\end{aligned} \tag{F.2}$$

In the special case $l = k$ the above formula becomes

$$\mathrm{tr}\{P^{(k)}, P^{(j)}, P^{(k)}\} = 2\delta^{kj}\mathrm{tr}\{P^{(j)}, P^{(j)}, P^{(j)}\} - \mathrm{tr}\{P^{(j)}, P^{(k)}, P^{(j)}\}. \tag{F.3}$$

Each of the three terms in the RHS of (F.2) can now be rewritten using the identity (F.3) with appropriate choice of indices, which gives us (4.4).

Let us now substitute (4.4) into the triple sum in the RHS of (4.3):

$$\sum_{j,k,l} \frac{h^{(j)}-z}{(h^{(k)}-z)(h^{(l)}-z)} \operatorname{tr}\{P^{(k)},P^{(j)},P^{(l)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\}$$

$$- \sum_{j,l} \frac{1}{h^{(l)}-z} \operatorname{tr}\{P^{(l)},P^{(j)},P^{(l)}\} - \sum_{j,k} \frac{1}{h^{(k)}-z} \operatorname{tr}\{P^{(k)},P^{(j)},P^{(k)}\}$$

$$+ \sum_{j,k} \frac{h^{(j)}-z}{(h^{(k)}-z)^2} \operatorname{tr}\{P^{(k)},P^{(j)},P^{(k)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\} - 2\sum_{j,k} \frac{1}{h^{(k)}-z} \operatorname{tr}\{P^{(k)},P^{(j)},P^{(k)}\}$$

$$+ \sum_{j,k} \frac{h^{(j)}-z}{(h^{(k)}-z)^2} \operatorname{tr}\{P^{(k)},P^{(j)},P^{(k)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\} - 2\sum_k \frac{1}{h^{(k)}-z} \operatorname{tr}\{P^{(k)},P^{(k)}\}$$

$$+ \sum_{j,k} \frac{1}{(h^{(k)}-z)^2} \operatorname{tr}\{P^{(k)},h^{(j)}P^{(j)},P^{(k)}\} - z\sum_k \frac{1}{(h^{(k)}-z)^2} \operatorname{tr}\{P^{(k)},P^{(k)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\} + \sum_k \frac{1}{(h^{(k)}-z)^2} \operatorname{tr}\{P^{(k)},A_1,P^{(k)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\} + \sum_j \frac{1}{(h^{(j)}-z)^2} \operatorname{tr}\{P^{(j)},A_1,P^{(j)}\}$$

$$= 2\sum_j \frac{1}{h^{(j)}-z} \operatorname{tr}\{P^{(j)},P^{(j)},P^{(j)}\} + \sum_j \frac{1}{(h^{(j)}-z)^2} \operatorname{tr}\{P^{(j)},A_1-h^{(j)}I,P^{(j)}\}, \quad \text{(F.4)}$$

where we used the identities $\sum_j P^{(j)} = I$, $\{P^{(k)},P^{(k)}\}=0$ and (3.3). Substituting (F.4) into (4.3) we arrive at (4.5).

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