Radon numbers for trees

Shoham Letzter *

February 22, 2016

Abstract

We consider P_3-convexity on graphs, where a set U of vertices in a graph G is *convex* if every vertex not in U has at most one neighbour in U.

Tverberg's theorem states that every set of $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into k sets with intersecting convex hulls. As a special case of Eckhoff's conjecture, we show that a similar result holds for P_3-convexity in trees.

A set U of vertices in a graph G is *free* if no vertex of G has more than one neighbour in U. We prove an inequality relating the Radon number for P_3-convexity in trees with the size of a maximum free set.

1 Introduction

Radon's classical lemma [8] states that every set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect. Tverberg [9] generalised this to partitions into more than two sets. Namely, every set of at least $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into k sets whose convex hulls have a point in common.

Inspired by this, Eckhoff conjectured in [3] that the situation is similar in general convexity spaces. A *convexity space* is a pair $(X, \mathcal{C})$ where X is a set and $\mathcal{C}$ is a collection of subsets of X, called *convex sets*, such that $\emptyset$ and X are convex, the intersection of convex sets is convex and the union of nested convex sets is convex. For example, the usual convex sets in $\mathbb{R}^d$ form a convexity space. We refer the interested reader to the book of van de Vel [10] for a thorough overview of convexity spaces. The *convex hull* of a set $S \subseteq X$, denoted by $H_{\mathcal{C}}(S)$, is the minimal convex set containing S, i.e. the intersection of all convex sets containing S. For a set S, a *k-Radon partition* (also known as a k-Tverberg partition) is a partition of S into k sets whose convex hulls have a point in common. A set is *k-Radon independent* (or k-r.i.) if it has no k-Radon partition. The k-Radon number (which is also known in the literature as the k-Tverberg number or k-partition number) of $(X, \mathcal{C})$ is the minimal number (if it exists) $r_k(\mathcal{C})$ such that every set $S \subseteq X$ of size at least $r_k(\mathcal{C})$ has a k-Radon partition. Eckhoff [3] conjectured that $r_k(\mathcal{C}) \leq (k-1)(r_2(\mathcal{C})-1)+1$ in every convexity

*Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK. Email: s.letzter@dpmms.cam.ac.uk

space. This conjecture has been proved in several convexity spaces including trees with geodesic convexity [6]. However, the general conjecture has recently been disproved by Bukh [1].

It would be useful for us to generalise the notion of a k-r.i set to multisets. This can be done in a natural way by considering partitions of multisets rather than sets. We define $\tilde{r}_k(\mathcal{C})$ to be the size of the largest k-r.i. multiset. Note that $\tilde{r}_k(\mathcal{C}) \geq r_k(\mathcal{C}) - 1$, with equality for $k = 2$ (as a 2-r.i. multiset is a set, i.e. no element can appear more than once). When $k = 2$ we often omit the prefix k, e.g., a 2-r.i. set may be called a r.i. set and we denote $\tilde{r}(\mathcal{C}) = \tilde{r}_2(\mathcal{C})$.

We shall study P_3-convexity in trees. For a graph G, a set U of vertices of G is P_3-*convex* or, briefly, *convex* if every vertex not in U has at most one neighbour in U. Equivalently, U is convex if it contains all middle vertices in the paths of length 2 between two vertices of U. P_3-convexity was first considered in the context of directed graphs and tournaments (see [4, 5, 7, 11]).

Throughout this paper graphs are always finite, simple and undirected. For a graph G, let $\tilde{r}_k(G)$ denote the largest k-r.i. multiset of G, and for a set $U \subseteq V(G)$, let $H_G(U)$ denote the convex hull of U in G.

As the first main result of our paper, we show that Eckhoff's conjecture holds for P_3-convexity on trees.

Theorem 1. *Let T be a tree, $k \geq 3$. Then $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$.*

Given a graph G, call a set $A \subseteq V(G)$ *free* if every vertex of G has at most one neighbour in A. Note that every free set in a graph G is also convex and the converse does not hold in general. Let $\tilde{\alpha}(G)$ be the size of a largest free set in G. It follows that $\tilde{r}(G) \geq \tilde{\alpha}(G)$. Our second main theorem answers a question posed by Dourado et al. [2].

Theorem 2. *Let T be a tree. Then $\tilde{r}_2(T) \leq 2\tilde{\alpha}(T)$.*

We shall show that this theorem is sharp in the sense that there are infinitely many trees for which we have equality.

The last inequality is not true in general as shown by the graph G_1 in Figure 1. Every two of the seven vertices of G_1 have a common neighbour, hence $\tilde{\alpha}(G_1) = 1$. It is easy to check that the set $A = \{2, 4, 6\}$ of vertices of G_1 is r.i. and that every set of 4 vertices of G_1 is not r.i., therefore $\tilde{r}(G_1) = 3$.

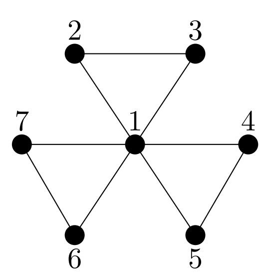


Figure 1: Graph G_1

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

2 Eckhoff's conjecture for P_3-convexity in trees

In this section, we prove Theorem 1. Recall its statement.

Theorem (1). *Let T be a tree, $k \geq 3$. Then $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$.*

Before turning to the proof we introduce some notation. For a graph G and a vertex $v \in V(G)$, define

$$\tilde{r}^*_k(G, v) = \max\{|R|\colon R \text{ is a } k\text{-r.i. multiset and } v \notin H_G(R)\}. \tag{1}$$

Proof of Theorem 1. We shall prove more than claimed in the statement of the theorem. Namely, we shall show that for every tree T the following assertions hold.

- $\tilde{r}^*_k(T, v) \leq (k-1)\tilde{r}^*_2(T, v)$ for every $v \in V(T)$,
- $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$.

Our proof is by induction on $n = |V(T)|$. Both statements are clear for $n \leq 2$.

Let T be a tree with $n \geq 3$ vertices. The first statement follows easily by induction using expression (2) for $\tilde{r}^*_k$ below. For a vertex $v \in V(T)$, let $v_1, \ldots, v_l$ be its neighbours, and for $i \in [l]$ let T_i be the component of v_i in $T \setminus \{v\}$. Then

$$\tilde{r}^*_k(T, v) = \max_{i \in [l]}\Big(\sum_{j \neq i} \tilde{r}^*_k(T_j, v_j) + \tilde{r}_k(T_i)\Big). \tag{2}$$

We now prove that $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$. Let R be a k-r.i. multiset of maximum size. If T has a leaf v which is not in R, let $T' = T \setminus \{v\}$. Then by induction,

$$|R| = \tilde{r}_k(T') \leq (k-1)r_2(T') \leq (k-1)r_2(T).$$

Thus we may assume that R contains each leaf of T at least once.

In the rest of the proof we consider two possible cases which will be dealt with in different subsections.

Case 1: There is a longest path $v_1, \ldots, v_m$ in T such that $\deg(v_2) \geq 3$

Let $z = v_3$, $y = v_2$ and $x_1, \ldots, x_l$ be the neighbours of y other than z. Note $l \geq 2$, and by the choice of $v_1, \ldots, v_m$ as a longest path, $x_1, \ldots, x_l$ are all leafs (see Figure 2).

Denote by s_i, $i \in [l]$, the number of appearances of x_i in R and by t the number of appearances of y in R. By our assumption that R contains every leaf at least once, $s_i \geq 1$ for every $i \in [l]$. As R is k-r.i. we have that $s_i, t \leq k-1$. Let $s = s_1 + \ldots + s_l$. We consider three cases according to the value of s.

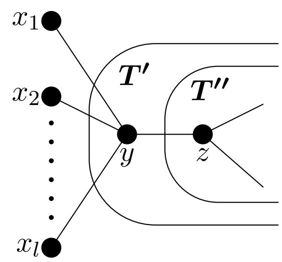


Figure 2: Case 1

Case 1.1: $s \le 2k-2$.

Let $\tau = \min\{s, 2k-2-s\}$, $\sigma = (s-\tau)/2$. Note that σ is an integer and $\tau + \sigma \le k-1$. Set $T' = T \setminus \{x_1, \ldots, x_l\}$ (see Figure 2). Let R' be the multiset obtained by adding σ copies of y to $R \cap V(T')$. Note $|R| = |R'| + s - \sigma = |R'| + \sigma + \tau \le |R'| + k - 1$.

Claim 3. *R' is k-r.i..*

Proof. We show that there exist sequences $a_1, \ldots, a_\sigma$ and $b_1, \ldots, b_\sigma$ satisfying the following conditions.

- $a_j, b_j \in [l]$ and $a_j \ne b_j$ for every $j \in [\sigma]$.
- $|\{j \in [\sigma] \colon a_j = i\}| + |\{j \in [\sigma] \colon b_j = i\}| = s_i$ for every $i \in [l]$.

Note that the existence of such sequences completes the proof of this claim. Indeed, suppose to the contrary that $R' = R'_1 \cup \ldots \cup R'_k$ is a k-Radon partition of R'. Obtain $R_1, \ldots, R_k$ by replacing each of the σ new copies of y with a distinct pair x_{a_j}, x_{b_j} where $j \in [\sigma]$. By the choice of a_j and b_j, we conclude that $R = R_1 \cup \ldots \cup R_k$ is a partition of R. Clearly $H_T(R_l) \cap V(T') = H_{T'}(R'_l)$ for every $l \in [k]$, hence this partition is a k-Radon partition of R, contradicting the choice of R as a k-r.i. set.

It thus remains to show the existence of such sequences. By induction on k, we show that if $s = s_1 + \ldots + s_l \le 2k-2$ and $s_i \le k-1$ for every $i \in [l]$ we can find two sequences satisfying the above. We assume $\sigma \ge 1$ or equivalently $s \ge k$, because otherwise there is nothing to prove. When $k = 2$ we thus have that without loss of generality $s_1 = s_2 = 1$, and we set $a_1 = 1, b_1 = 2$. If $k \ge 3$ assume that $s_1 \ge s_2 \ge \ldots \ge s_l$ and let $a_\sigma = 1, b_\sigma = 2$. Now set

$$s'_i = \begin{cases} s_i - 1 & i \in \{1, 2\} \\ s_i & \text{otherwise} \end{cases}$$

Note that $s' = s'_1 + \ldots + s'_l \le 2k-4$ and $s'_i \le k-2$ for every $i \in [l]$ (otherwise $s_3 \ge k-1$ and $s_1 + s_2 + s_3 \ge 3(k-1) > 2(k-1)$, a contradiction). Also, either $\sigma' = 0$ (in which case we are done), or $\sigma' = s' - (k-2) = s - (k-1) - 1 = \sigma - 1$. The proof may now be completed using the inductions hypothesis for $k-1$. □

Using Claim 3 we conclude by induction that

$$\tilde{r}_k(T) = |R| \le |R'| + (k-1) \le \tilde{r}_k(T') + (k-1) \le (k-1)(\tilde{r}_2(T') + 1).$$

The following claim completes the proof of Theorem 1 in Case 1.1.

Claim 4. $\tilde{r}_2(T) \geq r_2(T') + 1$.

Proof. Let S' be a r.i. set in T' of maximum size. Set

$$S = \begin{cases} S' \cup \{x_1\} & y \notin S' \\ (S' \setminus \{y\}) \cup \{x_1, x_2\} & y \in S' \end{cases}$$

Note that $|S| = |S'| + 1$. We shall show that S is r.i, thus proving the claim. Assume to the contrary that there exists a partition $S = A \cup B$ with $H_T(A) \cap H_T(B) \neq \emptyset$. Without loss of generality, $x_1 \in A$.

Consider the following three possibilities.

$\boldsymbol{y \notin S'}$**.**

Set $A' = A \setminus \{x_1\}$. Note that the following conditions hold.

- $S' = A' \cup B$ is a partition of S'.
- $H_T(A) = \begin{cases} H_{T'}(A') \cup \{x_1\} & z \notin H_{T'}(A') \\ H_{T'}(A') \cup \{x_1, y\} & z \in H_{T'}(A') \end{cases}$
- $H_T(B) = H_{T'}(B)$ and $y \notin H_T(B)$.

Therefore $H_T(A) \cap H_T(B) = H_{T'}(A') \cap H_{T'}(B) = \emptyset$, a contradiction.

$\boldsymbol{y \in S'}$ **and** $\boldsymbol{x_1, x_2 \in A}$**.**

Let $A' = (A \setminus \{x_1, x_2\}) \cup \{y\}$. Then $H_T(A) = H_{T'}(A') \cup \{x_1, x_2\}$ and $H_T(B) = H_{T'}(B)$. As before we reach a contradiction.

$\boldsymbol{y \in S'}$ **and** $\boldsymbol{x_1 \in A, x_2 \in B}$**.**

As S' is r.i. and $y \in S'$ we conclude that $z \notin H_{T'}(A \setminus \{x_1\}) \cap H_{T'}(B \setminus \{x_2\})$. Without loss of generality, $z \notin H_{T'}(B \setminus \{x_2\})$. Set $A' = (A \setminus \{x_1\}) \cup \{y\}$. As before $H_T(A) \subseteq H_{T'}(A') \cup \{x_1, y\}$ and $H_T(B) = H_{T'}(B) \cup \{x_2\}$. This leads to a contradiction to S' being r.i..

□

Case 1.2: $s = 2k - 1$

Define T' as before, and let R' be the union of $R \cap V(T')$ with a copy of x_1 and $k-1$ copies of y.

Claim 5. *R' is k-r.i..*

Proof. Replacing s_1 by $s_1 - 1$ returns us to the setting of Claim 3. Following the same arguments we obtain this claim. □

Set $T'' = T' \setminus \{y\}$ and $R'' = R' \cap V(T'')$ (see Figure 2 above). Then $z \notin H_{T''}(R'')$ as otherwise we can partition R' into k parts, $k-1$ of which contain y, and the last contains both x_1 and z. y is contained in all the parts of this partition, contradicting the fact that R' is k-r.i.. Thus

$$\tilde{r}_k(T) = |R| = 2k - 1 + |R''| < 3(k-1) + \tilde{r}^*_k(T'', z) \leq (k-1)(3 + \tilde{r}^*_2(T'', z)).$$

The following claim completes the proof of Theorem 1 in Case 1.2.

Claim 6. $\tilde{r}_2(T) \geq 3 + \tilde{r}_2^*(T'', z)$.

Proof. Let S'' be a r.i. set of T'' satisfying $z \notin H_{T''}(S'')$. We shall show that $S = S'' \cup \{x_1, x_2, x_3\}$ is r.i. thus proving the claim (note that $l \geq 3$, so S is well defined). Assume that we have a Radon partition $S = A \cup B$. Without loss of generality $x_1, x_2 \in A$. Set $A' = (A \cap V(T')) \cup \{y\}$, $B' = B \cap V(T')$. Then $H_T(A) \cap H_T(B) = H_{T'}(A') \cap H_{T'}(B')$. Claim 6 follows from the following claim.

Claim 7. *Let T be a tree, $v \in V(T)$ and S a r.i. set in T satisfying $v \notin H_T(S)$. Let $T_{v \leftarrow u}$ denote the tree obtained from T by adding a new vertex u and connecting it to v. Then $S \cup \{u\}$ is r.i. in $T_{v \leftarrow u}$.*

Proof. We prove the claim by induction on $|V(T)|$. The claim clearly holds when T has at most one vertex. Let $T' = T_{v \leftarrow u}$ and $S = A \cup B$ a partition of S. We show $H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = \emptyset$.

Let $v_1, \ldots, v_l$ be the neighbours of v in T. Let T_i be the component of v_i in $T \setminus \{v\}$ and denote

$$S_i = S \cap V(T_i),\ A_i = A \cap V(T_i),\ B_i = B \cap V(T_i).$$

Suppose first that $v_i \notin H_{T_i}(A_i)$ for every $i \in [l]$. Then $H_{T'}(A \cup \{u\}) = H_{T'}(A) \cup \{u\}$ and $H_{T'}(B) = H_T(B)$. Thus, as S is r.i., $H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = H_T(A) \cap H_T(B) = \emptyset$.

We can now assume that without loss of generality $v_1 \in H_{T_1}(A_1)$. As $v \notin H_T(S)$, this means that $v_i \notin H_{T_i}(S_i)$ for every $i \geq 2$. As S is r.i., $v_1 \notin H_{T_1}(B_1)$. Thus

$$H_{T'}(A \cup \{u\}) = H_{T_1}(A_1) \cup \{u\} \cup (\bigcup_{j \geq 2} H_{(T_i)_{v_i \leftarrow v}}(A_i \cup \{v\}))$$
$$H_{T'}(B) = \bigcup_{i \geq 1} H_{T_i}(B_i) = H_{T_1}(B_1) \cup (\bigcup_{i \geq 2} H_{(T_i)_{v_i \leftarrow v}} B_i).$$

Therefore

$$H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = \bigcup_{i \geq 2} (H_{(T_i)_{v_i \leftarrow v}}(A_i \cup \{v\}) \cap H_{(T_i)_{v_i \leftarrow v}}(B_i)).$$

The claim follows using the induction hypothesis with T_i, $i \geq 2$. □

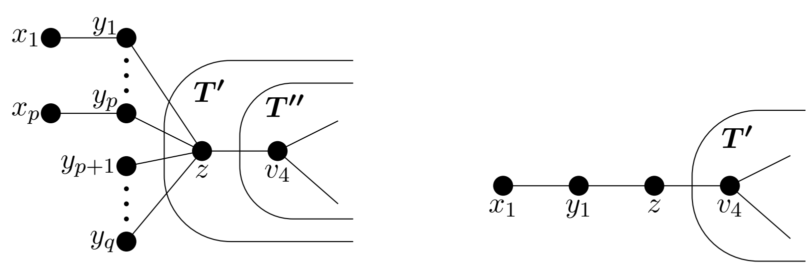
□

Case 1.3: $s \geq 2k$

Similarly to Claim 3, we can conclude that the multiset obtained by adding k copies of y to $R \cap V(T')$ is k-r.i., which is obviously a contradiction.

Case 2: $\deg(v_2) = 2$ in every longest path $v_1, \ldots, v_m$ of T

Fix a longest path $v_1, \ldots, v_m$ in T. Denote $v_3 = z$ and note that each of its neighbours other than v_4 is either a leaf or has degree 2 and is adjacent to a leaf (by the choice of the longest path and the



(a) Case 2

(b) Case 2.1

Figure 3: Case 2

definition of Case 2). Let $y_1, \dots, y_p$ be the neighbours of z other than v_4 which have degree 2 and $y_{p+1}, \dots, y_q$ the neighbours of z other than v_4 which are leafs. Let $x_1, \dots, x_p$ be the neighbours of $y_1, \dots, y_p$ which are leafs respectively (see Figure 3a).

Denote by s_i, $i \in [p]$, the number of appearances of x_i in R; t_i, $i \in [q]$, the number of appearances of y_i in R and u the number of appearances of z in R. Let $t = t_1 + \ldots + t_q$. As in the previous case, we conclude from the fact that R is k-r.i. that $t \leq 2k - 1$. We consider three cases.

Case 2.1: $q = 1$

Then $t_1 + \min\{s_1, u\} \leq k - 1$ (otherwise obtain a k-Radon partition of R by putting a copy of y_1 in t_1 sets, and a copy of x_1 and z in the other $k - t_1$ sets). Thus

$$s_1 + t_1 + u = t_1 + \min\{s_1, u\} + \max\{s_1, u\} \leq 2(k - 1).$$

Let $T' = T \setminus \{x_1, y_1, z\}$, $R' = R \cap V(T')$ (see Figure 3b). Then

$$\tilde{r}_k(T) = |R| \leq |R'| + 2(k - 1) \leq \tilde{r}_k(T') + 2(k - 1) \leq (k - 1)(\tilde{r}_2(T') + 2).$$

The proof of Theorem 1 in Case 2.1 follows from the following claim.

Claim 8. $\tilde{r}_2(T) \geq \tilde{r}_2(T') + 2$.

Proof. Let S' be a r.i. set in T'. Set $S = S \cup \{x_1, y_1\}$. It is easy to verify that S is r.i.. □

Case 2.2: $t \leq 2k - 2$

As before set $\tau = \min\{t, 2k - 2 - t\}$ and $\sigma = (t - \tau)/2$ (then $t - \sigma \leq k - 1$). Let $T' = T \setminus \{x_1, \dots, x_p, y_1, \dots, y_q\}$ (see Figure 3a) and let R' be the multiset obtained by adding σ copies of z to $R \cap V(T')$. Then as in Claim 3, R' is k-r.i. and thus

$$\tilde{r}_k(T) = |R| = |R'| + s + t - \sigma \leq \tilde{r}_k(T') + (p + 1)(k - 1) \leq (k - 1)(\tilde{r}_2(T') + p + 1).$$

The proof of Theorem 1 in this case follows from the following claim.

Claim 9. $\tilde{r}_2(T) \geq \tilde{r}_2(T') + p + 1$.

Proof. Let S' be r.i. in T'. Set

$$S = \begin{cases} S' \cup \{x_1, \ldots, x_p, y_1\} & z \notin S' \\ (S' \setminus \{z\}) \cup \{x_1, \ldots, x_p, y_1, y_2\} & z \in S' \end{cases}$$

One can show that S is r.i. similarly to the proof of Claim 4. □

Case 2.3: $t = 2k - 1$

Let $T' = T \setminus \{x_1, \ldots, x_p, y_1, \ldots, y_q\}$ and $T'' = T' \setminus \{z\}$ (see Figure 3a). Let $R'' = R \cap V(T'')$ and let R' be the multiset obtained by adding $k-1$ copies of z and a copy of y_1 to R''. As in Case 1.2, R' is k-r.i. in T and R'' is k-r.i. in T'' with $v_4 \notin H_{T''}(R'')$. Thus

$$\tilde{r}_k(T) = |R''| + s + 2k - 1 \leq \tilde{r}^*_k(T'', v_4) + s + 2k - 1 \leq (k-1)\tilde{r}^*_2(T'', v_4) + s + 2k - 1. \quad (3)$$

As $s_i \leq k - 1$ for every $i \in [p]$ we have that $s \leq (k-1)p$. If $s < (k-1)p$ we obtain

$$\tilde{r}_k(T) \leq (k-1)(\tilde{r}^*_2(T'', v_4) + p + 2).$$

And the proof of Theorem 1 in this case follows from the claim below.

Claim 10. $\tilde{r}_2(T) \geq \tilde{r}^*_2(T'', v_4) + p + 2$.

Proof. If S'' is r.i. in T'' with $v_4 \notin H_{T''}(S'')$ then similarly to the proof of Claim 6, $S = S'' \cup \{x_1, \ldots, x_p, y_1, y_2\}$ is r.i. in T. □

Thus we may assume $s = (k-1)p$ i.e. $s_1 = \ldots = s_p = k - 1$.

Claim 11. $t_1 = \ldots = t_p = 0$.

Proof. Assume otherwise, then without loss of generality $t_1 \geq 1$. Let $\phi = k - t_1$. Similarly to the proof of Claim 3 we will show the existence of sequences $a_1, \ldots, a_\phi$ and $b_1, \ldots, b_\phi$ satisfying the following conditions.

- $a_j, b_j \in [2, q]$ and $a_j \neq b_j$ for every $j \in [\phi]$.
- $|\{j \in [\phi] \colon a_j = i\}| + |\{j \in [\phi] \colon b_j = i\}| \leq t_i$ for every $i \in [2, q]$.

This leads to a contradiction as we can then obtain a k-Radon partition of R by putting a copy of y_1 in t_1 of the sets, and putting a copy of x_1 and a pair y_{a_l}, y_{b_l} in each of the other $k - t_1$ sets. y_1 will be in the intersection of the convex hulls of the sets (here we use the assumption that $s_1 = k-1$ so this is indeed possible).

If $t_i \leq k - t_1 - 1$ for every $i \in [2, q]$, we proceed as in Claim 3 to prove the existence of such sequences. Otherwise, let i_0 be such that $t_{i_0} \geq k - t_1$. Note that

$$\sum_{j \neq 1, i_0} t_i = t - t_{i_0} - t_1 \geq 2k - 1 - (k-1) - t_1 = k - t_1.$$

Thus in this case we can choose $a_1 = \ldots = a_\phi = i_0$ and $b_1, \ldots, b_\phi \in [2, q] \setminus \{i_0\}$ to satisfy the requirements. □

Using Claim 11 it follows that $2k-1 = t = t_{p+1} + \ldots + t_q$. As $t_i \le k-1$ for every $i \in [q]$, we conclude that $q - p \ge 3$.

Claim 12. $\tilde{r}_2(T) \ge \tilde{r}_2^*(T'', v_4) + 3 + p$.

Proof. Let S'' be a r.i. set in T'' with $v_4 \notin H_{T''}(S'')$. Let $S = S'' \cup \{x_1, \ldots, x_p, y_{p+1}, y_{p+2}, y_{p+3}\}$. It is easy to see that S is r.i. in T. □

Recalling inequality 3, we obtain

$$\tilde{r}_k(T) \le s + 2k - 1 + (k-1)\tilde{r}_2^*(T'', v_4) \le (k-1)(p + 3 + \tilde{r}_2^*(T'', v_4)) \le (k-1)\tilde{r}_2(T).$$

This completes the proof of Theorem 1.

□

3 An upper bound on the Radon number in terms of $\tilde{\alpha}(T)$

This section is devoted to the proof of Theorem 2. We remind the reader of the statement.

Theorem (2)**.** *Let T be a tree. Then $\tilde{r}_2(T) \le 2\tilde{\alpha}(T)$.*

Recall that a set A of vertices is called free if every vertex in the graph has at most one neighbour in A and $\tilde{\alpha}(T)$ denotes the size of a largest free set in T. Given a graph G and a vertex $v \in V(G)$, define

$$\tilde{\alpha}^*(G, v) = \max\{|A| \colon A \text{ is free and } v \notin A\}.$$

Proof of Theorem 2. We prove a stronger statement than what is claimed in the theorem. We shall show that for every tree T the following assertions hold.

- $\tilde{r}^*(T, v) \le 2\tilde{\alpha}^*(T, v)$ for every vertex $v \in V(T)$.
- $\tilde{r}(T) \le 2\tilde{\alpha}(T)$.

We prove these statements by induction on $n = |V(T)|$. Both statements clearly hold for $n \le 3$.

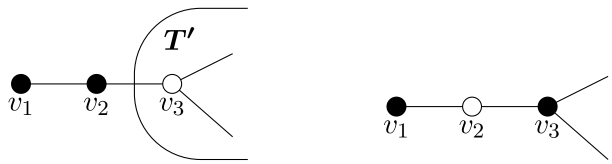
To prove the first statement, let $v \in V(T)$ and denote by $v_1, \ldots, v_l$ its neighbours. For every $i \in [l]$ let T_i be the connected component of v_i in $T \setminus \{v\}$. It is easy to see that

$$\tilde{\alpha}^*(T, v) = \max_{j \in [l]}\{\sum_{i \ne j} \tilde{\alpha}^*(T_i, v_i) + \tilde{\alpha}(T_j)\}.$$

Note the similarity to expression (2) from the previous section. It thus follows by induction that $\tilde{r}^*(T, v) \le 2\tilde{\alpha}^*(T, v)$.

We now proceed to proving that $\tilde{r}(T) \le 2\tilde{\alpha}(T)$. Let R be a r.i. set of maximum size in T. As in the proof of Theorem 1, we can assume that R contains all leafs of T. The *brothers* of a leaf v are the leafs in distance 2 from v. Then in particular, every leaf has at most 2 brothers, as no vertex of T can have more than 3 neighbours in R.

The following claim will be useful for the rest of the proof.



(a) Case 1a (b) Case 1b

In this figure and the following ones a black vertex is in R, a white vertex is not in R and for a grey vertex it is unknown if it is in R.

Figure 4: Cases 1a, 1b

.

Claim 13. *Let T be a tree. There exists a free set $A \subseteq V(T)$ of size $\tilde{\alpha}(T)$ satisfying that for every leaf $v \in V(T)$ either v or one of its brothers is in A.*

Proof. Let A be a free set in T of maximum size, v a leaf in T and u its only neighbour. If $v \in A$ we are done. Otherwise, by the maximality of A, $A \cup \{v\}$ is not free. As u is the only neighbour of v, there is a neighbour $w \neq v$ of u which is contained in A. If w is a leaf, we are done. Otherwise, set $A' = (A \backslash \{w\}) \cup \{v\}$. Then A' contains v and is free and has size $\tilde{\alpha}(T)$. Continuing similarly will result in a free set of size $\tilde{\alpha}(T)$ with the property that for each leaf either it or one of its brothers is in the set. □

We consider three cases concerning longest paths in T. Note that the theorem can be easily verified if the longest path in T has at most 3 vertices, thus we assume that a longest path in T contains at least 4 vertices. We consider each case in a separate subsection.

Case 1: There is a longest path $v_1, \ldots, v_m$ such that the component of v_4 in $T \setminus \{v_5\}$ has no leaf in distance 3 from v_4 with brothers.

We consider six cases.

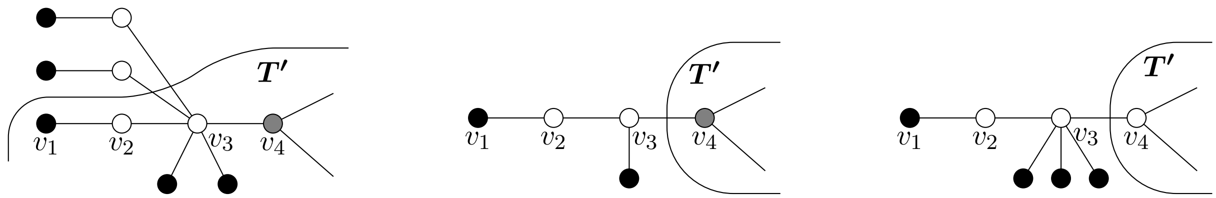
(a) $v_1, v_2 \in R$.
Set $T' = T \setminus \{v_1, v_2\}$ (see Figure 4a), $R' = R \cap V(T')$.
R' is r.i. in T' and $v_3 \notin H_{T'}(R')$. Thus, by induction,

$$\tilde{r}(T) = |R'| + 2 \leq \tilde{r}^*(T', v_3) + 2 \leq 2(\tilde{\alpha}^*(T', v_3) + 1).$$

Note that $\tilde{\alpha}^*(T', v_3) + 1 \leq \tilde{\alpha}(T)$, because if $A' \subseteq V(T') \setminus \{v_3\}$ is free then $A' \cup \{v_1\}$ is free in T. Therefore $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$ in this case.

(b) $v_1, v_3 \in R$ (see Figure 4b).
Set $R' = (R \backslash \{v_3\}) \cup \{v_2\}$. It is easy to see that R' is r.i. and it follows from Case 1.1 that $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$.

We can now assume that the above two cases do not occur. Consider the neighbours of v_3 other than v_4. Each such neighbour either is a leaf, or has degree 2 and its other neighbour is a leaf (using the fact $v_1, \ldots, v_m$ is a longest path and that we are in Case 1). Let S_i, $i \in \{1, 2\}$, be the set of



(a) Case 1c (b) Case 1d (c) Case 1e

Figure 5: Case 1c, 1d, 1e

neighbours of v_3 other then v_4 with degree i. Note that $v_2 \in S_2$, and by our previous assumptions: $S_1 \subseteq R$, $S_2 \cap R = \emptyset$. In particular $|S_1| \leq 3$. Consider the remaining four cases.

(c) $|S_2| \geq 2$.
Let T' be the component of v_3 in $T \setminus (S_2 \setminus \{v_2\})$ (see Figure 5a). Then

$$\tilde{r}(T') \geq \tilde{r}(T) - (|S_2| - 1),$$

as $R \cap V(T')$ is r.i. in T' and $R \setminus V(T')$ contains only the leafs which are neighbours of vertices in $S_2 \setminus \{v_2\}$. Furthermore

$$\tilde{\alpha}(T) \geq \tilde{\alpha}(T') + (|S_2| - 1).$$

To see this, let $A' \subseteq V(T')$ be a largest free set in T' containing v_1 (recall Claim 13). Then $v_3 \notin A'$ and the set obtained by adding the leafs which are neighbours of the vertices in S_2 to A' is free in T. Hence, by induction,

$$\tilde{r}(T) \leq \tilde{r}(T') + |S_2| - 1 < 2(\tilde{\alpha}(T') + |S_2| - 1) \leq 2\tilde{\alpha}(T).$$

We can now assume that $S_2 = \{v_2\}$.

(d) $|S_1| \leq 1$.
Let T' be the connected component of v_4 in $T \setminus \{v_3\}$ (see Figure 5b). Then $\tilde{r}(T') \geq \tilde{r}(T) - 2$, as $R \cap V(T')$ is r.i. in T', and $v_2, v_3 \notin R$ (otherwise consider Cases 1a,1b). Also $\tilde{\alpha}(T) \geq \tilde{\alpha}(T') + 1$, because if $A' \subseteq V(T')$ is free in T' then $A' \cup \{v_1\}$ is free in T. We obtain

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(\tilde{\alpha}(T') + 1) \leq 2\tilde{\alpha}(T).$$

(e) $|S_1| = 3$.
Let T' be as in the previous case (see Figure 5c) and set $R' = R \cap V(T')$. Then $v_4 \notin H_{T'}(R')$ and $|R| = |R'| + 4$. Also $\tilde{\alpha}(T) \geq \tilde{\alpha}^*(T', v_4) + 2$, because if $A' \subseteq V(T') \setminus \{v_4\}$ is free, then $A' \cup \{v_1, v_2\}$ is free in T. Hence

$$\tilde{r}(T) \leq \tilde{r}^*(T', v_4) + 4 \leq 2(\tilde{\alpha}^*(T', v_4) + 2) \leq 2\tilde{\alpha}(T).$$

(f) $|S_1| = 2$.
Set $T' = T \setminus \{v_1, v_2\}$ (see Figure 6a). If T' contains a free set of maximum size A' such that $v_3 \notin A'$, then $A' \cup \{v_1\}$ is free, so in this case $\tilde{\alpha}(T) \geq 1 + \tilde{\alpha}(T')$ and

$$\tilde{r}(T) \leq 1 + \tilde{r}(T') < 2(1 + \tilde{\alpha}(T')) \leq 2\tilde{\alpha}(T).$$

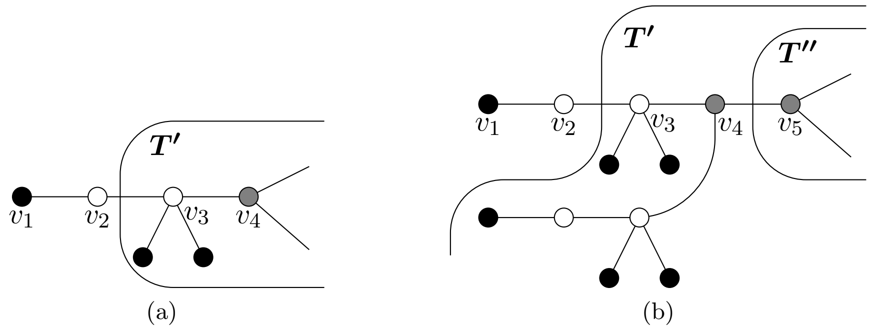


Figure 6: Case 1f

Therefore we may assume that v_3 is contained in every largest free set of T'. Let S be the set of neighbours of v_4 other than v_5 and for $v \in S$ let T_v be the connected component of v in $T \setminus \{v_4\}$.

We need the following claim.

Claim 14. *T_v has depth 2 as a tree rooted in v for every $v \in S$.*

Proof. Let $v \in S$, $v \neq v_3$ (note that the claim is clear if $v = v_3$). By the choice of $v_1, \dots, v_m$ as a longest path in T, T_v has depth at most 2. We now show that T_v has depth at least 2, i.e. v has neighbours which are not leafs. Let A' be a free set of maximum size in T'. If v has no neighbour in T_v which is not a leaf, then $(A' \setminus \{v_3\}) \cup \{v\}$ is also a free set of the same size in T', contradicting our previous assumption. □

If for some $v \in S$, T_v is not isomorphic to T_{v_3} (as rooted trees at v,v_3 respectively), by changing the selected longest path to go through v instead of v_3, we go back to one of the previous cases. Thus we may assume that the trees T_v, $v \in S$, are all isomorphic to T_{v_3}.

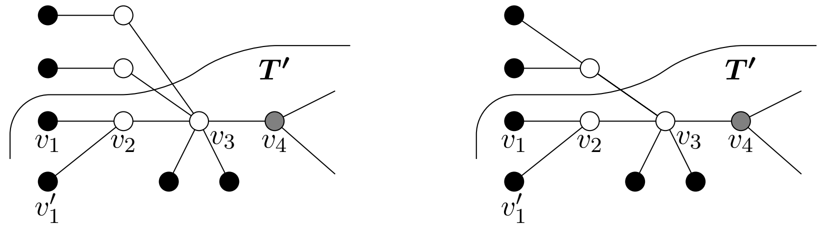
Let T'' be the component of v_5 in $T \setminus \{v_4\}$ (see Figure 6b), $R'' = R \cap V(T'')$. Then $|R \setminus R''| \leq 3|S| + 1$ as for each $v \in S$, $|R \cap V(T_v)| = 3$ and possibly v_4 in R. Note also that $\tilde{\alpha}(T) \geq \tilde{\alpha}(T'') + 2|S|$, because the union of a free set of T'' with the leafs in distance 3 from v_4 and their neighbours is free in T. Therefore

$$\tilde{r}(T) \leq \tilde{r}(T'') + 3|S| + 1 \leq 2\tilde{\alpha}(T'') + 4|S| \leq 2\tilde{\alpha}(T).$$

Case 2: Case 1 does not hold, and there exists a longest path $v_1, \dots, v_m$ such that the component of v_4 in $T \setminus \{v_5\}$ has no leafs in distance 3 from v_4 with more than one brother.

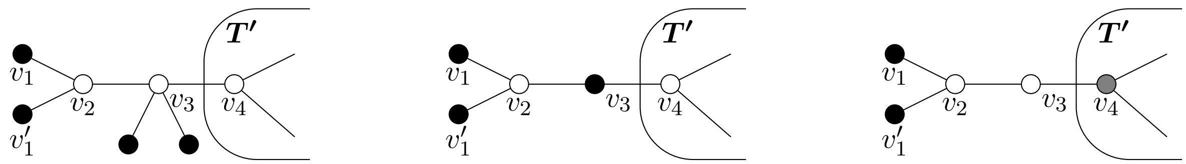
Choose the longest path such that v_1 has exactly one brother v_1'. Then $v_1, v_1' \in R$ and as R is r.i., $v_2 \notin R$. We consider the neighbours of v_3 other than v_4. Note that they can be of degrees 1, 2 or 3 only and that if they have degree 2 or 3 the other neighbours are leafs. Let S_i, $i \in \{1, 2, 3\}$, be the set of neighbours of v_3 other than v_4 with degree i. Consider the following six cases.

(a) $S_2 \neq \emptyset$.
Let T' be the component of v_3 in $T \setminus S_2$ (i.e. remove all neighbours of v_3 of degree 2, see Figure



(a) Case 2a (b) Case 2b

Figure 7: Cases 2a, 2b



(a) Case 2c (b) Case 2d (c) Case 2e

Figure 8: Cases 2c, 2d, 2e

7a). Then, by induction

$$\tilde{r}(T) \leq \tilde{r}(T') + 2|S_2| \leq 2(\tilde{\alpha}(T') + |S_2|) \leq 2\tilde{\alpha}(T).$$

The last inequality follows from the fact that a maximum free set in T' can be assumed to contain v_1 (see Claim 13), so it does not contain v_3 and we can add the $|S_2|$ leafs that were discarded to obtain a free set in T.

We now assume $S_2 = \emptyset$.

(b) $|S_3| \geq 2$.
Set T' to be the component of v_3 in $T \setminus (S_3 \setminus \{v_2\})$ (see Figure 7b). R contains $2(|S_3| - 1)$ of the discarded vertices, thus, as in the previous case,

$$\tilde{r}(T) \leq \tilde{r}(T') + 2(|S_3| - 1) \leq 2(\tilde{\alpha}(T') + (|S_3| - 1)) \leq 2\tilde{\alpha}(T).$$

Hence we can assume $|S_3| = 1$. Note that $|S_1| \leq 2$, since R is r.i..

(c) $|S_1| = 2$.
Set T' the component of v_4 in $T \setminus \{v_3\}$ (see Figure 8a). Then

$$\tilde{r}(T) \leq 4 + \tilde{r}^*(T', v_4) \leq 2(2 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}(T),$$

since $v_4 \notin H_{T'}(R \cap V(T'))$ (R is r.i.) and the union of a free set in $V(T') \setminus \{v_4\}$ with $\{v_1, v_2\}$ remains free.

(d) $S_1 = \emptyset$ and $v_3 \in R$.
Choose T' as in the previous case (see Figure 8b). Again $v_4 \notin H_{T'}(R \cap V(T'))$ and similarly

$$\tilde{r}(T) \leq 3 + \tilde{r}^*(T', v_4) < 2(2 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}(T).$$

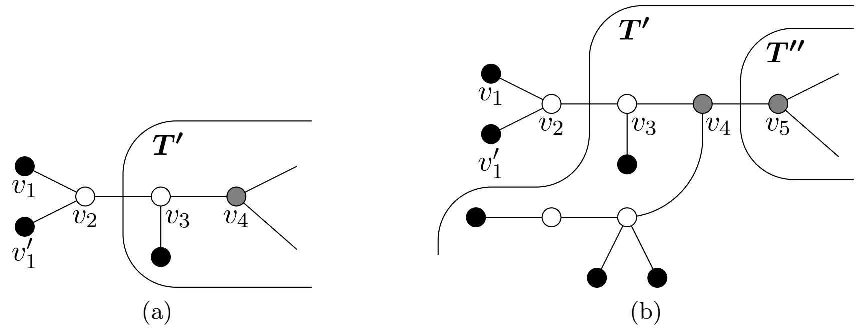


Figure 9: Case 2f

(e) $S_1 = \emptyset$ and $v_3 \notin R$.
Again set T' as before (see Figure 8c). Here R contains only 2 of the discarded vertices and we can add v_1 to any free set of T' to obtain a free set of T. Thus

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(\tilde{\alpha}(T') + 1) \leq 2\tilde{\alpha}(T).$$

(f) All previous cases do not hold, i.e. $|S_1| = |S_3| = 1$ and $S_2 = \emptyset$.
Set $T' = T \setminus \{v_1, v_1', v_2\}$ (see Figure 9a). If T' contains a maximum free set A' with $v_3 \notin A'$ then $A' \cup \{v_1\}$ is free in T and thus

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(1 + \tilde{\alpha}(T')) \leq 2\tilde{\alpha}(T).$$

Therefore, we may assume that every maximum free set of T' contains v_3. As in Case 1f, let S be the set of neighbours of v_4 different from v_5 and for $v \in S$ define T_v to be the component of v in $T \setminus \{v_4\}$. Similarly to Claim 14, T_v has depth 2 as a tree rooted at v for every $v \in S$. If T_v is not isomorphic to T_{v_3} or to the graph in Case 1f (as rooted trees), by changing the longest path to go through v, we can continue as before. (Note that in all but the present case and Case 1f, we did not consider other neighbours of v_4).

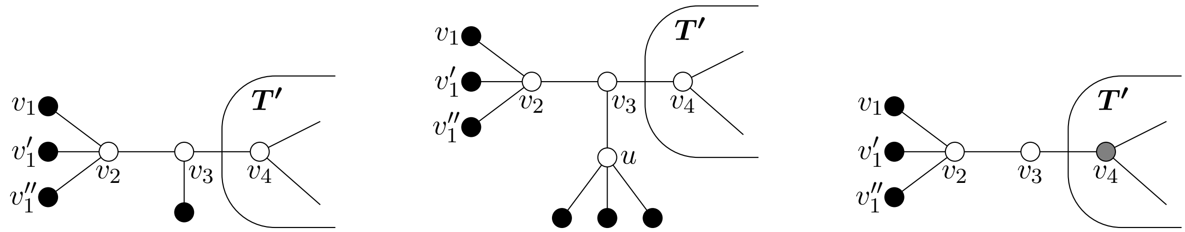
Set T'' to be the component of v_5 in $T \setminus \{v_4\}$ (see Figure 9b). R contains three vertices of T_v for every $v \in S$ and possibly it contains v_4 as well. Also $\tilde{\alpha}(T) \geq \tilde{\alpha}(T'') + 2|S|$, as we can add two vertices from each T_v to a free set of T'' to obtain a free set. Thus

$$\tilde{r}(T) \leq \tilde{r}(T'') + 3|S| + 1 \leq 2(\tilde{\alpha}(T'') + 2|S|) \leq 2\tilde{\alpha}(T).$$

Case 3: For every choice of a longest path $v_1, \ldots, v_m$ the connected component of v_4 in $T \setminus \{v_5\}$ has a leaf with 2 brothers in distance 3 from v_4.

We choose the longest path such that v_1 has 2 brothers v_1' and v_1''. Then $v_1, v_1', v_1'' \in R$, so $v_2, v_3 \notin R$.

Similarly to Cases 2a,2b, we can assume that the neighbours of v_3 other than v_4, have at most three neighbours different than v_3, all of which are leafs. By the choice of $v_1, \ldots, v_m$ as a longest path, the neighbours of v_3 other than v_4 are either leafs or have degree 4 and are neighbours to 3 leafs. Thus, as R is r.i., v_3 can have degree 2 or 3 only. Set T' to be the component of v_4 in $T \setminus \{v_3\}$. We consider seven possible cases.



(a) Case 3a (b) Case 3b (c) Case 3c

Figure 10: Cases 3a, 3b, 3c

(a) v_3 has degree 3 with the only neighbour other than v_2 and v_4 being a leaf (see Figure 10a). Then $v_4 \notin H_{T'}(R \cap V(T'))$. Note that $\tilde{\alpha}(T) \geq \tilde{\alpha}^*(T', v_4) + 2$, as we can add v_1 and v_2 to a free set in $V(T') \setminus \{v_4\}$ to obtain a free set. Thus

$$\tilde{r}(T) \leq \tilde{r}^*(T', v_4) + 4 \leq 2(\tilde{\alpha}^*(T', v_4) + 2) \leq 2\tilde{\alpha}(T).$$

(b) v_3 has degree 3 with the only neighbour other than v_2 and v_4, u, having three neighbours which are leafs (see Figure 10b).
Then again $v_4 \notin H_{T'}(R \cap V(T'))$, and $\tilde{\alpha}(T) \geq 3 + \tilde{\alpha}^*(T', v_4)$, as we can add v_1, v_2 and a leaf which is a neighbour of u to a free set of T'. Thus

$$\tilde{r}(T) \leq 6 + \tilde{r}^*(T', v_4) \leq 2(3 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}^*(T).$$

In the remaining cases we assume that v_3 has degree 2. Let T'' be the component of v_5 in $T \setminus \{v_4\}$.

(c) T' has a maximum free set A' with $v_4 \notin A'$ (see Figure 10c).
Then $A' \cup \{v_1, v_2\}$ is free and

$$\tilde{r}(T) \leq 3 + \tilde{r}(T') < 2(\tilde{\alpha}(T') + 2) \leq 2\tilde{\alpha}(T).$$

We may now assume that every maximum free set of T' contains v_4. Let S be the set of neighbours of v_4 other than v_3 and v_5.

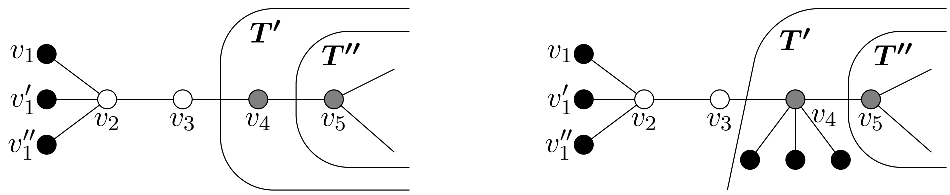
Claim 15. *The vertices in S are leafs in T.*

Proof. Let $v \in S$, and T_v the component of v in $T \setminus \{v_4\}$. Then T_v has depth at most 2 as a tree rooted in v (by the choice of $v_1, \ldots, v_m$ as a longest path). Let A' be a free set of maximum size in T', then $v_4 \in A'$. If v has a neighbour in T_v, u, it is either an endevertex, or all of its neighbours except for v are leafs. Then $(A' \setminus \{v_4\}) \cup \{u\}$ is free in T', a contradiction. Thus v has no neighbours in T_v, i.e. it is a leaf in T. □

Clearly, the claim implies $|S| \leq 3$.

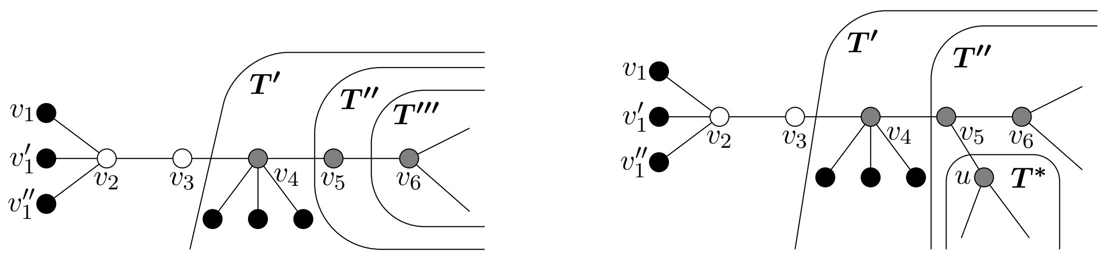
(d) $S = \emptyset$ (see Figure 11a).
Then $|R \cap (V(T) \setminus V(T''))| \leq 4$ and we can add v_1, v_2 to a free set of T''. Thus

$$\tilde{r}(T) \leq 4 + \tilde{r}(T'') \leq 2(2 + \tilde{\alpha}(T'')) \leq 2\tilde{\alpha}(T).$$



(a) Case 3d (b) Case 3e

Figure 11: Cases 3d, 3e



(a) Case 3f (b) Case 3g

Figure 12: Cases 3f, 3g

We can assume now that $S \neq \emptyset$.

(e) There is a free set A'' of maximum size in T'' with $v_5 \notin A''$ (see Figure 11b).
Then $|R \cap (V(T) \setminus V(T''))| \leq 6$ and the union of A'' with v_1, v_2 and a leaf from S is free, thus

$$\tilde{r}(T) \leq 6 + \tilde{r}(T'') \leq 2(3 + \tilde{\alpha}(T'')) \leq 2\tilde{\alpha}(T).$$

Thus we can assume that every largest free set in T'' contains v_5.

(f) v_5 has degree 2 in T.
Let T''' be the component of v_6 in $T \setminus \{v_5\}$ (see Figure 12a). Then, as in the previous case,

$$\tilde{r}(T) \leq 6 + \tilde{r}(T''') \leq 2(3 + \tilde{\alpha}(T''')) \leq 2\tilde{\alpha}(T).$$

(g) v_5 has a neighbour $u \neq v_4, v_6$.
Let T^* be the component of u in $T \setminus \{v_5\}$ (see Figure 12b). The following claim can be proved similarly to the proofs of Claims 14, 15, using the above assumptions.

Claim 16. *T^* has depth 3 as a tree rooted in u.*

By considering a longest path going through u instead of v_4, we can assume that the component of u in $T \setminus \{v_5\}$ satisfies the same conditions as the component of v_4. However, in this case T'' has a leaf in distance 2 from v_5, a contradiction to the assumption that every maximum free set of T'' contains v_5.

□

sharpness of Theorem 2

The following example shows that Theorem 2 is sharp.

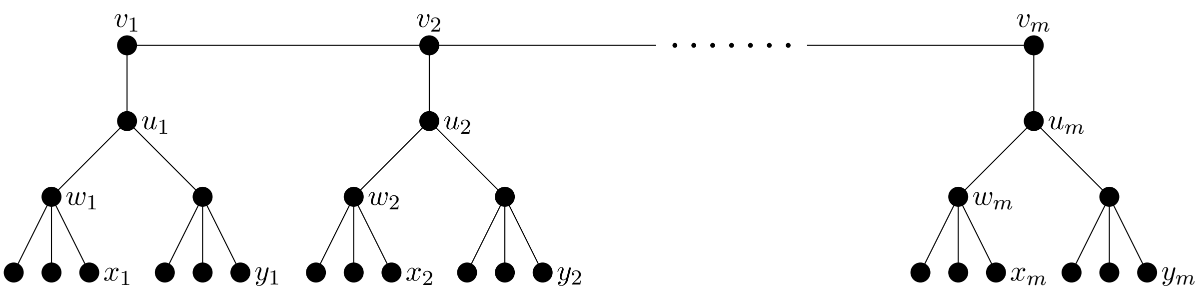


Figure 13: Sharpness of Theorem 2

This is sequence of trees T_m, $m \geq 1$, with $10m$ vertices. $\tilde{r}(T) \geq 6m$ (the set of all leafs is r.i.). Let A be a free set of T_m with maximum size. We can assume that A contains the leafs $x_1, \ldots, x_m, y_1, \ldots, y_m$. Thus $u_1, \ldots, u_m \notin A$. Also, A contains at most one of the 3 neighbours of u_i for each $i \in [m]$. Hence $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_m\} \cup \{w_1, \ldots, w_m\}$ is a free set of maximum, size, so $\tilde{\alpha}(T) = 3m$. By Theorem 2, $\tilde{r}(T) \leq 2\tilde{\alpha}(T) = 6m$. Thus $\tilde{r}(T) = 6m = 2\tilde{\alpha}(T)$.

4 Concluding Remarks

In this paper we proved two results about the Radon number for P_3-convexity in graphs. It may be interesting to consider these problems for general graphs. Regarding Theorem 1, it is still an open problem to determine whether Eckhoff's conjecture holds for P_3-convexity in all graphs. We showed that the inequality $\tilde{r}(G) \leq 2\tilde{\alpha}(G)$ from Theorem 2 does not hold for all graphs G, but it may still be the case that a similar but weaker inequality holds in general. Furthermore, for both results, it would be interesting to characterize the trees for which the results hold with equality.

Acknologdements

I would like to thank my supervisor Béla Bollobás for reading an early version of this paper and helping to greatly improve its presentation. I would also like to thank the anonymous referees for carefully reading this paper.

References

[1] B. Bukh, *Radon partitions in convexity spaces*, Preprint, arXiv:1009.2384.

[2] M. C. Dourado, D. Rautenbach, V. Fernandes dos Santos, P. M. Schäfer, J. L. Szwarcfiter, and A. Toman, *An upper bound on the P_3-Radon number*, Discrete Math. **312** (2012), no. 16, 2433–2437.

[3] J. Eckhoff, *The partition conjecture*, Discrete Math. **221** (2000), no. 1-3, 61–78.

[4] P. Erdős, E. Fried, A. Hajnal, and E. C. Milner, *Some remarks on simple tournaments*, Algebra Universalis **2** (1972), 238–245.

[5] P. Erdős, A. Hajnal, and E. C. Milner, *Simple one-point extensions of tournaments*, Mathematika **19** (1972), 57–62.

[6] R. E. Jamison-Waldner, *Partition numbers for trees and ordered sets*, Pacific J. Math. **96** (1981), no. 1, 115–140.

[7] J. W. Moon, *Embedding tournaments in simple tournaments*, Discrete Math. **2** (1972), no. 4, 389–395.

[8] J. Radon, *Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten*, Math. Ann. **83** (1921), no. 1-2, 113–115.

[9] H. Tverberg, *A generalization of Radon's theorem*, J. London Math. Soc. **41** (1966), 123–128.

[10] M. L. J. van de Vel, *Theory of convex structures*, North-Holland Publishing Co., Amsterdam, 1993.

[11] J. C. Varlet, *Convexity in tournaments*, Bull. Soc. Roy. Sci. Liège **45** (1976), no. 11-12, 570–586.