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Valuing portfolios of interdependent real options under exogenous and endogenous uncertainties☆

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ABSTRACT

Although the value of portfolios of real options is often affected by both exogenous and endogenous sources of uncertainty, most existing valuation approaches consider only the former and neglect the latter. In this paper, we introduce an approach for valuing portfolios of interdependent real options under both types of uncertainty. In particular, we study a large portfolio of options (deferment, staging, mothballing, abandonment) under conditions of four underlying uncertainties. Two of the uncertainties, decision-dependent cost to completion and state-dependent salvage value, are endogenous, the other two, operating revenues and their growth rate, are exogenous. Assuming that endogenous uncertainties can be exogenised, we formulate the valuation problem as a discrete stochastic dynamic program. To approximate the value of this optimisation problem, we apply a simulation-and-regression-based approach and present an efficient valuation algorithm. The key feature of our algorithm is that it exploits the problem structure to explicitly account for reachability – that is the sample paths in which resource states can be reached. The applicability of the approach is illustrated by valuing an urban infrastructure investment. We conduct a reachability analysis and show that the presence of the decision-dependent uncertainty has adverse computational effects as it increases algorithmic complexity and reduces simulation efficiency. We investigate the way in which the value of the portfolio and its individual options are affected by the initial operating revenues, and by the degrees of exogenous and endogenous uncertainty. The results demonstrate that ignoring endogenous, decision- and state-dependent uncertainty can lead to substantial over- and under-valuation, respectively.

1. Introduction

A fundamental issue in real options analysis and decision-making under uncertainty is how to account correctly and adequately for the multiple sources of uncertainty occurring in most practical real-life situations. In these situations it is generally assumed that the effective sources of uncertainty are purely exogenous and, as such, are independent of both the actions taken by the decision maker and the state of the underlying system affected by these decisions. For example, in the case of investment in a new wind farm, the wind farm's performance depends on factors such as location, time of day and the wind turbines' characteristics; however, parameters such as the wind speed, and consequently the amount of power generated, are independent of the investor's decision of whether to build the wind farm or not. Likewise, if the amount of power generated by such a wind farm is sufficiently small and/or the relevant wholesale electricity market to which the power is sold is comparatively large, then the underlying wholesale price of electricity, and consequently the investor's potential revenues are also independent of the investor's decision.

There are, however, many practical situations in which the relevant sources of uncertainty are endogenous, i.e. dependent on the decision maker's actions or the underlying system's state, or both. In the case of the wind farm example, if the above-mentioned conditions are violated, i.e. if the new wind farm is sufficiently large and/or the electricity market relatively small, then the introduction of a new wind farm will affect the wholesale price

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of electricity and hence the investor's future revenues. Similarly, although the "off-the-shelf" cost of new wind turbines may be known and a feasibility study may provide a construction cost estimate, the actual cost of building a new wind farm will not be known until the investor actually builds it. During the building process, the investor learns and reveals the wind farm's true capital cost. If the investor wants to sell the wind farm at the end of its lifetime, in the absence of a second hand market, the resale value will depend on its "state", which may include such factors as its lifetime, asset value, wear and tear, and decommissioning cost.

Despite the prevalence of exogenous and endogenous sources of uncertainty in many real-life situations, there remains a need for a unified approach that accounts for both when real options analysis is used to evaluate practical investment problems. Including both types of uncertainty in a real options approach has rarely been studied in the related literature (Ahsan & Musteen, 2011). Although portfolio of real options approaches have been applied when there is only exogenous uncertainty, there is a need to include both types because that enables decision-makers to manage the two uncertainty types simultaneously (Otim & Grover, 2012). Some authors have therefore suggested that future work should examine the interactions between different sources of uncertainty and the portfolio's individual options, e.g. see Tiwana, Keil, and Fichman (2006) and Li, James, Madhavan, and Mahoney (2007). More recently, a critical review of Trigeorgis and Reuer (2017) has suggested four extensions, three of which are addressed here: portfolios of interdependent real options, multiple sources of uncertainty, and endogenous resolution of uncertainty through learning.

This paper introduces a valuation approach for portfolios of interdependent real options under exogenous and endogenous sources of uncertainty. Studying the problem of a sequential and partially reversible investment project, we consider a portfolio of options to: defer investment; stage investment; temporarily halt expansion; temporarily mothball the operation; and permanently abandon the project during either construction or operation. In the problem studied here, the portfolio's value is affected by four underlying uncertainties. Of these, the project's actual cost to completion and its salvage value are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the operating revenues and their growth rate evolve exogenously. Similar to Maier, Polak, and Gann (2018), we use an influence diagram to graphically model the interdependencies between the portfolio's real options and mathematically translate these into a set of constraints. The constraints and the stochastic processes describing the uncertainties' dynamics are then integrated into a multi-stage stochastic optimisation problem which is formulated as a stochastic dynamic program.

Our decision model is a stochastic dynamic, discrete-time (Markovian) model: the transition of the state S_t of the underlying system at time t to state $S_{t+\Delta}$ after a time increment Δ is driven by our decisions as well as by the random processes describing the uncertainties. Here we distinguish between *exogenous* and *endogenous* sources of uncertainty. Modelled as stochastic Markovian processes, the evolution of endogenous uncertainties depends on the decision maker's strategy or the system's state, or both, while those of exogenous uncertainties are unaffected by decisions and states. Compared to standard real option models, models with decision- or state-dependent random variables are much more difficult to solve by simulation-and-regression methods since it is generally impossible to use random deviates which have been sampled once at initialisation. However, as shown in Supplementary Material C, it is sometimes possible by a reformulation termed *exogenisation* to use the same fixed set of random deviates even for endogenous uncertainty. In this paper it is therefore assumed that endogenous uncertainty can be exogenised. To approximate the value of this optimisation problem, we use an extension of standard simulation-and-regression methods (e.g. see Cortazar, Gravet, & Urzua, 2008; Glasserman & Yu, 2004; Longstaff & Schwartz, 2001; Nadarajah, Margot, & Secomandi, 2017; Tsitsiklis & Van Roy, 2001) whose basic structure and principles are described in Supplementary Material D.

The main contributions of this work are in the following three areas:

(1) Our model extends the standard models in several ways: (i) standard models generally assume a single time step, meaning that the time evolves from t to $t+1$ after a decision has been made at t, but here the decisions $a_t \in \mathcal{A}_{S_t}$ imply the time delay Δ_h, i.e. the moment in time for the next decision is $t+\Delta_h$. This makes our model more flexible as it allows us to address problems with multiple time steps; (ii) unlike standard problems, here the random variables ξ_t appearing in the transition function depend on the state of the system S_t. To enable computational tractability, however, we show how exogenised random factors ε_t can be used instead by assuming that the ξ_t can be written in the form $\xi_t(\omega) = f(S_t(\omega), \varepsilon_t(\omega))$, where $\varepsilon_t(\omega)$ is independent of $S_t(\omega)$ when following a sample path ω (see Supplementary Material C); (iii) we explain how the parametric regression model can be made dependent on the state S_t to account for the circumstance that some basis functions of the parametric model are impossible for some states.

(2) We present an extended algorithm to account for complexities induced by the extended model. First, compared to standard algorithms for problems with only exogenous uncertainty, the incorporation of the decision-dependent uncertainty results in an additional path-dependency. We therefore propose a forward induction procedure in which the resource state space generation is interleaved with the Monte Carlo sampling steps of the information state space generation. Secondly, we include *reachability* in our forward pass to account for the circumstance that some resource states may not be reachable, or only in a subset of sample paths. This is a key feature as it enables us to design an efficient backward approximation algorithm that considers only reachable resource states and the set of paths in which they can actually be reached. Thirdly, we describe how the structure of the problem to be solved can be exploited through dynamically and appropriately adapting the set of basis functions used in the parametric model in order to avoid numerical inaccuracies related to 1(iii).

(3) We demonstrate the applicability of our approach and perform a set of detailed numerical analyses using an illustrative example of an urban infrastructure investment in London. We first conduct a reachability analysis to investigate the complexity of the problem in terms of the number of both resource states and sample paths, and show that the presence of the endogenous, decision-dependent uncertainty generally leads to an increase in algorithmic cost and a decrease in simulation efficiency. Subsequently, we investigate the sensitivity of the value of the portfolio and its individual options to the initial level of annual revenues, as well as to the degrees of exogenous and endogenous uncertainty. We illustrate that the availability of real options is more valuable for low values of initial revenues, and that the portfolio is substantially more valuable than individual options. We also illustrate that the portfolio value increases monotonically in both the exogenous and the endogenous, state-dependent uncertainty, but that there is a non-monotonic effect with respect to the endogenous, decision-dependent uncertainty. More importantly, this work shows that ignoring decision- and state-dependent uncertainty can lead

to substantial over- and under-valuation, respectively, and also provides the reasons for this.

The rest of this article is organised as follows: Section 2 reviews the relevant literature with an emphasis on the operational research as well as on the finance and management literature. Section 3 describes the investment problem by specifying both the portfolio of interdependent real options (Section 3.1) and the set of uncertainties (Section 3.2) considered in this work. In Section 4 we present the modelling and valuation approach together with the simulation-and-regression-based valuation algorithm (Section 4.3). The approach and the algorithm are then applied to the real-case of a district heating network expansion investment in the London borough of Islington (Section 5). Results are presented and discussed in Section 5.4. Finally, some concluding remarks and suggestions for future research are provided in Section 6. Additional information regarding the exogenisation of endogenous uncertainty, the basic algorithm, input data and illustration of sample paths is provided as supplementary material.

2. Literature review

The classification of uncertainties into exogenous and endogenous has received considerable attention in different branches of literature, and importantly in the operational research as well as in the finance and management literature. With regard to the former, to the best of our knowledge, the work of Jonsbråten, Wets, and Woodruff (1998) was the first to classify the formulation of stochastic programs into "standard" formulations with decision independent random variables and "manageable" formulations, in which the distribution of the random variables is dependent on decisions. Calling the former "exogenous uncertainty" and the latter "endogenous uncertainty" (Goel & Grossmann, 2004), Goel and Grossmann (2006) specified the way in which decisions can affect the stochastic process – which describes the evolution of an uncertain parameter (Kirschenmann, Popova, Damien, & Hanson, 2014) – by presenting two types of endogenous uncertainty. The first is when the decision alters the probability distribution, whereas the second relates to the decision affecting the timing of uncertainty resolution, a process often described as information revelation.

Considering the above specification of endogenous uncertainties, several relevant works have appeared in the operations research literature over the last few decades. As for the first type of endogenous uncertainty, Pflug (1990) was the first to take into account decision-dependent probabilities in a stochastic optimisation problem by considering a controlled Markov chain where the transition operator depends on the control, i.e. the decision. Other relevant articles related to this type are in the context of stochastic network problems (Held & Woodruff, 2005; Peeta, Salman, Gunnec, & Viswanath, 2010), global climate policy (Webster, Santen, & Parpas, 2012) and natural gas markets (Devine, Gabriel, & Moryadee, 2016). By contrast, the second type of endogenous uncertainty has received considerable more attention in the literature. The first work related to this type was (Goel & Grossmann, 2004), which presented a stochastic programming approach for the planning of an investment into a gas field with uncertain reserves represented through a decision-dependent scenario tree. Other relevant works include the optimisation of R&D project portfolios (Solak, Clarke, Johnson, & Barnes, 2010) and pharmaceutical clinical trial planning (Colvin & Maravelias, 2010; 2011).

Moreover, several works have incorporated both the second type of endogenous uncertainty and exogenous uncertainty in the formulation of stochastic programmes. For generic problem formulations and solution strategies see the rather theoretical works of Dupačová (2006), Goel and Grossmann (2006), and Tarhan, Grossmann, and Goel (2013). Recent advances and summaries over existing computational strategies have been presented by Grossmann, Apap, Calfa, Garca-Herreros, and Zhang (2016) and Apap and Grossmann (2017). However, although almost all publications of this branch of literature refer to the classification and specification of Jonsbråten et al. (1998) and Goel and Grossmann (2006), respectively, Mercier and Van Hentenryck (2011) argued that problems in which merely the observation of an uncertainty depends on the decisions, but the actual underlying uncertainty is still exogenous (= second type of endogenous uncertainty) should be classified as "stochastic optimization problems with exogenous uncertainty and endogenous observations".

Unlike the operational research literature, the finance and management literature appears to be rather ambiguous, even somewhat inconsistent when it comes the classification of uncertainties. Indeed, although the importance of taking this distinction into account has been widely recognised in this branch of literature, especially in works related to the field of real options (Bowman & Hurry, 1993; Folta, 1998; Li, 2007; Oriani & Sobrero, 2008), there is no clear and widely accepted definition. For example, Pindyck (1993) distinguishes between technical and input cost uncertainty while noting their different effects on investment decisions as these incentivise investing and waiting, respectively. Building upon this distinction, McGrath (1997) called for a third form of uncertainty that lies in-between. Furthermore, McGrath, Ferrier, and Mendelow (2004) refers to the exogenous and endogenous resolution of uncertainty through the passing of time and learning, respectively. By contrast, Van der Hoek and Elliott (2006) took note of uncertainties that are state-dependent rather than dependent on the option holder's decisions.

Various researchers have applied real option approaches to valuation problems with both exogenous and endogenous uncertainty. Generalising the work of Roberts and Weitzman (1981), Pindyck (1993) evaluated a staged-investment with technical (endogenous) and input cost (exogenous) uncertainty using a finite difference method. Other relevant articles considered both types of uncertainty in the context of information technology investment projects (Schwartz & Zozaya-Gorostiza, 2003), patents and R&D projects (Schwartz, 2004; Schwartz & Moon, 2000), pharmaceutical R&D projects (Hsu & Schwartz, 2008; Pennings & Sereno, 2011), product platform flexibility planning (Jiao, 2012), and nuclear power plant investments (Zhu, 2012). However, according to Miltersen and Schwartz (2007), the algorithms of Miltersen and Schwartz (2004), Schwartz (2004), Hsu and Schwartz (2008), and Zhu (2012), which are plain extensions of the basic algorithm of Longstaff and Schwartz (2001) for single American-style options, "cannot easily handle temporary suspensions of the" investment project nor isolate the options' values. Also, these works considered only the abandonment option, rather than a real options portfolio. With regard to state-dependent uncertainty, Sbuelz and Caliari (2012) studied the influence of state-dependent cashflow volatility on the investment decisions related to corporate growth options, whereas Palczewski, Poulsen, Schenk-Hopp, and Wang (2015) examined optimal portfolio strategies under stock price dynamics with state-dependent drift.

Nevertheless, these real option approaches are rather inflexible and restricted in terms of the size of the real options portfolio, the number and types of uncertainties as well as the valuation method applied. This paper takes a fundamentally different approach by introducing a framework for valuing portfolios of real options under exogenous and endogenous uncertainties. In particular, we study an investment problem with several types of real options (deferring, staging, mothballing, and abandoning), two exogenous uncertainties (operating revenues and their growth rate), and two endogenous uncertainties (decision-dependent cost to completion and state-dependent salvage value). Using an illustrative example of a district heating network in

London, we provide portfolio insights and find that the portfolio value increases monotonically in both the exogenous (revenue) and the endogenous, state-dependent (salvage value) uncertainty, but that the endogenous, decision-dependent (cost to completion) uncertainty has a non-monotonic effect. This effect is largely due to the availability of abandonment options, whose values – enabled by partial reversibility – are directly and indirectly driven by state- and decision-dependent uncertainty, respectively. Most notably, we show that, in general, ignoring the former results in under-valuation, whereas ignoring the latter leads to over-valuation, thereby highlighting the importance of accounting correctly for uncertainty.

3. The investment problem

In this section, we present the investment problem studied here by specifying both the portfolio of interdependent real options and the set of underlying uncertainties.

3.1. Portfolio of interdependent real options

We study the problem of a decision maker wanting to determine the value of a sequential and partially reversible investment in a project whose stage-wise expansion (development) can be deferred, temporarily halted and/or abandoned altogether, and, once operating, whose cash flow generating asset can be used until the end of the asset's project life in T_3^{max} time periods, temporarily mothballed and/or abandoned early.

Representing the set of flexibilities as a portfolio of interdependent real options, the portfolio's single, well-defined options are:

(a) Option to defer investment: Instead of starting immediately at time 0, the decision maker may choose to defer the start of the expansion until the expiration of the right to undertake this investment in T_1^{max} time periods, without incurring any direct costs associated with deferring.

(b) Option to stage investment: As the development takes time to complete, the decision maker can invest at a rate of $0 < C_t \leq I^{max}$ in period t as long as the remaining investment cost K_t at the beginning of period t is greater than 0 – i.e. while the construction is not yet completed –, where I^{max} and K_0 are the maximum rate of investment and the initial (expected) cost of completion, respectively.
(i) Option to temporarily halt expansion: If conditions turn out to be unfavourable, the decision maker can halt the expansion (i.e. set $C_t = 0$) at a cost of $C^{d,h}$, maintain the halted expansion for a total of T_2^{max} time periods at a periodic cost of C^h, and, if desirable, resume development at a cost of $C^{h,d}$.
(ii) Option to abandon the project during construction (i.e. when $K_t > 0$): Whether developing or halted, the project can be permanently abandoned at any given point in time t for the salvage value X_t, which is assumed to contain any costs that abandonment during construction involves.

(c) Option to temporarily mothball the operation: If operation of the asset becomes uneconomic, the decision maker can mothball the operating asset at a cost of $C^{o,m}$, maintain the mothballed asset at a periodic cost of C^m, and, if conditions become favourable again, reactivate the asset at a cost of $C^{m,o}$.

(d) Option to abandon the project during operation (i.e. when $K_t = 0$): Whether operating or mothballed, the decision maker retains the right to permanently abandon the project at any time t for its salvage value X_t, which is assumed to contain all costs related to abandoning during operation.

The above described individual real options are well-known and have been widely examined in the real options literature – such as "time to build" effects in Majd and Pindyck (1987) –, for an overview see Trigeorgis (1996).

3.2. Characterisation of uncertainties

This study considers four sources of uncertainty – also referred to as stochastic factors or random variables – denoted by K_t, V_t, μ_t and X_t, representing the project's actual cost to completion at time t, the revenues (net cash flow) generated by operation in period t, the growth rate of revenues in t and the salvage value at time t, respectively. The first and the fourth uncertainty are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the dynamics of the second and third factor are exogenous. While the choice of stochastic factors obviously depends on the specific investment problem at hand, our choice, which is sufficient for the purpose of this work, covers several relevant and widely applicable stochastic factors, so is important for many practical applications where the sources of uncertainty are exogenous and endogenous. Unlike previous studies, which have considered these uncertainties mostly in isolation, here we consider the four uncertainties jointly since they are relevant to most projects' major phases including construction (cost to completion), operation (revenues), and decommissioning/disposal (salvage value). Note that the consideration of a stochastic growth rate allows us to model random variations in the general economic conditions and adds complexity to the problem, enabling us to both demonstrate the capability and test the robustness of our proposed valuation approach.

The four stochastic factors are described by discrete-time random walks with drift, in a general form by:

$$M_{t+\Delta} = \varphi_t M_t + f_t(M_t, \theta_1)\Delta + \sigma_t(M_t, \theta_2)\sqrt{\Delta}\varepsilon^m_{t+\Delta}, \tag{1}$$

where φ_t is a discounting multiplier, f_t is the drift function, Δ is the time step, σ_t is the diffusion function, and $\varepsilon^m_{t+\Delta}$ is the driving zero-mean process. Note that for endogenous stochastic factors, the parameters θ_1 or θ_2, or both depend on the decision or state, or both. The driving process $\varepsilon^m_{t+\Delta}$ is always Gaussian white noise (GWN), i.e. a standard normal random variable whose increments are iid, but drivers for different stochastic factors may be correlated.[1] Table 1 summarises the stochastic factors considered here.

The dynamic of the project's actual cost to completion, K_t, depends on the rate of investment, $0 \leq C_t \leq I^{max}$, chosen by the decision maker, and is given by:

$$K_{t+\Delta} = K_t - C_t\Delta + \sigma_k\sqrt{C_t K_t \Delta}\,\varepsilon^k_{t+\Delta}, \tag{2}$$

where σ_k is the degree of technical uncertainty. The above equation is a discrete approximation of the controlled diffusion process proposed by Pindyck (1993). As analytically shown by Pindyck (1993), Schwartz and Zozaya-Gorostiza (2003) and referred to as "bang-bang policy" by Schwartz (2004), the optimal rate of investment is either 0 or I^{max}, i.e. $C_t \in \{0, I^{max}\}$, because the processes (2) and (3)–(5) are uncorrelated.

The revenues received at time t for operation between t and $t+\Delta$, V_t, and their rate of growth, μ_t, evolve exogenously according to:

$$V_{t+\Delta} = e^{-\kappa_v\Delta}V_t + (1 - e^{-\kappa_v\Delta})V_0(1 + \mu_t t) + \sigma_v\sqrt{\frac{1 - e^{-2\kappa_v\Delta}}{2\kappa_v}}\,\varepsilon^v_{t+\Delta}, \tag{3}$$

[1] This, of course, does not change the exogenous and endogenous dynamics of uncertainties.

Table 1
Summary of stochastic factors considered in this study.

Description	Factor	Defining eq.	Dynamics	Driving process[a]
Cost to completion	K_t	(2)	Decision-dep.	GWN, independent of (3)–(5)
Operating revenues	V_t	(3)	Exogenous	GWN, correlated with (4) and (5)
Growth rate	μ_t	(4)	Exogenous	GWN, correlated with (3) and (5)
Salvage value	X_t	(5)	State-dep.	GWN, correlated with (3) and (4)

[a] Gaussian white noise.

$$\mu_{t+\Delta} = e^{-\kappa_\mu \Delta}\mu_t + (1 - e^{-\kappa_\mu \Delta})\bar{\mu} + \sigma_\mu \sqrt{\frac{1 - e^{-2\kappa_\mu \Delta}}{2\kappa_\mu}}\varepsilon^{\mu}_{t+\Delta}, \tag{4}$$

where σ_v and σ_μ are the standard deviations of changes in V_t and μ_t, respectively, as well as κ_v and κ_μ are positive mean reversion coefficients that describe the rate at which the corresponding factors converge to their linear trend, $V_0(1+\mu_t t)$, and long-term average, $\bar{\mu}$, respectively. The nested model (3)–(4) is similar to the discrete versions of Schwartz and Moon (2001), who also used an *Ornstein–Uhlenbeck process*[2] to describe the evolution of μ_t. For the evolution of V_t, however, we apply a trending Ornstein–Uhlenbeck model with stochastic linear trend adapted from Lo and Wang (1995), which is more realistic than both the revenue dynamics in Schwartz and Moon (2001) and the geometric mean reversion with deterministic exponential trend (i.e. $V_0 e^{\bar{\mu}t}$) considered by Metcalf and Hassett (1995).

The state-dependent salvage value obtained for abandoning the project at time t, X_t, is a function of the expected asset value at time t, Z_t, which is a deterministic function of the state S_t (see (6)), and of a *homoscedastic* noise term (i.e. error independent of the state), which is considered to be random. The salvage value process is described by:

$$X_{t+\Delta} = Z_{t+\Delta} + \sigma_x Z_{t+\Delta}\varepsilon^{x}_{t+\Delta}, \tag{5}$$

where σ_x is the standard deviation of X_t. Unlike the existing approaches that allow for stochastic salvage (or abandonment) values such as Myers and Majd (1990), Adkins and Paxson (2017), which assume these values evolve exogenously, we introduce a state-dependent salvage value as suggested in Van der Hoek and Elliott (2006), thereby represent one of the many practical situations in which the salvage value depends on endogenous factors (Trigeorgis, 1993). It is important to note that by "state" we actually mean its "resource" component (see Section 4.1), rather than its "information" component, because the latter's three stochastic factors given by (2)–(4) are, of course, state-dependent too because Markovian.

4. Methods

This section contains the modelling of the investment problem as a multi-stage stochastic decision problem, the formulation of the valuation problem as a discrete stochastic dynamic program, and the description of the valuation algorithm applied. A summary of the notation used is presented in Appendix A.

4.1. Modelling

The flexibilities available to the decision maker when having the portfolio of interdependent real options of Section 3.1 are shown by the influence diagram in Fig. 1. It contains nine nodes of which five are decision nodes and four are terminal nodes, as well as 18 transitions that link these nodes. The set of nodes, decision nodes and transitions is given by $\mathcal{N} = \{1, 2, \ldots, 9\}$, $\mathcal{N}^d = \{1, 3, 5, 6, 8\}$ and $\mathcal{H} = \{1, 2, \ldots, 18\}$, respectively, and the duration of transition $h \in \mathcal{H}$ is Δ_h time period(s). To help understand the intuition behind Fig. 1 see the influence diagram for a comparatively simple American-style option in Maier et al. (2018).

The state of the investment project at time t is written as:

$$S_t = (\underbrace{t, N_t, T_t, Q_t}_{R_t}, \underbrace{K_t, V_t, \mu_t, X_t}_{I_t}), \tag{6}$$

where $N_t \in \mathcal{N}$ is the node at time t; T_t is the time left at t to defer investment/halt expansion/use the developed asset; Q_t is the amount invested up to time t; and K_t, V_t, μ_t and X_t are as defined in Section 3.2. The first four variables of S_t are part of the resource state R_t, whereas the information state I_t is made up of the problem's four random variables, two of which are exogenous and two are endogenous.

To each decision node $n \in \mathcal{N}^d$ we associate binary (0–1) variables a_{th} in such a way that $a_{th} = 1$ indicates that transition h is made at time t and 0 otherwise. It is clear that the action space $b^D(N_t)$, which represents the set of outgoing transitions of node N_t, is given by

$$b^D(N_t) = \begin{cases} \{1,2,3\}, & \text{if } N_t = 1, \\ \{4,5,6,7\}, & \text{if } N_t = 3, \\ \{8,9,10\}, & \text{if } N_t = 5, \\ \{11,12,13,14\}, & \text{if } N_t = 6, \\ \{15,16,17,18\}, & \text{if } N_t = 8, \\ \{\}, & \text{otherwise.} \end{cases} \tag{7}$$

The decision variables $a_t = (a_{th})_{h \in b^D(N_t)}$ must satisfy the feasible region $\mathcal{A}_{S_t}$, which describes the set of feasible transitions given S_t and is defined by the following constraints:

$$\begin{cases} \sum_{h \in b^D(N_t)} a_{th} = 1, & \forall N_t \in \mathcal{N}^d, \quad (8) \\ a_{t1} T_1^{max} < T_t + T_1^{max}, & \quad (9) \\ a_{th} T_t = 0, & \forall h \in \{3, 12, 16\}, \quad (10) \\ a_{t5} K_t = 0, & \quad (11) \\ (1 - a_{t5} - a_{t7}) K_0 < K_t + K_0, & \quad (12) \\ a_{th} T_2^{max} < T_t + T_2^{max}, & \forall h \in \{6, 9\}, \quad (13) \\ (1 - a_{th}) T_3^{max} < T_t + T_3^{max}, & \forall h \in \{12, 16\}, \quad (14) \end{cases}$$

where $a_{th} \in \{0,1\}, \forall h \in \mathcal{H}$. The meaning of these constraints is as follows: (8) enforces that exactly one transition is made at a decision node; (9) and (13) ensure the investment can be deferred and the expansion halted, respectively, only if there is enough time left; (10) makes sure the development opportunity can only expire at $T_t = 0$ but not before, and, together with (14), these constraints make sure the developed project is completed at $T_t = 0$; and, finally, (11) ensures that the asset's operation can only begin if $K_t = 0$, at which point the developed asset has to be abandoned due to (12) if not operated.

The transition function, which is generically written as $S^M(S_t, a_t, W_{t+\Delta_h})$, describes the evolution of S_t from t to $t + \Delta_h$

[2] This simple mean-reverting process is more realistic than a geometric Brownian motion process in problems that involve natural gas and electricity price uncertainty (such as district heating networks) given that the underlying price processes in general exhibit mean reversion.

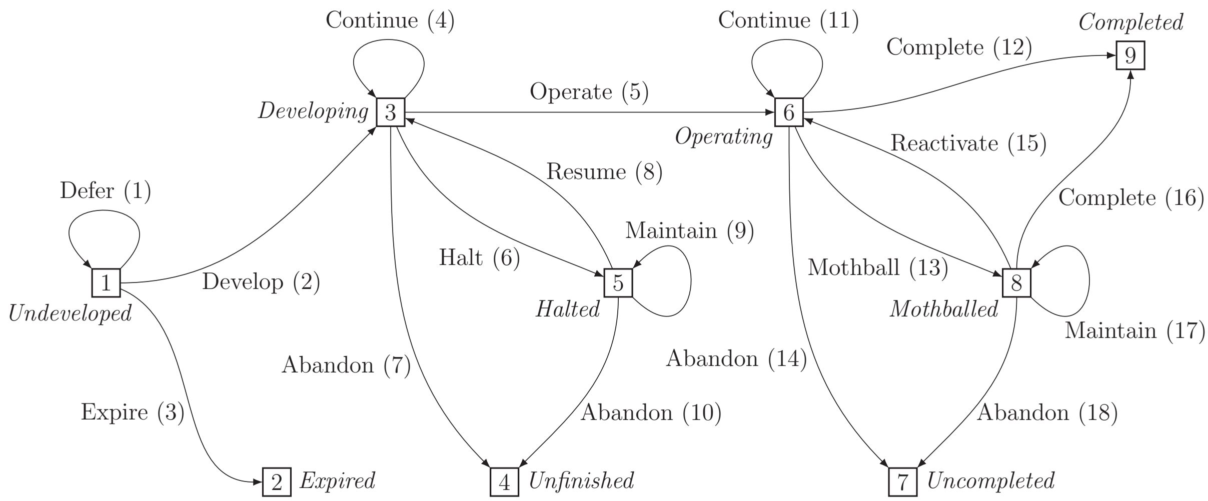


Fig. 1. Flexibilities provided by portfolio of interdependent real options.

after having made decision a_t with respect to $\mathcal{A}_{S_t}$ and learned new information $W_{t+\Delta_h}$. It is composed of the resource transition function $S^R(\cdot): R_t \to R_{t+\Delta_h}$ and the information transition function $S^I(\cdot): I_t \to I_{t+\Delta_h}$. With regard to the former, the transition of t is trivial as it simply evolves to $t+\Delta_h$; the transition of N_t is implicitly given by the adjacency matrix (not shown here) of the directed graph $(\mathcal{N}, \mathcal{H})$ underlying the influence diagram; the transition of T_t is given by:

$$T_{t+\Delta_h} = \begin{cases} \max\{T_t - \Delta_h, 0\}, & \text{if } a_{th} = 1, h \in \mathcal{H}_1, \\ T_2^{max}, & \text{if } a_{t2} = 1, \\ T_3^{max} - \Delta_5, & \text{if } a_{t5} = 1, \\ T_t, & \text{otherwise,} \end{cases} \tag{15}$$

where $T_0 = T_1^{max}$ and $\mathcal{H}_1 = \{1, 6, 9, 11, 13, 15, 17\}$; and the transition of Q_t is given by:

$$Q_{t+\Delta_h} = \begin{cases} Q_t + I^{max}\Delta_h, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ Q_t, & \text{otherwise,} \end{cases} \tag{16}$$

where $Q_0 = 0$. In contrast to the deterministic transitions of the variables of R_t, the information state variables evolve generally stochastically according to:

$$K_{t+\Delta_h} = \begin{cases} \max\Big\{K_t - I^{max}\Delta_h \\ \qquad + \sigma_k\sqrt{I^{max}K_t\Delta_h}\varepsilon^k_{t+\Delta_h}, 0\Big\}, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ K_t, & \text{otherwise,} \end{cases} \tag{17}$$

$$V_{t+\Delta_h} = e^{-\kappa_v\Delta_h}V_t + (1 - e^{-\kappa_v\Delta_h})V_0(1 + \mu_t t) + \sigma_v\sqrt{\frac{1 - e^{-2\kappa_v\Delta_h}}{2\kappa_v}}\varepsilon^v_{t+\Delta_h}, \tag{18}$$

$$\mu_{t+\Delta_h} = e^{-\kappa_\mu\Delta_h}\mu_t + (1 - e^{-\kappa_\mu\Delta_h})\bar{\mu} + \sigma_\mu\sqrt{\frac{1 - e^{-2\kappa_\mu\Delta_h}}{2\kappa_\mu}}\varepsilon^\mu_{t+\Delta_h}, \tag{19}$$

$$X_{t+\Delta_h} = Z_{t+\Delta_h}(S_{t+\Delta_h}) + \sigma_x Z_{t+\Delta_h}(S_{t+\Delta_h})\varepsilon^x_{t+\Delta_h}, \tag{20}$$

where $Z_t(S_t)$, the expected asset value at time t, is given by:

$$Z_t(S_t) = \begin{cases} -\alpha I^{max}, & \text{if } N_t = 3, K_t > 0, \\ \gamma Q_t, & \text{if } N_t = 3, K_t = 0, \\ -\beta I^{max}, & \text{if } N_t = 5, \\ \gamma Q_t e^{-\zeta(T_3^{max} - T_t)}, & \text{if } N_t = 6, \\ \delta Q_t e^{-\zeta(T_3^{max} - T_t)}, & \text{if } N_t = 8, \\ 0, & \text{otherwise,} \end{cases} \tag{21}$$

where $\alpha \geq 0$ and $\beta \geq 0$ define the expected abandonment cost when *Developing* or *Halted*, respectively; $\gamma \geq 0$ and $\delta \geq 0$ are pay-out ratios determining the expected asset value when *Operating* or *Mothballed*, respectively; and ζ is the periodic depreciation rate.

Lastly, the pay-off function is represented by:

$$\begin{aligned} \Pi_t(S_t, a_t) = & -I^{max}(\Delta_2 a_{t2} + \Delta_4 a_{t4}) + V_t(a_{t5} + a_{t11}) \\ & + X_t(a_{t7} + a_{t10} + a_{t14} + a_{t18}) + X_t(a_{t12} + a_{t16}) \\ & - C^{d,h}a_{t6} - (C^{h,d} + I^{max}\Delta_8)a_{t8} - C^h\Delta_9 a_{t9} \\ & - C^{o,m}a_{t13} + (V_t - C^{m,o})a_{t15} - C^m\Delta_{17}a_{t17}, \end{aligned} \tag{22}$$

where the first two terms on the right-hand side represent the cost for developing and the income from operations, respectively; the second line's terms represent the net income from abandoning and completing, respectively; the third line contains costs related to halting, maintaining and resuming (during development), respectively; and the last line's terms represent the cost of mothballing, the net income from reactivating and the maintenance cost when mothballed, respectively. Note that, for simplicity, it is assumed that completing the project – by making either transition 12 (when *Operating*) or transition 16 (when *Mothballed*) – results in a pay-off of the salvage value X_t, which thus represents the project's residual value.

4.2. Valuation problem

The value of the portfolio of interdependent real options at time 0 given state S_0, $G_0(S_0)$, is obtained by solving the following multi-stage stochastic optimisation problem:

$$G_0(S_0) = \max_{(a_t)_{t\in\mathcal{T}}} \mathbb{E}\Big[\textstyle\sum_{t\in\mathcal{T}} e^{-rt}\Pi_t(S_t, a_t)\Big|S_0\Big], \tag{23}$$

where $S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$, $a_t = (a_{th})_{h\in b^D(N_t)}$, $a_{th} \in \{0,1\}$, $a_t \in \mathcal{A}_{S_t}$, $\mathcal{T}$ is the set of decision times, $S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h})$, and r is the discount rate.

Applying Bellman's well-known "principle of optimality", the stochastic optimisation problem in (23) can be solved recursively, with the stochastic dynamic programming (SDP) recursion for calculating the optimal value of being in state S_t given by:

$$G_t(S_t) = \max_{a_t}\Big(\Pi_t(S_t, a_t) + \mathbb{E}\big[e^{-r\Delta_h}G_{t+\Delta_h}(S_{t+\Delta_h})\big|S_t, a_t\big]\Big) \tag{24}$$

$$\text{s.t.} \quad a_{th} \in \{0, 1\}, \qquad \forall h \in b^D(N_t), \tag{25}$$

$$a_t \in \mathcal{A}_{S_t}, \tag{26}$$

$$S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h}), \quad \forall h \in b^D(N_t), \tag{27}$$

where $W_{t+\Delta_h} = \left(\varepsilon^k_{t+\Delta_h}, \varepsilon^\nu_{t+\Delta_h}, \varepsilon^\mu_{t+\Delta_h}, \varepsilon^x_{t+\Delta_h}\right)$ describes the information that arrives between time t and $t+\Delta_h$. The aim is then to determine $G_0(S_0)$, given the boundary (or terminal) condition $G_t(S_t) = 0, \forall t \in \mathcal{T}, N_t \in \mathcal{N} \setminus \mathcal{N}^d$.

4.3. The simulation-and-regression-based valuation algorithm

In order to approximate the value of the portfolio of interdependent real options characterised by the SDP recursion (24)–(27), we implement an extended simulation-and-regression-based algorithm, which differs from previous ones by the following details:

- The decisions $a_t \in \mathcal{A}_{S_t}$ imply the time delay Δ_h, that is the moment in time for the next decision is $t+\Delta_h$;
- The random variables ξ_t appearing in the transition function depend on the state of the system. However, it is assumed that they can be written in the form $\xi_t(\omega) = f(S_t(\omega), \varepsilon_t(\omega))$, where $\varepsilon_t(\omega)$ is independent of $S_t(\omega)$. Thus the random factors ε_t can be considered as exogenous (see Supplementary Material C);
- Some basis functions are impossible for some combinations of states and actions.

Furthermore, our proposed algorithm is both a generalisation and formalisation of the solution procedures offered by Miltersen and Schwartz (2004), Schwartz (2004), Hsu and Schwartz (2008), Zhu (2012), which are plain extensions of the algorithm for single American-style options proposed by Longstaff and Schwartz (2001). A somewhat similar, but more special algorithm was introduced by Maier et al. (2018). While our algorithm also consists of an induction procedure with a forward and a backward pass as in standard simulation-and-regression methods (e.g. see Cortazar et al., 2008; Glasserman & Yu, 2004; Longstaff & Schwartz, 2001; Nadarajah et al., 2017; Tsitsiklis & Van Roy, 2001), the procedure's individual steps need to be extended in a number of ways to account for the complexity of the extended model. See Appendix B for a description of the solution procedure's steps in which we assumed, for the sake of simplicity, that $\Delta_1 = \Delta_2 = \Delta_4 = \Delta_6 = \Delta_8 = \Delta_9$ and $\Delta_5 = \Delta_{11} = \Delta_{13} = \Delta_{15} = \Delta_{17}$.

The forward induction procedure generates the discrete state space $\mathcal{S}_t$ through "exploration" of the resource state space $\mathcal{R}_t$ and simulation (Monte Carlo sampling) of the information state space $\mathcal{I}_t$ for all $t \in \mathcal{T}$. However, in contrast to standard methods, where the resource state space can be generated independently of the information state space, in our forward pass these have to be interleaved. This is because, in addition to the path dependency of R_t due to the sequential decision process underlying the portfolio of real options, now both R_t and I_t are path-dependent due to the decision-dependent cost to completion, K_t. In fact, whether a resource state and its corresponding information state can be reached at time t (and are therefore part of $\mathcal{R}_t$ and $\mathcal{I}_t$, respectively) does not solely depend on the sequence of decisions made up to this point, but also on how K_t evolves stochastically; for instance, it might be that a particular R_t can be reached in only a subset of paths denoted by Ω_{R_t}, where $\Omega_{R_t} \subseteq \Omega$ and Ω is the set of all sample paths. Moreover, since the stochastic cost to completion can be directly translated into a stochastic time to completion, the decision times in $\mathcal{T}$ are also path-dependent.

As a strategy in our procedure to overcome the curse of dimensionality related to both $\mathcal{I}_t$ and the outcome space (for a discussion see Maier et al., 2018; Nadarajah et al., 2017), whenever needed we approximate the conditional expectation in (24), which represents the continuation function

$$\Phi_t(S_t, a_t) = \mathbb{E}\left[e^{-r\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) \middle| S_t, a_t\right], \tag{28}$$

by the following continuous, finite-dimensional function ("the parametric model"):

$$\hat{\Phi}_t^{L_{S_t}}(S_t, a_t) = \sum_{l=0}^{L_{S_t}} \hat{\alpha}_l(S^R(R_t, a_t))\phi_{S_t l}(I_t), \tag{29}$$

where L_{S_t} is the model's dimension; $\{\phi_{S_t l}(\cdot)\}_{l=0}^{L_{S_t}}$ are called basis functions (or features), which depend only on I_t and not the full S_t; and the coefficients $\left(\hat{\alpha}_l(S^R(R_t, a_t))\right)_{l=0}^{L_{S_t}}$ are obtained by the least-squares regression in (B.4). Unlike the parametric models of standard simulation-and-regression methods, here L_{S_t} and $\phi_{S_t l}$ depend on S_t. This dependency enables us to reduce the model's dimension if $N_t = 1$ ($N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\}$) by omitting functions of K_t and X_t (K_t) in the regression since these stochastic factors are constant or non-existent in these situations, thereby avoiding numerical and implementation issues. Importantly, the parametric model (29) is determined separately for each relevant and feasible decision a_t, given state $S_t = (R_t, I_t)$, whilst taking into account the set of paths Ω_{R_t} in which R_t can actually be reached. By contrast, in standard algorithms for problem with only exogenous uncertainty every R_t can be reached along each path $\omega \in \Omega$.

The valuation procedure shown in Algorithm 1 applies a backward induction to approximate the value of the stochastic dynamic program (24)–(27). Starting at $t = \max \mathcal{T}$ and moving backwards to $t = \min \mathcal{T} \setminus 0$, for each state $S_t \in \mathcal{S}_t$ perform the following three steps: (i) approximate (28) by both (29) and (B.4) separately for all feasible a_t that do not lead to a terminal node, otherwise set them to 0 (*lines 3–9*)[3]; (ii) compute the pathwise optimisers $\hat{a}_t(\omega)$ for all $\omega \in \Omega_{R_t}$ in which R_t can be reached (*line 11*); (iii) using these pathwise optimisers, determine the approximation $\bar{G}_t(S_t(\omega))$ for each path $\omega \in \Omega_{R_t}$ (*line 12*). At $t = 0$, we have $(K_0, V_0, \mu_0, X_0) = (K_0(\omega), V_0(\omega), \mu_0(\omega), X_0(\omega))$, so we can simply calculate the value of (28) by taking averages of the pathwise continuation values over all $|\Omega|$ paths, and make optimal decisions based on these average values, giving $\bar{G}_0(S_0)$ (*line 17*). Importantly, using the reachability analysis from the forward pass, in the above three steps only paths $\omega \in \Omega_{R_t}$ in which resource states R_t can be reached are used, rather than the full set of paths Ω. An illustration of the main steps of this valuation approach when in state S_t at t is given by Fig. 6.2 of Maier (2017).

4.4. Numerical accuracy and simulation efficiency

While standard simulation-and-regression approaches in general give a lower bound on the optimal solution since the continuation function is approximated by a finite-, and usually low-dimensional function, the quality of this approximate solution depends on a range of factors (see, e.g., Fabozzi, Paletta, & Tunaru, 2017; Nadarajah et al., 2017 and the literature therein). For example, considering a one-dimensional setting and polynomials as basis functions in the parametric model, Glasserman and Yu (2004) examined the relationship between the number of simulated paths ($|\Omega|$) and the number of basis functions (L), and showed that the required $|\Omega|$ for ensuring worst case convergence increases exponentially in L. Under general assumptions and considering shifted Legendre polynomials, Stentoft (2004) proved convergence in a multi-dimensional setting if both $L \to \infty$ and $|\Omega| \to \infty$ provided that $L^3/|\Omega| \to 0$. Cortazar et al. (2008) have shown that taking advantage of the problem structure and carefully choosing an appropriate set of basis functions (e.g. call and put

[3] In many practical applications we can correct for obviously incorrect approximations of the continuation function, e.g., by simply bounding the approximation using an appropriate deterministic bound to ensure non-negativity.

Algorithm 1: Approximation of optimal value of problem (24)–(27).

Data: All the above
Result: $\bar{G}_0(S_0)$

for $t = \max\{\mathcal{T} \setminus 0\}$ **do**
  forall $S_t \in \mathcal{S}_t$ **do**
    forall $a_t \in \mathcal{A}_{S_t}$ **do**
      if $a_{th} = 1, h \in \{3, 7, 10, 12, 14, 16, 18\}$ **then**
        $F_t(S_t(\omega), a_t) \leftarrow 0, \forall \omega \in \Omega_{R_t}$
      else
        Use both (29) and (B.4) to determine: $F_t(S_t(\omega), a_t) \leftarrow \hat{\Phi}_t^{L_{S_t}}(S_t(\omega), a_t), \forall \omega \in \Omega_{R_t}$
      end
    end
    forall $\omega \in \Omega_{R_t}$ **do**
      Compute pathwise optimisers: $\hat{a}_t(\omega) \leftarrow \underset{a_t(\omega) \in \mathcal{A}_{S_t(\omega)}}{\arg\max} \left\{ \Pi_t\left(S_t(\omega), a_t(\omega)\right) + F_t\left(S_t(\omega), a_t(\omega)\right) \right\}$
      Approximate optimal portfolio value along each path ω:
      $\bar{G}_t\left(S_t(\omega)\right) \leftarrow \Pi_t\left(S_t(\omega), \hat{a}_t(\omega)\right) + e^{-r\Delta_h} \bar{G}_{t+\Delta_h}\left(S^M\left(S_t(\omega), \hat{a}_t(\omega), W_{t+\Delta_h}(\omega)\right)\right)$
    end
  end
  $\mathcal{T} \leftarrow \mathcal{T} \setminus t$
end
At $t = 0$, $S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$, determine: $\bar{G}_0(S_0) \leftarrow \max_{a_0 \in \mathcal{A}_{S_0}} \left\{ \Pi_0(S_0, a_0) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{-r\Delta_h} \bar{G}_{\Delta_h}\left(S^M\left(S_0, a_0, W_{\Delta_h}(\omega)\right)\right) \right\}$

options on the expected spot price (Andersen & Broadie, 2004; Nadarajah et al., 2017), rather than simply using high-order polynomials of information state variables as in Glasserman and Yu (2004) and Stentoft (2004), allows one to substantially reduce the required L for a given level of accuracy, and is computationally more efficient.

However, while the accuracy of the approximation and hence the quality of the lower bound can be improved by choosing the set of basis functions appropriately, here the algorithm's simulation efficiency – in terms of actually utilisable sample paths – depends on the Monte Carlo sampling steps. Unlike standard approaches, where the number of sample paths available at each step of the valuation procedure is chosen in advance ($=|\Omega|$) and remains constant within the backward pass, in our extended algorithm the number of paths in which resource states can actually be reached ($|\Omega_{R_t}|$) is generally not known in advance and varies across resource states.[4] Moreover, although disregarded by Miltersen and Schwartz (2004), Schwartz (2004), Hsu and Schwartz (2008), and Zhu (2012), the additional path-dependency caused by the decision-dependent uncertainty K_t may result in $|\Omega_{R_t}| \ll |\Omega|$, which means that the simulation efficiency is reduced as $|\Omega_{R_t}|/|\Omega| \ll 1$. To avoid potential effects on the accuracy of the approximation, it may be necessary to increase $|\Omega|$ in order to ensure sufficiently large values $|\Omega_{R_t}|$ (in addition to adapting L_{S_t} appropriately). Future work might therefore investigate the computational complexity of such an extended approach in terms of convergence and efficiency, and explore the development of a duality-based, upper bound algorithm to provide performance bounds.

5. An illustrative example

This section provides details about the numerical example, describes the computational implementation of our valuation algorithm, and presents and discusses the results.

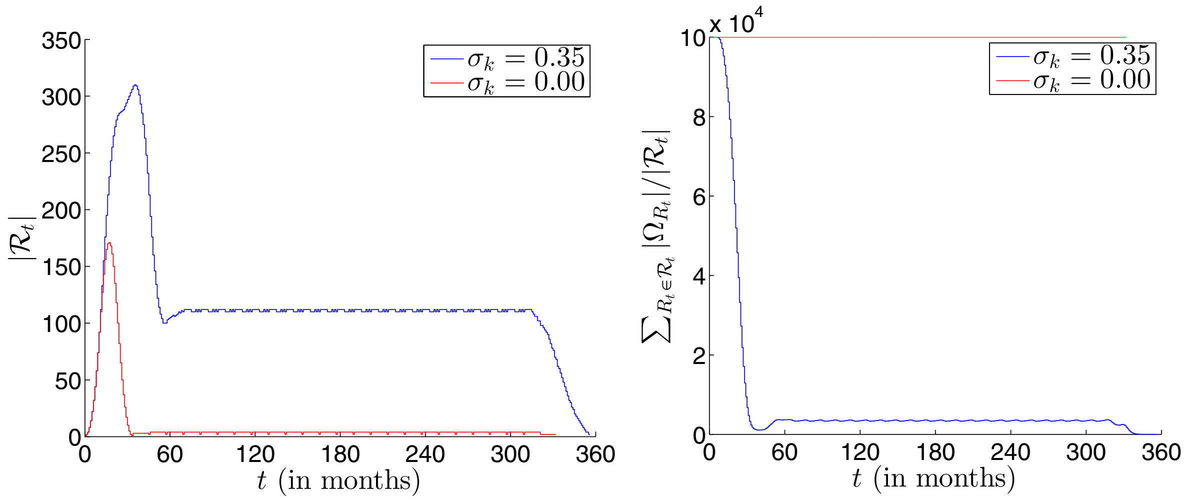
[4] A different approach is to use the simulated evolution of K_t to determine the probability distribution describing the probability that construction will be completed given Q_t (see Pennings & Sereno, 2011).

5.1. Expansion of district heating network

We consider the real case of an investment into the expansion of the district heating network in the London borough of Islington. We focus here on the development of the network's "north extension", as identified in a recent report (Grainger & Etherington, 2014) which investigated the development of a borough-wide network on behalf of the local council. It should be noted, however, that their economic assessment is based on simple temporal discounting in a deterministic setting and does not account for time to build nor the project's residual value. According to this report, the capital expenditure of this expansion and the initial, annual operating revenues are estimated at £9.94 millions (K_0) and £564,600 (V_0), respectively. The report also noted that the asset can be used for up to 25 years (i.e. T_3^{max}=300). The interest rate, used to discount monetary values, is 3.5% per year (i.e. $r = 3.5\%/12$), as recommended by HM Treasury (2011). In addition, we assume the following: a maximum rate of investment of £1.0 million per month (I^{max}); the possibility of deferring development/halting expansion for up to one year (i.e. $T_1^{max} = T_2^{max}$=11); and the following durations of transitions (in months): $\Delta_h = 1, \forall h \in \{1, 2, 4, 6, 8, 9\}$; $\Delta_h = 12, \forall h \in \{5, 11, 13, 15, 17\}$; and 0 for the remainder of the transitions. A summary of the input data is given in Supplementary Material E.

5.2. Generated state space and utilised basis functions

The discrete state space was generated by applying the forward induction procedure described in Section 4.3 (and Appendix B) and using the data of Section 5.1. More specifically, 100,000 paths ($|\Omega|$) were generated to describe the stochastic evolution of the four factors K_t, V_t, μ_t and X_t for all $t \in \mathcal{T}$ (see Supplementary Material F for an illustration of five sample paths). With regard to the parametric model in (29), we apply as basis functions $\phi_l(\cdot)$ polynomials of the information state variables as well as both call and put options on the expected value of these variables partially based on



(a) Number of reachable resource states at every time $t \in \mathcal{T}$.

(b) Average number of paths in which resource states are reachable.

Fig. 2. Impact of introduction of decision-dependent uncertainty on the evolution of both the number of reachable resource states (left) and the average number of paths in which these are reachable (right) over t.

(Andersen & Broadie, 2004; Cortazar et al., 2008; Nadarajah et al., 2017). In case $(N_t = 3 \wedge K_t > 0) \vee N_t = 5$, we use a constant term, the four information state variables, polynomials of degree two (i.e. the squares of each variable and their cross products), polynomials of degree three, as well as the value of call and put options on the expected value of each variable and the square of this value. Otherwise, if $N_t = 1$ ($N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\}$), as mentioned in Section 4.3, we can reduce L_{S_t} by eliminating all the functions of K_t and X_t (K_t) because $K_t = K_0$ and X_t is non-existent ($K_t = 0$), so these variables do not add any information value to the least-squares regression. This selection of the set of basis functions $\phi_l(\cdot)$ means that:

$$\phi_{S_t l}(I_t) = \begin{cases} \phi_l(V_t, \mu_t), & \text{if } N_t = 1 \quad (\Rightarrow L_{S_t} = 18) \\ \phi_l(K_t, V_t, \mu_t, X_t), & \text{if } N_t = 3, K_t > 0, \quad (\Rightarrow L_{S_t} = 51) \\ \phi_l(V_t, \mu_t, X_t), & \text{if } N_t = 3, K_t = 0, \quad (\Rightarrow L_{S_t} = 32) \\ \phi_l(K_t, V_t, \mu_t, X_t), & \text{if } N_t = 5, \quad (\Rightarrow L_{S_t} = 51) \\ \phi_l(V_t, \mu_t, X_t), & \text{if } N_t \in \{6, 8\}. \quad (\Rightarrow L_{S_t} = 32) \end{cases} \tag{30}$$

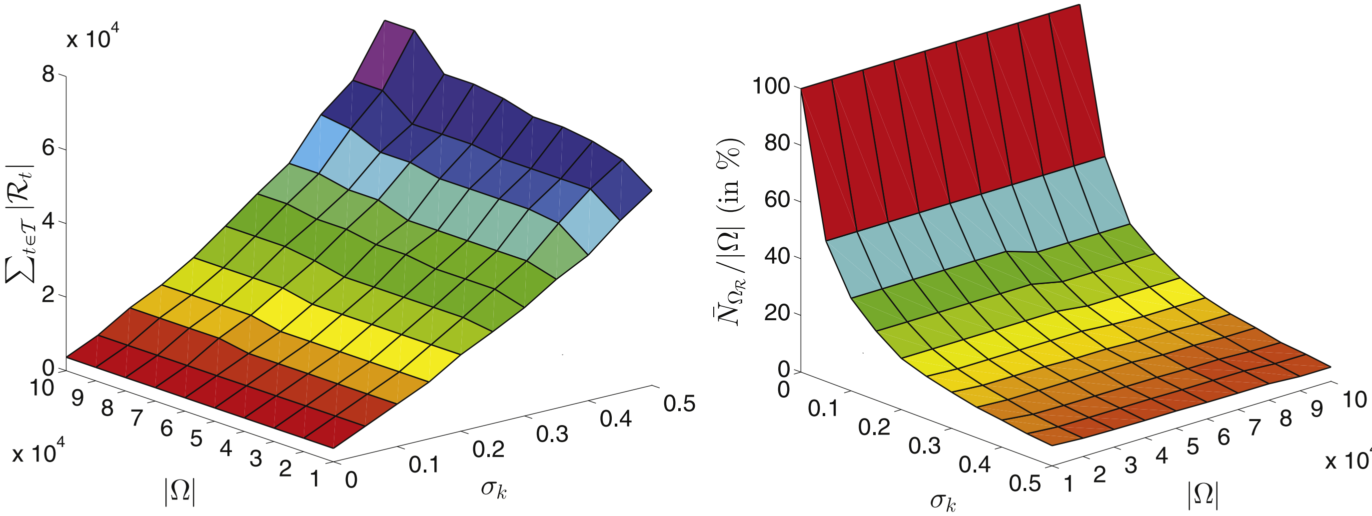
To avoid numerical problems the basis functions were properly scaled before performing the least-squares regression based on a singular value decomposition (SVD) algorithm. The solution procedure described in Appendix B was implemented in MATLAB.

5.3. Reachability analysis

To support the claim made in Section 4.3 that resource states may not be reachable in every simulation path, this subsection numerically analyses the impact of the decision-dependent uncertainty in the context of reachability. Figs. 2(a) and (b) show the number of resource states reachable at each time t, $|\mathcal{R}_t|$, and the average number of paths in which resource states can be reached at $t \in \mathcal{T}$, $\sum_{R_t \in \mathcal{R}_t} |\Omega_{R_t}|/|\mathcal{R}_t|$, respectively. As can be seen, the number of reachable resource states is substantially higher in almost every point in time as σ_k increased from 0.00 to 0.35. In fact, the total number of reachable resource states, $\sum_{t \in \mathcal{T}} |\mathcal{R}_t|$, increased more than tenfold[5] (from 3635 to 41,815), highlighting the algorithmic complexity introduced by the decision-dependent uncertainty K_t. At the same time, the introduction of the decision-dependent uncertainty resulted in a sharp decline in the average number of paths in which resource states are reachable, with an almost elevenfold decrease – from 100,000 to 9002 paths – in the average number of paths available for each reachable resource state, given by $\bar{N}_{\Omega_{\mathcal{R}}} = (\sum_{t \in \mathcal{T}} \sum_{R_t \in \mathcal{R}_t} |\Omega_{R_t}|)/(\sum_{t \in \mathcal{T}} |\mathcal{R}_t|)$. While such a decrease in the number of paths generally reduces the complexity associated with solving both the least-squares regression (B.4) and the integer programs (*line 11* of Algorithm 1), and thereby might counteract the overall increase in computational efforts, it has a potentially adverse impact on the accuracy of the parametric model fit, as discussed in Section 4.4.

In addition to this twofold effect, the introduction of the decision-dependent uncertainty has important implications for the nature of our simulation-based approximation procedure. Figs. 3(a) and (b) report the impact of both the degree of the decision-dependent uncertainty (σ_k) and the number of sample paths generated ($|\Omega|$) on the total number of reachable resource states, i.e. $\sum_{t \in \mathcal{T}} |\mathcal{R}_t|$, and on the ratio of the average number of paths in which resource states are reachable to the number of paths generated, i.e. $\bar{N}_{\Omega_{\mathcal{R}}}/|\Omega|$, respectively. It can be seen that if there is no decision-dependent uncertainty ($\sigma_k = 0$) then the total number of reachable resources states is independent of the number of generated paths, and the average number of paths available for each reachable resource state, $\bar{N}_{\Omega_{\mathcal{R}}}$, is equal to $|\Omega|$, so each resource state R_t can be reached in $|\Omega|$ paths, i.e. $|\Omega_{R_t}| = |\Omega|$ for all $R_t \in \mathcal{R}_t$. However, while the total number of reachable resource states, as expected, increased in both $|\Omega|$ and σ_k, the ratio $\bar{N}_{\Omega_{\mathcal{R}}}/|\Omega|$ decreased not only in the latter, but, somewhat paradoxically, also in the former. In fact, in our analysis we found that in more than 65% of the 90 relevant cases the ratio $\bar{N}_{\Omega_{\mathcal{R}}}/|\Omega|$ actually decreased in $|\Omega|$ and remained virtually constant in the remainder of the cases.

[5] At the same time, the computational effort – in terms of elapsed (wall-clock) time – required to solve the valuation problem increased by approximately 165%. All computations were performed on a desktop computer with Intel Core i7-3770 CPU (3.40 GHz), 24 GB RAM, and Windows 7 Enterprise (64-bit OS).



(a) Total number of reachable resource states.

(b) Efficiency of Monte Carlo sampling.

Fig. 3. Impact of both the degree of the decision-dependent uncertainty (σ_k) and the number of sample paths ($|\Omega|$) on algorithmic complexity (left) and simulation efficiency (right).

Table 2
Value (in £millions) of investment project with and without real options portfolio as well as value of individual real options for different levels of initial annual revenues.

Annual revenue (£m) V_0	Value without options (–)	Value of option to Defer (a)	During expansion: Value of option to Halt (b-i)	Abandon (b-ii)	Stage (b)	During operation: Value of option to Mothball (c)	Abandon (d)	Switch (c,d)	Value with portfolio of options (a,b,c,d)
0.40	0*	0	0	0	0	0	0	0	0*
		(-)†	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.45	0*	0	0	0	0	0	0	0	0.000042
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.50	0*	0.0006	0	0	0	0	0	0	0.1448
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.55	0.5868	0.0760	0.0045	0.1321	0.1618	0	0.1643	0.1643	0.9110
		(12.95)	(0.77)	(22.51)	(27.57)	(0)	(28.00)	(28.00)	(55.24)
0.5646	0.8586	0.0702	0.0043	0.1112	0.1419	0	0.1412	0.1412	1.1438
		(8.18)	(0.50)	(12.95)	(16.53)	(0)	(16.44)	(16.44)	(33.22)
0.60	1.5178	0.0556	0.0040	0.0754	0.1029	0	0.0978	0.0978	1.7296
		(3.66)	(0.26)	(4.97)	(6.78)	(0)	(6.45)	(6.45)	(13.96)
0.65	2.4487	0.0353	0.0035	0.0442	0.0658	0	0.0590	0.0590	2.5838
		(1.44)	(0.14)	(1.80)	(2.69)	(0)	(2.41)	(2.41)	(5.52)
0.70	3.3797	0.0161	0.0029	0.0260	0.0433	0	0.0361	0.0361	3.4621
		(0.48)	(0.08)	(0.77)	(1.28)	(0)	(1.07)	(1.07)	(2.44)
0.75	4.3106	0.0008	0.0027	0.0153	0.0306	0	0.0222	0.0222	4.3588
		(0.02)	(0.06)	(0.36)	(0.71)	(0)	(0.51)	(0.51)	(1.12)

Note: the sets of transitions available in the different settings are as follows: $\mathcal{H}^- = \{2, 3, 4, 5, 11, 12\}$ in (–); $\mathcal{H}^- \cup \{1\}$ in (a); $\mathcal{H}^- \cup \{6, 8, 9\}$ in (b-i); $\mathcal{H}^- \cup \{7, 10\}$ in (b-ii); $\mathcal{H}^- \cup \{6, \ldots, 10\}$ in (b); $\mathcal{H}^- \cup \{13, 15, 16, 17\}$ in (c); $\mathcal{H}^- \cup \{14, 18\}$ in (d); $\mathcal{H}^- \cup \{13, \ldots, 18\}$ in (c,d); and $\mathcal{H}$ in (a,b,c,d).
* No investment.
† Numbers in parentheses represent the value of the option(s) as a percentage of the value without options.

This means that, in contrast to simulation-based approaches for standard problems, generating more sample paths will in general not equally increase the number of actually utilisable paths when addressing problems with such decision-dependent uncertainty.

5.4. Results and discussion

In order to illustrate the extent to which the profitability of the district heating investment project depends on the initial value of the annual revenues, V_0, Table 2 shows the sensitivity of the value of different portfolio configurations to varying levels of V_0. As can be seen, for values of V_0 of £0.50 millions and below, the value of the investment project without options – configuration (–) – is 0. This is because the expected NPV of the project is −£2.2060 millions, −£1.2751 millions, and −£0.3441 millions for values of V_0 of £0.40 millions, £0.45 millions, and £0.50 millions, respectively, so the optimal "now-or-never strategy", which does not take any flexibility into account, is to leave the project undeveloped. The same strategy is optimal for the project with the portfolio of options (a,b,c,d) for the lowest value of V_0 under consideration. However, for levels of V_0 of £0.45 millions and £0.50 millions, the value of the project with the options portfolio (a,b,c,d) is positive, reflecting the substantial value of having the flexibility provided by the portfolio of interdependent real options. Interestingly, in the first case, although the portfolio with all options achieves a positive value there is no individual option that provides sufficient added value on its own (i.e. in isolation), whereas in the case $V_0 = £0.50$ millions, having the option to defer alone – configuration (a) – also results in an economically viable project.

As can be seen from Table 2, beginning at a V_0 of £0.55 millions, the values of both the project without any flexibilities and almost all portfolio configurations are positive.[6] In most cases the value of the project with portfolio (a,b,c,d) is considerable larger than without options (–), revealing the significant added value that is obtained by considering such a complex portfolio. While the values of the project without any options and the portfolio with all options both increase in V_0, the values of almost all of the individual options in isolation show a different trend. Indeed, the values of the options to defer (a), to halt (b-i), and to abandon the project during construction (b-ii) and operation (d) are decreasing in V_0, meaning there is less value in deferring, halting, and abandoning as the value of initial annual revenues increases. This is because the annual revenues, although still uncertain (i.e. stochastic), revert now to a linear trend that is shifted upwards, so their level is generally higher, which makes deviating from the static now-or-never strategy, and consequently the flexibility provided by individual real options less valuable. For all values of V_0 under consideration, the option to temporarily mothball the operation – configuration (c) – is of no value because the simulated values of V_t are always positive, making mothballing an economically unattractive option. Also, as is apparent from this table, and in line with the real options literature, the values of the portfolio's individual options are generally non-additive since the value of the portfolio (a,b,c,d) does not equal the sum of the values of its individual options (a), (b) and (c,d).[7]

The effects of the degrees of exogenous and endogenous uncertainty on both the value of the portfolio of options and the comparative performance of the portfolio's individual options are particularly important for understanding the influence of different underlying uncertainties. In order to illustrate these effects for the exogenous annual revenues, V_t, and the endogenous, decision-dependent cost to completion, K_t, Fig. 4 shows for $C^{o,m} = C^{m,o} = C^m = 0$ the way in which the standard deviation of changes in revenues, σ_v, and the degree of technical uncertainty, σ_k, affect the value of the investment project. While the effects of changes of σ_v on the value of the project without options is negligible, the value of the portfolio is generally increasing in σ_v, particularly steep for higher levels of σ_v and it seems the increase is more pronounced for lower values of σ_k. This monotonic increase in project values results from the flexibilities provided by the portfolio of real options, which allow a decision maker to limit downside risk and exploit the upside potential of increased annual revenues, as compared to the negligibly affected value of the investment project without options, which applies a static now-or-never strategy.

On the other hand, increasing σ_k from 0 to 0.05 (i.e. introducing some construction cost uncertainty) results in a sharp decline in values of the investment project, but the decline is smaller for the project with the portfolio of real options. The reason for this sharp decline is mainly due to the increase in actual cost of completion caused by the introduction of technical uncertainty, but also because of the discretised investment expenditures.

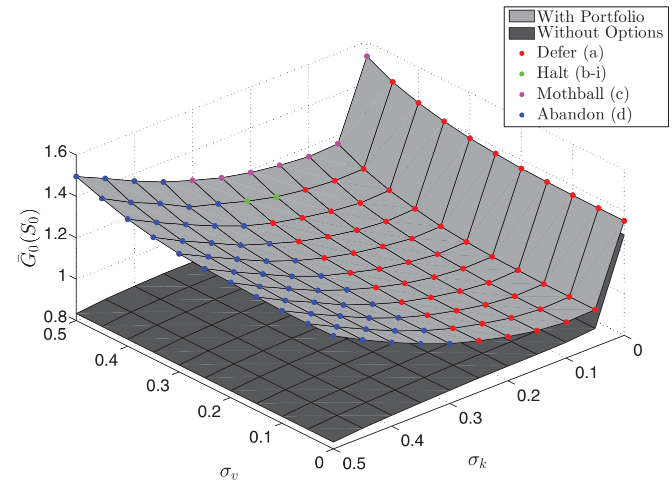


Fig. 4. Value of investment project with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_k) uncertainty.

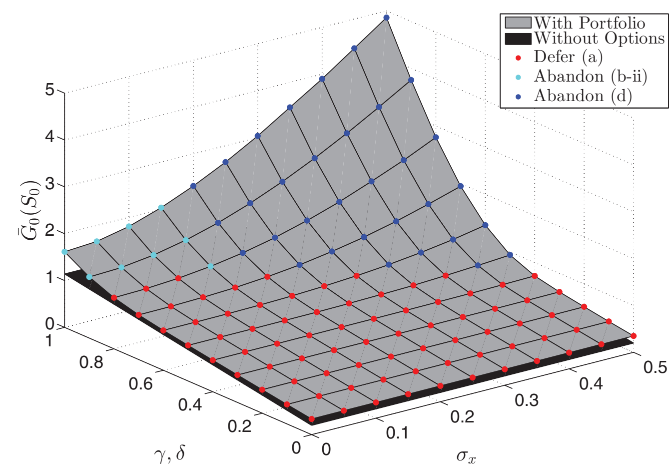


Fig. 5. Value of investment project with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x.

Unlike the investment project without options, whose value is always decreasing in σ_k, beginning at a σ_k of 0.1, the value of the portfolio is increasing in σ_k. This somewhat unexpected non-monotonic behaviour is because the flexibility provided by the portfolio, particularly by its option to abandon during operation (d), allows one to partially reverse the investment by recovering increased investment expenditures in situations with high values of σ_k, thereby taking advantage of relatively high state-dependent salvage values. This seems to explain why option (d) is the portfolio's most valuable individual option when the degree of technical uncertainty (σ_k) is high, whereas in most other situations, the option to defer (a) is the portfolio's most valuable option. Interestingly, for high values of σ_v, there are even situations in which options (b-i) and (c) are most-valuable, reflecting the ability of such a complex portfolio of real options to manage exogenous and endogenous uncertainties simultaneously in a wide range of uncertain environments.

To show the effect of the endogenous, state-dependent salvage value, X_t, on investment decisions, Fig. 5 shows the extent to which the value of the investment project is affected by the pay-out ratios γ and δ as well as by the standard deviation σ_x. The value

[6] As expected, the deterministic NPV of £2.1m reported by Grainger and Etherington (2014) for V_0=£0.5646m is larger than the expected NPV we obtained for the project without any flexibility since they assume there is no construction cost uncertainty, thereby overvalue the district heating investment.

[7] Here, interactions between the portfolio's individual real options result in the sum of individual option values being greater than the value of the portfolio.

of the project without options – where X_t is received as residual value when completing the project after 25 years of operation – is positive for all parameters under consideration. Furthermore, its value increases virtually linearly in (γ, δ) because of the linear dependence of the expected asset value, Z_t, on (γ, δ), but is practically unaffected by changes in σ_x simply because the expected value of X_t does not change. Although the value of the project with the portfolio of options is always greater than the value of the project without options, the difference remains relatively constant for low values of (γ, δ) and for both low σ_x and moderate (γ, δ), with the option to defer (a) being the portfolio's most valuable individual option in these situations. As can be seen, however, for high expected asset values and fairly high yet risky salvage values, the portfolio considered here is capable of extracting considerable value from flexibilities, especially from abandoning the project during either construction (b-ii) or operation (d). The above results therefore highlight the importance of applying such a portfolio of real options approach when there is both exogenous and endogenous uncertainty.[8]

6. Conclusions

This paper presents an approach for valuing portfolios of interdependent real options under both exogenous and endogenous uncertainties. We illustrate this approach by valuing the expansion of a district heating network in London. Unlike existing valuation approaches, which have considered only exogenous uncertainty or rather inflexible and restricted portfolios, this work has studied a complex yet practical real options portfolio under conditions of four relevant sources of uncertainty. The portfolio's options were to defer investment, to stage investment, to temporarily mothball the operation, and to permanently abandon the project. Two of the underlying uncertainties, decision-dependent cost to completion and state-dependent salvage value, were endogenous, whereas the other two, operating revenues and their growth rate, were exogenous. We have extended standard models in several ways in order to address this complex investment problem. In our extended model we considered the possibility of multiple time steps, and we made the parametric model state-dependent to account for the fact that basis functions are impossible for states in which the values of the corresponding decision- or state-dependent factors are constant or non-existent. Computational tractability was enabled by assuming that it is possible to exogenise endogenous uncertainty.

We have presented an extended simulation-and-regression-based algorithm to approximately solve the valuation problem. In our algorithm's forward induction procedure, the resource state space generation is interleaved with the Monte Carlo sampling steps of the information state space generation because of the additional path-dependency resulting from the presence of decision-dependent uncertainty. The key insight underlying our algorithm is that some resource states may not be reachable or only in a subset of sample realisations. Therefore, our algorithm's forward pass crucially includes reachability. We have demonstrated the applicability of our modelling approach and algorithmic strategy using an urban infrastructure investment in London. The reachability analysis showed that the presence of the endogenous, decision-dependent uncertainty has adverse impacts on algorithmic complexity and simulation efficiency. We also showed how our approach can be used to isolate individual options' values and provided insights into which types of options are most useful. As expected, the availability of real options is more valuable for low values of initial revenues and the portfolio is substantially more valuable than individual options. Of these, we found that abandoning during operation provides the highest individual value with mothballing being of no value.

We have also investigated the way in which the value of the real options portfolio is affected by the degrees of exogenous and endogenous uncertainty. The sensitivity analysis demonstrated that the portfolio value increases monotonically in both the exogenous and state-dependent uncertainty, but we made the surprising observation that the decision-dependent uncertainty has a non-monotonic effect, which is due to the availability of abandonment options. Most notably, our numerical analysis demonstrates that ignoring endogenous, decision- and state-dependent uncertainty can lead to substantial over- and under-valuation, respectively, thereby highlighting the importance of correctly accounting for sources of uncertainty. The illustrative example shows that our approach is flexible and powerful, and could be used without difficulty to value more complex portfolios and their individual real options under both types of uncertainty. Future work will explore ways to model the dynamics of other sources of endogenous uncertainty as well as investigate how these can be integrated into the valuation framework presented here. Other promising extensions of our framework could include the consideration of risk aversion (Chronopoulos, De Reyck, & Siddiqui, 2011) and, especially in the context of district heating networks, competition and market power (Virasjoki, Siddiqui, Zakeri, & Salo, 2018).

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Appendix A. Nomenclature

Table A1 contains a summary of most of the notation used in this work.

Appendix B. Solution procedure

The forward induction procedure consists of the following steps:

1. Starting at time 0 and using (17), sample $|\Omega|$ paths of K_t conditional on $a_{0,2} = 1$ and $a_{t4} = 1$ until $K_t(\omega) = 0, \forall \omega \in \Omega$, where $\Delta^{con}(\omega) = \{\min t : K_t(\omega) = 0\}$ and $\mathcal{T}^{con} = \{\Delta^{con}(\omega) : \omega \in \Omega\}$ denote the construction time in path ω and the set of construction times, respectively.
2. Determine the set of decision times, $\mathcal{T}_n$, for all decisions nodes $n \in \mathcal{N}^d$:

[8] It is important to note that while we have not included an example with only exogenous uncertainty (V_t, μ_t) – i.e. without endogenous uncertainty (K_t, X_t) –, the combined impact of K_t and X_t can be identified from Fig. 4 at (σ_k, σ_v)=(0,0.10) and Fig. 5 at $(\sigma_x, (\gamma, \delta))$=(0,0.70), respectively.

Table A1
Summary of notation.

Sets and indices	
$\mathcal{N}$	Set of nodes, $\{1,\dots,N\}$
$\mathcal{N}^d$	Set of decision nodes, $\mathcal{N}^d \subset \mathcal{N}$
$\mathcal{H}$	Set of transitions, $\{1,\dots,H\}$
t	Time index, $t \in \mathcal{T}$, where $\mathcal{T}$ is the set of decision times
$\mathcal{S}_t$	State space at time t
$\mathcal{R}_t$	Resource state space at time t
$\mathcal{I}_t$	Information state space at time t
ω	Sample path, $\omega \in \Omega$, where Ω is the set of all sample paths
Ω_{R_t}	Set of sample paths in which R_t is reachable, $\Omega_{R_t} \subseteq \Omega$
l	Index of summation, $l = 0,\dots,L_{S_t}$, used to specify the lth dimension of the parametric model, where $l = 0$ refers to a constant term
Parameters	
Δ_h	Duration of transition $h \in \mathcal{H}$
r	Discount rate
$\phi_{S_t l}(I_t)$	A basis function (or feature) that extracts information from I_t
L_{S_t}	Dimension of parametric model given that we are in state S_t
Variables	
S_t	State at time t, so that $S_t = (R_t, I_t)$
R_t	Resource state variable
I_t	Information state variable
a_{th}	(Binary) decision at time t for transition h, so that $a_t = (a_{th})_{h\in b^D(N_t)}$
a_t	Action (or decision) at time t
$\mathcal{A}_{S_t}$	Feasible region when in S_t at time t
$\alpha_l(S^R(R_t, a_t))$	Regression coefficient (or weight) when we are in resource state R_t at time t and take action a_t
W_t	Exogenous information that first becomes known at time t
Functions and mappings	
$b^D(N_t)$	Set of outgoing transitions of node N_t
$S^M(S_t, a_t, W_{t+\Delta_h})$	Transition function, giving state $S_{t+\Delta_h}$ given that we are in state S_t, take action a_t (i.e. make transition h), and then learn $W_{t+\Delta_h}$, which is revealed between t and $t+\Delta_h$
$S^R(R_t, a_t)$	Resource transition function, giving resource state $R_{t+\Delta_h}$ given that we are in resource state R_t and take action a_t (i.e. make transition h)
$\Pi_t(S_t, a_t)$	Payoff at time t given we are in state S_t and take action a_t
$G_t(S_t)$	Value of portfolio of real options when in state S_t at time t
$\bar{G}_t(S_t)$	Approximation of $G_t(S_t)$
$\Phi_t(S_t, a_t)$	Continuation value at time t when in state S_t and taking action a_t
$\hat{\Phi}_t^{L_{S_t}}(S_t, a_t)$	Approximation of $\Phi_t(S_t, a_t)$

$$\mathcal{T}_n = \begin{cases} \left\{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T_1^{max}\right\}, & \text{if } n = 1, \\ \left\{\tau_1 + \Delta_1(1+i+2j+m) : \tau_1 \in \mathcal{T}_1, i,j,m \in \mathbb{Z}_{\geq 0}, \Delta_1(1+i+j) \leq \max \mathcal{T}^{con}, \right. & \\ \qquad \left. \Delta_1(j+m) \leq T_2^{max}\max(0, \min(1,j))\right\}, & \text{if } n = 3, \\ \left\{\tau_1 + \Delta_1(2+i+2j+m) : \tau_1 \in \mathcal{T}_1, i,j,m \in \mathbb{Z}_{\geq 0}, \Delta_1(1+i+j) < \max \mathcal{T}^{con}, \right. & \\ \qquad \left. \Delta_1(1+j+m) \leq T_2^{max}\right\}, & \text{if } n = 5, \\ \left\{\tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(1+j) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i,j \in \mathbb{Z}_{\geq 0}, \right. & \\ \qquad \left. \Delta_1 i \leq T_2^{max}, \Delta_5(1+j) \leq T_3^{max}\right\}, & \text{if } n = 6, \\ \left\{\tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(2+j) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i,j \in \mathbb{Z}_{\geq 0}, \right. & \\ \qquad \left. \Delta_1 i \leq T_2^{max}, \Delta_5(2+j) \leq T_3^{max}\right\}, & \text{if } n = 8. \end{cases} \tag{B.1}$$

3. Generate the possible resource state space $\mathcal{R}_{nt}$ for each decision node n and time t:

$$\mathcal{R}_{nt} = \begin{cases} (t, 1, T_1^{max} - t/\Delta_1, 0), & \text{if } n = 1, t \in \mathcal{T}_1, \\ \left\{(t,3,T,Q) : \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, \right. & \\ \qquad \tau_1 < t, \Delta_1 \leq Q/I^{max} \leq \Delta^{con}(\omega), \exists \omega \in \Omega, & \\ \qquad \left. 0 \leq T_2^{max} - T \leq \max(t - \tau_1 - 2\Delta_1, 0)\right\}, & \text{if } n = 3, t \in \mathcal{T}_3, \\ \left\{(t,5,T,Q) : \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, \right. & \\ \qquad \tau_1 < t, \Delta_1 \leq Q/I^{max} < \Delta^{con}(\omega), \exists \omega \in \Omega, & \\ \qquad \left. \Delta_1 \leq T_2^{max} - T \leq \max(t - \tau_1 - \Delta_1, \Delta_1)\right\}, & \text{if } n = 5, t \in \mathcal{T}_5, \\ \left\{(t,6,T,Q) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, \right. & \\ \qquad T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - \Delta_5, & \\ \qquad \left. T \bmod \Delta_5 = 0, i \leq T_2^{max}\right\}, & \text{if } n = 6, t \in \mathcal{T}_6, \\ \left\{(t,8,T,Q) : \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, \right. & \\ \qquad T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - 2\Delta_5, & \\ \qquad \left. T \bmod \Delta_5 = 0, i \leq T_2^{max}\right\}, & \text{if } n = 8, t \in \mathcal{T}_8. \end{cases} \tag{B.2}$$

4. For all $R_t \in \mathcal{R}_t, t \in \mathcal{T}$, where $\mathcal{R}_t = \bigcup_{n \in \mathcal{N}^d} \mathcal{R}_{nt}$ and $\mathcal{T} = \bigcup_{n \in \mathcal{N}^d} \mathcal{T}_n$, compute the set of paths Ω_{R_t} in which $R_t = (t, n, T_t, Q_t)$ is reachable:

$$\Omega_{R_t} = \begin{cases} \Omega, & \text{if } n = 1, \\ \left\{\omega \in \Omega : t - \tau_1 - T_2^{max} + T_t \leq \Delta^{con}(\omega), Q_t / I^{max} \leq \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\right\}, & \text{if } n = 3, \\ \left\{\omega \in \Omega : t - \tau_1 - T_2^{max} + T_t < \Delta^{con}(\omega), Q_t / I^{max} < \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\right\}, & \text{if } n = 5, \\ \left\{\omega \in \Omega : \Delta^{con}(\omega) = Q_t / I^{max}\right\}, & \text{if } n \in \{6, 8\}. \end{cases} \tag{B.3}$$

5. Use (18) and (19) to sample $|\Omega|$ paths of V_t and μ_t, respectively.
6. Use (20) and (21) to sample $|\Omega|$ realisations of X_t.

The backward induction procedure is shown by Algorithm 1, with the optimal values of the coefficients $\left(\alpha_l(S^R(R_t, a_t))\right)_{l=0}^{L_{S_t}}$, given R_t and a_t, in *line 7* determined by (B.4).

$$\left(\hat{\alpha}_l(R_{t+\Delta_h})\right)_{l=0}^{L_{S_t}} = \underset{(\alpha_l(\cdot))_{l=0}^{L_{S_t}}}{\arg\min} \left\{ \sum_{\omega \in \Omega_{R_t}} \left[e^{-r\Delta_h} \bar{G}_{t+\Delta_h}(S_{t+\Delta_h}(\omega)) - \sum_{l=0}^{L_{S_t}} \alpha_l(R_{t+\Delta_h}) \phi_{S_t l}(I_t(\omega)) \right]^2 \right\}, \tag{B.4}$$

where $R_{t+\Delta_h} = S^R(R_t, a_t)$ and $S_{t+\Delta_h}(\omega) = (R_{t+\Delta_h}, I_{t+\Delta_h}(\omega))$.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2019.01.055.

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