Frege’s Logicism: Getting an insight into what we Grasp.

An Inquiry into the Intended Relation between the Content of Our Arithmetic, and the Content of Frege’s Logically Perfect Language.

A Thesis for the Degree of Doctor of Philosophy

by

M. R. J. Jennings

Philosophy Department
University College London
University of London
Abstract

This thesis is an inquiry into whether Frege intended his logicist reduction to be an explication of the content of ordinary arithmetic, or whether he conceived of it as a replacement, or indeed as something in between those two possibilities. In other words, did Frege seek to capture in Concept Script the Sinn and Bedeutung of our ordinary arithmetical language? Or did he seek to capture none, or just one of those things—Bedeutung? On the face of it, Frege himself is not altogether transparent on the matter: he appears at best unclear, and at worst inconsistent.

I argue that Frege sought to uncover both the Sinn and Bedeutung of our arithmetical language. This in itself is not a new thesis, though few subscribe to it. Those who do subscribe to it argue that, for Frege, we have a defective or partial grasp of the content of ordinary arithmetic, which deficiency Frege sought to make good by constructing a logically perfect language in which the full contents of ordinary arithmetic are revealed. I reject the view that our grasp is defective, this on exegetical and non-exegetical grounds. Distinguishing between defective understanding, and defective grasp, I claim that Frege took our grasp to be full, if to some extent tacit and unperspicuous.

The methodology of my inquiry lies in the main on a reappraisal of Frege's text. But it lies also on the fruitfulness of the claim that the doctrine of the recovery of tacit grasp informs Frege's logicism. Not only does this doctrine help to make plausible the above claim regarding the relation between our arithmetic and Concept Script arithmetic. It helps to resolve other issues, which arise in the course of arguing for that claim. These other issues are of an independent interest. One such issue concerns what Frege meant by the claim that, by means of fruitful definitions, analytic truths can extend our knowledge, and how this relates to the notion of carving up a proposition already grasped. Another issue concerns whether and in what sense we should regard Frege's logicism as epistemologically motivated.
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# Abstract


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Introduction

§1. Frege held the logicist thesis that arithmetical truths are derived logical truths. He believed that by means of definitions of the central concepts of arithmetic,¹ presented in purely logical terms, and by means of a small number of logical axioms, and logical rules of inference, he could show that the simplest truths of arithmetic are theorems of logic.² To this end he constructed a logically perfect language: his Concept Script. By its means he attempted to exhibit a logically perfect proof: a system in which each inferential step would clearly be shown to be gapless, and in which the dependencies and linkages between truths would be transparent. Thereby, Frege believed, would the nature of the truths, upon which arithmetic depends, be shown beyond doubt to be of logic alone.

An important, but neglected, question regarding this project concerns Frege’s understanding of the relation between the content of our ordinary arithmetical language, and the content of the Concept Script. At first sight, Frege’s intentions appear to be unclear. For instance, passages regarding the nature of his definitions will strike the reader now as suggesting that definitions were intended to be stipulations of a quite different content, now as suggesting that they were intended to be explications, then again as suggesting that they were meant to be a curious admixture of the two. Preoccupied with the task of constructing and executing the crucial proofs that would vindicate his logicist claim, Frege neglected to be clear, and one might say consistent, about this important question. Nevertheless, by closely examining his texts, I believe that we can unearth a fairly definitive answer.

The question is important because an answer to it would allow us properly to assess the significance that the success of Frege’s logicism would have had. It would be informative to be shown that the content of our ordinary arithmetical terms were of a purely logical nature. It would explain what makes it possible for our ordinary arithmetical practice to have the widest possible application: as logic applies to all thought, so number applies to all that is

¹ Frege sometimes speaks of defining with respect to both content and their expressions. I shall sometimes do so too.
² He failed of course, because axiom V helped to generate an inconsistency.
thinkable—everything thinkable being countable (vagueness of concept terms notwithstanding: see below). And of course it would be a revelation of no inconsiderable magnitude to be shown that what we ordinarily refer to in using our arithmetical terms are abstract, logical objects. Moreover, if Frege’s logicism had succeeded, and if he had intended it to be a revelation about the contents of our minds—our beliefs and judgements—then a further significance would have lain in the possibility of explaining how our arithmetic could be known *a priori*. This account would have flowed from our having gained a clear insight into our arithmetical thoughts. For if those thoughts were of a purely logical nature, it might appear that our judgements would be justified in virtue of our having a clear grasp of their contents. Their self-evidence would provide us with the resources for an immediate recognition of their truth. In summary, the explanatory value of Frege’s logicism would, in the first instance, turn on the light it would cast on what we ordinarily grasp.

But many commentators there are who would discount this—hereafter ‘hermeneutic’—view. They would deny that Frege sought to disclose part or all of what, in virtue of our arithmetical practice, is already in our epistemic possession. Instead, it is claimed that Frege intended Concept Script to represent a radical reconstruction of the content of our ordinary arithmetic—hereafter the ‘revolutionary’ view.

I have so far been using ‘content’ to mean any one or both of the categories deployed by Frege after 1891: namely *Sinn* and *Bedeutung*. For Frege, *Bedeutung* is the denotation of a term, supposing it to have a denotation at all, which he took to be either an object or function. He characterizes the latter two categories as complete (saturated), and incomplete (unsaturated) respectively, whose corresponding terms he likewise characterizes as complete (proper names, including sentences) and incomplete (function, including concepts and relation expressions). The *Sinn* of a sentence is a thought or proposition. The *Sinne* of non-sentential terms are ‘parts’ of the thought.³ Frege maintains that we apprehend the realm of *Bedeutung* only by means of grasping *Sinn*—specifically, by means of grasping thoughts. *Sinne* are possible ways of

³ There is a controversy in Frege scholarship regarding how Frege should be understood regarding the notion of a thought having parts. We shall touch on this in chapters four and five. For the
regarding the Bedeutung; and as there are numerous ways of regarding a Bedeutung, so there are different Sinne corresponding to the same Bedeutung.

The revolutionary hypothesis is the claim that neither the Sinn nor the Bedeutung of ordinary arithmetical terms—if such they have—is preserved in Frege's Concept Script. The hermeneutic hypothesis is the contrary claim: Sinn or Bedeutung was preserved. Prior to 1891, Frege had used the term 'judgeable content', 'conceptual content', or just 'content' without the Sinn-Bedeutung distinction firmly, if at all, in mind. He appears to have used the first two terms to mean states of affairs, and the latter term to mean the parts of these states of affairs. Rather than discuss this early period in terms of whether Frege sought to preserve the contents of ordinary arithmetical language, I shall inquire whether, on looking back at his early works, having drawn the formal distinction, Frege would see himself as having uncovered the Sinn or Bedeutung of our terms.

Clearly, if the content of Concept Script is not that of ordinary arithmetical language, then it is not of our arithmetic that we speak when our subject matter is Concept Script arithmetic. On one version of the revolutionary view, our grasp of the content of ordinary arithmetical language is seriously defective. We think that what we grasp is determinate, that our subsentential terms refer, and that our sentences have truth-value, but in this we are deluded—revisionism. This is not to say that, so construed, Frege's logicism would have been without explanatory value. But it would not have been explanatory of our arithmetical practices. Frege would perhaps have shown that, contrary to Kant, an arithmetic modeled on our own practices, is analytic a priori. And he would be able to explain its universality. But he would not have enlightened us about what was already in our epistemic possession.

One consideration in favour of the revolutionary hypothesis is that unless more is said, the hermeneutic interpretation would be counter-intuitive. This is because the 'content' of the Concept Script, and that of ordinary arithmetic,

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5 Where I use 'content' in this context, I shall mean Sinn or Bedeutung, or both.
appear to be radically different from each other. Certainly, on reflecting on what I grasp when engaged in ordinary arithmetical practice, I do not immediately recognize that its content is the same as that found in Frege's logically perfect language. Yet there are many passages in the latter's work that can be read in a way that would support hermeneutism. Although the revolutionary exponent realizes this, he will point out that the same passages can be reconstrued in favour of his interpretation. Be that as it may, even supposing there is an indeterminacy regarding textual evidence, it might be said that we should appeal to the most charitable interpretation; we should attribute to Frege the least counter-intuitive reading of his logicist project. In that case, it might be said that appeal to how things appear to the thinker would, surely, count in favour of the revolutionary hypothesis. In addition, one might appeal to Frege's frequent remark that we have a defective understanding of even the simplest arithmetical truths. On the revisionist view of this remark, the reason why we do not recognize that the thoughts contained in the Concept Script are our ordinary arithmetical thoughts is because, when we contemplate what is within our ken, we find that there is nothing precisely there.

§2. There are a number of versions of the hermeneutic hypothesis. One such concedes that the Sinn of ordinary arithmetical language is different from that of Concept Script; but it argues that, nevertheless, Frege sought to preserve the Bedeutung—'weak hermeneutism'. Another variant, inspired no doubt by the alleged counter-intuitiveness mentioned above, holds that, although Sinn and Bedeutung are preserved, our grasp of the Sinn is defective. We have only a partial grasp of what is captured in Concept Script, and it is for this reason that we are unable to recognize sameness of both Sinn and Bedeutung—'mild hermeneutism'.

In the following chapters, I argue that, despite its initial plausibility, the revolutionary hypothesis is the less convincing position. But so too, I argue, are the above two variants of the hermeneutic hypothesis. In particular, advocates of the views outlined so far overlook an important thesis that appears to underlie Frege's work, one which helps to deflect the alleged counter-intuitiveness that seems, at first sight, to bode ill for the hermeneutic interpretation. Central to this
underlying thesis is the notion to tacit grasp. A stronger hermeneutic reading is made possible once we read Frege's remark about defective understanding in light of this notion. According to this view, not only are the contents of Concept Script and that of ordinary arithmetical language the same; we fully grasp this content prior to Concept Script. The difference in understanding given the construction of Concept Script is simply that, while we have all along fully grasped the content, certain features of that grasp were unperspicuous to us, partly because our grasp was to some extent tacit. The attainment of perspicuity in this respect was therefore a central feature of Frege's logicist project. The thesis that Frege sought to uncover both the Sinn and Bedeutung of our ordinary terms and that our grasp of their Sinn was full, if to some extent tacit and unperspicuous, I call 'strong hermeneutism'. The aim of this thesis is to lend support for this reading.

The kind of tacit grasp that, I think, at least implicitly informs Frege's foundational project has several properties. To appreciate the doctrine we need to bear in mind a number of further features of Sinn. First, Sinn effects a reference relation to a corresponding Bedeutung. Second, this relation exists independently of our use of any item of language to which the Sinn becomes attached. Third, Sinn gets attached to our terms in virtue of our use of that term. Fourthly, in the case of arithmetical practice, our grasp of Sinn appears to be assured, not merely by our use of the relevant symbols, but also in virtue of certain arithmetical procedures or practices being integral to our rational lives. It is with this background in mind that we should approach the notion of tacit and unperspicuous grasp. Its main features seem to be the following.

(i) It is possible fully to grasp the Sinn of a term without our ever having articulated the structure of the Sinn in question, if it has one—if it is non-primitive—i.e. without our ever having defined the term to which the Sinn is attached. We shall see that, from the outset, this phenomenon appears to have informed the very spirit of Frege's Grundlagen.

(ii) Where a term is introduced to a thinker by means of a definition, the thinker will thereafter invariably not have the Sinn in consciousness when using that term. In the case of using definitions in proofs, even where there is familiarity

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6 The points in the paragraph are not uncontroversial, see chapter 1.
with the definiens, the thinker will from the start replace the definiens with its definiendum, and make use of the latter, more simple symbol, to represent the structured Sinn. In other words, even in cases of considerable familiarity with the Sinn of a term, the thinker will leave its Sinn in abeyance; it will not be before the thinker's mind when, by means of the symbol, he engages in thought.

(iii) Frege thinks that ordinary usage is enough for the thinker fully to grasp the Sinn, but not enough to have complete understanding of it. In the case of arithmetical terms, to grasp a Sinn fully it is enough that the thinker be competent in arithmetic. It is not required that he be aware of the reference relation being effected by virtue of the Sinn in his possession.

(iv) Nor is it necessary for the grasp of Sinn that the thinker should, on becoming conscious of the Sinn of the term, immediately recognize that the Sinn in question in fact belongs to the term.

   In the case of non-sentential, non-logically simple terms: Frege appears to think that a perspicuous grasp is attained given familiarity with its definition. In the case of a logical simple, perspicuity of grasp is had where the thinker recognizes its non-definability or primitiveness. In these cases, we have features of understanding that are constitutive of perspicuity of grasp, but which are independent of conditions for grasp per se.

§3. The hermeneutic hypothesis rests on a presupposition that a variant of the revolutionary hypothesis—revisionism—would deny. The presupposition is that, for Frege, there is a determinate mathematical reality independent of our ability to frame definitions, and independent of their use in true sentences and theories. According to the revisionist, the defective definitions and understanding of which Frege speaks, signals Frege's view that we have not carved-up content—mathematical reality—in a sufficiently precise way that would allow us to entertain truth-evaluable thoughts. In a word, the content of all our ordinary arithmetical terms are vague or indeterminate, which, for the purposes of rigorous science, Frege endeavoured to make precise by means of his definitions. Definitions carve-up mathematical reality, but not at the joints, since it has none; and in doing so we construct concepts. For the revisionist, then, definitions are a means of concept acquisition.
In chapter one, I consider and reject this variety of the revolutionary hypothesis. This leaves us with two ways in which to underpin the hermeneutic hypothesis. Either we can read Frege as a platonist-realist, or as a platonist-idealist. According to both positions, mathematical reality is independent of our ability both to frame definitions and to construct theories, and that this reality consists in abstract entities, items that exist non-spatially and non-temporally. On both views, mathematical reality, for Frege, is to be uncovered by our definitions, rather than constructed by them. Conceptual analysis affords an opportunity for genuine discovery. The difference is that, for the platonist-idealist, what we discover is in some sense constructed by the mind. In particular, mathematical reality is constructed by the mind as a precondition for the possibility of thought and inquiry. There are good reasons to remain undecided about which of the two platonist views Frege held. However, it is suggested that the balance of favour, given the text, probably lies with the realist reading.

In chapter two, I turn to two further ways of approaching the revolutionary thesis, both of which, unlike the revisionist, accept that content is determinate independently of our ability both to construct definitions and to use of them in a theoretical system. But like the revisionist, one version sees especial significance in Frege’s use of stipulative definitions: the assignment of meaning to empty signs. It argues that our pre-Concept Script arithmetical terms were semantically defective: they failed to satisfy Frege’s criterion of adequacy for a term to have a Bedeutung. The other version of the revolutionary claim is that even if our ordinary arithmetical language is semantically in order, Frege’s logicist project was not a quest to uncover unique and hidden objects. The first version just described can be neutral on this latter question.

One way of arguing for either of these versions is to claim that Frege’s logicism is principally a mathematical project, and that, qua mathematician, Frege did not intend his definitions to preserve Bedeutung. For example, exponents of the latter version would point to passages in Frege’s work that might indicate that he would grant various equally tenable ways of reconstructing arithmetic. I show that the passages in question can be read as compatible with the view of numbers as hidden and unique objects. Moreover, I show that there is sufficient textual evidence to support the claim, not only that
Frege sought to uncover the *Bedeutung* of our ordinary arithmetical terms, but that this enterprise, though surprising from a more modern perspective, was perfectly in keeping with the spirit of mathematical inquiry current in Frege's time.

In chapter three, I inquire into whether, if Frege did seek to uncover the hidden *Bedeutung* of our arithmetical terms, which he took to be unique, he did not go further and seek to capture the *Sinn* of those terms as well. After consideration of passages that would suggest that he did not, I examine the two possible views available thereafter: mild and strong hermeneutism. I argue that the former view—that our grasp is defective or partial—rests on not having sufficiently distinguished between the notion of understanding and the notion of grasp. More specifically, I argue that the knowledge attained of a concept by means of explicative definitions is constitutive of a greater understanding of that concept, rather than constitutive of a greater grasp: it does not, for instance, effect the transition from a partial to full grasp. According to strong hermeneutism, the success of Frege's logicism would have afforded us knowledge of the central concepts of arithmetic that would be constitutive of a more perspicuous grasp, a grasp that was in any case full.

In chapter four and five, I consider an objection to strong hermeneutism. This is that, in *Grundlagen* at least, Frege took his definitions to be of a kind that extend our knowledge: they had contents that could play the role of informative identity statements. In that case, if the definiendum is an ordinary arithmetical term, and the definiens the result of its analysis, then it would seem that the contents would be different. This of course would accord with weak hermeneutism. Alternatively, the informativeness in question might be accounted for in terms of the transition from partial to full grasp, thus supporting mild hermeneutism. The notion of recovery of tacit grasp, and the idea of attaining greater perspicuity as distinct from greater grasp, allows us to deflect the objection.

While both chapters four and five deal with the above kind of objection, they at the same time deal with separate issues that are of an independent interest. Running in conjunction with the central theme of chapter four is the question what Frege meant by the connection between fruitful definitions and
the extension of knowledge, particularly with respect to analytic truths. The supplementary theme of chapter five is the question what Frege meant in *Grundlagen* §64 by carving up a thought by means of an equivalence relation in order to arrive at something conceptually new, and how this is compatible with certain other things that he says. I show that strong hermeneutism—particularly given its feature of tacit grasp—can help to clarify and resolve some of the controversies surrounding this topic, and that our discussion of them further supports and clarifies the notion of tacit grasp.

Finally, chapter six offers a further reason why, apart from additional textual evidence, we should prefer the strong hermeneutic claim to both the revolutionary claims and the hermeneutic variants. This touches on the notion of explanatory value mentioned earlier. A central value of Frege's logicism, I argue, and to my mind his principal motivation, is epistemological. Frege sought not only to show that *our* arithmetical truths are analytic; he furthermore aimed to show how they could be known *a priori*: *i.e.* that the justification of *our* judgements of the truths of arithmetic need involve no reference to experience. The exponent of the revolutionary hypothesis must account for this anomaly; though they try to play down the problem, they fail satisfactorily to do so.

Furthermore, I show in chapter six that there is a tension with this epistemological construal of the value of Frege's logicism, and the mild hermeneutism. If our grasp is partial or indeterminate, it is unclear on what grounds Frege could have taken himself to be warranted in believing that he had said something informative about certain of our propositional attitudes involving arithmetical propositions. Finally we bring out the connection between attaining a perspicuous grasp and the epistemological motivation adumbrated above. In particular, it is shown that a perspicuous grasp of some truths of our arithmetic is sufficient to access justification for our judgements. In a word, understanding the propositions of arithmetic alone—*viz.* having a sufficiently perspicuous grasp—allows us to recognize beyond rational doubt that they are true.
Chapter 1

Platonism Vs. Non-Platonism.

§1.1. Introduction. In seeking to prove his logicist claim that the laws and central concepts of arithmetic are of a purely logical nature, and thus that the truths of arithmetic are purely logical truths, was Frege attempting to reveal hidden features of the content of our ordinary arithmetical language? If so, was it just the reference (Bedeutung) of our ordinary terms that he sought to explicate and lay bare in his Concept Script? Or were their sense (Sinn) also meant to be captured? Or was Frege only trying to show that a scientifically more rigorous, and structurally equivalent arithmetic could replace the content of our ordinary arithmetical language, which language he, in any case, took to be seriously defective?

We call an affirmative answer to the first and second question the ‘weak and mild-strong hermeneutic view’ respectively; and an affirmative answer to the third question, the ‘revolutionary view’. Accordingly, the mild-strong hermeneutic claim holds that ordinary arithmetic is Concept Script arithmetic in disguise: both the Sinn and Bedeutung of our ordinary number sentences are preserved and laid bare in a logically perfect language. On the weak hermeneutic view, the disguise is only partial. Only Bedeutung is preserved. According to all varieties of the hermeneutic interpretation, the imperfections of ordinary arithmetical language to which Frege so often alludes pertain not to its semantics—not to the realm of content: Sinn or Bedeutung—but to our grasp. For the weak and mild varieties, the grasp is partial. On the strong hermeneutic claim, there is simply a lack of perspicuity in an otherwise full, if to some extent tacit, grasp of our everyday arithmetical propositions. In other words, the propositions and truths formally presented in Frege’s Concept Script were intended to be more perspicuous renderings of the contents of our ordinary arithmetic, contents we ordinarily grasp. In contrast to the hermeneutic views, the revolutionary view
holds that Frege designed Concept Script as a replacement of our old arithmetical language and its content. By building arithmetic anew, he would ensure against the defects of the old language.

Of course a presupposition operative in the hermeneutic hypothesis\(^1\) is that, for Frege, arithmetical entities are there to be discovered; they exist independently of our stipulative definitions and of the role the defined terms play in inferential reasoning. To many readers of Frege, arguing for this claim—hereafter the ‘hermeneutic presupposition’—would hardly seem necessary. Surely, many will say, Frege’s writings make clear that he believes in the existence of a third realm in which non-temporal-spatial mind-independent-entities—functions and objects—reside?\(^2\) Other commentators too, for whom the reading of Frege as a realist-platonist is disputable, will subscribe to the hermeneutic presupposition. Typically, they will take the mathematical realm as something to be discovered, and at the same time take the contents of it to be a product of the mind in some Kantian sense\(^3\): hereafter the ‘platonist-idealists’. Still other commentators reject the hermeneutic presupposition outright. This more radical approach holds that, for Frege, there are no such entities—no \textit{Bedeutung}—independent of the stipulation and use in a system of scientifically admissible definitions.\(^4\) Accordingly, Frege’s frequent remarks about faulty definitions and explanations signal that our arithmetical language had no \textit{determinate} content; what content it had was literally defective. The possibility of discovering mathematical content prior to our linguistic activities is thus ruled out—\textit{revisionism}.

\(^1\) In what follows I shall use ‘hermeneutic hypothesis’ to refer to all three versions.


\(^3\) See below.

\(^4\) I'll explain this below. Its two main advocates are S. Wagner, and J. Weiner.
Of course one need not be a revisionist to hold the revolutionary view. But one must be a revisionist to reject the hermeneutic presupposition. One might subscribe to a platonist reading—and accordingly a realm of the discoverable—yet hold that pre-Concept Script language had no determinate content; or that it is quite simply different from that found in the Concept Script. We will consider these alternatives in the next chapter. In this chapter our purpose is to reject the revisionist reading. It is to be conceded that there may be some ambiguity concerning whether the mathematical entities of which Frege speaks are, for him, in some sense mind-dependent. But this should not lend credence to the radical anti-platonist reading. At most the ambiguity should occasion some uncertainty whether we are to read Frege as platonist-realistic, or platonist-idealist; either of which positions can underwrite the hermeneutic hypothesis, weak or strong.

§1.2. Frege’s Alleged Platonist-Realism. By means of arguing for the hermeneutic presupposition, I begin by setting out various ways in which one might regard Frege as a platonist-realistic.

Construed as a platonist, Frege believed that what we refer to in our mathematical discourse are functions of various levels and types—which, he says, are founded deep in the nature of things—and abstract objects: entities that exist neither in space nor time. Construed as a realist, Frege regards these items as existent independently of the mind. For example, thoughts about these items are determinately true or false independently of our recognition of the truth-value that the thought might have.

Frege’s platonist-realistic declarations seem at first sight to be as numerous as they are unambiguous.

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5 At least as regards the abstract realm.
6 Translations of the Philosophical Writings of Gottlob Frege (TPW), p. 41.
7 Cf. Grundgesetze, 1964, vol. i, pp. 1-16. Here Frege speaks of a domain of the objective that is distinct from the actual.
8 Frege nowhere considers the possibility or otherwise of thoughts whose truth-value is in principle beyond our capacity to ascertain.
1. Frege says that logic presupposes that "... words are not empty, that sentences are expressions of judgements, that we are not merely playing with words." So statements of the form \((\exists x) (x = N)\), where \(N\) is a singular term, are trivially true. Presupposed here is that what we refer to and quantify over exists. That is, Frege takes mathematical discourse as factive, the expression of literal truths that say something about mathematical entities. The truths of number theory are about functions, for example concepts; and certain objects or extensions, for example numbers. "The botanist means to assert something just as factual when he gives the Number of a flower's petals as when he gives their colour." Yet numbers are neither perceptible, nor are they mental entities. For numbers, and other extensions, as well as functions, exist non-spatially and non-temporally. They are thus causally inert and immutable. They cannot therefore reside in the realm of the mental or physical. So we should not equate the objective with what is 'handleable or spatial or actual'.

2. Frege's remarks about the existence of thoughts are of a similar hue and further illustrate his belief that abstract entities not only exist, but are mind-independent. "What is grasped, taken hold of, is already there and all we do is take possession of it ... [It does not come into existence as a result of [any mental] activities." Having similarly characterised thoughts as abstract, and declared their mind-independence, we are told that

'A third realm must be recognized. Anything belonging to this realm has it in common with ideas that it cannot be perceived by the senses, but has it in common with things that it does not need an owner so as to belong to the contents of his consciousness.'

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9 In a piece written around 1884.
10 Grundlagen, §26, p. 34.
12 Grundlagen §61, p. 71; §45, p. 58; §58, pp. 69-70.
13 Grundlagen §26, p. 35.
14 'It is of the essence of a thought to be non-temporal, non-spatial.' Cf. 'Logik', p. 135, p. 137, PW.
15 G. Currie has argued that Frege's platonist-realism holds only with respect to thoughts. 'Frege on Thoughts', p. 235, Mind (1980); 'Frege's Realism' (1978), p. 220, Inquiry 21. I'll discuss this in §1.4.
16 'Logik' (1897), p. 137, PW.
'A person sees a thing, has an idea, grasps or thinks a thought. When he grasp or thinks a thought he does not create it but only comes to stand in a certain relation to what already existed—a different relation from seeing or having an idea'.

Frege likewise attributes mind-independence to numbers. Number statements are true independently of whether anyone takes them to be true. In speaking of the statement '3 is a prime', for instance, Frege says

'What we want to assert in using that proposition is something that always was and always will be objectively true, quite independently of our waking or sleeping, life or death, and irrespective of whether there were or will be other beings who recognize or fail to recognize this truth'.

And in Grundlagen he says

'The mathematician cannot create anything at will, anymore than a geographer can; he too can only discover what is there and give it a name'.

3. Not only are numbers said to be objective and mind-independent, they are compared with what he takes to be real things, like the earth. Frege is yet more explicit concerning whether numbers are real (wirklich) albeit abstract in nature.

'I heartily share [Cantor's] contempt for the view that in principle only finite numbers ought to be admitted as real. Perceptible by the senses they are not,'
nor are they spatial ... and if we restrict the real to what affects our senses ...
then naturally no number of any of these kinds is real.\textsuperscript{21}

4. Frege begins the positive part of \textit{Grundlagen} with the epistemological question 'How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?'\textsuperscript{22} His immediate answer is to explain how we can obtain a criterion by means of which we can determine whether the object in question is a number. Part of the explanation is captured by the movement in thought from the truth that there are just as many Fs as Gs, understood as the obtaining of a one-one correlation between two concepts, to the truth that the number of Fs is the same as the number of Gs. If we have established that sentences in which numerical singular terms occur are true, and if the truth of a sentence is determined by the referents of its parts, then we can infer that numbers exist.\textsuperscript{23} In later writings, Frege speaks of a logical faculty, \textit{logische Fähigkeiten},\textsuperscript{24} associated with axiom V,\textsuperscript{25} by means of which we apprehend logical objects, like numbers. Reason and the grasp of certain truths, Frege believed, gives us access to abstract objects.

If there are logical objects at all—and the objects of arithmetic are such objects—then there must also be a means of apprehending, of recognising, them. This service is performed for us by the fundamental law of logic that permits the transformation of an equality holding generally into an equation.\textsuperscript{26}

'By means of our logical faculties we lay hold upon the extension of a concept, by starting from concepts.'\textsuperscript{27}

Moreover, one might argue that, since all concepts and other functions are non-spatial-temporal mind independent entities, and since terms of logic denote these,

\textsuperscript{22} \textit{Op. cit.} §62, p.73.
\textsuperscript{23} To read the application of the context principle as an indication that Frege was concerned to justify taking abstract singular terms at face value, and of justifying ascribing reference to them, is not without controversy, as we will see below. Dummett subscribes to this reading, 'Frege as a Realist', p. 83; cf. C. Wright's \textit{Frege's Conception of Numbers as Objects}, p. 25.
\textsuperscript{24} \textit{Grundgesetze}, vol. 2, §74.
\textsuperscript{25} $\forall x (Fx \leftrightarrow Gx) \leftrightarrow \{x: Fx\} = \{x: Gx\}$.
\textsuperscript{26} \textit{TWP}, p. 161.
\textsuperscript{27} 'On Schoenflies: Die Logischen Paradoxien der Mengenlehre' (1906), p. 181, \textit{PW}.
as well as logical objects, then, given that all knowledge involves the laws of logic, all knowledge is partly about these entities.  

Finally, it might be argued that Frege conceived of his platonism as having an explanatory value. There are five ways in which this might be so.

Firstly, items in the abstract realm contribute to part of an explanation of the content of our terms and of the truth of our sentences. That mathematical discourse is factive, and consists in the main in truths, is because there are such things as abstract entities that render our thoughts about them true.

Secondly, the realm of the abstract is deployed to explain the shareability of thoughts, or communication. What is subjective is mind-dependent; it is temporary and private (e.g. ‘mental pictures, formed from the amalgamated traces of earlier sense impressions’). If numbers were ideas the truths of mathematics would not be shareable or communicable, as plainly they are. And if everything existent were identified with the contents of one’s consciousness, with ‘ideas’, disputation would be impossible; both agreement and disagreement would collapse into the expression of thoughts that no-one but the owner of the ‘ideas’ would be able to grasp. ‘If the number two were an idea, then it would have straight away to be private to me only. Another man’s idea is, ex vi termini, another idea.’ But a ‘... number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is.’

Thirdly, the realm of the abstract is what explains the eternality of mathematical truths. If the language of mathematics were about subjective ideas, then there would be no eternal truths in mathematics. The truths would change according as the mental or physical conditions on which they depend would change. But the truths of mathematics do not change; they are non-temporal.

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29 Grundgesetze vol. i, p. 17 (p. xxix).
30 Grundlagen, p. v.
34 Grundlagen, p. vi; §27, p. 37.
Fourthly, platonism explains objective truth. If what we refer to were ideas, then truth would not be independent of us.\textsuperscript{35} A proposition would cease to be true once we cease to think it.\textsuperscript{36} And what is true for me would not be true for you, though both of us might be right.

Fifthly, platonism explains the truths of logic. The laws of logic are the laws of truth, which are factive statements: they state what is. From the laws of logic, stating what is, Frege says that the laws of thought can be discerned. That is, the authority of the laws of thought, normative laws stating how we ought to think if we are to grasp true propositions, and indeed if we are to think at all, is to be accounted for by the laws of logic. Since the laws of truth make ontological claims, the laws of thought are given a metaphysical grounding.\textsuperscript{37}

Clearly, if the above traditional platonic reading can be sustained, so too can the hermeneutic presupposition be. By the weak thesis, Frege's definitions are meant to capture and explicate the self-same \textit{Bedeutungen} as we have, unbeknownst to ourselves, all along been referring to when engaged in ordinary arithmetical practice, using ordinary arithmetical terms. That we have been doing so is because we grasp the \textit{Sinn} of the term; it being in virtue of \textit{Sinn} that the entity in question is determined. By the strong hermeneutic thesis, not just the \textit{Bedeutung} but also the self-same \textit{Sinn}, again unbeknownst to the thinker, is captured by the definition. It is plain that both these versions of the hermeneutic hypothesis rest on the presupposition that there are determinate entities, the \textit{Sinn} and \textit{Bedeutung} of our terms, referred to irrespective of whether the terms have been defined. But some Frege commentators there are who would deny this kind of independence between our capacity to define and the existence of the entity of the term defined. Not only would they altogether reject the platonist reading given above. They would claim that Frege's logicism was about constructing a structurally equivalent arithmetic to replace our own. Let us now consider this counter claim.

§1.3. The Revolutionary View and the Hermeneutic Presupposition. To begin with, Frege says that a non-primitive term is admissible in science only if the Sinn of the term determines sharp boundaries of application,\(^{38}\) and that failing this condition the term lacks Bedeutung—hereafter ‘the principle of completeness’ (*Grundsatz der Vollständigkeit*).

1. ‘All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp, that it should be determined, with regard to every object whether it falls under that concept or not.’ \(^{39}\)

2. ‘If we represent concepts in extension by areas on a plane, this is admittedly a picture that may be used only with caution, but here it can do us good service. To a concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all round, but in places just vaguely faded away into the background. This would not really be an area at all; and likewise a concept that is not sharply defined is wrongly termed a concept.’ \(^{40}\)

Since this condition is rarely, if at all, met in ordinary arithmetic language, it might be claimed that, for Frege, arithmetical terms, being either ill-defined, or not defined at all, even although they are non-primitive, lack Bedeutung.\(^{41}\) This view might be supported by some of Frege’s further remarks on definitions. Definitions, he says, are symbolic devices used for laying down the meaning of an empty sign. In particular, we are told that ‘gleichzahlig’ is to be regarded as an empty symbol: ‘I must ask that this word be treated as an arbitrarily selected symbol, whose meaning is to be gathered, not from its etymology, but from what

\(^{38}\) *PW*, p. 179; *Nachgelassene Schriften (NS)*, pp. 193-194, Felix Meiner Verlag, Hamburg. One reason for his insisting on the completeness principle is that only thereby, Frege thinks, can the law of excluded middle be adhered to, which he takes to be necessary if we are to capture or formulate laws of a science. If something fails to display a sharp boundary, it cannot be recognized in logic as a concept, just as something that is not extensionless cannot be recognized in geometry as a point, otherwise it would be impossible to set up geometrical axioms. Cf. ‘The Laws of Inertia, p. 133, *CP*. Furthermore, the constraint is a consequence of the functionality of concepts: essential to functions is that for any argument the function should yield just one value. \(^{39}\) *Grundlagen*, p. 87. Cf. ‘On Schoenflies: Die Logischen Paradoxien der Mengenlehre’ (1906), p. 180, *PW*.


is here laid down'. Furthermore, we are told that 'it must be noted that for us the concept of number has not yet been fixed'. Accordingly, Frege would not have intended his definitions to capture the \textit{Bedeutung} of ordinary arithmetic language, since those terms had none. Nor, arguably, is \textit{Sinn} preserved. On the revisionist view now to be discussed, it would not do to say that we refer to these items despite the faulty definitions. There is no determinate realm of \textit{Bedeutung} independent of scientifically admissible definitions, independent of Frege’s Concept Script; no determinate entities to which our pre-Concept Script terms could be even assigned. According to S. Wagner and J. Weiner, Frege takes fulfillment of the completeness condition to be sufficient for being a concept. That is, the existence of a concept is exhausted by the existence of a precise description of some distinction: exhausted by the distinction of what does and does not fall under the concept, the description. ‘[T]he notion of a term’s having a \textit{Bedeutung} consists in nothing more than the sharp boundaries for the application of the term and the indication of the epistemological roots of truths expressed by sentences in which the term appears.’\textit{ Mutatis mutandis} singular terms that are constructed from concept terms and definite description operators. It must be demonstrated that one and only one thing meets the description specified by the concept word. Frege does not rely on the idea of a third realm, construed as a platonic sphere, with the idea of an interaction with its members, because none is required. To

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42 \textit{Grundlagen} §67, p. 79. The first quote of this paragraph reads, ‘Hiergegen ist zu bemerken, dass fur uns der Begriff der Anzahl noch nicht feststeht, sondern erst mittels unserer Erklärung bestimmt werden soll’. Thiel, §63, p. 63.
43 See below; \textit{Grundlagen} §63, p. 74.
45 This claim is explicit in S. Wagner, in ‘Frege’s Definition of Number’, p. 14, \textit{Notre Dame Journal of Formal Logic} (1983), 24; cf. Weiner, 1984. I think the same claim is implicit in H. Sluga’s \textit{Gottlob Frege} (1980), where he argues that the \textit{Bedeutung} of our terms exist only in systems, and his noting (rigly) that, for Frege, arithmetic lacked a system prior to Frege’s Concept Script, p. 158, p. 140, pp. 130-136. For an explicit denial that, for Frege, functions are mind-independent non-spatial-temporal entities, see Weiner’s ‘Burge’s Literal Interpretation of Frege’ p. 592, \textit{Mind} (1995b).
47 Weiner, 1990, p. 190. ‘Epistemological roots’ here means that a term must be defined from primitive terms or be themselves primitive. The primitiveness of the thing itself, she says, will make plain whether it is logical, empirical, or intuitive. Cf. op. cit. P. 205.
grasp a proposition, the Sinn of sentences in indicative and interrogative mood, is to impose structure on formless entities. This is the role of our definitional or analytical activities. But since our definitions are defective so too is our grasp. Thus ‘arithmetic as it stands lacks fully determinate content, for reasons related to failures of reference’, so that Frege would be ‘forced to admit something like partial grasp of contents ...’  

It would follow that ‘[t]he thoughts expressed [by our arithmetical sentences] are defective—they are insufficiently precise to determine truth-values,’ and thus that the hermeneutic hypothesis tout court is false.

It will be objected that Frege speaks, as we have seen, of what is objective as mind-independent: for example, that the mathematician, like the geographer, can only discover what is there and give it a name. The plausibility of the revisionist claim about what Bedeutung amounts to, depends in part on the plausibility of the claim that it is not the structure of the reality that determines concepts. If the world were already divided up into physical and abstract objects, then interaction with them would be required in order to give them a name. But, say the revisionists, the structure of the world, for Frege, is determined by concepts—distinctions—the thinker creates and imposes. ‘[T]he world does not dictate where the borders of the North Sea are drawn, whether anyone draws any borders for any seas, or even whether the concept of being a sea is formulated ...’

‘[T]he physical world is not articulated—we impose structure on it ...’. In other words, in speaking of concepts, Frege makes no assumption that reality is carved at the joints—e.g. that it is divisible into natural kinds—because for him there are no joints. What is ‘independent of the mind is structureless contents’. True, it

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48 Grundlagen, p. 88.
52 J. Weiner, op. cit. p. 204.
53 Weiner, 1995a, p. 365; cf also her 1990, p. 184. On Weiner’s view of Frege, reality is not cut at the joints because, for Frege, there are no joints. To borrow her metaphor, reality is rather like an uncut cake, it has no structure ‘the world no more has joints than a cake does.’ Op. cit. 1995a, pp. 366-67.
54 Cf. Wagner, op. cit. p. 12.
will be said, a geographer, for example, discovers things and gives them names. But an antecedent structure is imposed nonetheless; and thus talk of interaction with mind independent objects is inappropriate. Indeed especially is this clear, we are told, in the case of the geographer. He needs a concept of a mountain; and he needs a co-ordinate system, a literal map, within which the objects and places can be located, and within which they can be recognized as the same again. The thinker imposes these two things; the world does not determine them. But once the concepts to be used have been decided upon, the geographer cannot create anything at will, but can only discover what is there and give it a name. So what exists depends on our conceptual capacity, here understood as the ability to construct 'precisely formulated arbitrary descriptions'. Hence the notion of discovery, of which Frege speaks, should be understood as hitting upon a possible distinction, one that can be formulated or expressed precisely. But that is not to say that concepts or objects are already there awaiting discovery.

The revisionist claim about the nature of Bedeutung, and of what it is for a term to have one, depends on a still further claim. Namely: Frege recognised no primitive proper names either for the *apriori* sciences or for the empirical sciences. This claim is important because of course if primitive names were recognized, then a reference relation would be required to supplement the conditions mentioned above for the admissibility of a term for science. That is, we would require an account of how words hook on to items of reality.

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56 Cf. Weiner, 1995a, p. 370. Similarly ‘[t]o say that unrecognized objects exist ... is to say that it is possible to mark off, in language, boundaries that have never actually been marked off by anyone. Emphasis on this idea of what precise formulations are possible, or can be drawn, is necessary if we are to avoid misreading a further remark of Frege’s’. That is, ‘The North Sea is not affected by the fact that it is a matter of our arbitrary choice which part of all the water on the earth’s surface we mark off and elect to call the “North Sea”’, *Grundlagen*, §26, p. 34. Cf. 1995a, pp. 367-68. Cf. also her 1990, pp. 171-75. Further defence of this view is given below.

57 That there are no primitive proper names in Frege’s *a priori* conception of science is I think clear. As J. Weiner notes, recognition by Frege of primitive proper names for the empirical science need not impugn the otherwise minimal requirements of admissibility of a term for the *a priori* sciences. Cf. 1990, p. 140, p. 198. For the view that Frege would not have permitted such names in the empirical sciences, see op. cit. pp. 191-224. Given the above view of the criteria of a term’s having Bedeutung, the absence of primitive proper names has a further consequence. This is that there can be no question as to whether we succeed in talking about the world. We can only fail in our descriptions of it. This is because either our judgements of statements containing the
The revisionists further support their claim about the nature of *Bedeutung*—and consequently their view of the nature of Frege's logicist reduction of number theory—by appealing to their construal of three methodological principles that guide Frege's project. This construal, they claim, better captures Frege's intentions than does the traditional platonist reading. Let us examine these principles.

§1.3.1. **Methodological Principle 1.** Consider first Frege's dictum that we should never lose sight of the distinction between object and concept. The revisionist denies that these categories are ontological. The distinction is meant neither as a classification of a certain mind-language-independent structured reality, nor as providing a ontological foundation for objectivity. Rather the point of the division is to meet the requirements for the apprehension of thought: to facilitate logical regimentation, no more, no less. To appreciate an assertion, to grasp a thought, is to appreciate elementary implications: that one statement can be asserted on the basis of another. In doing so, we are led to recognize inference patterns, which in turn compel recognition of logically significant parts of a sentence. For example, our appreciation of the inference pattern involved in Leibniz's law—that from 'Fa' and 'a = b', to 'Fb'—leads to the recognition that statements be segmented into such logically significant parts as proper names and first-level predicates.

There are two reasons why one might think that this logical significance exhausts the content of the categories. The first reason turns on asking whether the categories of object and function could serve to classify a 'mind independent
reality'. For the platonist, it is because numbers are objects—complete or saturated entities—that we use proper names to stand for them, and why proper names should have the logical character that reflect the kind of thing they stand for. It is because something is a concept, an incomplete or unsaturated entity that we use incomplete or unsaturated expressions to stand for them. It is because the furniture or structure of reality is fixed independently of language and thought that our language and thought segment respectively into the categories of object-expression and function expression, and their corresponding saturated and unsaturated Sinn. Syntax supervenes on the logical forms of reality, not the other way round. But essential to this picture, it is claimed, is that we could get our classification wrong.\(^6\) Otherwise ‘we could not think that we could use predicates for the categories in classifying things in reality’. The trouble, we are told, is that we cannot err in this way. There is no ‘intelligible application of the category terms themselves to items in different categories’.\(^6\) You cannot think that a thing belongs to a category that it does not belong to. Frege’s notion of logical category is bound up with the kind of argument places that an expression can have, there being logical places where only objects can stand, and other logical places where only concepts can stand.\(^6\) Because of this, Frege says ‘I do not want to say it is false to say concerning an object what is said here concerning a concept; I want to say its impossible, senseless to do so.’\(^6\) What belongs in the argument place of ‘(...) is an object’ is a saturated expression. To put a concept expression, ‘a dog’ in its place would not be to classify a concept as an object. Concepts are unsaturated, incomplete entities, and so too their expressions. So the result would not be a sentence, but ‘(...) is a dog is an object’; no thought would have been expressed. Indeed given the connection between the category distinctions and


\[^6\] Cf. op. cit. p. 170.

\[^6\] ‘Foundations of Geometry 1’ (1903), p. 281, p. 282, in CP; cf. ‘Comments on Sense and Meaning’ (1892-95), p. 120, PW.

\[^6\] ‘On Concept and Object’ (1892), p. 189, CP, ‘[T]he word “concept” itself is, taken strictly, already defective, since the phrase “is a concept” requires a proper name as grammatical subject; and so strictly speaking, it requires something contradictory, since no proper name can designate a concept; or perhaps better still, something nonsensical.’, ‘On Schoenflies: Die logischen Paradoxien der Mengenlehre’ (1906), pp. 177-78, PW.
argument places '( ) is an object' is true of anything—anything you say is an object is an object.

So too is it impossible to make a mistake in classifying concepts. Functions cannot be picked out with a name; nor, if picked out by other means can we give them a name. 'A is a concept' and 'A is not a concept' (where 'A' is a concept expression), do not express thoughts: two incomplete expressions alone do not yield a sentence. And to think the thought expressed by 'The concept horse is a concept easily attained' is to think about an object, not a concept. The analogue predicate of '(...) is an object' is of second level, and can only be shown by some such pattern as:

\[
(\forall x) (x \text{ is a dog } \rightarrow x \text{ is a dog}); \\
(\forall x) (x \text{ is a tree } \rightarrow x \text{ is a tree}).
\]

Let '(\forall x) (\Phi x\rightarrow \Phi x)' be the expression for the kind of second-level function we refer to when using the above pattern. Again, we cannot use this expression to misclassify objects as belonging to the category of concepts. We would only get nonsense were we to try to fill the argument place with an object. But fill it with any first-level function expression, and we will always get a true sentence.

The other reason against treating the logical categories as an ontological distinction turns on the claim that, for Frege, all theories must be stateable in his Concept Script. If that is so, then there could be no metaphysical theory. The aim of Frege’s philosophical work, unlike a scientific work, could then not be objective theorizing, not the establishment of truths. In light of this, it is suggested that we regard his

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65 Cf. Wittgenstein,'s Tractatus, 5. 4731. On the other hand, while 'A horse is a concept' is a legitimate expression, it is no help in classification, since it doesn’t say of some concept that it is a concept. It says (\forall x) (Hx \rightarrow Cx), which is false. Whatever falls under the concept horse (H) is an object, and cannot then be a concept (C), and whatever is concatenated with 'is a concept' is an object.

66 Cf. Frege’s ‘On Concept and Object’ (1892), p. 45, in TPW.
68 Cf. J. Weiner, 1995a, p. 593; 1995b, p. 376; 1990, p. 69. ‘Frege’s logical notation is meant to express all content of any statement that has significance for inferences in which it figures.’ Op. cit. ‘Burge’s Literal Interpretation of Frege’ (1995b), p. 593. ‘Frege is committed to the belief that all scientific claims should ultimately be expressible in Begriffsschrift’. Her reasoning seems to be that laws of logic are applicable to all scientific reasoning, 1990, p. 70. It is also based on a remark of Frege’s in Grundlagen §26 (see below §1.3.1.) that what is objective is what is subject to laws and what is expressible in words.
philosophical remarks as elucidations rather than objective statements of facts,\textsuperscript{71} whose purpose is to acquaint us with the workings of the Concept Script. Consider again Frege's notion of objecthood and concepthood. One cannot state in the Concept Script that, for example, objects and functions are totally different from each other.\textsuperscript{72} The distinctions cannot be stated; they can only be shown—by use of the terms in his Concept Script. Furthermore, to treat Frege's remarks on the category distinctions as other than elucidations, would be to commit him to an incoherent view. '(...) is an object' and '(...) is not a concept' would be literally true for all arguments that could fill the gaps. But if '($\forall x$) ($x$ is not a concept)' is true, then we cannot take seriously Frege's claim that statements of number are assertions about concepts.\textsuperscript{73} And if '($\forall x$) ($x$ is an object)' is true, and all concepts are objects, then the predicate 'is an object' is empty, and thus too the claim that numbers are objects.\textsuperscript{74}

Accordingly, Frege's remarks on the notion of concepthood and objecthood do not simply miss his thought. It is unclear that there is a thought to miss.\textsuperscript{75} In saying that the categories can only be shown,\textsuperscript{76} we must think of what is shown as what is internal and constitutive of Concept Script, not as what is external to it. Again what fixes logical structure is not a reality outside Frege's logical perfect language: reality is internal to language.\textsuperscript{77} That logical structure is part of what makes it possible for a sentence to be about something. For that reason, the logical order or structure—the distinctions between


\textsuperscript{72} Cf. Weiner, 1995a, p. 374, p. 375, p. 376; Cf. also her 1990, p. 257. If we can say of an object that it is non-temporal-spatio and objective, the same cannot be said of a concept, and vice versa. In ($\forall g$) ($g$ ($)$ is non-spatio-temporal and objective), the gap represents the logical space in which only an object name or first-level variable can stand. Taking the first-level variable we get ($\forall g$) ($g$ ($x$) is non-temporal-spatio objective). But this is simply universal generalization, ($\forall x$) ($\forall g$) ($g$ ($x$) is non-spatio-temporal and objective). It says that for any argument, the value of a function at that argument is non-spatio-temporal and objective, which clearly is not true: e.g. the value of the function, mother of $x$.


\textsuperscript{74} Op. cit. p. 256.


\textsuperscript{76} Weiner, 1990, p. 275.

\textsuperscript{77} This would not be to say that Frege cannot be wrong about his Concept Script. But if a Concept Script does go wrong that is not because it fails accurately to reflect certain structures of external reality. It is because it diverges from 'its own own inevitable inner structure', the preconditions of the sayable, which the Concept Script has failed to make clear. Cf. C. Diamond, 1984, p. 169.
function and object, between first and second order functions, and so on—cannot itself be what a sentence is about.  

§1.3.2. Methodological Principle 2. In further support of the revisionism, consider next Frege’s methodological principle that we should always distinguish between the objective and subjective. Frege draws the distinction in terms of subjective and objective ideas, where subjective ideas are governed by the psychological laws, (e.g. of association), and where ‘objective ideas’, which are non-sensible, belong to logic. According to the revisionist, objectivity here means no more than what is governed by laws. As such no reliance is made on metaphysically more substantial entities. Otherwise put, Frege’s notion of objectivity is grounded in reason rather than in ontology. Several early passages of Grundlagen might appear to bear out this view.

‘[T]he foundation of objectivity cannot indeed lie in the sense impressions, which as an affection of our mind is wholly subjective, but, as far as I see, only in the reason (nur in der Vernunft).’

Frege refers here to the inappropriateness of using subjective ideas to explain the success of human practices, like communication, and our possession of scientific knowledge. What ensure objective intersubjectivity—grasping the same thought or truths—are the laws of thought or reason. Elsewhere too Frege seems to qualify what he means by objectivity when he omits in the following to state that the ontological is constitutive of its foundation.

‘[W]hat is objective is what is subject to laws (ist darin das Gestzmässige), what can be conceived and judged (Begriffliche, Beurtheilbare), what is expressible in words (was sich in Worten ausdrücken lässt).’

‘... I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of the reason, for

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78 In this sense it might be said that ontological categories supervene on logical categories. Cf. T. Ricketts, 1986a, p. 84, p. 87, p. 89, p. 90; cf. 1986b, p. 180; cf. also his ‘Frege and the logico-centric predicament’, Nota 19 (1985), pp. 5-6.

79 Again, those that are ‘of a sensible pictorial character’, which are ‘different in different men’.


81 §27.
what are things independent of reason? To answer that would be as much as
to judge without judging, or to wash the fur without wetting it."82

Further indication of Frege's view of objectivity lies, it is said, in his
remark that the starting point of his logic is judgement and its content.83 Since, for
Frege, judgement is the recognition of truth, and that this is manifested in
assertion, one might say that Frege starts his logic from basic facts about
assertion. Intrinsic to assertion is its being assessable as correct or incorrect. This
is manifested in the possibility of agreement and disagreement, which a shared
background in logic makes possible: again, we share an appreciation of
elementary implications, so that by constructing chains of reasoning we can
discern agreement or disagreement, or mere difference of assertion. In contrast to
assertion is the venting of inner mental states, like pleasure and pain. There can
be no agreement or disagreement here, since there is no thought to be shared.
Thus it is allegedly this contrast, between the essential features of assertion and of
venting of inner states, whose foundation lies in the distinction between laws of
thought and psychological laws, that the contrast between the objective and
subjective consists.84

In that case, the doctrine of the primacy of judgement would show that the
direction of explanation is the reverse of what a platonist interpretation might
claim. It is from our knowledge of the norms of thought that we explain the laws
of logic, whose statements are, pace the realist construal, not about denizens of a
third realm.85 Facts about assertion are taken as Frege's starting point.86 There are
no further, more fundamental, elements that explain the contrast between the
subjective and objective. For this reason, it will be said that Frege's talk of the
shareability of thoughts serves merely to bring out these distinctive features of

83 'Boole's Logical Calculus and the Begriffsschrift' (1880-81), pp. 16-17, PW; NS, pp. 17-19.
86 Most of the points in this paragraph are subscribed to by Ricketts. See for example, his 1986b,
p. 173, and his 1986a, p. 66, where he holds that Frege's claims about the objectivity of judgement
were not intended as factual claims. So there's no question of Frege's alleged platonism as
explanatory of objectivity.
judgement and assertion. Hence it is in the shareability of a thought, in its intersubjectivity, that its objectivity might be said to reside.

Revisionists readily concede that Frege says that thoughts are independent of anyone actually grasping them. A property of a thought is inessential 'if it consists in, or follows from, the fact that this thought is grasped by a thinker'. But these revisionists deny that this means more than that the mind-independence of a thought is constituted by the possibility of our grasping them. And that possibility need be grounded by no more than the formulation of sharp concepts, and conformity with logical laws. For example, the thought, All whales are mammals, consists of sharp concepts, whales, and mammals, and the operation of condition and universal quantifier: (Vx) (Wx → Mx). Moreover, Frege took axiom V to be necessary and sufficient to yield the extension of any concept. Also Frege says that

'\[t\]he being of a thought may also be taken to lie in the possibility of different thinkers grasping the thought as one and the same.'

'... [A] thought consist[es] in the possibility of being grasped by several people ...

Is it not thus clear, the revisionist will ask, that the sort of access we can have to thoughts is part of their nature; and clear too that no third realm is relied on?

Finally, we saw that allegedly there can be no philosophical theory, pace the platonist view, that could provide an explanation of the objectivity of scientific practice and the authority and universality of logic. Again, if all theories must be stateable in his Concept Script, his philosophical remarks, which cannot be so stated, should be construed not as literal truths, but as elucidations. Of course if Frege’s remarks about the subjective (e.g., sense impressions) admit of objective formulation, then they are communicable, and so objective. But they are not objective. His remarks do not describe subjective objects. They are elucidatory

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88 'The Thought' (1918-19), p. 371, CP.
90 'Negation' (1918-19), p. 376, CP.
of the idea of 'linguistic expressibility and systematic science'. In that case, the predicate 'is objective' is true of everything, making its use vacuous. Hence, the essential distinctions between the subjective and objective, like the categories, object and concept, are not themselves stateable. Literal truths about them cannot be expressed, because there are none.

§1.3.3. Methodological Principle 3. So far in this section we have drawn attention to the sense in which apparently ontological notions have been construed as internal to Frege’s Concept Script; in particular, how the notions of object and function are supervenient on syntax. It might be said that nowhere is the sense of this ‘internality’ clearer than in Frege’s final methodological principle, the context principle; in particular, in its use in explaining existential claims.

Frege urges that we should never ask for the meaning of a term in isolation but only in the context of a proposition; and he later says that only in the context of a proposition does a term have a meaning. On the revisionist reading, the context principle supports the claim that an explanation of how we succeed in talking of numbers involves no notion of reference understood as a relation that hooks our words onto (say) metaphysically substantive objects. Suppose, for example, that the truth-value of a sentence is assigned indirectly, via an assignment of a semantic value or referent to its logically significant parts, one that involves a mental association, a relation between a word and an extra-linguistic entity, a relation regarded as apprehended independently of language. It

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97 As it is nowadays called.
98 Grundlagen §62, p. 73. ‘Nur im Zusammenhange eines Satzes bedeuten die Wörter etwas.’ (Thiel, §62, p. 71). In this connection H. Sluga says ‘it seems to me ... that ... Frege rejected ontological questions as groundless dogmatic metaphysics’ (p. 236, 1977). See also his, ‘Frege and the Rise of Analytic Philosophy’, p. 482-484, Inquiry, 18, 1975, for more on how the notions of Bedeutung and the categories of concept and object echo this way of viewing the context principle. Cf. also his ‘Frege as Rationalist’ (1976), p. 32, in Studies on Frege, ed. M. Schirm, (Stuttgart, 1976); and his 1980, §4, p. 140. Frege scholars differ as to whether Frege retained the principle after Grundlagen; also on whether its application is limited to abstract singular terms. The above account takes it as having been a pervasive and enduring principle in Frege’s work. It seems to me that Frege did retain the principle. He appears to rely on it, for example, in his 1903 explanation of the real numbers.
could then be in terms of that relation that the meaning of abstract terms might be explained. It might explain our conception of the thing allegedly denoted, and might explain the very act of reference, how we succeed in talking of objects, how the mind hooks onto extra-linguistic reality. The context principle precludes this type of picture. It precludes a meta-semantic perspective. Frege deploys the principle to make the point that lack of any mental association need not prejudice the belief that our sentences are factive. ‘It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition taken as a whole has a Sinn; it is this that confers on its parts also their content.’

To see what in part this means, consider whether, and if so to what, the proper name ‘3’ might refer, granted that it is a genuine proper name. According to the context principle, to show that ‘3’ refers to an object—one that satisfies the characteristics that numbers must satisfy—we need only show that the sentence in which it occurs is true. In this mathematical case, proof is required. One might proceed by first laying down explicit definitions of ‘3’, ‘1’ and ‘>’. From these we could prove, amongst other things, that 3 is greater than 1, and that 3 is the number that belongs to the concept is greater than or equal to 2 and less than or equal to 4. If we have shown thereby that 3 is greater than 1, and have granted that ‘3’ is a singular term, then we could not consistently deny that the number 3 exists.

A variant application of the context principle would be to deploy contextual definitions rather than explicit definitions. Suppose we wish to explain the content of a singular term and that no explicit definitions are available. The explanation might proceed indirectly by defining the content of the sentence as a whole in which the term occurs. This might be done via another sentence that does not contain the singular term in question. We could lay down that the two

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99 Grundlagen, §60 p. 71.
100 As already noted, Frege had not formally drawn the distinction between Sinn and Bedeutung in Grundlagen. But we can assume that, looking back on Grundlagen, he would have regarded the context principle as applying to both Sinn and Bedeutung. For of course in asking in §60 how numbers are to be given to us (Wie soll uns denn eine Zahl gegeben sein), he is asking after the Bedeutung of number words; and Bedeutung, for Frege, is never given barely, but always in some...
sentences share the same truth-conditions, and maintain that the logical form of each sentence is as its surface form suggests. So for example, in an initial illustration of the context principle, Frege stipulates that (A) 'The number of Fs = the number of Gs' is true just in case the circumstances expressed by (B) 'There are just as many Fs as there are Gs' obtain. So if we have determined the truth-value of (A) by means of (B), and if we grant that the logico-syntactic structure suggested by the surface form of both (A) and (B) are left intact, then, again, we cannot consistently deny that there are numbers.

In either of these two examples of the context principle, the possibility of reference is said to require no explanation. The question 'What does “the number one” refer to' simply reduces to the question of whether any sentence in which that term occurs is true. Similarly, to ask whether 'the number one' refers to any thing is to ask whether 'there is such a thing as the number one' is true. In other words, questions in the formal mode are reduced to, and receive their answers in, the material mode: again the general context principle, operative in Grundlagen, permits no distinction between the meta-language and object language. This means that there is no philosophical inquiry into abstract objects, or into the meaning of terms, nor into the relation between such objects and us, how we refer to them. To suppose that there is room for such an inquiry would be to violate the context principle. It would be to require that the meanings of words be asked for independently of the context of the sentence in which they occur. Moreover, we might see Frege's context principle as applying not just to abstract singular terms, but to all terms and contexts. In general terms, this means that any existential inquiry into numbers, or any other entity, must be internal to the language of the particular discourse in question, and a matter for the relevant science to answer.

Insofar as the assignment of truth-value is explained in any of the above two ways, one might conclude that no interaction occurs between the referents of the sentential parts and the thinker. So given that such sentences are true and that the logical form mirrors the semantic parts, the referential parts must be construed way, and this by way of Sinn.
as less than full-blooded. Our number terms, while taken as referential, should at the same time be taken only as objective abstractions, 'phantoms due to the refractive power of the linguistic medium' as a façon de parler. Noteworthy too is that Frege is taken to qualify his claim that the numbers are self-subsistent objects. ‘Self-subsistence’, he says, does not mean that these number words 'signify something on their own when removed from the context of a proposition. It means only that the use of these number-words as predicates is excluded'.

In sum: suppose that the following are fair representations of Frege. (a) The determinateness of reality depends on imposition of concepts. (b) The existence of concepts depends on the possibility of forming sharp distinctions. (c) Frege admits no primitive proper names. (d) The context principle was a pervasive and enduring feature of Frege's work. Then the context principle can be construed thus. Not only is no reference relation active in explaining the semantics of our abstract singular terms. Any notion of a reference relation, construed as a relation between the thinker and an independently existing object, and moreover one that would explain how we succeed in talking about objects, is illegitimate. There can be no substantive metaphysical question, no philosophical question, about extra-linguistic objects and how words hook onto these items. This is because there is no gap between reality and our cognitive capacity to

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103 Cf. Sluga, 'Frege's theory of the objectivity of numbers, value ranges, functions, etc., was never intended as an ontological theory', 1976, p. 29, p. 37, p. 43. See below for an elaboration and variation of this type of view.

104 *Grundlagen* §60, p.72.

105 Sluga advocates this view in his 1975, p. 478. But no one has illustrated the position more extensively than has Weiner, 1990. Dummett has argued that Frege abandoned the context principle on the ground that, after 1891, Frege assimilated sentences to proper names, and that this implied that there was no logically significant distinction between them. But even if sentences are regarded as proper names, they have special significance that distinguishes them from ordinary proper names: they name the true, and truth is what we all aim at. Cf. Sluga, 1977, p. 238, cf. T. Burge, 'Frege on Truth' (1986), p. 114, pp. 123-125, in L. Haaparanta and J. Hintika (1986).

formulate distinctions in language. On this reading of the context principle, if a term has been suitably defined; if it occurs in a true sentence; if that sentence has been verified by the procedures appropriate to its subject matter: then the term has \textit{Bedeutung}—at least in the above qualified sense of \textit{Bedeutung}.

Clearly on this view, the hermeneutic presupposition cannot be sustained. Nor then can the hermeneutic hypothesis itself be. Frege could have intended to capture neither the \textit{Sinn} nor the \textit{Bedeutung} of our ordinary arithmetical terms since prior to Frege's definitions these terms lacked a fully determinate content.\textsuperscript{107} This is not to say that, for Frege, the pre-Concept Script arithmetic had no content. While mathematically it was in order, semantically it was—from a philosophical point of view—defective. Although it sufficed for their purposes, mathematicians and layman alike had been reasoning with the 'mere shells of arithmetic propositions'.\textsuperscript{108} 'linguistic forms without definite meanings which a grasp of the subject matter would provide'.\textsuperscript{109} On this view, grasping is a process of concept formation, and concept formation is a process of defining, while all attempts at defining the central concepts of arithmetic had failed prior to Frege's system. Again, if it is true that, for Frege, '[t]he definition of number is an act of constructing a new truth which makes numbers of certain extensions'.\textsuperscript{110} then the hermeneutic hypothesis is false.

\section*{§1.4. Against Revisionism.} Let us now respond to the above challenge and show that the hermeneutic presupposition remains unhinged. We can do so by considering the platonist-idealist reading, and by showing that some of the evidence in support of the revisionist claim can be used to support this alternative platonist interpretation, while any other alleged evidence against the hermeneutic hypothesis can simply be deflected.

On the platonist-idealist reading, definitions capture what is pre-theoretically already there to be discovered. There are determinate concepts and

\textsuperscript{107} Cf. Wagner, op. cit. p. 15.
\textsuperscript{108} Op. cit. A similar view is indicated by Weiner's view that, because of faulty definitions, thoughts involving them are literally defective, cf. 1990, p. 248.
\textsuperscript{109} Wagner, 1983, p. 6.
objects, independent of our ability to provide suitable definitions for our terms, some of which at least are nonetheless mind-dependent, products of reason. Reason constructs its own mathematico-logico objects and concepts prior to our discovery of them, prior to our mathematical experiences. It does so because possession of mathematico-logico objects and concepts is a precondition for being the kind of rational agents that we are (and presumably because there is no other way they can be acquired). In particular, reason constructs logical objects in order to make rational inquiry possible; the disposition and capacity for inquiry being itself part of what it is to be mindful.

One possible illustration of Frege as platonist idealist, at least as regards mathematics, is in terms of axiom V. Frege thinks of himself as having discovered this ‘law’. By way of analyzing the thoughts we ordinarily have, he takes himself to have uncovered what practioners of the mathematical sciences have always made use of, but about which they have until now been only dimly aware. Early in *Grundlagen*, Frege says that reflection on what we do by instinct reveals the ‘hidden processes of nature’, the laws (of number theory) by means of which we must think if we are to be rational agents. What we have been unconsciously aware of is the principle that the obtaining of an equivalence relation between objects and concepts indicates that there is something between them that we can identify as the same. Take an equivalence relation that partitions a set into disjoint non-empty subsets, where the identity holds between members of each subset. The law has us reorganize or carve up content in such a way as to construct (create?) the objects: sets or extensions. This then is the kind of procedure captured by axiom V; it is what ‘instinct’—the instinct of reason?—has engaged in all along. Instinctive subdoxastic applications of the axiom have had us create such objects as the extensions of equinumerous concepts. In this connection might we construe Frege’s remark that ‘the charm of arithmetic lies in

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111 M. Resnik has suggested that Frege should be read as a platonist idealist, though he omits a good deal of the details regarding how such a view would work. Cf. his *Frege and the Philosophy of Mathematics* (1980), p. 166.
its rationality', in 'reason's study of itself'. The point is echoed much later in 'Der Gedanken' where Frege further remarks, though not without an important qualification, that logic and mathematics is an investigation of the mind. 'Neither logic nor mathematics has the task of investigating minds and contents of consciousness owned by individual men. Their task could perhaps be represented rather as the investigation of the mind; of the mind, not of minds.' Similarly might we understand Frege's remark that in arithmetic we are concerned with objects given directly to reason and as its nearest kin.

Thus it cannot be part of the platonist-idealist reading, if it is to be exegetically sound, that the construction of logical objects is in any way 'a mental act', conscious or otherwise—'a product of an inner process'. (Nor, as already said, is it connected, in any constitutive sense, with the role of definitions, whereby stipulations are a means of constructing determinate Bedeutung). That would be to reduce the axioms of logic to psychological laws. The central concepts and objects of arithmetic are parts of the furniture of what it is to be rational or mindful. As such the numbers are independent of mental acts: while they are the products of reason, it is partly because these objects exist that certain mental acts are possible—e.g., the grasp and judgement of thoughts. But though constitutive in part of 'the mind', it does not follow that the basic laws of arithmetic, and the objects and concepts involved, are immediately accessible as items of knowledge or as items to be grasped. As Frege makes plain, identifying them can involve much intellectual labour spanning many epochs.

Now whether or not the platonist-idealist reading is plausible, the hermeneutic presupposition itself—that definitions capture what is pretheoretically already determinately there—cannot, I think, be seriously gainsaid. It is true that Frege speaks of definitions as arbitrary stipulations, and of terms whose meaning has yet to be laid down by such a device. But we should

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112 Grundlagen §105, p. 115.
113 CP, p. 369.
114 Grundlagen §105, p. 115.
116 Grundlagen, p. vii; also 1891.
117 See pp. 22-23 above for Frege's view of definition as arbitrary stipulation.
not think that, for Frege, what is laid down was not already there independent of the arbitrary stipulation.\textsuperscript{118} The point is suggested, I believe, by the following remarks.

1. '[T]he concept of number, as we shall be forced to recognize, has a finer structure than most of the concepts of the other sciences, even though it is one of the simplest in arithmetic.'\textsuperscript{119}

2. 'Such differences [between types of thought] exist only in the degree of purity and independence of psychological influence and of the aid of language ... for the expression of thought, and somewhat further on the fineness of structure of the concepts; but precisely in this respect mathematics might be surpassed by no other science, not even those of philosophy.'\textsuperscript{120}

3. 'Since the subject matter of mathematics recedes into the background and is dominated by thought more than in the other sciences, and since mathematical ideas have been developed into a richer and more subtle structure than elsewhere, this science is especially suited to serve as a basis for epistemological and logical investigations.'\textsuperscript{121}

The first two passages come at the outset of \textit{Grundlagen}, well before any attempt is given to lay down the \textit{Sinn} and \textit{Bedeutung} of our arithmetical terms. And as intimated in the last passage, it is the task of logic, taken as an epistemological enterprise, to lay bare the nature and structure of these contents. This is the role of definitions. As such, Frege must take the refineness and determinateness of the structure of our central arithmetic notions to be independent of our definitions, independent of our linguistic capacities.\textsuperscript{122}

Furthermore, a consequence of the revisionist view is that Frege took mathematicians to be mistaken in thinking that they had proved the truth of

\textsuperscript{118} Nor should we think that, for Frege, what is stipulated is not what was already referred to prior to the definition. As we will show in chapter 3, Frege's stipulations can be seen as laying down what the referent of one's ordinary terms are.

\textsuperscript{119} \textit{Grundlagen}, p. iv.

\textsuperscript{120} Op. cit. pp. iii-iv.


\textsuperscript{122} As is well known, Frege operates with another conception of definition, that of explication of terms already in use but hitherto undefined. This touches on the issue of the relation between ordinary arithmetic language and that of Concept Script language. As said, our central concern here in this chapter, however, is to establish the precondition for the hermeneutic view. This depends on showing that there are, for Frege, determinate entities to be captured by definitions.
arithmetical propositions, or even grasped true thoughts. This consequence alone should make us wary, to say the least, of the revisionists’ way of supporting the revolutionary view.\textsuperscript{123} Certainly there is no suggestion in the following passage that, when engaged in arithmetical practice, Weierstrass grasps defective thoughts.

'Weierstrass has a sound intuition of what number is and working from this he constantly revises and adds to what should really follow from his official definitions. In so doing he involves himself in contradictions and yet arrives at true thoughts, which, one must admit, come into his mind in a purely haphazard way. \textit{His sentences express true thoughts, if they are rightly understood. But if one tried to understand them in accordance with his own definitions, one would go astray.}\textsuperscript{124}

Frege here makes clear that it is the definition that is faulty, not the thought, which, note, is said to have a truth-value. In saying this, he makes clear the independence of thought—and by corollary its constituent \textit{Sinne}, and what, if anything, they refer to—from definitions. It would not do to reply that in the above passage truth is mentioned merely to signal that Weierstrass is warranted in certain of his inferences and assertions.\textsuperscript{125} Truth, for Frege, is not that kind of thing. Nor do Frege’s numerous other critiques of mathematicians’ definitions suggest that, in laying down precise definitions and in constructing a system, we make \textit{Bedeutung} determinate for the first time. Thus, we might say, on behalf of the platonist-idealist that definitions lay bare the contours of the already existing concepts and objects that we have created all along from the primitive stages of our arithmetical development.\textsuperscript{126}

\textsuperscript{123} Their way being to undermine the hermeneutic presupposition.
\textsuperscript{124} 1914, p. 222, \textit{PW}. My emphasis.
\textsuperscript{125} This appears to be how Weiner responds to the, surely odd, consequence of her view that for Frege ordinary arithmetical language was deplete of truths, despite numerous passages where true propositions are mentioned (cf. 1990, p. 123). She does not, by the way, mention the passage just cited.
\textsuperscript{126} One further advantage of this reading is that it puts into perspective a further remark of Frege’s not yet mentioned. ‘If, as shown, we do not find the concept of magnitude in intuition, but create it ourselves, then we are justified in trying to formulate its definition so as to permit as manifold an application as possible, in order to extend the domain that is subject to arithmetic as far as possible.’ ‘Methods of Calculation based on an Extension of the Concept of Quantity’ (1874), p. 57, \textit{CP}. On the other hand, this is a very early work, which precedes all Frege’s more philosophical writings. So he may simply have changed his mind on reflecting more carefully on the nature of numbers.
What, then, of the claim that the reason why Bedeutung is dependent on our linguistic capacities is because, for Frege, reality itself is essentially structureless, that it is not carved at the joints, but is determinate only in the context of the definitions—distinctions—we care to impose? Well, one can speak of ‘reading something out of a jumble or mixture of things’ (was man aus einem Gemenge heraus liest), without implying that the world is an inarticulated, structureless jumble. One can speak of the world as a Gemenge, as a mixture out of which many different structures can be discerned, without implying that we impose linguistic-dependent concepts formed by us on an otherwise indeterminate, inchoate reality. It is true that what counts as non-primitive objects—i.e., those formed from concepts—do not conform to our common notion of an object, if there is such a notion. But so much the worse for our pre-theoretic intuition. Recognition of the objectivity of a concept denoted by, for example, ‘The Northern most half of an ornamental carving on my chair’ need not mean that the reality of this distinction is dependent on the potentiality of our descriptive resource. Recognition of such concepts need not have us cast doubt on whether, for Frege, reality is intrinsically ordered. Not least because Frege himself says that what we read out of the Gemenge is already there. ‘[W]hat we see into or single out from amongst other things is already there and does not come into existence as a result of [conceiving, understanding, of grasping a thought].’\footnote{Logik’ (1897), p. 137, PW.} And it is in this connection that we should read Frege’s remark that ‘a mathematician can no more create anything at will than can the geographer; he too can only discover what is there and give it a name’.\footnote{Op. cit.} Contrary to revisionists, definitions are a tool for the discovery of the essential structure of concepts, not a tool for their invention.

Take next the claim that, apropos the context principle, Frege qualifies the sense in which he understands an abstract object as self-subsistent.\footnote{Cf. Grundlagen §96, p.108; Grundgesetze, p. 23.} We noted

\footnote{\begin{quote}
‘The self-subsistence that I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a proposition, but only to preclude the use of such words as predicates or attributes, which appreciably alters their meaning.’ Op. cit.
\end{quote}}
the claim that the context principle precludes a meta-semantic perspective, and
that, while from a logical point of view number terms are proper names that stand
for objects, from a metaphysical point of view, there are no real objects for them
to stand for. We can grant that the context principle precludes the notion of a
relation between word and extra-linguistic correlate, conceived as a mental
association or conception had outside of language. Again, the context principle
abjures any further, metaphysical, inquiry once singular terms occur in true
sentences and true theories. But that need not imply that no relation whatever, no
interaction with abstracta, exists: for example with the extensions of
equinoymous concepts. Concepts and objects are, for Frege, independent of our
linguistic capacities: in particular, independent of Frege’s principle of
completeness. Rather than see Frege as indicating a non-ontological view, we
might see him as saying that, generally speaking, terms do not have content—*Sinn*
or *Bedeutung*—independently of the context in which they occur. Their content
consists in part in their contribution to the content of the whole sentence, and no
contribution is made outside that context.\(^\text{131}\) In light of this, both the platonist-
realist and platonist-idealist alike can say that, given the context principle,
interaction with abstracta is subsequent to our having grasped true thoughts whose
constituent *Sinne* include those of abstract singular terms. It is in virtue of *Sinn*
that a reference relation to extra-linguistic entities is effected. It is because of it
that we can talk about, and thus interact with, abstract entities.

One might further illustrate the view of Frege as platonist idealist—in
pursuit of a hermeneutically conceived logicism—by conceding that the logical
categories (function and object) are likewise themselves parts of the structure of
mind, rather than features of a mind-independent reality. That is, the truth of the
hermeneutic presupposition need not preclude the view that the logical categories
supervene on our modes of thinking, and possible syntax. But if content is to be
determinate independently of definitions—definitions being a means of getting at
their structure, a means of discovery—then it cannot also be true that, on

\(^{\text{131}}\) Cf. Dummett’s *Frege: Philosophy of Language*, p. 4, 1981a.
discovering this structured content—say, that numbers are the extensions of equinuerous concepts—we impose on it our mind-dependent logical categories, dividing an already structured content into concepts and objects, into a form that it did not already have. Frege can have no notion of structure save in terms of his logical categories. The structures must be of objects and functions. In that case, whether or not structure is mind imposed, it must precede our definitions and conscious cognitive activity. Accordingly, for Frege, the mathematical world as it appears to us is not the world as it is in itself. On this view, the mind receives raw non-conceptual inputs from the perceptual sense, or from an intuitive faculty, or from the faculty of reason, whereupon it constructs a world of objective appearance by those structures that are part of the constituents of the mind—a world which, independently of the mind’s categories and principles, is of structureless content. The central concepts and objects of number theory, as well as the laws of that theory, would likewise be constructed by the mind and at the same time exist determinately prior to our discovery and articulation by means of definitions. But whether this generalized platonist-idealist view is Frege’s is simply unclear, there being insufficient evidence. Still, if such a view could be established, then of course it would not be incompatible with the hermeneutic presupposition.

Be that as it may, it might be wondered whether the latter version of Frege as platonist-idealist has sufficient content to distinguish itself adequately from the platonist-realist reading. This question becomes sharpest when we bear in mind that, for Frege, the basic arithmetical concepts and objects, like the logical laws, are to be regarded as eternal and immutable, as well as constitutive features of mindfulness. The concept of mindfulness is not, for Frege, dependent on being realized by thinkers;\textsuperscript{132} the laws and its constituents that comprise the concept of mindfulness transcend instances. Pending further clarification of the alleged distinction one could regard axiom V in accordance with the platonist-realist construal. It is true that various of Frege’s remarks, that objectivity is founded on reason, that objectivity is not independent of judgement, nor of what can be

\textsuperscript{132} Anymore than, say, Pythagoras’s theorem is dependent on its being grasped or discovered.
conceived, may seem to fit uncomfortably with the platonist-realist view. But this impression may be dispelled once we read them as an epistemic rather than a constitutive claim. Accordingly, Frege is not saying that the logical objects are constituted from elements of our rational faculty. He is not saying that what is objective arises out of our judgements. He is saying that nothing could count for us as objective unless a thought concerning it is possible, unless the thing could be conceived, unless in some transcendental sense ideal judgements were possible.

Let us return to the revisionists. The question of mind independent eternal and immutable truths is particularly pressing for them. According to J. Weiner, Frege is expressing the view that the justification of arithmetical truths—existence proofs—involves no facts about us or about our subjective ideas, no psychological facts. But this is implausible. Certainly Frege wished to exclude such facts from proofs, but that is not enough to explain the above (platonist) remarks. Frege specifically had the eternality of thoughts in mind. He believed that we stand in relation to them, that judgement involves a two-place relation between a belief and an immutable entity. Similarly does the act of grasping involve a relation to a non-spatio-temporal object. Frege would have said none of these things if he had taken the \textit{Sinn} of a sentence to be founded ultimately on linguistic-dependent distinctions.

What, now, of the claim that there are, for Frege, no literal philosophical truths? And how might we explain his remark, not yet mentioned, that 'I take it as a sure sign of a mistake if logic has need of metaphysics and psychology—sciences that require their own logical first principles'. Is not this a further reason for saying that Frege’s philosophical pronouncements are elucidations: \textit{viz.} for denying that they are literal objective statements of truths? No. Frege is here abjuring the claim that the science of logic, far from the mother of all science—the basic foundational science of all science—could itself be founded on a further, still more basic, science: namely, metaphysics. Such a metaphysic, or

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135 Ibid. p. 369, p. 371.
136 'Logik' (1897), pp. 137-38, \textit{PW}.

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psychology, could never extricate itself from use of the laws it seeks to explain. This is why it could not be supplanted as a foundational science. Still, that is not to say that, since such a metaphysic would be subject to laws, metaphysics itself could not ground the truth of those laws. Room is still available for an ontological foundation to play an explanatory role.\footnote{Grundgesetze, p. 18.}

At any rate, Frege is not here denying the possibility of a science of metaphysics. He is denying only its primacy as a science of first principles. So although Frege's logical categories, and other philosophical claims, cannot be stated in his own Concept Script, that may simply be one of its defects, explained in terms of Frege's single-minded pursuit of his logicist goal. Suppose, however, that he would deny that his philosophical pronouncements are expressible in any Concept Script. Well, he might think that there are such true thoughts nevertheless: for example, that objects and functions are totally different from each other. Although we have here a logically improper use of language; although there is no way of saying what we are trying to say in a logically impeccable language: there may yet be such a thought to aim at, and moreover one that is true. Such thoughts would be ineffable. And any attempt to state the distinction between objects and functions would fail.\footnote{Cf. Diamond, 1984, 'I find it difficult to be tolerant about the ascription of such nonsense to Frege (or about the same ascription in the case of Wittgenstein). When there is no way of saying properly what we are trying to say, what we come out with is in fact a kind of nonsense, and corresponds to no ineffable truth' (ibid. p. 181).}

As she points out, Wittgenstein's remarks about throwing away the ladder are not altogether clear. 'Whoever understands me eventually recognizes [my propositions] as nonsensical, when he had used them—as steps—to climb up beyond them. (He must, so to speak, throw away the ladder after he has climbed up it)' \textit{Tractatus} (6. 54). Diamond thinks that these remarks apply to the point that some features of reality cannot be put into words: to the very idea of what can be shown. Cf. 'Throwing away the Ladder' (1979), p. 7. As already said, it is in this way that Weiner thinks we should understand Frege's philosophical remarks; and according to Diamond, it is this way that we should understand Wittgenstein's remark that in \textit{Tractatus} he is not trying to put forward philosophical doctrines, since this cannot be done—a view she also takes to be Frege's. It is not the case that after we have thrown away the ladder what remains is a 'logical form of reality', some essential feature of reality, which we cannot say or think is there. We
be shown nonetheless. That they would be shown would be because of an external relation between the logical characteristic of things, and the logical features that are taken to be part of the mode of expressions. But, then, what of the claim that all theories must be stated in a Concept Script? Frege nowhere explicitly says this. It is true that early in *Grundlagen* he says that what is objective is subject to laws. But again, it might be claimed that, even though there can be no linguistically expressible thoughts about the logical categories, still those thoughts are subject to laws.\textsuperscript{140}

§1.5. Closing Remarks. We have sought to undermine revisionism, and in particular its corollary that, for Frege, the contents of the reconstructed number theory, those of his logicist system, were radically distinct from whatever content number theory had prior to the reconstruction. And we have pointed to some ambiguity about whether Frege can be regarded as platonist-realist or platonist-idealistic. What is important for the hermeneutic hypothesis is that there be room for the view that there are determinate thoughts and other items to be discovered prior to our grasp of them. For unless Frege believed that arithmetical thinkers make shareable discoveries about determinate entities, which they have already been referring to, that entities are already there awaiting disclosure by analysis, then the hermeneutic hypothesis is false. We have cited passages replete with allusions to the possibility of discovery; of concepts, objects and thoughts whose richness of structure await analysis; and we have deflected conflicting interpretations.

So let us grant for the purposes of what follows that, for Frege, there is a realm of determinate logico-arithmetic items to be discovered. It is quite another matter of course whether Frege believed that the definitions of Concept

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\textsuperscript{140} Frege appears to suggest that we can refer to concepts by 'the concept' despite the defect. It is just that, taken literally, the expression does not so refer. See for example his 1914, p. 239, *PW.*
Script captured the *Bedeutung* of our ordinary arithmetical terms. As said, revisionism is only one way of arguing for the revolutionary claim. And we have not yet argued that our ordinary arithmetical terms have *Bedeutung*. Nor have we argued that they are unique. We have argued only that there are determinate *Bedeutung* for our terms to refer to. Let us then turn to these two further features of the hermeneutic hypothesis, that our terms have a *Bedeutung*, and that they are unique.
Chapter 2
Capturing *Bedeutung*

§2.1. Introduction. According to the weak hermeneutic thesis, Frege intended his Concept Script to capture only the *Bedeutung* of our ordinary arithmetical terms. According to the _mild-strong_ hermeneutic thesis, he intended that their *Sinn*_ too be preserved. The revolutionary thesis denies both these claims. There are three ways of arguing for the revolutionary claim, and in the last chapter we considered and rejected one of these: revisionism. Revisionists claim that (i), for Frege, ordinary arithmetical language lacked determinate content; (ii) that this was because a sentence having truth evaluable content is dependent on adequate definitions, which, for Frege, were wholly lacking prior to his logicist system. In this chapter, I consider the two other ways of arguing for the revolutionary thesis. Unlike revisionists, advocates of both approaches concede that, for Frege, there is a determinate content independent of our capacity to make linguistically expressible distinctions. Like revisionists, one of these approaches relies on placing special significance on Frege’s use of stipulative definitions. The claim is that pre-Concept Script language was defective: its terms failed to satisfy the criterion of adequacy for a term to have a *Bedeutung*. The other approach is to claim that, even if pre-Concept Script language was semantically in order, certain of Frege’s other remarks tell against the hermeneutic hypothesis. In particular, it claims that these remarks belie the view that Frege’s logicism was a quest to uncover the nature of hidden and unique objects, which analysis of ordinary arithmetical terms was meant to disclose.

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1 As in the last chapter, I shall use ‘hermeneutic hypothesis’ to mean both the mild and strong version. Recall that the mild-strong distinction denotes the state of the thinker’s grasp. Mild-hermeneutism claims that, while Frege sought to preserve the *Sinn* and *Bedeutung* of our ordinary arithmetical language, our grasp of its *Sinne* was partial or defective. By contrast, strong-hermeneutism is the claim that our grasp is full, if to some extent tacit, and unperspicuous.
§2.2. Does Pre-Concept Script Arithmetical Language have a Determinate Content?

After having opened his *Grundlagen* with the question what is the meaning of 'the number one?', and after having there exposed to ridicule a number of likely responses, Frege writes

'... is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one that is apparently so simple? Small hope, then, that we will be able to say what number is. If a concept fundamental to a mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome; especially as we will hardly succeed in finally clearing up negative numbers, or fractional or complex numbers, so long as our insight into the foundation of the whole structure of arithmetic is still defective.'

Much later, he says

'As you know, I have made many efforts to get clear about what we mean by the word 'number'. It is suggestive, but I think no more than suggestive, that Frege in the above uses the plural and singular possessive adjectives in referring to arithmetic as 'our science' and of 'its objects', and uses the definite article in 'the foundations of the whole structure' and 'the number one'. It suggests that he thinks that the numbers are unique objects, that arithmetic has a unique foundation, that it is a science consisting of a body of truths, and that it is 'our' science, since some at least of its truths are grasped and employed by us. Moreover, the first passage does not say that we altogether lack an insight into the foundation. It says only that it is defective. The defect, in my view, is that we are at a loss properly to explicate the nature of what we refer to; we fail to provide satisfactory definitions.

By contrast, one might say that Frege thinks that our defective insight, shown in our failure to provide adequate definitions, stems from the fact that, as he sees it, our arithmetical terms fail to refer. The 'scandal' in question is in part that we are unaware of these shortcomings. True, Frege speaks of 'our science'...

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2 *Grundlagen*, p. ii. On numerous other occasions too Frege refers to the deficiencies of previous attempts by others to define the central concepts of arithmetic. See for example, the introduction, and chapters one and two of *Grundlagen*, and 'On Mr. H. Schubert's Numbers' (1899), p. 249, *CP*.

3 'Frege to Zsigmondy' (undated) xviii/1, p. 176, *Gottlob Frege: Philosophical and Mathematical Correspondence (BW)*, 1980.
and 'the foundation'. But he may simply be highlighting that these are assumptions we make (he may also think that it is a useful way of speaking). By asking after the meaning of our terms, he emphasizes our misunderstanding, which he thinks is as serious as it is widespread. And when he emphasizes that we do not know what we refer to when using the most basic arithmetical terms, he may be trying to rid us of these assumptions. He may be suggesting that our ignorance stems from semantic defects widespread in our language. After all, one might say, our ordinary arithmetical terms fail to meet his criteria for having a Bedeutung.\(^4\) And in crucial places in Grundlagen, Frege speaks of definitions as stipulations of meanings of terms hitherto without meaning. ‘When we are concerned with truth, ... we have to reject concept-words whose demarcation is indistinct. Of every object it must be determined whether it falls under the concept or not; a concept-word that does not satisfy this requirement has no Bedeutung.’\(^5\) Similarly, in Begriffschrift Frege says that sentences containing vague expressions lack a ‘judgeable content’.\(^6\) Also much later Frege says ‘[i]t is self-evident that what is given a name (sign) must be determined by the definition. A word without a determinate meaning has no meaning so far as mathematics is concerned’.\(^7\)

Moreover, looking back on Grundlagen, having drawn the distinction between Sinn and Bedeutung, Frege appears to regard those defects of our terms as likewise impacting on their having Sinn. For he asks, ‘[h]as the question “Are we still Christians?” really got a Sinn, if it is indeterminate whom the predicate “Christian” can truly be ascribed to, and who must be refused it?’ Frege gives a seemingly unambiguous answer in the negative. This he does when showing that to restrict the domain of quantifiers is to violate the principle of completeness. For suppose that, on defining the term for the addition operator, we stipulate that

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\(^4\) As noted in chapter 1. T. Burge has urged against construing Frege as taking virtually all words in ordinary discourse to be vague. Vagueness, he thinks, is treated as a special case with the most prevalent cases of defective understanding identified as imprecision and fluctuating usage of terms. Cf. ‘Frege on Sense and Linguistic Meaning’ (1990), pp. 35-37, in The Analytic Tradition (1990), eds. D. Bell and T. Cooper. But imprecision is also a failure of the sharpness constraint, and it has all the apparent repercussions, now to be discussed, that linguistic vagueness has.

\(^5\) Nachgelassene Schriften (NS), p. 133.

\(^6\) 1879; §27, CN.
our variables range only over numbers, rather than over all objects.¹ Then by contraposition the proposition expressed by ‘If a is a number and b is a number then \(a + b = b + a\)’ can be transformed into the proposition expressed by ‘If \(a + b\) is not equal to \(b + a\), and a is a number, then b is not a number’. Not only do we violate our own stipulation by breaking out of the restricted domain ‘... our antecedent clause ‘if \(a + b\) is not equal to \(b + a\)’ is without Sinn, assuming that the sign of addition has not been completely defined’.² To be sure, the view that ordinary arithmetic lacks Sinn and Bedeutung is not that our ordinary arithmetical language lacks content tout court. For proponents of this view impute to Frege a distinction between conventional significance or meaning, and Sinn and Bedeutung. The latter are technical philosophical notions and the preserve of a properly constructed, scientific language. According to the first two approaches to the revolutionary view, Frege thinks that ordinary arithmetical language has only conventional significance—which arises from linguistic norms of usage—but that this content is insufficiently precise or structured to determine a truth-value.

The trouble with the above view is that it takes out of context Frege’s remarks regarding the sharpness constraint of definitions. In doing so, it exaggerates the consequence of not meeting the constraint. That Frege did not think that faulty definitions imply that our terms literally lack Sinn and Bedeutung is suggested in his answer to the formalist’s claim that arithmetical terms lack content. Frege notes that the very application of number theory is made possible because of its propositional content. ‘... [A]n arithmetic with no thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded

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¹ ‘Foundations of Geometry II’ (1906), p. 303, CP.
² Frege took logic to have a maximal generality of application. Accordingly, he took quantifiers to range over an unrestricted domain. So, for example, first order quantifiers range over all objects; second-order quantifiers range over all properties; third-order quantifiers range over all properties of properties, and so forth.
³ ‘Frege on Definitions’, pp. 149-50, from Grundgesetze, vol. ii, §65, in Translations of the Philosophical Writings of Gottlob Frege (TWP), eds. P. Geach and M. Black. As already said, one reason for Frege’s sharpness constraint is that unless complied with it would be ‘impossible to lay down precise laws for them. The law of excluded middle is really just another form of the requirement that the concept should have a sharp boundary’. §56, p. 139, TPW.
to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing ...?\(^{10}\)

What, then, is the right way to view Frege's stricture on definitions? Frege says of the function of ordinary language that

'... it is essentially fulfilled if people engaged in communication with one another connect the same thought, or approximately the same thought, with the same sentence. For this it is not at all necessary that the individual words should have a *Sinn* and *Bedeutung* of their own, provided only that the whole sentence has a *Sinn*. Where inferences are to be drawn the case is different: for this it is essential that the same expression should occur in two sentences and should have exactly the same meaning in both cases. It must therefore have a meaning of its own, independent of the other parts of the sentence.'\(^{11}\)

The passage clearly indicates that Frege believed that when we use ordinary arithmetical language we grasp the same thoughts. But in other ways, it is not altogether clear at first sight what Frege has in mind here. Surely he does not really believe that in ordinary use of language we grasp thoughts and communicate without the use of inferences, however elementary. According to the passage, when we use the pre-Concept Script sentences '0 < 1', and '1 < 2' we share the same thoughts, namely that 0 < 1, and 1 < 2—namely by inference. So surely Frege would not deny that from these thoughts we can, and ordinarily do, grasp the thought that 0 < 2. But that we can do so is because of the common occurrences of the individual *Sinn*. If communication is effected, then the thinkers take the thought as having a determinate logical form. That is a condition for the thought to have *aboutness*. So for example, the thinkers would take the above thoughts to be about objects, numbers, standing in a relation of inequality. This must be because the thinkers read off the structure of the thought, which indicates its aboutness, from the structure of sentence. So the logically significant units of the sentence will have a *Sinn*. How, then, should we

\(^{10}\) §65, p. 167, *TWP*.

\(^{11}\) 'Frege to Peano' (29.9.1896), p. 115, *BW*.  

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understand the claim that the individual words, taken on their own, lack Sinn and Bedeutung?

The first thing to note is that the principal target of Frege's criticism in the above passage is that of conditional definitions. With this type of definition, the same word can mean different things in different sentences, and the parts of the different sentence signal what meaning the term in question has. As Frege goes on to say, regarding the Sinn of an individual term, '[i]n the case of incompletely defined words there is no ... independence: what matters in such a case is whether the case at hand is the one to which the definition refers, and that depends on the other parts of the sentence'. So the same word would have a shifting Sinn determined by the sentence in which it occurs. If we relied entirely on the linguistic expression to know what the Sinn was, it would be as if it had none. We can clarify this claim in terms of Frege's logically perfect language. From the perspective of his Concept Script, the non-primitive thoughts would be grasped from definitions, axioms and rules of inference. From that perspective, Frege takes our grasp of a thought, and its constituents to be in virtue of our use of the logically rigorous language. Terms must be rigorously expressed because we draw inferences partly on the basis of them; and which thought we grasp in the proof system we take to be determined by the definitions, axioms, and rules of inference. So if the latter three items are not impeccably laid down, then we cannot take ourselves to have proved or grasped the thought intended. This is why Frege says 'logic can only recognize concepts whose boundaries of application are clearly laid down because only thereby can precise laws be set up'. Unless a term is associated with a unique Sinn and Bedeutung, several such Sinn and Bedeutung are likely to be attached to it. From the point of view of logic—from the point of view of usage in proof—the indefiniteness would make the term of little utility. From that perspective, it would be as much as to say that the term has no Sinn and Bedeutung at all. Hence we should see Frege's strict

\begin{footnotesize}
\begin{enumerate}
\item[13] 'Frege to Peano', p. 114, BW.
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adequacy constraint on definition in the context of the demands and purposes of ideal proof. On laying down the principle of completeness, Frege says ‘All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp ...’. But that is not to say that, from our everyday perspective, outside the demands of a rigorous proof system and the pursuit of a rigorous science, our thoughts and their constituent terms do not have a Sinn.

The following passage, cited in the last chapter in favour of the hermeneutic presupposition, can be used to support this claim that ordinary arithmetical language has a determinate content: Sinn and Bedeutung. As already said, Frege’s actual remarks should make us wary of attributing to him the belief that mathematicians were mistaken in thinking that they had ever reasoned soundly, or in believing that they had never grasped true arithmetical thoughts. Indeed, a closer look at the passage in question suggests how further we should understand the consequence of those linguistic defects born of faulty definitions:

Weierstrass has a sound intuition of what number is and working from this he constantly revises and adds to what should really follow from his official definitions. In so doing he involves himself in contradictions and yet arrives at true thoughts, which, one must admit, come into his mind in a purely haphazard way. His sentences express true thoughts, if they are rightly understood. But if one tried to understand them in accordance with his own definitions, one would go astray (my emphasis).

What is suggested here is, again, that Frege believed that the thinker refers to the Bedeutung of pre-Concept Script terms despite the linguistic defects, despite the faulty understanding carried by the definitions. In the surrounding text from which the above passage is taken, Frege says that the reference of a thought is determined by the referents of its parts. In that case, Frege would take Weierstrass to be referring to numbers. Finally neither in this passage, nor in many others like it, in which he speaks of pre-Concept Script arithmetical terms as

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15 Grundlagen, p. 87. My emphasis.
16 Namely: that there is a determinate arithmetical content independent of our capacity to draw distinctions or formulate definitions. See chapter 1.
17 ‘Logik in der Mathematik’ (1914), p. 222, Postumous Writings (PW).
having *Bedeutung*, is there evidence that Frege had in mind some loose sense of the term ‘*Bedeutung*’—e.g., conventional significance or linguistic meaning.\(^\text{18}\)

§2.3. **Definition as Stipulation Vs. Definition as Analysis: The Role of Definition and the Motivation of Frege’s Logicism.** What, now, of the presence of stipulative definitions in Frege’s foundational programme? Clearly this type of definition is *prima facie* incompatible with the hermeneutic hypothesis. The issue can be usefully addressed, I think, in light of a further doubt about the hermeneutic hypothesis. This doubt arises when considering the question of what motivated Frege’s logicist project.

By asking after Frege’s motivation I mean: what was it about that project—*viz.* showing that the truths of arithmetic are derivable exclusively from purely logical axioms (primitive truths) and definitions that utilize only terms of logic—that, for Frege, promised to make it worthwhile? According to Paul Benacerraf,\(^\text{19}\) the sole motivation of Frege’s logicism was mathematical. An assumption associated with this claim is that, being primarily a mathematical project, Frege would not have been interested in seeking to preserve reference of our ordinary mathematical terms. *Qua* mathematician, Frege sought to construct arithmetic anew from scientifically sound foundations, and that *qua* mathematician he was simply not concerned with such philosophical issues as the preservation of *Sinn* and *Bedeutung* between ordinary arithmetical terms already in use and those of his Concept Script.\(^\text{20}\) Frege’s use of stipulative definitions perfectly accords with this way of construing his logicism. It is a tool for the radical reconstruction of arithmetic.

One thing wrong with this view of Frege’s motivation is that it overlooks the role and importance of another kind of definition employed in Frege’s inquiries. Frege used analytic definitions: *viz.* analysis of terms with determinate meaning that had been introduced without being defined. Use of this type of

\(^{18}\) Pace Weiner, 1990, p. 112.


\(^{20}\) Ibid. p. 63.
definition better accords with a historically more accurate picture of the real impetus behind Frege's investigations. A closer examination of the climate of mathematics in Frege's time shows that mathematicians' conception of definitions was by and large informed by an essentialist presupposition and by the view that definitions capture otherwise hidden features of the subject matter already referred to. What emerges is an integrated picture of Frege as mathematician-philosopher. Of course this leaves us *prima facie* with the problem how to reconcile the use of two quite different conceptions of definition. Clearly, a meaning for a term cannot be stipulated if it already has one. Nor can an analysis of a term's *Sinn* and *Bedeutung* be had if it has none. We will address this after we have laid out the issues behind the question what promised to make Frege's logicism worthwhile.

§2.3.1. Beneacerraf's Reading of Frege's Motivation and Stipulative Definitions. At first sight, the motivation for Frege's foundationalist project may seem clear. After all, do not the following two passages amount to a manifesto for reforming mathematical thinking, for instituting greater rigour into mathematical practice? And is it not clear that the point of rigour was to attain cogency and consistency of reasoning, as well as certainty?

1. '... essential for mathematics ... is the recognition of its close connection with logic ... This much everyone would allow that any inquiry into the cogency of a proof or the justification of a definition must be a matter of logic. But such inquiries simply cannot be eliminated from mathematics, *for it is only through answering them that we can attain to the necessary certainty*.'

2. '...it must still be borne in mind that the rigour of the proof remains an illusion, even though no link be missing in the chain of our deductions, so long as the definitions are justified only as an afterthought, by our failing to come across any contradictions. *For this reason I have felt bound to go back rather further into the general logical foundations of our science ...*'

Such is the line taken by Paul Benacerraf in a reading that might be further supported by the first two sections of *Grundlagen*. There Frege appears to align

himself with a tradition associated with the likes of Cauchy, Dedekind and Weierstrass, whereby rigour was pursued in order to help overcome real (not just hypothetical) cases of inconsistency and to ensure cogency and certainty in mathematical reasoning.

'The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials.'

'Proceeding along these lines, we are bound eventually to come to the concepts of Number and to the simplest propositions holding of positive numbers, which form the foundation of the whole of arithmetic.'

In emphasizing Frege's motivation as that of securing consistency and cogency of reasoning in mathematics, Benacerraf has argued that Frege's logicism is only incidentally a philosophical project. I leave aside what precisely 'incidentally' might mean here. For his view will, at any rate, strike some as odd given that, shortly after the passage just cited, Frege declares that 'Philosophical motives too have prompted me to inquiries of this kind': namely that of showing that arithmetic truths are analytic a priori. But Frege, we are told, has reconstrued these notions in such a way that they no longer belong to philosophy. The question how we are to classify arithmetical truths given these

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24 'Frege: The Last Logicist', p. 34.
25 Ibid. p. 23.
26 Grundlagen §3.
27 Ibid. p. 23, p. 24, p. 26, p. 32, p. 34. Traditionally, an analytic truth is one that is true solely in virtue of the meaning of terms, or as Hume put it, in virtue of the connection between ideas. See his Enquiries Concerning Human Understanding, §iv, part 1, p. 25, ed. L. A Selby-Bigge, 3rd edition, Clarendon Press, Oxford. Analyticity was, and still is, envisaged, by some, as a means to explaining a priori knowledge. Warrant for believing the truths of such propositions allegedly flows from our understanding of the propositions or meaning of the terms used to express the propositions. Recall that for Frege, a truth is a priori if it is derivable from general laws, which neither needs, nor admits of proof. If the proof relies essentially on 'appeal to facts', by which Frege means appeal to truths which cannot be proved and are not general because they contain assertions about particular objects, then the truths are synthetic. And whether the truth is synthetic or analytic depends respectively on the degree of generality of the laws on which they rest. These ideas will be discussed more fully in chapter six, when I discuss a possible connection between, on the one hand, analyticity and Frege's alleged epistemological motivation, and on the other hand the hermeneutic hypothesis. What matters here is that, as Benacerraf stresses, Frege's way of understanding these notions appear to differ from later epistemic conceptions.
distinctions has become a mathematical one. It is a matter of ‘finding the truth of a proposition and following it up right back to the primitive truths’.28

‘For even though the concepts concerned may themselves belong to philosophy, yet, as I believe, no decision on these questions can be reached without assistance from mathematics—though this depends of course on the sense in which we understand them.’29

Benacerraf thinks that the latter clause ‘though this depends of course on the sense in which we understand them’ is especially telling. We are not to understand analytic truths as philosophers later conceived them; not as propositions true in virtue of their meaning. They ‘concern ... not the content of the judgement but the justification [proof] for making the judgement’.30 Similarly are the notions of a priori and a posteriori divorced from traditional epistemological inquiries. A proposition is a priori if it is derivable from general laws, but a posteriori if it rests on a singular rather than general statement. Nor apparently is the kind of proof (justification) in question epistemological. What is sought after is insight into the kind of dependence these truths have one upon the other, not insight into relations among beliefs in a foundation (or in a coherent set).

I postpone discussion of whether Benacerraf is right in his understanding of Frege’s notions of the analytic a priori until chapter six.31 The above epistemological issues can be temporarily subtracted in presenting an integrated picture of Frege as mathematician-philosopher. Before outlining this view, it will be useful to see how, in responding to Benacerraf, other interpreters have also misunderstood the kind of conception of definition that, I think, informs Frege’s work.

These commentators profess to being puzzled at the idea that Frege’s logicism was motivated by a mathematical point of view. Some of them maintain either that Frege’s Grundlagen is without a mathematical motivation, or that what

28 Grundlagen, §3, p. 4.
31 I take up these issues when, continuing our discussion of his motivation, we discuss Frege’s alleged epistemology. It will transpire that Frege has a pre-modern, Leibnizian conception of the a priori, which is indeed epistemological.
Frege took to be its motivation is unconvincing. There are at least four reasons for this view. First, it is claimed that neither the mathematician nor the competent layman would claim to be other than certain apropos the veracity of the simplest arithmetic propositions. At any rate, it is not clear that any proof could improve upon these ordinary and \textit{prima facie} well-founded convictions of ours. Second, it is hardly clear that there is a threat of inconsistency. For one thing, most mathematicians do not even attempt to define positive whole numbers, and even where a definition is attempted it is, as Frege himself notes, frequently omitted from use in proof. For another thing, it is \textit{prima facie} implausible to suppose that lest proof be given, arithmetic might show itself to be inconsistent. The third reason for thinking that Frege's logicism is either without a mathematical motivation, or without a convincing one, is that the tradition with which Frege appears to have associated himself had come to an end by the time Frege had arrived on the scene. In effect, Frege had misunderstood the mathematical tradition; which, if true, might explain why his work was not generally well received by the mathematical community.


33 Well founded because our familiarity with these truths, and our countless successful applications, makes them seem right. These are considerations that Frege himself notes. 34 \textit{Ibid.} p. 306.

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Fourth, it is unclear that the mathematical objectives identified as Frege’s accord with the tradition of rigour to which Frege aligns himself. For on one reading, the movement to rigorize analysis stemmed from problems quite different from any that could be said to lie behind Frege’s logicism. Let me illustrate this latter point.

On the one hand, the demand in mathematics for sharper definitions of concepts arose in order to overcome inconsistencies involving the concepts in question, or because the lack of sharper definitions prohibited the solution of problems; and to prevent possible future instances of them. On the other hand, there is the view that, rigour alone was not enough to motivate the need for sharper definitions of concepts—e.g., that of limit, continuity, convergence, and the derivative—and that the real driving force was the need to solve quite different mathematical problems. According to this view, Cauchy’s concern to define the above concepts was pragmatically motivated. He himself relied on less than rigorous methods—for example, he used the notion of infinitesimal—where these served his purpose of solving technical problems. Cauchy required the clarification of the above concepts because the algebra and geometry inherited from Euler and d’Alembert and Lagrange was insufficient to determine whether the sum of an infinite series of continuous functions is itself always continuous. This was worth knowing because it laid the way to assess the merits of various techniques for solving partial differential equations. Though the notion of an infinitesimal proved useful to Cauchy in helping to solve technical problems, anomalies eventually arose, which Weierstrass sought to eliminate in order to

epistemological, motivation. It is a search to establish how we can know the truths of arithmetic a priori. Consider Frege’s attempt to delineate clearly the benefits that will result from answering the questions he has posed. These are epistemological benefits that embody the ideals of mathematical apriorism. When Frege emphasizes the possibility of complete clarity and certainty in mathematical knowledge, he is advancing a picture of mathematics that is almost irrelevant to the working mathematician.

The Kitcher view thus tries to turn the Benacerraf view on its head, arguing that the point of Grundlagen was solely philosophical rather than mathematical. The kind of philosophical logicism attributed to Frege involves the bringing together of a number of disparate features of Frege’s work in a way that makes Frege’s logicism an extension and revision of Kant’s epistemology. Again, this is a matter for chapter six.

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By ‘infinitesimal’ was meant a quantity smaller than any fraction but greater than zero.
have reliable methods for his work on elliptic function theory. Finally Dedekind sought to define continuity in order to establish the existence of limits—Cauchy's appeal to geometric analogies having proved faulty. On either way of reading the reasons behind the pursuit of rigour, it is not obvious how Frege's own quest for rigour, and thus his pursuit of sharper definitions, fits with this tradition; and many commentators have concluded that Frege was wrong to have aligned himself to it. As Kitcher observes

> When we disentangle the factors which led to the Weierstrassian rigorization of analysis, we find a sequence of local responses to mathematical problems ... Frege was wrong to portray himself as continuing the nineteenth century tradition.\(^{39}\)

An assumption of the Kitcherian reading of Frege's motivation is that virtually all nineteenth century mathematicians saw rigorization of analysis as a means to solving technical problems. On that assumption, Frege's logicism has no place within this tradition, try though he did to align himself to it. But there are two rather stark flaws behind Kitcher's view. He reasons that no mathematician of Frege's time believed that the truths of arithmetic were in danger of falling into contradiction; so that it's not the case that the securing of certainty is Frege's mathematical motivation. Thus from a mathematician's perspective, Frege's logicism is without a point. The first flaw is to think that a mathematical motivation must be in the mainstream of research. The second flaw is to think that Frege, himself of course no inconsiderable mathematician, was mistaken about his motivation—given what Kitcher takes to have been the main area of research in mathematics at that time. As it happens, in the passage cited from *Grundlagen* section 1, Frege was not aligning himself to the tradition outlined above. As we will now see, more was going on in mathematics in Frege's time than the Kitcherian makes out.\(^{40}\)


\(^{39}\) Cf. 1984, pp. 269-70.

\(^{40}\) I leave the issue of Frege's apparent concern with inconsistency unresolved here. I will return to it in chapter six.
§2.3.2. Frege’s Essentialism and Analytic Definitions. The trouble with misidentifying the mathematical climate in which Frege worked is that one fails thereby to appreciate an important feature of Frege’s conception of definitions. This conception was informed by a research project in mathematics that in turn was guided by metaphysical considerations. The connection between mathematical research and philosophical speculation existed in Frege’s time on a spectrum on which he stood, though by no means alone, at one extreme. Frege himself indicates in various of his works that affinities exist between mathematics and philosophy, and that it is both possible and desirable that there be further cooperation between these two disciplines. Here, for example, he calls for others to join him on his end of that spectrum.

'It may well be that the cooperation between these two sciences, in spite of many démarches from both sides, is not so flourishing as could be wished and would, for that matter, be possible.'

A good definition was not simply a matter of being usable in proof. Mathematicians typically saw their definitions as answerable to the nature of the things denoted by our ordinary mathematical terms. As I shall show, the nature of mathematical entities was often seen to be unclear or hidden and in need of being revealed in the definitions if the notions were to be properly understood and if their potential fruitfulness was to be maximized.43

Before saying more about this, there should be no doubt that Frege did operate with the explicative sense of definitions: analytic definitions. Witness, his response to Newton, Hobbes, and Locke’s view that numerical formulae are unprovable and immediately self-evident,44 and to Hankel and Leibniz’s view that the concept of number is indefinable. ‘If the general inclination is, on the whole, to hold that Number is indefinable, that is more because attempts to define it have

41 For example, Riemann, a distinguished mathematician if ever there was one, allowed philosophical concerns to influence the direction of his work. In his Habilitationschrift he explicitly rejects the idea that geometrical knowledge is a priori. Cf. G. Nowak, in History of Modern Mathematics, eds. Rowe and McCleary. Grassmann is another mathematician whose work was philosophically motivated. He was interested in ascertaining the nature of space and reasoning. Hankel, Heine, Bolzano, Dedekind are other examples.
42 Grundlagen, p. v.
43 See below for what more precisely I have in mind.
failed than because anything has been discovered in the nature of the case to show that it must be so." Furthermore, when at the outset of Grundlagen Frege asks after the meaning of ‘number one’, it seems he has in mind the idea reiterated in 1914 that ‘We have a simple sign with a long established use. We believe that we can give a logical analysis of its Sinn, obtaining a complex expression that in our opinion has the same sense’. In any case, if by ‘definition’ Frege meant simply arbitrary stipulations, he would not have spoken of justifying definitions, or indeed of some being false. Also, something like analytic (explicative) definitions are operative in Begriffsschrift, as a paper written to cast light on the latter work makes clear. This same paper indicates not only that Frege worked with this type of definition; it indicates that in general mathematicians did so too.

‘All these concepts have been developed in science and have proved their fruitfulness. For this reason what we may discover in them has a far higher claim on our attention than anything that our everyday trains of thought might offer has. For fruitfulness is the acid test of concepts, and scientific workshops the true field of study for logic.’

But what exactly did he take definitions to preserve? I think it is clear that Frege sought to uncover at least the Bedeutung of our mathematical terms, and that he took his contemporary mathematicians to have a similar conception. For in his ‘Review of Husserl’, Frege says that the definitions of mathematicians are intended to preserve Bedeutung: ‘what matters to the mathematician is the thing itself, the Bedeutung of words’. This view would strike most contemporary mathematicians as odd. Not so for mathematicians of Frege’s time. To see this, we need to bear in mind that a number of interrelated debates formed the backdrop of Frege’s mathematico-philosophical research, many of which touched on ontological questions, particularly in geometry. The issues relevant to our purpose emerge after the 1820s in the wake of radical changes in our understanding of geometry, especially given the renewed interest in projective

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47 ‘Boole’s Logical Calculus and the Concept-script’ (1880-81), p. 33, PW. My emphasis.
48 1894, p. 200, CP.
The projective school of geometry began to take space to consist in many more points and lines than traditional Euclidean geometers had believed. With the introduction, for example, of 'lines at infinity' and 'imaginary points', whose coordinates could be taken to be complex numbers, questions were raised about the conceptual basis of complex numbers, whether it was geometric or algebraic (see below). The controversy became part of a more general debate about whether geometry was about space, whose main advocate was Plücker, or whether it was, as Riemann and Grassmann believed, about certain structures presented algebraically. One instance of this type of research programme was to distinguish and isolate the algebraic and geometric content from the geometry inherited from Riemann. That it was unclear how Riemann's work should be classified was because results taken to be arithmetic in nature were proved using reasoning taken to be geometric. For this reason much of this type of research involved reproving theorems. But of the several perceived advantages in attaining

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49 Two kinds of development are relevant here. Traditionally, geometry was thought of as a priori and necessary as well as being about extension and quantity. The first development occurred with respect to the latter feature of that conception, i.e. that geometry was a quantitative science. This was undermined by the rediscovery and development of projective geometry: the study of those geometrical properties that remain invariant under certain transformations: i.e. properties that involve the notion of magnitude (e.g. angle, length, volume, area). See, e.g. E. Nagel’s ‘Geometry and the Formal Development of Logic’, esp. pp. 195-99. For the other development in geometry see main text and note 35 below.

In Frege's day, analytic expressions in geometry were complex expressions taken to refer to such objects as points and line segments, whereas nowadays analytic geometry is taken to be the science of tuples of numbers with relations defined on them. (Here 'analytic expressions' are those used in analytic geometry, where analytic geometry, in contrast with synthetic geometry, is characterized by use of the nowadays familiar coordinate system and coordinate equations, as against traditional Euclidean methods of proof in synthetic geometry.) Cf. J. Tappenden, 'Geometry and Generality in Frege's Philosophy of Arithmetic', p. 324, in Synthese vol. 102, no. 3, 1995a.

50 By 'complex number' I mean of course a number consisting of two parts, a real and an imaginary part, commonly denoted by 'a + bi', where 'a' and 'b' range over the domain of real numbers (i.e. the integers, rationals and irrationals), and 'i' refers to $\sqrt{-1}$. With the introduction of imaginary points came the question whether geometry could be characterized as the study of extension or space.

51 In particular there was the question of what grounded generalized geometries—a geometry is generalized by the free choice regarding what the elements of space were taken to be, e.g., lines, planes, spheres: whether they were grounded in spatial objects or abstract manifolds (what nowadays is regarded as point sets). Cf. Tappenden, ibid. p. 328.

52 Cf. Tappenden, ibid. p. 324. Another issue concerned the conceptual priority of arithmetic over geometry. Namely, could numerical coordinates be introduced into a synthetically characterized space without circularity? A further issue concerned Plücke's approach to general geometry versus that of Grassman and Riemann's.
new proofs of old truths, none, it should be said, concerned the promise of certainty since nowhere were the truths of the theorems in doubt. Rather the advantages included (i) a better understanding of why certain techniques work or why certain theorems were true; (ii) an insight into the fact behind the theorem; (iii) demarcation of the appropriate method of proof, reflecting whether the reasoning was geometric or arithmetic; (iv) a demarcation of the extent of generality.\textsuperscript{53}

A feature underlying this type of inquiry was the search for the right or appropriate conceptual setting—the right ontological territory—in which the mathematical notion under investigation could be seen to belong. It was a type of research project that sought greater insight into the structure and nature of mathematical concepts, many of which were long in use. It was in effect a quest for a greater insight into the semantics of one’s mathematical terms. Connectedly, the idea of a hidden conceptual setting, or one not fully revealed, was used to explain the notion of change and continuity within mathematics. In the context of projective geometry, for example, many geometers took the above developments not as an alternative to Euclidean geometry, or as overturning it, but as one ‘growing organically’ out Euclidean geometry, resulting from having discovered hidden aspects of geometry’s ontological territory. As M. Wilson has pointed out, ‘... the projective geometers saw the additions as revealing the “true world” in which geometrical figures live. The familiar figures such as circles or spheres have parts that extend into the unseen portions of six dimensional complex space, so that when we see a Euclidean circle, we perceive only a portion of the full figure (nineteenth century geometers liked to claim that we see the full shape of geometrical figures in the manner of the shadows of Plato’s cave)’.\textsuperscript{54}

This is not to say that essentialism was by any means the only presupposition, or even the dominant one, in mathematical research. We know from the predilection of some mathematicians towards formalism, that this was not so. But it further illustrates how misleading it would be to try to downplay the philosophical issues


of Frege's logicism by emphasizing its mathematical side. Historically, the two are inextricably intertwined.

In summary, new proofs of theorems were required because of a perceived lack of clarity and distinctness of understanding of these theorems or propositions consequent on our having not reflected upon these constituent notions in the context of their appropriate conceptual setting—the nature of their ontological territory. What this view in part amounted to was the belief that the meaning of our mathematical notions was determined not simply by arbitrarily stipulating how they should be defined. Frege's conception of definition, informed by this essentialist thesis, which he inherited from his mathematical background, is one in which the real meaning of a mathematical term was taken to be determined by the conceptual environment in which the mathematical concept was thought naturally to belong.$^{55}$ Frege's mathematical climate encouraged the belief that there is more to be learned about what lies in our epistemic possession. When we ordinarily refer to a circle we refer to an entity some of whose essential properties elude us but which should be captured in our definition of the concept.

This kind of concern was common throughout Frege's own career. As a graduate student he sought to establish the reason why arithmetic has greater generality than geometry, and as such was connected to the interests just described.$^{56}$ The kind of research in mathematics just outlined, including Frege's graduate work, naturally extends to a motivation of Frege's logicism overlooked by Benacerraf and the Kitcherian: the question of what grounds the nature of the

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$^{55}$ This kind of investigation into the semantics of our terms is prominent also in the works of Bolzano. It is also prominent following Berkeley's famous complaint in the Analyst of the unclarity involved in our understanding of the central concepts used in Calculus. The well-known view of Cantor and Dedekind that numbers are the creation of the human mind, and the views of other mathematicians concerning the nature of numbers considered in Grundlagen is further testimony to this claim that mathematicians sought to explicate the essence of our arithmetic concepts. I shall develop this theme further in §4 below.

$^{56}$ For example, in 'On Geometrical Representation of Imaginary Forms in the Plane' we read 'When we consider that the whole of geometry rests on axioms that derive their validity from the nature of our intuitive faculty, we seem well justified in questioning the sense of imaginary forms, since we attribute to them properties which not infrequently contradict all our intuitions.' On the next page we read, '[s]ince we can perform the same operations with these complex numbers as with real ones, we can infer a set of geometrical propositions from these imaginary points of intersection which we could also infer from the real ones. It is now of the greatest importance to
generality of arithmetic. In his ‘Formal Theories of Arithmetic’, for example, Frege speaks of the reluctance to accept irrational and complex numbers, and notes that, while this was overcome by geometric representations, proofs based on purely geometric axioms concealed the true state of affairs of the truths involving these types of numbers. Similarly, in Grundlagen he states that the aim of proof is not merely to attain certainty, but also to reveal the true nature of things. It was hard to explain the generality of arithmetic when the conceptual setting was taken to be geometric: spatio-temporal. By contrast, the basis of the generality of arithmetic could be made plain, if one took the ontological territory of arithmetical language to consist entirely of purely logical entities.

It is in the context of the above mathematico-philosophico picture that we can best see Frege’s pursuit of definitions of arithmetical terms long in use but not yet properly defined, and that of making good our ‘defective insight into the whole structure of arithmetic’. Witness, in particular, Frege’s remark in the introduction to Grundlagen, apropos definitions.

‘Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off irrelevant accretions that veil it from our eyes.’

What can we hope to achieve by attending to those concepts, like number, that have proved useful to science, those concepts that have long been grasped but hitherto not defined? We hope to discover their nature and structure, their pure form: to gain a greater understanding of why mathematics works as it does.

A way of reconciling Frege’s use of two apparently incompatible kinds of definition now suggests itself. The first two chapters of Grundlagen show that different thinkers associate different meanings with their arithmetical terms.

find out when a proposition which holds for real forms can be carried over to imaginary ones*. (1873), pp. 1-2, in CP.

57 J. Tappenden 1995a details the extent of this continuation. My use of that connection is somewhat different from his, however. Demopolous subscribes to a similar view of the history of mathematics. Neither of these writers use their research to make the central points I make in this chapter.

58 1885, p. 117, CP.

59 Ibid. p. xi, §2, §3, §4, §17.

60 Grundlagen, p. vii.
Moreover, Frege later complains that '[d]efinitions are set up, but it doesn't enter the author's head to take them seriously and to hold himself bound by them. So there is nothing to place any check on our associating quite unwittingly a different meaning with a sign or word'. Stipulative definitions are introduced after analysis of pre-Concept Script content has been carried out; its purpose is to ensure that henceforth what is to be associated with arithmetic terms is the content made explicit by conceptual analysis. A stipulative definition lays down the structure and nature of the concept revealed by examining the real *Sinn* of our ordinary arithmetic terms. By revealing the conceptual setting, we explain the success of our universal application of arithmetical reasoning.

§2.4. Frege and the Uniqueness of Numbers. Anyone wishing to deny that Frege took his definitions to be articulations of the structure and nature of the concepts and objects denoted by our ordinary arithmetical terms must explain away the passages cited above. Although these detractors would, I think, have a tough row to hoe, there are certain things they could say.

Firstly, they might grant that Frege believed that pre-Concept Script arithmetical language was semantically in order. It expressed true thoughts, and so our ordinary terms had *Bedeutung*. Secondly, they might grant that, *apropos* the conceptual setting view, Frege saw his definitions as answerable to—and taken to capture—a determinate mathematical reality onto which our ordinary arithmetical language was already mapped, despite our defective understanding. Nevertheless, they may reply that this essentialist reading is not incompatible with the view that the denotations of our function and object terms were far from unique to that mathematical reality. This is the third approach to the revolutionary view mentioned at the outset.

So for instance, in speaking of 'achieving knowledge of a concept in its pure form' Frege might have had in mind merely the articulation of certain general features of our mathematical practice or usage of a mathematical term,

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61 1914, p. 242, *PW*. Also, witness the variety of attempted definitions Frege examines in various of his work, e.g. the earlier parts of *Grundlagen*.
which any of a variety of possible definitions could capture. One feature, for example, would be that the definition of a concept of number must be such as to make clear the maximal generality of application of that concept, the fact that number is applied not just to physical things but to all things thinkable. So it must involve an operator taking sortal concepts as arguments. It must also involve a one-one mapping.

To be sure, there are passages in *Grundlagen* and *Grundgezete* that, prima facie, tell against the idea that Frege was seeking to disclose the nature of unique and otherwise hidden objects, the numbers. In particular, Benacerraf and Bynum have cited the following two passages as conclusive evidence for the non-uniqueness view.

(A) 'I believe that for "extension of the concept" (umfang des Begiffes) we could write simply "concept". But this would be open to two objections: That this contradicts my previous statement that the individual numbers are objects, as is indicated by the use of the definite article in expressions like "the number two" and by the impossibility of speaking of ones, twos, etc. in the plural, and also by the fact that the number constitutes only an element in the predicate of a statement of number: that concepts have identical extensions without themselves coinciding.

'I am, as it happens, convinced that both these objections can be met; but to do this would take us too far afield for present purposes.

I assume that it is known what the extension of a concept is.'

Is it not clear, as the above advocates of the revolutionary view believe, that Frege is here acknowledging the possibility of constructing an alternative arithmetic, and thus rejecting the claim that numbers are unique objects? Nor is this all. Having reiterated his view that cardinal number is an extension whose members are concepts that stand in a relation of one-one correspondence, Frege qualifies it with the remark

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62 By 'the numbers' I shall mean the positive integers. I will indicate otherwise when referring to other kinds of number.

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(B) "This way of getting over the difficulty cannot be expected to meet with universal approval, and many will prefer other methods of removing the doubt in question. I attach no decisive importance to bringing in the extensions of concepts at all".  

(The 'difficulty' referred to here is Frege's belief that the initial contextual definition 'The n:F(x) = the n:G(x) <-> F(x) = G(x)' could not be used in all cases to determine the truth-value of a sentence of the form 'n: F(x) = a', where 'a' is a singular term, e.g., 'Julius Caesar', that occurs outside the context of the definition. See below.)

As it happens, neither of these remarks threatens the hermeneutic hypothesis. Regarding the first passage, unless more is said about how Frege is using the term 'concept' here, such a straightforward reading would make these passages, as well as much else that Frege says in Grundlagen, appear rather odd. To see this, consider the two possible objections, anticipated by Frege, to the substitution of 'concept' for 'extension of the concept'. First, it would contradict his thesis that numbers are objects as indicated by the definite article. Second, it would contradict his view that number terms cannot stand in their own right as predicates, as in 'Solon is one', but only as part of a predicate as in 'Solon is one of several wise men'. Frege is confident that he can meet these objections. But note that it is these anticipated objections—the aforementioned 'contradiction'—not the thesis that numbers are objects, that he thinks can be overcome. Frege did not think that his criteria of objecthood were dispensable. So if in the above passage Frege really thought that we could take number not as an object (extension) but as a concept, then he could not have believed that the two possible objections were surmountable. To read the passage as evidence that

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64 Grundlagen §68, p. 80, fn. 1.  
65 Grundlagen §107, p. 117.  
66 The number of things of the kind F is the same as the number of things of the kind G if and only if there are just as many Fs as there are Gs, where 'just as many as' means there is a one-one correspondence between the two concepts or kinds of things Fs and Gs. See Grundlagen, §§65-66.  
67 As he points out, if it were a legitimate predicate then we could infer from 'Solon is one' and 'Thales is one' to 'Solon and Thales are one'. Frege is also aware of course that some contexts of use may seem to suggest that these expressions are legitimate. But in these cases 'one' is not being used as a number one, but in the sense of something being unitary. Cf. Grundlagen §29.
Frege is contemplating that numbers could be regarded as concepts is to attribute to him the view that to take numbers to be concepts would not be to contradict the thesis that numbers are objects. We have already seen (in chapter one) that Frege insists on the sharpest possible distinction between concept and object. So how could he have thought that substitution of ‘concept’ for ‘extension of concept’ need not contradict his view that numbers are objects? Surely he could have done so only by taking the former term to mean what the latter term means. In fact, in ‘Über Begriff und Gegenstand’, Frege refers back to the first passage just cited from *Grundlagen*, and says that in the expression ‘the number that belongs to the concept F is the extension of the concept equinumerous with the concept F’ one could replace ‘extension of the concept’ by ‘concept’.

‘If he [Kerry] thinks ... that I have identified concept and extension of a concept, he errs. I merely expressed my view that in the expression “the number that belongs to the concept F is the extension of the concept numerically equivalent with the concept F” one could replace the words “extension of the concept” by “concept”. Notice carefully here that this word [the word “concept”] is combined with the definite article. Besides, this was only an incidental remark on which I based nothing.’

So an explanation for Frege’s indifference regarding the possible substitution is that he took the expressions ‘the concept F’ and ‘the extension of the concept of F’ to have the same *Bedeutung*.

Why did Frege contemplate the substitution of one term for the other? There are several possible explanations. One reason might be that he thought that ‘the concept F’ would better accord with the sensibilities of his fellow mathematicians. At the same time, the substitution would accord with his insistence that classes are derivative from concepts—that predication is epistemologically and logically prior to abstract objects. By and large, mathematicians of Frege’s day disparaged traditional logic, to which the term

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68 P. 48; *Kleine Schriften (KS)*, p 72.
69 Moreover, this captures the idea that elements are fixed or delimited only through concepts, elements being what fall under the concept.
'extension' belonged, believing it imprecise and unfruitful, and preferred instead the term 'set': i.e. to think of functions extensionally. (Arguably, Frege saw concepts as functions as early as 1879.) To the traditional logician, on the other hand, the term 'the concept in extension' expressed simultaneously the notion of an extension, taken as an object, as well as an aspect or characteristic mark of the corresponding concept, the intension. Logicians regarded the intension as having semantical priority over extension because the former fixes or determines the latter. Also, they regarded it as having epistemic priority since one could think of an extension only through its intension. By contrast, mathematicians of the time, who were beginning to define numbers in terms of sets, tended not to appreciate the idea in traditional logic that intension had explanatory priority. Thus Frege may have seen 'The concept F' as the happier expression for the notion traditionally expressed by 'the concept in extension'. Moreover, Cantor sometimes used the expression 'the concept ...' to refer to abstractions; that is, more or less as Frege used 'the extension of the concept'. As Frege notes 'the difference that Mr. Cantor writes 'General concept' (Allegemeinbegriff) where I write 'extension of the concept' seems inessential ...'. So Frege's suggested

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71 Here two functions are taken to be identical if they have the same values for the same arguments.

72 But only much later, of course, did he see concepts as functions whose values were truth-values.

73 Or at least it was regarded as having these kinds of priority. For these two points see for example, J.S. Mill’s *A System of Logic*.

74 But this, according to Frege, helped to make the conceptual foundations of the notion of set correspondingly unclear.

75 KS, p. 164; cf. *Grundlagen*, p. 80. With Frege’s insistence of conceptual priority of concepts over abstract objects, he could avoid the errors that he had accused the mathematicians of. For instance, by taking sets in terms of a list of membership, one could not account for the null set. Yet the notion of a concept under which nothing falls is natural and familiar. Cf. 'Boole's Logical Calculus and the Begriffsschrift' (1880-81), p. 34, *PW*; *NS*, p 38; *Grundlagen*, p. 30, p. 59, p. 64; *KS*, p. 105; ‘Über Begriff und Gegenstand’ (1891), p. 102 (KS, pp. 206-07). Also, the notion of the infinite is not justified by logical addition; whereas this need not be so, if the notion of a completed totality is seen as deriving from a definite concept denoted in thought. Cf. 'Boole’s Logical Calculus and the Begriffsschrift' (1880-81), *PW*, p. 34; *NS*, p. 38; *KS*, p. 164; *Grundgesetze*, p. 31; cf. *Grundlagen*, p. 98.
substitution might be a tacit acknowledgement of the equivalence of Cantor’s definition.\textsuperscript{76}

At any rate, even if Benacerraf and Bynum were right about their interpretation of passages (A) and (B) cited above, which they are not, this would be insufficient to count against the hermeneutic claim. It could still be the case that Frege sought to explicate the propositions that we grasp when engaged in ordinary arithmetic judgements. The passages they cite would be compatible with the claim that Frege was uncertain about what the \textit{Bedeutung} of our arithmetical terms were. But that Frege had doubts need not be in dispute. After all, was it not just such a state of uncertainty that helped to motivate his logicism in the first place? Frege’s willingness to place no great store in his definitions may simply reflect this uncertainty, as well as an open mind. That open mindedness would come in the wake of having himself not yet achieved the more perspicuous understanding, which he believed could be attained on completion of a system of proof that was at once the most simple and comprehensive. In fact, Frege did contemplate treating numbers as concepts, though not in the passages mentioned by Bynum and Benacerraf. He did so following the realization that his system was inconsistent. Doubtful whether numbers really were objects, he considered that they might be second-level functions.\textsuperscript{77} But again, this is not incompatible with the hermeneutic hypothesis.

Still, would not Frege have granted that the proposed alternative constructions of arithmetic current in his day were, up to a point, already being used to model our arithmetical practices? For example, there was the attempted construction of an arithmetic in the manner of J.S. Mill’s physical notions (agglomerations of physical things). Accordingly, would not Frege grant that the system would work within the bounds of the physical? \textit{Mutatis mutandis} for those constructions from purely temporal notions (e.g. sets of moments of time) or geometric notions (e.g., sets of points, lines, or planes). In other words, our arithmetical judgements would remain true or valid up to a point, that point being

\textsuperscript{76} I owe the latter points of this paragraph to T. Burge, 1984, p. 22.

\textsuperscript{77} ‘Aufzeichnung für Ludwig Darmstaeder’ (1919), pp. 256-57, \textit{PW}.
determined by the nature of the definitions preferred at the time in the construction of the proxy arithmetic.

An examination of Frege's critique of rival theories makes plain that he did not think that these proposed alternatives, even with their restrictions on generality, were workable. The reason is clear. Although Frege believed that we could form some concepts by abstraction from particulars,\textsuperscript{78} he rejected the claim that we could arrive at objects (including sets) by this means, and further rejected the claim that the kind of concept arrived at would be that of number. '... [S]uppose' he says, 'that we do, as Thomae demands, "abstract from the peculiarities of the individual members of a set of items", or "disregard, in considering separate things, those characteristics that serve to distinguish them". In that event we are not left, as Lipschitz maintains, with "the concept of the Number of things considered"; what we get is a general concept under which the things in question fall.'\textsuperscript{79} Furthermore, try as we might to form a set of units we would fail, thinks Frege, since the process of abstracting from the properties of particulars cannot provide us with a plurality. Frege also offers reasons against thinking of the number one as a unit. For example, E. Schröder had argued that the things to be counted had first to be brought under the concept of unity, because each thing to be counted had to be regarded as a unit. But this would be to treat 'one' as a first-level property. Its unworkability is shown, Frege says, by the fact that whereas we can say that 'Thales was wise', and that 'Solon was wise', and can reason that 'Thales and Solon were wise', we cannot reason in a similar way with the predicate 'is one' that 'Thales and Solon are one'.\textsuperscript{80} Nor can we think of the concept of number as a set of things, 'a physical agglomeration', instead of a set of units, since there can be a difference of number with no corresponding physical difference.\textsuperscript{81} Nevertheless, Frege assumes that we succeed in our arithmetical practices; that theorists and layman alike are competent in their

\textsuperscript{78} Op. cit. §34, p. 45; cf. 'On Mr. H. Schubert's Numbers' (1899), pp. 253-54, \textit{CP}.
\textsuperscript{79} \textit{Grundlagen} §34, p. 45.
\textsuperscript{81} Op. cit. §25, pp. 32-33. 'One pair of boots may be the same visible and tangible phenomenon as two boots. Here we have a difference in number to which no physical difference corresponds; for two and one pair are by no means the same thing, as Mill seems oddly to believe.'
arithmetic reasoning and that their applications are as wide as conceptual thought. Both layman and theorist alike successfully apply the concepts of arithmetic to what is in space and time, as well as to what is not in space and time. Again, Frege must think that we do not have a perspicuous conception of what precisely we refer to. More on that later.

Given Frege's own notion of an extension, might he nonetheless have accepted a variety of ways of defining numbers as objects? It is nowadays well known that within set theory a number of logically equivalent and equally acceptable possibilities exist for set-theoretic reductions of numbers. E. Zermelo and J. Von Neumann offer two such accounts of arithmetic. Both their systems use the same construction of zero and the number one but diverge thereafter: where '0' is defined as the null set; and '1' as the set that contains the null set as its sole member: the unit set of the null set. So for instance, Zermelo\(^2\) defines each number except zero as the unit class of its predecessor: i.e. '2' refers to the unit set of 1, '3' to the unit class of 2, etc, thereby yielding the series \(\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\). By contrast, Von Neumann\(^3\) takes each number except zero as the set of all its predecessors. '2' refers to the set whose members are 0 and 1; '3' to the set whose members are 0, 1, and 2, etc. yielding the series \(\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\).

The possibility of a multiplicity of set-theoretic reductions\(^4\) accords of course with an important observation of Dedekind's about \(\omega\)-sequences. This is that any well ordered-series of objects—objects ordered into a progression by \(\leq\)—constitutes a suitable basis for a model for the interpretation of the theory of natural numbers: \(i.e.\) for satisfying the Dedekind-Peano axioms of arithmetic.\(^5\)

\(^4\) Benacerraf has argued that the multiplicity of set-theoretic reductions helps to show that numbers cannot be sets. If numbers were sets, then there would be some principled way to identify which sets they are. But this there cannot be since each type of reduction is equally adequate as a foundation for arithmetic. See his 'What numbers could not be' in *Philosophical Review*, 1965, rept. in *Selected Readings in the Philosophy of Mathematics* 1983, 2nd edition, eds. P. Benacerraf and H. Putnam.
\(^5\) The axioms are (where 'S' means successor, 'N' means number):

1. \(N(0)\);

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According to Richard Heck, Frege not only in some sense accepted Dedekind's observation, but he anticipated Dedekind's proofs that support the observation. Heck's evidence is that, in *Grundgesetze*, Frege proved the categoricity of his basic laws—where a set of axioms is categorical if any two relational structures satisfying them are isomorphic. If the set of axioms is categorical then, arguably, it completely characterizes the structure of these relational structures. And according to Dedekind's view—a structuralist view—natural numbers have no intrinsic (defining) properties outside the structure (or simply infinite set). There is, allegedly, nothing more to numbers than the properties stated in the Dedekind axioms, properties had purely in virtue of their ordering or their relations to each other. As Resnik puts it

> 'In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics, that is, the entities which our mathematical constants, and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure.'

Let us be clear that Frege was no structuralist. So if Heck is right in his observation, then it is apt to ask why Frege was not a structuralist. One reason

\[
2. \quad \mathbb{N}(x) \rightarrow \mathbb{N}(S(x));
3. \quad \neg(\exists x [\mathbb{N}(x) \wedge 0 = S(x)]
4. \quad \mathbb{N}(x) \wedge \mathbb{N}(y) \wedge S(x) = S(y) \rightarrow x = y
5. \quad 0 \in X \wedge (\forall x) [\mathbb{N}(x) \wedge x \in X \rightarrow S(x) \in X] \rightarrow (\forall x) (\mathbb{N}(x) \rightarrow x \in X)
\]

(5) is of course the principle of mathematical induction. To put it another way, if 0 has a property p, and if any number has this property, then its successor has it also, and thus all numbers have P.

Cf. Richard Heck's 'Definition by Induction in Frege's Grundgesetze der Arithmetik', in *Frege's Philosophy of Mathematics*, ed. William Demopolous. What is in effect a categoricity theorem is formulated in light of Frege proof of the infinity of natural numbers, where he seeks to establish a one-one correspondence between the number of natural numbers and a series of other objects. The number that belongs to the concept natural number is then aleph_0.

Frege’s basic laws are essentially equivalent to the Dedekind axioms.

A relational structure is taken to be a set—more specifically an ordered n + 1 tuple—consisting of a set and one or more relation that has the set as its field (the union of its domain and range). In <ω, R₁, ..., Rₙ>, the first member, ω, is the set of 'natural numbers', the other members are the relations that order the members of ω. We say that two relational structures <A, R₁, ..., Rₙ>, and <B, S₁, ..., Sₙ> are isomorphic if they have the same form. That is, there is the following kind of one-one correspondence: for each of the relations R, members of A stand in relation R if and only if the corresponding elements of B stand in the corresponding relation S to each other.

concerns the order in which the structuralist explains natural number, and the relation cardinal number has to that order.90 The structuralist’s definition of number is principally of ordinal numbers, and according to him, ordinals explain the natural numbers.91 Given a structuralist account of ordinals, and some such instance92 as \{1,...,n\}, we can explain the concept of cardinal number in terms of a one-one correspondence between the set \(x\) of things to be counted and the ordinal set of natural numbers. For example, ‘The set of \(X\) has cardinal \(n\)’ would mean that \(X\) is equipollent with the set \{1,...,n\}.93 Thus in explaining natural number, structuralists give ordinal number explanatory priority over the concept of cardinal number. Since natural number is explained in terms of ordinals, and since cardinal number, with its use in counting, involves relations external to the elements of the simply infinite system, the concept of cardinal number is not included in the definition of natural number.

Frege’s preferred explanatory order is the other way round. He explains natural number in terms of cardinal number, and does so because he thinks that the cardinal concept has conceptual priority: it is explanatorily more basic. The concept of cardinality, Frege thinks, is central to the generality of our applications of arithmetic, and that it is this aspect of our arithmetical practice that he wishes to explain—stressing as he does that the definite descriptions of the form ‘the \(n\): \(F\)’ apply to any sortal concept term ‘\(x\) \(\Phi\)’.94 What Frege demands, in the first instance, is an account of the \(S\)inn of ‘the \(n\): \(F\)’.95 This is because he thinks that it is in virtue of the \(S\)inn of an arithmetical sentence—in virtue of expressing thoughts—that arithmetic has application. So it is partly in virtue of the \(S\)inn of

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90 Cardinal numbers being those answering to the question ‘how many such and such kind are there?’
91 The positive and negative whole numbers.
92 ‘Some such instance’ because, as noted above, any well-ordered sequence would suffice for the structuralist.
94 To any well-defined sortal concept. Cf. Grundlagen §46. See ‘Frege to Anton Marty’ (1882), BW. The point is a consequence of Frege’s view that number statements are assertions about concepts: ‘... die Zahlangabe eine Aussage von einem Begriffe enthalte.’ Thiel, §46, p. 60.
95 As noted in the first chapter, this way of putting matters may sound anachronistic, since Frege had not formally distinguished between \(S\)inn and Bedeutung. However, looking back on the issue, it is in terms of the distinction that Frege would have regarded the above.
our arithmetical terms that arithmetic has maximal generality of application.\textsuperscript{96} Any definition of number that fails to include this feature of \textit{Sinn} would be inadequate. Indeed, this was a not infrequent criticism Frege had of his rivals.\textsuperscript{97} The structuralist views of numbers given rise to by Dedekind’s characterization of the structure of natural numbers would have been no less a target.

So Frege’s awareness, if any, of something akin to Dedekind’s Categoricity Theorem need not imply that Frege believed that arithmetic could be reconstructed without a unique \(\omega\)-sequence, a unique domain of objects for the \textit{Bedeutung} of our terms. He could well have thought, for example, that he could construct one \(\omega\)-sequence from purely logical objects, and another \(\omega\)-sequence from extensions whose members are thoughts, where both contain infinite members, and between which a one-one mapping obtains. But that need not mean that Frege believed that arithmetic could be equally well constructed from extensions whose members are \textit{Gedanken}. Frege set out to explain the generality of arithmetic. He observed that only the truths of logic had an equal generality, so that if the truths of arithmetic were truths of logic, only purely logical entities could be their truth makers.

But might not Frege have recognized the possibility of permutations of logical objects? If the question is did he do so, then an affirmative answer seems unlikely. For Frege was not seeking to construct a new arithmetic. He was trying to account for the success of our already existing general arithmetical practices, which would, he believed, be a revelation about the nature of what our terms refer to. He had already convinced himself that there was just one arithmetic; that numbers were unique objects. Indeed, the Julius Caesar problem appears to be evidence of this fact.\textsuperscript{98} The Julius Caesar problem is the consequence of Frege’s demand that a definition of (say) number must afford us the capacity to discern

\textsuperscript{96} \textit{Grundgesetze}, vol. ii, §91, ‘Frege against the Formalists’, p. 67, in \textit{TWP}.

\textsuperscript{97} Cf. ‘On Formal Theories of Arithmetic’ (1885), pp. 112-21, \textit{CP}. See also \textit{Grundlagen} §14, §19, §40.

whether any object, not presented as a number, is in fact a number. As far as I can see, this demand has no relevance in a purely mathematical context. The demand is motivated from a philosophical, rather than mathematical, point of view. It is premised on the assumption that numbers are unique objects. In that case, the question, Is Julius Caesar a number?, would arise in the context of alternative ω-sequence. But it would not do so if Frege had believed that any ω-sequence would suffice. In that case, Frege’s prior conviction that numbers are unique and hidden objects would have directed him away from considering possible alternative constructions within his logical theory. It was a conviction that continued long after his abandonment of logicism.

‘The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis—a geometrical one in fact—so that mathematics in its entirety is really geometry.’

From the beginning to the end, Frege regarded his foundation project as a hermeneutic inquiry.

§2.5. Closing Remarks. We have now seen that the mathematical context in which Frege worked can be used to counter an assumption that informs some commentators’ conception of Frege’s logicism. The assumption was that Frege’s motivation is mathematical and only incidentally philosophical, and that Frege’s definitions, being those of a mathematician’s, were not intended to capture the Bedeutung of our mathematical terms. I have tried to undermine that assumption by questioning the implied distinction between Frege’s purely technical and philosophical achievements, as well as by adducing some neglected passages of his work. I have emphasized that in general mathematicians prior to and during the nineteenth century not uncommonly saw themselves as seeking to uncover the hidden essence of certain mathematical notions. They did so in order better to understand problems and techniques in which those notions figured, and I have suggested that Frege too had engaged in just such a project. In seeking to define

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99 Grundlagen §66.
100 ‘Numbers and Arithmetic’ (1924-25), in PW. My emphasis.
terms already in use, mathematicians saw the meaning of these terms as constitutively dependent on the conceptual setting—i.e. the nature of a concept's ontological territory—of the mathematical notions whose explications were being sought. The nature of this territory was often unclear or hidden and needed to be revealed in the definitions if the potential fruitfulness of the notions were to be maximized. I have further argued that this 'hidden essentialism', arguably to be found in Frege's work, is more full-blooded than others might be tempted to believe it to be. That is to say, Frege believed that there were unique logical objects to be discovered that best account for our arithmetical practices.

Having made probable the weak hermeneutic thesis, let us now turn to a defence of the stronger versions. Common to the latter is the claim that, by means of his definitions, Frege attempted not only to capture the *Bedeutung* of our ordinary arithmetic terms, but also their *Sinn*. 

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Chapter 3

Explicating the Sinn of our Number Terms.

§3.1. Introduction. It was argued in the last chapter that, contrary to the revolutionary view, Frege’s logicist project involved the attempt to capture the Bedeutung of our ordinary arithmetical terms: the weak hermeneutic claim. In this chapter, we consider whether Frege, at any stage, took himself to be capturing their Sinn also. There are, to be sure, passages in his work that would suggest that he did not, and for this reason many Frege scholars subscribe either to the revolutionary or the weak hermeneutic view. I shall argue that the passages in question are by no means decisive, and that certain of Frege’s other views, as well as his practice, would warrant a stronger hermeneutic reading. One of two positions might be adopted once this stronger reading is accepted. On the one hand, one might argue that if Frege did seek to preserve the Sinn of our ordinary terms, then our grasp of them must have been partial. On the other hand, one might argue that Frege sought to uncover a full, yet in some respects tacit, grasp of our ordinary arithmetical concepts. It is to be shown that the former position rests on an assumption, which I reject, about where Frege would draw the line between knowledge that is constitutive of a grasp of a proposition and its constituents, and knowledge that pertains to our understanding of those items. On the strong hermeneutic view, argued for here, what was defective was not our grasp, but our understanding of the nature of the thing grasped: the Sinn.

First, however, a word about the notion of Sinn. Frege identifies it as having three functions, but otherwise says little about it.¹ Firstly, Sinn explains the possible cognitive value afforded by sentences whose respective constituent terms have the same Bedeutung or denotation. Here the notion of Sinn is captured

¹ See especially ‘Frege to Jourdain’ (undated) VIII/12 [xxi/12] p. 79, Frege’s Mathematical and Philosophical Correspondence (BW). ’Über Sinn und Bedeutung’ (1892), CP; ‘Aufzeichnung uber Sinn und Bedeutung’ (1892/1895), PW; ‘Funcktion und Begriff’ (1891), TWP.
by the metaphor of modes of presentation (*Arts die gegeben-seins*): ways of thinking or conceiving of a *Bedeutung*. Secondly, *Sinn* fixes or determines *Bedeutung*: for each *Sinn* there is at most one *Bedeutung*, which is the semantic value or denotation of our terms. This latter connection, between *Sinn* and *Bedeutung*, exists independently of the apprehending mind: it is an abstract, eternal relation, and part of the foundation of reason. Thirdly, *Sinn* is the denotation of expressions in oblique contexts. Only the first two functions will figure in what follows.

The *Sinn-Bedeutung* distinction was not officially drawn until 1891, and until then Frege had used the terms ‘judgeable content’, ‘conceptual content’, or just ‘content’. One might think that therefore we should reformulate our question to take account of this earlier period. However, in a letter to Husserl, which looks back on *Grundlagen*, Frege remarks ‘[w]hat I used to call judgeable-content is now divided into thought and truth value’. In light of this, we might think that in dividing judgeable content into the two components, truth-value and thought, Frege had hitherto confusedly combined both the later categories. This impression might be reinforced by the fact that the terms ‘*Sinn*’ and ‘*Bedeutung*’ occur in *Grundlagen*. Of course this by itself might indicate simply that Frege had used both terms as synonyms for the everyday notion of linguistic meaning. But equally it might indicate that Frege had some hazy awareness of the distinction later made systematic. In support of the latter claim, we might note that, in the same letter to Husserl of 1891, Frege acknowledges that he had used these terms in this earlier work, and offers some clarification. Noting four places where ‘*Bedeutung*’ should be substituted for ‘*Sinn*’, he leaves in tact, and perhaps thereby sanctions, the other occurrences. Be this as it may, what matters to our

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2 As argued in chapter 1.
3 ‘In sect. 97 I should now prefer to speak of “having a Bedeutung” instead of “having a Sinn”. Elsewhere, too, e.g. in sects 100, 101, 102, I would now often replace “Sinn” by “Bedeutung”.”
4 In *Grundlagen* §16. p. 22; Thiel, pp. 29-30, Frege raises a question about the possibility of informative identity statements, and in §67, p. 78 he offers a reply that resembles his later notion of modes of presentation. It is noteworthy that, while identity is construed meta-linguistically in *Begriffsschrift*, in *Grundlagen* it is treated objectually. I postpone a discussion of this until chapter
inquiry is not so much whether, in Grundlagen, Frege operated with some partial conception of the Sinn-Bedeutung distinction. More important is how, on looking back on that work with the distinction in mind, Frege would have regarded the definitions contained therein. On re-reading Grundlagen, would he have seen his definitions as preserving the Sinn of our ordinary arithmetical terms, or simply as capturing their Bedeutung?

§3.2. The Weak Hermeneutic View. Some writers—for example, R. Grossman, M. Beaney, and M. Dummett—have argued that, while Frege intended Grundlagen-definitions to capture the Bedeutung of our ordinary arithmetical terms, they were also intended to be informative identity statements. If Grundlagen-definitions were meant to be informative, then this would suggest that they express differences in Sinn: differences in Sinn being what explain differences in cognitive value, viz. between those identity statements that could extend our knowledge and those that could not. In that case, the thesis that Frege intended to preserve Sinn would be false, and the weak hermeneutic view true.

What I want to discuss in this chapter, however, is a variant of this hermeneutic view. In order to state it, we should recall that Frege’s conception of Grundlagen-definitions appears to be the result of confusing two types of definition. On the one hand, he says ‘The definition of an object does not, as such, really assert anything about an object, but only lays down the meaning of a symbol’. On the other hand, Grundlagen-definitions cannot be stipulations of content tout court to new terms. For one thing, he speaks of definitions as

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4 See Beaney ibid. p. 125, p. 140, p. 143, p. 152, p. 252. Beaney thinks that this is confirmed in Frege’s response to Husserl’s version of the paradox of analysis (see below).

5 Compare Frege’s example ‘a = a’, and ‘a = b’, where both the latter is true.

6 The view just outlined involves a large number of complex issues, which I shall discuss in the next two chapters.

7 Grundlagen §68, p. 78.
admitting justification, and of being either true or false. For another, the whole spirit, and also often the letter, of Grundlagen bespeaks of concern with the content of our arithmetic, of our ordinary statements. Frege seeks to prove these statements. To this end, he seeks to make perspicuous their logical form, and to provide an analysis of their constituent concepts. Whether conceived as explications of the Sinn or Bedeutung, Grundlagen-definitions are in some sense explications of the content of our ordinary arithmetical terms.

‘... [W]e very soon come to propositions that cannot be proved so long as we do not succeed in analysing concepts that occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number that has to be either defined or recognized as indefinable. This is the point that the present work is meant to settle. On the outcome of this task depends the decision as to the nature of the laws of arithmetic.'

The advocate of the variety of weak hermeneutic view now to be discussed would respond to the presence of this admixture of different types of definition in the following way. The analytical or explicative variety uncovers the Bedeutung of our ordinary terms: terms not introduced by way of definition. Nevertheless, both Sinn and Bedeutung are stipulated to terms lexically identical to our ordinary arithmetical terms. The reason is that our grasp of the Sinn of ordinary arithmetical terms is so defective that analytical work cannot be seen as reproducing their original Sinn. Instead quite different Sinne are laid down, which are not recognized by the thinker as belonging to his ordinary arithmetical terms. The stipulative definition lays down how our terms are to be understood henceforth, thus ridding us of any defective grasp. And they stipulate that the definiens and definiendum are to have the same Sinn and Bedeutung. I'll say more about defective understanding in a moment.

According to this view, then, at no point did Frege take himself to have uncovered any part of the Sinn of our ordinary arithmetical terms. Of course one might concede that, at some stage, Grundlagen-definitions were seen to preserve the original Sinn, however defectively grasped, but argue that this explicative

10 Grundlagen §4, p. 5; Thiel, §4, p. 16.
view of Grundlagen definitions was later abandoned. In other words, one might grant that Frege began with the mild-hermeneutic claim, but insist that he later resorted to a weaker version. At first sight, two of Frege's papers, 'Review of Husserl' and 'Logik in der Mathematik' of 1914, would seem to confirm that, supposing Frege did see Grundlagen-definitions as explications of Sinn, he indeed later changed his mind. In the latter paper, Frege distinguishes two kinds of definitions, stipulative (aufbauende) and explicative (zerlende). Whereas Aufbauende definitions stipulate a Sinn and Bedeutung for an empty sign, the zerlende variety seek to lay bare the Sinn and Bedeutung of terms already in use, terms not originally introduced by way of definition. Given the second interpretive option—viz. that Frege changed his mind about preserving Sinn—the zerlegende variety of definition is arguably reminiscent of Grundlagen-definitions.

'In the development of science it can ... happen that one has used a word ... over a long period under the impression that its Sinn is simple until one succeeds in analyzing it into simple logical constituents. By means of such analysis, we hope to reduce the number of axioms; for it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalyzed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur.'

There should be no doubt about the similarity, striking as it is, between this passage and that quoted before it from Grundlagen. Nor, I think, should there be much doubt that, on looking back at Grundlagen, Frege regarded Grundlagen-definitions as explications of Sinn. Still, some doubt there might be. For according to the view that Frege never intended to preserve Sinn—the first interpretive option—Grundlagen-definitions are not quite the zerlende variety of 1914. Zerlende or explicative definitions, as described at that time, purport to provide an analysis of the Sinn and Bedeutung of terms already, perhaps for long, in use but not introduced by definition. The advocates of the first option will say that, because of defective grasp, the analysans of Grundlagen-definitions

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11 'Logik in der Mathematik' (1914), p. 209, PW.
uncovered only the *Bedeutung* of the analysandum. (On the other type of weak hermeneutic view, discussed in the next chapter, the terms of the analysans and analysandum differ in *Sinn*; definitions are treated as informative identity statements.) So the passage just cited is not reminiscent of the counter-part passage of *Grundlagen* cited earlier in this section; and thus there was no change of mind. *Grundlagen* definitions never were meant to be explications of *Sinn.* But this view, I take it, is unconvincing. *'Logik in der Mathematiks'* surely was intended as a clarification of his foundational project, particularly the passage on definitions just cited.

This latter point is further exemplified by another passage from the same work of 1914. In it Frege attempts to address the question of how we can know that an explicative definition is successful: how can we know that the analysans and analysandum have the same *Sinn.* Indeed this passage in question, cited below, might be used to support the first option for the advocate of the kind of weak hermeneutic view discussed in this chapter. As said, this view grants that, at some point, Frege conceived of definition as uncovering ordinary *Sinn,* but argues that he later changed his mind. Accordingly, the change of mind would have arisen in connection with the requirement that definitions be uniformly eliminable in proofs: a definiens must be replaceable by its definiendum, or *vice versa.* The rule of proof involves transforming a definition into a judgement of identity. Frege's justification for this rule is that a thinker is warranted in asserting the identity of *Sinn* only where the thinker immediately recognizes that the two sides of the identity have the same *Sinn.* Frege takes this to be a criterion for the success of *zerlende* definitions, and acknowledges that, on this score, doubts about his definitions are likely to arise. In that case, the rule of proof will not apply *vis-à-vis* *zerlende* definitions. To circumvent the problem, Frege

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13 1914, p. 210, *PW.*
14 The acknowledgement is implicit given that *'Logik in der Mathematik'* was aimed at clarifying his foundational inquiries. As we saw in the last chapter, it is explicit in *Grundlagen* §69.
15 1914, p. 210, *PW.*
proposes to treat all definitions as *aufbauende*. We are to regard the definiendum as if empty and simply stipulate its *Sinn*.\(^\text{16}\)

‘... there is no need at all to answer the question concerning the *Sinn* in which—whatever it may be—this sign [A] had been used earlier. In this way we court no objections. However, it may be felt expedient to use sign A instead of sign B. But if we do this, we must treat it as an entirely new sign that had no *Sinn* prior to the definition. We must therefore explain that the sense in which this sign was used before the new system was constructed is no longer of any concern to us, that its *Sinn* is to be understood purely from the constructive definition that we have given. In constructing the new system we can take no account, *logically speaking*, of anything in mathematics that existed prior to the new system. Everything has to be made anew from the ground up.’\(^\text{17}\)

Of course, while it might be true that this passage is meant to clarify Frege’s notion of definition operative in *Grundlagen*, it might well support a still further claim, that Frege changed his mind about the whole hermeneutic hypothesis. That is to say, while at some point Frege envisaged *Grundlagen*-definitions to uncover the content—*Sinn* or *Bedeutung*—of our ordinary arithmetical terms, the passage just cited would suggest that he had changed his mind on both counts. But care is needed in construing Frege’s remark that ‘we can take no account ... of anything in mathematics that existed prior to the new system’. We should bear in mind the qualification ‘logically speaking’. Frege in the surrounding text clearly does try to take account of the content of ordinary arithmetical terms. It is just that ‘logically speaking’ he cannot do so. That is, he cannot prove that the analysans and the analysandum of an ordinary arithmetical term have the same *Sinn*. Nor can he as yet show that the claim that they do express the same *Sinn* is self-evident. But that does not mean that Frege had at this point changed his mind about the

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\(^{16}\) M. Dummett notes this, but is at times more circumspect. For example, he speaks of a clear sense replacing a cloudy one and says ‘we should eschew any pretence that the cloudy sense corresponds to the clear sense in any precisely stateable manner’. Cf. ‘Frege and the Paradox of Analysis’, p. 18, 1987, in 1991a. He also speaks of this kind of activity as being what, from 1914, fruitful analysis consisted of for Frege. Op. cit. p. 19. Moreover, he also believes that Frege doesn’t take as problematic the question whether definiendum and definiens have the same *Sinn*. Op. cit. p. 20. As will be clear in this chapter and the next, I reject all these claims.

\(^{17}\) 1914, p. 211. My emphasis.
hermeneutic hypothesis. So let us in this chapter continue to grant that Frege sought to preserve Bedeutung. Zerlende definitions, as deployed in Grundlagen, are to be seen as laying down at least the Bedeutung of a pre-analytic arithmetical term for a term lexically identical to it. Be that as it may, a number of Frege scholars would take the above passage as evidence that either Frege never did regard his logicist project as the attempt to capture the Sinn of our ordinary arithmetical terms; or that, if he did, he later changed his mind.

1. 'Although there is no intention to depart too radically from common usage, in the end the analyses are answerable not to the meanings of words, but to the nature of things'.

2. 'What logical analysis ... does is not explicate an antecedently given sense, but determine what the sense of a sign is to be.'

3. 'It is ... astonishing that ... Frege could have come so to deprecate the conceptual analysis that had formed so large a part of Grundlagen as to deny the very possibility of conceptual analysis save in rare and unproblematic cases.'

Looked at another way, however, the textual evidence adduced thus far against the mild or strong hermeneutic view, and in favour of the weak variety, falls somewhat off the mark.

Firstly, it should be borne in mind what all versions of the hermeneutic view must concede. The contents of aufbauende definitions of arithmetical terms are not simply conjured out of a hat. They are the results of the analysis of

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18 Again, for more on the claim that Frege sought to preserve the Bedeutung of our ordinary terms see chapter two.

19 1996, p. 149, p. 150. Beaney says that that this presupposes realism. We saw in chapter one that it need not do so if some sort of platonist transcendental idealism can be made plausible. Cf. Weiner, 1990, 'The definition does not consist of an analysis of what was previously understood ... and it should be viewed as nothing more nor less than an abbreviation.' Op. cit. p. 125; cf. p. 120, p. 121, p. 225, p. 266. Also, E. Picardi takes the passages cited above as evidence that, for Frege, definitions need not preserve Sinn. Indeed she regards Frege in 'Logik in der Mathematik' as taking an unexpectedly conventionalist turn. See her 'Frege on Definitions and Logical Proof', p. 230. Cf. M. Beaney, 1996, p. 147.


concepts already grasped, the fruit of much labour, often spanning hundreds of years.

(A) 'What one calls the history of concepts is really a history either of our knowledge of concepts or of the *Bedeutung* of words. Often it is only after much mental work, which may have lasted for centuries, that man at last succeeds in recognizing a concept in its purity (*einen Begriff in seiner Reinheit zu erkennen*), in stripping off the irrelevant accretions that conceal it from the mind's eye. ... Do the concepts, as we approach their supposed sources, reveal themselves in peculiar purity? Not at all; we see everything as through a fog, blurred and undifferentiated.'

Given the context of this passage, one thing about it, at least, is clear. The knowledge of a concept of which Frege speaks is the knowledge laid down in definitions of arithmetical terms. We saw earlier that Frege himself from 1891 would read *Grundlagen* in light of the *Sinn-Bedeutung* distinction: namely as containing those latter two categories, albeit confusedly. Now recall how the following passages, taken from *Grundlagen* and 'Logik in der Mathematik', appear to echo each other.

',... we very soon come to propositions that cannot be proved so long as we do not succeed in analysing concepts that occur in them into simpler concepts or in reducing them to something of greater generality.'

'In the development of science it can ... happen that one has used a word ... over a long period under the impression that its *Sinn* is simple until one succeeds in analyzing it into simple logical constituents. By means of such analysis, we hope to reduce the number of axioms ...'

Given this comparison, it should be clear, in part, how Frege himself would later read passage (A). It is not simply of *Bedeutung* that he is speaking; it is also that by means of which we have access to it: its *Sinn*. *Sinn* of a concept is a constituent of a thought, and the means by which we lay hold of the concept. The definition articulates our grasp of the *Sinn*. As such, it is an articulation of our

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22 *Grundlagen*, p. vii; Thiel, p. 8. Both T. Burge, 1990, pp. 38-41, and M. Beaney, 1996, p. 148, cite this passage in support of their view that, for Frege, we have only a partial grasp of the *Sinn* of arithmetical entities.

23 *Grundlagen* §4, p. 5; Thiel, §4, p. 16.

24 1914, p. 209.
knowledge of the concept in its purity: the concept as it is in itself. Moreover, we saw in the last chapter that \( \textit{aufbauende} \) and \( \textit{zerlende} \) definitions have compatible purposes in Frege’s work. Given what we have said in this and the preceding chapter, we can say that the function of a stipulative definition is twofold. It addresses the question of failure of the completeness principle \( \textit{vis-a-vis} \) our ordinary terms. And it helps to circumvent uncertainties about whether the explanans has the same \( \textit{Sinn} \) as the explicandum.

A further point to be made here is that there is something of a double standard inherent in the above weak-hermeneutic view. This arises from reliance on the passage in ‘Logik in der Mathematik’ cited above in which Frege speaks of building up the science anew. It is said that this helps vindicates their view: definitions do not capture the \( \textit{Sinn} \) of ordinary terms; instead they are answerable only to the nature of things—the \( \textit{Bedeutungen} \). But if the passage in question were decisive against my claim about \( \textit{Sinn} \), it is to be wondered why it would not also be decisive against the view that definitions preserve only the \( \textit{Bedeutung} \) of our ordinary arithmetical terms. I shall return to this point later.\(^{25}\)

I shall now begin to show that a still more serious difficulty attends their view. We are to consider further whether the passage from ‘Logik in der Mathematik’ cited earlier, does indeed mark a change of mind about definitions as explications of \( \textit{Sinn} \). To answer this, let us step back awhile and elaborate our account of the presence of \( \textit{aufbauende} \) definitions. For it is their presence in the latter paper that prompts the view in question—that if \( \textit{Grundlagen} \)-definitions were seen to preserve \( \textit{Sinn} \), then Frege later changed his mind. But it is not, I believe, simply the alleged textual evidence that informs the variety of weak hermeneutic view being considered here. The view appears to rely also on a doctrine—the \textit{transparency thesis}—commonly attributed to Frege. The \textit{transparency thesis} states that:

\(^{25}\) I develop the point in chapter six to show why Frege would have preferred the strong variety of hermeneutism.
1. Anyone who grasps the Sinn of two expressions must thereby know whether or not they are the same.\textsuperscript{26}

Frege himself never actually states this thesis. Despite that, the alleged warrant for attributing it to him seems to have two sources. The first is the criterion of analytic definitions already mentioned:

2. We shall be able to assert that the defining expression has the same Sinn as that of the term it purports to explain only when it is self-evident.\textsuperscript{27}

The second source is Frege’s criterion for sameness of thought:

3. Two sentences express the same thought just in case anyone who recognizes the one as true must immediately recognize the other as true.\textsuperscript{28}

In the rest of this section, I shall argue for the following three propositions. First, the weak hermeneutic reading of the above passage from ‘Logik in der Mathematik’ is warranted only given the transparency thesis. Second, the criterion of analytic definitions and criterion for sameness of thought are distinct from the transparency thesis, and should not be used in its support. Thirdly, the transparency thesis cannot be attributed to Frege without amendment, but that this would render the thesis insufficient to support the weak hermeneutic claim.

Let us begin by seeing how we should read the significance of the criterion of analytic definitions.

Clearly, what lies at the heart of the above passage from ‘Logik in der Mathematik’ is Frege’s acknowledgement that his definitions might appear strange and arbitrary. This is why he admits that they might fail the criterion of analytic definitions. Second, the transparency thesis, and should not be used in its support. Thirdly, the transparency thesis cannot be attributed to Frege without amendment, but that this would render the thesis insufficient to support the weak hermeneutic claim.

Let us begin by seeing how we should read the significance of the criterion of analytic definitions.

Clearly, what lies at the heart of the above passage from ‘Logik in der Mathematik’ is Frege’s acknowledgement that his definitions might appear strange and arbitrary. This is why he admits that they might fail the criterion of analytic definitions. And it is in part why P. Benacerraf and others reject both the mild and strong hermeneutic view:

‘... Mathematical definitions do not standardly reflect preexisting synonymies.

... consider Cantor’s Theory of Transfinite Numbers. Frege praises the theory


\textsuperscript{27} Cf. 1914, 210, \textit{PW; Nachgelasene (NS),} p. 227.

\textsuperscript{28} ‘Kurze Übersicht meiner logischen Lehren’ (1906), p. 197, \textit{PW; NS,} p. 213.
as extending knowledge but takes Cantor gently to task for having appealed to
"the rather mysterious 'inner intuition'" (Grundlagen 98) Frege then goes on
to add, "For I think I can anticipate how his two concepts [following in the
succession, and Number] could have been defined" (Grundlagen 98). Surely,
whatever Frege may be claiming here, he is not claiming that Cantor
overlooked an appeal to preexisting synonymies which he, Frege, thinks he
can produce. 29

Incidentally, there is an oddity here in arguing from a non-standard case, like
Cantor's introduction of transfinite numbers, to the view that Frege's definitions
do not 'standardly' reflect pre-existing synonymies. Our central question concerns
the standard case. This includes terms relating to use of finite numbers and
associated concepts, not transfinite numbers, which Frege takes Cantor to be
introducing for the first time. The question is whether Frege sought to explicate
the Sinn of our ordinary arithmetical terms.

I take it that Frege intended that the Sinn of his terms be shown by their
definitions. 30 Clearly, neither the layman nor expert takes himself consciously to
'associate' the same thing as Frege does with ordinary arithmetical terms. Since
Frege is aware of this, he would concede Benacerraf's point. This is evident from
passages in which he considers rival views. These theorists have selectively
attended to the arithmetical concept they wish to explicate; but they associate
quite different definitions with the commonly shared signs of ordinary arithmetic.

1. 'Obviously each of these [mathematicians] attaches a different Sinn to the
word 'number'. So the arithmetics of these three mathematicians must be
quite different. A sentence from the first mathematician must have a quite
different Sinn from the equivalent-sounding sentence of the second
mathematician.' 31

2. '... Even if the form of words is the same, the thought expressed must be
quite different. Now it is striking that the sentences of these fundamentally
different sciences, each of which is called arithmetic, are constituted by
precisely the same words. And it is even more striking that the practitioners
of these sciences have no inkling that their sciences are fundamentally

30 For some defence and clarification of this, see below.
31 1914, p. 215, PW
different. They all believe that they are doing arithmetic, and the same arithmetic at that, the same number theory, although what one of them is calling a number has no resemblance at all to what another is calling a number.\textsuperscript{32}

It is important to realize that in these two passages Frege is simply entertaining a hypothesis. The hypothesis is that if the Sinn of the terms for the central concepts of arithmetic, as laid down by definitions, were different; and if these different Sinne could account for the different arithmetical practices: then the rival theorists would, unbeknownst to themselves, be engaged in a quite different science—the definitions in each case would determine a quite different Bedeutung. The passage cited echoes an earlier one: '... if everyone had the right to understand by this name ['one'] whatever he pleased, then the same sentence about one would mean different things for different people—such sentences would have no common content.'\textsuperscript{33} Equally important is to realize that Frege thinks that the alternative definitions do not determine arithmetical entities.\textsuperscript{34} Furthermore, it is important also to bear in mind that, for Frege, the thinker's association of these defective definiens with arithmetical terms does not prevent the thinker from engaging in arithmetical practice. Hence their successful engagement has nothing to do with their definitions. For Frege believes that our common arithmetic can be accounted for only in terms of grasping a particular body of true thoughts. This would explain why he entertains the objection in a hypothetical spirit. Behind it is the assumption that we all grasp a common content.\textsuperscript{35} The same assumption is reiterated when he says that as a community of thinkers we have knowledge of science and that this depends on our having a common grasp of the same shareable thoughts.\textsuperscript{36}


\textsuperscript{33} Grundlagen, p. i.

\textsuperscript{34} Cf. 1914, p. 217; cf. Grundlagen, §24. See also chapter 2 above.

\textsuperscript{35} This is especially apparent in the introduction and first two chapters of Grundlagen.

\textsuperscript{36} Cf. 'Der Gedanke' (1918-19), p. 362, p. 371; 'Über Sinn und Bedeutung', p. 160, CP; cf. 'Frege to Jourdain', BW. 'One can hardly deny that mankind has a common store of thoughts, which is transmitted from generation to another.' 'Über Begriff und Gegendstand' (1892), p. 46, fn *, in TWP.
At first sight, the following passage would appear flatly to contradict my interpretation of the last two citations (1-2), and lend still further support to the weak hermeneutic reading. Here Frege characterizes his opponents as reaching blindly here and there, struggling but failing to get hold of the same *Sinn*.

'Sinn appears in such a foggily blurred manner that when they make to get hold of it, they reach for it in the wrong place. One reaches perhaps erroneously to the right, the other to the left; and so they do not get hold of the same thing, although they wanted to. How thick the fog must be for this to be possible!'\(^{37}\)

Care is needed in understanding what failing to 'get hold of the same thing' means here. It should be borne in mind that it is the expert mathematicians who are the object of Frege's ridicule. In particular, it is their attempted definitions, rather than their inability to grasp even the simplest arithmetical thought, that is parodied. The alternative interpretation would be to say that, for Frege, Weierstrass fails to grasp the *Sinn* of 'Number' or of the numerals. But that would be absurd. Frege would have maintained no such thing. The metaphor of reaching now to the right, now to the left, alludes, I suggest, not to the failed attempt to grasp the *Sinn* of terms whose definitions are here scrutinized, but to the failed attempt to exhibit that *Sinn*. These mathematicians are striving to gain a clear insight into what they already grasp. They are seeking to explicate, to make explicit, its nature and structure. They too reflect on what they already grasp, but their definitions, their attempted articulations, taken literally, would reveal a quite different *Sinn*. Because of their faulty definitions, they fail to lay hold of, fail to make overt or articulate, what is already in their possession.

Given the above analysis, I would be happy to concede, therefore, that the last three passages cited indicate that Frege's definitions do not capture or preserve what is, in one respect, associated with our ordinary arithmetical terms, that other thinkers associate different things. For this objection is rendered otiose once we realize that 'associating a *Sinn* with a term' can mean several things. If

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Frege concedes that we grasp a common store of thoughts whilst engaged in arithmetical practice, then the association between arithmetical terms and the requisite *Sinn* must be other than conscious. In fact, Frege thinks that even where we have a term introduced by an *aufbauende* definition, the association of *Sinn* with the definiendum, the term subsequently in use, remains in one's unconscious. What is in consciousness when we engage in mental activity is often not *Sinn*, but abbreviated symbols and ideas associated with it. The reason *Sinn* is often in abeyance is because we simply do not have the mental capacity to keep it in consciousness.

‘When we examine what actually goes on in our mind when we are doing intellectual work, we find that it is by no means always the case that a thought is present to our consciousness which is clear in all its parts. For example, when we use the word ‘integral’, are we always conscious of everything appertaining to its *Sinn*? I believe that this is only very seldom the case. Usually just the word is present to our consciousness ... if we tried to call to mind everything appertaining to the *Sinn* of this word, we should make no headway. Our minds are simply not comprehensive enough ... It follows from this that a thought ... is in no way to be identified with a content of my consciousness.’

Here then is another way of reading Frege’s remarks about expert mathematicians failing to get hold of the same *Sinn*. They fail to bring it forth into consciousness.

The last three passages indicate something curious about Frege’s notion of *Sinn vis-à-vis* the nature of *Bedeutung*. On the one hand, he thinks that when his rivals use ‘one’, they refer to the number one (as Frege understands it in his Concept Script). That they do so must be in virtue of the *Sinn* that corresponds to ‘one’. On the other hand, Frege clearly thinks that the *Sinn*, which the theorist takes himself to associate with ‘one’, is neither the *Sinn* captured in Frege’s definition, nor such as could determine the real—and for Frege the only—number

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38 1914, p. 209, *PW*. In ‘Sources of Knowledge of Mathematics and the mathematical natural Sciences’ (1924-25), p. 269, *PW*, Frege says ‘The connection of a thought with one particular sentence is not a necessary one; but that a thought of which we are conscious is connected in our mind with some sentence or other is for us men necessary.’ The point to note here is that, for Frege, thoughts can be in our possession without being in our consciousness.

39 ‘1 is the number that belongs to the concept “identical with 0”.’ Op. cit. §77, p. 90.
one. Grant for the moment that the Sinn consciously associated with ‘one’ by any of his rivals determines some Bedeutung or other. To take an example, the idea associated in the mind of the Millian might be ‘the characteristic manner in which [a particular] agglomeration is made up of, and may be separated into, parts’. Or for another theorist, what one associates with the same numeral might be geometric points. Then Frege must think that ‘one’, as used by rival theorists, has two Sinne. One consciously associated with it; the other belonging to it in virtue of the term’s arithmetical use and attached unconsciously. Frege must therefore think that, unbeknowst to the user, the same term may refer simultaneously to more than one Bedeutung. Nevertheless, Frege believed that only one such Bedeutung could account for the rival’s success in grasping truth-valued arithmetical thoughts.

What we have said lends further clarification to what Frege meant by saying that, in constructing a new system, we should not take into account what others associate with the same terms. On my reading, Frege is indicating three things. First, he is circumventing possible uncertainty regarding whether the explicans and explicandum of a term has the same Sinn. Second, he wishes to cast out the Sinn consciously associated with arithmetical terms—notably by theorists—that Frege either takes to be irrelevant to arithmetic, or takes to miss the logical structure of their Bedeutung. Noteworthy in this respect is that Frege is careful in his critique not to concede to his opponents that they have true beliefs

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40 In ‘Negation’ Frege says ‘It is certainly praiseworthy to try to make clear to oneself as far as possible the Sinn one associates with a word’, p. 381, CP. Here Frege may well have had in mind any of the two senses of ‘associate’.
42 Frege thought that one could not justify postulating abstract objects by making lists. To do so is ‘very arbitrary and in actual thinking without significance’. ‘Boole’s Logical Calculus and the Begriffsschrift’ (1880-81), p. 34, PW; NS, p. 38. But as seen in chapter two, many mathematicians did understand their set theoretic constructions in this way. So unless they grasped the right (Frege’s) Sinn, they failed to refer, or even to think. At the same time, however, we have seen that Frege took these mathematicians to grasp true arithmetic thoughts. Hence we can infer that Frege took them to grasp the Sinne displayed by the latter’s definitions.
43 See Chapter 2 above.
44 What I have in mind pertains to the so-called conceptual priority constraint said to govern Frege’s definitions. An example of this would be to try to define cardinal equivalence (just as many as) in terms of the concept of cardinal number, as indeed Husserl did. I shall discuss this
about different arithmetical concepts. The underlying presumption is that there are *Sinne* possessed by us all and that our apprehension of them is such that we are prone systematically to predicate properties of them that they simply cannot have. Otherwise we would not share the same body of truths; we would not have a common arithmetic. Thus it seems to be Frege's view that *Sinn* may belong to a term and may be grasped whenever that term is used without that *Sinn* being consciously associated with it. If that is so, then Frege could hardly have subscribed to the *transparency thesis*. It would have contradicted his claim that our arithmetical practice is the manifestation of our grasp of a shared body of thoughts. At most, he would have insisted on the caveat that *Sinn* be grasped with a certain degree of perspicuity. The same caveat would apply to his *criterion for sameness of thought*. In that case, the analytic definitions are untouched by the *transparency thesis*: the latter simply does not apply. Moreover, it is clear that once we amend the transparency thesis with the above perspicuity requirement, then the thesis is left with little strength to support the weak hermeneutic claim.

Third, it is clear that the *transparency thesis* is distinct from the *criterion of analytic definitions* (CAD). Even if we have doubts about whether an explicans and its explicandum have the same *Sinn*, that does not imply that they differ in that respect. So without the caveat of perspicuity, the transparency thesis is false. The former thesis (CAD) concerns the conditions for asserting sameness of *Sinn*. Since thinkers ordinarily associate different items with their arithmetical terms, these conditions are rarely met. In other words, the decision to substitute stipulative definitions for explicative is prompted by an important technical issue. This is that a doubt might arise about the validity of the rule of proof that permits the elimination of definitions—the replacement of definiens and definiendum. Doubt about this rule of proof would impact on recognition of the validity of the deductive reasoning in question. So at one level, Frege's substitution of

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kind of case in the next chapter.

45 1914, pp. 216-17, *PW*.

46 These definitions would likewise do what we said above: make good the defects of ambiguity and incompleteness.
aufbauende for zerlegende definitions is a precautionary move; it is to eliminate the above eventuality. It is a move that he would perhaps have made even if most people were unlikely to doubt the identity.

§3.2.1. A Further Objection. Let us now consider Frege’s other paper said to undermine either the mild or strong hermeneutic view: his review of Husserl’s Philosophie der Arithmetik 1. We can bring a fresh perspective to it, given our analysis above. In this review, Frege considers an objection Husserl raises against taking definitions to preserve Sinn. A point to note at the outset is that, on the face of it, Husserl’s criticism is based on reading the tenor of Grundlagen-definitions as an attempt to preserve Sinn of ordinary arithmetical terms. Even more significant is that Frege nowhere gainsays Husserl’s reading. Frege here paraphrases the objection, which is a version of the paradox of analysis:

‘A definition is ... incapable of analysing the Sinn, for the analysed Sinn just is not the original one. In using the word to be explained, I either think clearly everything I think when I use the defining expression: we then have the ‘obvious circle’; or the defining expression has a more richly articulated Sinn, in which case I do not think the same thing in using it as I do in using the word to be explained: the definition is then wrong. ... This reveals a split between the psychological logicians and mathematicians. What matters to the former is the Sinn of the words ... whereas what matters to the latter is the thing itself: the Bedeutung of words.’

As Frege later makes clear, the objection just raised applies to both (explicative) zerlegende and (stipulative) aufbauende definitions. As such neither types of definition should be seen to preserve Sinn, if Husserl’s objection is accepted.

But is it accepted? In answering this question, it is important to see that the objection has several parts, and that Frege’s responses to them can be read in different ways. The central part of the objection is that Frege’s definitions do not preserve Sinn. It would have been extremely odd of him to accept this part of the objection. Only a year earlier in Grundgesetze did he explicitly state that the two parts of a definition—the definiens and definiendum—have the same Sinn, a view
he endorsed in writing subsequent to the review of Husserl's book. The oddity would be that, whereas the review of 1894 would signal a change of mind, his later work would signal a reversion, made entirely without explanation, to the view of definitions as having the same Sinn.\textsuperscript{48} If there was a change of mind in the review, then one reason for his subsequent reversion, apparent in his ‘Logik in der Mathematik’ of 1914, is clear. To accept Husserl’s objection would commit Frege to the view that the thought expressed by ‘A’ is different from the thought expressed by ‘B’, where the only difference between A and B is that in the one there occurs a definiens, in the other the definiendum. Recall that Frege believed that definiendums are psychologically indispensible: without them we could not conduct proofs.\textsuperscript{49} In that case, Husserl’s objection would entail that, in deductive reasoning, the thought that one sets out to prove is not the same thought as occurs in the final line of the deduction. Hence to accept Husserl’s objection would be to block Frege’s logicist proofs. A change of mind therefore looks unlikely.

Fortunately, a different reading of Frege’s response to Husserl’s criticism is possible. Frege says that if one ‘think[s] clearly everything’ when using the word to be explained, then one explains nothing by offering a definition, since what one has clearly in mind and what one offers as a definition is identical. We can take ‘think clearly’ to mean being able to bring the definiens of the term fully into consciousness. Also, we can take it to mean the ability to recognize immediately that the two expressions have the same Sinn. Suppose, then, that the Sinn is not fully in consciousness. Let a recovery be of the kind that Frege thinks

\textsuperscript{47} 'Review of E. G. Husserl, Philosophie der Arithmetik I' (1894), pp. 199-00, \textit{CP}.

\textsuperscript{48} Dummett notes another oddity: that it ignores the requirement, stressed in \textit{Grundlagen}, that definitions respect conceptual priority. Dummett appears to take conceptual priority rightly to mean explanatory priority, but says little else by way of clarification. Part of my understanding of it is as follows. Frege took there to be a hierarchy of truths, which, with respect to some of them, his logicist project was intended to lay bare. Doing so would reveal an order of dependencies and linkages of truths, whereby a finite set of basic, unprovable, self-standing truths, is disclosed. That so, a concept A is conceptually prior to concept B if A is an essential component of a truth that is more basic than a truth of which B is an essential component. Given this construal, two points are noteworthy. First, a concept cannot be conceptually prior to another concept, if both concepts are essential components of primitive truths. Second, since Frege retained his interest in identifying the objective system of dependencies among truths, he must have retained his concern to respect conceptual priority.

\textsuperscript{49} 1914, p. 209, \textit{PW}.

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takes much time and intellectual work. Then to bring the fully articulated *Sinn* to one’s attention would show that the definition has explanatory value. As seen, this idea of what it is to ‘think clearly’ is apparent in the passage already cited from ‘Logik in der Mathematik’. Recall that there Frege spoke of the *Sinn* of ‘integral’ as lying in abeyance in one’s unconscious, and of the psychological necessity for us to think with sentences containing the definiendum of a term rather than its definiens. So Frege accepts this part of Husserl’s criticism. *Zerlegende* definitions would be explanatorily pointless if the *Sinn* is already clearly grasped: it would be to explain one thing in terms of itself.\(^{30}\)

This brings us to the second of the two parts to Husserl’s objection. This is that we have something different in consciousness when we entertain either the definiens or the definiendum, making the analysis wrong. Again Frege can be seen to agree with Husserl. Take a thinker who at time \(t_1\) reasons with sentence \(S_1\). Suppose that an analysis of a definiendum in \(S_1\) occurs between \(t_1\) and \(t_2\). Let the same thinker at time \(t_2\) reason with sentence \(S_2\), where the difference between \(S_1\) and \(S_2\) is that the definiens has been substituted for the definiendum. Clearly Frege would accept that what occurs in the thinker’s consciousness would be different. As he says later in a letter to Husserl, ‘judged psychologically, the analyzing proposition is of course always different from the analysed one …’.\(^{31}\) Elsewhere, we have seen another reason why he thinks this: namely that all manner of bogus associations would be attached to the definiendum. A further reason is that in \(S_1\), the *Sinn* of the definiendum is absent from consciousness. This part of Frege’s response shows what we know already. It reiterates his lack of interest in what precisely and actually occurs in consciousness when a thinker is engaged in arithmetical practice—a matter indeed for the ‘psychological logicians’.

\(^{30}\) It is therefore wrong to say, as Dummett does (Op. cit. p.143), that Frege is unfair to Husserl in ignoring the paradox of analysis. On my reading, Frege accepts the paradox as so formulated. However, he does not think that it applies where perspicuity of grasp is lacking. I discuss the paradox further in the next two chapters.

\(^{31}\) ‘Frege to Husserl’, Vii/3 [xix/3] 1906, p. 67, *BW*. Dummett takes the letter as indicating Frege’s denial that definitions of an existing term should exactly capture the *Sinn* already attached to that
Finally, the passages cited above make it indisputable that Frege was concerned with the question whether the old and new languages of arithmetic share the same *Sinn*. Our question in this section has been whether the two key passages cited signal a negative answer. I have suggested that it would be a *non sequitur* to infer that it does. Even at face value, Frege can be seen neither to deny nor to claim that his definitions capture *Sinn*. And even if he did have persistent doubts, his decision to treat the results of analysis as the definiens of *aufbauende* definitions is compatible with some degree of belief that the 'analytic' identities excavated in the *Grundlagen* investigations are sound. It is perhaps significant that Frege nowhere says that his decision to treat the purportedly 'analytic definitions' as *aufbauende* stem from his own doubts about the identity of *Sinn* between the analysans and analysandum. At the very most, these passage signal that his belief falls short of certainty. In subsequent chapters, I will give further reason to think that Frege had sought to capture *Sinn* of our ordinary terms. Moreover, I shall show that he had good reason to believe that he had succeeded.

§3.3. The Mild Hermeneutic View and Partial Grasp. Having now dealt with those passages that *prima facie* support the weak hermeneutic view, let us consider next whether we should accept the mild or strong versions, starting with the former. According to the mild claim, the very most that we can attribute to Frege is the view that, prior to Frege's investigations, we had only a partial or defective grasp of arithmetical thoughts. This view is to be distinguished from that to which one version of the revolutionary view—radical revisionism—is committed. It is also to be distinguished from that to which the advocate of the weak version is committed. As seen in chapter one, the revisionist holds that while ordinary sentences of arithmetic have some kind of content, it is insufficiently determinate to fix a truth-value—they are mere propositional shells. The view of partial grasp that distinguishes the mild-hermeneutic position is that while semantically our arithmetical sentences are in order, our grasp of the *Sinn* of
those sentences is not. The exponent of this view believes that the definition supplements and completes the grasp. To the weak hermeneuticist, the Sinn of our numerals is different, whether fully or partially grasped, and that new Sinne are instituted.\footnote{We will discuss this view in the next chapter.}

To be sure, throughout Grundlagen and elsewhere, Frege is much exercised by what he takes to be our defective understanding of the central concepts of number theory. Two passages in particular might appear to vindicate the mild hermeneutic stance, the second of which we have already had occasion to cite.

1. ‘How is it possible, one may ask, that it should be doubtful whether a simple sign has the same Sinn as a complex expression if we know not only the Sinn of the simple sign, but can recognize the Sinn of the complex one from the way it is put together? The fact is that if we really do have a clear grasp of the Sinn of the simple sign, then it cannot be doubtful whether it agrees with the Sinn of the complex expression. If this is open to question although we can clearly recognize the Sinn of the expression from the way it is put together, then the reason must lie in the fact that we do not have a clear grasp of the Sinn of the simple sign, but that its outlines are confused as if we saw it through a mist. The effect of logical analysis of which we spoke will then be this—to articulate the Sinn clearly.'\footnote{1914, p. 211.}

2. ‘What one calls the history of concepts is really a history either of our knowledge of concepts or of the Bedeutung of words. Often it is only after much intellectual work, which may have lasted centuries, that man finally succeeds in discerning a concept in its purity, in removing the irrelevant accretions that conceal it from the eye of the mind. … Do the concepts, as we approach their supposed sources, reveal themselves in peculiar purity? Not at all; we see everything as through a fog, blurred and undifferentiated.'\footnote{Grundlagen, p. vii; cf. ‘The Laws of Inertia’ (1885), p. 133, CP. Both T. Burge and M. Beaney cite this passage in support of their view that, for Frege, we have only a partial grasp of basic arithmetical notions.}

Concerning the second passage, Frege thinks that our defective understanding is evident from our failure to supply definitions of our ordinary arithmetical terms. \textit{Real} definitions of the concepts of arithmetic will be those given in purely logical
the assumption being that the terms in which the definition is couched will reflect the purely logical nature and essential structure of the concept in question. So knowledge of a concept in its 'purity' (Reinheit) is knowledge of its real definition. But definitions are not just a way of presenting a Bedeutung. From the perspective of Sinn-Bedeutung, they are also a way of displaying the Sinn—the means by which that Bedeutung is given. Thus Bedeutung is given by grasp of the Sinn, which itself is shown by the definition. Since the definition displays the concept's corresponding Sinn, one might believe that knowledge of a definition is required for a full grasp, and thus believe that the 'history of ... our knowledge of concepts' refers to the history of our grasp. Accordingly, it is a defect of our grasp of the central concepts of arithmetic that, prior to Frege's logical investigations, we can say so little about what putatively we refer to when using numerals. If we do grasp the Sinn of the basic vocabulary of number theory, and if Frege's definitions do capture their Sinn, then our grasp of them must fall short of being complete. Thus without knowledge of definitions, 'we see everything as through a fog, blurred and undifferentiated'.

Appeal to the second passage might appear to bear out this claim that the notion of partial grasp was present in Frege's deliberations. Accordingly, despite some differences between these passages, in essence they refer to the same kind of phenomenon: lack of full grasp. One difference is that the latter kind of case might occur without the former. That is, the question of the sameness of Sinn between the analysans and analysandum could be raised without an analysis taking 'centuries' and without 'considerable intellectual work' (grösse geistige Arbeit). It is less certain whether the converse is true: analysis taking much time and effort

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55 'Real definition' here contrasts with 'nominal definition': the latter would correspond to explanations of how the term is to be used in ordinary arithmetic context.

56 Frege seems to confirm this way of reading the passage in a later passage: 'Unlike the author, I see no great need for being able to talk about the history of the development of a concept, and find that there is good reason to avoid this phrases. If we said instead ... 'history of the grasp of the concept', this would seem to me much more to the point...'. Cf. his 'The Law of Inertia' (1885), p. 133, CP.

yet being instantly recognized as correct once the work is complete. Whether or not this is true, it is because of this kind of lack of clear grasp, referred to in this second passage, that the kind of case referred to in the first passage occurs.

Quite what lack of clear grasp means, however, is another matter. It seems that Frege could have meant one of at least two things. Apart from perhaps meaning partial grasp, Frege could be intimating a lack of perspicuity of a kind that leaves unaffected the degree to which the *Sinn* in question is grasped. Doubts about whether it means partial grasp might arise when considering the connection between the last two citations. About the second passage, there can be no doubt that Frege is intimating that our understanding of the *Bedeutung* of our ordinary terms is defective. But whether he also thinks that our grasp of the *Sinn*, by means of which we apprehend the concept, is defective in the sense of making it less than full, is less certain. For one thing, it is unclear that he thinks that an inability to define a term entails partial grasp. It may simply indicate defective understanding. For another thing, it is unclear what he could take partial grasp to amount to anyway. To explore these issues, let us first consider why one might read the last two passages cited as indicating partial or incomplete grasp rather than simply defective understanding.

One might claim, on behalf of the mild hermeneutic view, that success of Frege's logicist project would bring an extension of usage of our arithmetical terms. So for example, on being able to prove the simplest truths of arithmetic, the thinker would have attained further inferential abilities, and these would, one might argue, constitute a greater grasp.58 There are, to be sure, passages in Frege's work that might seem to support the view that Frege subscribed to an inferential semantics: viz. the view that *Sinn* is individuated by the kinds of inferences that the thinker, given full grasp, is disposed to make. But these remarks are few, and anyway insufficient by themselves to show that Frege did indeed seek to individuate *Sinn* in terms of its inferential properties. The most that one can say, I

think, is that Frege’s aim, in *Begriffsschrift*, was to isolate that part of judgeable content relevant to influencing the validity of inferential reasoning. But of course to isolate the inferential properties of a judgeable content is one thing; to identify or explain the judgeable content in terms of these properties is another. Frege might have believed simply that the judgeable content explains the inferential potential, rather than the other way round. A further doubt arises given Frege’s official distinction between *Sinn* and *Bedeutung*, and his taking thoughts to be individuated by their cognitive potential. And we have already seen that Frege might have read the distinction as already implicit in *Grundlagen*.

Besides, even if one were to concede that Frege did subscribe to an inferential semantics, one still might doubt that a thinker’s mastery of Frege’s logicist system would constitute an extension of the thinker’s grasp of the central concepts of arithmetic. On this view, a full grasp of (say) the concept of cardinal number will consist in part in the thinker’s possession of a set of dispositions to use the concept in arithmetical judgements. The trouble here is that the central arithmetical concepts deployed in the logicist proofs are not deployed in any arithmetic context. No applications of ‘*Nx:* *Fx*’—no counting procedures—nor any judgements involving pure mathematics, are made in the logicist proof. It would not do to say that in teaching us that the central concepts of number theory are of a purely logical character, the logicist proofs would have taught us something relevant to a fuller grasp, e.g. that they have a maximal generality as part of their nature. Surely, any competent practitioner of arithmetic, be he expert or layman, knows already that the basic concepts of arithmetic have the greatest possible range of application. Connectedly, Burge says that a full grasp, for Frege, is had where we have a definition that settles all questions about the application of

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59 §12, ‘In a judgement I consider only that which influences its possible consequences. Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated’.

60 For more on cognitive value, see the next chapter.

61 Of course one might say that Frege in his earlier period was an inferential semanticist, but that he changed his mind given the distinction between *Sinn* and *Bedeutung*. But whether Frege was accordingly a mild hermeneutic theorist in the earlier period will depend on a further point now to be made.
the term—a question that ‘a genuine sense had to settle’. Again, the thinker already knows, if only tacitly, that the concept of number can be applied wherever a sortal concept can be meaningfully specified; he already knows that anything thinkable can be counted—he does not need Frege’s Concept Script to teach him that. Unless more is said, the view that arithmetically competent thinkers have only a partial grasp of arithmetical propositions would appear to be counterintuitive.

It might be replied that a full grasp of arithmetical concepts goes beyond the arithmetical contexts. Fully to grasp these concepts is to be able to use them beyond the measuring of magnitudes, and beyond the quantifying of instances, if any, of sortal concepts. But it is unclear that this is true. And even if it were true, it would be unclear whether, on acquiring knowledge or insight into the logical linkage of truths, we do in fact learn that the basic concepts of arithmetic possess greater inferential properties than were imagined hitherto. For arguably the dispositions for this extended use were already in place. It is plain that many of the philosophers and mathematicians whom Frege criticizes sought to define the central concepts, as well as to prove the simplest truths of arithmetic by their means. That they did so was because they suspected that the simplest truths of arithmetic were not primitive truths. But as already said, I do not believe that Frege subscribed to an inferential semantics. Besides, the option is counterintuitive. We would not ordinarily say that if a thinker lacks the disposition to use an arithmetical concept in non-arithmetical contexts, then his grasp of that concept is defective.

Another argument in favour of the partial grasp thesis is to claim that use of number terms does not determine what their Sinn is. According to M. Dummett and T. Burge, for example, however great our mastery of conventional usage of mathematical terms, however complete our grasp of conventional significance associated with these terms, this, for Frege, is not enough for a thinker properly to

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*Burge, 1984, p. 6.*
grasp the *Sinn* belonging to them. Frege himself does not explicitly say this. Yet does he nevertheless signal it in his practice? To answer this, let us ask why he might have held this view. One reason, which Burge seems to have in mind, is that reflection on use would produce an array of incompatible accounts: witness the first two chapters of *Grundlagen*. But a closer examination of that work shows that matters are less straightforward. Indeed, it shows that Dummett and Burge’s claim is the wrong way round. Part of what makes Frege’s critique of alternative theories appear effective—at least *prima facie*—lies precisely in showing that those rival accounts do not conform to our ordinary usage. The central fault of these accounts is not that too much reliance is given to the examination of a thinker’s dispositions. On the contrary, our dispositions are his guide. The fault is rather that the examination of those dispositions does not proceed in the right way. The right way to proceed is to reflect on the use of one’s terms—on one’s arithmetical practices—within the right conceptual framework. Frege’s three methodological principles partly constitute that framework. This is why he insists that they be adhered to when engaged in conceptual analysis of what one ordinarily grasps. It is in the light of his logical theory, together with selective attention to our ordinary practices—on our dispositions—that Frege attains his insights. It is by this means that he discovers such features as the unrestricted generality of application of our terms, that numbers belong to concepts, that numbers are objects, as well as the centrality, and conceptual priority, of one-one correspondence between extensions of first level concepts. Once these features are brought to the fore, it seems to Frege that, pending a more plausible alternative, numbers could not but be extensions of equinumerous concepts. Others too will see this, Frege hopes, if only they would have their reflections on the ordinary usage of arithmetical terms be guided by the methodological principles expounded in *Grundlagen*.

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64 See chapter 1.
Finally, even if the partial grasp view were right, which I doubt, it would still be wondered why partial grasp is entailed by the fact, if it is one, that we cannot articulate the *Sinn* of a term from examination of our full mastery of its conventional usage. I see no compelling reason why the absence of the definition in question would impair our grasp. So I see no compelling reason to think that knowledge of a *real* definition would fill the gap between full and partial grasp.

§3.4. The Strong Hermeneutic View and Full Tacit Grasp. According to my strong hermeneutic view, there were at least some periods during Frege’s foundational inquiry when he sought to translate both the *Sinn* and *Bedeutung* of our arithmetical language into his Concept Script. Furthermore, this view involves the claim that there is, for Frege, a set of arithmetical practices or usage of terms that is constitutive of the full grasp of certain mathematical concepts. If my hermeneutic view is right, then Frege recognized no difference between the layman and the expert regarding the fullness of their grasp. Hence there would indeed be a distinction implicit in Frege’s work between defects of perspicuity in one’s grasp, and partial grasp itself.

Implicit too would be a threefold distinction between kinds of knowledge of a concept. Firstly, there would be knowledge or understanding of a concept that is constitutive of its grasp. Secondly, there would be knowledge of a concept that has no such bearing, but is nevertheless constitutive of greater perspicuity of that concept. Thirdly, there would be knowledge of a concept that is irrelevant both to our grasp and to its perspicuity. To be sure, a thinker needs some understanding of a concept in order to grasp it. But one’s understanding certainly need not be complete in order to grasp it fully. Nor need a thinker be able to articulate that piece of understanding—whatever it happens to be—that is constitutive of his grasp.\(^{65}\) Advocates of the mild-Hermeneutic view appear to have conflated these distinctions regarding what knowledge of a concept entails.

\(^{65}\) Competence in ordinary arithmetical practice would surely suffice.
Knowledge of a concept entails greater understanding of a concept; but greater understanding need not entail greater grasp.

Prior to Frege's logical investigations, a thinker had only partial or incomplete understanding of even the simplest concepts of arithmetic, but had full epistemic possession of the thought, or thought component, under investigation. Since arithmetical concepts (Sinn) are, on my reading, fully grasped, there seems to be no alternative but to say that what would have been achieved is the recovery or articulation of what we had all along only tacitly apprehended. The defects of understanding of arithmetical concepts, which stem from partial knowledge of the concept, are of the kind that relates to our lack of perspicuity of grasp. Hence one could say that insight or discovery of hidden inferential properties of the concept of number would be partly what it is to have a clear and distinct grasp of the judgeable contents. Making good our partial understanding can afford greater perspicuity of grasp.

But what more precisely is tacit grasp, and how plausible a view is it?

Concerning the first question, it seems to me that there are three kinds of tacit grasp relied on in Frege's investigations. Common to all three types is the presence of a distinction, already noted, between conscious and unconscious grasp.

1. The first, most ordinary type occurs where a term has been introduced by definition. As already noted, Frege thinks that 'we simply do not have the mental capacity to hold before our minds a very complex logical structure so that it is equally clear to us in every detail'. Frege thinks that tacit grasp is both

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66 ... I should like to claim that I have given precise verbal expression to principles and methods which up to now have been used only tacitly.' 'Mr. Schubert's Numbers' (1899), p. 250, CP. Unfortunately, this passage is not as definitive as it may seem. The citation is taken from a longer passage in which Frege pretends, in a spirit of mockery and sarcasm, to be a formalist like Schubert. But the passage is suggestive nonetheless. For what is mocked of course is the formalism, not the claim that tacit grasp actually occurs. As such Frege's remarks show a significant awareness of the phenomenon central to strong hermeneutism.

67 1914, p. 222, PW; NS, pp. 239-40.
useful and inevitable (given how we are), but that it proves troublesome given the	hird type below.\footnote{It can hinder as well as facilitate mathematical progress. As we shall see in the final chapter, it can also hinder us where our purpose in constructing a system is epistemological.}

2. The second kind occurs where the *Sinn* was not acquired through definition. In this case, as in the last, the thinker has some awareness, however hazy, of the *Sinn* as it would be presented by the definiens. One example is provided by Frege’s analysis of the *Sinn* of ‘just as many as’, or that of mathematical induction. Of the former, the *Sinn* is given in terms of one-one correspondence whose structure he lays bare as: \((\forall x) \ (Fx \rightarrow (\exists y) \ [Gz \land (\forall z) \ (Rxz \leftrightarrow z = y)]) \land (\forall y) \ (Gy \rightarrow (\exists x) \ [Fx \land (\forall w) \ (Rwy \leftrightarrow w = x)])\)\footnote{Grundlagen §72, pp. 84-85. Here the first clause states the condition for the concepts \(F\) and \(G\) to be in a relation of many-one while second clause states the condition for the concepts to be in a relation of one-many. Where the conjunction of these two conditions obtain, the concepts stand in a relation of one-one.} Frege takes as granted that we fully grasp ‘just as many as’—not least because that grasp is manifest in our ability to count, which is nowhere in doubt. While it is, for many, an unarticulated grasp, once it is articulated, once Frege’s definition of one-one correspondence is given, there is likely to be, given sufficient familiarity, a recognition of its correctness, an element of ‘aha … yes of course!’\footnote{About this, Frege was of course mistaken. Husserl, for example, did not recognize the putative sameness of *Sinn* between the cardinal equivalence ‘just as many’ and ‘one-one mapping’. According to Husserl, ‘just as many’ means that if you were to count the Fs and also the Gs you would get the same number. Furthermore, he believed that the obtaining of a one-one mapping between the Fs and the Gs merely guarantees that the number of Fs is the same as the number of Gs. Moreover, Waismann would object that numerical equivalence between two sets merely guarantees that the elements could be correlated, not that such a relation actually obtains. Cf. Introduction to Mathematical Thinking (1951), pp. 108-09 (Hafner Publishing Company, London).} In another example, Frege makes his interest in this kind of tacit grasp explicit. He remarks that people all along have grasped axiom V but that their attempts to describe it are faulty—e.g., they take functions as standing on either side of identity. “The expression is different from ours, but all the same here too we have an equality holding generally transformed into an equation (identity).”\footnote{Grundgesetze vol. ii, §146, ‘Frege on Definition’, p. 160, in TPW.} Further, speaking on the notions of class, set, manifold, he says ‘… we may well suppose that what mathematicians call a set (etc) is nothing other than an extension of a concept,
even if they have not *always been clearly aware of this*.\textsuperscript{72} In neither case does Frege fault our grasp of these notions. What are faulted are the attempted proofs and definitions, which faults arise in the absence of a perspicuous grasp.

3. An instance of the third kind of tacit grasp is apparent in the case of Frege’s definitions of the *Sinn* of ordinary numerals. One will wonder whether it is really plausible to suppose that we have a full tacit grasp of the *Sinn* of ‘0’ as: the extension of the concept *equinumerous to the concept not identical to itself*. Particularly is this so if we compare it with some of the other examples of tacit grasp just illustrated. For instance, it will be wondered whether if this definition really did represent the recovery of a full tacit grasp there should not be some recognition of its being right. What distinguishes this third kind of grasp is the absence of any awareness, however unconscious, that this is what we had all along grasped when using the term ‘0’. Moreover, there need be no recognition following the unconscious recollection. Still, this does not mean that Frege despaired of there ever being an element of recognition *vis-a-vis* his definition of the numerals.\textsuperscript{73} Indeed there is evidence that he believed that the light would gradually dawn on the practitioner given sufficient familiarity.\textsuperscript{74}

Finally, it might be said that the notion of full tacit grasp would commit Frege to certain counter-intuitive consequences, and that this would be a reason against attributing that notion to him. If Frege believed that the basic concepts of arithmetic were tacitly grasped, why should not that mean that more complex mathematical concepts were also involved? Indeed, what is to stop us from thinking that all mathematics is already in our possession? Where, if at all, would he draw the line? I think that, for Frege, there is a line, and that tacit grasp does not commit him to saying that all mathematics is already in our epistemic possession. But where he would draw it, I do not know. It is a remarkable fact that, for all its complexity, mathematics is built up from a fairly small set of elementary arithmetical concepts. But that we tacitly grasp these concepts need

\textsuperscript{73} ... or regarding Axioms V, which falls also into this category.
\textsuperscript{74} We discuss this in chapter six.
mean no more than that we have all along grasped a set of fairly elementary set of propositions. This would suggest that there are perhaps, for Frege, two kinds of mathematical discovery. One kind is that whereby constituents of propositions long grasped but hitherto undefined are reflected upon and whose nature and structure is laid bare. The other is that whereby basic concepts and propositions are combined in new and interesting ways, and produce, for example, the theory of calculus. In short, it is one thing fully to grasp, if on occasion unconsciously, the basic concepts of mathematics. It is another thing to bring these concepts and propositions into rich and complex combinations, as in the articulation of the theory of differential calculus.75

§3.5. Closing Remarks. We have been arguing towards the view that Frege’s logicist project was the attempt fully to translate the Sinn and Bedeutung of our ordinary arithmetical language into Concept Script. Furthermore, we have argued that one way to defend this hermeneutic view is to see that the doctrine of the recovery of full, if to some extent tacit, grasp, informed his project. While we do not have full understanding of, for example, the central concepts of arithmetic we nevertheless have a full grasp of them. Frege and his rival theorists sought to articulate this grasp. A number of considerations favoured our interpretation. Firstly, by distinguishing between conscious and unconscious association of Sinn with a term, by drawing attention to the notion of dual reference of a term, I deflected the complaint that the Sinn displayed by Frege’s definition is not what we ordinarily associate with our ordinary arithmetical terms. Secondly, I brought to attention the fact that Frege appears to attribute false beliefs to his rival theorists rather than true beliefs about different entities. Thirdly, I showed how passages that appear at first sight to tell against the mild or strong hermeneutic claim can on further reflection be used in its support—especially if the notion of

75 As it happens, I believe that Frege would probably include other concepts of mathematics among those already in our possession prior to our formal initiation into mathematics. It may not be implausible to say that our judgements involving the varying velocities of objects in the world, e.g. judgments involved when trying to cross a busy road, contain the concept of instantaneous rates of change, that is the first order derivative.
full tacit grasp, rather than simply partial grasp, is deployed. Fourthly, I argued that insufficient attention is given to the distinction between knowledge or understanding of a concept that is and is not constitutive of our grasp of that concept. In doing so, we shed some light on what tacit grasp is for Frege. Finally, we defended this part of the strong hermeneutic thesis against the milder version, which claims that our grasp, for Frege, was partial or defective. We shall have more to say about the latter claim in chapter six when we consider other aspects of Frege's motivation. In the meantime, we need to consider further arguments against weak hermeneutism, and further arguments in favour of the strong variety. In particular, we turn to the view mentioned at the outset of this chapter: that Frege's definitions were meant to be informative identity statements.
§4.1. Introduction. We have been arguing that Frege attempted fully to translate our ordinary arithmetical language into Concept Script. According to this, the strong-mild hermeneutic, thesis, the terms of Frege's logicist proof were meant to capture both the \textit{Sinn} and \textit{Bedeutung} of our ordinary arithmetical terms. The difference between strong and mild hermeneutism concerns the state of the thinker's grasp prior to the translation. The mild view holds that our grasp was partial or incomplete. The strong view holds that our grasp was full, if to a large extent unperspicuous and tacit. In the last chapter, we considered whether Frege intended his definitions to capture solely the \textit{Bedeutung} of our ordinary terms, rather than their \textit{Sinn} as well, and we offered reasons for believing that Frege subscribed to strong hermeneutism. But mostly our concern there was negative: to deflect a not uncommon reading of two passages, from 'Logik in der Mathematik' and 'Frege to Husserl 1906', that \textit{prima facie} tell against our preferred variety of the hermeneutic hypothesis.

In this chapter, we consider two further objections, one of which we mentioned \textit{en passant} in chapter three. The first, and most important, of these rests on the observation that Frege regarded \textit{Grundlagen}-definitions as informative. The observation, if accurate, might have one of two explanations, either of which would undermine strong hermeneutism. Let the \textit{Grundlagen}-definitions be explicative; and let it be true that they are indeed informative. Then one explanation might be that, where, for example, the definition is of a singular term, the items flanking the identity sign must, in some respect, have been regarded, by Frege, as differing in content. In that case, Frege could not have been seeking to explicate the \textit{Sinn} of our ordinary arithmetical terms. Hence

\footnote{Another example would be of biconditionals, flanked by two sentences in which would contain}
strong hermeneutism would be false. An alternative explanation might be that the informativeness resides not in a difference in the content of the analysans and analysandum, but rather in the differences in the thinker's grasp. Here the informativeness resides in the transition from a partial to full grasp. The definition imparts new information to the thinker about the constituents of the analysandum: information constitutive of a greater grasp. This latter explanation of the informativeness of *Grundlagen*-definitions would of course perfectly accord with the mild-hermeneutic thesis, which sees the difference between ordinary arithmetical language and Concept-Script arithmetic not in terms of content—*Sinn* and *Bedeutung* are preserved—but in terms of different kinds of grasp. But it would refute my strong hermeneutic hypothesis: that the translation, and the analysis that precedes it, recovers a full, but in large measure tacit, grasp of arithmetical propositions, and thereby makes good a defective understanding of these propositions.

One reason for thinking that *Grundlagen*-definitions are informative arises from what Frege actually says. Recall first that Frege has two conceptions of definitions. On the one hand, he remarks that

'[s]uch definitions merely serve to effect a superficial simplification by fixing an abbreviation. Once converted into a judgement they can be regarded as analytic since all that can come out again is what was put into the new symbols.'

'Such definitions' are what he also calls definitional identities. In the first instance they are simply declarations about the content of symbols, whether their content is explicative or not. Once the declaration is made, the definition is transformed into an identity statement: *i.e.* it is no longer a declaration about the content of a symbol; it is an assertion about the identity of contents. Since the content of the right-hand side of the identity, the definiens, is the same as that of

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either the definiens or definiendum of a function expression.

2 *Begriffsschrift* §24, CN. My emphasis.

3 For example, a definitional identity, for Frege, would be the metalinguistic statement—"The Number of F" is the extension of the concept equinumerous concept F. Once this stipulation has been made, the definitional identity can be transformed into, *The Number of F is the extension of the concept equinumerous to the concept F*—an identity statement in the object-language.
the left-hand side, the definiendum, the statement is analytic or analytically true. Note that this latter notion of analyticity seems to invoke Kant's criterion:

'A true judgement of the form 'A is B' is analytic if the predicate B is contained in the subject term.'

On the other hand, Frege distinguishes between 'more fruitful' and 'less fruitful definitions', and identifies the latter as having informed Kant's view that analytic propositions do not extend our knowledge. By contrast, Frege claims that his own 'more fruitful' definitions show that, on the contrary, analytic propositions, of which transformed definitional identities were a paradigm case, could represent a cognitive advance—hereafter the pivotal connection.

'The conclusion we draw from [the more fruitful definitions] extend our knowledge ... and yet are ... analytic.'

On describing Grundlagen-definitions as fruitful, Frege says that we can prove with the definition what cannot be proved without it.

'... [W]e very soon come to propositions that cannot be proved so long as we do not succeed in analysing concepts that occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number that has to be either defined or recognised as indefinable.'

'Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless.'

A number of commentators have maintained that Frege took the analysans of Grundlagen-definitions to be conceptually richer than its analysandum. If the

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4 This is not at all to suggest that, for Frege, the subject-predicate form was any real part either of the logical structure of the thought or of any part of the logical syntax of the Concept-Script. See below.
5 *The Critique of Pure Reason*, A6-7/B10. For Kant this is both a necessary and sufficient condition for analyticity. For Frege it is only a sufficient condition.
6 In other words, I shall mean by 'pivotal connection' whatever the connection is between Frege's fruitful definitions and extension of knowledge, in particular extensions of knowledge involving analytic propositions.
7 *Grundlagen* §88.
10 R. Grossman, 'Frege's very idea of a reduction of arithmetic to logic rests, as it were, on the notion of informative identity statements.' *Reflections on Frege's Philosophy*, (1969), p. 4; cf. M. Dummett, 'Frege and the Paradox of Analysis', p. 25, in *Frege and Other Philosophers* (1990a); *The Interpretation of Frege's Philosophy* (1981b), p. 339, p. 327; *Frege's Philosophy of
extension of knowledge in question cannot be attained by the definiendum but
only by means of its definiens, the equation between the definiendum and
definiens would indeed seem to be informative. In general, then, Frege is
allegedly committed to the view that two thoughts, whose single difference is that
the one is expressed with a sentence containing a definiendum, and the other by a
sentence containing its definiens are type distinct. On that difference rests the
notion that, given fruitful definitions, analytic judgements can extend our
knowledge.

A further consideration in support of this first objection (to strong
hermeneutism) is independent of what precisely the connection between fruitful-
definition and extension of knowledge turns out to mean. For *prima facie* it just
seems evident that Frege’s definitions are informative. Consider the transformed
definitional identity:

(1) \(0 = \text{the number that belongs to the concept not identical to itself.}\)

Consider also a similar transform of Frege’s definition of number:

(2) \(\text{The number that belongs to the concept } F = \text{the extension of the concept}
\text{equinumerous with the concept } F.\)

In that case, given (1) and by substitution of the variable in the right-hand side of
(2), we get a definitional equivalence between the right-hand side of (1) and the
right-hand side of the following:

(3) \(\text{The number that belongs to the concept not identical to itself = the}
\text{extension of the concept equinumerous with the concept not identical to}
\text{itself.}\)

Not only might (1)-(3) by themselves strike the thinker as informative; we can
derive what appears to be a still further definitional equivalence from them:

(4) \(0 = \text{the extension of the concept equinumerous with the concept not}
\text{identical to itself}.\)

Again the relevant possible corollaries are as before. Either the contents of the
terms flanking the identity are in some sense different. Or the contents are the

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11 A version of this point, used in another context, can be found in R. Grossmann, 1969, pp. 251-
same, and the grasp of some of the constituents of a least one side of each identity statement is new. Either way, the strong hermeneutic thesis is undermined.

The second, and, it will emerge, less important, objection, arises from observing that, after Grundlagen, Frege no longer deploys the connection between fruitful-definition and extension of knowledge; nor does he again use ‘analytic propositions’ in this earlier context. According to M. Dummett and E. Picardi, abandonment of the latter notion would further support the view that Frege’s definitions never had the objective of reproducing the Sinn of our ordinary arithmetical terms.¹²

It is to be found that none of the above objections need be sustained.

§4.2. Analytic Propositions, Extension of Knowledge, and Frutiful Definitions. Let us start by securing greater purchase on the pivotal connection—the connection between fruitful-definitions and extension of knowledge.

As is well known, one of the ways in which Kant understands the notion of an analytic proposition is that P is analytic if its predicate-concept is contained in its subject-concept. So understood, he denies that analytic propositions extend our knowledge, since if F is the subject-concept of P, whose analysans is $f_1 \land f_2 \land f_3 \land f_i \ldots \land f_n$, and if the analysans of the predicate-concept G is some subset of those $f_i$ properties, then a thinker who understands the subject-concept of P cannot, by coming to know the truth of P, subsequently extend his knowledge: for he already knows that P by knowing F.¹³ Hereafter the ‘containment view’.

By contrast, a proposition, S, enlarges our knowledge, for Kant, if the source of S’s predicative material is not part of the other constituents of S. For Kant, this left just one possible source: intuition.

¹³ These remarks apply whether the definitions are stipulative or explicative.
As Frege points out, one problem with Kant's way of understanding analytic propositions is that it leaves out singular and existential judgements. Another problem is that it is not equivalent to the second way in which Kant understands analyticity, and this makes for a potential incompatibility between the two criteria. The second way is that P is analytic if its denial is self-contradictory: if it involves the law of non-contradiction. Whereas singular and existential judgements would not be classified as analytic on the first criterion, they would be so classified on the second criterion if their truth were shown to rest on laws of logic. Moreover, they would be candidates for extending knowledge in the above sense of that term, whereas universal affirmative judgements would not be—because, again, in the latter case, the concept in the predicate would be contained in the subject; but not in the former case, of singular and existential judgements, since they have no subject concept.

Frege identified the fault of Kant's classification in the importance assigned to the subject-predicate model of representing the structure of a thought and sentence. Having rejected this model, Frege showed how to unify Kant's criteria of analytic truths by bringing together two of their key features: their key features being, for one criterion, the notion of definition, and for the other criterion, the notion of a logical law. A proposition is analytic, Frege proposed, if it could be proved from (1) purely logical laws and (2) from definitions of its constituent parts in purely logical terms. That is, Frege ceased on the importance both of definitions, which informed the containment criterion, as well as that of a logical law, which informed the other criterion of analyticity. By insisting that definitions be given in purely logical terms, he forged the connection between the two conceptions. For he saw that lying at the heart of Kant's criteria is the insight that a truth in question is analytic if it rests solely on logic, and of course Frege's definitions were meant to help show that this is in fact the case with arithmetical truths.

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14 E.g. ‘7 > 4’, and ‘There is a prime number between 4 and 7’, respectively. Cf. Grundlagen, and 'The aim of 'Conceptual Notation', p. 91, CN.
15 By retaining these two features, Frege, like Kant, was able to maintain that an analytic truth is
The rejection of the notion of subject-predicate model allowed not only for a more inclusive criterion of analyticity. Its removal helped to show that the candidate truths of analyticity (e.g. those of arithmetic) could meet the newly unified criteria. It did so quite simply because what replaced the old model was a more precise representation of the structure of propositions; not the least among which were those propositions transformed from definitions (see below). This in turn allowed for the greater representation of valid inferences. The model in question is, of course, that of representing logical syntax as having a function-argument structure.\footnote{As Frege says, the judgeable content of the form }\Phi(a)\text{ can be regarded as the value of a second level concept }\psi(\alpha)\text{ for the concept }\Phi(x)\text{ as argument; or as the value of }\Phi(x)\text{ for the object }A\text{ as argument. }Begriffsschrift\ \S10, \textit{CN}. \text{In general, if what is regarded as replaceable is of level }n, \text{ what remains is then of level }n + 1. \text{ Put in terms of predication, where predication of objects is first-level, objects are of level zero, so that predication of these properties or relations are of second level, as predication of these must then in turn be of one level higher up, and so on. As pointed out below, the primary importance of this lies with its greater scope for representing valid inferences. Given }\{\Phi(a), \Phi(b), \exists x \; F(x), \forall y \; F(y)\}, \Phi(a)\text{ is regarded as decomposable into the first-level predicate '}F(x)\text{' with object as argument 'a'. Given }\{\Phi(a), \Phi(b), \exists \phi(\alpha)\}, \text{ we can regard '}F(x)\text{' as decomposable into first-level predicate '}F(x)\text{' and a second-level predicate '}\psi(\alpha)\text{'.}

To see one advantage of rejecting the subject-predicate model consider the following example:

1. Everything killed himself.
2. Cato killed Cato.
3. Something killed Cato.

The validity of this argument can be accounted for on Frege's argument-function model, but not on the subject-predicate model. On the latter model, we can account for the validity of the move from 2 to 3, but not from 1 to 2. From 2 to 3, the same predicate, 'killed Cato' is applied to different subjects. But from 1 to 2, different predicates are applied, 'killing himself' and 'killed Cato'. On Frege's model, however, the same two-place relation-term '... killed ...' is applied in each step. Cf I. Rumfitt's 'Frege's Theory of Predication', \textit{Philosophical Review} (1993).

\footnote{Grundlagen \textsection\textsection88. My emphasis.}
quantifier by disjunction. Together with other ways of listing characteristics, 
namely by use of negation and the conditional, one would have at hand the 
elements for representing the most complex of concepts. The criticism is rather 
that the Kantian (or Boolean) method lays down elements of the definition in 'no 
special order'. Many concepts have structure involving quantifiers and other 
operators of differing scope. Yet no language before Concept Script was 
sufficiently equipped to represent this special ordering. Ambiguity of scope can 
mean failure to distinguish between a true and false sentence.\(^\text{18}\) Also unless 
differences in scope are made clear, the validity of reasoning involving a 
definitional transform, or any other sentence, will not be represented, precluding it 
thereby from use in proof. This problem became pressing in mathematics of the 
nineteenth century. On the one hand, nowhere are sentences involving multiple 
generality more common than in mathematical discourse. On the other hand, 
mathematicians demanded greater rigour in proofs, but lacked sufficient 
techniques. Frege's Concept Script provided the means. Consider the sentence, 
\(F\) is everywhere continuous at \(a = (\forall x) (\forall y) (y > 0 \rightarrow \neg (\forall z) (z > 0 \rightarrow \neg (\forall w) 
(|w| \leq z \rightarrow |f(x + w) - f(x)| \leq y)))\).\(^\text{19}\) Because the scope differences of the 
quantifiers are clear, so too are the truth-conditions of the sentence, and thus also 
its contribution to the validity of the proof. Clear too is the irrelevance of the 
subject-predicate model to the logical syntax of this sentence.

More importantly for our purposes, Frege identified this more fruitful 
method of defining as warranting the view that, pace Kant, analytic truths could

\(^{18}\) For example 'Every boy loves some girl' is ambiguous as it stands. It says either that 'Every 
boy loves the same girl', or that 'Every boy loves some girl or other'. Depending on what one 
intended to say, it could be regarded either as \((\forall x) [B(x) \rightarrow (\exists y) G(y) \land xLy]\), where \([B(\xi) \rightarrow (\exists y) 
G(y) \land xLy]\) is a first level concept, taken as an argument for \((\forall x)\); or as \((\exists y) (Gy \land (\forall x) (Bx \rightarrow 
xLy))\), where the existential quantifier takes the remaining part as argument.

\(^{19}\) Frege extols his own method of forming concepts, emphasizing the use of the quantifier, negation and 
conditional as central to it. 'The content is to be rendered more exactly than is done by verbal language. 
There is only an imperfect correspondence between the way the words are concatenated and the structure of 
the concepts.' See his 'Booles Rechnende Logik und die Begriffschrift' (1880-81), pp. 12-13, PW. For a 
useful survey of the kind of concept-formation identified here as 'less fruitful' see H. Sluga's 'Frege 
Against the Booleans' (1987), esp. p. 83, p. 84, p. 87, p. 89. However, H. Sluga's paper falls short of 
making clear the kind of contrast Frege is drawing between 'more fruitful' and 'less fruitful definitions'.
extend our knowledge. Referring back to the definitions proved in earlier sections of *Grundlagen*, Frege, in the following three *pivotal extracts*, draws a contrast between Kantian analytic propositions and his own, cognitively more powerful, variety.

'What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others.'

Here again, in his reference to 'simple list', contrasted with his reference to *elements organically connected*, Frege is advertising the more fruitful method of definition: namely as residing in greater representation of structure, particularly of scope differences, whether of quantifiers or other truth-functional operators.

Frege continues

'A geometrical illustration will make the distinction clear to intuition. If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this ... we ... use the lines already given in a new way for demarcating an area [Fn: Similarly, if the characteristics are joined by “or”.] Nothing essentially new, however, emerges in the process.'

The section from which this pivotal passage is extracted makes clear, I think, that Frege has in mind here the containment view of analytic propositions, and the method of definition that had accompanied, or indeed informed, it. Hereafter, however, a swift switch occurs from discussion of the containment variety to the merits of the Fregean type. Frege indicates the latter below with reference to inference and proof.

'But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as

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synthetic; and yet they can be proved by purely logical means, and are thus analytic.\textsuperscript{22}

Unfortunately, these three (pivotal) passages, replete as they are with metaphor, do not, as they stand, make clear the contrast Frege seeks to draw here between, on the one hand, analytic propositions that, given fruitful definitions, do extend knowledge, and on the other hand, analytic propositions that, given the less promising variety, do not. In particular, one will still wish to ask what precisely is meant here by ‘extension of knowledge’; as well as ask what it is about quantifier-variable structure—a central characteristic of the ‘more fruitful method of defining’—that makes extension of knowledge possible. For instance, does the remark ‘something new emerging’ pertain, as many Frege scholars have maintained, to a process of arriving at something conceptually new—hereafter the ‘inflated sense’ of extension of knowledge? And what precisely would that mean? Or does Frege have in mind a more deflated sense of ‘extension of knowledge’?

Consider, first, what the latter possibility might mean. Analytic propositions do not extend our knowledge, Kant claimed, because they fail to comply with the condition:

\textit{Proposition, S, extends our knowledge if S’s predicated material is not part of the other constituents of S.}

Frege reconstructs the notion of analyticity in terms of logical proof,\textsuperscript{23} involving the requisite kind of premises. In that case, he could claim that some analytic propositions comply with the above Kantian condition in the following deflated or minimalistic sense:

\textit{Proposition S is an extension of our knowledge if it is inferred from a set of premises, included among which are ‘fruitful definitions’, and such that while S is not a member of that set, the elements of S are elements\textsuperscript{24} of the propositions of that set.}

\textsuperscript{22} Op. cit.

\textsuperscript{23} Nothing radical is intended here by my use of ‘reconstructs’. Frege of course takes himself to be explicating what he takes Kant to have had (albeit dimly) in mind. See Grundlagen, §3, fn. 1; §88, fn. 1. Precisely what we are to make of that claim is controversial. We take up the issue in chapter six when we consider Frege’s alleged epistemology.

\textsuperscript{24} I use ‘element’ here neutrally with respect to whether Frege had a merelogical view of the structure of propositions or judgeable content.
Clearly this construal of Frege's use of 'extension of knowledge' meets Kant's condition. The epithet 'extension of knowledge' would apply to analytic propositions because, given 'fruitful definitions', the source of the predicated material would be shown to stem from other propositions alone—and because these propositions would be derivable exclusively from logical laws. But notice that no presumption is made here regarding the grasp of concepts not grasped prior to the deductive proof, or prior to any analysis of the content of our ordinary arithmetical language. If we could substitute the terms of, e.g. the sentence '7 + 5 = 12', for definitions in purely logical terms and could prove the proposition by means only of these definitions and logical laws, then that proposition would be analytic. More, it would represent an extension of knowledge—but in the above minimalist sense. No presumption would be made here about the 'conceptually new' because none would be required.

But just for that reason might this minimalist notion of extending knowledge be criticised. For prima facie, Frege in the pivotal extracts wishes to understand the pivotal connection—i.e. between fruitful definitions and extension of knowledge—in terms of the idea that analytic propositions can be informative, that they can yield the conceptually new: the inflated sense of the pivotal connection. As just seen, he says

(1) 'We ... use the lines already given in a new way for demarcating an area [Fn: Similarly, if the characteristics are joined by “or”.] Nothing essentially new, however, emerges in the process. (2) But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. (3) What we shall be able to infer from it, cannot be inspected in advance; (4) here, we are not simply taking out of the box again what we have just put into it.'

(Because the purport of this passage is not, to my mind, immediately clear, and for ease of exposition, I have divided the passage into four parts.) First, we have a contrast in sentences (1)-(2) between 'nothing essentially new' emerging and 'what was not previously given at all'. But in what, precisely, does this contrast consist? It is not altogether clear that Frege does have 'new concept' in mind,
however one might understand the term ‘new’. Here ‘boundary lines’ refers to the extension of concepts, given in this case by definitions. By use of disjunction or conjunction, new concepts can be formed: we combine the boundary lines in new ways, yet without altering the existing lines of any single concept. If we use only these old methods, then the boundary lines will not reflect with sufficient clarity the structure of the concepts identified as the more fruitful: e.g. the ancestral relation, continuity, concept of limit, and number. The reason is that given above. Without Frege’s quantifier-variable technique, the boundary lines will not represent scope differences, and other important structure. The true boundary lines will not have been ‘previously given at all’. Accordingly, what is ‘essentially new’ need mean no more than that of rendering of the logical structure hidden by old methods of demarcating boundary lines. Hence, if we read the sentences of (2) in this way, then, modulo further consideration, my minimalist construal of ‘extension of knowledge’ need not be unfaithful to the spirit, nor indeed to the letter, of the above pivotal extracts.

It may appear, however, that my interpretation of sentences (1)-(2) fails to cohere with lines (3) and (4). In (3) Frege says ‘[w]hat we shall be able to infer from [the fruitful definition] cannot be inspected in advance’. And in (4), he says ‘... we are not simply taking out of the box again what we have just put into it’. Surely, it will be said, these two lines speak of the inflated sense of ‘extension of knowledge’? But let us look more closely. In lines (3)-(4), Frege continues his contrast with the containment view of analytic proposition begun in lines (1)-(2). Given analytic propositions of the containment variety, we can inspect in advance what can be inferred from it, at least to a certain extent. If we know what is contained in the subject-concept \( f_1 \ldots f_n \), then we know from our understanding of it what can be deduced from it. (If the subject term is A, then we can infer that ‘A is \( f_1 \)’, ‘A is \( f_2 \)’, ‘A is \( f_1 \)’, and so forth.) Line (4) gives the reason: we are merely taking out of the box—the subject term—what we have just put into it. But this is not possible with Frege’s variety of analytic propositions. Firstly, there is no box

\[25\] We cannot of course know in advance all that can be inferred from it.
out of which to take what one has just put into it—Frege’s propositions are not of that form. Secondly, we cannot inspect in advance what can be inferred from a Fregean proposition alone; because to make such an inspection we require not one proposition, but several. Again, my minimalist reading of ‘extension of knowledge’ is not incompatible with the pivotal extracts.

But what, if anything, is the connection—alluded to in line (3)—between our alleged lack of foresight regarding what is inferable from a more fruitful definition, and the advantage of Frege’s method of defining identified above? And why might the connection matter? The advantage of Frege’s method identified above was that it could represent more concepts more thoroughly than could the Boolean method. On the inflated sense of extending knowledge, the connection in question lies with the grasp of new concepts, and that is why it matters. On the minimalist reading, by contrast, no such connection is there, nor need it be. And it is not there. To see this, we should recognise that lines (3) and (4) are apt to mislead. Taking line (3), suppose we have two analytic propositions, (A) and (B), both converted from a definitional identity, where (A) involves only Boolean methods, and (B) involves Frege’s more fruitful approach. Let them be used in separate proofs. Now proposition (A) might be of enormous complexity, prohibiting us from seeing what is derivable by its means. Indeed, the proof involving (B), consisting of nested generality, might be of considerably less complexity than the proof in which (A), the Boolean counterpart, occurs. We may well be able to see immediately, or even after some contemplation, the inferential possibilities availed by the definition, given other premises. Of course, given a suitably astute cognitive agent, this might be true of the most taxing definition, be the definition Fregean or Boolean. After all, Frege himself says

‘No doubt these propositions are in a way contained covertly in the whole set taken together, but this does not absolve us from the labour of actually extracting them and setting them out in their own right’.26

My point is that, taken as a general remark about what is inferable given a fruitful definition, line (3) fails to capture what is crucial to the notion of fruitful
definition. The remark appears to rest on an observation about our psychological limitations.

The passage just cited returns us to line (4), in particular, to its potential to mislead. For the passage appears to conflict with the claim, suitably broadened, that ‘... we are not simply taking out of the box again what we have just put into it.’ For any analytic propositions that extends knowledge precisely does depend on taking something out of the box—now the set of propositions—whose contents we supplied on constructing the proof system. Unless of course the qualifier ‘simply’ carries some special, but as yet unspecified, weight. So if we do not read (3) and (4) simply in terms of the contrast set out above, they would appear to be both misleading and false. Moreover, they will appear to be of little importance to the pivotal connection—between a fruitful-definition and extending knowledge. Again, fruitful definitions enable us to prove truths otherwise unprovable. This is because Frege’s method permits a more comprehensive representation of the structure of certain concepts than does the Boolean method. This by itself secures an extension of knowledge, given our minimalist reading of that notion. Pending the viability of alternative readings, it appears that that is all that is essential to the pivotal extracts. The rest, I suggest, is misleading rhetoric. As we shall now see, our construal of ‘extension of knowledge’ would take us a step forward in rebutting an objection, outlined above and considered below, to our strong hermeneutic claim. First let us further consider the inflated construal of the pivotal extract.

§4.3. The Value of Deduction and Analysis: An Inflated Interpretation. M. Dummett has suggested that the pivotal extracts are a response to a two pronged challenge. The first challenge is to explain how deductive reasoning from purely

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26 *Grundlagen* §17, p. 23. My emphasis.
27 See p. 122.
28 Of course its purpose might be to persuade a sceptical audience of the merits of analytic propositions.
29 This is explicit in, for example, his ‘The Value of Analytic Propositions’, and his *Frege: Philosophy of Mathematics* (1991b), p. 42. J. Tappenden endorses this way of looking at the
logical notions can yield epistemically valuable truths: viz. *extend knowledge* in the robust sense. The second challenge is taken to be implicit in Frege's overall response to the general question of how deduction can yield new truths, and concerns the validity of deductive reasoning. The validity of an argument is recognised by the common structures between the premises and conclusion. To account for the validity of deductive reasoning, we need to say something to the effect that knowledge of the premises, in particular knowledge of their logical form, could carry with it knowledge of the conclusion. But while its validity, if any, can be accounted for along these lines, it would not explain how the unexpected might arise in the course of the deductive proof.\(^{31}\)

The problem of the justification of deduction has of course a counterpart at the level of conceptual analysis—the paradox of analysis—and in his recent book, Michael Beaney claims that Frege is concerned to meet the problems that analysis presents. The paradox of analysis can be formulated as a challenge to say how 'fruitful definitions', which Beaney interprets (in part) as the analysis of concepts already grasped,\(^{32}\) can be both correct and of 'epistemic value': viz. extend our knowledge in the inflated sense.\(^{33}\) The analysans will be a correct analysis of the analysandum just in case they have the same content, but will be informative just

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\(^{30}\) This is not to say that knowledge is closed under entailment—that if I know P, then I know the truth of all the propositions entailed by P. It is simply that the warrant for our belief in the truth of the conclusion is carried both by the warrant for our belief in the truth of the premises, and by our warrant for our belief about the rules of inference.

\(^{31}\) Our concern here is with the first challenge, rather than with the question how deduction can be valid and at the same time be epistemically valuable.

\(^{32}\) As implied above, definitions can be fruitful whether analytic or stipulative. Cf. Dummett, 1981b, p. 339. Dummett's reason for thinking this differs from mine. For myself, the essence of a fruitful definition resides in its quantifier-variable structure (cf. Tappenden, 1995b, p. 456), particularly regarding scope differences. For Dummett, it seems to reside in a three-way connection between the *primacy of judgement* thesis, decomposition, and the presence of quantifier-variable structure, where all three features contribute to an account of conceptual discovery or acquisition. See below. Frege himself explicitly speaks of defining concepts that have already proved fruitful in science. Thus, as Frege uses the term 'fruitful definitions' he has analytic definitions specifically in mind.

\(^{33}\) Beaney, 1996, p. 127, p. 139. Beaney argues that a central purpose of Frege's work was to show how logic can be of epistemological value, that is, extend our knowledge. Op. cit. p. 123; also cf. pp. 118-19. As will emerge my only reservation about this claim resides in how 'extension of
in case the content is different. But if the analysis seems informative, then it should likewise seem incorrect; and if it is correct, then it should be uninformative. So put, the paradox seems to threaten the very possibility of the fruitfullness of conceptual analysis, and threatens to render analytic propositions all but banal.

Assuming these challenges to be Frege’s, the Dummettian interpretation of his resolution rests on emphasizing the combination of two doctrines:

(i) The primacy of judgements over their parts;
(ii) The multi-analysability of thoughts.

The primacy thesis (i) is in opposition to epistemological atomism: the traditional view that we first come by concepts or particulars and only subsequently form judgeable contents, or thoughts. According to Frege, this gets matters the wrong way round. ‘... I do not begin with concepts and put them together to form a thought ... ; I come by the parts of the thought by analysing the thought.’ Concerning (ii), the thoughts we grasp can be analysed in various ways: ‘I do not believe that for every judgeable content there is only one way in which it can be decomposed.’ ‘... [A] thought can be analysed in different ways.’

On the Dummettian view, Fregean analytic propositions—and thus deductive proof—are valuable in the way that the Kantian variety is not, because Frege’s more sophisticated method of definitions involves decomposing a thought in a way that yields new concepts. Even though the thought to be decomposed

knowledge’ should be understood.

34 By ‘Dummettian’ I shall mean anyone who subscribes to the solution that Dummett presents below to the challenges just described. As subsequent footnotes will show, the interpretation to be adumbrated can be found in much of Dummett’s work on Frege. The essentials of the view can be found also in Beane, 1996; E. Picardi, ‘Note on Dummett and Frege on Sense-Identity’, European Journal of Philosophy (1993). J. Tappenden, 1995b. While Tappenden endorses the Dummettian view, ibid. p. 432, there are some respects where there is an affinity with my view (see below).


36 ‘Boole’s Rechnende Logik und die Begriffsschrift’ (1880-81), PW.

37 See below for an elaboration.


39 ‘Frege to Anton Marty’ (1882), p. 101, BW.

40 ‘Uber Begriff und Gegenstand’ (1892), CP.
might be fully grasped, the apprehension of that thought does not require the grasp of the new concept arrived at by decomposing it—full-blooded decomposition. Such a concept is not in general a constituent of that proposition. For we do not have to have grasped that concept in order to grasp the proposition from which it was extracted.41

'Frege's 'fruitful definitions' start from a given proposition and yield not the concepts originally thought in it but new concepts.'42 To this extent '... we are engaged in a creative activity, splitting up the [thought] in a new way.'43

The new concept, extracted from a proposition already grasped, gives rise to a new proposition, which may prove to be analytic a priori, so that the attainment of this new conceptual information can represent an extension of knowledge."44

So for example, to grasp the content of 'Cato killed Cato' we do not, says Dummett, have to have the concept of suicide. To grasp the Sinn of 'Cato committed suicide', on the other hand, we do have to have the concept of suicide: we have to grasp its definition. At the very least one must know tacitly that 'Cato killed Cato if and only if Cato committed suicide'.45 In order to understand 'Cato killed Cato' we need only know the content of the name 'Cato' and what it is for person x to kill person y. We arrive at a concept by noticing other sentences with a similar pattern. In the case of the concept of suicide, we notice that both occurrences of 'Cato' can be replaced by other names, after which we can lay down the definition that 'a committed suicide' is to be equivalent to 'a killed a'.

Similarly if we regard '13' as replaceable in '13 >1 (V n) (n divides 13 → n = 13 ∨ n = 1)' we arrive at the concept represented by 'ξ >1 (V n) (n divides ξ → n = ξ ∨ n = 1)', and thereafter stipulate a definitional abbreviation 'is prime'. Thus we can arrive at a new thought, e.g., 5 is prime.46 Again we do not, says Dummett, have to have the concept of primality to grasp the thought expressed by

41 For the constituent-component distinction, and the thesis that the thought composed is not the same as the thought that results from the decomposition, cf. e.g. 1981b, p. 340.
42 Beaney, 1996, p. 129.
45 Here I assume that Frege rules out namesakes.
the sentence from which the concept expression (the definiens) is extracted. We need understand only the simple constituents that make up the semantic structure of the sentence, viz. what analysis reveals as the structure of the thought: *i.e.* ‘>’, ‘divides’, ‘=’, the two connectives, ‘and’ and ‘if’, and the universal quantifier. For the Dummmettian, it is this idea that lies behind the *pivotal extracts* quoted at the outset. In particular, it is what is meant by

> 'more fruitful determination of concepts draw boundary lines that were not previously given at all. What we shall be able to infer from them cannot be predicted in advance; we are not in this case simply taking out of the chest what we had put into it.'

Given the extraction of the new concepts in the course of the deductive process, new propositions can be formed. As this points to how deduction yields unexpected results, so the validity of the reasoning is secured, in part at least, given that the new concepts were already present in the premises.

Analogously, M. Beaney claims that decomposition, so construed, forms part of an initial response to the paradox of analysis mentioned above. The challenge at the level of conceptual analysis was to say how definitions can be both correct and of epistemic value. To the extent that it is the same content that gets split up, the analysis is legitimate, and the result is an analytic truth. And to the extent that a new concept is yielded, the process is fruitful. In light of this, Beaney too reads line (3) of the *pivotal extracts*—'what we shall be able to infer from [the fruitful definition] cannot be inspected in advance'—as alluding to

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48 *Grundlagen* §88. My emphasis.
49 Dummettian model of fruitful-definitions, applied to deduction, would work in something like this way:

1. Either Jupiter is larger than Neptune and Neptune is larger than Mars, or Mars is larger than Neptune and Neptune is larger than Jupiter.
2. \((\forall x)(\forall y)(\forall z) (y \text{ is larger than } x \land x \text{ is larger than } z \lor z \text{ is larger than } x \land x \text{ is larger than } y)\).
3. \((\forall x)(\forall y)(\forall z) (x \text{ is intermediate in size between } y \text{ and } z) \leftrightarrow (\forall x)(\forall y)(\forall z) (y \text{ is larger than } x \land x \text{ is larger than } z \lor z \text{ is larger than } x \land x \text{ is larger than } y)\).
4. \((\exists x) (x \text{ is intermediate in size between Jupiter and Mars})\).

50 Beaney, 1996, p. 139.
something conceptually novel that can be extracted from a proposition containing
the analysandum. Thus, *apropos Grundlagen*-definitions, Beaney’s Frege
envisioned the analysans as conceptually distinct from the analysandum, and hence
took them to be informative identity statements, as well as analytic.

This response to the paradox of analysis was, Beaney argues, further
clarified in the wake of Husserl’s criticism that Frege’s definitions do not preserve
the sameness of content.\(^{51}\) In particular, Frege clarified what he had meant all
along by ‘sameness of content’, and that this latter notion figured in his initial
response to the paradox. Now according to Husserl, evidence for the claim that
the definiens and definiendum do not capture the same content is that they do not
reflect the same knowledge involved in each case. Beaney claims that Husserl’s
point prompted Frege to the following considerations. If ‘sameness of content’
means that the same conceptual resources are involved—‘epistemic
equivalence’—then it cannot be legitimate to speak of decomposition in terms of
splitting up the same content in new ways.\(^{52}\) If splitting up the same content
yielded a new concept, then we would not have epistemically equivalent contents.
But if we had epistemically equivalent contents, then the result of the
decomposition could not be fruitful—could not extend our knowledge.\(^{53}\) Again, if
the results of decomposition are fruitful, in the above ampliative way, then the
contents cannot be epistemically equivalent. Yet unless we can talk of ‘splitting
up the [same] content in a new way ... the whole analytic basis of Frege’s project
collapses’.\(^{54}\) Frege’s response, says Beaney, was to deploy the distinction between
*Sinn* and *Bedeutung*. Definitions are correct insofar as the definiens and
definiendum have the same *Bedeutung*; and they are of epistemic value insofar as
the definiens and definiendum differ in *Sinn*.\(^{55}\)

We have, in chapter three, already argued that Frege’s response to Husserl
should not be read as conceding that the definiens and definiendum differ in *Sinn*.

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\(^{51}\) Op. cit. p. 141. See also chapter three above.
\(^{53}\) Not at least in the inflated sense. The *minimalist* sense is simply overlooked by the Dummettian.
There is another reason why we should not do so. Indeed Beaney himself offers it as a criticism of Frege. The problem is that, unless definiens and definiendum have the same Sinn, we cannot discriminate between identity statements that are suitable definitions and those identity statements that are not. To take Beaney's example:\textsuperscript{56}

1. The direction of line a is the extension of the concept parallel to line a.
2. The direction of line a is the extension of the concept parallel to line b

Let it be that line a and line b are parallel. Then identity statements (1) and (2) have the same Bedeutung: the extensions referred to on the right-hand side are the same. Yet whereas (1) is an analytic truth, (2) is synthetic.

Finally, suppose that Frege had indeed responded to the paradox of analysis initially as just described. That is, the analysis is legitimate and analytic if the 'same content' is split up, and informative if new concepts are yielded. That would show that he was concerned with the legitimacy and analyticity of the analysis. In that case, it is unlikely that Frege would have deployed the Sinn-Bedeutung distinction as a refinement of the initial response in the way that Beaney claims. For it is evident that to do so would undermine what originally he was trying to secure: the analyticity of his definitions. I'll return to Beaney's interpretation below.

Let us take stock. We have seen it argued that the early connection between fruitful definition and extending knowledge involved the claim that certain informative statements are definitional transforms. The repercussions of this claim, if true, are twofold. Either Frege did not seek fully to capture the content of our ordinary arithmetical terms. Or, Frege took the content of our grasp prior to the logicist reduction to be different from the content of our grasp subsequent to it. As against this, I offered my minimalist view of the pivotal connection. If my latter construal can be defended against the Dummettian rival, then the threat to my strong hermeneutic claim, posed by full-blooded decomposition, would be considerably weakened. So let us turn now to some criticisms of Dummettian decomposition.

\textsuperscript{56} Op. cit. p. 141.
§4.3.1. Full-Blooded Decomposition: Some Doubts. Dummett has claimed that to gainsay his reading of decomposition would leave fruitfulness of definitions and deduction unexplained, thus making the principle of decomposition pointless thereby.\(^{57}\)

1. ‘... [Frege’s] account of the fruitfulness of deductive reasoning is clear. *It depends on* the fact that we can impose a pattern on a complex sentence; we can therefore notice relations between sentences and between the thoughts which they express *of which we had in no way to be aware in grasping those thoughts*.\(^{58}\)

2. ‘*The whole point* of regarding the [decomposition] of the original sentence as a means of attaining the concept was that a grasp of the content of that sentence did not require us to see it as split up in that way, *nor, therefore, to have the concept of a prime number*’.\(^{59}\)

3. ‘What made [decomposition] a process of concept-formation was the fact that a grasp of the concept revealed by [decomposition] was not integral to an apprehension of the content being analysed.’\(^{60}\)

4. ‘... on this fact *depends* the conviction that definitions can and should be fruitful which Frege voices in the *Grundlagen*.\(^{61}\)

To briefly summarise the Dummettian view: the conclusions we draw from a deductive proof extend our knowledge, since new concepts will have been discovered and extracted from thoughts,\(^{62}\) even though those thoughts may already have been fully grasped—*full-blooded* decomposition. The above four passages are intended to capture the essence of what explains this. To that extent they are meant to be equivalent.

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\(^{57}\) Dummett actually uses the term 'analysis' in some of the passage quoted, which I have changed for 'decomposition'. This is the terminology he uses in 1981b, which I retain in order not to confuse the reader.

\(^{58}\) All emphasis in these four quotations is mine. Cf. ‘Alternative Analyses’, 1981b, p. 291; cf. also ‘Synonymy’, 1981b, p. 339: ‘But the whole point of Frege’s account of the formulation of incomplete expressions is that such an expression may occur in a sentence for the understanding of which we do not need to discern its presence; we arrive at it by decomposition of the sentence’.


\(^{60}\) Op. cit. p. 297. By ‘content’ it is clear that Dummett means a Fregean thought.


\(^{62}\) This is fairly explicit in 'Alternative Analysis', 1981b, pp. 290-91.
Yet citation (1) can be construed as saying something quite different from what the other three passages say. Central to what is explicitly stated in 2-4 is that a new concept, one not constitutive of our (full) grasp of some proposition, is nevertheless a feature of that proposition and can be extracted from it by decomposition. But (1) makes no explicit reference to grasping new concepts. In fact, one could read (1) as saying that we notice relations between thoughts for the first time; awareness of which is not, perhaps, necessary to our grasp of those thoughts.\(^\text{63}\) That is, we could grant that the fruitfulness of deduction ‘depends’ on the fact that, by imposing patterns in the way that decomposition permits, we can notice relations between thoughts for the first time. But that does not ‘depend’ on the idea of extracting new concepts, as the Dumettian understands it.\(^\text{64}\) Thus there is a slide in his use of ‘depends’ as it occurs in (1) and (4).\(^\text{65}\) Moreover, the presence of the slide would indicate that the value of deduction, and the notion of fruitful definitions, might be independent of the inflated sense of extracting ‘new’ concepts.

To pursue this point, notice that Dummett appears to conflate further ideas surrounding full-blooded decomposition. On the one hand, the leading idea of passages 1-4 seems to be:

\[
\text{(A) the extraction of a concept not hitherto grasped prior to the } \\
\text{decomposition of the thought in which it is contained.}
\]

For Dummett says ‘We can come by a new (complex) predicate, one of which we have previously had no conception, and with it, its [\textit{Sinn}], by decomposition of a sentence whose [\textit{Sinn}] had already been grasped’.\(^\text{66}\)

On the other hand, the central idea of passages 1-4 might be:

\(^{63}\) But as said, Dummett intends more than this.  
\(^{64}\) Op. cit. §17.  
\(^{65}\) The use of ‘depend’ is also implicit in the other two passages, (2 and 3) and so the slide is implicated there as well. It cannot be said, on Dummett’s behalf, that the first quote represents a change of mind for Dummett. Passages like (2-4) can be found in Dummett’s work written either before or after (1).  
\(^{66}\) My emphasis.
(B) here a new concept is one that, while not part of our grasp of the propositions from which it was extracted, was grasped prior to the decomposition.

For elsewhere Dummett says '[b]y doing this [decomposing thoughts], we arrive, perhaps for the first time, at the functional expression, and likewise, at the concept.' Other remarks confirm that Dummett has both notions in mind: 'It does not matter a great deal for the fruitfulness of a definition whether the notion being made precise by it is being introduced for the first time or has long been hazily familiar ... .' For in either case 'the advance of knowledge proceeds by forming new concepts by way of decomposition ...'. So for Dummett, whether or not a definition is explicative is not crucial to whether it is fruitful. What, for Dummett, is essential to the fruitfulness of a definition is its connection with decomposition.

But how can we form new concepts and thereby advance our knowledge, if the concept, acquired by decomposition, was already familiar to the thinker? Dummett does not say what 'new concept' or 'advance of knowledge' means in this case. Of course, a logically impeccable proof of some propositions $S$ can be said to advance our knowledge where $S$ is thereby established for the first time. But we should not wish to say that the proof was an advance of knowledge in the sense of being a conceptual advance. Prima facie, (B) is incompatible with Dummett's inflated sense of extending knowledge. While we can account for what 'new concept' means in terms of (B), in doing so we lose Dummett's intended connection with 'advance of knowledge'. All we are left with is our minimalist sense of extending knowledge. Yet to concede to my minimalist reading in connection with (B), would weaken the claim that full-blooded decomposition applies to (A).

To try to forge the connection between the inflated sense of extending knowledge and decomposition, let us modify (B) to mean:

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(C) concepts formed from the concatenation of propositional fragments, some of which may be already 'hazily familiar'.

A close look at (A) and (C) might suggest that, even from a nonexegetical point of view, the Dummettian reading is not altogether plausible. The first case, (A), is where a 'new' concept is extracted from a single proposition. Dummett is surely right to note in passage (2) that we do not have to regard (say) the sentence '13 >1 ∧ (∀n) (n divides 13 → n = 13 v n = 1)' as split up in the way indicated (taking '13' as replaceable) in order to grasp the content or thought expressed. And so he is probably right also to say that we do not already have to have the concept of primality in order to understand that sentence. For the understanding of a sentence does seem to depend on some grasp of how it is constructed, or at least a grasp of its structure. Doubtless it is also true, apropos passage (2), that we have to have an understanding of the simple constituents indicated above. And it is probably true too that, as indicated in citation (1), we do not need (for grasping thoughts) to be 'aware' both of the ways in which a thought can be decomposed and of the ways in which the thought relates to other thoughts in accordance with these different decompositions. Still, why infer that, for Frege, we do not have to have the concept in order either to extract it or to understand the judgeable content from which the extraction is to take place? No more need that claim be true, than the claim that we cannot grasp a concept without knowing how to define it. At any rate, are we really to imagine that, for Frege, our grasp of the thought expressed by '13 >1 ∧ (∀n) (n divides 13 → n = 13 v n = 1)' did not involve the concept of primality? Are we really to deny that, for Frege, a grasp of the concept of suicide or the concept of killing oneself is implicated in our understanding of the proposition Cato killed Cato? Surely not. More plausible is that, for Frege, our grasp of Cato killed Cato involves at least a tacit grasp of the concept of suicide. For the grasp of killing surely involves some understanding of the concept anything killing anything. Similarly with our grasp of the thought expressed by '13 >1 ∧ (∀n) (n divides 13 → n = 13 v n = 1)'; the concept of

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69 To use Frege's metaphor. Since Dummett only consider cases of extracting a concept from a
primality is at least tacitly grasped. Of course these are just examples (Dummett’s) of decomposition. But it is incumbent on the Dummettian to offer an instance where the inflated sense of extending knowledge, engendered by full-blooded decomposition, is exemplified.

What, now, of the second case: (C)? That is, ‘new concepts’ are arrived at from the concatenation of propositional fragments, some of which fragments may be ‘hazily familiar’. Here it might appear that attempts to forge a connection between the inflated sense of knowledge extension and decomposition fare better. But what exactly would (C) involve? Frege would not permit the formation of a new complex predicate simply by the concatenation of different predicative material. To do so would violate the primacy of judgement thesis. Instead he would first combine the thoughts that contain the predicative fragments. The ‘new’complex predicate would then be extracted from the newly conjoined complex thought. But what would motivate the concatenation of the formulae? Clearly, the apprehension that it would be fruitful to do so. Yet since the fruitfulness in question is the formation of a ‘new concept’, such a concept is presupposed in the very point of conjoining the formulae. Suppose, however, that this presupposition is absent. Then we are back where we started. We have a case of decomposition of a single thought for the attainment of a ‘new concept’, and no reason as yet to assume that our grasp of the ‘new concept’ is not implicated in our grasp of that very thought.

In case one is still not convinced by my response, a further, simple, observation should suffice to make plain that there is something amiss with the Dummettian notion of decomposition. First, I do not rule out the abstract possibility of Dummett’s account. I simply do not think that it is necessary to account for what Frege had in mind in the pivotal extracts. A general way of

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70 To be sure, there are several passages in which the term ‘new concept’ occurs, which I shall explain in due course. One reason for momentarily suppressing it is to highlight the point that, even if Frege was tempted by something like the Dummettian view, he need not have been. For Frege’s explanatory needs could be served by my more modest, minimalist proposal about how the pivotal connection, between extending knowledge and fruitful definitions, could be understood.
seeing this is to note that the history of mathematics tells a quite different story from the one envisioned by the Dummettian. This is true whether the core of the Dummettian account is (A) or (C). What the history of mathematics shows is that a concept or technique is first discovered and that only afterwards—'often after much intellectual labour'—are proofs of them finally found. A challenge to the Dummettian is for them to cite an instance where decomposition, as they envision it, has actually occurred. It will be found that there is none. Certainly there is little in the pivotal extracts to suggest that Frege envisioned any historically unprecedented mode of conceptual discovery. Frege was interested in ascertaining the nature of a body of truths already known: that of arithmetic. Frege's references to the concepts of *extension of knowledge*, and fruitful definitions, were made with the case of arithmetical truths in mind. We should bear this in mind, if we are not to lose sight of what Frege meant. Doing so, we see that whatever epistemic value was to be found in his logicist proof, the body of analytic truths, must be independent of the kind of conceptual novelty envisaged in either (A) or (C). For Frege believed that a deduction of even the most obvious truths (those of arithmetic) was epistemically worthwhile. By construing the significance of decomposition in terms of the inflated sense of extending knowledge, the Dummettian is in danger of missing this simple point. Again, I refer the reader to my minimalist construal of 'extension of knowledge':

*Proposition S is an extension of our knowledge if it is inferred from a set of premises, included among which are 'fruitful definitions', and such that while S is not a member of that set, the elements of S are elements of the propositions of that set.*

§4.3.2. **Full-blooded Decomposition: A Diagnosis.** Contrary to the Dummettian reading, the sole point of decomposition is, in my view, to exhibit the relations exemplified by valid deductive reasoning. There are other passages, however, that might be taken to gainsay this point, and in turn to lend support to the Dummettian view.

71 Frege's famous refrain.
'I start out from judgements and their contents, and not from concepts ... I only allow the formation of concepts to proceed from judgements. If, that is, you imagine the 2 in the content of possible judgement

\[ 2^4 = 16 \]

to be replaceable by something else, by (-2) or by 3 say, which may be indicated by putting an x in place of the 2:

\[ x^4 = 16, \]

the content of possible judgement is thus split into a constant and variable part. The former, regarded in its own right but holding a place open for the latter, gives the concept '4th root of 16'.

We may now express

\[ 2^4 = 16 \]

by the sentences '2 is a fourth root of 16' or 'the individual 2 falls under the concept "4th root of 16" or belongs to the class of 4th roots of 16'. But we may also just as well say '4 is a logarithm of 16 to the base 2'. Here the 4 is being treated as replaceable and so we get the concept 'logarithm of 16 to the base 2':

\[ 2^4 = 16. \]

The x indicates here the place to be occupied by the sign for the individual falling under the concept. We may now also regard the 16 in \( 2^4 = 16 \) as replaceable in its turn, which we may represent, say, by \( x^4 = y \). In this way we arrive at the concept of a relation, namely the relation of a number to its 4th power.\(^72\)

Indeed the Dumettian would claim that this citation vindicates their case. It is, to be sure, natural to take this type of decomposition, which figures large in Frege's logicist proof, as a means of grasping 'new concepts'. The passage states that we start from judgeable contents and subsequently arrive at concepts from decomposition. But it is noteworthy that Frege does not explicitly mention 'new concepts' here. Nor is there any reason as yet to think that 'arriving at concepts' is synonymous with 'new concepts'. Besides, even if they were synonymous, 'new concepts' permits of too weak a construal to vindicate the inflated sense of decomposition.

\(^72\) 'Boole's Logical Calculus and the Concept-Script' (1880-81), pp. 16-17, *PW*. 

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To illustrate, the judgeable content expressed by \(2^4 = 16\), can be decomposed in such a way as to say at least one of three different things. To regard \(4\) as replaceable would be to regard the judgeable content as saying: (1) the concept \(\text{logarithm of 16 to the base 2}\) is true of the object 4. Thus I could infer \((\exists x) (x \text{ is the logarithm of 16 to the base 2})\). Relative to (1), I could say two 'new' things, as well as draw two 'new' inferences, given the original judgeable. If I decompose \(2^4 = 16\) so that '2' and '16' are replaceable, we would have the statement: (2) '2 and 16 stand in the relation of a number to its fourth power'. I could then infer \(\left(\exists x \right) \left(\exists y \right) (x^4 = y)\). Or, regarding the first level concept expression \(x^4 = 16\) as extractable from \(2^4 = 16\), we could regard the latter as saying: (3) '2 is the 4th root of 16', and thus infer \(\left(\exists x \right) (x^4 = 16)\). Relative to each other, the three statements (1)-(3) all involve the predication of something 'new'. That is, what is explicitly said in each case is different. Yet relative to their common thought, \(2^4 = 16\), nothing is new, since that judgable content or thought is about all three things.

Let us turn to the alleged connection between the primacy of judgement thesis and the notion of fruitful definitions, as understood by Dummett. The connection is essential to their account. But given my minimalist reading of 'extension of knowledge', it is of little importance. Indeed the root of the Dummettian misunderstanding seems to lie in having overstated the importance of the primacy of judgement in explicating the pivotal connection.\(^3\) Common to all Frege's examples of 'fruitful definitions' is the presence of structures that involve quantifiers of differing scope. So Frege's definitions are fruitful because by their means we can reach proofs that we could not reach by other means. By contrast, the key to the Dummettian notion of fruitful definition is full-blooded decomposition: from a fully grasped proposition we can subsequently decompose it to extract a 'new concept'. As seen, the primacy of judgement thesis, which underlies decomposition, is the counterpart of epistemological atomism. Both these theses have an epistemological and ontological component. First, in contrast

\(^3\)This reliance is explicit in Dummett and Beaney. Of the latter see esp. 1996, p. 130, p. 141, p.
to atomism, the primacy thesis holds that concepts and objects are grasped only in the context of thoughts. Secondly, again in contrast to atomism, it claims that concepts, because of their unsaturatedness, exist only in the context of being saturated. That is, concepts exist only in combination with complete entities. But clearly these differences are compatible with the view that whatever is arrived at by decomposition is grasped simultaneously with the grasp of the thought decomposed—it need not be subsequent to it, contrary to what full-blooded decomposition requires.

An epistemological atomist holds that concepts and objects are first grasped, and that only subsequently are they combined in judgement to yield a thought. But clearly, an epistemological atomist could still champion the merits of decomposition and fruitful definitions, at least as I construe these latter two notions. Again, the principal point of decomposition is to exhibit the relations exemplified by valid deductive reasoning. Fruitful definitions make possible the explicit representation of certain inferences that cannot be represented by the Boolean variety. Whether those fruitful definitions are conceived of as stipulative or not, their merits are available to the atomist.74

§4.4. The Value of Deduction and Fregean Analytic Propositions. One of the central questions of this chapter is whether Frege, at any stage, saw the Grundlagen-definitions, as informative identity statements.75 If the answer is in the affirmative, then clearly it must be because he took our grasp of the definiendum to be different in content from our grasp of the definiens. Then contrary to a central theme of our overall thesis, Frege did not regard his logicist project as involving a full translation of our ordinary arithmetical language. We

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74 Dummett and Beaney claim additional support for their understanding of what 'fruitful definition means. This is Frege's remark that splitting up a judgeable content in various ways 'has nothing to do with the conceptual content, but is solely a matter of our way of regarding it'. Begriffsschrift, §9, CN. 'Frege and the Paradox of Analysis', 1991a, pp. 25-26; 'Alternative Analysis', 1981b, p. 282; 'The Value of Analytic Propositions', 1991b, p. 41; cf. also Beaney, 1996, p. 130. I believe that the latter writers have misconstrued what the Begriffsschrift reading means. See Baker and Hacker's Frege: Logical Excavations (1984), p. 159, pp. 310-11. Cf. H. Sluga 1980, p. 86.
have until now examined one kind of consideration in favour of the view that the
definiens and definiendum do differ in content. This consideration centred on
Dummettian decomposition, which turned on (i) construing the pivotal extracts in
a particular way, and (ii) on connecting the primacy of judgement thesis with full-
blooded decomposition. Even if, however, the cogency of my criticisms regarding
the Dummettian is granted, the strong hermeneutic view might still be rejected.
For after all, it does seem, on the face of it, that the Grundlagen-definitions do
differ in content; or that what we grasp is different. Let us call this the
‘informative view’.

I want to assess the informative view by addressing Dummett’s remark,
noted earlier, about the consequences of rejecting his view. This was that unless
the inflated sense of extension of knowledge is subscribed to, the epistemic value
of deductive proof—and of analytic truths—could not be accounted for. Although
I reject this, my alternative position, now to be described, might seem to play into
the hands of the informative view. My alternative position relocates the epistemic
value of analytic propositions and deduction as having two sources. One of these
sources is the presence in Frege’s early and later work of the notion of modes of
presentation or determination. Of course it is just this idea that Frege takes to
explain how true identity statements, or other sentences whose Bedeutung are the
same, can be informative. I shall argue, however, that statements of identities that
arise from definitional identities are an exception.

Firstly, it is clear that either the notion of modes of determination or modes
of presentation is present in Frege’s early work. Clear too is what role it plays.
It addresses the question of how deductive reasoning—and thus analytic truths—
can be epistemically valuable. Frege broaches this latter issue early in
Grundlagen:

75 Or biconditionals, where the two sentences involve a definition of a predicate.
76 Whereas modes of presentation are grounded in Sinn, ‘modes of determination’ are grounded on
metalinguistic distinctions. For our purposes, it does not matter which of these two notions Frege
was working with at this time. If he was working with Sinn, then no account is given until 1891.
Incidentally, with respect to post 1891, it is useful to think of modes of presentation, now grounded
in Sinn, as applying to those cases where the Sinn of a term has a denotation, whereas modes of
determination apply to terms with no denotation as well.
'But this view ... has its difficulties. Can the great tree of the science of number as we know it, towering, spreading, and still continually growing, have its roots in bare identities? And how can the empty forms of logic come to disgorge so rich a content?'

The first question might be put thus. It is widely believed that, as analytic judgements are uninformative, so pure logic is barren and sterile. By contrast, it is widely believed that both the content of arithmetic is rich and fruitful, and that judgements involving them, can support valuable extensions to our knowledge. How then, Frege is asking, if the truths of arithmetic are so rich in content, while those of logic are apparently so bereft, can these truths be the same?

The second question is a challenge to show how the empty forms of logic can disgorge the rich content of arithmetic. No affirmative answer to the first question can be given without showing how the content of this great tree of knowledge can be brought forth out of itself (aus sich heraus suchen inhalt gewinnen), how it can be disgorged from the empty forms of logic. How can we do that if logic apparently has not the content that the truths of mathematics are commonly believed to have? We have already seen the answer to this second question. It lies in combining the Boolean operators with quantifier-variable structure. It is to show that logic is far richer than is commonly believed.

The first question was how arithmetical truths can be logical truths when, on the face of it, logic is barren and sterile, bereft of content, whereas arithmetic truths are rich in content. The question alludes to the notion of mode of

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77 Grundlagen §16, p. 22; Thiel, §16, pp. 29-30. My emphasis.
78 Tappenden, 1995b, p. 432, provides a stark example of how the above quote can be misconstrued. We do not need the Dummettian interpretation of decomposition to answer the two questions just raised.
79 Note that, in Grundlagen §16, the concept of extension of knowledge, if it is there at all, relates to a different usage than in the pivotal extract. In the pivotal extract, we have the minimalist notion. In Grundlagen §16, by contrast, Frege is asking how, since logic is considered barren and sterile, there could be a body of propositions, a body of knowledge, like that of arithmetic, if it consisted in mere logical truths. That is, how, if arithmetic is just logic, is knowledge of an arithmetical concept to be had in the first place? For to logic belongs a 'legend of sterility', a barren and sterile terrain (op. cit. p. 22; Thiel, §16, pp. 29-30), whereas to arithmetic belongs, one might say, a legend of fecundity, a landscape of rich conceptual harvest. At any rate, these two issues, or ways of understanding extension of knowledge, are clearly distinct from the Dummettian's full-blooded understanding.
That it does so appears to be confirmed in the following passage, which occurs later in *Grundlagen*

> ‘If ... we were to adopt this way out, we should be presupposing that an object can only be given in one single way ... All identities would amount simply to this, that whatever is given in the same way is to be reckoned as the same ... This, however, is a principle so *obvious and unfruitful* as not to be worth stating ... Why is it ... that we are able to make use of identities with such significant results in such diverse fields? Surely it is ... because we are able to recognise something as the same again even although it is given in a different way.’

In a letter to Peano, Frege reiterates the concern aired in the above citation by saying that without modes of presentation, we would be unable to explain how mathematics could consist in other than bare and trivial identities.

What, then, of our initial, but more pressing, question? Is there a genuine conflict between the presence of modes of determination or presentation in Frege’s logicist system, and the strong hermeneutic view *vis-à-vis* the alleged informativeness of definitional identities? To see that there is not, we need to introduce the other of the two sources of the epistemic value of deduction and analytic propositions mentioned above. This further source lies with the clarificatory status of analytic propositions that Kant himself extolled, but whose possibilities Frege saw anew. The clarificatory status of these propositions turns

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80 Another (crucial) sense in which this is so is of course that logic is meant to comprise, and is able to yield to us, logical objects.

81 ... again, where the modes are grounded either metalinguistically or by *Sinn*.

82 §67, p. 78. My emphasis. The ‘way out’ in question concerns how one might solve the Julius Caesar question: i.e. whether we should stipulate that an object could be given in only the way stipulated in the definition.

83 I’m not saying, however, that this passage suggests that Frege believed in *Sinn* at this point in his career.


There are two ways in which *Sinn* complements the early fruitful-definition-extension of knowledge axis. Firstly, as argued in chapter 1, no primitive proper names are envisaged in Frege’s work. Thus given that all objects are the values of functions, we can understand modes of presentation in terms of Frege’s function-arguments model: that same object can be the value of different functions. Cf. Backer and Hacker, *Frege: Logical Excavations*, pp. 301-02. Secondly, not all such functions will involve (fruitful) definitions, since not all functions are definable. Thus given that not all modes of presentation involve fruitful definitions, some analytic truths that extend knowledge elude the account given in the pivotal extract.
on the idea, touched on in chapters two and three, of recovering full tacit grasp. Frege believed mathematics to be awash with instances in which, although we have full possession of arithmetical propositions (including all their constituents), we are not always clearly conscious or able to disclose all that we think in apprehending these propositions. The logicist proof provides a more perspicuous rendering of the propositions already in our ken. Fruitful definitions show, in another way, how analytic propositions and deduction can be epistemically valuable in the following sense:

1. It could provide a more thorough analysis of concepts, affording us an insight into (a) its complexity, and (b) into how the parts of the concept are organically connected.
2. These greater results of analysis bear still further fruit given their role in deductive proof, yielding to us an insight into (c) how the truths of propositions are connected; and (d) an insight into which truths depend on which other truths.

To have clarity of grasp of a concept is in part for the thinker to be able either to provide or recognise an analysis of it. That is, he must recognise or be able to lay bare its structure and to know its nature, where to know its nature is to grasp the bounds of its validity. If the concept in question is a logical simple (an indefinable), then the thinker must be able to recognise that it is, as well know its nature. To have distinctness of grasp is to have the above kind of grasp with respect to the environment—the ontological territory—in which the concept in question belongs. That is, the thinker must have clarity of grasp of those concepts with which the principal concept in question is most immediately linked. So for example, when I grasp the concept of cardinal number, the distinctness of that grasp is made apparent by exhibiting the clarity with which I apprehend such surrounding concepts as successor and the individual numbers—concepts without which possession of the principal concept would not be had.

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85 The metaphor is used more than once by Frege. See for example, Grundlagen §88, p. 100. As seen, Frege uses the term to express the view that his definitions capture the inner structure or essence of the concept yielded.
86 The extent of its generality or application.
87 It is unclear whether, in saying that Kant’s method of definition is one of the least fruitful, Frege ever meant that they were unfruitful. If he did so, and if he thought that, as a consequence, Kant
It should now be clear why I do not think that Frege intended *Grundlagen*-definitions to be informative. Let us distinguish between identity statements that are informative and those that are in another sense enlightening. The former depend on their failure to meet Frege’s individuation principle for the sameness of thought. This is that a thinker could without irrationality take conflicting propositional attitudes towards two thoughts. But this principle does not fail where *Grundlagen*-definitions are concerned: i.e. in which one thought contains the definiens, and another thought contains its definiendum. At least it does not do so if the *Sinne* of the two terms are grasped with the kind of perspicuity outlined above. Where clarity and distinctness is lacking, the principle of individuation does not apply.

This way of collapsing the informative view may meet with resistance given Grossman’s examples of definitions as informative identity statements. For it might be said that the above remarks apply to any one of the following three statements taken separately.

1. ‘0 = the number that belongs to the concept not identical to itself.’
2. ‘The number that belongs to the concept not identical to itself = the extension of the concept equinumerous to the concept not identical to itself.’

Took analytic propositions to be of little epistemic value, then he was mistaken. Conceptual analysis was, for Kant, no less than for Frege, by no means a conceptually worthless enterprise.

‘... [W]e recognize better, i.e., more distinctly and clearly and with greater awareness, what we already knew (Logik Blomberg, p. 131; cf. Critique of Pure Reason, A5-6/B9).’

‘If only we knew what we know ... we would be astonished by the treasures contained in our knowledge (Wiener Logik, p. 843).’

Here Kant makes clear that definitions can be fruitful because they facilitate the recovery of tacit knowledge. Furthermore, for Kant, analytic propositions express the fruits of this kind of conceptual analysis, making the role of analytic judgements clarificatory (*Erklärungsurteile*), rather than ampliative judgements (*Erweiterungsurteile*), where the latter *extend our knowledge* in our *minimalist* sense. This aspect of the epistemic value of analytic judgements is proportional to the fecundity of analysis—the structural richness of the concept recovered—and the use in proof to which the result of analysis is put. In the *pivotal extract*, Frege says that Kant and others had underestimated the value of analytic propositions *qua* their role as *Erklärungsurteile*. If part of the value of deduction lies in the attainment of a more perspicuous rendering of what we ordinarily grasp, then, unbeknownst perhaps to Frege, there was far more agreement between himself and Kant about the epistemic value of analytic truths.

88 ‘Two sentences A and B may stand in such a relation to one another that anyone who recognizes the content of A as true must also immediately recognize that of B as true, and conversely, where it is assumed that there is no difficulty in grasping the content of A and B.’ ‘Kürze Übersicht meiner logischen Lehren’ (1906), p. 197, *PW; NS*, p. 213.
3. '0 = the extension of the concept equinumerous to the concept not identical to itself.'

It is true of course that the substitutions we can make from (1)-(3) do seem to be paradigm cases of the connection between *Sinn-Bedeutung* and extending knowledge. Fortunately, the impression is mistaken. What we have on the right-hand side of (1) is a partial representation of the left-hand. Similarly with the left-hand side of (2). Only (3) provides a full representation. The essential difference between (1)-(2), and (3) is of course the absence of ‘number’ in (3). To get the full representation, Frege expects us to appeal to his definition of the concept of cardinal number: *extension of the concept equinumerous to the concept F*. He leaves it to the reader to make the definition of ‘0’ fully explicit by suitable substitutions involving his definition of the concept of number. The point of the partial representation, I take it, is to remind the reader that it is to a concept that a number attaches without at the same time being a property of second level.

Finally, if the points made in this section are right, then Beaney’s characterisation of the connection between decomposition and the paradox of analysis is misplaced. The paradox would arise, for Frege, in the way presented by Beaney, only given an assumption about the nature of definitions. This is that definitions are a certain kind of informative identity statement: namely analytic statements *par excellence* that *extend our knowledge* in the inflated sense. But this overlooks two principal points. First, by ‘extension of knowledge’, Frege may have had in mind my *minimalist* construal. Secondly, while Frege’s definitions are doubtless epistemically valuable, they enlighten us, they arguably do so without adding to our conceptual repertoire. Thus, if Frege had my version of analysis and decomposition in mind, then he would also have been unconcerned

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89 It is tempting at first to make play on Frege’s actual wording. Expressions he uses when announcing his definition of number differ from those used when announcing his definition of zero. In the former he says ‘Ich definire ...’, whereas in the latter case, he says ‘Ich erkläre ...’. So at first sight, Frege in the latter case might be simply explaining zero in terms of the kind of concept that the number belongs to. However, later in the same section he uses the term ‘Definition’: ‘Ich hätte zur Definition der 0 jeden andern Begriff nehmen können, unter den nichts fällt.’. Thiel, *Grundlagen* §74.
with Beaney’s reading of the paradox of analysis. Beaney’s criticisms, adumbrated above, would not therefore apply.

§4.5. Proving that Our Arithmetic Truths are ‘Analytic’: A Change of Mind? Let us turn, finally, to the second objection to the strong hermeneutic view mentioned at the outset. The objection in question is prompted by the observation that, after Grundlagen, Frege no longer speaks of the connection between fruitful definition and extending knowledge—the pivotal connection. Indeed, he appears to renounce it: hereafter the ‘critical passages’.

1. ‘No definition extends our knowledge. It is only a means of collecting a manifold content into a brief word or sign, thereby making it easier for us to handle. This and this alone is the use of definitions in mathematics.’
2. ‘... it is not possible to prove something by means of a definition alone that would be unprovable without it. When something looks like a definition really makes it possible to prove something that could not have been proved before, then it is no mere definition but must conceal something that must either be proved as a theorem or accepted as an axiom.’

In particular, Frege no longer speaks of analytic propositions vis-à-vis arithmetical propositions, and of how the former can extend our knowledge. As it is expounded in Grundlagen, this early pivotal connection is, arguably, inextricably linked with analytic propositions. Moreover, some of these latter propositions, those of arithmetic, would owe their name ostensibly to the fact that they involve explications of ordinary arithmetical terms as constituents. So on speaking of fruitfulness, Frege, it seems, had ‘analytic definitions’ in mind. As already seen, on one construal, a thinker could enlarge his knowledge by arriving at the explicans, even though he had already grasped the explicandum. Thus, if Frege renounced the early pivotal connection—between fruitful definitions and analytic propositions that allegedly extend our knowledge—then that might indicate that he had renounced analytic definitions as well. Conversely, if he had ceased searching for analytic definitions of our arithmetical terms, preferring a more

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90 ‘Foundations of Geometry’ (1903), CP, p. 274; cf. 1914, p. 208, PW.
91 1914, p. 208, PW.
constructive variety, that might explain the absence of the pivotal connection after Grundlagen.

Alternatively, suppose that analytic definitions were inessential to the early extension of knowledge axis, as expounded in Grundlagen, and that my minimalist construal of extension of knowledge is right. Then there would be good reason for thinking that the pivotal connection was retained. Prior to discovering the inconsistency in his system, Frege could still maintain the hypothesis that arithmetical truths were analytic: that they were derivable from logical laws and definitions, albeit non-explicative, involving only logical constituents. But, then, the hermeneutic hypothesis would be false.

On the other hand, it seems likely that the early pivotal connection, as expounded in Grundlagen, is inextricably linked to analytic definitions. For without the latter, a good deal of the explanatory value of the logicist reduction would be lacking. In chapter three, we explained Frege's use of stipulative definitions in terms of uncertainty regarding the correctness of his attempted explications. Frege's reservations about asserting the correctness of explicative definitions—preferring, as he does, to treat the yielded content as stipulative—would invoke similar reservations about explicitly referring again to the early pivotal connection. But as said, uncertainty need not mean a change of mind. Frege had put forward a hypothesis: that arithmetical truths were analytic. The criterion was to prove the former truths solely from the elements of logic, including definitions, which, if the system of proof was to have explanatory value, were to be 'analytic' of our ordinary arithmetical terms. At any rate, until the hypothesis was made good, there would be little point in reiterating the early pivotal connection.

If that explains Frege's reticence, what explains the critical passages? Grundlagen-definitions are fruitful, we saw, because they make possible a derivation of a proposition that would not be possible without it. But given the

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... and following the discovery, if Frege had given-up the claim that numbers are object, and instead taken them to be second-level concepts.

I shall elaborate on this claim in chapter six.
critical passages, it is, as E. Picardi notes, 'impossible to appreciate in which sense it can be said that analytical propositions enlarge our knowledge'. For a definition, in the critical passages, is simply a linguistic device. Their point is one purely of psychological convenience. We introduce a simple sign for a more complex one, making the execution of proof easier for us. To Dummett this means that, Frege in the critical passages is 'betraying earlier insights' since he had 'insisted on the fruitfulness of good definitions on the crucial role played by them in the advance of our mathematical knowledge'. As seen, Dummett's account of the pivotal connection depends on the claim that the thought yielded by decomposition is distinct from the thought decomposed. The critical passages mark a change because definitions are now said to consist in mere stipulation of sameness of Sinn and Bedeutung for the definiens and definiendum. Once the simple expression gets its meaning, the 'definition' is turned into an identity statement, which, because both sides of the identity express the same Sinn, does not extend our knowledge. Thus since the new simple sign adds nothing new, 'definitions' are not necessary for a system.

Note that the critical passages would imply what I have suggested all along. The predicate arrived at by decomposition is part of our grasp of the thought from which it was extracted. Moreover, as we saw earlier, there is, given my interpretation, neither a conflict between the 'new' and 'old' extension of knowledge axis, nor a change of mind regarding the 'old' axis. Not only do the early pivotal connection and the connection between Sinn and Bedeutung and extending knowledge coalesce. The underlying idea of Sinn, mode of presentation or mode of determination, is present in both Begriffsschrift and Grundlagen; it there supplements the earlier axis in accounting for the value of deductive reasoning. Again, the Dummettian overlooks the minimalist sense of extending knowledge.

94 'Frege on Definition and Logical Proof' (1988).
95 'Synonymy', 1981b, p. 338.
96 For this reason, Dummett criticises Frege for being unfaithful to his own earlier doctrines. But Frege never said that their Sinne were distinct.
But according to Tappenden, the pivotal connection does conflict with the later Sinn and Bedeutung-extension of knowledge axis; and this forced a 'reorganization' of the earlier notion of extending knowledge. It is this, apparently, that explains critical passages. The conflict is that, following the Sinn-Bedeutung distinction, extension of knowledge revolves around grasping different thoughts, whereas the earlier axis revolves around splitting up the same content in different ways. 'Inferring one sentence from another with the same content can extend knowledge in Frege's sense if the best proof leading from one to the other involves quantifier inferences.' Yet given the later Sinn-Bedeutung distinction, '[i]f the inference from one sentence to another extends knowledge, the sentences have different [Sinne].' That so, 'rather than 'fruitful concepts-definitions' supporting transitions that 'draw new boundaries' [yield new concepts] it is', says Tappenden, 'differences of 'knowledge value' [different Sinn] that establish which judgements extend knowledge'. Because of this, the later Sinn-Bedeutung-extension of knowledge axis leaves 'no room for a distinct 'drawing new boundaries story' as the key to analytic judgements extending knowledge'. As said, Tappenden endorses the Dummettian account. So looking back on Grundlagen-definitions from the perspective of the Sinn-Bedeutung distinction, Frege would regard the sentence, part of whose content is arrived at by decomposed, as informative relative to the sentence whose content was decomposed.

It is unclear, from what Tappenden says, that the later Sinn-Bedeutung-extension of knowledge axis does preclude the fruitful-definitions story as 'the key to analytic judgements extending knowledge'. What makes a judgement that enlarges knowledge analytic, Tappenden suggests, is that the inference 'A' to 'B' involves the same content. No: it depends on the nature of the premises from which 'B' is best derived. There is good reason why extension of knowledge in Grundlagen does not depend on inference from one sentence to another sentence

with the same content. A genuine inference, for Frege, involves the movement from one truth to another truth, and thus a movement from one proposition to a different proposition. Yet, if ‘A’ and ‘B’ have the same content, that would undermine the inferential move that helps to underpin the early extension of knowledge story. Nor is there a conflict given the Sinn-Bedeutung-extension of knowledge axis. ‘A’ and ‘B’ can represent an extension of knowledge if, for example, their predicates differ in Sinn. Again, whether they are analytic depends on the status of the premises from which they can be derived. Mutatis mutandis sentences of the form ‘a = b’.

What, then, is the significance of the critical passages? Their surrounding context would suggest that Frege is emphasising a particular conception of definitions. On the one hand, they have an epistemological importance. To this category belong the properties of ‘fruitfulness’. In Begriffsschrift, emphasis is put on the explicit rendering of the logical syntax of language. Without it, proof would elude us, as would an account of the validity of certain inferences. Frege never overlooks this conception of ‘definition’. ‘The insight [that definition] provides into the logical structure is’ says Frege, ‘not only valuable in itself but also is a condition for insight into the logical linkage of truths’.\(^{101}\) ‘A definition in arithmetic that is never adduced in the course of a proof, fails of its purpose’.\(^{102}\) On the other hand, there is the subject-independent or logical point of view, emphasised in the critical passages. That the concepts of arithmetic are purely logical notions; that the truths of arithmetic are theorems of logic; that the logical structure of these concepts and truths, and the dependencies and linkages of the latter belong to arithmetic: all these facts are independent of our apprehension of them. The epistemological achievement of definitions is of no importance from this point of view, from how things are independent of our recognition of them.

\(^{101}\) ‘Foundations of Geometry (1906), p. 302, in CP.
\(^{102}\) ‘Notes for Ludwig Darmstaeder’ (1919), p. 256, PW. Frege appears to revert to this way of speaking of definitions in the same paper in which he distinguishes Aufbauend from Zerlegung. Cf. 1914, p. 212, PW.
is in this sense that definitions are logically inessential.\textsuperscript{103} Beyond that, no change, nor any conflict, need be signalled by the critical passage.

§4.6. Closing Remarks. In this chapter, we have considered two types of objection to our strong hermeneutic thesis. (1) \textit{Grundlagen}-definitions are informative: \textit{i.e.} the content of two sentences, whose sole difference is the occurrence, respectively, of the definiens and definiendum, are distinct. (2) Frege ceased to deploy the \textit{Grundlagen} notion of analytic, and that this signals either that he never intended fully to capture the content of our ordinary arithmetic terms, or that he changed his mind. We have rejected (2) outright, and have adduced some considerations in favour of rejecting (1). This is not to say, however, that these responses are enough to vindicate my strong hermeneutic interpretation. This is partly because of a passage in \textit{Grundlagen} §64, not yet discussed, in which Frege speaks of carving up content in order to yield new concepts. The reason I have suppressed it until now is because the doctrine of carving up content involves independent problems that, I believe, the notion of the recovery of tacit grasp can help solve. For this reason, §64 warrants separate treatment. Let us then continue our defence of strong hermeneutism by turning to this passage.

\textsuperscript{103} I owe this observation to J. Proust, although she does not consider the notion of tacit grasp, nor does she use her observation in the way I have used it here. See her \textit{Question of Form: Logic and Analytic Propositions from Kant to Carnap} (1989), p. 123.
§5.1. Introduction. In the previous three chapters, we showed how the notion of full tacit grasp helps to cast light on the nature of Frege's logicism. That is to say, we showed that one could regard Frege's foundational investigations as involving a strong hermeneutic thesis. This is the claim that the *Sinn* and *Bedeutung* of our ordinary arithmetical terms are fully translated into the Concept Script, and that hitherto we lacked a clear and distinct grasp of ordinary arithmetical concepts, but that this grasp was, nevertheless, full or complete, if to some extent tacit. In the last chapter we considered and deflected a number of passages that appeared to undermine this hermeneutic claim: in particular, passages that one might construe as indicating that definitional identities are informative identity statements. As noted earlier, there is one further passage, *Grundlagen* §64, not yet discussed that might, on the face of it, favour this type of objection to my thesis. For in *Grundlagen* §64, when setting out to define cardinal number, Frege notes that a judgeable content (or thought) can be carved up in such a way as to yield a new concept. As also noted earlier, I have so far suppressed discussion of this passage because it involves independent problems that warrant separate examination. In this chapter I shall show, not only that the alleged threat to the strong hermeneutic claim can be addressed, but that the further problems specific to the passage in question can be resolved in a way that helps illuminate and lend support to the notion of full tacit grasp.

In *Grundlagen* §64, Frege introduces a technique for splitting up a thought that differs, both in method and purpose, from that introduced in *Begriffsschrift*. As seen in the last chapter, the role of the latter is to extract functions in order to exhibit the relations exemplified by valid deductive reasoning. By contrast, Frege envisaged the principal role of the technique introduced in *Grundlagen* §64
initially as that of individuating or introducing abstract objects. Frege insisted, that proper use of the singular terms for these objects requires our having a means of recognising the object as the same again. In other words, our grasp of the content of the singular term involves a grasp of the concept of which the object in question is an instantiation. The point is reflected of course in the fact that in each case, the singular terms utilise a description of the kind of object purportedly referred to. Given a grasp of the function denoted by 'the direction of ...' we at the same time grasp the concept exemplified by the values of the function. So to introduce an abstract singular term is simultaneously to specify a corresponding sortal concept: the kind of object apparently referred to. Frege illustrates the strategy thus.

'The judgement ‘line a is parallel to line b’, or, using symbols, a // b, can be conceived as an identity. If we do this, we get the concept of direction, and say: “the direction of line a is identical with the direction of line b”. Thus we substitute the symbol // by the more general symbol =, through removing what is particular to the content of the former and distribute it between a and b. We carve up the content in a way different from the original, and this yields us a new concept.'

1 Difference in method is not important to us here. Unlike the procedure described in Grundlagen, §64, what is yielded by the Begriffsschrift method is suggested by the structure of the sentence: we simply consider any part of it as replaceable. Common to both methods, however, is that the processes begin with a complete thought: see below.

2 Grundlagen §64.

3 Ibid. p. 72, p. 74. Of course, Frege decided against taking ‘F=G ↔ Nx:Fx = Nx:Gx’ as a contextual definition. He rejected it because he believed that contextual definitions alone could not tell us the truth value of identity statements in which one or both of the original complex singular terms is replaced by one of another sort, e.g. ‘Nx:Fx = q’. Instead Frege explicitly defined the cardinality concept: a class whose members are concepts that stand in a one-one relation both to each other and to the concept in whose extension they belong (Op. cit. §68, pp. 79-80). The original proposal, ‘Nx:Fx = Nx:Gx ↔ F=G’, is then derived from the above definition, and thus treated as a theorem rather than axiom of logic (Op. cit. §73).

So is our topic of this chapter not forestalled from the outset? It is well to be clear about what Frege did and did not reject here. D. Bell offers a useful insight into the matter by suggesting in another connection that, ‘... we need to distinguish carefully between two claims that can be made on behalf of such a procedure. On the one hand, ... transformation can be construed as a means of specifying or generating new concepts, whereas on the other hand, in the use of so-called ‘definition by abstraction, the process has been assigned the function of introducing or individuating abstract objects. Frege himself insists, rightly, that while the former use is both unobjectionable and valuable, transformation by itself is nevertheless quite incapable of fulfilling the latter function, that is, of providing determinate identity conditions for abstract objects.’ (Cf. ‘Objects and Concepts,
In the case of natural numbers taken as abstract objects, Frege takes the criterion in question to be a one-one correlation; and takes statements of the form 'there are just as many Fs as Gs'—'F=G'—to mean that such a relation obtains. Starting with the thought F=G, represented as involving the equivalence relation, just as many as, we reorganise or reconstitute it into the form Nx:Fx = Nx:Gx—the number of Fs = the number of Gs—and thereby extrapolate the corresponding sortal concept.

Let us call any two constituent sentences embedded by the following biconditionals a 'Fregean Pair', and this kind of contextual explanation 'transformation-strategy'.

\[ 'N': \text{Nx:Fx = Nx:Gx} \iff F=G. \]
\[ 'D': \text{The dir(a) = the dir(b) } \iff a // b. \]
\[ 'A': \{x: Fx\} = \{x: Gx\} \iff \forall x (Fx \iff Gx). \]

Whether, however, we conceive of the strategy as a mode of concept acquisition or as an abstraction procedure that introduces abstract objects, there are, on the face of it, a number of problems with it.\(^5\) One such problem is that it relies on doctrines that together constitute the following incompatible trinity.

1. **Sameness of thought**: The sentences flanking the biconditional express the same thought in a different way.
2. **Logical form**: The different surface syntax of each of the two sentences are taken at face value, mirroring the different logico-semantic form of the thought in each case.
3. **Explanatory priority**: We explain our grasp of the constituent terms of the left-hand sentence of the biconditional, as well as explain how to determine

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\(^4\) I shall also sometimes refer to 'N' as 'Hume's Principle', as it is standardly called.

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There are further considerations in support of Bell's point. Frege says in *Grundlagen* §57 that 'our concern here is to arrive at the concept of number in a way that is usable in science'. Thus when, finally, he rejects the proposal of §64, he is not denying that we can come by the concept by the carving up process. He is rejecting the idea that the method yields us a definition of the concept that meets his standards of rigour. Besides, Frege nowhere says that a non-logically simple concept is grasped only by defining it. Moreover, it is grasp of the concept by means of the contextual explanation that makes intelligible the eventual proposal to explicitly define it, since reference to the concept is made in the proposal. (See below for more on this.) Finally, the fact that Frege made the proposal regarding N might suggest that, pace strong hermeneutism, he envisioned that the definition would take the form of an informative identity statement. **
the truth of that sentence, in terms, respectively, of our grasp of the thought as represented on the right-hand side, and of our knowledge of how its truth is determined.

The alleged incompatibility of these doctrines arises in various ways depending on which of several ways we might try to maintain *sameness of thought*[^6]. We might claim that one member of a Fregean pair is but a disguised way of saying what the other member says. But to do so would violate *logical form*. That, in turn, would have one of two defeating consequences. To take ‘\(Nx:Fx = Nx:Gx\)’ as a disguised way of saying what ‘\(F=G\)’ says, would undermine Frege’s claim that numbers are abstract objects. Conversely, to take ‘\(F=G\)’ as an abbreviated way of saying what ‘\(Nx:Fx = Nx:Gx\)’ says would violate *explanatory priority*.

The second problem is that *sameness of thought* would seem to undermine the very motivation of the transformation strategy, as well as *explanatory priority*. Frege envisaged the contextual explanation of \(N\) as a way of disclosing a ‘new’ concept—that of cardinality—from an analysis of the thought represented by ‘\(F=G\)’[^7]. The thinker grasps the thought of the form \(F=G\) prior to his having discovered the concept by analysis. In that case, how could the operation described in *Grundlagen* §64 yield anything new? It would seem that we already grasp the concept in virtue of our grasp of the thought to be analysed—the *motivation problem*. So no transformation process—required by *explanatory priority*—need occur.

Commentators typically identify these problems as depending on the thesis that a thought is literally composed out of its parts: *strong compositionality*. Accordingly, it is suggested that we should not understand the transformation strategy in terms of rendering the thought of the form \(F=G\) into the form \(Nx:Fx = Nx:Gx\), whereby we excavate, as it were, the concept of cardinal number allegedly lying within ‘\(F=G\)’ awaiting disclosure by analysis. In general, it is denied that, of any *Fregean Pair*, the parts, taken literally to compose the one member of the

[^5]: Too many to deal with them all here.
[^6]: We will see more fully below what the content of these doctrines is and why I think that Frege held them.
[^7]: I will clarify and argue for this claim below.
pair, are the parts that compose the other member and conversely. Instead while there is a thought common to both ‘F=G’ and ‘Nx:Fx = Nx:Gx’, critics propose that it be construed as coarse grained. It is one that determines a common state of affairs, but which contains neither the constituent Sinne of ‘F=G’, nor those of ‘Nx:Fx = Nx:Gx’.

It is to be argued that this proposal undermines an essential element of the transformation strategy. Frege sought to explain ‘Nx:Fx = Nx:Gx’. We can identify three components of this explanatory task: (i) to ascertain that upon which its truth depends; (ii) to determine its truth; and (iii) to explain how we can arrive at its content—how we can grasp it. According to the above proposal, the thinker moves from our grasp of ‘F=G’ to the common—more coarse grained—content, whose carving or reconstitution makes for a re-description in terms of ‘Nx:Fx = Nx:Gx’. The trouble is that this proposal makes mysterious how anything is imparted thereby, and hence blocks (iii). Strong compositionality is a way of explaining the conceptual resources required for understanding a thought: the constituent Sinne supply this resource. What the proponents of the proposal object to is the suggestion that sameness of thought means that the conceptual resources for understanding the sentences ‘Nx:Fx = Nx:Gx’ and ‘F=G’ are the same. But if the ‘parts’ of Nx:Fx = Nx:Gx are not in some sense ‘parts’ of the common thought—viz. if the conceptual resources are not the same—then it cannot be from a grasp of the coarse grained notion of thought that the thinker acquires the requisite conception or grasp of what is represented by ‘Nx:Fx = Nx:Gx’. It is, moreover, unclear where else, for Frege, the explanation could come from. In that case, Frege would have required a finer grained notion of thought than the above proposal allows.

Having the finer notion of content, on the other hand, would imply that we already have some grasp of the concept sought after; and so it would seem that we are back where we started. The transformation strategy, envisioned in

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8 The proposal would solve the second problem, concerning the motivation of the Grundlagen strategy.
9 It is another matter whether strong compositionality provides a necessary explanation, and also another matter whether Frege subscribed to strong compositionality; see below.
Grundlagen §64, appears therefore to be either uninformative or without explanatory force. But this diagnosis, I shall argue, rests on a false dilemma. Fregean analysis is indeed a matter of yielding what is already ‘part’ of the thought, and moreover, the success of this kind of analysis depends on it. I collapse the dilemma by showing that commentators misconstrue certain of the doctrines surrounding the contextual explanation. In particular, we find that the common conception of what explanatory priority and ‘new concept’ mean, as well as sameness of thought, is defective. Pivotal to our solution is, again, the doctrine of the recovery of tacit grasp.

§5.2. The Trinity of Incompatible Doctrines Behind Grundlagen §64. Numbers cannot be physical aggregates or physical properties; nor can they be the products of human mental process; they can only be one kind of thing: abstract objects. This claim, argued for in Grundlagen, opens Frege to the charge that, since, as Kant claimed, it is only through sensibility that objects are given to us, and since we have no kind of idea of number, our numerical terms lack content. Frege’s response is that it does not matter whether we are able to establish an extra-linguistic mental association with an expression. It does not matter that we can have no encounter with the referent, or have any other causal commerce with it, or indeed whether we can imagine the referent. ‘It is enough if the sentence as a whole has a Sinn; it is through this that its parts obtain their content.’ To know what category a thing belongs to, and to know whether there is a correspondence between the symbol and the world, it is sufficient to know to what logical category the term in question belongs, and to know the truth of the sentence in which that term occurs. In light of this type of strategy, Frege’s task is to explain how (i) a sentence ‘Nx: Fx = Nx: Gx’ gets to have the thought of the form

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10 Frege refers to this doctrine in Grundlagen §89.
11 This concern appears to be one of the girders running from op. cit. §58 to his final answer in §69.
that the surface syntax of the sentence would suggest it has, and moreover to explain (ii) how we can determine that the thought is true.\textsuperscript{14}

Concerning (i), Frege, on my reading, relies on the doctrine that the same thought can be analysed (zerlegt) in distinct ways: \textit{multi-analysability}.\textsuperscript{15} This involves the claim that the different surface syntactic structures of the sentential representations mirror the different logico-semantic structures of the thought. There should be little doubt that Frege took the constituents of ‘N’—‘Nx:Fx = Nx:Gx’ and ‘F=G’—to be an instance of \textit{multi-analysibility}. For on explaining how to arrive at a criterion for numbers, Frege says

> we must define the \textit{Sinn} of the sentence
> the number that belongs to the concept F is the same as the number that belongs to the concept G;
> \textit{i.e.} \textit{we must render} \textit{(wiedergeben)}\textsuperscript{16} the content of this sentence in another way, without use of the expression
> ‘the number that belongs to the concept F’.

This way, we give a general criterion for the identity of numbers.\textsuperscript{17}

Talk here of ‘reproducing’ or ‘rendering’ the \textit{Sinn} of the sentence certainly suggests that, for Frege, the same thought can be regarded now as represented by a sentence—‘Nx:Fx = Nx:Gx’—involving an identity sign, flanked by complex singular number terms that purportedly refer to the aforementioned abstract

\textsuperscript{14}Pivotal to Frege’s response is the thesis that, when we apprehend things, we do so mediated by language and thought. This thesis becomes explicit in later work, following the distinction between \textit{Sinn} and \textit{Bedeutung}, e.g. in ‘Der Gedanke’, \textit{CP}. In his earlier work, the thesis is suggested by his strategy for answering ontological and epistemological questions concerning how, if numbers are what he says they are, we can be given them:
> ‘How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To explain the \textit{Sinn} of a sentence in which a number word occurs.’. Op. cit. §62. My emphasis.


\textsuperscript{16}In this context: giving the \textit{Sinn} or content in another way.

\textsuperscript{17}‘In unserm Falle müssen wir den \textit{Sinn} des Satzes
> ‘die Zahl, welche dem Begriffe F zukommt, ist dieselbe, welche dem Begriff G zukommt’ erklären; d. h. wir müssen den Inhalt dieses Satzes in anderer Weise wiedergeben, ohne den Ausdruck
> ‘die Anzahl, welche dem Begriffe F zukommt’ zu gebrauchen.” \textit{Grundlagen} §62; my emphasis.
objects, now as represented by a sentence—'F=G'—involving an expression for an equivalence relation and its associated relata—in this case concept expressions. There is another reason for thinking that the above example of the transformation strategy exemplifies multi-analysability. This is that Frege initially envisaged Hume's Principle as analytic of the concept number. This required that 'Nx:Fx = Nx:Gx' and 'F=G' have the same content at the level of thought, whilst retaining logical form.

At any rate, there can be little doubt that Frege held sameness of thought in this case. For one thing, it seems likely that he believed that thoughts with the same truth-conditions are the same. In Grundgesetze, for example, he says, '[t]he Sinn of [a sentence]—the thought—is the thought that [the truth] conditions are fulfilled'. And there can be no doubt that Frege took the sentential constituents of 'Nx' to have the same truth-conditions. For another thing, Frege states elsewhere that the Fregean Pair '(Fx ↔ Gx) ↔ {x:Fx} = {x:Gx}', 'expresses the same Sinn but in a different way'. Moreover, he took two thoughts to be the same just in case anyone who recognizes the one as true (false) immediately recognizes the other as having the same truth-value. As Dummett notes, since Frege took this thesis (immediacy criterion) to be both necessary and sufficient for sameness of thought, and took the Fregean Pairs to satisfy the criterion, 'it can be presumed to have influenced him in supposing that [they are] synonymous forms of sentences'.

So given sameness of thought, and logical form, Frege could show how to determine the truth of 'Nx:Fx = Nx:Gx' by determining the truth of 'F=G'. Furthermore, given the notion of rendering (wiedergeben) the same thought in different ways—multi-analysability—he could transform F=G into the form of

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19 'Funktion und Begriff' (1891), p. 27, TWP. Elsewhere too, Frege says that the same thought can be expressed in different ways. Cf. pp. 142-143, p. 149, 'Logik' (1897), PW.
20 Two propositions A and B possess the same sense (express the same thought) iff 'anyone who recognizes the content of A as true must straight away [ohne weiteres] also recognize that of B as true, and conversely, anyone who recognizes the content of B must immediately [unmittelbar] also recognize the content of A (equipollence)', NS, p. 213.
21 1991b, p. 171.
Nx:Fx = Nx:Gx—disclosing the concept of number by an analysis of F=G—and, it was hoped, explain how we come to grasp the content of ‘Nx:Fx = Nx:Gx’.

The kind of explanatory order alluded to here—explicating ‘Nx:Fx = Nx:Gx’ in terms ‘F=G’—consists not in the claim that the truth of one member of a Fregean pair is dependent on the other pair. Their having the same truth-conditions of course ensures their co-dependency in this respect. The claim is that we cannot grasp the content of ‘Nx:Fx = Nx:Gx’, or ‘The dir (a) = the dir (b)’, nor determine their truth-value, unless we can grasp or know the criterion made explicit by ‘F=G’, and ‘a // b’, respectively. So in a proof, our explanation of ‘Nx:Fx = Nx:Gx’, or ‘The dir (a) = the dir (b)’ must be in terms of their counterparts. Otherwise we risk presupposing what we are trying to prove. While we cannot have any sort of idea of number, we can in many cases intuit the truth of instances of ‘F=G’. Similarly, we can intuit that a // b is the case. Thus priority, together with the technique of rendering the same thought in a different way, forms part of an explanation of how parts of Nx:Fx = Nx:Gx can be imparted to a thinker given his grasp of the thought in the form F=G. Thereafter, if we have determined the truth of ‘F=G’, then we can infer that the sentence ‘Nx:Fx = Nx:Gx’ is also true, and thus infer that ‘(∃y) (y = Nx:Fx)’. Any further objection about the existence of abstract objects—or about the content and truth of ‘Nx:Fx = Nx:Gx’—would, Frege believed, be misguided; indeed, question begging.

‘... even if the Earth is really not imaginable, it is at any rate an external thing, occupying a definite place; but where is the number 4? It is neither outside us nor within us ... but the only conclusion to be drawn from that is, that 4 is not a spatial object, not that it is not an object at all. Not every object has a place.’

As already noted, there are problems with the strategy. Clearly, Frege cannot, as a way of maintaining sameness of thought, treat one side of the biconditional ‘Nx:Fx = Nx:Gx ↔ F=G’ as a disguised way of saying what the other side says. Whichever constituent sentence we take as having a

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22 Cf. *Grundlagen* §62, p. 71; Thiel, §62, p. 73.
23 §64, p. 75.
24 Cf. §64, pp. 72-73; Thiel, §64, p. 75.
masquerading surface form, we thereby undermine *logical form*. And to do so would be to play into the hands of the sceptic. On the one hand, by taking ‘\(\neg x : F(x) = \neg x : G(x)\)’ as a disguised way of saying what ‘\(F = G\)’ says we violate Frege’s doctrine that numbers are objects. On the other hand, by taking ‘\(F = G\)’ as an abbreviated way of saying what ‘\(\neg x : F(x) = \neg x : G(x)\)’ says we violate the *explanatory priority*. We thus forego our (non-circular) explanation of how we arrive at the content of ‘\(\neg x : F(x) = \neg x : G(x)\)’, and also of how we determine its truth.

The sceptic urges that we treat the left-hand side of a Fregean Pair as a disguised way of saying what the right-hand side says. He does so not simply because he finds no grounds for believing in the existence of abstracta. More immediately, he charges that \(N^*\) and \(D^\ast\), given *logical form*, are ontologically inflationary: the left-hand side commits us to an ontology not implicated by the right-hand side. Frege would simply reject this point. While the grammatical structures of the members of a Fregean Pair are taken at face value, one cannot regard the right-hand member as entirely blameless in this respect. It is at least potentially misleading. For given *sameness of thought*, the ontological commitments of that side are not simply what the grammatical structure of that sentence would indicate. As Wright puts it in response to the sceptic, where ‘\(a / b\)’ has epistemological priority, ‘The dir(a) = the dir(b)’ has ‘ontological priority’: the right-hand side has a hidden ontological commitment—that implied by the left-hand side. On clarifying ‘ontological priority’, Wright makes a stronger claim yet.

‘What is there to prevent us saying that, since the left-hand side does contain an expression referring to a direction, it is the apparent lack of reference to a direction on the right-hand side which is potentially misleading, or ‘mere surface grammar?’ Rather, what we have on the right-hand side is a sentence which

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25 Concerning the claim that abstract entities exist—often referred to, perhaps misleadingly, as nominalism.
26 Or, given the claim that both sides of ‘\(N^*\)’, or ‘\(D^\ast\)’, have the same truth-conditions. See below.
27 Wright and Hale characterise \(N^*\) and \(D^\ast\) as a way of justifying our belief that the abstract singular terms that flank the identity sign in ‘\(\neg x : F(x) = \neg x : G(x)\)’ and ‘The dir(a) = the dir(b)’, refer to mind-independent abstract objects. I think that this is how Frege regarded them. See chapter six.
28 Implied given *logical form, sameness of thought*, and the truth of an instance of Hume’s Principle.
achieves a reference to a direction without containing any particular part which so refers."^{29}  

Wright has since retracted this view, claiming it implausible, if not absurd.^{30}  For anyone who knows no terms for directions or numbers, but who uses the vocabulary of the right-hand side of N" or D", nevertheless unwittingly refers to directions or numbers (hidden reference view). Yet reference to a particular object through use of a sentence presupposes an ‘identifying thought of that object’:\(^31\) it is to bring ‘the thought-of object under the relevant sortal concept’. That of course is ruled out if, with respect to N", or D", we know only the thought expressed by ‘F=G’, or ‘a // b’, since at that stage—given explanatory priority—we have no concept of direction or concept of cardinal number. But ex hypothesi, the thoughts are the same. So a thinker could not understand ‘a // b’, or ‘F=G’, without a grasp of the concept of direction, or the cardinality concept, respectively. But then if we already grasp the thought, albeit in a different form, it would seem that we must already be in possession of the said concept yielded by the carving. Hence Frege is committed to the hidden reference view. Moreover, it is to be wondered therefore how by analysing it—splitting it up in a different way—we can yield anything new. Again: sameness of thought appears to undermine explanatory priority, as well as the motivation of the Grundlagen-strategy as a means of acquiring a new concept: motivation problem.

As already said, critics take these problems to depend on strong compositionality. So if they are right, then it would seem that Frege would have had little choice but to modify sameness of thought—assuming he understood the latter thesis in this way. As Hale puts it, ‘if a statement involves reference to objects of a certain kind that is a feature of its sense’.^{32}  So if members of the same

\(^{29}\) 1982, p. 32. My emphasis. Wright takes this aspect of platonism to be explicit in Frege. Dumett endorses this interpretation.  
\(^{31}\) C. Wright, 1988, p. 459; 1990, p. 89.  
\(^{32}\) Cf. 'Grundlagen §64', Proceedings of the Aristotelian Society (1997), p. 249. If one sentence involves reference to a particular object that another sentence does not, then given that ‘reference is not an eliminable aspect of thought’ the Fregean platonist—and thus Frege himself—would be
Fregean Pair effect different reference relations, then they express different thoughts in the *strong compositional* sense. Yet since the Fregean Pairs meet the *immediacy criterion*, the latter violates *strong compositionality*. This would suggest that therefore while the criterion is a necessary condition, it is not sufficient. To deny this point, it is claimed, would be counter-intuitive. For one thing, *priority*—indeed the very motivation of the transformation strategy—requires that the constituent thoughts of a Fregean Pair be distinct. For another, someone could, we are told, understand the thought expressed by ‘a // b’, without as yet being capable of understanding the thought expressed by ‘The dir(a) = the dir(b)’. For instance, it is said that one might lack a grasp of a part of the thought expressed by one sentence (say, the concept of number) without lacking any part of the thought expressed by the other sentence (that involving a one-one relation).

§5.3. Modifying *Sameness of Thought*. Given the *hidden reference view*, we might say that while reference is made to lines rather than to directions when judging the thought expressed by ‘a // b’, we nevertheless commit ourselves to the existence of directions in so judging. As such ‘it cannot’, says Wright, ‘be a necessary condition for two sentences that share the same truth conditions that if one sentence involves reference to an object then the other sentence also involves reference to that object’. ‘The most we can say is that it entails that the object in question exists.’ Thus Wright espouses a coarser grained notion of sameness of thought.

mistaken to identify the members of a Fregean Pair as expressing the same thought. Cf. Wright, 1988, p. 459; 1990, p. 90.


34 Again: two propositions A and B possess the same sense (express the same thought) iff ‘anyone who recognizes the content of A as true must straight away [ohne weiteres] also recognize that of B as true, and conversely, anyone who recognizes the content of B must immediately [unmittelbar] also recognize the content of A (equipollence)’ (NS. p. 213.).


36 This is Wright’s response. As he puts it, we should question the assumption that the semantic role of a singular term is one exclusively of reference: while every use of a singular term is existentially committing, not all are referential. Cf. 1988, p. 458; cf. 1990, p. 89, respectively.


38 Op. cit. It follows that Wright would agree with Dummett that since for the Fregean the *Sinn* of a sentence is a function of the sense of its parts the mereological view *vis-à-vis* the Fregean pairs is
content: ‘F=G’ and ‘Nx: Fx = Nx: Gx’ have the same truth-conditions, same ontological commitments, but distinct constituent Sinne. The principle of individuation proposed is that two sentences have the same non-compositional content—‘weak sense’—just in case they depict the same state of affairs, which any coarser-grained conception would fail to do. Accordingly, the (revised) contextual explanation would take something like the following form:

'We introduce 'the direction of line a = the direction of line b' as a (content (=weak sense)-preserving) redescriptions of the state of affairs consisting in the two lines being parallel, whilst simultaneously stipulating that its logical form is precisely what its surface syntax suggests.

Since that shared content is available to one who understands the statement about parallelism it can be appreciated without possession of the concept of direction. (That concept does not, therefore, lie hidden within the shared content, awaiting disclosure by analysis.)

Each of the two sentential representations—say, ‘Nx: Fx = Nx: Gx’ and ‘F=G’—express a different thought, since each have different constituent Sinne. However, both represent, in their different ways, the same state of affairs. They do so

untenable. Ibid. p. 250. So for Wright, as for Dummett, in order to grasp the thought expressed by 'a // b' a thinker is not required to grasp the concept of direction.


In contrast to Hale's more cautious view, Sluga and Wagner reject the claim that Frege ever subscribed to strong compositionality, understood as the claim that the Sinn of a whole sentence is literally composed of constituent parts. Fregean thoughts, for Wagner, are to be distinguished from what he calls propositions (1983, p. 8). Fregean thoughts, unlike propositions, are unstructured bearers of truth-value, containing no constituent parts, such as concepts and objects (or in later terminology: the senses of terms that refer to functions and objects (ibid.)). On this reading propositions arise from these contents. They do so as a result of structures that the mind formulates and imposes in order to represent judgeable contents to itself. In particular, these structures are arrangements of concepts and terms such as are necessary if thought is to be possible. Accordingly, the logical grammar of the Begriffsschrift is to be regarded as a specification of the possible structures with which we can and must think contents (ibid. p. 12). So as with Hale's reading, neither the relation of equinumerosity, or the cardinality concept are to be found in the content.

It would follow that in order to grasp the thought contained in 'Nx: Fx = Nx: Gx', I do not have to have the concept of cardinal number. Similarly, in order to grasp the thought of the sentence 'The dir (a) = the dir (b)' I do not have to have the concept of direction. For since the thought of 'F = G' is the same as that of the first sentence, by grasping the content of the former I thereby grasp the thought of the latter. Mutatis mutandis the sentence pairs involving parallelism and direction. Since we do not, on Sluga and Wagner's reading, already possess prior concepts we must acquire them through analysis, this being equivalent to definition (ibid. p. 9). The carving process is, for Sluga and Wagner, as it is for Hale (1997, p. 252), a means of acquiring concepts that are not already part of the judgeable content (Hale: weak sense/thought) lying on the right of the biconditional.
because they have in common a coarse grained thought, which mediates between
the thinker and the common state of affairs. The two thoughts that flank the
biconditional, are different representations, or carvings, of it—‘F=G’ and ‘Nx:Fx
= Nx:Gx’ being simply different ways it is grasped.

'By accepting the equivalence we come to know the content—but not yet the
(fully compositional) sense—of its left hand side. But if we further accept
this as having the logical form Frege intends, and so discern in it a genuine
occurrence of the familiar identity predicate, we shall be led to recognize the
expressions flanking it as singular terms which, provided they have reference
at all, stand for objects of a kind whose identity-conditions are given by the
equivalence. The new sortal concept of direction may then be introduced as
applying to objects of that kind.'  

It is not, to my mind, altogether clear how Hale's proposal could work. What is unclear is how the thinker is meant to effect the transition from grasping
the coarse grained content, as represented by ‘F=G’, to grasping the same coarse
grained thought as represented by ‘Nx:Fx = Nx:Gx’. The problem is that the
coarse-grained thought does not, as it were, contain the constituent Sinne that
helps make up the thought of the sentences of the Fregean Pair. This means that
F=G can impart no conception of what can be yielded by the carving. And that
makes unclear what, for Frege, would explain the thinker's grasp of 'Nx: ... x'.
Consider again the stages involved in the Hale-Wright proposal. First, by
accepting the equivalence, the thinker 'comes to know the content—but not yet
the (fully compositional) sense—of its left-hand side'. Thereafter '... we further

41 1997, p. 252.
42 My emphasis. Hale, 1997, p. 252; cf. 1994, p. 129. The position taken up by Hale is one that
would of course require more for understanding a sentence than knowledge of its content (Hale's
'weak sense'). This is because one can allegedly grasp the content of 'The direction of line a is
parallel to line b' without grasping the concept of direction; and because one must grasp the
corresponding of direction to understand that sentence in which the content (weak sense) is purportedly
contained. Dummett says of this that 'No hint of any such distinction is present, however, in the
writings of Frege's middle period, nor, indeed, in the less systematic discussions of his early
period' (1991b, p. 173). In fact the option in question has long been regarded by a number of
Frege scholars, e.g. H. Sluga and S. Wagner, as an integral part of Frege's theory of judgement. It
is not clear from his paper that Hale is aware of this. See Sluga, 1980, pp. 90-95, pp. 134-37;
Wagner, 1983, fn. 42.
43 This by itself seems wrong. Surely, on Hale's own account, the thinker comes to know the
course grained content simply by knowing F=G, regardless whether the thinker is ever introduced
to N".
accept this as having the logical form Frege intends.’ My point is that \( F = G \) must impart some intuitive significance of what the content of ‘\( \text{N}x: Fx = \text{N}x: Gx \)’ would be. Yet if it does, then that must be because of a common fine-grained Sinn. But if it does not, then in what can our understanding of the identity sentence consist: hereafter the understanding problem?\(^4\)

To clarify my worry, it is worth separating this concern from another, connected concern that we might have about the transformation strategy. According to S. Wagner, all that ‘\( \text{N}x: Fx = \text{N}x: Gx \leftrightarrow F = G \)’ defines is an unstructured binary predicate, ‘The number of ... = the number of ...’. Since the terms of the predicate are undefined, its composition is not built up from identity and terms.\(^5\) The semantic parts of the left-hand side are of a simple predicate and two concept variables. So at this stage, we grasp merely a disguised form of what the right-hand says. Still, ‘\( \text{N}x: Fx = \text{N}x: Gx \)’ has intuitive significance. Its significance lies in suggesting the possibility of an equivalent to ‘\( F = G \)’.\(^6\) ‘It shows that if we were to have number terms, we could use them to restructure the apparently non-arithmetical content of ‘\( F = G \)’. What ‘\( \text{N}x: Fx = \text{N}x: Gx \)’ provides ‘... is a dummy structure which shows how genuine number terms would function.’\(^7\) But whence derives this intuitive significance? We are, after all, in a state of numerical innocence. It might now seem that the thinker is in the dubiously coherent position of proposing to define a concept, where the proposition that expresses the proposal makes reference to a concept of which, \textit{ex hypothesi}, he has, as yet, no conception or grasp. Granting this problem, we might say, with second-level language in mind, that ‘\( \text{N}x: Fx = \text{N}x: Gx \leftrightarrow F = G \)’

\(^4\) Moreover, it is not altogether clear what this coarse-grained notion of thought could be. The problem is that it fulfils incompatible functions. The coarse-grained notion of thought, if it is meant to be Fregean at all, must be such as to determine the state of affairs, its \textit{Bedeutung}, given that the state of affairs is itself structured. But it is not clear how it could do this if it is not already structured.


\(^7\) Op. cit.
implicitly defines the concept of a cardinality function: it allows us to grasp the type of concept to be defined.

But this way out is at best unnecessary, and at worse unworkable. According to the proposal, concept-acquisition in *Grundlagen* is a matter of hitting upon the right definition. In that case, the second-level concept, the concept of a cardinal concept, might be no exception. If it is definable, then we face the same problem, assuming it is a problem, as had in the original case. We need to explain what makes coherent the proposition that this second level concept be defined. The procedure for arriving at this (second-level) concept must have intuitive significance. Then presumably this also would be because the contextual procedure for arriving at the concept of a cardinal function implicitly defines a third-level concept: viz. the concept of a concept of a cardinal function.

But what is to prevent the problem from regenerating at each level of abstraction? One way of course might be if the concept of a cardinal number is primitive. But it will be wondered whether this proposal would not impact on Frege’s logicism. For would not the admission of a primitive arithmetical concept make its nature unclear? Even if it would not, Frege would perhaps have rejected the claim on two other grounds. First, the explanatory value of any deductive theory is inversely proportional to the number of its primitive truths relative to the number of non-primitive truths. Yet the proposal would commit Frege to as many primitive concepts as there are definable ones. So for any theory, the number of primitive truths would match the number of corresponding non-primitive truths. This would weaken the theory’s explanatory value. Second, it would be unclear on what grounds we could claim that the higher level concept in question is primitive, given that its lower level counterpart is not.

At any rate, the proposal of positing a concept of a cardinal function is unnecessary. The proposal stemmed from the observation that the very

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48 This suggestion might then be used as one way of explaining the point of §64 given that Frege eventually rejected the attempted contextual definition in favour of an explicit definition of cardinal number. For while the attempted contextual definition is eventually rejected—because it cannot by itself determine the objects which fall under it—it might nevertheless be said to represent a crucial transitional stage in the acquisition of that concept. Cf. Wagner, 1983, p. 10.
proposition to define a concept contextually—‘Nx:Fx = Nx: Gx ↔ F≡G’—makes reference to the concept to be defined. But the observation is mistaken. No reference to any particular concept need be made. Once the thinker knows that an equivalence relation obtains, he need make no appeal to any type of concept, or to the specific concept in question, in order to make the strategy intelligible. The thinker knows from his grasp of the concept of equivalence relations that there is something shared by the associated concepts—or objects—that flank the relation. He knows that a transition can be made involving identity and complex singular terms, where the latter involves a functor, which isolates what is shared. So at the initial stages of engaging in carving up the thought in its original form, the thinker is equipped with (a) a general knowledge of the logic of equivalence relations and (b) his grasp of the thought in its original form or representation. If we understand a sentence involving an equivalence relation, and we understand that what is expressed is true, then we also understand that there is something about the relata that is the same. Like identity, equivalence relations are reflexive, transitive and symmetric. But unlike identity, there may be predicates true of some of the objects that stand in the equivalence relation that are not true of the other objects that stand in the same equivalence relation. In order to isolate what is the same from what is different between them, we use the concept of identity, which Frege takes to be already known, together with our knowledge of the original equivalence. The resulting singular terms, which flank the identity sign, will have a compositionality that reflects the commonality identified and isolated in the carving-up process. So the thinker would succeed in the transition from a sentence expressing an equivalence to one of identity: e.g. ‘The $F = G$’.

But this overlooks Bell’s point mentioned at the outset of this chapter (see fn. 3). The Grundlagen-strategy yields the cardinality concept but not yet the objects to which it applies.

§63.

More generally, we can introduce new objects in the following way. We take an equivalence relation and its domain, a set of already given objects over which the relation ranges. We can then use the equivalence relation to partition this domain of objects into disjoint, nonempty subsets, i.e. where no member of a subset is a member of another subset of the domain. Our new objects will be the equivalence classes. Identity can then be predicated of members of each of the subsets or equivalence classes. For example, we take equinumerosity as our equivalence relation, one ranging over the domain of concepts; we then use this relation to partition the domain into non-empty disjoint subsets of concepts. So for example with the second-level relation we form the
Hence the problem is not how we make the carving process intelligible without making reference to the concept to be defined. The worry—to return to Hale and Wright—is what explains our grasp of the *Sinn* of ‘The φ F = the φ G’, once the carving has taken place. My challenge to those who would modify *sameness of thought* is to explain in what one’s understanding of the identity sentence ‘The φ of Fs = the φ of Gs’ would consist, or how they would reach it. It is incumbent on them to say what it is about (i) the thinker’s grasp of F=G (or a // b), (ii) his acceptance of *logical form* and *explanatory priority*, and (iii) his acceptance of the stipulated equivalence, that could possibly enlighten the thinker about how he might know what the content of ‘The φ …’ is. Given that *ex hypothesi* the cardinality concept (or direction) is no part of the thought of the form F=G (or a // b), thus no part of his conception—given that F=G can impart no intuitive significance regarding it—that challenge, I claim, cannot be met.

In their very recent discussion of implicit definitions in general, Hale and Wright have shown some awareness of something like my concern. They consider two possible solutions. The first, if successful, would show how understanding is achieved but would defeat the point of implicit definition. We could explain how implicit definitions impart understanding, if we could find some independent and canonical way of identifying the meaning. But then possession of these independent resources would be sufficient for explicit definitions. The only alternative, they claim, is that ‘implicit definitions must leave room for the capacity of such explanations to invent meanings’. The contextual explanation sets up a language game, a pattern of use of newly

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concept, *concept equinumerous with the concept F*, where F is a variable, and use this as a condition of membership for each subset. Numbers are then identified with these sets. This in effect is what Frege does in later passages of *Grundlagen*. The domain is carved up by use of a bijective relation into disjoint non-empty subsets of concepts. This was later captured by axiom V: The value-range of f = the value-range of g iff for every a, f(a) = g(a).


53 Op. cit. p. 11. Hale says that the concept in question is ‘produced by stipulating that the two sentences are to have the same truth-value and exploiting the overt syntax of the former to yield a new way of determining that truth value’ (1994, p. 130, fn. 5). But this, I claim, is simply not enough.
introduced terms, and this suffices for the creation or invention of a new (non-Fregean) concept. As such, we are told that

1. 'it makes no sense to doubt that there is a meaning taken on by the defining expression ... because it has ... been fully explained in the very process that creates it.'

2. '... [This] explanation can only consist in the fact that the implicit definition determines ... a pattern of use which is fully intelligible ...

There are, it seems to me, at least two problems with this (essentially Wittgensteinian) approach. The first, less serious for the non-exegetes of Frege, is that this manoeuvre is wholly unFregean. It is, for Frege, the grasp of independently existing concepts (Sinne) that determines the pattern of usage; it is this that constitutes the content of the newly introduced terms: not the other way round. Sinne, not usage, grounds our understanding. Our purpose here is to find a Fregean solution. It will be one that avoids the second, more serious, problem, now to be described, which the Wittgensteinian approach entrains.

Let us begin with the case of D^m. We are concerned to account for what the understanding of D^m consists in. As Hale-Wright rightly note, the thinker knows that the sentence ‘The [a] = the [b]’ can be used whenever he knows that a // b, and knows that he can engage in the inferential liaisons proper to the use of sentences involving singular terms. For the thinker accepts ‘The [a] = the [b] a // b’, and logical form. Still, it is far from obvious that this language game suffices for understanding, even supposing that we do invent concepts by creating a pattern of use in the way outlined. This is not to deny that the language game might fully explain how the sentence in question gets a Sinn; or indeed to deny that it gets the right one. To that extent, we can agree with citation 1. But there appears to be a slide from (1) to (2). From an encounter with the parallel lines //, the thinker is meant to factorise what is common between them. Unless

56 For Frege, assigning Sinn to a word or sentences appears to be analogous to fixing a Bedeutung. As the latter case may give rise to whether a unique reference has been assigned, so might the same question be asked, pari passu, of the former. P. Horwich raises this question as a general problem for implicit definitions. See his ‘A priori, Implicit Definition’ (2000). Neither P. Horwich, nor Hale-Wright, thinks that this is a problem given a use-model of determining meaning

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his knowledge of mathematics is fairly rich, he will be at a loss about what to extract—or what to associate with the singular terms he is about to introduce. He will still form an identity sentence. But without sameness of thought—without sameness of conceptual resources—it is unclear that the thinker would know what he has isolated by way of $D^\varphi$. Recall that, on the Hale-Wright proposal, the right-hand side lacks the intuitive significance *apropos* the possible left-hand carving. In that case, the thinker will have isolated blindly, as it were, not knowing what ‘The $\phi$ of line (...)’ represents. A considerable amount of time may lapse before finally he does arrive at an understanding of the singular terms. This is because, given the Hale-Wright notion of coarse content, some additional mathematical knowledge might be required. As his knowledge of mathematics grows, the thinker may realise that the language game has determined an identity sentence (‘The $\phi$ of line (a) = the $\phi$ of line b’) of several meanings. ‘The $\phi$ of line (...)’ might mean *the gradient of line* (...), or *the possible perpendicular bisector of line* (...), or *the direction of* ..., and so forth. It is not obvious why the thinker should be taken to grasp, the direction of line (...), prior to a grasp of the other candidate operators. Indeed it is not obvious that the direction operator would occur to the thinker. In short, given the Hale-Wright proposal, the implicit definition will have ‘invented’ too many meanings, with no reason as yet to think that what is grasped on setting-up the language game is the concept (direction) that Hale-Wright think would be grasped.

Let me elaborate my worry, and begin to suggest a Fregean solution. First consider Frege’s other examples of carving, as well as some examples of my own.

and understanding. I show below that, in certain cases, their confidence is misplaced.

57 The problem being described here is similar to, but not the same as, the Julius Caesar Problem. The problems are similar in that both involve indeterminacy of reference. However, the JC-problem is about indeterminacy of reference to objects. Frege took this to arise even though we have some grasp of the corresponding sortal concept: the JP-problem is aimed at definitions of sortals. The indeterminacy described above arises with respect to the function denoted by the operator ‘The $\phi$ of ...’. The kind of indeterminacy envisaged here could, arguably, arise even if the indeterminacy at the lower level did not.

58 The first three are Frege’s. See *Grundlagen* §64, p. 73; Thiel, §64, p. 75.
1. 'Triangle A is similar to triangle B $\iff$ The shape of A is identical to the shape of B'.
2. 'Plane A is parallel to plane B $\iff$ The orientation of Plane A is identical to the orientation of plane B'
3. 'Geometrical forms A and B share the property of collination $\iff$ The $\Phi$ of A is identical with the $\Phi$ of B'.

Other examples of this type of transition are available of course from more everyday applications of carving up.

4. 'Mary is just as old as John $\iff$ The age of Mary is the same as the age of John'.
5. 'Mary is just as large as John $\iff$ The size of Mary is the same as the size of John'.
6. 'Benedict is just as competent at improvising and sight-reading as William $\iff$ The musicianship of William is the same as the musicianship of Benedict'.
7. 'A is chromatically similar to B $\iff$ The redness of A is the same as the redness of B'.

In all these examples, it is straightforward to see how the acquisition of new concepts proceeds, as well as how a unique one is isolated. We are guided in our analysis by (a) our grasp of the tools of logic, including our grasp of the concept of equivalence relations, identity, and the distinction between concept and object; and (b) our grasp of the thought as represented in the form of the left-hand side of each biconditional 1-7. But crucially, we are further guided by being able selectively to attend to the state of affairs to be carved, and this affords us some conception—some intuitive significance—of what there is to be acquired. The thinker understands that what there is to be isolated (by use of the identity sign) is to be found simply by contemplating the state of affairs in question. For example, in the case of the equivalence relation, similarity, it is clear that, by attending to the commonality between triangles A and B, the thinker sees for himself that what they have in common is what subsequently we have called 'shape'. Here the commonality is intrinsic to the triangles (contrast such extrinsic similarities as being on the same table, both being rotated about the same x or y axis). We should not say that the concept of shape is no part of the thought, *Triangle A is*
similar to the triangle $B$, to be split up in a new way—or at least not deny that it is part of the conceptual resources required for its grasp. Neither should we say that we fail to have, in some sense, a full grasp of the concept of shape. Nor should we say that such a grasp is had other than from our grasp of the thought represented in terms of the similarity of the triangles (or other instances of shape).

Depending on which of two possible mathematical contexts Frege had in mind, we can see that $D''$ is analogous to those cases in our list above, and can see therefore how the concept of direction could be acquired. The context in question is projective geometry. In projective geometry, what is common to two parallel lines is that the lines meet at so-called ‘points of infinity’. From the perspective of the intuitor, the lines do indeed appear to meet. Think of yourself as standing in the middle of a long flat desert road that visibly extends to the horizon from where you stand. From your perspective, looking directly along the road to the horizon, the edges of this road—the parallel lines—would appear to meet at a point on the horizon: viz. ‘the point at infinity’. Since this is simply how the lines appear to us, the term ‘point at infinity’ or ‘direction’ is used to distinguish it from the other (concrete) points. It is this context of projective geometry, I believe, that Frege has in mind when, in *Grundlagen*, having illustrated the way that the thought $a \parallel b$ can be carved up, he complains that mathematicians have tended to violate priority.

Given our description of the relation between parallel lines and point at infinity one can readily see why a mathematician, unconcerned with priority—i.e. specifically with explaining how we grasp concepts—would have no objection to defining parallel lines in terms of points at infinity.

‘Often, of course, we conceive of the matter the other way round, and many authorities define parallel lines as lines whose directions are identical. The proposition that “straight lines parallel to the same straight line are parallel to one another” can then be very conveniently proved by invoking the analogous proposition about things identical with the same thing. Only the

\footnote{In projective geometry, ‘points at infinity’ and ‘direction’ are taken to be synonymous. Cf. Frege’s ‘On Geometric representation of Imaginary Forms in the plane’ (1873), p. 1, \textit{CP}.}

\footnote{Other commentators have pointed out that projective geometry may be what Frege had in mind. However, their observation is not used in the way I use it. See Baker and Hacker, Frege: Logical Excavations, and more recently, M. Wilson, ‘The Royal Road to Geometry’ (1992), and J. Tappendum, 1995b.}
trouble is that this is to reverse the true order of things. For everything geometrical must be given originally in intuition ... The concept of direction is only discovered at all as a result of a process of intellectual activity which takes its start from the intuition.\textsuperscript{61}

It is hard to see what other kind of mathematician Frege has in mind here than the projective geometer. As far as I can ascertain, no other type of mathematician would so define parallel lines.

Moreover, to deny that Frege had projective geometry in mind is to convict him of the rather stark error mentioned above. If non-projective geometry is the right context, and if Frege sought to assert merely that 'The $\phi$ of line (a) = the $\phi$ of line (b) $\leftrightarrow$ a // b', then he should have known that we violate the uniqueness constraint thereby.\textsuperscript{62} It is true that in projective geometry, and also in non-projective geometry, two lines have the same gradient just in case they are parallel. But in the former context, a conception of direction, unlike the concept of a gradient,\textsuperscript{63} is (at least) implicit in our grasp of a // b. It is this that guides us in what to isolate. With that context we get the (right) intuitive significance of what to factorise from a // b, which is lacking in the non-projective context. So granting that Frege did have projective geometry in mind, we can see clearly how, as with the other seven cases in our list, we can arrive at the 'new' concept—in this case, direction—through analysis of the thought of the form 'a // b'. Imagine that someone is attending to parallel lines in the context of projective geometry. He seeks to isolate what these parallel lines have in common. Clearly, the point at infinity or direction is there for the intuitor to pick out. The source of his conception of what to isolate lies in his very grasp of a // b. With his

\textsuperscript{61} Op. cit. §64, p. 75.

\textsuperscript{62} Connectedly, it is commonly supposed that the reason Frege switches from talking about carving up $F=G$ to talk about a // b in §64, is because the latter provides a simpler illustration of what he means. Cf. Hale, 1994, p. 125. It is unclear that the switch is to anything simpler. The difference between the two cases is not one of complexity or simplicity, but simply in the levels involved. Regarding the identity statements embedded in $N^*$ and $D^*$, the first of course involves a second level function that takes first level concepts for objects as values, whereas the latter involves a function of one level below. But regarding the carving procedure, the one example is neither more nor less simple than the other is. More importantly, the reason Frege switches examples is, I believe, because he wants to show philosophers that $N^*$ is intended to be analogous to $D^*$, where the latter is understood in the way I've just outlined. See below.

\textsuperscript{63} But not, I take it, that their perpendicular bisectors would be the same. However, they would share the same angle: namely that subtended by the lines joining their common point.
understanding of how to construct complex singular terms, he can thereafter speak of the direction of line a being identical to the direction of line b.\textsuperscript{64}

What, now, of $N^?$ Admittedly, it is hard to conceive how, once the equivalence is known to obtain, the thinker could fail to arrive at a unique Sinn by the carving process. The reason is not that he creates the concept, though indeed he might. It is that the obtaining of $F=G$ imparts an intuitive significance of what can be factorised: $F=G$ shares the same conceptual resource as the thought carved up. Let there be (say) two instances of Fs and Gs, the Fs—$a_1$, and $b_1$—in one group, the Gs—$a_2$, and $b_2$—in another. Each group stands side by side within the thinker’s perceptual vicinity. The thinker immediately apprehends that there is a one-one mapping between these two types of things. He sees that the instances of the Fs before him are paired with the instances of the Gs, also before him, and conversely, without remainder. In what would the intuitive significance consist? Well, he would see that the multitude or magnitude of the one set or group is the same as that of the other group. He would simply see that this is so. It would be rather odd if this were not the case. Yet this just is to see that the groups or sets have the same cardinality. To see that a one-one mapping obtains is \textit{thereby} to have some conception that the two sets share the same cardinality. Would we not say that the thinker lacked a full grasp of the one-one mapping if he did not have this conception of what to factorize? If he did not grasp the commonality between the instances of Fs and Gs, we would have a gap between use and understanding. The language game \textit{à la} Hale-Wright would effect a determinate use of ‘$N_x:F_x = N_x:G_x$’ without the right kind of understanding. As with those geometric applications above, the function, denoted by the singular term forming operator, is clearly already part of the original thought awaiting explicit disclosure by the thinker’s selective attention, and the logico-syntactic manoeuvres. Our conception of what to carve-up consists in our having a tacit grasp of the cardinality function. This is part of what we grasp on apprehending the state of affairs through the thought represented by ‘$F=G$’.

\textsuperscript{64} We can also see that, pace Hale and Wright, the ontological commitments are what the grammatical structure of both sentences would suggest.
In order to defend this interpretation of the transformation strategy, we need to reconsider the problems associated with the trinity of doctrines, since it might be argued that my proposal runs head on into them.

§5.4. Objections 1: Motivation of the Strategy. The first objection might be that I have undermined the motivation of the carving-up process, and moreover, gone against what Frege actually says. This objection touches of course on the claim considered in the last chapter: that Frege’s definitions are transformed into informative identity statements. In other words, if Grundlagen §64 is well motivated, then a new concept is yielded in the carving process. In that case, the resulting sentence is informative, thus undermining strong hermeneutism.

The motivation of carving-up the thought F=G is to show how we arrive at the concept of cardinal number. In particular, it is to show how our grasp of the concept of cardinal number can be explained in terms of our grasp of the thought of the form F~G. As Frege puts it in Grundlagen §64, we carve up the content in a different way and arrive at a ‘new’ concept. But I have suggested that the legitimacy of this strategy depends on the thesis that we already implicitly grasp the concept to be acquired. That is, we should reconsider what ‘new’ means. It is true that, if by ‘new’ Frege meant that we arrive at a concept of which we had hitherto no conception whatever, then my construal is unfaithful to Grundlagen §64. But that is not, I believe, what Frege meant by ‘new’. And even if he did understand ‘new’ in this way, my construal of the Grundlagen §64 would by no means render the carving up process pointless. There are at least two further ways of understanding the motivation. Even if it transpires that, on the most literal reading of Grundlagen, I am somewhat unfaithful to Frege’s actual words, I am not unfaithful to their181 spirit. At any rate, going against the letter, rather than spirit, of what he actually says, would be but a small price to pay given the greater coherence gained.

The abstract sortal concept arrived at through Frege’s contextual procedure can be seen to be ‘new’ in the following two senses.
1. In \( \mathbb{N} \), we reconceptualise what is explicitly expressed by the first sentence of each Fregean pair. Here 'reconceptualising' means making explicit the cardinality function only tacitly grasped. It involves the thinker in selectively attending to that background conception involved in grasping the thought in its original form, and in such a way as to attain to full cognizance of it. The element of newness lies in the way in which the cardinality function is possessed. The thinker's new way of possessing it means that the thinker can, for the first time, use it in acts of predication, reference and conscious reasoning.

2. The second sense in which an element of newness is retained concerns the distinction between the cardinality function and the concept of cardinality. At the initial stages of our numerical innocence, our grasp of the former is implicit in our more perspicuous grasp of the thought represented as explicitly about equinumerous concepts. Our grasp of the cardinality function, I suggest, arises simultaneously with our grasp of the cardinality concept. Thereafter, during the course of the analysis, and as we form singular terms, we come to grasp more transparently the concept of the things the cardinality function yields as values. It would be implausible, I take it, to hold that we grasp a function, if only tacitly, without having a conception of its range. Yet to have a conception of its range just is to have a grasp of the sortal concept.

Hence we have not undermined the motivation of *Grundlagen* §64. We have simply understood it in a way that past commentators have not. Furthermore, in doing so, we have dispelled the worry about informative identity statements vis-à-vis strong hermeneutism left outstanding from the previous chapter.

§5.4.1. Objection 2: *Explanatory Priority*. Critics might further claim that our reconstruction violates *explanatory priority*. Our proposal is that the concept of cardinal number is grasped, if only tacitly, on apprehending the thought in the form '\( F=G \)'.\(^{65}\) Yet *priority*, as traditionally construed, demands that we come by

\(^{65}\) This, recall, is the dilemma that Dummett, Hale and Wright have been concerned with.
the concept only by transforming the thought, $F = G$, into the thought, $N_x:F_x = N_x:G_x$, and that it is only through this transformation that the concept is grasped.

What is essential to Frege's *priority*, I take it, is what we might call 'conceptual parasiticity'. This is that we cannot grasp the concept of cardinality without a grasp of the thought, $F = G$. That is, we must have a grasp of what it is for a one-one mapping between first-level concepts to obtain. However, the objection in question only arises given a narrow, albeit widespread, reading of what priority means. This is that the grasp of the cardinality concept is *subsequent* to our grasp of $F = G$. By understanding *priority* in terms of conceptual parasiticity, we allow that a tacit grasp of the cardinality function may be had simultaneously with our grasp of a one-one correlation obtaining between first-level concepts.

Such indeed is what Frege himself, I think, might have had in mind when responding to Husserl's view that we often just learn what numbers are by counting. According to Husserl, Frege's *explanatory priority* gets matters the wrong way round. Grasp of the cardinality function is prior to grasp of the equinumerosity relation. If we ask an infant how many apples there are on the table, he is unlikely to proceed by first establishing that a one-one mapping obtains between the concept apple and some other concept, and secondly by carving up the content in order to assign numbers to concepts. More likely the infant will simply specify some number directly by counting. But as Frege remarks, in counting these objects directly, we grasp a one-one correlation between the things counted and the numerals used in counting: it is a precondition for counting. Here it would not be implausible to read Frege as relying on the idea that the grasp of the number concept is simultaneous with the grasp of the relation of equinumerosity obtaining between concepts. At any rate, Frege himself should have no objection to my notion of conceptual parasiticity. The traditional construal of *priority* involves a temporal notion—*viz.* the grasp of the 'new' concept is subsequent to the carving—whereas the notion of conceptual parasiticity makes plain that the former is inessential.
§5.4.2. Objection 3: The Absurdity of Unwitting Reference. Thirdly the critic will object that my interpretation of Grundlagen §64 encounters the same problem as C. Wright's initial interpretation faced. Again, the alleged problem arises given the *sameness of thought*—qua strong compositionality. According to Wright, the latter commits one to the 'absurd' view that a thinker can unwittingly refer to an object even though the thinker lacks the ability to identify it. If 'a // b' and 'The dir(a) = the dir(b)' express the same thought, then I refer to directions when I express 'a // b': hidden reference view. Yet the whole point of carving was to arrive at the concept of direction. Without the latter sortal concept, I cannot identify the things to which, given *sameness of thought*, I am allegedly referring.

Part of this objection depends on the traditional construal of *explanatory priority* criticised above: that the sortal concept in question is arrived at subsequently to the grasp, and carving up, of F~G. On my proposal, we already grasp the concept of number, albeit tacitly, when we grasp the thought of the form F~G. Moreover, the objection overlooks a certain feature of Fregean *Sinn* mentioned in the introduction of this thesis. Fregean *Sinne* are platonic entities; one of whose features is that a reference relation obtains between itself and its *Bedeutung*. This relation of reference is essential to *Sinn*; it is also mind independent: it exists independently of our capacity to engage in mental acts, like judging, predicating and referring. Accordingly, the act of referring results in part from our having an explicit grasp of this reference relation; in particular from our having selectively attended to what it is that we grasp, and from our having possession of the requisite symbol—e.g. a singular term. When we grasp the thought in the form F~G, we tacitly grasp in varying degrees of perspicuity the cardinality function and concept of number. Given *explanatory priority*, this means that parts of Nx:Fx = the Nx:Gx are also parts of F~G. So to tacitly grasp the *Sinn* of the concept of number is to be in possession of a reference relation to the appropriate *Bedeutung*. But that need not involve the thinker in the above act

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67 As argued for in chapter 1.
of referring. The important distinction here, then, is between a reference relation, borne by *Sinn*, and the act of referring, borne by a mental act. Tacit grasp of a concept is insufficient for the thinker to refer to any of its instances; not least because the feature of selective attention to the object of thought, the kind of thing referred to, is lacking.

§5.5. *Sameness of Thought Revisited.* Finally, let us bring together and comment on what we have said about *sameness of thought*.

Dummett has noted, rightly, that in *Grundgesetze* Frege does not repeat his words used in ‘Funktion und Begriff’ that the Fregean Pair (A") ‘∀x (Fx ↔ Gx)’ and ‘{x:Fx}={x:Gx}’ ‘express the same *Sinn* but in a different way’. But he conjectures, wrongly, I think, that Frege must have changed his mind. ‘We must presume that by 1893 [Frege] had come to acknowledge to himself that the thesis he had so vividly expressed in *Grundlagen* for the [Fregean Pairs D" and N"] ... had been an aberration incompatible with his other doctrines’.\(^1\) One such doctrine, we are told, is that the *Sinn* of part of a sentence is part of the thought expressed by the whole, construed as *strong compositionality*, a thesis taken to be Frege’s on the grounds that:

‘The names, whether simple or themselves composite, of which the name of a truth-value consists, contribute to the expression of the thought, and this contribution of the individual [component] is its *Sinn*. If a name is part of the name of a truth-value, then the *Sinn* of the former name is part of the thought expressed by the latter name.’\(^2\)

The problems Dummett alludes to are those of the *incompatible trinity*.\(^0\) But as it happens, these problems are in fact independent of *strong compositionality*. Without my interpretation of the transformation strategy, the same problems

\(^{68}\) 1991b, p. 176. It is noteworthy that Frege nowhere says that a Fregean Pair is no longer to be regarded as expressing the same thought. Dummett is aware of this, but alleges that ‘Frege was never very good at confessing past errors’ (op. cit. p. 176.).

\(^{69}\) Cf. *Grundgesetze*, vol. i, §32. It is unclear, to my mind at least, that this passage does provide sufficient grounds for attributing *strong compositionality* to Frege.

\(^{70}\) Hale too regards *strong compositionality* as incompatible with the transformation strategy. The problems associated with the trinity of doctrines, he claims, ‘rely on ... the ... claim ... that the sense of a complex expression is actually composed of the senses of its constituents ...’. As seen, Hale and Wright seek to solve some of the problems by rejecting *strong compositionality*. (Unlike
would arise given a function-argument model that precludes the claim that constituent *Sinne* are literally parts of the composition of the thought: *weak compositionality*. We could understand ‘part of a thought’ as part of what is constitutive of the identity of a thought but deny that the whole is actually composed of the parts. What generates the incompatible trinity—apart from the traditional construal of *priority* and *motivation*—is the thesis that the grasp of the whole depends on the grasp of its ‘parts’. Yet if that thesis applies at all, as surely it does, it applies to both models. If we add the *understanding problem* to the incompatible trinity, and if we reject my interpretation, then we get the same dilemma as the one we started with.

**First horn.** If each member of a Fregean pair does, *à la weak compositionality*, represent the same thought, then trivially whatever is essential to the identity of the thought is essential to the identity of the other. Either the concept arrived at through the carving up process is essential to the identity of the thought expressed by the two types of sentences, or it is not. The concept of direction is essential to the identity of the thought contained in ‘The dir(a) = the dir(b)’. Again, if it is essential to the one, and if they represent the same thought, it is essential to the other. In that case, *explanatory priority*, and the very motivation of the strategy—both taken as traditionally construed—are undermined.

**Second horn.** Alternatively, given that the motivation—again as traditionally construed—is to arrive at a ‘new’ concept, the thought expressed by ‘Nx:Fx = Nx:Gx’ must involve a concept absent from the other. It would follow

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Dummett, Hale rightly hesitates to attribute *strong compositionality* to Frege, 1997, p. 249.)

Of *strong compositionality*, Dummett says that ‘...[it] means nothing’, he says, ‘if it does not mean that a grasp of the thought depends on a grasp of that constituent sense’ (1991b, p. 176). But this could be true of *weak compositionality*, *i.e.* where the thought is not composed out of its parts. Dummett would deny that this thesis applies to the function-argument model. At the level of thought, function and argument would map us to its values, the *Sinn* of a sentence. These cannot, he claims, be literal parts composing the thought, since, given this model, there would be many more ways of yielding the same thought. Moreover, functions are understood in terms of the values they yield. So for example, to define a function is to specify the values yielded and for what arguments. But that, complains Dummett, would be to grasp the thought in advance of grasping the parts. But the complaint is mistaken. It is true that we cannot define a function, so conceived, without first grasping its values, in this case the thought. But that is not to say that we cannot grasp the function in advance of grasping its value.

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that someone could understand the sentence ‘F=G’ and yet be incapable of understanding the sentence ‘Nx:Fx = Nx:Gx’, for the reason that the thinker lacks the cardinality concept. Moreover, given weak (strong) compositionality, together with logical form, our Fregean Pair involves different Sinne, and therefore express different thoughts. In that case, we have the understanding problem.

Most commentators reject sameness of thought, as traditionally construed, because it is taken to be counter-intuitive: the parts of the thought, Nx:Fx = Nx:Gx, it is alleged, are different from the parts of the thought, F=G—even if the thoughts do pass the immediacy criterion. Judging the latter defective, Dummett has proposed amending it to: two sentences express the same thought just in case anyone who grasps the thought expressed by the one sentence thereby grasps the thought expressed by the other sentence. But to do so, Dummett claims, is to block room for analysis: the motivation problem again.

In response, I believe that it is not unlikely that Frege’s criterion already involves Dummett’s qualification. In any case, the Fregean Pairs meet the amended immediacy criterion, given the presence of tacit grasp. To grasp the thought of ‘a // b’ is to have some grasp of the concept of direction. Mutatis mutandis the other cases. In N, if someone recognises that there are just as many books on the table as there are pens, he tacitly knows that the number of the one kind of thing—the multiplicity or magnitude of one group—is the same as the number of the other kind of thing—the multitude or magnitude of the other group. If he were not to know that, then no amount of carving up the thought, F=G, would enable him to become numerically literate. Again, we would have use without understanding. Indeed, it would be far from clear that the thinker had fully grasped F=G.

Furthermore, pace Dummett, analysis is not blocked by the qualified immediacy criterion. As said, analysis in this case is a means of rendering a

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73 While necessary, it is not sufficient. The reason, says Hale, is ‘that it leads to violations of compositionality ...’. 1997, p. 254.
75 Again: two thoughts are the same just in case anyone who recognises the one as true immediately recognises the other as true, for the reason that someone who grasps the one thought thereby grasps the other.
thought in such a way as to attain a clear and distinct grasp of what is tacitly, though fully, grasped. In the case of N", for example, analysis is a means whereby the thinker becomes numerically literate—able to refer to, or quantifier over, and predicate things of, numbers conceived as objects, as well as to engage in certain kinds of conscious reasoning.

In diagnosing the dilemma, we need to distinguish *strong compositionality* from a further, connected, thesis: *isomorphism*. This is that the structure of a sentence mirrors the structure of a thought. That Frege held this thesis is not in dispute here. What I would dispute is the claim that it conflicts with *multianalysability*. Any conflict would depend on taking *isomorphism* to mean that a thought has a unique structure and that a single sentence exhaustively reflects it, so that any sentence with a different structure would represent a different thought.76 This way of regarding *isomorphism* prohibits the notion of ‘polymorphous thoughts’,77 a notion central to Frege’s explanation of how we grasp numerical content, and thus necessary to Frege’s variety of logicism. Since the stronger reading of *isomorphism* is neither of these things, then, given our weaker reading, we should reject it.78

Finally, important to *sameness of thought*, apropos a Fregean Pair, is not just that each of a constituent thought should have the same truth-conditions, same ontological commitments. We need also to say (i) that the same conceptual resources (same *Sinne*) are involved in grasping the thoughts of both sentences, and that (ii) what corresponds to these conceptual resources at the level of *Bedeutung* figure in the truth conditions. The importance of (ii) is the following.

76 ‘To say that the sense of a sentence is composed out of the sense of its constituent words is to say, not merely that, by knowing the sense of the words, we can determine the sense of the sentence, but that we can grasp that sense only as the sense of a complex which is composed out of parts in exactly that way; only a sentence which had exactly that structure, and whose primitive constituents corresponded in sense pointwise with those of the original sentence, could possibly express the very same sense.’ Dummett, 1981a, pp. 378-79; cf. 1991b, pp. 152-53.

77 To use of phrase from Harold Hodes’s ‘The Composition of Fregean Thoughts’ (1982).

78 Harold Hodes (ibid) rightly argues against attributing *strong isomorphism* to Frege in favour of polymorphous thoughts. Not only is the attribution of strong isomorphism to Frege unsupported by the text, it would commit Frege an ‘unpalatable multiplicity of distinct thoughts and of ambiguities in sentences expressing these thoughts’ (op. cit. p. 162). However, he thinks of the same thought as being composed out of different *Sinne*, p. 162, p. 171, p. 172, p. 173. I reject this since it would give rise to the *understanding problem*. We should instead see a thought as
It allows us to isolate those conceptual resources involved in grasping a thought that are plausibly not part of the identity of that thought—or whose (corresponding *Bedeutung*) cannot be said to be part of the truth conditions—from those resources that are. To deny (i) is to lapse into the *understanding problem*. Accordingly, we might say that while ‘\(Nx:Fx = Nx:Gx\)’ and ‘\(F=G\)’ say different things, because each sentence is explicitly about different things,\(^9\) the thoughts involved are the same:*\(^8\) ‘[they] express the same Sinn but in a different way’.

Hence we should regard Frege as having worked with a notion of content more fine grained than rival interpreters have supposed. It is more fine-grained to the extent that the concept sought after and arrived at on the left-hand side of the biconditional is part of the content of the original sentence, awaiting to be disclosed by analysis. Thus contrary to Dummett, it is unlikely that Frege changed his mind about the *sameness of thought*—not at least if what is essential to that claim is sameness of conceptual resources. Not only was there no need to do so; it would have made mysterious how the process could succeed.

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\(^7\) As Frege said of the two sides of what was to become axiom V. Cf. ‘Function and Concept’ (1891), p. 27, *TWP*. Cf. Beaney, 1996, 244-45.

\(^8\) Hale and Wright would take the right hand side of (1) and (2) as instantiating the kind of coarse-grained content that they have in mind. While the two left-hand side thoughts are different, given Strong Compositionality, they have a common content (taken at the level of thought).

1. ‘There are husbands \(\leftrightarrow (\exists x) (\exists y) (x \text{ is male } \& y \text{ is female } \& x \text{ is married to } y)\),
2. ‘There are wives \(\leftrightarrow (\exists x) (\exists y) (x \text{ is male } \& y \text{ is female } \& x \text{ is married to } y)\).

That they differ strongly compositionally, ‘follows’ from the fact that ‘someone might wonder whether there are husbands iff there are wives, without wondering whether there are husbands iff there are husbands’. But equally it could follow from Strong (or Weak) Compositionality, together with our reconstrual of *Isomorphism*. The different sentences reflect different structures of the same thought, allowing the thinker to attend to, have propositional attitude toward, certain combinations of constituent *Sinn* of the same thought, and not others.

Hale maintains that the difference in compositional *Sinn* arise because the left-hand sentences are formed in a different way, albeit from the same basic predicates—those discernible in the right-hand sides. Clearly the way we form the sentences is different. We extract different complex predicates from the right-hand, forming a sentence by existential closure, the predicates respectively:

1. \(\exists y (\xi \text{ is a male } \& y \text{ is female } \& \xi \text{ is married to } y)\)
2. \(\exists x (x \text{ is male } \& \xi \text{ is female } \& x \text{ is married to } \zeta)\)

But it is implausible, I think, to deny that a grasp of the right-hand side involves the grasp of the concept of husband and concept of wife. The left-hand would tell the lie against those who would substitute *sameness of thought* for a more coarse-grained notion, since it is claimed that the right-hand side instantiates the kind of notion required for carving. See Hale, 1997, p. 253.
Chapter 6
Frege’s Epistemological Logicism.

§6.1. Introduction. I have been arguing that Frege’s logicism was an attempt to show that ordinary arithmetic was Concept-Script arithmetic in disguise, and that we in some sense fully grasp the body of propositions that ground our arithmetical practice, despite the disguise. That is, Frege’s project was not, contrary to popular belief, an attempt to show how our arithmetic could be replaced by an arithmetic more perfectly formed, and one founded on purely logical notions. What we find in Frege’s Concept-Script is what we ordinarily grasp. That interpretation is made considerably more plausible, I argued, once we recognise a hitherto neglected doctrine that underlies Frege’s project: the recovery of tacit grasp. In this light, we said that, for Frege, our grasp is partly explicit, partly tacit, but in any case full, and that Concept-Script represents a more perspicuous rendering of the otherwise familiar arithmetical propositions we ordinarily possess: a thesis we called ‘strong hermeneutism’.

There were other exegetical advantages in bringing out the doctrine of the recovery of tacit grasp, apart from reinforcing the strong hermeneutic claim. For instance, we saw in the last chapter that the doctrine allows us better to understand what is involved in carving-up judgeable content as described in Grundlagen §64. And in the chapter before that, we saw how the doctrine could help us to lay bare the connection Frege drew between fruitful definitions, and extensions of knowledge, specifically regarding analytic propositions. In this, the final chapter, I offer still further support for the hermeneutic hypothesis, and point out an advantage that the strong version of this hypothesis has over its variants. In doing so, I draw on a debate regarding the sense, if any, in which Frege’s project was epistemological. The advantage in question concerns the explanatory value of Frege’s project. In particular, strong hermeneutism offers one way, amongst others, in which we might regard Frege’s logicism as an epistemological project. I show this by bringing out a connection in Frege’s work between the recovery of full tacit grasp and the attainment of a priori knowledge of arithmetical truths.
Again, it is argued that Frege sought to show how a priori justification of our judgements, whose contents are our ordinary arithmetical propositions, could be achieved by attaining to the kind of perspicuity of grasp he, arguably, set out to achieve.

I show that attaining to this kind of perspicuity meant showing that ordinary arithmetical concepts are purely logical in nature, and that ordinary arithmetic truths are purely logical truths. It meant defining what could be defined and proving what could be proved. And it meant revealing the self-evident truths on which the truths of ordinary arithmetic depend, and seeing in what their self-evidence consists. One way in which Frege would have us to see the nature of these truths is by having us see the consequence to the thinker should their truth be denied. The consequence of so doing, Frege observes, would be nothing less than the breakdown in judgement and rationality. For the laws of logic were in part constitutive of rationality and judgement; they were constitutive of the mind: 'mind' in the sense of an abstract and eternal ideal, not in any psychological sense of that term. Given this, to attain an insight into the structure and nature of the truths of arithmetic—a perspicuous grasp—was to attain a privileged reason for taking those propositions to be true. What would make the reasons privileged would be that they were at the same time reasons for those propositions' being true.

So a further advantage of the strong hermeneutic reading lies in bringing out a possible explanatory value of Frege's logicist project. Now according to the milder hermeneutic claim, we have for Frege, only a partial grasp of the Sinn of ordinary arithmetical terms: it is indefinite, indeterminate. The trouble is that this makes it unclear how Frege could have taken himself to show that our ordinary judgements or beliefs are analytic a priori. If our grasp is so defective, then it is to be wondered how we are to attribute to the ordinary thinker those propositional attitudes represented in Concept Script. Of course, a similar problem besets the various revolutionary theses. Advocates of the revolutionary theses try to down play the extent to which Frege aimed at explicating our arithmetic. That they fail to do so became apparent in chapter three; it becomes still more apparent when we consider Frege's project as epistemologically conceived. These problems just
described do not arise, of course, for strong hermeneutism, since, according to it, our grasp is full and thus determinate.

There is, however, a potential obstacle regarding the above advantage claimed for our strong hermeneutic reading. This is that a number of commentators take exception to characterising Frege's logicism as epistemological, rejecting as they do the view that Frege was concerned with the justification of belief or judgement. They see Frege's logicism instead as an investigation into the nature of arithmetical truths—or what kind of truths could be used in reconstructing arithmetic. On this view, although Frege spoke of 'justification' he meant proof, understood principally as the demonstration of dependencies between truths. Accordingly, proof was not a way of exhibiting relations between beliefs in some kind of foundational structure. Moreover, advocates of this anti-epistemic view might make two further claims. First, Frege's use of the predicates 'a priori' and 'a posteriori' is divorced from the epistemological content we ordinarily take them to have. Second, such an epistemological project would have transgressed Frege's psychologism. Other commentators argue that, while Frege's logicism is epistemological, his concern extends no further than to inferential justification, that he had nothing substantial to say about justifying our judgements of primitive truths. We show that, although Frege's remarks on the epistemology of primitive truths are sparse, what he does say, both implicitly and explicitly, suffices to undermine the anti-epistemic view.

The plan of this chapter is as follows. Section two offers an outline of some of the main elements of epistemology that I believe informed and helped motivate Frege's logicism. At the same time, some of these considerations help further to support the general hermeneutic hypothesis. In section three, I elaborate on the further advantage—the explanatory value—of the strong hermeneutism over the other possible readings of Frege's logicist project. Then in sections four and five, I provide an exposition and response to the main objections to the epistemological view argued for in this chapter. Finally in section six I offer an elaboration of this view, and emphasise the connection between attainment of a perspicuous grasp of our ordinary arithmetical truths—the recovery of tacit

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1 They were at the same time mind-independent and eternal. See chapter 1.
grasp—and what we can take Frege to have maintained regarding the source of justification for our judgements of primitive truths.

§6.2. Elements of Frege’s Epistemological Logicism. Perhaps one of the plainest indications, at least at first sight, that Frege’s logicism was epistemologically conceived, is his announcement that he wishes to show that the truths of arithmetic are *a priori* analytic. Frege understood these latter notions in terms of the deepest reason or ground (*Grund*) that could provide the justification for making a judgement about a proposition’s truth-value.

‘These distinctions between a priori and a posteriori, synthetic and analytic, concern ... the justification for a judgement. ... When a proposition is called a posteriori or analytic in my sense, this is ... a judgement about the profoundest reason upon which is based the justification for holding it to be true (*worauf tiefsten Grunde die Berechtigung des Fürwahrhaltens befuht*).’

Traditionally, an *a priori* justification of a belief or judgement is independent of experience; and where the judgement, or proposition involved in the judgement, is analytic, the source of the *a priori* justification is said to lie in understanding the content alone. For example, justification for believing the proposition that all bodies are extended might reside in the grasp of that content, since to deny its truth would involve a self-contradiction—where ‘body’ means, e.g. ‘a figure extended in space’. Similarly, Frege believed that we could achieve an *a priori* justification simply by having a proper grasp of the content of arithmetic. We are justified in believing arithmetical truths since denial of them would lead to a similar violation of the laws of logic. That, at least, is what I shall argue.³

Another element in Frege’s epistemology is that his logicist system takes a traditional Euclidean form: it is conceived as an axiomatic or foundational system of truths and knowledge. For guiding Frege’s logicist project is his belief that arithmetic has a foundational base: a small number of axioms on which the truths

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²*Grundlagen* §3, p. 3; Thiel, §3, p. 15.
³ This is not to say that Frege thought, as the logical positivists did, that arithmetical truths were analytic in the sense of being true in virtue of meaning. Rather it is to say that epistemic warrant was arrived at, for Frege, purely from our understanding of the contents. Again, the purpose of this section is simply to lay down the main elements of Frege’s alleged epistemology. I give a fuller explanation of them in section six after I have deflected some objections to their epistemological interpretation.
of arithmetic depend. What in part makes this project epistemological is his belief that in the execution of a gapless proof that proceeds from axioms according to purely logical rules of inference, our judgements involving these foundational or primitive truths would confer justification on our judgements involving the derived arithmetical truths. As Frege says, justification of an arithmetical judgement is a matter of ‘finding the proof of the proposition and following it back to the primitive truths’.

'I became aware of the need for a conceptual notation when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. Only after this question is answered can it be hoped to trace successfully the springs of knowledge upon which this science thrives. Even if this question belongs largely to philosophy, it must still be regarded as mathematical. The question is an old one: apparently it was already being asked by Euclid.'

Note the, at least apparent, presumption that it is our mathematics that he has in mind, not some radical or even partial replacement.

Frege’s conception of an axiom warrants some spelling out since it differs markedly from an understanding of an axiom widespread today. For Frege, axioms are general, self-evident truths, where the latter notion involves dual-properties: what we might call a ‘cognitive’ and ‘non-cognitive’ element. Both these elements are captured in the claim that ‘... it is part of the concept of an axiom that it can be recognised as true independently of other truths.’ The cognitive element is that property of the truth that makes it possible for a rational thinker to recognize from a grasp of the content alone that the proposition in question is true. ‘[T]he truth of a logical law is immediately evident of itself, from

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4 On pursuing the primitive, cf. §64; cf. pp. 173-74, pp. 289-90, CP. Frege also recognised logical simples or indefinables, the constituents of a definiens, which themselves were taken to be self-evident in an objective sense (see below). Cf. ‘The Law of Inertia’ (1891), p. 133, CP; ‘Foundations of Geometry II’ (1906), p. 302, CP.
5 Cf. ‘Frege to Huntington’ (circa. 1902), p. 57, BW.
6 Grundlagen §3, p. 4.
7 ‘On Mr. Peano’s Conceptual Notation and my own’ (1897), p. 235, CP (my emphasis); cf. ‘Frege to Hilbert’ (27.12.1899), p. 37, BW.
8 For example, axioms might be thought of as neither true nor false. They might be regarded as rules, or implicit definitions. Or they might be taken to be apt for truth-value, but capable of being false. Moreover, their status as axioms might be regarded as merely a matter of which truths we decide to take as the starting point of a proof.
9 ‘On Euclidean Geometry’ (1899-1906), p. 168, PW.
the Sinn of its expression.'\textsuperscript{10} The non-cognitive element is that property of the proposition that makes its truth depend on no other truth.\textsuperscript{11} These elements, I believe, lie behind Frege's remark that such truths "neither admit nor require proof".\textsuperscript{12}

Something about both these elements can be gathered from his observation that

"After deserting for a time the old Euclidean standards of rigour, mathematics is now returning to them, and even making efforts beyond them ... in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that earlier passed as self-evident (selbverständlich).\textsuperscript{13}

Our impression of a truth's obviousness, our conviction that a proposition can be just seen to be true, our impression that no proof is possible: this is not enough to qualify the truth as self-evident. Furthermore, just as a proposition might appear obvious to us without being self-evident—e.g. the simplest of arithmetical truths—so might the converse be true. A truth might be self-evident even though it does not immediately strike the thinker as obvious. Thus for example, Frege says of his definitions of number that they will 'perhaps be hardly self-evident at first'. Something similar is suggested, I think, by his initial qualms about axiom V, as when he says that "... it is not as self-evident as one would wish for a law of logic".\textsuperscript{14} Why might a proposition not be self-evident at first? Surely because the thinker may lack a perspicuous grasp of it. Otherwise put, attaining to sufficient

\textsuperscript{10} 'Compound Thoughts' (1923-25), p. 405, CP.

\textsuperscript{11} Frege acknowledges that 'unprovable' can mean unprovable relative to a system of true and certain thoughts. A true and certain thought can be an axiom if selected by a thinker as a starting point in a proof. While in this case it is unprovable, in another system the same true and certain thought could be a theorem, and thus provable. Cf. Begriffsschrift §13, CN; cf. 1914, p. 205, PW. Nevertheless it is clear in the above that primarily Frege meant something else by axiom: 'recently, a vicious confusion has arisen over the use of the word 'axiom'. I therefore emphasize that I am using this word in its original meaning.' 'A new Attempt at the Foundations of Arithmetic' (1924-25), p. 278, PW.

\textsuperscript{12} "... die selber eines Beweises weder fähig noch bedürftig sind ... '"Grundlagen', p. 15 (Thiel); p. 4 (Austin).

\textsuperscript{13} Grundlagen, §1, p. 1; my emphasis. Für Vieles wird jetzt ein Beweis gefordert, was früher für selbverständlich galt, p. 13. Cf. §90, p. 102 'Often ... the correctness of such a[n] [inferential] transition is immediately self-evident to us, without our ever becoming conscious of the subordinate steps condensed within it.' Cf. 1914, p. 204, p. 205, PW.

\textsuperscript{14} Also, in Grundgesetze, pp. 3-4 (p.vii), Frege acknowledges that readers might have doubts about axioms V. Moreover, he was aware that other logicians rejected his system of logic. Cf. Grundgesetze, vol i, p xi; p. 7 (1967); cf. 'On the Foundations of Geometry I' (1903), p. 273 ff, CP; 'Sources of Knowledge of Mathematics and natural Sciences' (1924-25), pp. 168-69, PW.
clarity of grasp may take much 'intellectual labour', so that what is self-evident in the cognitive sense might not coincide with self-evident in the non-cognitive sense of this notion. It is in this context that we should understand Frege observation of our becoming 'aware of the logical justification for what we think'. I shall say more about this later.  

It was with this conception of a basic or primitive truth that Frege sought to reconstruct arithmetic. His inquiry into the nature of the truths of arithmetic arose, however, not (principally at least) from a concern that the 'moral conviction' that the truths of the simplest numerical propositions were self-evident might desert us. Nor did Frege fear that appearances of countless successful applications of arithmetic might turn out to be illusory.

'The aim of proof is ... not merely to place the truth of a proposition beyond doubt, but to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? The further we pursue these inquiries, the fewer become the primitive truths to which we reduce everything.'

It was not certainty in the face of present or possible future doubt that Frege sought; the stability of this certainty was granted. Rather he sought its source. Frege believed that it could be found in a small number of self-standing truths, and that it was from these that one's experience of obviousness should stem. The most

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15 'Logik' (1879/91), p. 1, PW.
16 Certainty is another dual-aspect property. There is certainty in the subjective sense, and there is a property of certainty intrinsic to the proposition in question, both of which can belong to non-primitive truths as well. A proposition is certain in both senses if it is such that anyone who has a sufficient grasp of it finds it to be beyond reasonable doubt. Frege, for instance, believed that arithmetical truths were certain. He alludes to the former sense of certainty in Begriffsschrift §1, where he suggests that degrees of certainty about a truth correspond to the different modes of its discovery, or different modes of arriving at the truth in question. Cf. T. Burge, 'Frege on Knowing the Foundations' (1992), p. 312.
17 I believe that the whole spirit of Frege's logicism, his search for the nature of the truths of our arithmetic, and of our possible knowledge of them, bespeaks of a search for the unprovability of certain of our thoughts.
18 Grundlagen, §2, p. 2 (my emphasis); cf. op. cit. p. 102; cf., Grundgesetze, vol. i, p. 4.
19 Nevertheless, Frege expresses concern that contradiction might be found, and of arithmetic collapsing. What Frege might have had in mind is that the systems constructed by other mathematicians and philosophers are threatened with collapse due to their lack of rigour, rather than the collapse of arithmetic. On the other hand, Frege might have had in mind arithmetic in the broader sense, including real analysis, rather than arithmetic simply of the natural numbers. After all, concerns about consistency and cogency of real analysis were behind the movement toward greater rigour referred to by Frege. Cf. Demopolous, 'Frege and the Rigourization of Analysis', pp. 76-77. But as we noted in chapter 2, concerns with consistency and cogency had been dealt with by the time Frege wrote Grundlagen.
perfect proof would be a revelation about the nature of the 'immovable boulder': an insight into the source of epistemic justification. Thereby would our mere 'moral convictions'—feelings of the obvious—be met by an ideal justification. Moreover, reference to an immovable boulder indicates, I believe, that Frege took our ordinary arithmetic as already founded upon a small number of truths whose features, then, it was his task to reveal.

This brings us to a further element in Frege's epistemology. It lies with his reliance on a threefold division of 'sources of knowledge' (Erkenntnisquelle): sensory, geometric, and logical. Frege also uses the term 'ground of knowledge' (Erkenntnisgrund). There is good reason to read 'source' as meaning 'ground' or 'reason': i.e. that by means by which our judgements of these truths or propositions can be justified.

'I call axioms propositions that are true but are not proved because our knowledge of them flows from a source very different from the logical source, a source which might be called spatial intuition.'

'I have set myself the goal of basing arithmetic on logic alone. For this it is essential to exclude with certainty everything derived from other sources of knowledge (intuition, sensible experience). And for this it is again necessary to produce for each proof a chain of inferences without gaps, so that every transition proceeds according to a known logical law.'

'What I regard as a source of knowledge is what justifies the recognition of truth, the judgement.'

We find similar characterisations in papers written following the collapse of his logicist system when, abandoning the view that numbers are logical objects, Frege continued his search for the ultimate ground of arithmetic. For instance, in one of his latest works, where he takes our arithmetical knowledge as having a geometric source, Frege speaks of an 'a priori mode of cognition' that does not have to 'flow

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21 ‘Frege to David Hilbert’ (27. 12. 1899), p. 37, BW.
22 ‘Frege to Huntington’ (circa 1902), p. 57, BW. Cf. ‘Frege to Anton Marty’ (1882), p. 100, BW; and ‘Frege to Richard Hônigswald’ (26. 4. 1925), p. 55, BW: 'And so it happened that after the completion of the Basic Laws of Arithmetic the whole edifice collapsed around me... Originally, I pointed out the sources of knowledge without further justification. In its terse succinctness it was to have a hard-hitting effect ...'.
23 'Sources of Knowledge of Mathematics and the mathematical and natural Sciences' (1924-25), p. 267, PW. My emphasis.
from purely logical principles as I originally assumed'. Moreover, it is clear that, prior to his awareness of the inconsistency, Frege understood axioms V as the means by which we apprehend or grasp logical objects: 'by means of our logical faculties we lay hold upon the extension of a concept.' Axiom V was meant to be an instance of the logical source of knowledge; and so it is clear that Frege's deployment of axiom V was intended to justify existential claims about logical objects. In addition, some of Frege's thoughts about the role of axiom V were earlier expressed in a way that, I believe, further signals his interest in pursuing a hermeneutic project: '[i]n arithmetic we ... come to know ... [objects] given directly to our reason and, as its nearest kind, utterly transparent to it.'

The final element in Frege's epistemology concerns his attitude to our knowledge of self-standing truths. Reason by itself could afford us knowledge of the primitive logical truths and inference rules, since reason is the 'logical source of knowledge'. Frege believed that once the basic truths had been clearly demarcated and deployed in a logically perfect proof, justification of our judgements involving them would become clear. Ultimately the warrant or justification of our judgements of all arithmetical truths, as well as the kernal of self-evident truths, would be explained simply in terms of our understanding the content of the latter primitive truths.

'The truth of a logical law is immediately evident from itself, from the Sinn of its expression.'

'... we can inquire, on the one hand, how we have gradually arrived at a proposition and, on the other hand, how it is finally to be most securely

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26 As we saw in chapters one and five, Frege is exercised by the question how numbers are given to us if we can have neither an idea nor intuition of them. There we also saw that this question and, Frege's answer to it, can be construed as an epistemological question regarding how a thinker might be justified in believing that numbers exist.

27 Cf. Grundlagen §105, p. 115; Thiel, p. 104.

28 As we saw in chapters one and five, Frege is exercised by the question how numbers are given to us if we can have neither an idea nor intuition of them. There we also saw that this question and, Frege's answer to it, can be construed as an epistemological question regarding how a thinker might be justified in believing that numbers exist.

29 'Compound Thoughts' (1923-26), p. 405, CP. By 'logical law', Frege has in mind a basic logical laws.
grounded ... the answer [to the second question] is connected with the inner nature of the proposition considered.\textsuperscript{30}

But how merely from our grasp of their contents can primitive truths confer \textit{a priori} justification on the judgements in question? With the grasp of the content of the self-evident truths the thinker would see that they were true, and moreover, would see that about this there could be no reasonable doubt. Again, we will say more about the connection between understanding and justification \textit{vis-à-vis} logical source of knowledge and primitive truths later.

Finally, Frege recognised that a major obstacle to the above epistemological goal lay in the lack of perspicuity with which we grasp the simplest arithmetical truths.\textsuperscript{31} We do not readily discern the logical structure of the basic concepts of arithmetic: we either take them as logically simple, or take them as definable but in the wrong way. Again, I think we have here another indication that the hermeneutic hypothesis (some variant at least) is on the right lines.

"In order that such an [epistemological] undertaking should succeed the concepts must naturally be conceived distinctly."\textsuperscript{32}

Clearly without an adequate insight into the nature of our arithmetical truths, Frege would be unable to show how, simply by grasping the content of a small number of primitive logical laws, the truths of our arithmetic could be known \textit{a priori}. He would be unable to show how, by this means, the most certain kind of knowledge could be had.

\section*{§6.3. An Advantage of Strong Hermeneutism: Explantory Value.} Before clarifying and defending the above picture of Frege's motivation, let us note an advantage, not considered in previous chapters, for advancing the strong hermeneutic reading of it. To start with, we need to be clear why it matters that the revolutionary hypothesis—\textit{viz.} that neither \textit{Sinn} nor \textit{Bedeutung} are preserved in Frege's logicist system—be rejected.\textsuperscript{33} If the revolutionary view were right,

\begin{itemize}
\item \textsuperscript{30} \textit{Begriffsschrift} (1879), \textit{CN}; my emphasis. As we shall see more fully below, Frege has in mind both inferential and non-inferential justification.
\item \textsuperscript{32} \textit{Grundgesetze}, vol. i, p. 128, \textit{TWP}.
\item \textsuperscript{33} As discussed in chapters 1 and 2 above.
\end{itemize}
then there would be a tension inherent in the very motivation of Frege’s logicism. As we have seen, here and in other chapters, Frege speaks of gaining an insight into the structure and nature of ordinary arithmetic, and of laying bear its epistemological foundations. He wished to show that our arithmetic was analytic a priori. If neither Sinn nor Bedeutung were meant to be captured, then, even if his logicist system were consistent, we could not say that Frege had shown that our arithmetic was thus and so. Some advocates of the revolutionary view are not unaware of this tension. For example, following Resnik, Weiner has argued that Frege sought to reconstruct the concept of number, and indeed the whole of arithmetic, and that questions about the relation between the reconstructed concepts were dismissed by Frege on the grounds that they could not be answered. For the purposes of his logicism, Sinn is regarded as what is relevant to determining inferences, and that to this extent ‘the sense of pre-Fregean sentences of arithmetic is included in the Fregean understanding of these sentences’. Beyond that, no further constraint of adequacy is required regarding the replacement of ordinary arithmetic for that found in Concept-Script.

But it is not true that the same inferences would be preserved under the two systems. To the radical revisionist, the sentences of Concept Script arithmetic have mere shells of propositions for their contents. So the inferences drawn would be between sentences. By contrast, the inferences drawn using Concept Script were to be between thoughts, the terms for whose constituents, like ‘one’, would be abbreviations for a logically complex sign. So whether his logicist proof is seen in terms of inferential linkages between sentence and their structures, or between thoughts and their structure, the difference between the

34 This is not to say that without the epistemological reading, Frege’s logicism would be unmotivated. It seems to me that simply proving that arithmetical truths are derived logical truths would, from a mathematical point of view, be sufficient to make the foundational project worthwhile. But I think that nonetheless Frege’s principal concern was to show that, contrary to Kant, our arithmetic is analytic a priori.

35 On Resnik’s reading, what we grasp prior to the reconstruction is too unclear to be compared to the content of Concept Script arithmetic, and that for this reason Frege’s interest in the analyticity of arithmetic turned solely on the nature of the reconstruction. Unfortunately, he does not explain what he takes to be the nature of the reconstruction. Frege and the Philosophy of Mathematics (1980), p. 185.

36 Op. cit. 124. She says that in general the Sinne of everyday expressions and those of Frege’s logically perfect language cannot be identical because the Sinne of everyday expressions are imperfect. Cf. p. 228, p. 264. It is surely not, for Frege, Sinne that are imperfect, but only the means of their expression. She appears to side between these two notions, see, e.g. p. 265, 1990.
structure of the inferences of those of Concept Script arithmetic and those of its unreconstructed counterpart, would be stark. Worse, Frege does not in fact recognise inferences from sentences whose contents are not fully propositional or without a truth-value: he believes they would be mere ‘pseudo-inferences’. Hence, if the revisionist is right, then it is unclear how Concept Script arithmetic could confer explanatory value.

One might think that another option would be to regard the status of Frege’s definitions as a mixture of the analytical and conventionalist. So for example, concerning Frege’s definition of cardinal number, Dummett maintains that it ‘ow[es] nothing to the received sense of the [ordinary] expression’ and that Frege takes a conventionalist stance at this juncture. Similarly with the Sinn of all our numerals. In practice, the definition of the cardinality operator, and thus that of cardinal number, serves two purposes. It fixes the Bedeutung uniquely; and it serves to yield the equivalence ‘N(x): F(x) = N(x): G(x) ↔ F=G’. In deriving the truths of arithmetic, it is this latter equivalence that is deployed, not the definition of the cardinality operator. According to Dummett, it is only this that embodies ‘the received sense of the expression “the number of ...”’. As such, he argues that

‘... if we grant the legitimacy of defining all terms for numbers by the use of that operator, Frege’s demonstration of the analyticity of those laws is in no way impugned by the admitted partial artificiality of the definition he gives of it’.

The trouble with this view is that Frege sought to show that the truths of arithmetic tout court, not merely the laws of arithmetic, were analytic, and that they can be known a priori. As we have seen, this meant showing that the common store of arithmetical thoughts, which Frege believed we share, are those that are translated into Concept-Script. Thus Dummett’s claim is at odds with this aim. And it is at odds with Frege’s remark, clearly analogous to the case of arithmetic thoughts, that ‘Pythagoras’ theorem expresses the same thought for all men, whereas everyone has his own mental ideas’.

Or indeed from false thoughts. See below.

1991b, pp. 178-79.


Cf. ‘Frege to Husserl’ (30.10.1906), p. 67, BW.
theorem, like other arithmetical thoughts we grasp, would not be translated into Concept-Script; only laws governing it would be. Hence Frege would have shown that a different theory is analytic, and known *a priori*.

Doubt about the explanatory value of Frege’s Concept Script might also beset the mild hermeneutic view. Recall that on this view, we have only a partial grasp of ordinary arithmetical thoughts. As pointed out in chapter three, there is some lack of clarity about what, its principal advocate, T. Burge, means by ‘partial grasp’. On the one hand, he believes that, for Frege, all mathematical understanding and usage of our ordinary arithmetical terms fail to determine a *Sinn* or *Bedeutung*. That is, conventional significance of a term fails to ‘fix’ its *Sinn* or *Bedeutung*, and that this is a consequence of vague understanding and usage of those terms. Despite this, there is, he thinks, a sense in which, for Frege, vague mathematical terms do ‘express’ definite *Sinn* and have a definite *Bedeutung*. This is in virtue of the fact that mathematical practice is founded on, and depends upon, them. It is ‘founded on a deeper rational than anyone has previously understood. To say, as Frege says, that “Number” does denote a concept and does express a *Sinn* is to say that the ultimate foundation and justification of mathematical practice supplements current usage and understanding of the term in such a way as to attach it to a concept and sense.’ We have in chapter three already offered reasons against Burge’s view. A further point against it is that it makes unclear whether the beliefs or judgements of the thinker, engaged in ordinary arithmetic, are the same beliefs or judgements had when the thinker engages with the contents of Concept Script. For it is compatible with Burge’s account that while our mathematical practice is grounded on the contents of Concept Script, still our ordinary arithmetical judgements are not the same as those had when engaged in Concept Script arithmetic. Accordingly, there is a consequent unclarity regarding the sense in which, given the mild-hermeneutic view, Frege could speak of having shown that our ordinary arithmetical beliefs and judgements are justified *a priori*. And Burge himself says that, for Frege, ‘[s]ometimes

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43 Ibid. p. 6.
actual usage is just too far removed from some conceptions to attribute those conceptions to users'.

There is a possible response available to the exponent of mild-hermeneutism. But while it would assuage the worry just mentioned, it would at the same time undermine the motivation for the mild claim in favour of my stronger variety. To state the response in question, let us first see why Frege would not be content with weak hermeneutism. One might think that, if the weak hermeneutic view were right, it would be unclear how Frege could have believed that he had preserved Bedeutung. Of course there might be no problem if, generally speaking, weak-hermeneutism involved a reductionist project of one empirical science to another. For we could appeal to perception in order to determine whether the Bedeutung of one's terms is preserved. But matters are otherwise where the Bedeutung of our terms is purportedly abstract in nature. Suppose Frege had granted that the Sinn of the number terms in his Concept-Script is not the same as the Sinn of its everyday counterpart. Would there then be any good reason to believe that what the newly assigned Sinn determines is the same Bedeutung as that referred to by our ordinary terms? One might say that, by seeking to preserve Sinn, Frege would have a method, however imperfect, for ascertaining whether our arithmetical truths are analytic a priori. He could then have at least some reason, however defeasible, for thinking that his Concept Script offered some explanatory value concerning our ordinary arithmetical practices.

I am unconvinced by the points just made. Three points will make clear why. First, we saw in chapter two that Frege sought to capture the unique objects and concepts that would account for the truth-value of our ordinary arithmetical thoughts. Indeed he believed that his definitions captured these unique Bedeutung. He did so because he believed that there was no satisfactory alternative to his account. Second, he believed that, in using numerals, we have an unconscious grasp of the Sinn that connects with the unique arithmetical Bedeutung. Third, it is implausible to suppose that different thinkers attach

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different arithmetical *Sinn* to these numerals: that is, different *Sinn* that would determine the same unique arithmetical *Bedeutung*, rather than some other entity. The reason is that grasping different *modes of presentation* is dependent on conscious awareness and selective attention to the corresponding *Bedeutung*. Two thinkers possess different mode of presentation if they regard the *Bedeutung* as determined in different ways. But different ways of regarding the *Bedeutung* is absent where the *Sinn* belonging to the numeral is grasped unconsciously. Given these three points, Frege’s warrant for believing that his definitions capture unique arithmetical *Bedeutung* would transfer to the further belief that the same definitions capture the numeral’s unique *Sinn*. In other words, the direction of explanation, regarding why Frege should have preferred strong (or mild) hermeneutism over the weak variety, is the reverse of that mooted in the previous paragraph. It was because he believed himself to have captured the unique *Bedeutung* that he believed, or could have believed, that his definitions recovered a unique *Sinn*—not the other way round. Now Burge too might move from warrant for believing that *Bedeutung* was preserved, to warrant for believing that *Sinn* too was preserved. But to do so would commit one to tacit grasp, with no good reason, as yet, to say that one’s grasp is defective.

At any rate, no such unclarity about the explanatory value of a successful foundational project need arise given the stronger counterpart to mild hermeneutism. On our view, imperspicuity of grasp relevant to Frege’s epistemolgy is manifest in several ways. First, there is lack of clarity regarding the structure of the arithmetical truths themselves. Here greater clarity involves understanding that the constituents even of the simplest of arithmetical propositions are definable, and of attending to them, having conscious awareness of their structure. Second, lack of clarity of grasp is apparent where the thinker is unclear about the nature of the truths. The thinker tacitly knows that arithmetical concepts have the greatest possible application, and thus that the truths of arithmetic have the greatest possible generality. There is a sense in which he already knows that they are logical truths. The logicist proof brings out this understanding. Then there is our grasp of the primitive logical truths themselves. Here too Frege’s explications bring out what is already involved in our grasp of the logical truths. But it does so in different ways. As we will see below, Frege
envisaged that with all but axiom V, there would be recognition—some unconscious recollection—that his explications bring out what is involved in our understanding of these truths. But in the former case, the kind of tacit grasp involved in our understanding of axiom V would be more akin to his explication of arithmetical concepts. For as with axiom V, recognition that such a definition brings out the content of what we ordinarily grasp may not be immediately forthcoming.

Finally, Frege appears to have believed not only that the truths of arithmetic depend on the truth of these axioms. There is evidence that, for Frege, part of our ordinary grasp of everyday arithmetical truths involved a grasp of these axioms. First, it is clear, I think, that Frege relied on the notion of full tacit grasp regarding axiom V. For Frege believed that our ordinary understanding and use of the notion of an extension presupposes that we do have full, if unconscious, grasp of axiom V—axiom V being what allegedly governs our use an extension. As we shall see, the argument structure for the introduction of axiom V, and others, serves to recover our tacit grasp, and to make it perspicuous. In the case of axiom V, what is disclosed is that corresponding to every concept is an object. We discover (uncover) what ‘mankind has done by instinct’, namely those logical procedures by means of which numbers are yielded, the axioms or laws of thought that constitute the essence of thinking. Second, Frege says that the other axioms are grasped during our ordinary engagement in arithmetical practice: ‘[e]very axiom that is needed must be discovered and it is just the hypotheses [i.e. that a transition to a new judgement is self-evidently correct] that are made tacitly and without clear consciousness that hinder our insight into the epistemological nature of a law.’

§6.4. A Non-Epistemological Logicism. With these considerations in mind, let us now begin to clarify and elaborate Frege’s alleged epistemology outlined in section two. I will do so first by considering three possible objections to it.

48 Cf. Grundesetze vol. ii, §147, pp. 160-61, TPW.
49 Grundlagen §2, p. 2.
50 Grundgesetze vol. i, p. 128, TPW.
The first objection concerns Frege's use of the *a priori*. A central part of the traditional conception of *a priori* knowledge is of course that justification of a belief or judgement is independent of experience. It is said, however, that Frege's characterisation of either the *a posteriori*, or the *a priori*, makes no explicit reference to experience. A truth or proposition is *a posteriori*, for Frege, if it cannot be proved 'without reference to facts (*auf Thatsachen*), i.e. to truths that cannot be proved and are not general since they contain assertions about definite objects (*bestimmten Gegenständen*). By contrast, a truth or proposition is *a priori* if it can be proved exclusively from general laws. So if Frege had no intention of including experience in the contrast between the *a priori* and *a posteriori*, then this would suggest that he had reconstrued these concepts in a way that removed them from epistemology.

The second objection is that we have misunderstood Frege's use of 'justification' and 'source of knowledge', and that a look at the central motivation behind his logicism will bear this out, as well as perhaps explain our misunderstanding of Frege's use of 'a priori'. Frege sought to establish the independence of arithmetic from geometry: to show that arithmetic rests on a purely logical structure free from spatial and temporal notions. Rigour of proof was to help vindicate the claim of autonomy, rather than to help fulfil an

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51 *Grundlagen* §3, p. 4; Thiel, §3, p. 15.
52 Op. cit. §3, pp. 3-4. Whether a truth or proposition is synthetic or analytic depends on the nature of these laws. A truth or proposition is analytic if the laws on which its proof depends is of a purely logical nature—laws having the greatest possible range of application—and if the definitions themselves involve only logical notions. A truth is *a priori* synthetic if the general law in question has a generality less than that of the greatest possible kind.
54 See W. Demopoulous's 'Frege and the Rigourization of Analysis' (1994), rept. in Demopulous (1995), pp. 68-88; and A. Coffa's 'Kant, Bolzano and the Emergence of Logicism' (1992), rept. in Demopulous (1995). According to Demopulous, Frege use of the notion of justification is to be understood in terms of establishing the claim that arithmetic is independent of intuitive notions. On this view, Frege's logicism is principally about establishing that arithmetic has the greatest generality, and to explain this makes this so.

Coffa view of logicism differs from Demopulous's. The broader movement to rid intuition from *a priori* sciences stemmed from the view that intuition was 'a cancer that has to be extirpated in order to make mathematical progress possible. Frege's logicism, says Coffa, is to be seen 'as a gigantic fly-swatting ominously surveying the whole field of arithmetic, ready to squash pure intuition as soon as it comes in. My version of logicism is therefore this: Bozano and his followers manoeuvred pure intuition out of analysis and into arithmetic, where Frege's fly-swatting finally finished it off' (ibid. p. 38).

For Benacerraf, Frege's use of the notion of justification is to be understood simply in connection with providing rigorous demonstration that the truths of arithmetic do not involve a contradiction.
epistemological aim of explaining \textit{a priori} justification in terms of a logical source of knowledge. If a basic principle of arithmetic—e.g. connectedness of the ancestral and mathematical induction—were found to depend on intuition, then, in order to establish that principle, appeal would have to be made to our knowledge of space and time. The principle would then depend for its proof on our knowledge of spatial and temporal notions. So construed, Frege's logicism is likened to the work of Bolzano, Dedekind and other 19th century analysts—the 'conceptualist movement'—who sought to free calculus and the theory of the reals from any dependence on the sciences of geometry and kinematics. Indeed logicism, understood within this mathematical tradition, was but one stage in a process aimed at establishing once and for all that 'conceptual representations'—those free from spatial and temporal notions—suffice for the construction of every 'pure \textit{a priori} science'.

Furthermore, use of 'epistemological' notions was not uncommon among mathematicians of the 19th century, believing as they generally did that, for example, we have \textit{a priori} intuitions of space and time, and that the study of space and time lay within the province of geometry and kinematics, and perhaps mechanics. But within this broad mathematical movement the notion of \textit{a priori} intuition was not in general used in connection with a justificatory source, but in connection with that whereby truths are grasped and concepts acquired. So concern with the nature of our geometric knowledge, and interest in the epistemological notions of intuition and the \textit{a priori}, was not in general motivated by any substantial epistemological thesis. \textit{Mutatis mutandis} Frege. To determine the nature of the primitive truths, Frege hoped simply to explain the universality of the truths of number theory: \textit{viz.} by showing that the 'source of knowledge' is other than intuition. But this did not mean identifying a source of epistemic justification. It only meant identifying the nature of the primitive truths. Similarly, when Frege speaks of justification of judgement he 'is concerned with substantive questions about the truths of the propositions in question' rather than 'with epistemological and metaphysical questions that arise in accounting for that

\footnotesize{\textsuperscript{55} To borrow a phrase from A Coffa, 1982, p. 33.  
\textsuperscript{56} Demopoulos, 1994, p. 75.}
Accordingly, 'justification' means showing the dependencies among truths, not showing how, by a means of a logically perfect proof, our beliefs acquire properties of epistemic warrant. Again, Frege wished to identify the basic or self-standing truths, and to exhibit their nature—not to exhibit a source of a priori justification for our arithmetical judgements.

'There may be a hierarchical structure to our beliefs, with the hierarchy representing the relation of foundation or justification that a person's beliefs may bear to one another: the relation of dependence that actually obtains and which may vary from person to person even though the related beliefs might themselves be close to identical. On some (e.g., foundationalist) views, beliefs do form such a structure; on others (e.g., holists), they do not. Frege is not concerned with such a relation, but with relations of dependence among the propositions themselves, whether or not they are believed and however those beliefs may be related to one another in the epistemic world of any individual.'

The third objection rests on the observation that any such interest in a priori justification would have transgressed Frege's anti-psychologism. 'It would require him to give an account of how we can and do know what we know—an account that would force him into a discussion of the conditions under which our beliefs constitute knowledge, a topic which he correctly perceived would involve certain psychological issues but which he (wrongly, I think) sweeps out with his anti-psychologistic broom.'

Let us now address these objections.

§6.5. **Reinforcing the Epistemological View.** To begin with, the mathematical reading of Frege's motivation hardly undermines an epistemological construal of the above notions. It is tendentious to use this alternative reading of Frege's motivation as a framework for interpreting the sense in which Frege deployed epistemological notions. The classification is simply too broad, the interpretation too impressionistic. No doubt some mathematicians, concerned with the autonomy of arithmetic, did understand the above epistemological notions in a minimal sense; but others may well have had a more embracing and substantial

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construal of them. Bolzano himself is a case in point. True, Bolzano’s overriding concern was to rid *a priori* science from intuition; but he nevertheless retained an interest in what justifies our judgement in the mathematical laws. His answer may not have been plausible, or well developed, or at all integral to his project, but that is not the point. Of course it need not be denied that the search for autonomy is an accurate description of the mathematical motivation of Frege’s logicist project. It may even have exhausted what, from a mathematical point of view, made logicism worth striving for. But Frege’s logicism is richer than this anti-epistemic view suggests. One difference between Bolzano and Frege in this regard is that, unlike the former, Frege’s epistemological quest was integral to his concern to establish the independence of arithmetic from geometry. One had to establish the autonomy of arithmetic *vis-à-vis* geometric notions in order to show that the source of our knowledge of arithmetic is independent from the spatio-temporal. Thus nothing about the mathematical movement of logicism tells against the epistemic reading of Frege’s own brand of logicism.

Let us, however, respond to the more specific claims: first the claim that Frege avoided conceiving his logicism as epistemological because not to do so would have involved psychologism. This objection is based on a misunderstanding of Frege’s anti-psychologism. Three features, I believe, characterise it. (i) One is the view common in the 19th century that a task of the logician is to describe the inferences we actually make in order to articulate ‘laws of association of ideas’. Against this, Frege criticises a theory of truth that he thinks (i) gives rise to, *viz.*: (ii) the view that truth is determined by consensus. As Frege saw it, this theory left no room for the distinction between normative and descriptive laws of the science of inference. What is normative would collapse into what is agreed to best describe how most of us actually think. Frege of course regarded this as a failure to recognise that truth is independent of human judgement. The other feature of Frege’s anti-psychologism, prevalent in *Grundlagen*, is seen against two common mistakes in semantic theories of the

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61 *Wissenschaftlehre*, p. 354.
62 *Grundgesetze*, vol. i, p. 16. See Sluga, 1980, chapter 1, for a view of the way in which the 19th century intellectual climate was dominated by scientific naturalism.
time. (iii) One is that of identifying meaning of a term with an idea; the other is in identifying abstract objects with ideas, where in both cases an idea is understood in a psychological sense, as mind-dependent and (for Frege at least) unshareable.\footnote{\textit{Ibid.} pp. 12-15.} Given these features, it is clear that if Frege did seek to exhibit that means by which arithmetical judgements could be afforded \textit{a priori} justification, it need not have involved him in transgressing his own avowed anti-psychologism.\footnote{\textit{Ibid.} p. 12, p. 18, p. 20. For more on Frege's distinction between the subjective and objective, see chapter 1 above.} For an account of how we can and do know the truths of arithmetic \textit{a priori}—showing what would constitute an \textit{a priori} justified belief or judgement—can be independent of the items (i-iii) mentioned above.

What, now, of the claim that Frege radically reconstrued the \textit{a priori}-\textit{a posteriori} distinction—a claim prompted in part by the observation that the concept of experience lends no content to the distinction. For one thing, the claim seems rather odd since it is unclear what the point of the reconstrual would have been. For another, there is good reason to believe that in fact Frege did have experience in mind when contrasting the \textit{a priori} and \textit{a posteriori}. Again, P is \textit{a priori}, for Frege, just in case a proof of P exists that does not depend on any basic 'facts' about determinate (bestimmten) objects: just in case there exists a proof of P that involves only general propositions as axioms. Referring back to\textit{ Grundlagen}, Frege says that he had earlier\footnote{Cf. Kitcher, 'Frege's Epistemology', \textit{Philosophical Review} (1979), p. 247.} made it probable that (i) 'arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition'.\footnote{TWP, p. 128.} Also on dividing the sources of knowledge into three kinds, Frege takes (ii) any source of knowledge that is independent of sense perception to be \textit{a priori}.\footnote{Cf. \textit{Grundlagen} \S 8, p. 22; Thié, p. 12. Also Frege's talk of 'facts' in \S 17, and also in \S 8, is clearly aimed at singular facts of sensory experience.}  

Taken separately, these two pieces of evidence invite two different accounts of Frege's characterisation of the \textit{a priori}. On one account, Frege is

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\begin{itemize}
\item \textit{'Sources of Knowledge of Mathematics and the mathematical natural Sciences'} (1924-25), p. 267, \textit{PW; NS}, p. 286; 'Numbers and Arithmetical' (1924-25), p. 286, p. 277, \textit{PW; NS}, pp. 296-67. In the latter, Frege links the empirical with sensory perception and contrasts this with the \textit{a priori} modes of cognition'.
\end{itemize}
offering a definition of experience. Accordingly, experience is a normative or justificatory function; it is that on which empirical knowledge depends, and on which a priori knowledge is independent. More specifically, an experiential cognition is the conception of an epistemically basic truth about a particular thing. So experience is whatever provides basic knowledge of particular objects. And any knowledge whose justificatory component replies on knowledge of particulars counts as empirical knowledge. Clearly, Frege's alleged definition of experience extends considerably beyond sensory experience; and also well beyond what we would ordinarily count a posteriori knowledge, since here empirical knowledge need not depend on sensory-experience. This account would cohere with (i) and would account for the characterisation of the a priori-a posteriori in Grundlagen.

If Frege did subscribe to this conception of experience, it might answer two questions about the route his project took following the discovery of an inconsistency in his system of logic. One such question is why he did not fall back on Peano's axioms as the small number of basic laws from which the known truths of arithmetic might be derived. Given the above conception of experience, to do so would have committed Frege to the view that arithmetical truths are a posteriori, since some of the axioms concern facts about particular objects: e.g., '1 is a number' and '1 is not the successor of any number'. The second question is why Frege did not subscribe to some form of epistemological platonism, whereby, for instance, we directly apprehend by way of mathematical intuition certain elementary truths about the numbers. Again, given Frege's alleged definition of

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70 This view is taken by Gideon Rosen and Jamie Tappenden in their 'Frege's Theory of Experience' 1996 (unpublished). They do not use these two passages as spring boards for the two possible accounts.

71 This can be contrasted with two other conceptions of experience. There is a phenomenological conception of experience: a sphere of introspection, the apprehension of what is 'directly present' to consciousness. Also there is a psychological or learning-theoretic conception of experience. Here experience is identified with the proximal input to the causal process of learning and conditioning.

72 T. Burge has noted that Frege's understanding of the a priori stems from Leibniz. 'Frege on Apriority', in New Essays on the A priori (2000), eds. Paul Boghossian and Christopher Peacocke, p. 13, p. 15, p. 18, p. 22, p. 25. In particular, Burge notes that Leibniz seemed to hold this broad view of experience, namely as awareness, intellectual or sensory, of objects, ibid. p. 28.

73 Ibid. pp. 18-19.

74 Ibid. pp. 17-18. There are independent reasons why Frege would have rejected a certain form of epistemological platonism. Frege insisted that all knowledge is propositional. He believed that when we apprehend things we do so only by grasping thoughts about them. Postulation of a quasi-perceptual faculty by means of which we achieve a non-propositional acquaintance with abstract particulars would have violated that latter principle.
experience, it would have committed him to the view that the truths of arithmetic are \textit{a posteriori}.

It might be said that the above account cannot be right since otherwise Frege would not have regarded the truths of geometry as synthetic \textit{a priori}: for Frege, our knowledge of geometric truths is grounded on spatial intuition,\cite{Grundlagen §12} which seem to involve singular representations.\cite{Rossen and Tappenden's 'Frege's Theory of Experience', p. 22. Cf. Burge, 2000, p. 17.} In other words, if these singular representations were of particular things, then Frege would have regarded geometry as \textit{a posteriori} rather than \textit{a priori}.\cite{Grundlagen §13, p. 20.} To be sure, these intuitions would have to be pure in Kant's sense: \textit{i.e.} not arising from sensory experience or observation. Nonetheless, they would be intuitions of objects, making our knowledge empirical. The objection would be mistaken, since Frege explicitly denies that intuition in geometry is of particular objects: 'the points or lines or planes which we intuit are not really particular at all'.\cite{Grundlagen, op. cit. p. 20 (Thiel); p. 20 (Austin).} The reason is that a geometric entity does not have any peculiarities of its own; and this allows them 'to stand as representatives of the whole of their kind'.\cite{Grundlagen §10.}

If Frege was here responding directly to the above kind of objection, then this would indicate that he did indeed have the above theory of experience in mind.\cite{Rosen and Tappenden believe that he was. Ibid. p. 22, p. 23.} But in fact Frege had another concern. This was to reinforce his view that an attempted explication of arithmetic in terms of geometric concepts would be mistaken: numbers being quite different from geometric entities since the former have certain intrinsic properties that the latter do not have.\cite{Grundlagen §10.} At any rate, Frege's thesis about the non-particularity of geometric objects is a thesis he would need to assert on independent grounds. This is because the view of spatial intuition as a singular representation is in apparent conflict with a quite different (rationalist) thesis: that general laws cannot be inferred from particular instances of it. That there is no conflict between Frege's rationalism and his view about the laws of geometry becomes apparent once we realise that the objects in geometry are not particulars at all.

\footnote{See fn. 101 for how Frege might have understood the epistemological role of spatial intuition.}
\footnote{See Grundlagen §12.}
\footnote{Grundlagen, §13, p. 20.}
\footnote{Op. cit. p. 28 (Thiel); p. 20 (Austin).}
\footnote{Rosen and Tappenden believe that he was. Ibid. p. 22, p. 23.}
\footnote{Grundlagen §10.}
There are other uncertainties about whether the above definition of experience informed Frege's understanding of the *a priori*. For instance, if Frege did subscribe to the definition, he must have regarded Kant as having an *a posteriori* theory of our knowledge of arithmetic. As Frege himself points out, Kant regarded such numerical propositions as $7 + 5 = 12$ as unprovable. These are basic claims about particular things, knowledge of which truths would be empirical. Similarly, in 'Der Gedanke' Frege offers an argument for the claim that a thinker, the bearer of ideas, cannot himself be an idea. But then the above characterisation would commit Frege to the view that the proposition, I am not an idea, is *posteriori* rather than *a prior*. But this surely is not what Frege had in mind.

As it happens, we do not need to take Frege to be offering a definition of experience in order to regard his understanding of the *a priori* and *a posteriori* as epistemological. Various of Frege's remarks, including the second of the two remarks already mentioned—where Frege says that any source of knowledge that is independent of sense perception is *a priori*—suggests another account of his characterisation of the *a priori*-*a posteriori* distinction. Moreover, it seems clear from the following passages that Frege takes 'empirical' to mean justification by means of sensory experience. If so, then it is probable that, where Frege speaks of determinate objects in his characterisation of the *a posteriori*, he has spatio-temporal particulars in mind.

'I first repeat earlier assertions of mine that I still regard as true. *Grundgesetze* 1. p. 1. Arithmetic doesn't need to appeal to experience in its proofs. I now express this as follows: Arithmetic does not need to appeal to sense perception in its proofs.'

"If we call a proposition empirical on the ground that we must have made observations in order to have become conscious of its content, then we are not"

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83 There is a remark that has gone unnoticed in this connection that at least might appear to indicate that Frege was aware that experience need not be understood in sensory terms. ‘... [T]he truths of geometry, in particular the axioms, are not facts of experience, at least if by that is meant that they are founded on sense-perceptions.’ See Frege’s notes to Jourdain’s article about him in *Nachgelassene*. The rephrasing mentioned in the first citation might also suggest that, at least at some point, Frege had toyed with the Leibnizian notion of experience.
using the word 'empirical' in the sense in which it is opposed to 'a priori'—where *a priori* is justification of judgement from general laws.\(^8^5\)

It is clear from the surrounding text what contrast Frege is drawing: observation as a means of acquiring content; and observation as a means of justification.

Three final points will suffice to conclude this discussion of how Frege understood the *a priori*. First, it seems that Frege regarded any justification that rests on general laws, and any justification that is independent from sensory-experience, as essentially equivalent. Second, the reason why Frege did not consider taking Peano axioms as the foundation of arithmetic is not because he feared commitment to empiricism. Assuming he considered the Peano option at all, he would have feared that it would make unclear whence derives the source of their justification. Finally, the claim that Frege had reconstrued the above traditionally epistemic notions, placing them outside epistemology, is false.\(^8^6\)

Let us turn to how we should regard Frege's use of the terms 'justification'. Frege aimed to chart an epistemological ideal. He aimed to set out an ideal system whereby the best possible justification for making judgements about the truth of mathematical propositions could be had. Discovering the true order of dependencies among the truths of arithmetic, showing how arithmetic is developed from a small number of primitive truths contributed to that end. It did so because the primitive truths were self-evident in both the objective and subjective sense. A perspicuous grasp would confer on the judgements the justificatory features of self-evidence. That is, with a perspicuous grasp we would simply recognise from the content alone that the proposition is true. I'll say more about how this is supposed to work in a moment. First it is necessary to support the view of justification as more than a matter of dependencies among truths.

Part of this view is reinforced by Frege's conception of inference. Frege insisted throughout his career that only from premises known to be true should anything be inferred. In a letter to Hugo Dingler, for example, he writes


\(^8^6\) Frege might been worried that if by 'particulars' is meant spatial-temporal objects, then the truths of geometry would fall outside his classificatory scheme. This peculiarity of excluding truths from his classification as a consequence of Frege's characterisation of the distinctions between *a posteriori* and *a priori*, and that between analytic and synthetic, is there any way. Ordinarily, we would regard the proposition *A triangle has three angles* as analytic, and thus *a priori*. The primitive logical laws are another example.
‘... [W]e can only infer something from true propositions. Thus if a group of propositions contains a proposition whose truth is not yet known, or which is certainly false, then this proposition cannot be used for making inferences. If we want to draw conclusions from the propositions of a group, we must first exclude all propositions whose truth is doubtful'.

Latter in the same letter Frege draws a distinction between inferences made on the basis of true premises, and ‘pseudo-inference’, inferences drawn from false thoughts.

‘Suppose we have arbitrarily formed the propositions “2 < 1” and “If something is smaller than 1, then it is greater than 2” without knowing whether these propositions are true. We could derive “2 > 2” from them in a purely formal way; but this would not be an inference because the truth of the premises is lacking. And the truth of the conclusion is no better grounded by means of this pseudo-inference than without it. And this procedure would be useless for the recognition of any truths.’

One might of course still doubt that Frege’s conception of inference is informed by a concern with epistemic justification. What Frege has in mind, it might be said, is the (modern) distinction between soundness and validity: ‘pseudo-inference’ more or less means mere validity, whereas Frege’s insistence on derivations from true premises to true conclusions alludes to soundness. Also one might urge that Frege’s conception of inference be seen in the context of his view of assertion. Accordingly, what Frege really had in mind was the contrast between premises that the thinker commits himself to believing, and premises taken merely as suppositions. That is, Frege wished to stress that inference must proceed from asserted premises, where of course asserted premises need not be true. In that case, why did not Frege say that inference could proceed from premises that we take to be true, rather than that they must proceed only from true premises? According to Dummett, Frege would regard ‘taking to be true’ as something objectionably psychological, whereas assertion was not.

But I very much doubt that Frege did anticipate the distinction between validity and soundness. But even if he did, he should not be criticised, as he has

\[^{87}\text{31.01 1917, pp. 16-17, BW. My emphasis.}\]
\[^{88}\text{Op. cit. My emphasis.}\]
\[^{90}\text{Dummett, Frege: Philosophy of Language (1981a), p. 313.}\]
\[^{91}\text{Ibid.}\]
been, for having too strong a conception of inference, insisting on soundness rather than validity, since Frege's epistemological conception of logicism informed his logic. At least it did as regards his conception of inference. The point is that Frege insisted not merely that inferences proceed from true thoughts; he insisted that the truth of the thoughts be known. Anything less would do little to help us know the conclusion. Moreover, it is mistaken to believe that Frege construed 'taking to be true' in terms of psychologism; it was not a factor in Frege's view of inference. Recognising a truth itself involves taking something to be true, which of course is a mental act: '... using the inference from the general to the particular, for which of course a mental act [geistige Arbeit]—that of inferring—is still always required'. I'll discuss later in the next section the notion of 'taking to be true' and its connection with psychology.

§6.6. Perspicuity of Grasp and Frege's Apriority. It might be argued that even if Frege did use 'justification' to mean something epistemically substantial, his interest was limited to inferential justification, so that his notion of a 'logical source of knowledge' had nothing to do with justification of primitive truths.

'The grounds that justify the recognition of a truth often reside in other truths that have already been recognised. But if there are any truths recognised by us at all, this cannot be the only form that justification takes. There must be judgements whose justification rests on something else, if they stand in need of justification at all.

'And this is where epistemology comes in. Logic has to do only with those grounds of judgement that are truths. To make a judgement because we are cognisant of other truths as providing a justification for it is known as inferring. There are laws governing this kind of justification, and it is the aim of logic to set out these laws of correct inference.'

According to M. Dummett, Frege's reason for acknowledging the distinction between inferential and non-inferential justification is to make clear that his interest in justification is limited to inferential truths, the interest of the logician,

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92 This accords with Frege's remark that 'The task of logic is to set up laws according to which a judgement is justified by others ...'. See '17 Key Sentences on Logic' (1906 or earlier), p. 175, PW.
93 'Logical Generality' (1923), p. 258, PW.
94 'Logik' (1880s), p. 3, p. 6, PW; 'Key Sentences on Logic' (1906 or earlier), p. 175, PW. Cf. Grundgesetze, p. xvii. 'The question why and with what right we acknowledge a law of logic to be true, logic can give no answer.'
rather than with non-inferential justification, the interest of the 'epistemologist'.
Moreover, after acknowledging that it would not be inconsistent with Frege's
other doctrines for Frege to have claimed that judgements based on sense
perception are justified by being so based, Dummett remarks that

'Admittedly, although it is by means of our logical faculties or our reason that
we acknowledge the truth of the fundamental laws of logic, Frege does not
say that they are justified by being directly apprehended by the reason, but
that they are incapable of justification at all.'

Neither the first passage just cited from 'Logik', nor its surrounding text, clearly
indicates that Frege's interest in justification is limited to the inferential. Nor
should we be distracted by Frege's having here reserved the word 'epistemology'
for non-inferential justification. Inferential justification is, as Frege here makes
clear, as much an epistemological notion. The point of the passage is, I believe,
simply to make clear that there are two kinds of epistemic justification in a
deductive system of proof. It is also, I think, to make clear that possession of
these epistemic properties is contingent on the justification of our judgements of
the first principles: the axioms, definitions, and rules of inference.

As for the second citation, it comes as something of a surprise to find
Dummett arguing in this way. As its surrounding text makes clear, Dummett
moves from Frege's claim that the primitive truths 'neither need nor admit of
proof', to the claim that Frege did not believe that they could be justified at all, let
alone by means of our logical faculties. Two points in particular stand out. First,
when Frege says that the primitive truths do not admit of proof he means that there
are no other more basic truths from which it can be derived. And when he says
that these laws do not stand in need of proof he means that these truths are self-
standing, that their truth depend on no other. He does not say that our judgement
of basic truths cannot admit of justification; he recognizes both inferential and
non-inferential justification, as we have just seen.  

While a truth may neither
need nor be capable of proof, still our judgements involving them may yet require,

nothing to say about non-inferential justification. His interest lies exclusively with the first species
97 Frege acknowledges both inferential and non-inferential justification in '17 key Sentences on
Logic' (1906 or earlier), p. 175, PW.
or at least admit of, justification. It would seem that Dummett has conflated Frege’s notion of proof (Beweis) and justification (Berechtigung).

Second, it is true that Frege never specifically says that our judgements of the primitive truths of logic are justified by reason in a way that one—say a realist—might speak of observational judgements as justified in terms of sense perception. But he comes very close to doing so. In Grundgesetze he confirms that it is by means of our logical faculties, by reason alone, that we recognize the truth of logical laws. Here the logical source of knowledge cannot simply be that by means of which we acknowledge its truth. For ‘what I regard as a source of knowledge is what justifies the recognition of truth, the judgement’. Nor is there any good reason for limiting that remark to inferential justification. Inference in Frege’s system cannot proceed save from known truths: that applies to primitive truths as well.

After saying that, for Frege, our judgements of primitive truths of logic are incapable of justification, Dummett seeks further to clarify why

‘Reason is not ... a true parallel to sense-perception: to ascribe reason to an individual just is to ascribe to him a capacity to grasp thoughts and to make judgements on the basis of other judgements from which they follow. Reason does not prompt us to do such things, but consists in our doing them, whereas sense-perception prompts us to make observational judgements rather than itself consisting in our making them.’

I leave aside for a moment the, to my mind, dubious claim that reason is clearly disanalogous to sense perception in that it does not prompt us to grasp thoughts and make judgements. I observe only that reason—if included to mean the act of grasp and selective attention—certainly does seem to have this feature analogous to our various modes of perception. True, reason consists, for Frege, in grasping thoughts and making judgements. But that hardly precludes reason from prompting us to make judgements, not just on the basis of other judgements, but also simply on the basis of our grasp of the content of the sentence on whose truth-value we aim to judge. If Dummett were right in his exegesis, then Frege’s not infrequent references to a logical source of knowledge would reduce to a banality: the observation (trivial in this context) that reason involves thinking, and that we

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98 ‘Sources of Knowledge’ (1924-25), p. 267, PW. My emphasis.
humans have the facility for this enterprise. Frege might be mistaken about many things, but he is never banal. No one would deny that reason involves thinking. And no one should deny that, for Frege, reason—the logical source of knowledge—involves a source of epistemic justification *apropos* inferential justification. The question is whether, for Frege, this epistemic role of reason extended to being a source of justification for our judgements of primitive truths. So far we have some indication that it did.

Now one might accept that Frege did indeed take ‘logical source of knowledge’ to apply to both inferential and non-inferential justification, but deny that he had anything further to say about the latter. We have already noted that his remarks on knowing the foundational truths of arithmetic are sparse. One not uncommon explanation for this is that Frege relied on Kant for an answer and saw no reason either to elaborate it or defend it.^^ On this view, for example, Frege saw Kant as having identified the following parts of an epistemological framework: (i) to identify a set of basic *a priori* statements or axioms, the justification in the belief of which being non-inferential; (ii) to explain the warrant conferring process that no experience can defeat; (iii) to identify *a priori* preserving rules of inference; (iv) to show that mathematical truths can be obtained from basic statements by means of these *a priori* preserving rules of inference. On this view, Kant’s principal interest lay in detailing item (ii), whereas Frege’s interest lay (i), (iii) and (iv).^^

As it happens, Frege had no reason to rely on Kant since Frege says more about our knowledge of primitive truths than other commentators would have us believe. Frege’s notion of a logical source of knowledge includes the idea that

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[^100]: Kitcher argues that Frege saw Concept Script as a psychologistic proof (ibid. p. 245). A psychologistic proof is distinguished by the kind of knowledge it is said to produce; and by the further feature that whether a true belief is an instance of knowledge is contingent on whether the belief is generated in the appropriate way (ibid, p. 243). The kind of proof in question is a sequence of symbols representing mental states or processes the engagement with which suffice to confer *a priori* warrant, these being transmitted by the purely logical definitions and rules of inference to yield *a priori* knowledge of the theorems. On Kitcher’s view, then, Concept Script represents the kind of mental activity required of a thinker if his beliefs are to be conferred *a priori* warrant, one of a kind necessary for the most certain kind of knowledge.
[^101]: If it can be said that Frege was a Kantian about the truths of geometry, then that would lend some support to the claim that Frege relied on similar transcendental elements of Kantianism in accounting for knowledge of arithmetical truths. Frege agreed with Kant that the ultimate justification for our judgements in geometric truths is founded in pure intuition (this role of pure
reason alone provides justification for our judgements in the primitive truths of his logicist system. As to what this would consist in, we might explore two possible elements, both of which involve prescriptive norms. A clue to the first might be had from the following.

‘If we step away from logic, we may say: we are compelled to make judgements by our own nature and by external circumstances; and if we do so, we cannot reject this law of identity, for example; we must acknowledge it unless we wish to reduce our thought to confusion and finally renounce all judgement whatever. I shall neither dispute nor support this view; I shall merely remark that what we have here is not a logical consequence. What is given is not a reason for something’s being true, but for our taking it to be true. Not only that: this impossibility of our rejecting the law in question hinders us not at all in supposing beings who do reject it; where it hinders us is in supposing that these beings are right in so doing, it hinders us in having doubts whether we or they are right. At least this is true of myself. If other persons presume to acknowledge and doubt a law in the same breath, it seems to me an attempt to jump out of one’s skin against which I can do no more than urgently warn them.’

The thought-experiment enables us to realise that our laws of logic are such that it is impossible for us rationally to reject them. So is not Frege using the outcome of this thought-experiment as providing an a priori justificatory reason for believing the primitive truths? Most Frege scholars deny that the passage just quoted forms any part of Frege’s views on non-inferential justification. Most urge that it be read simply as part of Frege’s crusade against psychologism, to which the notion of ‘a reason for taking law of logic to be true’ belongs.

intuition is expressed in Grundlagen §12). For Kant the mind imposes its structure on the physical and mathematical realm; and it is in terms of this imposition that he explains the a priori. So the justificatory resource of an a priori cognition comes from cognitive capacities that contribute towards the cognition. Reflection on the nature of cognitive faculties provides a source of warrant. So for example, in mathematics, the faculty of intuition contributes images, imaginative constructions, not arrived at through sensory experience, making them pure intuitions. There is, to my mind, insufficient evidence that Frege understood the pure a priori in Kant’s way. And there is no evidence that, for Frege, we owe our knowledge of the primitive truths of arithmetic, the logical axioms, to ‘the cognitive source that produced them’ (to borrow a phrase from J. Proust, Analytic Propositions from Kant to Carnap, p. 148). By contrast, I argue below that, for Frege, a priori warrant is had from a property (self-evidence) of primitive truths.

102 Grundgesetze, vol. i, p. 15.
It can be argued, however, that Frege’s phrase ‘reason for taking something to be true’ is not a reference to the kind of psychologism discussed in the surrounding text of the above passage. There Frege remarks on ‘psychological laws of taking-to-be-true’, understood as the causes behind taking something to be true. But when he speaks in the passage just quoted of a reason for our taking something to be true he has, one might think, something else in mind. According to this view, a three-fold distinction is at work in the passage and its surrounding text: (i) reason for P being true; (ii) reason for taking P to be true; (iii) law or cause for taking P to be true.

"With the psychological conception of logic we lose the distinction between the grounds that justify a conviction and the causes that actually produce it. This means that a justification in the proper sense is not possible; what we have in its place is an account of how the conviction was arrived at, from which it is to be inferred that everything has been caused by psychological factors." Also, opponents overlook the fact that the notion of ‘taking or holding P to be true’ figures in Frege’s very definition or explanation of the a priori, a posteriori, the analytic, and the synthetic. Classifying P as having any of these latter properties ‘is a judgement about the ultimate ground for holding [taking P] to be true’. Finally, on speaking of stepping away from logic, Frege might be entering what he had earlier identified as the realm of epistemology, of non-inferential justification. By noting that the outcome of the thought-experiment is not a logical consequence we might see Frege as giving notice of this difference. So it is not unlikely that ‘reasons for taking something to be true’ belongs to the category of non-inferential justification.

In any case, the anti-epistemic view of the above citation leaves unexplained why Frege should have engaged in the thought-experiment, since, if he had simply wanted to make the distinction between psychologism and logic, he says too much. It also leaves unexplained why Frege should say ‘I shall neither dispute nor support this view’? Why the neutrality, if the passage merely signifies an attack on psychologism—assuming for the moment that Frege is expressing

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106 Cf. *Grundlagen* §3.
107 ‘Logik’ (1897), p. 147, *PW*. This distinction is explicit in ‘Der Gedanke’ (1918-19), pp. 351-52, *CP*; cf ‘Logik’ (1879-1891), p. 4, *PW*. Others writers have also noticed the distinction. See
neutrality here? Why not dispute virtually every word of it? Let us be clear what ‘I shall neither dispute nor support this view’ means. At first sight, it seems puzzling. For one thing, Frege appears to accept what he at the same time appears to profess neutrality about. He accepts that given how we are constituted we cannot but acknowledge our logical laws to be true. And he accepts that given how we are, we cannot but think that other beings, compelled by their different natures to reject our laws, would be wrong to do so. Of course one point of the passage is to stress that, *pace* the psychological logician, it is not our physical constitution and our external environment that makes logical laws true. Still, he accepts that the thought-experiment has some epistemological import. It does so because it is one way of seeing that the laws of logic are preconditions for the possibility of thought or rationality. We cannot conceive of rationality save in terms of these laws. A world in which these laws were rejected would be a world that we would have to judge as one of madness. That we can conceive of rationality, and that we can conceive of what makes rationality possible, even if we ourselves do not always exemplify it perfectly, is a reasonable indication that the ultimate truths upon which arithmetic depend are indeed true. So why say ‘I shall neither dispute nor support this view’ given that he at the same time acknowledges its epistemic import? One explanation might be that, in *Grundgesetze*, he wishes to say no more about these justificatory considerations, since non-inferential justification is not his concern of there. ‘I shall merely remark that what we have here is not a logical consequence.’

There is perhaps alternative explanation. Frege believes that the justificatory judgements involving the primitive truths—the taking to be true—can flow from our grasp of what makes them true, and this from our grasp of their self-evident content alone. ‘The assertion of a thought that contradicts a logical

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108 Also in *Grundlagen* (§14, p. 21) Frege argues that we cannot deny the truths of arithmetic without foregoing the possibility of thought. Was not *Grundlagen* §14 enough to show that we know the truths of arithmetic *a priori*? He probably thought that it was. But Frege appears to think that despite the realisation that it is impossible rationally to doubt arithmetical truths, still they might yet involve some intuitive features (cf. *Grundgesetze*, vol. i, p. 3. In that case, §14 might suffice to show only that arithmetical truths are known *a priori*. More would be required to show that we know these truths purely from a grasp of the content of the proposition, since we require a logicist proof to show that the content is analytic. Another reason for not being satisfied with §14 would be that it says nothing about whether numbers are logical objects.
law can indeed appear, if not nonsensical then at least absurd; for the truth of a logical law is immediately self-evident of itself from the sense of its expression'. But there is no guarantee that the truth will strike us as self-evident. Perspicuity of grasp may take some intellectual work to achieve. This point is brought out in the way Frege introduces the axioms (and rules of inference). His method of introduction might be seen as a way of eliciting a transparency of grasp necessary for seeing that a thinker cannot but judge that the axioms are true. Mutatis mutandis the role of the above thought-experiment.

But given sufficient perspicuity, the thinker need not consider his nature and his external environment to access the justificatory source for judgements of primitive logical laws. In the case of the axioms, all but axiom V are introduced by arguing from truth conditions of logical constants to the truth of the axioms. Of Axiom 1, for example, Frege says,

‘By §12, \((\Gamma \rightarrow (\Delta \rightarrow \Gamma))\)

could be the False only if both \(\Gamma\) and \(\Delta\) were the True while \(\Gamma\) was not the True. This is impossible; therefore

\[-/(a \rightarrow (b \rightarrow a)),\]’

where in §12 the material conditional is introduced as a function with two arguments. It ought to be stressed that introducing these axioms in this way is not a way of proving that they are true. It is not a demonstration that they rest on more intrinsic basic truths: e.g. from the law of non-contradiction, and truths about the way the material conditional maps truth-values onto truth-values. As seen, Frege makes clear in Grundlagen that the logical laws ‘neither require nor admit of proof’. Nor is there sufficient evidence that he had changed his mind by the time he wrote Grundgesetze. Rather Frege is presenting an informal explanation to show that from a mere grasp of their content we cannot but judge them to be true.

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109 In ‘Compound Thoughts’ (1923-25), p. 405, CP.
110 Cf. Burge, 1998, p. 316. He too takes the view that understanding alone of the content of primitive truths suffices for the justification of our beliefs in them. Cf. ibid. p. 312, p. 317, p. 320, p. 326, p. 338; cf. also his 1992, p. 645. One difference between Burge’s view and my own concerns his view of what grasping a proposition consists in for Frege. Burge attributes to Frege the view that to grasp a proposition is to grasp its inferential liaisons, so that the content of the proposition consists in these inferential properties. On my view, however, Frege holds that to grasp a proposition merely enables the thinker to see what these inferential liaisons are. Moreover, it appears that Burge’s inferential view is in tension with the view that the source epistemic justification of our judgements of the primitive truths of logic lies merely in entertaining the primitive truth.
111 Grundgesetze, vol. i, §18, p. 69.
The source of warrant for believing or judging the truth of these laws lies in their self-evident content. Anyone with a perspicuous grasp of these laws cannot but judge that they are true since in the very act of considering their falsity one sees that they cannot be. From a clear grasp of a primitive truth of logic we see that to entertain its falsity is rationally impossible. One simply sees that the axioms are true, and that they could not be otherwise. To put the point another way, one sees that these laws are part of the furniture of rationality and of mindfulness; one sees what grounds the normativity of the laws of thought. So a thinker cannot but be justified in believing the primitive truths. It is partly in this sense that our judgements of them are, for Frege, justified by being directly apprehended by reason, by a logical source of knowledge, by our logical faculties.

A perspicuous grasp of the axioms consists in an insight into what makes the 'boulder immovable'. The insight is that the boulder is a set of primitive logical laws that help constitute mindfulness; herein lies the authority of the laws of thought, the prescriptive norms. Awareness of the precondition of thought is, however, but one of two elements, one of two prescriptive norms, awareness of which helps to constitute the kind perspicuity of grasp in question. Awareness of the precondition of judgement is another. 'Any one who has once acknowledged a law of truth has by the same token acknowledged a law that prescribes the way in which one ought to judge ... .' Hence knowledge of what makes the 'boulder immovable' serves as an a priori justification for our judgements about them. In having a greater clarity of grasp of the laws of thought we see that the 'laws of truth are ... boundary stones set in an eternal foundation, which our thought can overflow, but never displace.' We see too that 'it is because of this that they have authority for our thought if it would attain to truth'. Judgement, for Frege, is the recognition of the truth of a thought. To judge is to advance from a thought to its truth-value. Reaching the truth is the aim, as well as a defining element—or constitutive feature—of the judging subject, the thinker. As such, the agent cannot but seek to conform to the prescriptions of the laws of thought, since not to do so would be to abrogate one's role of judging agent.

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112 *Grundgesetze*, vol. i, p. 15.
As against this, it will be replied that as statements of the laws of physics and geometry are no less descriptive than the laws of logic, so the former type of law is no less normative than are the latter type. 'We could, with equal justice, think of the laws of geometry and laws of physics as laws of thought or laws of judgement.' Yet we would not, the objection goes, normally think of statements of laws of physics and geometry as providing epistemic justification for our judgements.

Clearly, it is because we recognise the truth of certain natural laws that we accept their prescriptivity; that we allow them to govern how we ought to think. We cannot accept a natural law as true and at the same time refuse to allow it to govern how we ought to think. It would be irrational to judge the law of gravity as true, yet thereafter refuse to accept the truths that follow from it. It would be tantamount to rejecting the laws of logic. This is because the laws of logic provide the prescriptivity of these other laws. So to accept the laws of logic is to accept the prescriptivity of natural laws, if these latter laws are accepted at all. Conversely, on accepting the normativity of natural laws we tacitly accept the laws of logic. So it is because the laws of logic are the source of all such normativity that we can take this source as an indication of their truth. For to recognise this source is to recognise a constitutive feature of the mind, in the non-psychologistic or abstract sense of 'mind'. Hence it is not a consequence of my interpretation that by accepting the prescriptivity of a natural law we have a ground for believing that law to be true.

§6.7. Closing Remarks. I have tried to show that, although Frege had no elaborately stated theory of a priori justification, a substantial such epistemology is nevertheless to be discerned in his logicist project. Frege's few explicit words about the justification of judgements of mathematical propositions should be read not as an indication that no such thesis informed his project, but rather that he believed that little more needed to be said about how we can know mathematical

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117 This is how I understand Frege's remark that logic and mathematics can be seen as an investigation of the mind, whereas psychology is an investigation of minds. 'Der Gedanke' (1918-19), p. 369, CP.
truths. His reticence indicates confidence rather than indifference. Anyone with an adequate grasp of the propositions is forced by rationality to see that they are true. The act of apprehension itself affords us *a priori* justification; reason alone make these truths known. The thought-experiment elicits the logical source of knowledge of the primitive truths of arithmetic. It brings out their status as preconditions for the possibility of rationality and judgement. Another way of bringing out the justificatory norm is by explicating the axioms. By selectively attending to the laws of logic we become clearer about their nature, namely their prescriptive character or normativity. Moreover, we have stressed that this in part is the explanatory value of Frege's logicism. We have said that this value resides in the relation to our ordinary arithmetic: namely, showing how we can know these *a priori* purely from an understanding of their contents. Thus we have a further recommendation for strong hermeneutism, and the doctrine integral to it, the recovery of tacit grasp, and, the making our grasp of arithmetical propositions more perspicuous.
Conclusion

I believe that we have uncovered an important, and neglected, doctrine behind Frege’s foundational project. That doctrine—the recovery of tacit grasp, and the attainment of greater perspicuity—informed Frege’s project from beginning to end. One reason why it has been neglected is that the doctrine itself is mostly only implicit in Frege’s writings, and because other aspects of his writings appear to be inconsistent with it. We have offered several reasons why, from an exegetical point of view, the doctrine is important. Principally, it has helped to throw light on, and helped to resolve a number of controversies surrounding Frege’s work. These controversies concern the interpretation of a number of doctrines that are integral to logicism as Frege saw it.

Our main point of focus has been to show that the doctrine helps us to understand the relation between the content of our pre-Concept Script arithmetical language, and that of Frege’s logically perfect construction. The question of that relation has itself, we found, been a neglected area of research. Many Frege scholars have simply taken for granted that Frege either held a revolutionary view, or held the weak-mild hermeneutic hypotheses. Exponents of these views get most of their impetus from Frege’s frequent announcements that our understanding of arithmetical truths, and of our arithmetical language, is defective. But Frege’s text is compatible with the view that defective understanding makes for an unperspicuous, rather than incomplete, grasp. I have argued that, for Frege, our grasp was already full, if partly tacit, and that his definitions and proofs were a way of making that grasp clear and distinct. Alternatives to this view, we found, were counter-intuitive, as well as insufficiently supported by the text. For example, I claimed that, in the context of arithmetical practice, it is counter-intuitive to say that our grasp is partial, or less than complete, or that only part of the Sinn is in our cognitive possession. The completion of Frege’s logicism would have made no difference to our arithmetical usage, pure or applied. It is in these latter contexts, I argued, that we should answer questions concerning the status of our grasp—i.e. whether it was complete or incomplete—prior to the envisioned success of Frege’s logicist project.
I argued that this interpretation of ours—that Frege sought to make our grasp perspicuous by uncovering what was tacit—is more consonant with Frege's text. Particularly was this so with respect to some of the explanatory value he appears to have believed that his logicist proof promised. On our reading Frege's logicism was principally an epistemological enterprise. It was to show how our arithmetical truths could be known a priori. We argued that justificatory properties are afforded to the thinker simply in virtue of his having attained to a sufficient degree of clarity and distinctness of grasp of the arithmetical propositions in question. This was what, for Frege, would make his logicism worthwhile. Such an advantage would be undermined, given the alternative interpretation.

Exponents of the more radical reading of Frege's project—the revolutionary hypothesis—readily admit that judgements made in the two arithmetics would be different in content. Their response was to deny that Frege sought to show how our arithmetical judgements are analytic a priori. Once again, we found this type of interpretation to be exegetically implausible. Certain exponents of the less radical alternatives—e.g. the partial grasp view—agreed that Frege did indeed seek to show that our arithmetic is analytic and knowable a priori. The trouble here, we argued, was that the notion of an incomplete, seriously defective grasp, makes it is unclear that Frege was saying anything explanatory informative about our judgements and beliefs—unclear that he had excavated the content of our ordinary arithmetical judgements.

We showed a further advantage behind the doctrine of tacit grasp and attainment of perspicuity. This concerned the controversy surrounding what Frege had meant in claiming that, because of fruitful definitions, analytic truths extend our knowledge. Many Frege scholars have identified the role of the primacy of judgement, and the decomposition of thoughts, as crucial to the explanation of what Frege intended. As seen, these scholars commonly argue that by decomposing thoughts a thinker can arrive at something conceptually new even though the thinker already fully grasps the thought decomposed. Fruitful definitions are the results of decompositions, and Frege's notion of extension of knowledge, apropos analytic truths, is understood in terms of arriving at these new concepts. We argued against this interpretation. We did so, once again, on exegetical and non-exegetical grounds. By contrast, we identified a different role
for decomposition as it occurs in proofs. We viewed it as a means of (i) recovering some instances of tacit grasp, (ii) exhibiting the relations between thoughts in order to recognize the validity of deductive reasoning, and (iii) thereby of eliciting greater perspicuity.

The issue of decomposition bore on our central inquiry concerning the relation between the content of ordinary arithmetical language and the language of the Concept Script. The view that 'extension of knowledge' meant arriving at new concepts through decomposition threatened our reading of that relation. This more 'full-blooded' view of decomposition, as we called it, would mean that definitional identities were typically informative. That is, the Sinn of the definiens differed from the Sinn of the definiendum. So in the context of arithmetic, Frege's explications of our terms did not, on this view, preserve their ordinary Sinn, contrary to our strong hermeneutic thesis. Hence our principal motivation for undermining the reading of decomposition as full-blooded as it occurs in proofs.

The same question regarding the nature of decomposition continued to be asked in connection with a separate discussion of Grundlagen §64—a passage in which Frege explicitly mentions the arrival of 'new concepts' by such means. One purpose of that discussion was to further defend our minimalist reading of what Frege meant in saying that, apropos analytic truths, fruitful definitions extend our knowledge. Another purpose was to offer a further perspective on how we should see the carving up process described in the above passage of Grundlagen, and how we should resolve a puzzle regarding it. The puzzle in question was that the carving up procedure described in that passage seemed to require incompatible doctrines: (i) sameness of thought; (ii) explanatory priority; (iii) preservation of logical form. How could both sides of, e.g. the biconditional \( Nx:Fx = Nx:Gx \iff F=G \) be the same thought, given that they have different logical forms? And how, if they are the same thought, could the right-hand side explain our grasp of the left-hand side; how can it explain how we come by anything new: viz. the cardinality concept? Again, the doctrine of the recovery of tacit grasp, and the attainment of perspicuity, helped to answer these questions. Moreover, by reexamining the content of (i-iii), we were able at the same time to clarify and defend this neglected doctrine.
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