Minimum saturated families of sets

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Abstract

We call a family $\mathcal{F}$ of subsets of $[n]$ *s-saturated* if it contains no s pairwise disjoint sets, and moreover no set can be added to $\mathcal{F}$ while preserving this property (here $[n] = \{1, \ldots, n\}$).

More than 40 years ago, Erdős and Kleitman conjectured that an s-saturated family of subsets of $[n]$ has size at least $(1 - 2^{-(s-1)})2^n$. It is easy to show that every s-saturated family has size at least $\frac{1}{2} \cdot 2^n$, but, as was mentioned by Frankl and Tokushige, even obtaining a slightly better bound of $(1/2 + \varepsilon)2^n$, for some fixed $\varepsilon > 0$, seems difficult. In this note, we prove such a result, showing that every s-saturated family of subsets of $[n]$ has size at least $(1 - 1/s)2^n$.

This lower bound is a consequence of a multipartite version of the problem, in which we seek a lower bound on $|\mathcal{F}_1| + \ldots + |\mathcal{F}_s|$ where $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are families of subsets of $[n]$, such that there are no s pairwise disjoint sets, one from each family $\mathcal{F}_i$, and furthermore no set can be added to any of the families while preserving this property. We show that $|\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \geq (s-1) \cdot 2^n$, which is tight e.g. by taking $\mathcal{F}_1$ to be empty, and letting the remaining families be the families of all subsets of $[n]$.

1 Introduction

In extremal set theory, one studies how large, or how small, a family $\mathcal{F}$ can be, if $\mathcal{F}$ consists of subsets of some set and satisfies certain restrictions. Let $[n] = \{1, \ldots, n\}$, let $2^{[n]}$ be the family of all subsets of $[n]$ and let $[n]^{(k)}$ be the family of subsets of $[n]$ of size k.

A classical example in the area is the study of *intersecting families*. We say that a family $\mathcal{F}$ is *intersecting* if for every $A, B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$. The following simple proposition, first noted by Erdős, Ko and Rado [9], gives an upper bound on the size of an intersecting family in $2^{[n]}$.

Proposition 1. *Let $\mathcal{F} \subseteq 2^{[n]}$ be intersecting, then $|\mathcal{F}| \leq 2^{n-1}$.*

This follows from the observation that for every set $A \subseteq [n]$ at most one of A and $\overline{A}$ (where $\overline{A} := [n] \setminus A$) is in $\mathcal{F}$. This bound is tight, which can be seen, e.g., by taking the family of all subsets of $[n]$ that contain the element 1. In fact, there are many more extremal examples (see [7]), partly due to the following proposition.

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Proposition 2. *Let $\mathcal{F} \subseteq 2^{[n]}$ be intersecting, then there is an intersecting family in $2^{[n]}$ of size 2^{n-1} that contains $\mathcal{F}$. In other words, if $\mathcal{F} \subseteq 2^{[n]}$ is a maximal intersecting family, then it has size 2^{n-1}.*

Indeed, suppose that $\mathcal{F}$ is a maximal intersecting family of size less than 2^{n-1}. Then there is a set $A \subseteq [n]$ such that $A, \overline{A} \notin \mathcal{F}$. By maximality of $\mathcal{F}$, there exist sets $B, C \in \mathcal{F}$ such that $A \cap B = \emptyset$ and $\overline{A} \cap C = \emptyset$. In particular, $B \cap C = \emptyset$, a contradiction.

There have been numerous extensions and variations of Proposition 1. For example, the study of t-intersecting families [15] (where the intersection of every two sets has size at least t) and L-intersecting families [3] (where the size of the intersection of every two distinct sets lies in some set of integers L). Such problems were also studied for k-uniform families, i.e. families that are subsets of $[n]^{(k)}$ (see e.g. [2] and [16]). A famous example is the Erdős-Ko-Rado [9] theorem which states that if $\mathcal{F} \subseteq [n]^{(k)}$ is intersecting, and $n \geq 2k$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$, a bound which is again tight by taking the families of all sets containing 1. Another interesting generalisation of Proposition 2 looks for the maximum measure of an intersecting family under the p-biased product measure (see [1, 6, 13, 10]). A different direction, which was suggested by Simonovits and Sós [18], studies the size of intersecting families of structured families, such as graphs, permutations and sets of integers (see e.g. [5, 14]).

Here we are interested in a different extension of Propositions 1 and 2. Given $s \geq 2$, we say that a family $\mathcal{F} \subseteq 2^{[n]}$ is *s-saturated* if $\mathcal{F}$ contains no s pairwise disjoint sets, and furthermore $\mathcal{F}$ is maximal with respect to this property. An example for an s-saturated family is the set of all subsets of $[n]$ that have a non-empty intersection with $[s-1]$. In 1974 Erdős and Kleitman [8] made the following conjecture, which states that this example is the smallest s-saturated family in $2^{[n]}$.

Conjecture 3 (Erdős, Kleitman [8])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be s-saturated. Then $|\mathcal{F}| \geq (1 - 2^{-(s-1)}) \cdot 2^n$.*

Note that by Proposition 2, Conjecture 3 holds for $s = 2$. Given a family $\mathcal{F} \subseteq 2^{[n]}$, define $\mathcal{F}^C = 2^{[n]} \setminus \mathcal{F}$, and $\overline{\mathcal{F}} = \{\overline{A} : A \in \mathcal{F}\}$. Then for every $s \geq 2$, if $\mathcal{F} \subseteq 2^{[n]}$ is s-saturated then $\overline{\mathcal{F}^C}$ is intersecting. Indeed, if $A \notin \mathcal{F}$ then $\overline{A}$ contains $s-1$ pairwise disjoint sets of $\mathcal{F}$, so if A and B are such that $\overline{A}$ and $\overline{B}$ are disjoint, then at least one of A and B is in $\mathcal{F}$, as otherwise $\mathcal{F}$ contains $2(s-1) \geq s$ pairwise disjoint sets, a contradiction. By Proposition 1, it follows that if $\mathcal{F}$ is s-saturated then $|\mathcal{F}| \geq 2^{n-1}$. Surprisingly, beyond this trivial lower bound, nothing was known. Moreover, Frankl and Tokushige [12] wrote in their recent survey that obtaining a lower bound of $(1/2 + \varepsilon)2^n$, i.e. a modest improvement over the trivial bound, is a challenging open problem. In this paper we prove such a result.

Theorem 4. *Let $\mathcal{F} \subseteq 2^{[n]}$ be s-saturated, where $s \geq 2$. Then $|\mathcal{F}| \geq (1 - 1/s)2^n$.*

In fact, Theorem 4 is a corollary of a multipartite version of the above problem. A sequence of s families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subseteq 2^{[n]}$ is called *cross dependant* (see, e.g., [11]) if there is no choice of s sets $A_i \in \mathcal{F}_i$, for $i \in [s]$, such that $A_1, \ldots, A_s$ are pairwise disjoint. We call a sequence of s families $\mathcal{F}_1, \ldots, \mathcal{F}_s$ *cross saturated* if the sequence is cross dependant and is maximal with respect to this property, i.e. the addition of any set to any of the families results in a sequence which is not cross dependant. Our aim here is to obtain a lower bound on the $|\mathcal{F}_1| + \ldots + |\mathcal{F}_s|$. Note that if $\mathcal{F}$ is s-saturated then the sequence given by $\mathcal{F}_1 = \ldots = \mathcal{F}_s = \mathcal{F}$ is cross saturated. Hence, a lower bound on the sum of sizes of a cross saturated sequence of s families implies a lower bound on the size of an s-saturated family.

A simple example of a cross saturated sequence $\mathcal{F}_1, \ldots, \mathcal{F}_s$ can be obtained by taking $\mathcal{F}_1$ to be empty, and letting all other sets be $2^{[n]}$. This construction is a special case of a more general family of examples which we believe contains all extremal examples; we discuss this in Section 3. Our next result shows that this example is indeed a smallest example for a cross saturated sequence. Furthermore, it implies Theorem 4 by taking $\mathcal{F}_1 = \ldots = \mathcal{F}_s = \mathcal{F}$.

Theorem 5. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_s \subseteq 2^{[n]}$ be cross saturated. Then $|\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \geq (s-1)2^n$.*

We have two different approaches to this problem, each of which can be used to prove Theorem 5. As the proofs are short, and Conjecture 3 is still open, we feel that there is merit in presenting both proofs here in hope that they would give rise to further progress on Conjecture 3.

Our first approach makes use of an interesting connection to correlation inequalities. Let us start by defining the *disjoint occurrence* of two families. Given subsets $A, I \subseteq [n]$, let

$$\mathcal{C}(I, A) = \{S \subseteq [n] : S \cap I = A \cap I\}.$$

The *disjoint occurrence* of two families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ is defined by

$$\mathcal{A} \,\square\, \mathcal{B} := \{A : \exists \textit{ disjoint } \text{sets } I, J \subseteq [n] \text{ s.t. } \mathcal{C}(I, A) \subset \mathcal{A} \text{ and } \mathcal{C}(J, A) \subset \mathcal{B}\}.$$

Note that when $\mathcal{A}$ and $\mathcal{B}$ are both increasing families (i.e. if $A \in \mathcal{A}$, and $A \subseteq B \subseteq [n]$ then $B \in \mathcal{A}$), $\mathcal{A} \,\square\, \mathcal{B}$ is the set of all subsets of $[n]$ which can be written as a disjoint union of a set from $\mathcal{A}$ and a set from $\mathcal{B}$. This notion of disjoint occurrence appears naturally in the study of percolation. Using it, one can express the probability that there are two edge-disjoint paths between two sets of vertices in a random subgraph, chosen uniformly at random, of a given graph.

Van den Berg and Kesten [4] proved that $|\mathcal{A} \,\square\, \mathcal{B}| \leq |\mathcal{A}||\mathcal{B}|/2^n$ for increasing families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ and conjectured that this inequality should hold for general families. This was proved by Reimer [17] in a ground breaking paper and is currently known as the van den Berg-Kesten-Reimer inequality.

Disjoint occurrence is surprisingly suitable for the study of saturated families. For example, if $\mathcal{F}$ is 3-saturated then it is easy to see that $\mathcal{F}$ is increasing, so $\mathcal{F} \,\square\, \mathcal{F}$ is the family of sets that are disjoint unions of two sets from $\mathcal{F}$, which is exactly the family $\overline{\mathcal{F}^C}$. This observation alone implies an improved lower bound on $|\mathcal{F}|$ using the van den Berg-Kesten-Reimer inequality. We obtain a better bound using a variant of this inequality, which was first observed by Talagrand [19], and later played a major role in Reimer's proof of the van den Berg-Kesten-Reimer inequality in full generality.

Our second approach is algebraic: we define a polynomial for each set in a certain family related to $\mathcal{F}_1, \ldots, \mathcal{F}_s$, and show that these polynomials are linearly independent, thus implying that the family is not very large.

2 The proof

Before turning to the first proof of Theorem 5, we introduce the correlation inequality that we will need. We present its short proof for the sake of completeness.

Lemma 6 (Talagrand [19])**.** *Let $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ be increasing families. Then $|\mathcal{A} \,\square\, \mathcal{B}| \leq \left|\overline{\mathcal{A}} \cap \mathcal{B}\right|$.*

Remark. Before turning to the proof of Lemma 6, we remark that the statement of Lemma 6 holds even without the assumption that the families $\mathcal{A}$ and $\mathcal{B}$ are increasing. Furthermore, an equivalent version of this played a major role in Reimer's proof [17] of the van den Berg-Kesten-Reimer inequality.

Proof. We prove the statement by induction on n. It is easy to check it for $n = 1$. Let $n > 1$ and suppose that the statement holds for $n-1$. Given a family $\mathcal{F} \subseteq 2^{[n]}$, denote by $\mathcal{F}_0$ the family of sets in $\mathcal{F}$ that do not contain the element n, and let $\mathcal{F}_1 = \{A \subseteq [n-1] : A \cup \{n\} \in \mathcal{F}\}$. In particular, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq 2^{[n-1]}$ when $\mathcal{F}$ is an increasing family.

We have

$$\begin{aligned} |\mathcal{A} \,\square\, \mathcal{B}| &= |(\mathcal{A} \,\square\, \mathcal{B})_0| + |(\mathcal{A} \,\square\, \mathcal{B})_1| \\ &= |\mathcal{A}_0 \,\square\, \mathcal{B}_0| + |\mathcal{A}_1 \,\square\, \mathcal{B}_0| + |\mathcal{A}_0 \,\square\, \mathcal{B}_1| - |(\mathcal{A}_1 \,\square\, \mathcal{B}_0) \cap (\mathcal{A}_0 \,\square\, \mathcal{B}_1)| \\ &\le |\mathcal{A}_1 \,\square\, \mathcal{B}_0| + |\mathcal{A}_0 \,\square\, \mathcal{B}_1| \\ &\le \left|\overline{\mathcal{A}_1} \cap \mathcal{B}_0\right| + \left|\overline{\mathcal{A}_0} \cap \mathcal{B}_1\right| \\ &= \left|(\overline{\mathcal{A}} \cap \mathcal{B})_0\right| + \left|(\overline{\mathcal{A}} \cap \mathcal{B})_1\right| \\ &= \left|\overline{\mathcal{A}} \cap \mathcal{B}\right|, \end{aligned}$$

where the first inequality holds because $\mathcal{A}_0 \,\square\, \mathcal{B}_0 \subseteq (\mathcal{A}_1 \,\square\, \mathcal{B}_0) \cap (\mathcal{A}_0 \,\square\, \mathcal{B}_1)$, and the second one follows by induction. □

We are now ready for the first proof of Theorem 5.

First proof of Theorem 5. Let $\mathcal{F}_1, \ldots, \mathcal{F}_s$ be cross saturated, where $s \ge 2$. Note that

$$\overline{\mathcal{F}_i{}^C} = \mathcal{F}_1 \,\square\, \ldots \,\square\, \mathcal{F}_{i-1} \,\square\, \mathcal{F}_{i+1} \,\square\, \ldots \,\square\, \mathcal{F}_s. \tag{1}$$

Indeed, for every $A \notin \mathcal{F}_i$, $\overline{A}$ contains a disjoint union of sets from $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}, \mathcal{F}_{i+1}, \ldots, \mathcal{F}_s$ and, conversely, any $A \in \mathcal{F}_1 \,\square\, \ldots \,\square\, \mathcal{F}_{i-1} \,\square\, \mathcal{F}_{i+1} \,\square\, \ldots \,\square\, \mathcal{F}_s$ cannot be in $\mathcal{F}_i$ by cross dependence. By Lemma 6, the following holds for every $i \ge 2$.

$$\begin{aligned} \left|\mathcal{F}_i{}^C\right| &= \left|\overline{\mathcal{F}_i{}^C}\right| \\ &= |\mathcal{F}_1 \,\square\, \ldots \,\square\, \mathcal{F}_{i-1} \,\square\, \mathcal{F}_{i+1} \,\square\, \ldots \,\square\, \mathcal{F}_s| \\ &\le \left|(\mathcal{F}_1 \,\square\, \ldots \,\square\, \mathcal{F}_{i-1}) \cap \overline{(\mathcal{F}_{i+1} \,\square\, \ldots \,\square\, \mathcal{F}_s)}\right|. \end{aligned} \tag{2}$$

Denote $\mathcal{G}_1 = \mathcal{F}_1{}^C$, and $\mathcal{G}_i = (\mathcal{F}_1 \,\square\, \ldots \,\square\, \mathcal{F}_{i-1}) \cap \overline{(\mathcal{F}_{i+1} \,\square\, \ldots \,\square\, \mathcal{F}_s)}$ for $i \ge 2$.

Claim 7. *$\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ for $1 \le i < j \le s$.*

Proof. Indeed, if $i = 1$ then $\mathcal{G}_1 \subseteq \mathcal{F}_1{}^C$ and $\mathcal{G}_j \subseteq \mathcal{F}_1$. Otherwise, if $A \in \mathcal{G}_i \cap \mathcal{G}_j$ with $2 \le i < j$ then A is the disjoint union of elements from $\mathcal{F}_1, \ldots, \mathcal{F}_{j-1}$, so in particular (as the sets $\mathcal{F}_l$ are increasing) it is the disjoint union of elements from $\mathcal{F}_1, \ldots, \mathcal{F}_i$. Furthermore, since $i \ge 2$, A is also the complement (with respect to $[n]$) of a disjoint union of sets in $\mathcal{F}_{i+1}, \ldots, \mathcal{F}_s$, i.e. $\overline{A}$ is the disjoint union of sets in

$\mathcal{F}_{i+1}, \ldots, \mathcal{F}_s$. But this means that $[n]$ is the disjoint union of sets from $\mathcal{F}_1, \ldots, \mathcal{F}_s$, a contradiction to the assumption that $\mathcal{F}_1, \ldots, \mathcal{F}_s$ form a cross saturated sequence. $\square$

It follows from (1), (2) and Claim 7 that

$$\begin{aligned} |\mathcal{F}_1| + \ldots + |\mathcal{F}_s| &= s \cdot 2^n - (|{\mathcal{F}_1}^C| + \ldots + |{\mathcal{F}_s}^C|) \\ &\geq s \cdot 2^n - (|\mathcal{G}_1| + \ldots + |\mathcal{G}_s|) \\ &\geq s \cdot 2^n - 2^n = (s-1)2^n, \end{aligned} \tag{3}$$

thus completing the proof of Theorem 5. $\square$

Our next approach is algebraic. Before presenting the proof, we introduce some definitions and an easy lemma. Let n be fixed and consider the vector space V (over $\mathbb{R}$) of functions from $\{0,1\}^n$ to $\mathbb{R}$. Note that this is a vector space of dimension 2^n. Given a subset $S \subseteq [n]$, let $P_S : \{0,1\}^n \to \mathbb{R}$ be defined by $P_S(x) = \prod_{i \in S} x_i$, where $x = (x_1, \ldots, x_n)^T \in \{0,1\}^n$, and let $x_S \in \{0,1\}^n$ be defined by $(x_S)_i = 1$ if and only if $i \in S$. The following lemma shows that $\{P_S : S \subseteq [n]\}$ is a linearly independent set in V (in fact, as V has dimension 2^n, it is a basis).

Lemma 8. *The set $\{P_S : S \subseteq [n]\}$ is linearly independent in V.*

Proof. Suppose that $\sum_{S \subseteq [n]} \alpha_S P_S = 0$, where $\alpha_S \in \mathbb{R}$, and not all α_S's are 0. Let T be a smallest set such that $\alpha_T \neq 0$. Note that $P_S(x_T) = 1$ if and only if $S \subseteq T$. Hence

$$0 = \sum_{S \subseteq [n]} \alpha_S P_S(x_T) = \sum_{S \subseteq [n], |S| \leq |T|} \alpha_S P_S(x_T) = \alpha_T,$$

a contradiction to the assumption that $\alpha_T \neq 0$. It follows that $\alpha_S = 0$ for every $S \subseteq [n]$, i.e. the polynomials $\{P_S(x) : S \subseteq [n]\}$ are linearly independent, as required. $\square$

We shall use the inner product on V which is defined by

$$\langle f, g \rangle = \sum_{x \in \{0,1\}^n} f(x) g(x). \tag{4}$$

It is easy to check that this is indeed an inner product; in fact, it is the standard inner product, if functions are viewed as vectors indexed by $\{0,1\}^n$.

We are now ready for the second proof of Theorem 5.

Second proof of Theorem 5. Let $\mathcal{F}_1, \ldots, \mathcal{F}_s$ be cross saturated, where $s \geq 2$. Given i and $A \in \overline{{\mathcal{F}_i}^C}$, recall that by (1), A can be written as the disjoint union of sets from $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}, \mathcal{F}_{i+1}, \ldots, \mathcal{F}_n$. For every such i and A, fix a representation

$$A = B \cup C, \tag{5}$$

where B is a disjoint union of sets from $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}$ and C is a disjoint union of sets from $\mathcal{F}_{i+1}, \ldots, \mathcal{F}_s$. Let

$$Q_{i,A}(x) = \prod_{j \in B} x_j \cdot \prod_{j \in C} (x_j - 1).$$

Let W_i be the family of polynomials $Q_{i,A}$, where $i \in [s]$ and $A \subseteq \overline{\mathcal{F}_i{}^C}$.

We shall show that the sets W_i are pairwise disjoint and that $W_1 \cup \ldots \cup W_s$ is linearly independent. This will follow from the following two claims, which state that each W_i is linearly independent and that W_i and W_j are orthogonal for distinct i and j.

Claim 9. *W_i is linearly independent for $i \in [s]$.*

Proof. Suppose that $\sum_{A \in \overline{\mathcal{F}_i{}^C}} \alpha_A Q_{i,A} = 0$, where $\alpha_A \in \mathbb{R}$ and not all α_A's are 0. Let A be a largest set such that $\alpha_A \neq 0$. Note that for every $A' \in \overline{\mathcal{F}_i{}^C}$, $Q_{i,A'}$ can be written as

$$Q_{i,A'} = P_{A'} + \sum_{S \subsetneq A'} \beta_{A',S} P_S,$$

where the values of $\beta_{A',S}$ depend on the representation of A' as in (5). Hence, by choice of A,

$$\begin{aligned} 0 &= \sum_{A' \in \overline{\mathcal{F}_i{}^C},\, |A'| \leq |A|} \alpha_{A'} Q_{i,A'} \\ &= \sum_{A' \in \overline{\mathcal{F}_i{}^C},\, |A'| \leq |A|} \alpha_{A'} (P_{A'} + \sum_{S \subsetneq A'} \beta_{A',S} P_S) \\ &= \alpha_A P_A + \sum_{|S| \leq |A|,\, S \neq A} \gamma_S P_S, \end{aligned}$$

for some $\gamma_S \in \mathbb{R}$. However, since the P_S's are linearly independent (by Lemma 8), we have $\alpha_A = 0$, a contradiction. It follows that W_i is linearly independent, as required. □

Claim 10. *W_i and W_j are orthogonal for $1 \leq i < j \leq s$.*

Proof. Let $A \in \overline{\mathcal{F}_i{}^C}$ and $A' \in \overline{\mathcal{F}_j{}^C}$, where $1 \leq i < j \leq s$. Write $A = B \cup C$ and $A' = B' \cup C'$ for the representations as in (5). Let $x \in \{0,1\}^n$. We claim that $Q_{i,A}(x) = 0$ or $Q_{j,A'}(x) = 0$. Indeed, if the former does not hold, then $x_i = 1$ for $i \in B$ and $x_i = 0$ for $i \in C$. Note that $B' \cap C \neq \emptyset$, because $\{\mathcal{F}_1, \ldots, \mathcal{F}_s\}$ is cross dependant. Hence, $x_i = 0$ for some $i \in B'$, which implies that $Q_{i,A'}(x) = 0$, as claimed. It easily follows that $\langle Q_{i,A}, Q_{j,A'} \rangle = 0$ (recall the definition of the inner product given in (4)), as required. □

It follows from Claims 9 and 10 that $W_1 \cup \ldots \cup W_s$ is linearly independent, hence it has size at most the dimension of V, i.e. at most 2^n. But $|W_i| = |\mathcal{F}_i{}^C|$, thus, as in (3)

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \geq (s-1)2^n,$$

as desired. □

3 Conclusion

There are two main directions for further research that we would like to mention here.

The first is related to the tightness of Theorem 5. As mentioned in the introduction, the result is tight, which can be seen by taking $\mathcal{F}_1 = \emptyset$ and $\mathcal{F}_2 = \ldots = \mathcal{F}_s = 2^{[n]}$. In fact, this is a special case of the following class of examples: let $\mathcal{F}_1$ be any increasing family in $2^{[n]}$, let $\mathcal{F}_2 = \overline{\mathcal{F}_1{}^C}$ and let $\mathcal{F}_3 = \ldots = \mathcal{F}_s = 2^{[n]}$. Then $|\mathcal{F}_1| + |\mathcal{F}_2| = 2^n$ and it is easy to check that any set in $\mathcal{F}_1$ intersect every set in $\mathcal{F}_2$. Therefore, every such example yields a cross saturated set of smallest size. Furthermore, it is easy to see that these are the only examples for which $\mathcal{F}_3 = \ldots = \mathcal{F}_s$. It seems plausible that these are the only possible examples (up to permuting the order of the families). This problem of classifying all extremal examples, interesting in its own right, may give a hint on how to further improve the lower bound of the size of s-saturated families.

The second, and seemingly more challenging direction, is to improve on Theorem 4. We proved that if $\mathcal{F}$ is s-saturated then $|\mathcal{F}| \geq (1-1/s)2^n$, where the conjectured bound is $\left(1 - 2^{-(s-1)}\right) 2^n$. We note that it is possible to improve the lower bound slightly, to show that $|\mathcal{F}| \geq (1 - 1/s + \Omega(\log n/n))\, 2^n$, by running the argument of the first proof more carefully in the case where $\mathcal{F}_1 = \ldots = \mathcal{F}_s = \mathcal{F}$; we omit further details. It would be very interesting to obtain an improvement of error term $1/s$ to an expression exponential in s. We hope that our methods can be used to make further progress on this old conjecture.

Let us mention here a general class of examples of s-saturated families whose size is $\left(1 - 2^{-(s-1)}\right) 2^n$. We do not know of any other examples of s-saturated families, and feel that it is likely that if the conjecture holds, then these are the only extremal examples.

Example 11. *Given $s \geq 2$, let $\{I_1, \ldots, I_{s-1}\}$ be a partition of $[n]$. For each $i \in [s-1]$, pick a maximal intersecting family $\mathcal{F}_i$ of subsets of I_i; in particular, by Proposition 2, $|\mathcal{F}_i| = 2^{|I_i|-1}$. Define $\mathcal{F}$ as follows.*

$$\mathcal{F} = \{A \subseteq [n] : A \cap I_i \in \mathcal{F}_i \text{ for some } i \in [s-1].\}$$

It is easy to check that $\mathcal{F}$ is s-saturated as a family of subsets of $[n]$ and that it has size $\left(1 - 2^{-(s-1)}\right) 2^n$.

Note that this class of examples contains the example that was mentioned earlier, of the family of subsets of $[n]$ that intersect $[s-1]$.

Finally, we note the following interesting phenomenon.

Proposition 12. *If Conjecture 3 holds for $s+1$, then it holds for s.*

Indeed, suppose that Conjecture 3 holds for $s+1$, and let $\mathcal{F} \subseteq 2^{[n]}$ be s-saturated. Define $\mathcal{G} \subseteq 2^{[n+1]}$ as follows.

$$\mathcal{G} = \mathcal{F} \cup \{A \subseteq [n+1] : n+1 \in A\}.$$

Note that $\mathcal{G}$ is $(s+1)$-saturated (as a subset of $2^{[n+1]}$). Hence, by the assumption that the conjecture holds for $s+1$, we find that $|\mathcal{G}| \geq (1 - 2^{-s})\, 2^{n+1}$. Note also that $|\mathcal{G}| = |\mathcal{F}| + 2^n$. It follows that $|\mathcal{F}| \geq (1 - 2^{-s})\, 2^{n+1} - 2^n = \left(1 - 2^{-(s-1)}\right) 2^n$, as required.

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