The Parametrically Excited Pendulum: a Paradigm of Nonlinear Systems

Michael John Clifford
Centre for Nonlinear Dynamics
University College London

September 1994

Thesis submitted for the degree of Doctor of Philosophy
Acknowledgements

Thanks to all who made this possible; the members of the Centre for Nonlinear Dynamics and its Applications for their expertise, conversation, and criticism (most of it constructive), Steve Bishop for supervising this project, Allan McRobie for several illuminating discussions, SERC for funding, and last but certainly not least my parents and Elaine for their support and encouragement.
Abstract

The parametrically excited pendulum is a simple nonlinear system that exhibits a plethora of nonlinear phenomena. The steady state solutions can be subdivided into hanging, inverted, rotating, and non-rotating solutions. These in turn undergo bifurcations such as symmetry breaking, saddle-node, period-doubling, sub-critical, super-critical, and catastrophic bifurcations. These bifurcations are studied in detail by a variety of analytical and numerical techniques including perturbation methods, harmonic balance, method of strained parameters, braid and knot theory, Melnikov energy methods, Runge-Kutta numerical integration, Poincaré sections, path following, bifurcation following, and cell mapping. For non-rotating solutions, the parametrically excited pendulum may be considered as a system which permits escape from a symmetric potential well under parametric excitation. This simple statement allows a large body of existing theory on escape systems to be applied directly to the pendulum. Parameter zones where no major non-rotating orbits exist are termed "escape zones", and are predicted by applying various analytical techniques. The results are compared with the numerically determined bifurcation diagrams, and are assessed in terms of engineering integrity and the fractal nature of basin boundaries. The implications for real physical systems are considered in terms of safe operating parameters. Braid and knot theory is applied to a particular horseshoe which is conjectured to exist in the invariant manifolds of the hilltop saddles for the system. This analysis provides considerable insight into subharmonic bifurcational activity, and the role this activity plays in the overall bifurcational structure. It is argued that this combined analytical and numerical approach gives a more complete picture of the dynamics of the parametrically excited pendulum than either method would alone.
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title Page</td>
<td>1</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>2</td>
</tr>
<tr>
<td>Abstract</td>
<td>3</td>
</tr>
<tr>
<td>Contents</td>
<td>4</td>
</tr>
<tr>
<td>List of figures</td>
<td>8</td>
</tr>
<tr>
<td>List of tables</td>
<td>13</td>
</tr>
</tbody>
</table>

## 1 Introduction

1.1 Analytical Techniques.                                      | 15   |
1.2 Numerical Methods.                                          | 17   |
1.3 The Poincaré Map.                                           | 18   |
1.4 Attractors.                                                 | 18   |
1.4.1 Fixed Point.                                              | 19   |
1.4.2 Limit Cycle.                                              | 19   |
1.4.3 Torus.                                                    | 20   |
1.4.4 Chaos.                                                    | 20   |
1.5 Basins of Attraction.                                       | 21   |
1.6 Locating Solutions.                                         | 22   |
1.7 Unstable Fixed Points.                                      | 25   |
1.7.1 Importance of Unstable Solutions.                         | 27   |
1.8 Path Following.                                             | 28   |
1.9 Bifurcational Behaviour.                                    | 29   |
1.9.1 Flip Bifurcation.                                         | 30   |
1.9.2 Fold Bifurcation.                                         | 31   |
1.9.3 Hopf Bifurcation.                                         | 32   |
1.9.4 Symmetric Bifurcations.                                   | 33   |
1.9.4.1 Super-critical Pitchfork Bifurcation.                   | 34   |
1.9.4.2 Sub-critical Pitchfork Bifurcation.                     | 35   |
1.9.4.3 Symmetry Breaking Bifurcation.                          | 35   |
1.9.5 Bifurcation Following.                                    | 37   |
1.9.6 Global Bifurcations.                                     | 37   |
1.9.6.1 Heteroclinic Saddle Connection.                         | 38   |
1.9.6.2 Homoclinic Saddle Connection.                           | 38   |
1.10 Nonlinear Dynamical Systems - a Numerical Approach.        | 39   |
List of figures

1.1 The parametrically excited pendulum. 40
1.2 Divergence of initial conditions. 40
1.3 Cell mapping. 41
1.4 Basins of attraction. 42
1.5 Saddle point with stable and unstable manifolds. 43
1.6 Stable and unstable manifolds. 44
1.7 Time series before and after flip bifurcation. 45
1.8 Characteristic multipliers and phase portrait for flip bifurcation. 46
1.9 Example of period doubling cascade. 47
1.10 Characteristic multipliers and phase portrait for fold bifurcation. 48
1.11 Characteristic multipliers and phase portrait for Hopf bifurcation. 49
1.12 Graphical iteration of the cubic map. 50
1.13 Graphs of \(x_{n+2} = x_n\) around super-critical pitchfork bifurcation. 51
1.14 Super-critical pitchfork bifurcation. 52
1.15 Sub-critical pitchfork bifurcation. 52
1.16 Graphs of \(x_{n+2} = x_n\) around symmetry breaking bifurcation. 53
1.17 Symmetry breaking bifurcation. 54
1.18 Heteroclinic saddle basin bifurcation. 55
1.19 Blue sky catastrophe. 55
2.1 Analytically calculated boundary to primary Mathieu zone. 63
2.2 Numerically calculated 1st and 2nd unstable zones. 64
2.3 Comparison between numerically and analytically calculated zone. 65

3.1 Bifurcation diagram for parametrically excited pendulum. 80
3.2 Locus of symmetry breaking bifurcation. 81
3.3 Approximation to symmetry breaking bifurcation. 82
3.4 Phase portrait for the undamped unforced pendulum. 82
3.5 Approximate escape locus. 83
3.6 Time history of strange attractor. 83
3.7 Poincaré section of half of strange attractor. 84
3.8 Destroyer saddle. 84
3.9 Bifurcation diagram for parametrically damped system. 85
3.10 Integrity curves. 86
3.11a Stable manifolds of hill-top saddles. 87
3.11b Safe basin. 87
3.12 Heteroclinic tangency. 88
3.13 Bifurcation diagram for parametrically excited pendulum. 89
3.14 Invariant manifolds. 90
3.15 Idealised invariant manifolds. 91
3.16 3-shoe. 91
3.17 Invariant manifolds. 92
4.1 Analytical approximation to upper stability boundary. 97
4.2 Analytical approximation to upper stability boundary. 98
4.3 Schematic bifurcation diagram. 99

5.1 Phase portraits of rotating orbits. 105
5.2 Bifurcation diagram for rotating orbits. 106
5.3 One quarter of Poincaré section for rotating chaos. 107
5.4 Bifurcation diagram for rotating orbits. 108

6.1 Poincaré section of tumbling chaos. 112
6.2 Divergence of initial conditions. 113
6.3 Largest Lyapunov exponent. 114
6.4 Total angular displacement plots. 115
6.5 Averaged power spectral density. 116
7.1 Braid diagram formed from time history. 139
7.2 Period-4 knot. 140
7.3 Link diagram. 141
7.4 Reidemeister moves. 142
7.5 Crossings sliding past each other. 143
7.6 Reduced Burau matrices. 143
7.7 Torus knots. 144
7.8 Decomposition of reducible orbit. 144
7.9 Moduli of eigenvalues. 145
7.10 Cubic map with two turning points. 146
7.11 Cubic map with period-3 orbit. 146
7.12 Action of the cubic map. 147
7.13 Symbolic tree structure. 147
7.14 Spiral 3-shoe. 147
7.15 Homoclinic tangle. 148
7.16 Homoclinic tangle with main pips. 148
7.17 Schematic homoclinic tangle. 150
7.18 Framed braid. 151
8.1 Idealised 3-shoe.
8.2 Location of period-3 orbit in 3-shoe.
8.3 Location of period-3 orbit in trellis.
8.4 Location of period-3 orbit in escape zone.
8.5 3-shoe operations.
8.6 Period-2 bifurcational subform.
8.7 Period-3 period doubling cascade.
8.8 Pseudo Anosov orbit.
8.9 Moduli of eigenvalues.

9.1 Period-2 orbit and close recurrent trajectory.
9.2 Some unstable rotating orbits.
9.3 Framed braid.

10.1 Experimental setup.
List of Tables

8.1 Crossings matrix for period-1 and period-2 orbits. 175
8.2 Crossings matrix for all period-3 orbits. 176
8.3 Word length for orbits up to period-7. 176
8.4 Alexander polynomials. 177
8.5 Orbit classification. 177
8.6 Topological entropy. 178
8.7 Linking numbers of implied orbits. 178

9.1 Number of (n,r) rotating orbits. 180
After familiarising the reader with the analytical and numerical techniques applied in the following chapters we begin a systematic analysis of the various categories of the pendulum motion. In chapter 2, the hanging solution is considered in terms of stability. Non-rotating solutions are considered in chapter 3, which are the main focus of attention in terms of engineering applications, whilst inverted solutions are covered in chapter 4. An initial attempt at classifying rotating solutions is made in chapter 5, with the special case of tumbling chaos covered more fully in chapter 6. The powerful techniques of braid and knot theory are explained in chapter 7, before being applied to the non-rotating orbits of the parametrically excited pendulum in chapter 8. Additional subharmonic rotating orbits are covered in part in chapter 9, whilst chapter 10 mentions some experimental observations. Chapter 11 ends with some conclusions, and suggestions for future work.
Chapter 1: Introduction

The parametrically excited pendulum of figure 1.1, a simple vertically forced rigid pendulum constrained to move in the plane is a paradigm for nonlinear systems in that despite its apparent simplicity, a wide range of possible solutions and bifurcational behaviour can be realised over a moderate range of operating parameters. The system can also be used as a physical demonstration [Leven et al. 1985] to give a convincing visual display of many nonlinear phenomena, including chaos. As well as the obvious mathematical attraction to studying the parametrically excited pendulum [Mawhin 1988], there is a strong interest to study this system within the engineering community, as the pendulum is a good rough approximation for many systems, from the behaviour of off-shore structures [Rainey 1977], crane barges [McCormick and Witz 1993], buildings in earthquakes [Housner 1963], to the response of Josephson junctions [Salam & Sastry 1985]. In all these applications, despite the obvious simplification in the modelling process, much of the behaviour of the system can be characterised at least qualitatively by the parametrically excited pendulum. Other pendulum models that have been studied include the pendulum with applied torque [Gwinn 1986, Blackburn et al. 1987], the horizontal forced pendulum [Bayly & Virgin 1992], and the spherical pendulum [Tritton 1986]. All the systems possess similarities, which can be expected both on the fundamental level that the systems are in some way physically similar, and also by a stronger topological basis that will be developed in chapter 3. These qualitative similarities make the study of the parametrically excited pendulum of even greater importance, and under certain conditions the pendulum may be considered as typical
Although the results contained here are only strictly applicable in a quantitative sense to the parametrically excited pendulum, it is possible to make more general statements about the qualitative behaviour of higher dimensional systems in the light of the insight gained into parametrically excited systems. For instance, many of the bifurcations observed in the simple parametrically excited pendulum may occur in higher dimensional systems, giving an understanding of the mechanisms of how stability may be lost, and hence how failure may occur for a wide range of engineering systems. The fundamental philosophy applied here is that the dynamics of a high dimensional engineering system in a noisy environment may be modelled at least qualitatively by a simple low dimensional model. This approach has many advantages; computationally the problem becomes easier and faster to simulate, powerful analytical methods may be applied directly, and instead of merely noting behaviour, the dynamics of the system may be understood in terms of why we observe what we do. By simplifying the problem, and consequently allowing a large number of independent methods to be applied simultaneously, the individual findings are reinforced, and additional insight may be gained into the underlying dynamical processes at work. Such a combined approach is carried out in the following chapters.
of an even larger class of systems with symmetric potential energy functions [Clifford & Bishop 1993]. Before we begin a systematic description of the bifurcational behaviour of the parametrically excited pendulum, we outline some analytical and numerical techniques used in the following chapters.

1.1 Analytical Techniques

A wide variety of analytical techniques exist for the analysis of nonlinear ordinary differential equations [Nayfeh & Mook 1979, Hayashi 1964]. These typically involve linearisation of the system to give an approximate solution, and then additional nonlinear terms are accounted for by small perturbations from the original linear solution. There are many problems associated with such an approach. The predicted solution may or may not be a good approximation to the system, and the only way of checking is to do some physical experimentation or numerical integration. Also, inadmissible sequences of bifurcations may be predicted as noted by Thompson [Thompson 1989]. The advantages of the analytical techniques as opposed to numerical methods are demonstrated when a general picture of the dynamics of the system is required rather than the detailed behaviour of one particular trajectory. This can produce a broad outline of the possible parameter regions of interest, and highlight areas where numerical study would be beneficial. Also, analytical methods can provide 'bounds' on the system [McRobie & Thompson 1993a]. For example, in the case of systems which permit escape from a potential well, it may be useful to develop a region where, if any trajectory enters, it must
escape to an attractor at infinity. This would make deciding whether a trajectory has escaped considerably easier when numerical integration is applied.

Historically, analytical methods are considerably older than numerical methods, and it could be expected that these methods would fall into disuse with the advent of greater computing power. However, the introduction of algebraic manipulation packages have in fact increased the use of analytical methods. Here, we will make use of two analytical techniques, harmonic balance, and the method of strained parameters. The latter will be considered in detail in chapter 2. Harmonic balance [Hayashi 1964] is perhaps the oldest analytical method, as it is based on the traditional method of solving linear ordinary differential equations by substituting in an assumed solution containing sinusoidal components with unknown amplitudes, and calculating the unknown coefficients. For nonlinear ordinary differential equations, an initial sinusoidal approximate solution is assumed, and the coefficients are equated. However, there will be terms with frequencies other than the assumed solution, and so a second approximate solution containing these terms is assumed, and any unknown coefficients are evaluated by substituting into the original equation. The process may be iterated to include a large number of terms to give a closer approximation to the true solution. The method has undergone many recent advances to improve the accuracy and speed of calculating solutions [Lsu et al. 1982, Wong et al. 1991, Ling & Wu 1987] which we will not consider further, as for our purpose all we require is a simple one term approximate solution to the behaviour of the system.
1.2 Numerical Methods

A nonlinear dynamical system can be described by a set of autonomous first order ordinary differential equations of the form:

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n \]  \hspace{1cm} (1.1)

Typically we will consider systems in three dimensions with time periodic forcing, and so equation (1.1) becomes

\[ \dot{x} = f(x,t), \quad x \in \mathbb{R}^2 \times S^1 \]  \hspace{1cm} (1.2)

where \( x \) is a vector.

Solutions to equation 1.2 can be calculated relatively easily by a numerical integration technique such as a high order Runge-Kutta algorithm. The inputs are the initial condition vector \( x_0 \), and starting time \( t_0 \). The behaviour is computed by calculating \( x \) at time \( t_0 + \Delta t \) where \( \Delta t \) is a small time step. The procedure is iterated to produce the flow \( x(t) \) of the equation. Typically we take 200 steps per period \( T \) of the applied forcing function. The numerical solution will consist of a transient and a steady state solution. Both are of interest to engineers as in the design of systems both the long term and short term behaviour are important. For example, if the amplitude of some system is to be kept within bounds, say in the case of a mechanical system to avoid collision with other components, the long term behaviour must be within these limits, but also the transient behaviour must not be too large. On the whole we will concentrate on the long term behaviour of systems, although transient behaviour will be partially covered in chapter 3.
1.3 The Poincaré Map

The stroboscopic Poincaré section is defined as an $n-1$ dimensional hypersurface transversal to the flow in the $n$ dimensional space spanned by $x$ [Guckenheimer & Holmes 1983]. This is obtained simply by sampling the flow at $t \mod T = 0$. A solution can then be described by the series of points at which the flow intersects the Poincaré section, $a_i, a_{i+1}, a_{i+2}, ...$ where $a_{i+1} = P(a_i)$, and $P(x)$ is the Poincaré map. This reduction of an $n$ dimensional flow to an $n-1$ dimension map allows the same phenomena and problems of the qualitative theory of ordinary differential equations to be studied in their simplest forms [Smale 1967]. A mapping may also be called a diffeomorphism. The Poincaré map can be computed numerically by integrating equation (1.2) between $t=0$ and $T$ in an integer number of steps.

1.4 Attractors

Once the transient behaviour has died down, the solution of equation (1.2) settles on to an attractor or steady state solution [Thompson & Stewart 1986]. Some possible attractors are discussed below.
1.4.1 Fixed Point

A fixed point is an attracting point in phase space. That is, all nearby trajectories will settle on to the fixed point. The classic example of a fixed point is the hanging solution of a simple damped pendulum. Any initial condition will produce a trajectory which decays to this hanging solution as time increases. The rate at which the oscillations die away will depend on the damping present.

1.4.2 Limit Cycle

If we introduce a small periodic horizontal force to the simple damped pendulum, we expect the pendulum to oscillate periodically. This is an example of a limit cycle, or periodic attractor. Taking a Poincaré section after the transient behaviour has died away will yield a number of points. If there is only one point, then the period of oscillation is identical to the period of the forcing - hence we have a period-1 attractor. If, however there are \( p \) points in the Poincaré section, then the period of oscillation is \( p \) times the period of the forcing. Hence we call the solution a period-\( p \) attractor or limit cycle. Limit cycles with \( p > 1 \) are sometimes called subharmonic solutions. Note that a fixed point of a map corresponds to a limit cycle in the flow. Superharmonics are also possible where the period of oscillation is a fraction of that of the forcing. However, these solutions will only give one point on a Poincaré section making it easy to confuse a superharmonic with a simple period-1 solution. Superharmonics will not be covered in detail for the systems considered.
1.4.3 Torus

If a system responds at an incommensurate frequency to the forcing frequency we will see a series of dots on the Poincaré map winding around a circle. Such behaviour requires \( n \geq 4 \) and will not be of further interest here, as the systems we consider have \( n \leq 3 \).

1.4.4 Chaos

An asymptotic steady state attractor which is not a fixed point, limit cycle, or torus is often called chaotic [Tufillaro et al. 1992]. This term is unspecific, but nevertheless is useful for categorising the behaviour of many nonlinear systems. The Poincaré map for a chaotic attractor is composed of an infinite set of points which are dense, and form a fractal set, as opposed to a noisy or random response which when sampled would fill large regions of phase space. Chaotic trajectories have many properties that distinguish them from noise. One is the exponential divergence of nearby initial conditions. Given two nearby initial conditions, we would expect the solutions to converge as the integration proceeds. However, for a chaotic system, this is not the case; the trajectories diverge exponentially, and soon become entirely uncorrelated. This is shown in figure 1.2. The consequence for real engineering systems is that any measurement error in specifying the initial states of a chaotic system will be magnified with time. In effect, predicting the medium term behaviour of chaotic systems is impossible. However, by considering the nature of the chaotic
attractor, we may be able to produce some bounds on the behaviour. Another feature of a chaotic system is that short portions of the time series may appear to be nearly periodic as the chaotic attractor contains many unstable periodic orbits [Eckmann & Ruelle 1985]. We will see how this property can be exploited in chapter 9.

1.5 Basins of Attraction

Given that a nonlinear system may possess more than one attractor, we need to have some measure of the relative importance of each attractor in terms of how likely we are to achieve each solution. The set of all initial conditions which lead to an attractor is termed the basin of attraction. Various measures have been proposed to measure the area of a basin [Soliman & Thompson 1989, McRobie & Thompson 1991, Schiehlen 1993], and will be discussed further in chapter 3.

Basins of attraction can be calculated numerically by dividing a section of phase space into a grid of starts and integrating each point until an attracting solution is achieved. The finer the grid, the more accurate the definition of the basin of attraction [Foale & Thompson 1991]. However, this procedure is lengthy, and can be improved considerably by a cell to cell mapping approach [Hsu 1987]. The Poincaré section is again divided into a grid of small cells. We simplify the procedure by assuming that the whole contents of a cell are mapped into another single cell. Figure 1.3 shows the technique. We start with an empty cell C1, and map the centre of this cell forward. The resulting point lands in cell C2. We re-centre, and map forward to cell C3 and so on. The procedure is continued until a
non-empty cell is visited. If the new cell is in the string of cells encountered, then we have located a limit cycle. The period of this limit cycle can be calculated by the number of iterations it takes to return to this cell. All cells \( C_1, C_2, \ldots \) are labelled as being in the basin of attraction of this new attractor. Alternatively, we have landed in a cell which has already been identified as being in the basin of another attractor that has previously been located. If this is the case, the cells \( C_1, C_2, \ldots \) are labelled as belonging to this attractor. The method has been carried out, and by colouring in the basins of different attractors accordingly, figures such as figure 1.4, where the basins of three competing attractors are shaded accordingly.

### 1.6 Locating Solutions

We have already noted that by allowing a trajectory of a system to settle after a number of iterates of the Poincaré map, attractors can be located. However, if the dissipation or damping of the system is low, such convergence will take a long time to occur. A more rapid method is the Newton-Raphson root finding method outlined below. The added attraction of this method is that unstable as well as stable solutions can be easily located.

Following [Foale & Thompson 1991], we again make use of the Poincaré map \( P(x) \). A period-\( p \) limit cycle \( x = x_0 \) will satisfy:

\[
P^p(x_0) - x_0 = 0 \tag{1.3}
\]
If we define the residual map $G(x_0)$ as

$$G(x_0) = P^p(x_0) - x_0 \quad (1.4)$$

we require the solution

$$G(x_0) = 0 \quad (1.5)$$

Differentiating equation (1.4) with respect to initial condition, $x_0$ gives:

$$D G(x_0) = D P^p(x_0) - I \quad (1.6)$$

$D P^p(x_0)$, the Jacobian [Thompson & Stewart 1986] of the Poincaré map can be calculated by either a simple finite difference method, or by variational equations.

If we write

$$\dot{x}_n = f_n(x), \quad x = (x_1, x_2, x_3, ..., x_n)^T \quad (1.7)$$

and differentiate with respect to initial condition $x_0$, we have:

$$\dot{D}_{x_0} \dot{x} = D_x f(x) D_{x_0} x \quad (1.8)$$

Define:

$$X = D_{x_0} x \quad (1.9)$$

Then equation (1.8) becomes:
\[ \dot{x} = D_x f(x) x \]  

(1.10)

\( D_x f(x) \) can be written in full as:

\[ Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \]  

(1.11)

and so by numerically integrating the Jacobian with initial conditions \( I \), we can calculate \( DP^p(a) \).

Using the Newton-Raphson root finding method, a better estimate of the fixed point of the map is given by:

\[ a_{i+1} = a_i - [DG(a_i)]^{-1} G(a_i) \]  

(1.12)

\[ a_{i+1} = a_i - [DP^p(a_i) - I]^{-1} G(a_i) \]  

(1.13)
1.7 Unstable Fixed Points

We have already indicated that the Newton-Raphson method can locate unstable solutions. An unstable solution is a steady state solution which a typical trajectory will depart from exponentially with time. For example, the simple damped unforced pendulum could be balance upside down, but any small perturbation would result in the pendulum toppling over. There are unstable limit cycles, unstable tori, and even unstable chaotic attractors. The stability of a particular solution $x(t) = x_i(t)$ can be assessed in terms of the moduli of the eigenvalues of the derivative matrix:

$$Df(x_i) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

(1.14)

If any of the moduli of the eigenvalues are greater than 1, then the solution is unstable, whilst a stable solution has all its eigenvalues contained inside the unit circle [Thompson & Stewart 1986]. The case of a fixed point with eigenvalues both inside and outside the unit circle is a saddle cycle.

For a two dimensional map, the saddle will have local stable and unstable manifolds $W^s$ and $W^u$ tangent to the eigenvectors of the derivative matrix evaluated at the saddle point. The complete stable and unstable manifolds are given by the set of points in the Poincaré section which map on to these local manifolds [Guckenheimer & Holmes 1983]. Consider the saddle shown in figure 1.5. A typical trajectory will approach and leave the saddle along a path described by the points $T_i$ where $i = 0,1,2,...$ However, a trajectory which starts on the stable manifold $W^s$ will approach the saddle
asymptotically by mapping along the points $S_i$. Conversely, the point $U_0$ is mapped away from the saddle along the unstable manifold. It is important to note that the stable and unstable manifolds are not trajectories, but the set of all points in the map which approach or leave the saddle asymptotically under the action of the mapping function. For this reason, the stable manifold is also known as the *inset*, and the unstable manifold as the *outset*.

The stable and unstable manifolds of a saddle cycle can be obtained numerically by firstly locating the unstable fixed point of the map as above, and calculating the eigenvectors of the Jacobian. The larger eigenvalue corresponds to the unstable manifold, and the smaller to the stable manifold. For the unstable manifold, a small step is taken along the linearised eigenvector, and the resulting initial condition is integrated through a desired number of cycles. A second step is then taken along the eigenvector, and the resultant initial condition is integrated as before. The procedure is iterated, with the size of the step along the eigenvector being determined by the distance between consecutive mappings of the previous two initial conditions. This method is considerably simpler than that used by Alexander [Alexander 1989], which involves interpolating between steps. It is necessary to take a variable step size as the expansion along the unstable manifold is not uniform. A similar procedure is followed for computing the stable manifold, but by integrating backwards in time. An example of stable and unstable manifolds calculated in this manner is shown in figure 1.6.
1.7.1 Importance of Unstable Solutions

At first glance, unstable solutions are unimportant to engineers; they are not attracting solutions, and so are not observed as solutions in real systems. However, unstable solutions are of vital importance for many reasons. Firstly, the unstable manifolds of saddles can form the boundaries between competing attractors. We have seen already the importance of calculating basin boundaries. Unstable solutions can be controlled by simple modern control techniques [Shinbrot et al. 1993], and by carefully choosing an unstable solution, the performance of a given system may be enhanced in some measurable way. Also, the unstable solutions form a skeleton around which trajectories wind [Artuso et al. 1990a,b], and by describing a few unstable solutions, the nonlinear system may be characterised and completely understood [Mindlin et al. 1990]. These startling claims will be expanded in chapter 7. Unstable solutions can also be followed as the parameters of a system are changed, at bifurcations and may stabilise. We will consider this further in section 1.8.
1.8 Path Following

As the parameters governing the behaviour of a system are allowed to vary, the solutions will generally evolve slowly except at bifurcations [Thompson & Stewart 1986]. A particular solution or fixed point of the Poincaré map can be followed by continuously applying the Newton-Raphson scheme at every small change in the parameter. However, the technique is further improved by predicting how the solution will evolve. This is achieved by redefining the residual map to include the variation of a parameter, \( \alpha \). Equation (1.14) becomes

\[
Df(x_0) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial \alpha} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial \alpha} \\
0 & 0 & 1
\end{bmatrix}
\]  

(1.15)

and the method is applied as before as shown by Foale and Thompson [Foale & Thompson 1991].
1.9 Bifurcational Behaviour

We have already seen that a nonlinear dynamical system can often be modelled (section 1.2) in terms of a low dimensional ordinary differential equation. Typically three dimensions are enough to characterise a large amount of the behaviour found in higher dynamical systems. Often in these systems we are concerned with qualitative changes of behaviour in the system due to the variation of a control parameter or parameters. This may be achieved by studying the bifurcational behaviour [Abraham & Shaw 1988]. The term bifurcation refers to the branching of one path of equilibrium to an alternate qualitatively different path or paths by varying a control parameter [Guckenheimer & Holmes 1983]. Bifurcation theory is vital to the understanding of nonlinear dynamical systems, and here the bifurcations are considered by varying one control parameter only. These are known as codimension one bifurcations. Further bifurcations are possible with the control of two or more parameters.

Consider the ordinary differential equation:

$$\ddot{x} + c\dot{x} + \alpha x^3 = F(t)$$  \hspace{1cm} (1.16)

where $x$ is a variable (displacement, angle, current, etc), $c$ is a damping constant, $\alpha$ is a parameter, $F(t)$ is a time dependant periodic external forcing function, and a dot denotes differentiation with respect to time, $t$.

The equation can be rewritten as a system of three first order differential equations:
\begin{align*}
\dot{x}_1 &= x_2 = f_1(x) \\
\dot{x}_2 &= F(t) - cx_2 - \alpha x_1^3 = f_2(x) \tag{1.17} \\
t &= 1
\end{align*}

Solutions of equation (1.17) can be located by the variety of numerical and analytical techniques considered earlier, and the stability of a particular solution \( x(t) = x_q(t) \) can be assessed in terms of the moduli of the eigenvalues of the Jacobian. As the control parameter is varied, the eigenvalues or characteristic multipliers change, and one may leave the unit circle at a bifurcation. The type of bifurcation produced depends crucially on where the characteristic multiplier leaves the unit circle, and the other solutions in the vicinity of the bifurcating solution. Some typical bifurcations that occur in the systems we shall be interested in are considered below.

1.9.1 Flip Bifurcation

The flip, or period-doubling bifurcation occurs when a stable limit cycle with period \( T \) becomes unstable giving rise to a stable limit cycle of period \( 2T \). This qualitative change in behaviour is not as drastic as it sounds since it may be achieved by slightly flattening every other peak in the time series shown in figure 1.7. This is an example of a subtle bifurcation [Abraham & Shaw 1988]. The term 'flip' indicates that the new solution oscillates about the old solution. A period-doubling bifurcation is shown schematically together with the characteristic multipliers in figure 1.8. We
see that at the bifurcation, a characteristic multiplier leaves the unit circle through -1. The old period 1 limit cycle becomes a saddle cycle, and since a typically close trajectory will oscillate about this saddle cycle it is an inverting or I-type saddle. A period-doubling bifurcation of a period T to a period 2T limit cycle may be followed by another period-doubling to a period 4T limit cycle and so on. This cascade of bifurcations has been studied by Feigenbaum [Feigenbaum 1980], and often occurs in nonlinear systems. A period-doubling cascade is shown in figure 1.9.

1.9.2 Fold Bifurcation

The fold, saddle-node, or tangent bifurcation is caused by the collision of a saddle cycle with a stable limit cycle, resulting in the annihilation of both the solutions. This is an example of a catastrophic bifurcation since the bifurcation leaves no stable solution behind [Abraham & Shaw 1988]. The consequences for nonlinear systems are that a small change in a parameter may produce a large change in behaviour i.e. from a small amplitude solution to a coexisting large amplitude solution, or worse still, to no steady state solution at all. Hence the term catastrophic bifurcation. This scenario is depicted in figure 1.10 together with the characteristic multipliers of both the stable and unstable solutions. At the bifurcation, one characteristic multiplier for both the stable and unstable limit cycle equals +1. The original saddle cycle had two positive characteristic multipliers; a typical approaching or departing solution would stay on the same side of the saddle cycle, making it a Direct or D-type saddle cycle. After the bifurcation there are no local steady state
solutions, so trajectories wander out of the section to a distant stable limit cycle, or go to infinity.

1.9.3 Hopf Bifurcation

So far we have only considered the cases where the one of the characteristic multipliers leave the unit circle along the real axis. If one of the characteristic multipliers is imaginary, then the other will be its complex conjugate. Hence it is possible for a pair of characteristic multipliers to leave the unit circle as a complex conjugate pair. This occurs when a fixed point bifurcates into a stable limit cycle at a Hopf bifurcation [Thompson & Stewart 1986]. This is shown in figure 1.11. However, for the systems we will be interested in, there will always be a periodic forcing term, and so fixed point solutions are inadmissible; the lowest dimension of a solution will be 1 since the phase space is $\mathbb{R}^2 \times S^1$ and not $\mathbb{R}^3$. This will become clearer when a specific example which looks like a Hopf bifurcation is studied in chapter 3.
1.9.4 Symmetric Bifurcations

Symmetric systems possess at least three additional codimension 1 bifurcations. Whilst some may argue that symmetry is somehow nongeneric, this is not a view shared with nature, and strong evidence exists of symmetric bifurcations in experimental systems [Stewart & Golubitsky 1993, Mullin 1993]. To study symmetric bifurcations, consider the difference equation:

\[ x_{n+1} = \alpha x_n^3 + (1 - \alpha)x_n \]  \hspace{1cm} (1.18)

with x defined over the interval \([-1, +1\]). Just as integrating a second order ordinary differential equation over one period of forcing produced a two dimensional mapping, equation (1.18) represents a one dimensional map of the interval \([-1, +1\]) to itself under the mapping:

\[ x_{n+1} = F(x_n) \]  \hspace{1cm} (1.19)

The advantage of studying explicit mappings like equation (1.18) as opposed to integrating nonlinear ordinary differential equations is speed of computation, and ease of analytical solution. Here, the stability of a solution \(x = x_0\) is determined by the derivative of \(F(x)\) evaluated at \(x = x_0\). If the derivative has a modulus greater than one then the solution is unstable, whilst a value less than one indicates a stable solution. Note that the derivative must be real, ruling out a Hopf bifurcation. Due to the symmetric nature of the problem, if \(x = x_0\) is a solution, then \(x = -x_0\) is also a solution.
The iteration of a one dimensional map can be carried out graphically as in figure 1.12. We start at point A with \( x_0 = -0.8 \), and take a line up to point B and read off \( x_1 = 0.352 \), next take a horizontal line across to the dotted line \( x = x^* \), and then vertically down to point C \( x_2 = -0.88154 \), and so on.

1.9.4.1 Super-critical Pitchfork Bifurcation

The cubic map equation (1.18) [Mullin 1993] has an obvious period-1 solution \( x = 0 \). The stability of this solution can be considered by evaluating the derivative at \( x = 0 \).

\[
\frac{dF}{dx_n} \bigg|_{x_n=0} = 3\alpha x_n^2 + 1 - \alpha = 1 - \alpha
\]

(1.20)

Hence for \( \alpha < 2 \) the solution is stable, and for \( \alpha > 2 \) the solution is unstable. At the bifurcation, the derivative is -1, and so we may expect the birth of a period-2 solution. To see this, graphs of \( x_{n+2} \) against \( x_n \) for values of alpha around the bifurcation value are shown in figure 1.13. Indeed we see that a stable symmetric period-2 fixed point results as the bifurcation value is passed. This is very similar to the flip bifurcation already considered, but since the initial period-1 solution is the trivial \( x = 0 \) solution, and a symmetric (non zero) period-2 solution results, this bifurcation is called a super-critical pitchfork bifurcation due to the picture produced from the bifurcating solutions shown in figure 1.14 [Mullin 1993]. The symmetric
period 2 solutions can be calculated by putting $x_{n+1} = -x_n$ into equation (1.18). After some manipulation we discard the trivial zero solution to be left with:

$$x^2 = \frac{\alpha - 2}{\alpha} \quad (1.21)$$

We see immediately that this solution is only possible for $\alpha > 2$, the value already obtained for the super-critical pitchfork bifurcation.

1.9.4.2 Sub-critical Pitchfork Bifurcation

A catastrophic bifurcation for symmetric systems may be obtained by swapping the stable and unstable solutions from the super-critical pitchfork bifurcation to give the scenario displayed in figure 1.15. Here an unstable symmetric period-2 solution collapses around the stable zero solution to leave only the unstable zero solution. This is termed a sub-critical pitchfork bifurcation.

1.9.4.3 Symmetry Breaking Bifurcation

Consider the symmetric period-2 solution obtained in equation (1.21). As the parameter $\alpha$ is further increased we may expect further bifurcations to occur. Following Mullin [Mullin 1993], and writing the symmetric solution as $x = \pm s$, we can evaluate the stability of the solution by the second derivative:
\[
\frac{d^2 F}{dx^2} \bigg|_{x=s} = \frac{dF}{dx} \bigg|_{x=s} = \frac{dF}{dx} \bigg|_{x=s} dx \bigg|_{x=s}
\]

(1.22)

Hence the second derivative must always be positive, ruling out a flip bifurcation to a symmetric period 4 solution. The only way to lose stability is for the second derivative to equal +1. The value of alpha for this bifurcation can be calculated from equation (1.21).

\[
\left[ \frac{dF}{dx} \right]_s^2 = 1
\]

\[
\therefore \left( \frac{3 \alpha (\alpha - 2)}{\alpha} + 1 - \alpha \right)^2 = 1
\]

(1.23)

\[
(2\alpha - 5)^2 = 1
\]

\[
\alpha = 3
\]

Plotting the graph of \(x_{n+2}\) against \(x_n\) around the bifurcation value in figure 1.16 shows that the unstable symmetric period two solution, S has produced two stable 'mirror image' period two solutions, A and B. This is termed a symmetry breaking since the stable symmetric solution is destroyed bifurcation. The bifurcation diagram for this event is shown in figure 1.17.
1.9.5 Bifurcation Following

Bifurcations can be followed in parameter space by adding the constraint that one of the eigenvalues of the Jacobian is $+1$, $-1$, or of modulus 1 for the respective bifurcations considered. The variational method cannot be used in this case and so we must resort to the finite difference method [Foale & Thompson 1991].

1.9.6 Global Bifurcations

Up to now, we have only considered local bifurcations. That is, bifurcations that occur due to periodic orbits colliding and losing stability. These bifurcations only affect trajectories in the neighbourhood of the attractors. In contrast, global bifurcations affect whole basins of attraction, and can be caused by heteroclinic and homoclinic saddle connections [Abraham & Shaw 1988]. A heteroclinic saddle connection is a connection between two saddles, whilst a homoclinic saddle connection is where the stable and unstable manifolds of a saddle are joined. A heteroclinic saddle connection is shown in figure 1.18, and a homoclinic saddle connection is shown in figure 1.19.
1.9.6.1 Heteroclinic Saddle Connection

A heteroclinic saddle connection can cause the basins of attraction of competing attractors to be suddenly altered. This is shown in figure 1.18. There are three competing attractors (not shown in the figure), with basins hatched accordingly. The grey basin is unaffected by the sequence of events leading up to the saddle connection, whilst the other two basins are dramatically altered by this event. Since the bifurcation only causes the basins of attraction to be altered as opposed to the nature of any attractor, the event is called a basin bifurcation [Abraham & Shaw 1988].

1.9.6.2 Homoclinic Saddle Connection

Homoclinic saddle connections are a rich source of bifurcational activity [Tufillaro et al. 1992]. Here we will show how a homoclinic saddle connection causes an attractor and large basin of attraction to be created out of the blue. This is an example of a blue sky catastrophe [Abraham & Shaw 1988]. Figure 1.19 shows the formation of a loop composed of the stable and unstable manifolds of the saddle. Any point on this loop must leave the saddle along the unstable manifold, and return to the saddle along the stable manifold. However, this must take an infinite amount of time. Any points within this loop must therefore remain within the loop for all time, creating a stable limit cycle as the loop breaks. This sequence of events can of course be reversed by decreasing the control parameter. Hence a stable limit cycle with a
large basin may be instantaneously destroyed leaving no attractor. Hence this is a catastrophic bifurcation.

1.10 Nonlinear Dynamical Systems - a Numerical Approach

We have developed the necessary tools for analysing the behaviour of nonlinear dynamical systems expressed as ordinary differential equations. These tools, when used carefully and systematically can provide an understanding of complex nonlinear phenomena. Here we outline the approach advocated by many dynamicists:

1) Carry out a cell mapping algorithm on a region of phase space that is likely to contain the relevant attractors. Repeat at various parameter values to locate stable solutions.

2) Given the stable solutions located by step 1, path follow these solutions through parameter space to produce typical response curves. Note any bifurcations encountered in path following.

3) Follow the bifurcations by adding the eigenvalue constraint to the minimisation process and produce a bifurcation diagram.

4) Carry out further cell mappings at parameter values that are relatively unexplored to locate additional attractors, and proceed as before.

This philosophy will be applied to the parametrically excited pendulum in chapters 2, 3, 4, 5 and 6, after which an alternative strategy will be employed based on a topological approach.
Figures for Chapter 1

**Figure 1.1:** The parametrically excited pendulum, mass $m$, length $l$, subject to periodic displacement $z(t)$, and constrained to move in the plane.

**Figure 1.2:** Two nearby initial conditions are integrated numerically over 100 cycles of forcing. After 55 cycles, the two trajectories are clearly distinguishable, indicating the divergence of chaotic trajectories.
Figure 1.3: Cell mapping procedure: the Poincaré plane is divided into a number of cells. The centre of cell C1 is mapped forwards to cell C2. The centre of cell C2 is mapped forwards to cell C3, and so on. Cell C7 maps back into cell C6 indicating a possible period-2 attractor. All cells in the string are then shaded indicating that they are in the basin of the period-2 attractor.
Figure 1.4: Example of basins of attraction found using a cell mapping algorithm with 200 x 200 cells. The different shaded regions indicate 3 competing attractors.
Figure 1.5: Saddle point with stable and unstable manifolds $W^s$, and $W^u$ respectively. Points $T_0, T_1, ...$ are successive mappings of a typical trajectory which approaches the saddle and then leaves. Points $S_0, S_1, ...$ are successive mappings of a trajectory along the stable manifold which approach the saddle exponentially, whilst points $U_0, U_1, ...$ are successive mappings of a trajectory along the unstable manifold which leave the saddle exponentially.
Figure 1.6: Stable and unstable invariant manifolds of saddle labelled $W^s$, $W^u$ respectively, obtained by the method outlined in section 1.7.
Figure 1.7: Time series before and after flip bifurcation. In the top figure, the control parameter $\mu < \mu_c$ where $\mu_c$ is the value at the bifurcation. In the bottom figure, the control parameter $\mu > \mu_c$. Note that after the bifurcation every other peak is slightly higher or lower than the previous peak indicating a period-2 solution.
Figure 1.8: Characteristic multipliers and phase portrait for flip bifurcation. Solid circles indicate stable solutions, and hollow circles indicate unstable or metastable solutions. The stable period-1 solution gradually loses stability as one of the characteristic multipliers leaves the unit circle at -1. After the bifurcation the period-1 solution is unstable; a saddle cycle, and a period-2 stable solution remains indicated by two solid circles.
Figure 1.9: Example of a period doubling cascade in the Logistic map.
Figure 1.10: Characteristic multipliers and phase portrait for fold bifurcation. Solid circles indicate stable solutions, and hollow circles indicate unstable or metastable solutions. As the parameter is changed, the stable and unstable solutions approach, and annihilate leaving no solutions. At the bifurcation there is a characteristic multiplier at +1.
Figure 1.11: Characteristic multipliers and phase portrait for Hopf bifurcation. Solid circles indicate stable solutions, and hollow circles indicate unstable or metastable solutions. As the parameter is changed, the stable fixed point becomes unstable and throws off a stable limit cycle. At the bifurcation, two characteristic multipliers leave the unit circle as a complex conjugate pair.
Figure 1.12: Graphical iteration of the cubic map. Starting at point A, we take a vertical line to point B, then a horizontal line to the line $x_{n+1}=x_n$, and vertically down to point C.
Figure 1.13: Graphs of $x_{n+2} = x_n$ for $\alpha = 1.9, 2, 2.1$ around the super-critical pitchfork bifurcation for the cubic map. As the bifurcation value is exceeded, a period-2 symmetric limit cycle is produced, and the zero solution becomes unstable.
Figure 1.14: Super-critical pitchfork bifurcation for the cubic map. The solution $x=0$ is stable for $\alpha < 2$, and unstable for $\alpha > 2$. A period-2 stable symmetric solution exists for $\alpha > 2$.

Figure 1.15: Sub-critical pitchfork bifurcation. The solution $x=0$ is stable for $\alpha > 2$, and unstable for $\alpha < 2$. A period-2 unstable symmetric solution exists for $\alpha < 2$. 

52
Figure 1.16: Graphs of $x_{n+2}=x_n$ for $\alpha=2.8$, 3, 3.2 around the symmetry breaking bifurcation for the cubic map. The symmetric period-2 solution $S$ becomes unstable, and throws off two stable mirror image period-2 solutions $A$ and $B$. 

---

53
Figure 1.17: Symmetry breaking bifurcation for the cubic map. Only one of the two period-2 antisymmetric solutions are shown to avoid confusion. The unstable symmetric solution is not shown.
Figure 1.18: Heteroclinic saddle basin bifurcation. Two of the three shaded basins of attraction change as the heteroclinic saddle connection is broken. Since there is no change in the attractors, this is called a basin bifurcation.

Figure 1.19: Blue sky catastrophe. The formation of a homoclinic loop creates a stable limit cycle ‘out of the blue’.
Chapter 2: Parametrically Excited Pendulum: Hanging Solution

The equation of motion of the parametrically excited pendulum can be written as:

\[ l^2 \dddot{\theta} + \frac{d}{m} \dot{\theta}' + l(z'' + g) \sin \theta = 0 \]  

(2.1)

(see figure 1.1).

where the pendulum subject to periodic displacement \( z(\tau) \) is of length \( l \), mass \( m \), and has linear damping \( d \), where a dash denotes differentiation with respect to time, \( \tau \).

By writing:

\[ z(\tau) = -Z \cos \Omega \tau , \quad t = \omega_0 \tau , \quad \omega_0 = \sqrt{\frac{g}{l}} \]  

(2.2)

equation (2.1) reduces to

\[ \ddot{\theta} + c \dot{\theta} + (1 + p \cos \omega t) \sin \theta = 0 \]  

(2.3)

where:

\[ c = \frac{d}{\omega_0 ml^2} , \quad p = \frac{Z}{\Omega^2 g} , \quad \omega = \frac{\Omega}{\omega_0} \]  

(2.4)

There is an obvious solution \( \theta(t) = 0 \) to this equation. Physically, this solution corresponds to the pendulum hanging vertically downwards. It remains to be seen
whether this solution is stable in the sense that a small perturbation from the hanging position will produce damped oscillations back to the zero solution, or be magnified. This can be predicted by a linearisation about $\theta = 0$. Noting that for small $\theta$,

\[
\sin \theta = \theta - \theta^3/3! + \ldots
\]

we take the first term in the expansion to get the approximate expression:

\[
\ddot{\theta} + c\dot{\theta} + (1 + p\cos \omega t)\theta = 0
\]  

(2.5)

Re-scaling time according to $\tau = \omega t/2$, and differentiating with respect to $\tau$ gives:

\[
\frac{\ddot{\theta}}{\omega} + \frac{2c}{\omega} \dot{\theta} + \left( \frac{4}{\omega^2} + \frac{4p}{\omega^2} \cos 2\tau \right) \theta = 0
\]  

(2.6)

For $c=0$, this reduces to the same form as the Mathieu equation as studied by a number of researchers [Jordan & Smith 1987, Hayashi 1964]:

\[
\ddot{u} + (\delta + 2e\cos 2\tau)u = 0
\]  

(2.7)

where

\[
\delta = \frac{4}{\omega^2}
\]  

(2.8)

and

\[
e = \frac{2p}{\omega^2}
\]  

(2.9)
Approximate solutions to the Mathieu equation are well known, and depending on the values of $\delta$ and $\epsilon$, the equilibrium may be stable or unstable. The stable and unstable zones are separated by periodic solutions. Mathieu unstable zones are centred at $\delta = 1, 4, 9, 16, \ldots$ and so on. These zones are often displayed on a graph of $\epsilon$ against $\delta$ called a Strutt diagram [Hayashi 1964]. The largest zone is centred at $\delta = 1$, and is called the primary unstable zone. To determine the boundaries of this zone, we make use of a perturbation technique; the method of strained parameters [Nayfeh & Mook 1979].
2.1 Method of Strained Parameters

We assume that $\epsilon$ is a small parameter, and expand $u$ as a function of time, $t$ and $\epsilon$, and also expand $\delta$ as a function of $\epsilon$ alone.

\[ u(t, \epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots \quad \text{(2.10)} \]

\[ \delta = \delta_0 + \epsilon \delta_1 + \epsilon^2 \delta_2 + \ldots \quad \text{(2.11)} \]

The two expressions are substituted into equation (2.9), and we equate like powers of $\epsilon$.

\[ e^0: \quad \ddot{u}_0 + \delta_0 \dot{u}_0 = 0 \]
\[ e^1: \quad \ddot{u}_1 + \delta_0 \dot{u}_1 = -\delta_1 \dot{u}_0 - 2u_0 \cos 2t \quad \text{(2.12)} \]
\[ e^2: \quad \ddot{u}_2 + \delta_0 \dot{u}_2 = -\delta_2 \dot{u}_0 - \delta_1 \dot{u}_1 - 2u_1 \cos 2t \]

For the largest zone of instability, we put $\delta_0 = 1$, and solve equation (2.12a) to get:

\[ u_0 = a \cos t + b \sin t \quad \text{(2.13)} \]

(2.12)

substituting the initial solution into gives:

\[ \ddot{u}_1 + u_1 = -\delta_1 (a \cos t + b \sin t) - 2(a \cos t + b \sin t) \cos 2t \]
\[ = a(-\delta_1 - 1) \cos t + b(1 - \delta_1) \sin t \]
\[ - (a \cos 3t + b \sin 3t) \quad \text{(2.14)} \]
Any terms in \( \cos(t) \) or \( \sin(t) \) on the right of the equation (2.14) are called secular terms, and if any non-zero secular terms exist then the solution to equation (2.14) will increase exponentially with time. To eliminate secular terms, we have two possibilities:

\[
\begin{align*}
\delta_1 &= -1 & b &= 0 \\
\delta_1 &= +1 & a &= 0
\end{align*}
\]  

(2.15)

This leaves:

\[\ddot{u}_1 + u_1 = -(a \cos 3t + b \sin 3t)\]  

(2.16)

Taking the first case, \( b = 0 \) gives:

\[u_1 = \frac{a}{8} \cos 3t\]  

(2.17)

By substituting (2.17) into equation (2.13) we get:

\[\ddot{u}_2 + u_2 = -a \left( \delta_2 + \frac{1}{8} \right) \cos t + \frac{a}{8} \cos 3t - \frac{a}{8} \cos 5t\]  

(2.18)

Again, for no secular terms, we require the coefficient of \( \cos(t) \) to be zero, so:

\[\delta_2 = -\frac{1}{8}\]  

(2.19)
Taking the alternative case, \( a = 0 \) gives:

\[
\begin{align*}
  u_1 &= \frac{b}{8} \sin 3t \\
  \end{align*}
\]  

(2.20)

by substituting into equation (2.13) we get:

\[
\begin{align*}
  \ddot{u}_2 + u_2 &= -b (\delta_2 + \frac{1}{8}) \sin t + \frac{b}{8} \sin 3t - \frac{b}{8} \sin 5t \\
  \end{align*}
\]  

(2.21)

Again, for no secular terms, we require the coefficient of \( \sin(t) \) to be zero, so:

\[
\delta_2 = -\frac{1}{8}
\]  

(2.22)

Hence, the two periodic solutions that separate the stable and unstable behaviour of the zero solution are described by:

\[
\delta = 1 \pm e - \frac{e^2}{8} + O(e^3)
\]  

(2.23)

Substituting the original variables gives a quadratic equation in \( \omega^2 \):

\[
\omega^4 + (\pm 2p - 4) \omega^2 + \frac{P^2}{2} = 0
\]  

(2.24)

which can be solved to give the primary unstable zone shown in figure 2.1. We note that due to the change of variables, the zone is centred around \( \omega = 2 \). Additional
unstable zones exist around $\omega = 1, 3/4, 1/2, 1/4, \text{ and so on.}$

### 2.2 Numerical Computation of Unstable Zones

The unstable zones for the parametrically excited pendulum can be calculated numerically by assessing the stability of the zero solution in terms of the eigenvalues of the Jacobian. By adding the eigenvalue constraint, and bifurcation following, the numerically exact unstable zones are shown in figure 2.2 and compared with the analytical approximation in figure 2.3. The effect of the nonzero damping is to lift the primary unstable zone away from the x-axis, and to shrink other unstable zones. This makes the primary unstable zone the most important zone in terms of its influence over a large parameter range.

### 2.3 Effect of Higher Order Terms

In reducing the parametrically excited pendulum to the Mathieu equation, we ignored higher order terms. The effect of specific additional terms is considered by Nayfeh and Mook [Nayfeh & Mook 1979], but in general, these nonlinearities bound solutions that would otherwise 'blow up' to infinity, producing stable limit cycles. The details of which limit cycles are created will be covered in depth in chapter 3, where the non-rotating behaviour of the pendulum will be considered in full.
Figure 2.1: Analytically calculated boundary to primary Mathieu unstable zone, with no damping term.
Figure 2.2: Numerically calculated 1st and 2nd unstable zones for the parametrically excited pendulum, with damping, $c=0.1$. 
Figure 2.3: Comparison between numerically and analytically calculated primary unstable zone for parametrically excited pendulum with damping, $c=0.1$. Analytical solution calculated by the method of strained parameters from the Mathieu equation is shown as a dotted line, whereas solid line is the numerically calculated solution obtained by a bifurcation following routine. The numerically calculated 2nd zone is also shown.
Chapter 3: Parametrically Excited Pendulum:

Non-Rotating Solutions

In many applications of the parametrically excited pendulum, the main interest going beyond this value is in achieving solutions that do not exceed $\theta = \pm \pi$, as might correspond to failure in some sense. We term such solutions non-rotating, for the obvious reason that the pendulum does not rotate completely about its pivot point, but just oscillates back and forth. It is useful to develop a method which can predict which parameter zones will inevitably lead to solutions which go 'over the top' after a finite number of cycles of applied parametric excitation. Such parameter regimes are termed 'escape zones'. This can be achieved by both analytical and numerical techniques. Under the conditions expressed, we have an 'escape' problem. Such a problem is common in the engineering and physical sciences, from the possible capsize of ships, the gravitational collapse of massive stars, and activation energies in molecular dynamics [Thompson 1989]. Much work has been undertaken on this subject, particularly in understanding escape from an asymmetric potential well under direct forcing, [Thompson 1989, Szemplińska-Stupnicka 1988, 1992a,b] and more recently on parametric escape, again from an asymmetric potential well [Soliman & Thompson 1992a,b]. We shall see that there is much in common with the escape from an asymmetric potential well, but also there are many subtle differences which are worth considering.

Initially, we produce a bifurcation diagram for the non-rotating orbits in the parametrically excited pendulum, considering only bifurcations around the primary Mathieu unstable zone centred around $\omega=2$. The bifurcation diagram was produced
3.1 Bifurcational Behaviour

An overview of the bifurcational behaviour can be obtained by considering the bifurcation diagram of figure 3.1 in the \((\omega,p)\) parameter space. Only the major bifurcations are displayed around the primary unstable zone centred at \(\omega=2\) (similar detail can be examined near other unstable zones). The line labelled H is where small perturbations from the equilibrium \(\theta=0\) will first result in an initial growth which was predicted by the linear theory in chapter 2. To the right of the point c, the equilibrium bifurcates into a stable symmetric period-2 solution at a super-critical pitchfork bifurcation, whilst to the left, the equilibrium becomes unstable at a sub-critical pitchfork bifurcation. Line A corresponds to a period-2 fold where the unstable period-2 solution from the sub-critical pitchfork bifurcation stabilises at a saddle-node bifurcation. Line S corresponds to a symmetry breaking bifurcation, where the symmetric period-2 solution splits into two stable anti-symmetric period-2 solutions, and line F represents the end of a pair of period-doubling cascades of the two anti-symmetric solutions, although only the first period-doubling is shown, since the cascade is very rapid. The event at the end of the period-doubling cascades will be investigated in further detail in section 3.3. Lines H and F form the boundary of the 'escape zone', the zone where no major stable non-rotating attractors exist. A typical trajectory for a system which has parameters in the escape zone will exceed
\( \theta = \pm \pi \) after a finite number of periods of applied parametric forcing. The features of the bifurcation diagram agree with those produced by Mullin [Mullin 1993], and Bryant and Miles [Bryant & Miles 1990c].

The bifurcation diagram figure 3.1 for non-rotating orbits in the parametrically excited pendulum is very similar to that produced by Soliman and Thompson for escape from an asymmetric potential well under parametric excitation [Soliman & Thompson 1992a,b] except of course for the symmetry breaking bifurcation which is not applicable. There is also considerable similarity to the bifurcation diagram for escape from an asymmetric potential well under direct excitation [Thompson 1989], except again for the symmetry breaking bifurcation, and additionally the super-critical pitchfork bifurcation, which does not occur.

3.2 Analytical Solution: Harmonic Balance

Following Capecchi and Bishop [Capecchi & Bishop 1994], we assume a solution of the form:

\[
\theta(t) = \theta_0 + A \cos(\omega t + \beta) \nu
\]  

(3.1)

where \( \nu = \frac{1}{2} \) corresponds to the principal unstable zone. By substituting equation (3.1) into the equation (2.5), the equation of motion of the parametrically excited pendulum, and equating terms of cosine, sine and the constant term we have;
\[ \frac{A\omega^2}{4} - [p(J_1(A) - J_3(A))\cos \beta \cos \theta_0 + 2J_1(A)\cos \theta_0] = 0 \]  

(3.2a)

\[ cA\omega - 2[p(J_1(A) + J_3(A))\sin \beta \cos \theta_0] = 0 \]  

(3.2b)

\[ \sin \theta_0 [pJ_2(A)\cos \beta - J_0(A)] = 0 \]  

(3.2c)

where \( J_n(x) \) is the Bessel function of order \( n \). These three simultaneous equations could be solved numerically to give an approximate analytical solution by a Newton-Raphson technique. However, more can be derived by considering the various solutions that are known to exist.

Equation (3.2c) can be split further into:

\[ \sin \theta_0 = 0 \]  

(3.3a)

symmetric solution

or

\[ pJ_2(A)\cos \beta - J_0(A) = 0 \]  

(3.3b)

asymmetric solution

For the symmetric solution \( \theta_0 = 0 \), equations (3.2a,b) reduce to:

\[ \left[ A\omega^2 - 16J_1(A) \right]^2 + c^2\omega^2A^2 - 16p^2[J_1(A) - J_3(A)]^2 = 0 \]  

(3.4)

where
\[ \bar{c} = c \frac{J_1(A) - J_3(A)}{J_1(A) + J_3(A)} \] (3.5)

In the case of asymmetric solutions, for \( c = 0 \); \( \beta = 0 \) and for light damping \( c \gg 0 \), so equation (3.4) becomes:

\[ pJ_2(A) - J_0(A) = 0 \] (3.6)

### 3.2.1 Bifurcation Criteria

The first criteria for approximating the escape zone relies on approximating two bifurcations that bound the zone. Szemplińska-Stupnicka [Szemplińska-Stupnicka 1988, 1992a,b] shows that for several escape equations these bifurcations are a fold, and the event at the end of a period-doubling cascade of the primary response that comes from the unforced equilibrium solution. The second bifurcation is hard to predict analytically, and so the first period-doubling, or symmetry breaking bifurcation for the case of symmetric systems is often sufficiently close to the final bifurcation to provide a reasonable conservative estimate. As we saw in section 3.1, the escape zone for the parametrically excited pendulum is indeed bounded by two such bifurcations.

Equations (3.4) and (3.6) can be solved numerically by a Newton-Raphson
procedure to give the parameters where both symmetric and anti-symmetric solutions exist in the \((\omega, p)\) plane; i.e. the locus of the symmetry breaking bifurcation. The results are good for \(\omega > 1.5\), but poor at low frequency when compared to the locus obtained by a bifurcation following technique. This is shown in figure 3.2 where the approximate analytic solution is compared with the numerically computed symmetry breaking bifurcation.

The sub-critical bifurcation can be determined by the usual vertical tangency condition:

\[
\frac{dA}{d\omega} = \infty \quad (3.7)
\]

This occurs at \(A = 0\), and so by differentiating and taking the limit as \(A \to 0\), equation (3.4) yields:

\[
[\omega^2 - 4]^2 + 4c^2 \omega^2 - 4p^2 = 0 \quad (3.8)
\]

which agrees with the result obtained in chapter 2 by analysing the linearised form of equation (2.1); the Mathieu equation. Hence, equation (3.8) gives a good approximation to the left hand boundary of the escape zone.

These two approximate bifurcation curves together give a good approximation to the escape zone when compared to the escape zone obtained via numerical integration in figure 3.3. The agreement is fairly good, considering the approximations employed in the analytical solution.
3.2.2 Critical Velocity Criteria

Another approximate criteria for escape is that proposed by Moon [Moon 1987], that escape/chaos occurs when the maximum velocity of the motion is near the maximum velocity on the separatrix, (that is the stable or unstable manifold which forms the basin boundary between escaping trajectories and solutions which remain in the potential well) for the phase plane of the undamped, unforced oscillator. For the undamped, unforced pendulum, the phase plane shown in figure 3.4 is composed of concentric ellipses and rotating orbits.

The maximum velocity on the separatrix is:

\[ \dot{\theta}_{\text{sep}} = 2 \]  \hspace{1cm} (3.9)

From the harmonic balance solution;

\[ \dot{\theta}(t) = -Av \omega \sin(\omega t + \beta)v \]  \hspace{1cm} (3.10)

\[ \therefore \dot{\theta}_{\text{max}} = Av \omega \]  \hspace{1cm} (3.11)

In practice, it is necessary to introduce a correction factor, \( \alpha \) so that escape occurs when \( \theta_{\text{max}} = \alpha \theta_{\text{sep}} \). From experimental results [Moon 1980] \( \alpha = 0.86 \) gives good agreement. From equation (3.9) and equation (3.11), we obtain:

\[ A = \frac{4\alpha}{\omega} \]  \hspace{1cm} (3.12)
Truncating the Bessel functions and averaging equation (3.4) gives [Capecchi & Bishop 1994]:

\[
\frac{\omega^2}{2} - 2 + \frac{A^2}{4} + \omega^2c^2 - p^2\left[1 - \frac{A^2}{12}\right]^2 = 0
\]  

(3.13)

where

\[c = \frac{(1 - A^2/6)}{(1 - A^2/12)}
\]

(3.14)

substituting equation (3.12) into equation (3.13) gives;

\[p^2 = \frac{(\omega^4 - 4\omega^2 + 8\alpha^2) + 4\omega^2c^2}{4(\omega^2 - \frac{4\alpha^2}{3})^2}
\]

(3.15)

For low damping;

\[p = \frac{3(\omega^4 - 4\omega^2 + 8\alpha^2)}{2(3\omega^2 - 4\alpha^2)}
\]

(3.16)

denoted by M

The curve obtained from the approximation equation (3.16) is plotted in figure 3.5, and again compared to the numerically computed escape zone.

73
The two analytical methods employed to give approximations to the escape zone both yield reasonable results. The bifurcation criteria method is the more accurate especially at high frequency, but the critical velocity criteria does have the advantage of producing a closed form analytical expression for the zone rather than needing to solve a series of nonlinear equations numerically. Whilst both methods give good results, it is also worth noting that although stable non-rotating solutions exist below the escape zone, their basins of attraction may be fractal, and subject to rapid basin erosion, a possibility that we investigate further in section 3.5.

3.3 Chaotic Blue Sky Catastrophe

The bifurcation at the end of the period-doubling cascades is termed a chaotic blue sky catastrophe [Abraham & Shaw 1988]. This is because the two chaotic attractors vanish instantaneously along with their basins of attraction into the blue, leaving no stable non-rotating solutions. The chaotic nature of the attractor is unclear from simply observing a time series (figure 3.6) since every other peak appears to be almost identical as would be for a period-2 orbit, but in fact every peak is slightly different. To see that the final attractor is indeed chaotic, we show a Poincaré section of half of one of the attractors shortly before the final event in figure 3.7 where the fractal nature of the attractor can be seen clearly. Chaotic blue sky catastrophes often occur as the result of the collision of an unstable periodic orbit with the chaotic attractor [Grebogi et al.
For example, in the escape from an asymmetric potential well, a period-6 unstable solution has been identified which is responsible for the disappearance of the final chaotic attractor [Stewart 1987]. Given that there are similarities in the parametrically excited pendulum and the escape from a potential well we hypothesise that a period-12 unstable solution may be responsible for the final event. We choose period-12 rather than period-6 since the fundamental solution which comes from the equilibrium solution is in this case of period-2, rather than period-1 in the case of escape from an asymmetric potential well under direct excitation. A Newton-Raphson approach was used around the boundary of the chaotic attractor, and indeed an unstable period-12 orbit was successfully located. As the forcing parameter is increased the unstable solution was path followed, and found to move closer to the chaotic attractor. This is shown in figure 3.8, where the unstable orbit is seen to collide with the chaotic attractor at a parameter value which is indistinguishable from that of the final event.

One fundamental question raised by the discovery of this unstable period twelve orbit is where did this orbit originate? It could not have come from the period-doubling cascades since these produce orbits of periods 2, 4, 8, 16, 32, and so on. Hence, the cell map, path following, and bifurcation following methodology applied missed additional solutions. Also, do other unstable orbits exist, and if so, how many, and of what periods? These questions will be answered in chapter 8.
3.4 Generic Features of Escape from a Symmetric Potential Well under Parametric Excitation

We have already seen that the parametrically excited pendulum has many similarities with other escape systems. Indeed from a series of detailed observations of other escape systems, it seems that the parametrically excited pendulum is a typical example of any system which permits escape from a symmetric potential well under parametrically excitation [Clifford & Bishop 1993]. The reason for this generic behaviour will be shown in section 3.6. Here, we consider a parametrically damped system described by equation (3.17).

\[ \ddot{x} + (1 + q \cos \Omega t) \dot{x} + x - x^3 = 0 \]  

(3.17)

The exact form of excitation and shape of potential well is distinct from the parametrically excited pendulum, but if we consider the bifurcation behaviour in figure 3.9, we see that the two bifurcation diagrams are qualitatively similar. H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of point c. S is the locus of a symmetry breaking bifurcation, and F represents the end of a pair of period-doubling cascades although only the first period-doubling is shown. The period-2 fold line A associated with the sub-critical bifurcation is also shown.

This similarity makes the study of the parametrically excited pendulum of
relevance to any system which permits escape from a symmetric potential well under parametric excitation.

3.5 Fractal Basin Boundaries

The emphasis on predicting the escape zone has concentrated on the stability of particular attractors. This approach can be somewhat misleading, as a stable attractor may possess a tiny basin of attraction, and so may not be realisable in a typically noisy engineering atmosphere. This has lead to a variety of integrity measures [Soliman & Thompson 1989, Schiehlen 1993, McRobie & Thompson 1991] which seek to quantify the relative 'safe' basin area; the set of initial conditions which lead to solutions which remain within the potential well for all time. An important phenomenon which occurs for systems that permit escape from a potential well under direct excitation is a dramatic sudden loss of safe basin area over a moderate increase in the forcing parameter. This is known as the 'Dover cliff' effect [Soliman & Thompson 1992d]. The basin area can be quantified by taking a window of initial conditions, and recording the percentage that remain in the potential well after a number of cycles of forcing. This measure is termed the integrity [Soliman & Thompson 1989]. Different values for the integrity can result from taking different windows of initial conditions, but the method is still useful in quantifying the relative basin area as a parameter is increased.

Three integrity curves are shown for the parametrically excited pendulum in
The rapid basin erosion is due to the fractal nature of the basin boundary. A detailed description of the mechanisms underlying rapid basin erosion is contained in [Soliman & Thompson 1992c,d, Lansbury et al. 1992], but suffice it to say that after the basin boundary becomes fractal, the safe basin can be dramatically eroded by fractal ‘fingers’ which push into the safe basin. The basin boundary is defined by the stable manifolds of the hilltop saddles at $\theta = \pm \pi$. This is illustrated in figure 3.11a,b where a grid of starts and equivalent stable manifolds of the hilltop saddles are shown. Two pairs of fractal fingers are clearly visible, pushing into the safe basin. The fractal nature of the basin boundary can be predicted by a Melnikov analysis [Koch & Leven 1985]. However, the usefulness of such analysis is debateable since the Melnikov curve does not accurately predict the onset of rapid basin erosion, but only predicts the initial intersection of the invariant manifolds. Other intersections of the invariant manifolds of the hilltop saddles are more important in terms of predicting bifurcations. For example, the particular heteroclinic tangency illustrated in figure 3.12 is a good estimate for the symmetry breaking bifurcation. This tangency is compared to the locus of the symmetry breaking bifurcation in parameter space in figure 3.13. We see that this tangency is a good indicator of the locus of the symmetry breaking bifurcation, especially at high frequency. We term this a heteroclinic rather than a homoclinic tangency since for the case of non-rotating trajectories we consider the unstable equilibria at $\theta = \pm \pi$ to be unique. Other
intersections may be important in estimating the final escape of all trajectories [McRobie 1992c, McRobie & Thompson 1992, 1993a, Yamaguchi & Tanikawa 1992].

3.6 Horseshoe Formation

As new intersections of the stable and unstable invariant manifolds of the hilltop saddles are created, horseshoes are formed [Smale 1967]. Once the forcing parameter is increased to a value well inside the escape zone, the manifolds have the topological form displayed in figure 3.14. The structure formed is topologically equivalent to that of figure 3.15, which in turn can be transformed into the three striped horseshoe shown in figure 3.16. Since the horseshoe has three stripes, we term it a 3-shoe. We note that this 3-shoe is distinct from the spiral 3-shoe considered by McRobie which is a generic feature of escape from an symmetric potential well. However, this 3-shoe is identical to that discovered by Gwinn and Westervelt in the pendulum with applied torque [Gwinn & Westervelt 1986]. Also, the parametrically damped system considered in section 3.4 has the same horseshoe as can be seen from the invariant manifolds of the hill-top saddles in figure 3.17. This simple observation allows us to reduce the bifurcational behaviour of the parametrically excited pendulum, the pendulum with applied torque, and the parametrically damped system to the creation process of the 3-shoe. This topological analysis is carried out in chapter 8.
Figure 3.1: Major features of the bifurcation diagram of the parametrically excited pendulum in the $(\omega, p)$ parameter space. H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of c. S is the locus of a symmetry breaking bifurcation, and F represents the end of a period-doubling cascade although only the initial period-doubling is shown since the cascade is very rapid. The period-2 fold line A associated with the sub-critical pitchfork bifurcation is also shown along with the ‘escape zone’.
Figure 3.2: Line N is the locus of the symmetry breaking bifurcation for the parametrically excited pendulum obtained by numerically following the bifurcation. Line S is the analytical approximation to the symmetry breaking bifurcation obtained by a Newton Raphson procedure from equations (3.4) and (3.6). The approximation is good at high frequency, but poor at low frequency.
Figure 3.3: Approximations to the symmetry breaking bifurcation, S and the subcritical pitchfork bifurcation, H, obtained by harmonic balance, together with the shaded numerically obtained escape zone for the parametrically excited pendulum.

Figure 3.4: Phase portrait for the undamped unforced pendulum.
Figure 3.5: Approximate escape locus, M, obtained by critical velocity criteria, together with the shaded numerically obtained escape zone for the parametrically excited pendulum.

Figure 3.6: Time history of strange attractor for the parametrically excited pendulum. From the time history, the attractor is indistinguishable from a period-2 attractor.
Figure 3.7: Poincaré section of half of strange attractor for the parametrically excited pendulum. Parameters are $\omega = 2$, $p = 1.3426$.

Figure 3.8: Poincaré section of half of strange attractor for the parametrically excited pendulum. Parameters are $\omega = 2$, $p = 1.3426$, along with six circled points of the so called 'destroyer saddle', a period-12 unstable solution.
Figure 3.9: Major features of the bifurcation diagram of the parametrically damped system in the $(\omega,q)$ parameter space. H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of c. S is a symmetry breaking bifurcation, and F represents the end of a period-doubling cascade although only the first period-doubling is shown. The period two fold line A associated with the sub-critical pitchfork bifurcation along with the 'escape zone' is also shown.
Figure 3.10: Integrity curves for the parametrically excited pendulum, where \( I \) is the percentage of initial conditions that remain in the potential well after 16 cycles of parametric excitation.
Figure 3.11a: Stable manifolds of hill-top saddles for the parametrically excited pendulum. Parameter values are $\omega=2$, $p=1$, $c=0.1$.

Figure 3.11b: Safe basin for parametrically excited pendulum. Parameter values are $\omega=2$, $p=1$, $c=0.1$. Scales are as in figure 3.11a.
Figure 3.12: Heteroclinic tangency of the invariant manifolds of the hill-top saddles for the parametrically excited pendulum. The invariant manifolds are labelled according to whether they come from the left, l or right, r saddle, and whether they are stable, s or unstable, u. Parameters chosen are $\omega=2$, $p=1.35$, $c=0.1$. 
Figure 3.13: Major features of the bifurcation diagram of the parametrically excited pendulum in the $(\omega, p)$ parameter space. $H$ is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of $c$. $S$ is a symmetry breaking bifurcation, and $T$ represents the locus of the particular heteroclinic tangency displayed in figure 3.12. The period two fold line $A$ associated with the sub-critical pitchfork bifurcation is also shown, along with the second unstable zone.
Figure 3.14: Invariant manifolds of the hill-top saddles for the parametrically excited pendulum $\omega=2$, $p=1.8$. The invariant manifolds are labelled according to whether they come from the left, $l$, or right, $r$, saddle, and whether they are stable, $s$, or unstable, $u$. 
**Figure 3.15:** Idealised invariant manifolds of figure 3.14.

**Figure 3.16:** Topological structure of the invariant manifolds depicted in figure 3.14. The unstable manifolds are labelled according to whether they come from the left, $l$ or right, $r$ saddle. The stable manifolds form the remaining vertical lines, whilst the two saddles are shown as solid circles. All the (unstable) non-rotating orbits are located in the shaded regions.
Figure 3.17: Invariant manifolds of the hill-top saddles for the parametrically damped system $\omega=2$, $q=1.4$. The invariant manifolds are labelled according to whether they come from the left, $l$, or right, $r$, saddle, and whether they are stable, $s$, or unstable, $u$. 
Chapter 4: Parametrically Excited Pendulum: Inverted Solutions

It is well known that the inverted position of the parametrically excited pendulum can be stable for certain values of forcing amplitude and frequency, as has been demonstrated both numerically and experimentally [Stephenson 1908b, Mullin 1993]. The stable zones can be predicted by linearising the equation of motion to the Mathieu equation. In addition, there is much in common with the non-rotating orbits discussed in chapter 3.

4.1 Analytical Prediction of Stable Zones

By substituting \( \phi = \pi - \theta \) into equation (2.5), we can describe inverted oscillations by equation (4.1).

\[
\ddot{\phi} + c \phi - (1 + p \cos \omega t) \sin \phi = 0 \tag{4.1}
\]

Again, by linearising about \( \phi = 0 \), and re-scaling time, we have a damped Mathieu equation (4.2).

\[
\phi'' + \frac{2c}{\omega} \phi' + \left( \frac{4}{\omega^2} + \frac{4p}{\omega^2} \cos 2\tau \right) \phi = 0 \tag{4.2}
\]

For \( c = 0 \), this reduces to the Mathieu equation as studied by a number of researchers [Jordan & Smith 1987, Hayashi 1964].
\[ \ddot{u} + (\delta + 2\varepsilon \cos 2t)u = 0 \quad (4.3) \]

where,

\[ \delta = -\frac{4}{\omega^2}, \quad \varepsilon = -\frac{2p}{\omega^2} \quad (4.4) \]

From the results for the primary Mathieu zone in chapter 2, the boundaries are given by:

\[ \delta = 1 \pm \varepsilon - \frac{\varepsilon^2}{8} + \mathcal{O}(\varepsilon^3) \quad (4.5) \]

Substituting equation (4.4) into (4.5) gives a quadratic in \( \omega^2 \):

\[ \omega^4 + (4 \pm 2p) \omega^2 - \frac{p^2}{2} = 0 \quad (4.6) \]

Only one solution to equation (4.6) gives a meaningful result, and the curve obtained from solving equation (4.6) is plotted in figure 4.1. The numerically obtained stable zone is also shown in figure 4.2 to compare with the analytical results. We see that line A is a fairly good indicator for the upper boundary of the stable zone. An alternative approach is used by Stephenson [Stephenson 1908a] to predict stable zones, but is only valid at high frequency.
4.2 Bifurcational Behaviour

The bifurcations of the inverted parametrically excited pendulum were determined by path following the unstable solutions located in the vicinity of the $\theta=\pi$ solution. The solution becomes stable as two mirror image unstable period-1 solutions collide with the unstable hill-top solution. The inverted solution then becomes unstable at a super-critical pitchfork bifurcation, leaving a symmetric period-2 solution. These two bifurcations form the boundaries of the hatched stable region in figure 4.2.

The symmetric period-2 solution in turn undergoes a symmetry breaking bifurcation, and the subsequent period-2 mirror image solutions period-double to possibly chaotic attractors before disappearing at catastrophic bifurcations. The bifurcations of the inverted parametrically excited pendulum are shown schematically in figure 4.3. These bifurcations are identical to those of non-rotating solutions in chapter 3, with the exception that the pitchfork bifurcation is always super-critical in this case. We hypothesise that there may be a 3-shoe in the invariant manifolds of the two symmetric (unstable) period-1 solutions which would account for this similarity. In effect, these two unstable period-1 solutions are analogous to the hill-top saddles discussed in chapter 3.
4.3 Stability of Inverted Pendulums

Linear theory can be developed to predict that inverted multiple pendulums can be stabilised. This has been developed by Stephenson [Stephenson 1909] and more recently the theory has been improved, and the stability of multiple pendulums has been demonstrated experimentally [Acheson 1993, Acheson & Mullin 1993]. In the case of multiple pendulums, the two bifurcations surrounding the stable zone are a 'buckling instability', and a slower 'falling' bifurcation [Acheson 1993]. We remark that in the single parametrically excited pendulum, these two bifurcations correspond to the super-critical pitchfork bifurcation, and the catastrophic collision of the two period-1 unstable solutions with the inverted solution, respectively.
Figure 4.1: Analytical approximation to the upper stability boundary $A$, of the inverted parametrically excited pendulum calculated by reduction to the Mathieu equation.
Figure 4.2: Analytical approximation to the upper stability boundary $A_1$ of the inverted parametrically excited pendulum calculated by reduction to the Mathieu equation. The shaded region is the numerically calculated stable zone.
Figure 4.3: Schematic bifurcation diagram for the inverted parametrically excited pendulum. Solid (dashed) lines represent stable (unstable) solutions. From left to right, the bifurcations are as follows: inverted solution becomes stable as two period-1 unstable solutions labelled P1 collide with unstable inverted solution. Next, the inverted solution becomes unstable at a super-critical pitchfork bifurcation leaving a stable symmetric period-2 solution. This period-2 solution then undergoes a symmetry breaking bifurcation leaving two mirror image asymmetric stable solutions, only one of which is shown. These period-2 orbits then rapidly period-double to possibly chaotic attractors before a catastrophic bifurcation.
Chapter 5: Parametrically Excited Pendulum: Rotating Solutions

If we consider the parametrically excited pendulum to have phase space $\mathbb{R} \times \mathbb{S} \times \mathbb{S}$ by identifying $\theta = \pi$ and $\theta = -\pi$, we see that rotating solutions may exist. Firstly we define a rotating orbit as a solution which goes beyond $\theta = \pm \pi$. Rotating orbits can be further subdivided into those which rotate in the same direction for all time, that is, $\dot{\theta}(t) > 0 \ \forall t$ or $\dot{\theta}(t) < 0 \ \forall t$, and those which change direction in the course of their rotation. We call the former solutions 'purely rotating orbits', and these will be covered exclusively in this chapter. The orbits which change direction loop around the equilibria $\theta = 0$, and $\theta = \pm \pi$, and can rotate with zero mean angular velocity, i.e. they can return to where they started from without the necessity of a toroidal phase space. Physically this could correspond to the pendulum performing two clockwise and then two anticlockwise revolutions. To clarify the distinction between rotating and purely rotating orbits, we show phase portraits of two period-3 orbits in figure 5.1. The first is a purely rotating orbit, whilst the second loops around the equilibrium $\theta = 0$ before rotating beyond $\theta = -\pi$.

Purely rotating solutions may be partially categorised according to their period, $n$, and the number of complete rotations made in $n$ periods of forcing, $r$. Hence we use the notation $(n,r)$ to describe a period-$n$ orbit which makes $r$ complete rotations in $n$ periods. For example, a period-3 solution which rotates twice in three periods of forcing will be termed a $(3,2)$ rotating solution. At first, solutions with $n \neq r$ seem to be unusual, but just as a nonlinear system can respond at a different frequency to an applied forcing function, so the pendulum can rotate at an average
angular frequency distinct from its dynamical period, which in turn may be distinct from the forcing frequency. Such subharmonic orbits will be discussed in greater detail in chapter 9. No distinction is made here between clockwise and anticlockwise rotating orbits, and given the symmetry of the system, if a clockwise rotating orbit exists, then an equivalent mirror image anticlockwise solution must also exist. For this reason, we will concentrate solely on clockwise rotating orbits. For a complete classification of rotating orbits it would be necessary to resort to braid and knot theory to consider the linking properties of rotating orbits with the equilibria at $\theta=0$, and $\theta=\pm \pi$, which is beyond the scope of this thesis.
5.1 Analytical Treatment

The harmonic balance procedure applied to non-rotating solutions in section 3.2 can also be applied to the rotating solutions. However, the resulting equations are inseparable, and do not give a very accurate representation of the behaviour [Capecchi & Bishop 1994]. We avoid using a more accurate higher order harmonic balance procedure as we follow our reasoning in chapter 1 that the strength of analytical methods lies in their ability to produce a rapid approximate solution. Since in this case this cannot be easily achieved, we resort to numerical analysis.

5.2 Bifurcational Behaviour

The bifurcation diagram in figure 5.2 was produced by cell mapping and path following solutions until bifurcations were encountered, which were then followed by the method outlined in chapter 1. Figure 5.2 shows the bifurcational behaviour of the clockwise purely rotating period-1 (1,1) solution. Line A is where the solution is created at a saddle-node bifurcation. Below line A no rotating period-1 solution exists. Line B is where the (1,1) solution period-doubles to a (2,2) solution at the beginning of a gradual period-doubling cascade to chaos. Line F is the second period-doubling, but is sufficiently close to be regarded as the final period-doubling. As with the non-rotating chaotic attractor, the chaos is only stable over a narrow parameter regime. A Poincaré section of the chaotic attractor is shown in figure 5.3. Soon after the final period-doubling, the rotating chaotic attractor loses stability at a chaotic
explosion, and the trajectories tumble backwards and forwards, rotating clockwise and then anticlockwise in a chaotic fashion which is further investigated in chapter 6. The purely rotating solution restabilises as the tumbling chaos becomes unstable and a very rapid period-doubling cascade in reverse leaves the original (1,1) solution stable in the period-1 zone surrounded by line U.

5.3 Subharmonic Orbits

It has already been stated that subharmonic orbits with $n \neq r$ exist. In fact, it appears numerically that every orbit with $r \leq n \ n = 1, 2, 3, \ldots$ exists, and furthermore is stable for some parameter regime. No orbit has been found with $r > n$, and so we comment that it is impossible for the parametrically excited pendulum to rotate with a mean frequency greater than that of the forcing, excluding perhaps transient behaviour. The higher the period, the smaller the stable parameter regime, but again, unstable orbits are easily controllable [Shinbrot et al. 1993], and so the existence of these orbits is of engineering interest. Furthermore, stable subharmonic orbits exist below line A in figure 5.2. That is, to get a rotating solution, line A is not the minimum forcing. For physical systems this has two consequences. If we require a rotating solution, we may be able to use less force for a given frequency to achieve a rotating solution. However, if we require a non-rotating solution, there is in general no easily determined maximum forcing amplitude that we must not exceed to avoid the possibility of ending up with a rotating solution.
Whilst the parameter ranges over which particular subharmonic rotating solutions are stable are small, these zones are considerably larger than for the non-rotating solutions [Clifford & Bishop 1994a]. Figure 5.4 shows two subharmonic purely rotating period-3 solutions labelled (3,1) and (3,3), which can coexist with the (1,1) solution, overlaid on the major bifurcation diagram. The subharmonics are bounded by saddle-node and period-doubling cascades. We note that especially at high frequency, the regions are quite broad. (3,2) solutions also exist, but are stable over much narrower parameter ranges. Details on how these orbits were located will be given in chapter 9.
Figure 5.1: Phase portraits of a purely rotating and rotating (3,1) orbit. The first is a purely rotating orbit, whilst the other makes an oscillation in the potential well around the equilibrium $\theta=0$ before rotating beyond $\theta=-\pi$. 
Figure 5.2: Bifurcation diagram for the clockwise purely rotating orbits in the parametrically excited pendulum. \((n,r)\) period-\(n\) orbits rotate completely \(r\) times in \(n\) periods. Line A is where a period-1 \((1,1)\) orbit is formed at a saddle-node bifurcation. This orbit period-doubles at line B to a \((2,2)\) orbit, the beginning of a gradual cascade to rotating chaos. The second period-doubling occurs at line F, which is sufficiently close to represent the end of the cascade before a subsequent bifurcation to tumbling chaos. The period-1 orbit restabilises at a rapid reverse period-doubling cascade in the region surrounded by line U.
Figure 5.3: One quarter of the Poincaré section of the purely rotating (not tumbling) chaotic attractor. Parameter values are $\omega=2$, $p=1.81$, $c=0.1$. This strange attractor is only stable over a very narrow parameter regime before a chaotic explosion to the tumbling chaotic attractor.
Figure 5.4: Bifurcation diagram for the clockwise purely rotating orbits in the parametrically excited pendulum. \((n, r)\) period-n orbits rotate completely \(r\) times in \(n\) periods. Line A is where a period-1 \((1, 1)\) orbit is formed at a saddle-node bifurcation. This orbit period-doubles at line B to a \((2, 2)\) orbit, the beginning of a gradual cascade to rotating chaos. The second period-doubling occurs at line F, which is sufficiently close to represent the end of the cascade before a subsequent bifurcation to tumbling chaos. The period-1 orbit restabilises at a rapid reverse period-doubling cascade in the region surrounded by line U. Additionally, two period-3 zones are shown. These are bounded by saddle-node and period-doubling bifurcations. Note that the \((3, 1)\) orbits can exist below line A. \((3, 2)\) orbits also exist, but are stable over a very narrow parameter regime between the \((3, 1)\) and \((3, 3)\) zones.
Chapter 6: Parametrically Excited Pendulum:

Chaotic Tumbling Solution

By far the most interesting, and also the commonest behaviour of the rotating parametrically excited pendulum, is the tumbling chaos mentioned briefly in the preceding chapter. Here, we investigate the apparently random, but strictly deterministic behaviour, by means of Lyapunov exponents and power spectra.

6.1 Tumbling Chaos

By far the most important behaviour for the rotating pendulum both in terms of the parameter range over which it is stable and also from a purely mathematical perspective is the tumbling chaos. The apparently chaotic nature of the tumbling attractor has been demonstrated experimentally with any sequence of left-right tumbles being observed [Leven et al. 1985]. This is the typical behaviour of the parametrically excited pendulum over a wide range of parameters, and thus warrants further consideration. First, we need to show that the attractor is truly chaotic, and not just a random process. One clear indication that the attractor is indeed chaotic is the fractal nature of the Poincaré section shown in figure 6.1. If the output was random, we would not expect the fine structure displayed in figure 6.1, but rather a uniform spread of points.

Another property of chaotic attractors is the divergence of initial conditions.
This can be seen in figure 6.2 for the tumbling chaos in the parametrically excited pendulum where two nearby initial conditions diverge completely after a number of periods of parametric excitation. These visual checks lack mathematical rigour, and so we calculate the Lyapunov exponents for the attractor.

6.2 Lyapunov Exponents

In the same way that the eigenvalues of the derivative matrix indicate the relative stability of periodic attractors, and the contraction of local phase space, (see chapter 1) so Lyapunov exponents are a measure of the expansion on the chaotic attractor. This is a quantification of the divergence of chaotic trajectories already discussed. For a three-dimensional chaotic attractor, there should be a zero Lyapunov exponent along the time axis, a positive exponent indicating expansion and divergence of nearby trajectories, and a negative exponent to retain the overall volume contraction for dissipative systems. Lyapunov exponents can be calculated from time history data by a number of algorithms, although often only the largest exponent can be calculated with any certainty. For details of how to calculate the Lyapunov exponents of a dynamical system see for example [Tufillaro et al. 1992].

Lyapunov exponents for the tumbling solution were calculated in a region thought to be chaotic (figure 6.3) The largest Lyapunov exponent remains roughly constant and positive over a large range of parameters indicating that the nature of the attractor does not change markedly over this range of parameters. There are also
regions where the largest non-zero Lyapunov exponent is negative, indicating a stable periodic solution. This indicates a periodic window in the chaotic attractor, a common feature of nonlinear dynamical systems.

So far, we have gone to great lengths to show that the behaviour of the tumbling pendulum is far from random. However, the pendulum does possess a random quality in terms of diffusive dynamics. For the pendulum with applied torque [Blackburn et al. 1987], and the horizontally forced pendulum [Bayly & Virgin 1992], the averaged time series produced both experimentally and numerically have $1/f^2$ power spectra identical to that of a random process. For the parametrically excited pendulum, four plots of total angular displacement against time shown in figure 6.4 were averaged, and the resultant power spectra is shown in figure 6.5. A linear approximation gives a slope of 2.05, and so the dynamics of the tumbling chaotic attractor has an approximately $1/f^2$ power spectra, which is the same as the dynamics of the other two pendulum systems, and identical to a diffusive random process.
Figure 6.1: Poincaré section of tumbling chaotic attractor for the parametrically excited pendulum. Parameters are $\omega=2$, $p=2$. 
Figure 6.2: Two chaotic trajectories with nearby initial conditions for the tumbling parametrically excited pendulum are integrated over a number of cycles of parametric excitation. The divergence of the two trajectories can be seen clearly after 55 cycles. Angular velocity rather than angular displacement is plotted to avoid the discontinuity of going past $\theta = \pm \pi$. 
Figure 6.3: The largest Lyapunov exponent is plotted for the tumbling parametrically excited pendulum over a parameter regime where the attractor is believed to be chaotic. The parameter $\omega$ is fixed at $\omega=2$. Periodic windows are evident where the largest Lyapunov is negative. A period-6 window exists around $p=2.37$, and a period-8 window around $p=2.6$. 
Figure 6.4: Four plots of total angular displacement for the tumbling parametrically excited pendulum.
Figure 6.5: Averaged Power Spectral Density curves for the four plots of figure 6.6. The negative slope = 2.05.
Chapter 7: Braids and Knots in Forced Oscillators

A three dimensional dynamical system may be analysed by braid and knot theory which yields both local and global information. In particular, a single periodic orbit can imply the existence of a countable infinity of other orbits. To make use of this powerful statement it is necessary to develop some elementary braid and knot theory in the context of dynamical systems.

7.1 Braids

First, consider a three dimensional dynamical system with some external forcing function of time, \( t \).

\[ \ddot{x} = f(x, \dot{x}, t) \]  

(7.1)

Generally, the phase space is \( \mathbb{R}^3 \), but if \( f \) is periodic in \( t \) with minimum period \( T \), then the phase space can be cut to give \( \mathbb{R}^2 \times \mathbb{I} \) where \( \mathbb{R}^2 \) is the Poincaré plane \((x, \dot{x})\) and \( \mathbb{I} \) is the interval \( 0 \leq t \leq T \). Any periodic orbit can be plotted on this phase space with the nature of the crossings indicated. Straightening out the strands gives a geometric braid, \( \beta \) [Birman 1974]. A braid formed from a period four orbit is shown in figure 7.1. For three-dimensional nonautonomous dynamical systems, all crossings of strands will be in the same sense since the strand with the lower value of \( \theta(t) \) before the crossing corresponds a higher value of \( \dot{\theta}(t) \), which is perpendicular to the plane and hence the lower strand passes over the other [McRobie & Thompson 1993b].
Hence all braids obtained in this manner will contain only left over right crossings, forming a positive braid. Braids with both positive and negative crossings may be formed by adopting a different set of coordinates as will be shown in chapter 9. These are called mixed braids.

7.1.1 Knots

A knot is a closed one dimensional curve embedded in a 3 dimensional manifold [Tufillaro et al. 1992]. Hence a knot can be formed by identifying the Poincaré planes at $t=0$ and $t=T$ to form a $\mathbb{R}^2 \times S$ phase space, closing the braid in the obvious order preserving manner to give a knot. The knot formed from the period-4 orbit is shown in figure 7.2.

7.1.2 Braidwords

It is convenient to be able to express a geometric braid by an algebraic sequence indicating the order in which the strands cross one another. The braid $\beta$ can be described by a braidword or word for short; an ordered description of the crossings that make up the braid [Birman 1974]. $\sigma_i$ corresponds to crossing the $i^{\text{th}}$ string over the $(i+1)^{\text{th}}$ string. Similarly $\sigma_i^{-1}$ corresponds to crossing the $i^{\text{th}}$ string under the $(i+1)^{\text{th}}$ string. For example, the braidword for the period-4 orbit shown in figure 7.2 is: $\sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1$. Note that the strings are renumbered after every crossing.
The Artin braid group $\beta_n$ is the group generated by positive and negative crossings in each of the columns on $n$ strings.

### 7.2 Orbit Intervention Principle

One important consequence from plotting periodic orbits as braids is that bifurcational precedence information can be obtained relatively easily. We saw in chapter 1 that a fold bifurcation involves the collision of two orbits of the same period; one stable, and the other unstable. In braid terms, we would see two similar braids approach each other, and coalesce at the bifurcation. For this to occur, there must not be any other orbits 'in the way' of the colliding pair. In particular, the pair must have identical linking properties with every other periodic (and for that matter, nonperiodic) orbit in the system [McRobie & Thompson 1993b]. If this is not the case, then the bifurcation cannot occur until the intervening orbits are destroyed, for to do so would violate the uniqueness theorem. A link diagram may be formed by the union of two braid diagrams. The linking number between two periodic orbits is equal to half the number of crossings in the link diagram for positive braids. For mixed braids, a positive crossing adds a half whilst a negative crossing subtracts a half from the linking number [Tufillaro et al. 1992]. A period-3 orbit and a period-1 orbit are shown in figure 7.3 with linking number 2. Linking number is equivalent to the relative rotations used by Solari and Gilmore [Solari & Gilmore 1988], where the relative rotation of two orbits is simply the number of crossings divided by the number of periods. For example, the two orbits in figure 7.3 have a relative rotation of $4/3$. 

119
7.3 Knot Equivalence

Since the choice of Poincaré section and hence the start of the knot is somewhat arbitrary, crossings can be transferred from bottom to top of the closed braid via conjugation [Holmes & Ghrist 1993] i.e. the knots \( \sigma_2 \sigma_3 \sigma_1 \sigma_2 \) and \( \sigma_1 \sigma_2 \sigma_3 \sigma_1 \) are conjugate. If one knot \( K_1 \) in \( \mathbb{R}^3 \) can be transformed into another knot \( K_2 \) without passing strings through each other or breaking the string, then the knots \( K_1 \) and \( K_2 \) are said to be ambient isotopic. More specifically, there exists a homotopy \( H_t: \mathbb{R}^3 \to \mathbb{R}^3 \), \( t \in I \) such that \( H_0 \) is the identity map, \( H_1 \) sends \( K_1 \) to \( K_2 \) and \( H_t \) is a homeomorphism \( \forall t \in I \) [Boyland & Franks 1989]. Reidemeister [Reidemeister 1932] showed that if two knots are ambient isotopic, then one can be transformed into the other by applying a sequence of Reidemeister moves. The three Reidemeister moves are shown in figure 7.4. If all three moves are necessary, then the knots are ambient isotopic. If only type 2 and 3 moves are necessary then the knots are regular isotopic. A new definition - positive isotopic has been proposed [McRobie & Thompson 1993b] if only type 3 moves are necessary. This is equivalent to regular isotopy for positive braids. The three Reidemeister moves have their equivalents in relations operating on the braid group \( \beta_n \) termed Markov moves [Tufillaro et al. 1992]. Since the number of strings (dynamic period) is invariant, Reidemeister move 1 is not allowed. The other two moves can be described by the relations:

\[
\begin{align*}
M2: & \quad \sigma_i \sigma_{i+1} = 1 \\
M3: & \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq n-2
\end{align*}
\]
Also, crossings can slide past each other as in figure 7.5.

\[ M4: \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2 \]

Two braids are \textit{braid equivalent} if one can be deformed into the other using \( M2, M3, \) and \( M4 \) and properties of the group. Deciding whether two braids are equivalent is termed the \textit{word problem}. This has been achieved for the braid group \( \beta_n \), but is a lengthy procedure. Two positive braids \( \beta_a, \beta_b \) are \textit{positive equivalent} \( \beta_a =_+ \beta_b \) if one can be deformed into the other using \( M3 \) and \( M4 \) alone. Furthermore, two positive braids \( \beta_a, \beta_b \) are \textit{positive cyclic equivalent} \( \beta_a =_{pce} \beta_b \) if one can be deformed into the other using \( M3 \) \( M4 \) and cyclic permutation of the symbols [McRobie & Thompson 1993b]. If two braids are positive cyclic equivalent, then the associated knots are positive isotopic, and a saddle-node bifurcation \textit{may} be possible. However, we still need to consider the linking with other orbits in the system. For braids obtained from time series, it is only necessary to solve the positive cyclic equivalence problem rather than the word problem. However, this problem is still lengthy, and the computational time rises with word length [Birman 1974]. Hence, it is more useful to have a series of quick tests to decide whether two braids are not positive cyclic equivalent rather than to categorically state that they are after lengthy calculations.
7.3.1 Word Length

Two periodic orbits cannot annihilate at a saddle-node bifurcation if their braids formed from their time series have different word lengths [McRobie & Thompson 1993b].

bifurcation possible $\implies$ positive equivalence $\implies$ positive cyclic equivalence $\implies$ same word length

Note that the implications are strictly one way, and just because two orbits have the same word length does not imply that they can annihilate. However, word length is very useful especially at higher period in classifying orbits of the same period into smaller groups which may be able to annihilate as we will see in section 8.5. For positive braids, word length is equivalent to exponent sum of the word [Holmes & Williams 1985] since negative crossings subtract whilst a positive crossing adds one to this quantity.

For a possible period doubling bifurcation to occur, it is necessary to construct an allowable cabling from the knot with the lower period [McRobie & Thompson 1993b]. To do this, each strand in the braid of the lower period orbit is split into three parallel strands. An arbitrary number of positive half twists is applied to one such trio. As a quick test though, a period-n orbit with L crossings will period-double into an orbit of period-2n with $4L + 2m + 1$ crossings where m is a positive integer. Also, the link diagram formed by the union of the two orbits will have $3(3L + 2m + 1)$ crossings.
7.3.2 Knot Polynomials

Knot polynomials are a more sophisticated way of distinguishing between braids of the same word length. There are a wide range of knot polynomials to choose from - the Alexander, Jones, Kaufman, HOMFLY, LMBH, Alexander-Conway, and so on [Alexander 1923, Jones 1985, Kauffman 1987, Brant et al. 1986, Conway 1970]. Each has its advantages in terms of differentiating between knot-types, and consequent disadvantages in ease of use and computational time. Only the Alexander polynomial will be covered in the following since no extra information is possible by using a more sophisticated polynomial in this case. The Alexander polynomial is the most well known, and oldest knot polynomial. It was introduced in the 1920’s [Alexander 1923], and based on branched coverings of the knot complement. The polynomial only distinguishes between ambient isotopic knots, but is simple to calculate by the reduced Burau representation [McRobie & Thompson 1993b]. Each crossing in the braidword is replaced by a square matrix $S_j(t)$ of dimension $n-1$. Multiplying these matrices together in order gives the reduced Burau matrix $S(t)$, and the Alexander polynomial, $A(t)$ is given by:

$$A(t) = \left| S(t) - I \right| / P(t)$$

where $$P(t) = 1 + t + t^2 + \ldots + t^{n-1}$$

It will be shown later that the reduced Burau matrix contains more information.

The matrices $S_j(t)$ are given in figure 7.6.
7.4 Classifying Knots

Knots, or rather underlying mappings, can be classified into three classes: finite order, reducible, and pseudo-Anosov [Boyland & Franks 1989]. Finite order knots are created by simple twists, and do not imply any complex dynamics. An important family of finite order knots are the torus knots. A \((p, q)\) torus knot is a knot which lives on the standard torus \(T^2\), winding around it \(p\) times in one direction and \(q\) times in the other. We set \(q < p\) with \(p\) and \(q\) relatively prime. A \((p, q)\) torus knot can be displayed as a positive braid on \(p\) strands [Holmes & Williams 1985]. Two torus knots are shown in figure 7.7 as examples. A \((p, q)\) torus knot has \(q(p-1)\) crossings. Also note that a \((p, 1)\) torus knot is ambient isotopic to the unknot by removing each crossing one at a time by a series of \((p-1)\) type 1 Reidemeister moves.

Reducible knots are created at period-doubling bifurcations, and can be represented as a combination of two finite order braids, one inserted inside the other. The period-4 braid shown in figure 7.1 can be decomposed into a \((2, 1)\) torus knot and a 2 strand unknot as in figure 7.8.

Pseudo-Anosov knots are irreducible, and imply that any map containing such an orbit must posses a chaotic orbit and hence a complete family of (unstable) periodic orbits [Boyland & Franks 1989]. 'A pseudo-Anosov orbit implies chaos' is the two dimensional equivalent of the famous 'period three implies chaos' statement for one dimensional maps [Li & Yorke 1975]. Strictly, a period-n orbit A is pseudo-
Anosov if a path \( \alpha \) from points \( a_i, a_j \in A \) is not homotopic in \( \mathbb{R}^2 \setminus A \) to its \( n \)th image [Boyland & Franks 1989]. Thus, global information about the map can be derived from the geometric braid formed from the time series of a single orbit. For instance, to prove that a chaotic orbit exists, all that is necessary is to find a pseudo-Anosov orbit in the time series.

### 7.4.1 Topological Entropy

A lower bound on the topological entropy of pseudo-Anosov orbits can be obtained by calculating the spectral radius of the reduced Burau matrix \( S(e^\theta) \). Loosely, the topological entropy of the associated map is greater than or equal to the logarithm of the moduli of the eigenvalues of \( S(e^\theta) \) as \( \theta \) varies from \(-\pi\) to \( \pi \) [McRobie & Thompson 1993b]. By plotting the moduli of the eigenvalues of \( S(e^\theta) \), a lower bound on the topological entropy of the orbit may be deduced. This procedure is carried out for the period five knot \( \sigma_4\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \) as an example in figure 7.9.
7.5 Orbit Forcing Theory

As has already been stated, a pseudo-Anosov orbit implies chaos. However, we can say considerably more. By considering the reduced Burau matrix we can calculate a minimum periodic structure [Boyland & Franks 1989]. That is, given a single orbit, what other orbits must be present in the associated map?

Theorem [Boyland & Franks 1989]: Suppose that trace(S(t)ⁿ) contains the term aⁿ for some i ≠ 0. Then for aᵢ ≠ 0, a fixed point of period n exists with linking number i.

For example: the reduced Burau matrix obtained for the period five orbit \(a^4a_2a_3a_1a_4\) is given below:

\[
S(t) = \begin{bmatrix}
0 & -t & 1 & 0 \\
0 & -t^2 & 0 & 1 \\
-t^3 & 0 & 0 & 1 \\
-t^4 & 0 & 0 & 0
\end{bmatrix}
\]

\[\text{Trace } (S(t)) = -t^2 \Rightarrow \text{one period-1 orbit with linking number 2.}\]

\[\text{Trace } (S^2(t)) = t^4 - 2t^3 \Rightarrow \text{one period-2 orbit with linking number 3,}\]

\[\text{and original period-1 orbit with linking number } 2 \times 2 = 4.\]

This procedure will be applied in section 8.8 to provide further bifurcational precedences for the parametrically excited pendulum.
7.6 Symbolic Dynamics

Consider again the cubic map:

\[ x_{n+1} = \alpha x_n^3 + (1 - \alpha) x_n = f(x_n) \]  \hfill (7.2)

By plotting \( f^p(x_n) \) versus \( x_n \) for \( p = 1, 2, 3 \ldots \) at a sufficiently large value of \( \alpha \) it is apparent that there are a large number of periodic points with period \( p \) which satisfy:

\[ f^p(x_n) = x_n \]  \hfill (7.3)

Given this complicated behaviour, it is useful to have some intuitive 'name' associated with each periodic point, and an ordering on the creation of these points as \( \alpha \) goes from 0 to 4. These two goals can be achieved by a symbolic dynamics approach.

7.6.1 Itineraries

The graph of \( f(x_n) \) versus \( x_n \) has three distinct regions, separated by the two turning points, A and B in figure 7.10. The regions are defined as follows:

Region 0: \( x_n \in [-1, -\frac{1}{2}] \)

Region 1: \( x_n \in (-\frac{1}{2}, +\frac{1}{2}) \)

Region 2: \( x_n \in [+\frac{1}{2}, +1] \)

where the brackets denote end closures in the normal manner. The dynamics of the
map can be coded in the following way. The address \(a(x)\) of \(x \in [-1, +1]\) is defined as:

\[
a(x) = \begin{cases} 
0 & \text{if } x \in [-1, -\frac{1}{2}] \\
1 & \text{if } x \in (-\frac{1}{2}, +\frac{1}{2}) \\
2 & \text{if } x \in [+\frac{1}{2}, +1]
\end{cases}
\]

The itinerary \(I(x)\) of \(x\) is defined as the bi-infinite symbol sequence:

\[I(x) = \ldots a(f^2(x)) a(f^4(x)) \ldots a(x) a(f(x)) a(f^2(x)) \ldots\]

This is the order that the regions are visited under successive applications of the map.

A periodic point with period-\(p\) will have an itinerary with a repeating sequence of \(p\) symbols. These symbols can be used to uniquely identify the periodic point. For example the period-3 point with the itinerary \(\ldots 012.012 \ldots\) can be labelled 012. The three points in the orbit are shown in figure 7.11. A period-\(p\) orbit will be comprised of \(p\) points. These points can be identified by a right hand shift operation on the itinerary of any point in the orbit. This is achieved by moving the decimal point one place to the right. Hence the period-3 point \(\ldots 012012.012012 \ldots\) maps to \(\ldots 012012.12012 \ldots\) which is the point 120. This in turn maps to 201, and then back to 012. This method of coding also lets us predict how many periodic points of any given period may be present. For \(\alpha=4\) we have a family of orbits based on three symbols without restraint. There will be \(3^p\) periodic points with period-\(p\), but some of these points may not have minimum period-\(p\). For instance, of the nine possible sequences of two symbols from the alphabet \([0,1,2]\) three periodic points are of period one, leaving six points in three pairs of orbits. These are noted below:

00 reduces to 0 a period-1 fixed point

11 reduces to 1 a period-1 fixed point
22 reduces to 2 a period-1 fixed point
01 maps to 10 a period-2 orbit
02 maps to 20 a period-2 orbit
12 maps to 21 a period-2 orbit

The number of periodic points and orbits up to period eight are given below:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3P</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td>729</td>
<td>2187</td>
<td>6561</td>
</tr>
<tr>
<td>minimum periodic points</td>
<td>3</td>
<td>6</td>
<td>24</td>
<td>72</td>
<td>240</td>
<td>696</td>
<td>2184</td>
<td>6480</td>
</tr>
<tr>
<td>minimum periodic orbits</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>18</td>
<td>48</td>
<td>116</td>
<td>312</td>
<td>810</td>
</tr>
</tbody>
</table>

Hence at period-8 we have 6561 periodic points, 6480 of which are of minimum period-8 (the others are of period-4, period-2, and period-1), giving 810 distinct period-8 orbits.

Not only does the symbolic encoding of periodic points and periodic orbits provide a convenient method of naming orbits, but it can also be used to give the relative location of the periodic orbits. To see this, it is necessary to consider the action of the map on the three intervals 0, 1 and 2. This is shown schematically in figure 7.12. Interval 0 is stretched over the whole interval [-1, +1], and is anchored at -1. Interval 2 is stretched over the whole interval [-1, +1], but is anchored at +1. The interval 1 is also stretched over [-1, +1], but it is folded so the point at the left hand extremity is mapped to the right, and the point at the right hand extremity is mapped to the left. We say that the intervals 0 and 2 are orientation preserving and interval 1 is orientation reversing. Loosely, if we start with two points in the same interval, if they are both in either of the orientation preserving intervals then the orbit with the higher value of $x_n$ will have the greater value of $f(x_n)$. However, if we start with two points both in the orientation reversing interval, then the orbit with the higher value of $x_n$ will have the lesser value of $f(x_n)$.
The relative locations of periodic points can be calculated by a hierarchical tree as in figure 7.13. At row 1, the three symbols are written in the obvious lexicographical order 012. In the next row, under each symbol, is written the three symbols in the same order if under an even symbol, and in the reverse order 210 if under a 1. Reading the names of period-$p$ orbits off at row $p$ in the tree gives the relative locations of all the minimum period $p$ points together with any other period-$p$ points. This is cumbersome if we want to locate orbits with large period, and so we develop an algebraic location scheme.

### 7.6.2 Invariant Coordinates

Just as the itinerary is the history of where an orbit is going, so the geography of the orbit can be described by an invariant coordinate [Tufillaro et al. 1992]. To calculate this, we write down a symbol sequence from the itinerary of the point, and the orientation. We define the orientation $o(x)$ as reversing; $o(x) = -1$ if the sum of the symbols up to the site in question is odd, and preserving; $o(x) = +1$ if the sum of the symbols up to the site in question is even. If the orientation is preserving, then the next symbol in the invariant coordinate is just the symbol in the itinerary, whilst if the orientation is reversing, then the next symbol in the invariant coordinate is the opposite of the symbol in the itinerary where 0 and 2 are opposites, and 1 is its own opposite. This procedure is considerably easier to use than to put into words. Consider the period two point with itinerary 12. We must write this as 1212. The invariant coordinate is calculated below as 1012:

<table>
<thead>
<tr>
<th>Site</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Itinerary</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$o(x)$</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>Invariant Coordinate</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

The orbits are ordered in the obvious lexicographical way with respect to the invariant
coordinate. The ordering of all period one and period two points obtained from the hierarchical tree is confirmed by the invariant coordinates calculated below:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>21</th>
<th>2</th>
<th>Itinerary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0000</td>
<td>0121</td>
<td>0202</td>
<td>1012</td>
<td>111</td>
<td>1210</td>
<td>2020</td>
<td>2101</td>
<td>2222</td>
<td>Invariant Coordinate</td>
</tr>
</tbody>
</table>

### 7.6.3 Kneading Sequence

So far we have named all the periodic points of the cubic map via an itinerary, and calculated the relative locations of these points by means of an invariant coordinate. It is also possible to derive an ordering on the periodic points - i.e. how all the periodic points are created. This may be achieved by calculating a kneading sequence, and has been carried out by Ringland and Tresser [Ringland & Tresser 1993]. The cubic map is an example of a (+-+) bimodal map; a one dimensional map with two points A, B, with $-1 < A < B < +1$ such that $f(x)$ is increasing on $[-1,A] \cup [B, +1]$, and decreasing on $[A,B]$. Another remarkable result is that any (+-+) bimodal map will have similar bifurcational behaviour, so for every periodic point in the cubic map, there will be an equivalent point in any (+-+) bimodal map. Such universal behaviour is common in nonlinear dynamics.
7.6.4 Pruning

The order in which orbits are created may also be described in terms of a pruning sequence. Working back from the family of orbits based on three symbols without restraint, it is possible to construct a set of rules that restrict the existence of branches of orbits. For instance, one rule may be to disallow any orbit with an itinerary containing the substring 00. This would disallow orbits 102001, 21001, 100, and so on. A series of similar rules may be applied in sequence to the full tree structure to provide a complete understanding of the order in which orbits are created and destroyed.

7.7 Horseshoe Formation in One Dimensional Maps

Consider again the action of the function $f(x)$ on the three intervals

- 0: $x_n \in [-1,-\frac{1}{2}]$
- 1: $x_n \in (-\frac{1}{2},+\frac{1}{2})$
- 2: $x_n \in [+\frac{1}{2},+1]$

for the cubic map equation (8.2). The action of the map is effectively to stretch the interval [-1, +1] to three times its original length, and then to fold it back on itself twice. This was shown graphically in figure 7.12. Hence points which are originally close together become uncorrelated as they are stretched apart and eventually folded into different sheets. This type of transformation was first noted by Smale [Smale 1967], who considered the simpler quadratic map defined by:
Here there are only two intervals, but the effect of the mapping is again to stretch and fold the interval back onto itself. This process is often called the \textit{bakers transformation} [Tufillaro \textit{et al.} 1992] as it is similar to the stretching and folding of dough in the making of flaky pastry. The \textit{horseshoe} shape formed by this process is fundamental to the understanding of nonlinear dynamical systems, and chaos. The divergence of chaotic trajectories noted in chapter 1 is graphically explained, and the analysis of many complex systems can be reduced to studying a particular horseshoe map [Smale 1967]. Returning to the structure formed by the cubic map, we term the particular horseshoe formed a 3-\textit{shoe} because of the number of stripes. Another 3-shoe configuration is possible by arranging the stripes in a spiral fashion as in figure 7.14. McRobie calls this structure a spiral 3-shoe [McRobie 1992a].

7.8 Horseshoe Formation in Two Dimensional Maps

The formation of horseshoes in two dimensional maps is closely linked to the invariant manifolds of saddle points. The local stretching is provided by the unstable manifold, which increases the distance between points, whilst the stable manifold ensures that the stretching is locally reversed to ensure the global area contracting properties are maintained. The folding is due to the tangles formed by homoclinic intersections between the stable and unstable manifolds. The intersecting manifolds divide the phase space into regions called \textit{eyes} and \textit{lobes}, which are amenable to
A pip, or Primary Intersection Point is a homoclinic intersection of the stable and unstable manifolds of a D-type saddle such that sections of the stable and unstable manifolds, $W^s$ and $W^u$ respectively from the saddle to the intersection point do not intersect up to the point considered. The area inside the loop formed by $W^s$ and $W^u$ is termed an eye. In addition, the pip is a main pip if locally extensions to the manifolds from the saddle beyond the pip are outside the eye, and an intermediate pip if locally extensions to the manifolds from the saddle beyond the pip are inside the eye [McRobie & Thompson 1991]. This is illustrated in figure 7.15. Main pips map to main pips, and likewise, intermediate pips map to intermediate pips, so forming two distinct sequences of homoclinic points. We label main pips as $x_i$ and intermediate pips $x_{i+\frac{1}{2}}$, where $i=\ldots,-2,-1,0,1,2,\ldots$. Also $x_{i+1} = G(x_i)$, where $G(x)$ is the associated map. The region enclosed by the curves $W^s[x_i,x_{i+\frac{1}{2}}]$ and $W^u[x_i,x_{i+\frac{1}{2}}]$ are called lobes. The mapping of lobes can be determined by considering the points on the boundary, i.e. the stable and unstable manifolds. Consider the homoclinic tangle in figure 7.16. Main pip $x_0$ maps to $x_1$, and intermediate pip $x_{\frac{1}{2}}$ maps to $x_1+\frac{1}{2}$. Hence, the section of stable manifold $W^s[x_0,x_{\frac{1}{2}}]$ must map to $W^s[x_1,x_1+\frac{1}{2}]$. Also the unstable manifold $W^u(x_0,x_{\frac{1}{2}})$ must map to $W^u(x_1,x_1+\frac{1}{2})$. Any point inside the loop formed by the two manifolds $W^s[x_0,x_{\frac{1}{2}}]$ and $W^u(x_0,x_{\frac{1}{2}})$ must map to the region inside the loop formed by the two manifolds $W^s(x_1,x_1+\frac{1}{2})$ and $W^u(x_1,x_1+\frac{1}{2})$, and so lobe $L_0$ maps to
lobe $L_1$.

Figure 7.17 contains the basis for the horseshoe structure. If we map the rectangle $ABCD$ forward twice, main pip $A$ maps to $A^2$, and intermediate pip $D$ maps to $D^2$. $B^2$ must lie on the stable manifold extended from the saddle beyond $A^2$, and $C^2$ must lie between $D^2$ and the saddle, again on the stable manifold. Hence, the rectangle $ABCD$ is deformed into the horseshoe structure $A^2B^2C^2D^2$ after two applications of the mapping function. Repeated mappings of the rectangle will produce more stretching and folding, giving rise to complicated dynamics, and possibly chaos.

7.9 **Horseshoe Formation in a General Nonlinear System**

The process of locating and identifying particular horseshoe structures from the invariant manifolds of saddles in the Poincaré section can be applied if there is a good idea of which saddle controls the underlying dynamics of the system. This is not always the case, and a more general method which is also useful for characterising the dynamics of real engineering systems where no explicit mathematical model exists, is to identify a *template* or *knot holder* [Holmes & Williams 1985]. This can be achieved by considering a few low order periodic orbits. A template is an expanding map on a branched surface which seeks to preserve the topological structure of all periodic orbits in the flow [Tufillaro *et al.* 1992]. It can be described by a finite crossings matrix, which in turn may be used to fingerprint the underlying chaotic attractor [Mindlin *et al.* 1990]. The method
has met with considerable success when applied to a variety of experimental systems [Fioretti et al. 1993, Lefranc & Glorieux 1993], and is a very useful and easily implemented technique.

Initially, we require the low order (unstable) periodic orbits. These can be obtained in a variety of ways; the Newton Raphson root finding technique discussed in chapter 1, or by the method of close recurrence. The latter is essential for experimental data, and also useful where there is a large region of chaos in a numerical system. The method is based on the principle of ergodicity [Eckmann & Ruelle 1985], that is, that a chaotic trajectory wanders arbitrarily close to every point in the chaotic attractor, including all the unstable periodic orbits. Thus a sample of a chaotic trajectory which behaves almost periodically for a finite number of cycles before wandering off to other regions of phase space indicates a nearby unstable periodic orbit [Tufillaro et al. 1992]. The actual periodic orbit can then be located by the Newton Raphson root finding technique, or just approximated by the relevant sample of the chaotic trajectory.

7.9.1 Constructing a Template

A template is equivalent to a framed braid of the period-1 orbits in the system; that is, a braid of the period one orbits with the local torsion of each orbit indicated [Tufillaro et al. 1992]. A framed braid of four period-1 orbits is shown in figure 7.18. The strands are thickened, and the local
torsion of each orbit is indicated by the number of half twists applied to each ribbon.

Following the convention adopted by Tufillaro [Tufillaro et al. 1992], the ribbons are numbered consecutively from left to right, starting with symbol 0. To convert the ribbon graph to a template, the ribbons are joined at a branch line, and the top and bottom are joined. The order in which the branches are layered can be determined from the template matrix which will be defined later, and can be transformed into a standard form which has the left hand branch at the bottom by a sequence of ribbon moves rather like type two Reidemeister moves.

The resultant branched semi-flow should contain all the knots in the original system whilst preserving all the topological invariants such as knot types, relative rotations, and local torsions. The template matrix for the template of figure 7.18 is given below:

\[
\begin{bmatrix}
+3 & +2 & +2 & +2 \\
+2 & +2 & +2 & +2 \\
+2 & +2 & -1 & -2 \\
+2 & +2 & -2 & -2
\end{bmatrix}
\]

Determining the knot types of all periodic orbits is now a simple task. These can be compared to orbits in the real system located by the method of close returns, which provides an effective method of validating or invalidating the template structure.
7.10 Topological Approach to Analysing Nonlinear Systems

In chapter 1 we developed a numerical approach to analysing nonlinear systems based on cell mapping, path following, and bifurcation following. Here we propose an alternative topological approach.

1) Choose a region of parameter space which contains a large amount of unstable periodic orbits - ideally a chaotic attractor.

2) Identify a particular horseshoe map either by constructing a template, or by analysing the invariant manifolds of a particular saddle.

3) Construct bifurcational precedence relationships by categorising the orbits in the horseshoe in terms of topological invariants such as knot types, linking numbers, and relative rotations.

4) Check results by identifying braids in the real system and comparing with the theoretical calculations.

5) Additionally, calculate topological entropies of any pseudo-Anosov orbits, and derive minimum periodic orbit structure.

This philosophy is applied to the parametrically excited pendulum in chapters 8 and 9.
Figure 7.1: Braid diagram formed from time history of period-4 orbit.
Figure 7.2: Period-4 knot from braid diagram of figure 7.1
Figure 7.3: Link diagram of period-1 and period-3 orbit. There are four positive crossings giving a linking number of +2, or a relative rotation of 4/3.
Figure 7.4: Reidemeister moves to show knot equivalence. Type 1 involves the removal of a fox curl by pulling the two ends taunt. Type 2 is simply lifting one string over the other, whilst type 3 can be achieved by sliding the strands over one another. All the moves are reversible.
Figure 7.5: Two crossings can slide past each other if they are at least two crossings apart.

\[
S_1(t) = \begin{bmatrix}
-t & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\quad S_2(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
t & -t & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S_3(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & t & -t & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\quad S_4(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & t & -t \\
\end{bmatrix}
\]

Figure 7.6: Reduced Burau matrices for a four strand knot.
Figure 7.7: Two examples of $(p,q)$ torus knots where $p$ is the number of strands, and $q$ is the number of complete rotations about the minor axis of the torus.

Figure 7.8: Decomposition of period-4 reducible orbit into a $(2,1)$ torus knot and a two-strand unknot.
Figure 7.9: Moduli of the eigenvalues of $S(e^{i\theta})$ $-\pi < \theta < \pi$ for a pseudo-Anosov knot.
Figure 7.10: Cubic map with two turning points A and B.

Figure 7.11: Cubic map, $\alpha=4$ with two turning points A and B, and period-3 orbit 012.
Figure 7.12: Action of the cubic map on the three intervals 0 1 and 2. Intervals 0 and 2 are simply stretched over the whole interval whilst interval 1 is twisted and stretched over the whole interval.

Figure 7.13: Symbolic tree structure for ordering the period-3 fixed points of the cubic map. At period-2, the ordering is 00, 01, 02, 12, 11, 10, 20, 21, 22. At period-3, the ordering is 000, 001, 002, 012, 011, 010, 020, 021, 022, 122, and so on.
Figure 7.14: Spiral 3-shoe with horizontal strips labelled $H_0$, $H_1$, $H_2$, and vertical strips labelled $V_0$, $V_1$, $V_2$ where strip $H_i$ maps to strip $V_i$ for $i=0,1,2$ under one iterate of the map.

Figure 7.15: Homoclinic tangle showing main pip, intermediate pip, lobe, and eye.
Figure 7.16: Homoclinic tangle showing main pips, and intermediate pips. Main pip $x_0$ maps to $x_1$, and intermediate pip $x_{-1}$ maps to $x_1$. Lobe $L_0$ maps to lobe $L_1$. 
Figure 7.17: Schematic homoclinic tangle showing the basis for horseshoe formation in two dimensional maps. Rectangle ABCD is stretched, and folded into the horseshoe structure $A^2B^2C^2D^2$ under two iterations of the map.
Figure 7.18: Framed braid of four period-1 orbits. All except the two circled crossings are positive, and the local torsion of each branch is also indicated.
Chapter 8: Analysis of Periodic Orbits in a 3-shoe

The three striped horseshoe, or 3-shoe shown in figure 8.1 described in chapter 3, is the key to understanding the bifurcational structure of non-rotating orbits in the parametrically excited pendulum. Here, we utilise the symbolic dynamics, braid and knot theory developed in chapter 7 to locate subharmonic orbits, produce bifurcational subforms, and prove that the pendulum is indeed chaotic.

8.1 Symbolic Description of Orbits

Just as in chapter 7, where we saw that the dynamics of a one dimensional map could be coded by a symbolic alphabet, so the dynamics of the non-rotating orbits in the parametrically excited pendulum can be coded using an alphabet of three symbols. We again define the address \( a(x) \) where \( x \) is now a vector \((\theta, \dot{\theta})^T\) of a point on the Poincaré section as:

\[
\begin{align*}
0 \text{ if } x \in H_0 \\
1 \text{ if } x \in H_1 \\
2 \text{ if } x \in H_2
\end{align*}
\]

The itinerary is now the bi-infinite symbol sequence:

\[
...h_3h_2h_1...h_3h_1h_2...
\]

where \( G^k(x) \in H_j \Leftrightarrow h_k = j \)

152
the history of all horizontal stripes visited. Since the horizontal strip $H_j$ maps to vertical strip $V_j$,

$$v_{k+1} = h_k$$

where $G^k(x) \in V_j \Leftrightarrow v_k = j$

and so the itinerary may be expressed identically as:

$$\ldots v_2 v_1 v_0 . v_1 v_2 v_3 \ldots$$

Again, we will be interested almost entirely in periodic orbits; that is, orbits with itineraries that consist of substrings of length $n$, where $n$ is the dynamical period. The itinerary of any periodic orbit can therefore be obtained by carefully observing the order of horizontal or vertical strips visited under repeated Poincaré mappings. Conversely, orbits can be located by constructing a pair of invariant coordinates.

### 8.2 Invariant Coordinates

Just as orbits in the one dimensional cubic map could be located by an invariant coordinate, we can locate orbits both in the idealised 3-shoe, and consequently in the real system by calculating a pair of invariant coordinates. In common with the cubic map, stripe 1 in the 3-shoe is orientation reversing whilst stripes 0 and 2 are orientation preserving. Hence, we can use precisely the same method to calculate invariant coordinates. We require two invariant coordinates to identify points. We define the $h$ coordinate as $x_0 x_1 x_2 x_3 \ldots$ and the $v$ coordinate as $y_0 y_1 y_2 y_3 \ldots$. The $h$ coordinate can be calculated directly from the itinerary using the following:
\[ x_k = h_k \quad \text{if } o(k) = +1 \]
\[ x_k = (2-h_k) \quad \text{if } o(k) = -1 \]

where the orientation

\[ o(k) = +1 \text{ if } \sum_{j=0}^{k} h_j \text{ is even} \]
\[ o(k) = -1 \text{ if } \sum_{j=0}^{k} h_j \text{ is odd} \]

For the \( v \) coordinate, we must first apply a left hand shift to the itinerary giving

\[ \ldots v_2 v_{-1} v_0 v_1 v_2 v_3 \ldots \]

and proceed according to:

\[ y_k = v_k \quad \text{if } o(k) = +1 \]
\[ y_k = (2-v_k) \quad \text{if } o(k) = -1 \]

where the orientation

\[ o(k) = +1 \text{ if } \sum_{j=0}^{k} v_j \text{ is even} \]
\[ o(k) = -1 \text{ if } \sum_{j=0}^{k} v_j \text{ is odd} \]

Assuming an equal spacing of orbits, the invariant coordinates can be digitised according to:

\[ x^{10} = \sum_{k=0}^{\infty} \frac{2x_k}{5^{k+1}} \]
\[ y^{10} = \sum_{k=0}^{\infty} \frac{2y_k}{5^{k+1}} \]

and then the orbit can be located on the idealised 3-shoe. This in turn allows us to guess the position of orbits in the trellis of invariant manifolds of the real system, and
to use this guess to locate the orbit precisely by a Newton-Raphson procedure. As an example, we carry out this procedure for the period-3 orbit 001.

8.3 Locating Period-3 Orbits

In an otherwise rigorous study Bryant and Miles [Bryant & Miles 1990c] failed to locate any stable (non-rotating) period-3 orbits for the parametrically excited pendulum. They commented that this was an unexpected result since both the pendulum with applied torque and the horizontally forced pendulum possessed period-3 orbits [Bryant & Miles 1990a,b]. Here we locate eight period-3 orbits, four of which are stable and discover why the traditional cell mapping, path following approach did not locate these solutions.

Take the orbit 001. This has points 001, 010, and 100. The invariant coordinates are calculated below:

<table>
<thead>
<tr>
<th>Point</th>
<th>x</th>
<th>y</th>
<th>x^{10}</th>
<th>y^{10}</th>
</tr>
</thead>
<tbody>
<tr>
<td>001</td>
<td>001221...</td>
<td>012210...</td>
<td>0.023810</td>
<td>0.119048</td>
</tr>
<tr>
<td>010</td>
<td>012210...</td>
<td>122100...</td>
<td>0.119048</td>
<td>0.595238</td>
</tr>
<tr>
<td>100</td>
<td>122100...</td>
<td>001221...</td>
<td>0.595238</td>
<td>0.023810</td>
</tr>
</tbody>
</table>

These points are located in the 3-shoe in figure 8.2. Using this information, a Newton-Raphson procedure successfully located the three points in the parametrically excited pendulum shown in figure 8.3. As we noted in chapter 1, unstable solutions
may bifurcate into stable solutions. To see if this was the case, the period-3 solution 001 was path followed, and found to stabilise at an inverse flip bifurcation before rapidly becoming unstable at a saddle-node bifurcation. Indeed for \( \omega = 2 \), the solution was only stable for \( 1.6741286 < p < 1.6741420 \). If we compare these values to the bifurcation diagram in figure 8.4, we see that these parameters are in the escape zone. Hence we have located a period-3 solution which was initially thought not to exist, and furthermore, the solution is stable in the escape zone. However, this stability is only over a very narrow parameter range, and the basin of attraction is tiny. Indeed, it is so small that a cell mapping algorithm with 200 x 200 cells and \( -\pi < \theta < \pi, -3 < \dot{\theta} < 3 \) failed to locate it. This explains the failure of the numerical techniques in chapter 3 to locate such solutions. This period-3 solution may not be realisable in a physical engineering system since the presence of noise would make locating it virtually impossible, but modern nonlinear control theory can stabilise such solutions even in the presence of noise [Shinbrot et al. 1993], making the discovery of engineering importance. All eight period-3 solutions were successfully located in the parametrically excited pendulum, and in the parametrically damped system, strengthening the 3-shoe hypothesis.

### 8.3.1 Higher Order Periodic Orbits

Now we are in a position to explain the existence of the period-12 orbit that was thought to be responsible for the chaotic blue sky catastrophe in chapter 3. This orbit was created at a saddle-node bifurcation and then rapidly became unstable at a
flip bifurcation. We exercise some caution, since there will be 44220 orbits of (minimum) period twelve in the 3-shoe, and so locating the correct orbit may prove to be difficult. Indeed, several other period-12 orbits were located close to the one discussed in chapter 3.

8.4 Developing a Crossings Algorithm

We have seen how the symbolic dynamics approach can locate stable and unstable orbits in the parametrically excited pendulum. However, we can say more about the bifurcational behaviour of the parametrically excited pendulum by considering the relative rotations of periodic orbits in the 3-shoe. We noted in chapter 7 that two orbits cannot annihilate at a saddle-node bifurcation if there is another orbit one another in the way. Hence two orbits A and B cannot annihilate if there is another orbit C which crosses A a different number of times than it crosses B in a given number of periods of forcing. This is equivalent to the relative rotations argument used by Solari and Gilmore [Solari & Gilmore 1988]. We develop an algorithm to calculate the number of crossings of a pair of orbits from their itineraries alone [Tufillaro et al. 1992, McRobie 1992a]. This can be achieved by considering the three operations that produce the 3-shoe described below, and shown in figure 8.5:

OP1: Stretch all strips vertically and shrink horizontally.

OP2: Bring strip H2 to the left.

OP3a: Bring strip H1 to the centre.
A symbolic algorithm can then be developed. The algorithm consists of three rules:

**Rule 1:** Align the itineraries. Change every symbol 2 to a 0 if it is opposite a 2, and to a 1 if it is opposite a 0. If a 2 is opposite a 1, change the 2 to a 1 and the 1 to a 0.

**Rule 2:** Locate a site where the two modified itineraries are different, and start from the next site on the orbit which had a 0 at the previous site.

**Rule 3:** Proceed along the modified itinerary until a 1 is encountered. At the next site swap orbits.

The number of crossings is the sum of symbols selected.

For instance, for the two period three orbits 001 and 120:

writing the itineraries gives:

\[ \ldots001.001\ldots \]

\[ \ldots120.120\ldots \]

Modifying according to rule 1:

\[ \ldots001.001\ldots \]

\[ \ldots110.110\ldots \]

Select a start and proceed (bold):

\[ \ldots001.001\ldots \]

\[ \ldots110.110\ldots \]

Which gives four crossings in six periods of forcing, or two crossings in three periods, a relative rotation of 2/3.
This procedure was carried out for all period-1 and period-2 orbits in table 8.1, and for all period-3 orbits in table 8.2. The diagonal elements are the local torsions, which are equal to the sum of 1's in the root of the orbit. The results were checked by considering the crossings of all orbits that were numerically located in the parametrically excited pendulum, and found to agree.

8.5 Bifurcational Precedences from Crossings Data

To construct bifurcational precedences, consider the crossings matrix of period-1 and period-2 orbits in table 8.1. Two orbits can only bifurcate if the rows are identical, with the exception of the diagonal elements. Immediately we see that the period-2 orbit 21 can bifurcate with 10 and 20. This corresponds to the symmetry breaking bifurcation. The symmetric orbit 02 cannot bifurcate with the zero solution 1 whilst any other period 2 orbit exists since:

\[
\begin{align*}
01 \times 1 &= 2 & \text{whilst} & 01 \times 02 &= 0 \\
12 \times 1 &= 2 & \text{whilst} & 12 \times 02 &= 0 \\
10 \times 1 &= 2 & \text{whilst} & 10 \times 20 &= 0 \\
21 \times 1 &= 2 & \text{whilst} & 21 \times 20 &= 0
\end{align*}
\]

where \( A \times B \) indicates the number of crossings of \( A \) and \( B \) in two periods.

Here we rule out a sub-critical pitchfork bifurcation since this scenario would require four period-2 orbits and there are only three period-2 orbits in the 3-shoe. To include this bifurcation we could consider a higher order horseshoe, but for our purposes this
is unnecessary as the extra period-2 orbit is the only orbit we have located that is not explained by the 3-shoe. Also, none of the period-1 solutions can bifurcate with each other whilst any period two solution remains.

We can now produce a bifurcational subform indicating the bifurcations that produce the period-2 orbits in figure 8.6. This confirms the bifurcation diagram produced by the methods outlined in chapter 2 apart from the sub-critical pitchfork bifurcation.
8.5.1 Period Three Bifurcational Precedences

The crossings matrix for period-3 orbits over three periods of parametric excitation is shown in table 8.2. From the crossings matrix, by considering the similarity of the appropriate rows, we can conclude that:

* (001,010,100) and (002,020,200) can bifurcate, along with their symmetric equivalents, (221,212,122) and (220,202,022).

Once these four orbits are destroyed,

* (110,101,011) and (120,201,012) can bifurcate, along with their symmetric equivalents, (112,121,211) and (102,021,210).

The eight period-3 orbits bifurcate in I-type and D-type saddle pairs, indicating saddle-node bifurcations (folds), and subsequent period-doubling cascades, shown schematically in figure 8.7. The bifurcations were shown to occur in the predicted order by again locating the solutions numerically, and path-following for the parametrically excited pendulum. This result confirms the location of stable period-3 solutions.
8.5.2 Period Doubling Cascade and Period Three Bifurcational Precedences

More information can be gained from considering the crossings between period-2 and period-3 orbits. Rather than producing a complete picture, we will consider the precedence between the period doubling cascade from the period-2 orbit 12 and the period-3 saddle-node (001,010,100), (002,020,200). The orbits in the period doubling cascade can be named by adopting McRobie’s conjecture [McRobie 1992b] that when a period-\(n\) parent orbit period doubles to a period-\(2n\) daughter orbit, the name of the daughter orbit can be constructed from the name of the grandparent and parent orbit as follows:

\[
D = GGP
\]

For instance, the period-doubling cascade from 12 is as follows:

\[
12 \Rightarrow 12121112 \
\Rightarrow 1112111212121112 \
\Rightarrow 121211121212111212121112 \
\Rightarrow 121211121212111212121112 \
\Rightarrow \ldots
\]

The crossings of the first five orbits in the cascade with the period-3 orbit 001 are given below:

\[
\begin{align*}
12 \times 001 & = \quad 2 \text{ crossings in 6 periods} \\
& \quad \text{relative rotation} = 2/6 \\
1112 \times 001 & = \quad 6 \text{ crossings in 12 periods} \\
& \quad \text{relative rotation} = 6/12 \\
12121112 \times 001 & = \quad 10 \text{ crossings in 24 periods} \\
& \quad \text{relative rotation} = 10/24 \\
1112111212121112 \times 001 & = \quad 22 \text{ crossings in 48 periods}
\end{align*}
\]
relative rotation $= \frac{22}{48}$

$121211121211121112121112 \times 001 = 42$ crossings in 96 periods

relative rotation $= \frac{42}{96}$

The relative rotation of the kth orbit in the cascade with 001 is the average of the previous two orbits. Since the relative rotation of the kth orbit and the (k-1)th orbit are different, then no orbit in the cascade can be destroyed in the presence of the period 3 orbit 001. This confirms that a pair of period three orbits are stable above the catastrophic bifurcation at the end of the period-doubling cascade, i.e. in the escape zone. However, in practice these orbits are only stable over a small parameter regime.

8.6 Braids of Periodic Orbits

The braid diagrams of periodic orbits can be drawn by considering the 3-shoe operations in figure 8.5. First, the $n$ Poincaré points in the orbit are located by calculating the h invariant coordinate, and the crossings of strands can be found by systematically applying the 3-shoe operations.

Word length can be used to separate orbits of the same period into subgroups which may annihilate at saddle-node bifurcations. For instance, at period-4 there are twelve orbits of word length 3, and 6 with word length 5. This division is even more useful at higher period, as can be seen in table 8.3 where all orbits up to period 7 are
classified according to word length. For instance, at period-7, there are seven subgroups with different word length.

As the period is increased, patterns emerge in the number of orbits of a given word length. For \( n > 3 \) there are \( 4(n-1) \) orbits with minimum word length \((n-1)\). For \( n > 4 \) there are 8 orbits with maximum word length. This maximum word length is given by:

\[
L(\beta_n)_{\text{max}} = \frac{1}{2} (n - 1)^2 + \frac{1 + (-1)^n}{2}
\]

The orbits with maximum word length can be named as a 3 symbol substring plus \((n-3)\) 1’s where the substring is one of the following: 010, 011, 020, 021, 201, 202, 211, or 212. For instance, at period-7, the orbits with maximum word length are 1111010, 1111011, 1111020, and so on.

8.7 Classification of Periodic Orbits by Alexander Polynomial

The Alexander polynomial discussed in chapter 7 was calculated using the reduced Burau representation for all the periodic orbits up to period-7. This divides all the orbits into groups that are ambient isotopic, but we can further separate the orbits according to dynamical period. The results are shown in table 8.4. The calculation of the Alexander polynomial yields no further information than the
crossings data for orbits up to period-6, but at period-7 the subgroups of orbits with
twelve crossings can be further divided into two by considering the Alexander
polynomials. Additional information can be gained by calculating the topological
entropy of the orbits by the method outlined in chapter 7. This allows us to classify
the orbits into torus, reducible, or pseudo-Anosov knots. The results are shown in
table 8.5. For the period-7 orbits with 12 crossings we see that 40 of the orbits are
pseudo-Anosov, and the remaining 16 orbits are (7,2) torus knots. We can now say
more about the orbits with minimum and maximum crossings. The orbits of period-n
with minimum crossings are (n,1) torus knots. The orbits with maximum crossings
are more interesting, since for odd periods the orbits with maximum crossings are
(n,(n-1)/2) torus knots, whilst for even periods, the orbits with maximum crossings
may be reducible or pseudo-Anosov, but not torus knots. We also note that none of
the period-3 orbits here imply chaos, and we need to go to period-5 before we find
pseudo-Anosov orbits that imply chaotic dynamics. Using this information, we
successfully located a suitable period-5 orbit by the invariant coordinate, Newton-
for
Raphson methods already outlined the parametrically excited pendulum. The time
history of the orbit is shown in figure 8.8, and by again calculating the Alexander
polynomial for this orbit we see that it is indeed pseudo-Anosov. This is an important
result since it proves that the non-rotating parametrically excited pendulum is chaotic
without the 3-shoe hypothesis being necessary.
8.8 Topological Entropy of Pseudo-Anosov Orbits

The topological entropy of all pseudo-Anosov orbits up to period-7 in the 3-shoe was calculated by the method outlined in chapter 7. We show graphs of the moduli of the eigenvalues of the matrix $S(e^{i\theta})$ with $-\pi < \theta < +\pi$ for all pseudo-Anosov orbits in figure 8.9, and hence calculate the topological entropy in table 8.6.

8.9 Orbit Forcing Theory

The number and linking numbers of implied orbits of a given period up to period-4 for each pseudo-Anosov orbit are given in table 8.7. This information can be used to provide further bifurcational precedences by considering which orbits in the 3-shoe are implied by the pseudo-Anosov orbits. For example, consider the period-5 pseudo-Anosov knot. One of the orbits with such a knot type is the orbit 22121. This orbit is even, and is formed at a subharmonic saddle-node bifurcation. The braid of the orbit can be calculated from an algorithm based on the 3-shoe operations discussed in section 8.3 as $\sigma_4\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$. The linking numbers of all implied orbits up to period-4 are given in table 8.8. We require one period-1 orbit with linking number 2, one period-2 orbit with linking number 3, and two period-3 orbits with linking numbers 4 and 5. Recalling that the linking number of two orbits is half the number of crossings in the equivalent number of periods, we can discover what orbits in the 3-shoe are implied by calculating crossings data for all possible orbits.
This is carried out for all orbits up to period-3 together with the pseudo-Anosov orbit under consideration below:

\[
\begin{align*}
22121 \times 0 &= 0 \text{ crossings in 5 periods; linking number} = 0 \\
22121 \times 1 &= 4 \text{ crossings in 5 periods; linking number} = 2 \\
22121 \times 2 &= 0 \text{ crossings in 5 periods; linking number} = 0 \\
22121 \times 12 &= 6 \text{ crossings in 10 periods; linking number} = 3 \\
22121 \times 01 &= 4 \text{ crossings in 10 periods; linking number} = 2 \\
22121 \times 02 &= 4 \text{ crossings in 10 periods; linking number} = 2 \\
22121 \times 001 &= 4 \text{ crossings in 15 periods; linking number} = 2 \\
22121 \times 002 &= 4 \text{ crossings in 15 periods; linking number} = 2 \\
22121 \times 011 &= 8 \text{ crossings in 15 periods; linking number} = 4 \\
22121 \times 012 &= 8 \text{ crossings in 15 periods; linking number} = 4 \\
22121 \times 021 &= 8 \text{ crossings in 15 periods; linking number} = 4 \\
22121 \times 022 &= 8 \text{ crossings in 15 periods; linking number} = 4 \\
22121 \times 112 &= 10 \text{ crossings in 15 periods; linking number} = 5 \\
22121 \times 122 &= 8 \text{ crossings in 15 periods; linking number} = 4 \\
\end{align*}
\]

Immediately, we can see that the only period-1 orbit with the correct linking number is the orbit 1. This is the zero solution. Note that this does not imply that the solution is stable, but just that it exists if the period-5 orbit 22121 exists. The only period-2 orbit with the correct linking number is 12. This orbit was formed at the symmetry
breaking bifurcation (see section 8.4), and so we can conclude that the orbit 22121 must form after the symmetry breaking bifurcation. Since the symmetry breaking bifurcation is close to the final chaotic event, we can infer that the odd period-5 orbit that bifurcates with 22121 may be stable in the escape zone. Moving to the period-3 orbits, the orbit 112 must exist since it is the only period-3 orbit with linking number 5, whilst five period-3 orbits have the linking number 4, and so any of these orbits could exist. However, from the period-3 bifurcations considered in section 8.5.1, we see that the orbit which bifurcates with 112 is 021, and this is indeed one of the period-3 orbits with linking number 4.
Figure 8.1: Idealised 3-shoe with vertical stripes V0, V1, V2, and horizontal stripes H0, H1, H2 such that stripe Hn maps to stripe Vn.

Figure 8.2: Location of period-3 orbit 001 in the idealised 3-shoe by invariant coordinates.
Figure 8.3: Location of period-3 orbit 001 in the trellis of invariant manifolds of the hilltop saddles for $\omega=2$, $p=1.8$, $c=0.1$ using the invariant coordinates as an initial guess for a subsequent Newton-Raphson algorithm.
Figure 8.4: Location in the escape zone of the stable period-3 orbit 001 in parameter space for the parametrically excited pendulum denoted by small circle. H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of point c. S is a symmetry breaking bifurcation, and F represents the end of a period-doubling cascade although only the initial period-doubling is shown. The period-2 fold line associated with the sub-critical bifurcation is also shown, along with the second unstable zone.
**Figure 8.5:** 3-shoe operations. OP1 stretches all the strips vertically and shrinks them horizontally. OP2 takes the bottom strip to the left, and OP3 takes the middle strip into the centre and applies a half twist.

**Figure 8.6:** Period-2 bifurcational subform for the 3-shoe.
Figure 8.7: Period-3 period-doubling cascade.

Figure 8.8: Period-5 pseudo-Anosov orbit located in the parametrically excited pendulum. Parameters are $\omega=2$, $p=2$. 
Figure 8.9: Graphs of the moduli of the eigenvalues of the matrix $S(e^g)$ for all the pseudo-Anosov orbits up to period-7 in the 3-shoe.
Table 8.1: Crossings matrix of all period-1 and period-2 orbits over 2 cycles. For example, the orbits 10 and 1 cross twice in two periods. Orbits 10, 20, and 21 can bifurcate since the relevant columns are identical apart from the diagonal elements which are the local torsions.
Table 8.2: Crossings matrix for all period-3 orbits over three periods.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>3⁰</td>
<td>3¹</td>
<td>8²</td>
<td>12³</td>
<td>16⁴</td>
<td>20⁵</td>
<td>24⁶</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6⁵</td>
<td>24⁶</td>
<td>40⁷</td>
<td>56⁸</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8⁸</td>
<td>36⁹</td>
<td>88¹⁰</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>12¹¹</td>
<td>56¹²</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8¹³</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>56¹⁴</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>24¹⁶</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8¹⁸</td>
<td></td>
</tr>
</tbody>
</table>

Table 8.3: Wordlength for orbits up to period-7 in 3-shoe. Entries are of the form N^L where there exist N orbits with wordlength L.
Table 8.4: Alexander polynomials for orbits up to period-7 in 3-shoe. Entries are of the form $N^L$ where there exist $N$ orbits with wordlength $L$.

<table>
<thead>
<tr>
<th>Alexander Polynomial</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3^1$</td>
<td>$3^1$</td>
<td>$8^2$</td>
<td>$12^3$</td>
<td>$16^4$</td>
<td>$20^5$</td>
<td>$24^6$</td>
</tr>
<tr>
<td>$1-t+t^2$</td>
<td></td>
<td></td>
<td></td>
<td>$6^5$</td>
<td></td>
<td>$24^6$</td>
<td>$40^7$</td>
</tr>
<tr>
<td>$1-t+t^2-t^3+t^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$8^8$</td>
<td>$36^9$</td>
</tr>
<tr>
<td>$1-t+t^3-t^5+t^6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$12^{11}$</td>
</tr>
<tr>
<td>$1-t-t^2-t^3+t^4-t^5+t^6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$16^{12}$</td>
<td></td>
</tr>
<tr>
<td>$1-t+t^3-t^4+t^5-t^6+t^8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$8^{13}$</td>
<td>$56^{14}$</td>
</tr>
<tr>
<td>$1-t+t^3-t^4+t^5+t^6+t^9+t^{10}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1-t+t^3-t^4+t^5-t^8+t^9-t^{11}+t^{12}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$8^{18}$</td>
</tr>
</tbody>
</table>

Table 8.5: Orbit classification for orbits up to period-7 in 3-shoe. Entries are of the form $N^{L_{\text{type}}}$ where there exist $N$ orbits with wordlength $L$, and type is $(p,q)$, red, or pA for a $(p,q)$ torus knot, a reducible orbit, or a pseudo-Anosov orbit respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^0_{(1,1)}$</td>
<td>$3^1_{(2,1)}$</td>
<td>$8^2_{(3,1)}$</td>
<td>$12^3_{(4,1)}$</td>
<td>$16^4_{(5,1)}$</td>
<td>$20^5_{(6,1)}$</td>
<td>$24^6_{(7,1)}$</td>
<td></td>
</tr>
<tr>
<td>$6^4_{\text{red}}$</td>
<td>$24^6_{\text{pA1}}$</td>
<td>$40^7_{\text{pA2}}$</td>
<td>$56^8_{\text{pA3}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8^8_{(5,2)}$</td>
<td>$36^9_{\text{red}}$</td>
<td>$88^{10}_{\text{pA4}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$12^{11}_{\text{red}}$</td>
<td>$40^{12}<em>{\text{pA5/16^{12}</em>{\text{red}}}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8^{13}_{\text{red}}$</td>
<td>$56^{14}_{\text{pA6}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$24^{16}_{\text{pA7}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$8^{18}_{(7,3)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 8.6: Topological entropy for all pseudo-Anosov orbits up to period-7 in the 3-shoe.

<table>
<thead>
<tr>
<th>Knot</th>
<th>Period</th>
<th>Entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>pA1</td>
<td>5</td>
<td>ln(1.7221)</td>
</tr>
<tr>
<td>pA2</td>
<td>6</td>
<td>ln(1.8832)</td>
</tr>
<tr>
<td>pA3</td>
<td>7</td>
<td>ln(1.9469)</td>
</tr>
<tr>
<td>pA4</td>
<td>7</td>
<td>ln(1.4317)</td>
</tr>
<tr>
<td>pA5</td>
<td>7</td>
<td>ln(1.6617)</td>
</tr>
<tr>
<td>pA6</td>
<td>7</td>
<td>ln(1.5562)</td>
</tr>
<tr>
<td>pA7</td>
<td>7</td>
<td>ln(1.5560)</td>
</tr>
</tbody>
</table>

Table 8.7: Linking numbers of implied orbits up to period-4 for all pseudo-Anosov orbits up to period-7 in the 3-shoe. For example, the period-5 knot pA1 implies that there exists one orbit of period-1 with linking number 2, one orbit of period-2 with linking number 3, and two orbits of period-3 with linking numbers 4 and 5.

<table>
<thead>
<tr>
<th>Knot</th>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>pA1</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>4,5</td>
<td></td>
</tr>
<tr>
<td>pA2</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>4,5</td>
<td>5,7</td>
</tr>
<tr>
<td>pA3</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>3,4,5</td>
<td>5,7</td>
</tr>
<tr>
<td>pA4</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td></td>
<td>6,7</td>
</tr>
<tr>
<td>pA5</td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>5,6,7</td>
<td>9,10</td>
</tr>
<tr>
<td>pA6</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>6,7</td>
<td></td>
</tr>
<tr>
<td>pA7</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td></td>
<td>11</td>
</tr>
</tbody>
</table>
Chapter 9: Rotating Subharmonic Orbits

In this chapter we locate additional rotating orbits which populate the tumbling chaotic attractor.

9.1 Locating Rotating Subharmonic Orbits.

The tumbling chaotic attractor is bounded by the unstable invariant manifolds of the hilltop saddles. However, unlike the easily visible 3-shoe that contained all the oscillatory orbits, the particular horseshoe which contains the rotating orbits formed by the manifolds is not easy to see. It is unfortunate that the horseshoe is not easily visible, since this would allow us to locate the unstable periodic orbits that populate the attractor by the methods of symbolic dynamics applied to non-rotating orbits in chapter 8. Instead we make use of the ergodicity of chaotic attractors mentioned in section 1.4.4 to locate subharmonic unstable rotating periodic orbits. To recap, sections of the chaotic trajectory will have strong recurrence properties, and over a short number of periods may appear to be periodic. This is the case if the chaotic trajectory is close to an unstable periodic solution. For example, in figure 9.1, the highlighted portion of chaotic time history for the tumbling chaotic attractor appears to be periodic around an unstable period-2 orbit. By considering longer portions of chaotic trajectory many periodic orbits can be approximated. If further precision is required, these recurrent sections of the chaotic time series can be used as the initial guess for a Newton-Raphson procedure. This was carried out for the parametrically excited pendulum, and some of the purely rotating periodic orbits located are shown.
in figure 9.2.

A table of all the \((n,r)\) rotating orbits up to period-3 located by this method is shown below:

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>4</td>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 9.1: Number of \((n,r)\) rotating orbits in the parametrically excited pendulum at parameters \(\omega = 2\), \(p = 2\), where \(n\) is the dynamical period, and \(r\) is the number of complete rotations.

The number of rotating orbits located is consistent with a four striped horseshoe. Additionally, there seems to be a pattern of the number of \((n,r)\) orbits emerging. We remark that this procedure is not exhaustive, and orbits not contained in the chaotic region of phase space may be overlooked. Additional unstable rotating orbits may exist at higher parameter values, but at the range of parameters that are considered here, this is unlikely given the relatively constant Lyapunov exponent calculated in chapter 6. Locating higher order periodic orbits is also possible, and orbits up to period-7 have been located by this technique. Rather than trying to locate all the higher order periodic orbits, we produce a template which should contain all the periodic orbits.
9.2 Template Analysis

The four (1,1) period one orbits can be used to construct a template. However, if we plot displacement in the normal manner we will produce discontinuous braids as the pendulum goes past $\theta = \pm \pi$ by the very nature of the phase space. To avoid this, we use a phase space transformation with new polar coordinates $(R, \theta)$. Also to lose the symmetry of the system and uniquely identify orbits which rotate in opposite directions we take:

$$R = e^\theta$$

Transforming back into $(x,y)$ coordinates gives:

$$x = e^\theta \sin \theta$$
$$y = e^\theta \cos \theta$$

Hence, we can plot $x$ against time, and the nature of any crossings can be determined by the value of $y$ which is perpendicular to the plane. In this case we do not get only positive braids since some of the crossings are negative. The nature of the crossings is determined by the two values of $y$ at each crossing, with the strand with the larger value passing over the other. This was carried out for the four period one orbits, and the resulting template is shown as a framed braid together with the crossings matrix in figure 9.3. This allows all braids of any given period to be calculated using the procedures in chapter 7. We avoid a complete analysis as little can be gained from this aspect of the template construction. However, the matrix serves to fingerprint the
chaotic attractor and allows the system to be compared to other chaotic attractors [Mindlin et al. 1990].
Figure 9.1: Comparison between an unstable period-2 solution and a close recurrent tumbling chaotic trajectory for the parametrically excited pendulum. Parameters are $\omega=2$, $p=1.85$. 
Figure 9.2: Some (n,r) unstable purely rotating orbits for the parametrically excited pendulum located by the method of close returns. Angular velocity is plotted against time over six periods of parametric forcing. Parameters are $\omega=2$, $p=1.85$. 
Figure 9.3: Framed braid and crossings matrix of rotating period-1 orbits for the parametrically excited pendulum. All except the two circled crossings are positive. The local torsion of each branch is also indicated.
The experimental results discussed in this chapter are limited by the lack of available instrumentation. Hence, there are no quantitative measurements, only observations of the possible solutions, and the bifurcations from one solution to another. However, these observations agree with the numerical results contained in earlier chapters, and the detailed experimental observations of others [Levin et al 1985, Mullin 1993]. In a real life situation, it may only be possible to observe behaviour, but this could provide information via time series data (see chapter 7), which in turn could serve to validate the numerical modelling. It is hoped to carry out further experiments in the future, and this is part of ongoing research at University College London.
Chapter 10: Experimental Observations

10.1 Experimental Rig

One of the reasons for studying the parametrically excited pendulum is that it can be used to give a convincing visual demonstration of chaos and other nonlinear phenomena. There have been several attempts to study the system by experimental observations [Skalak & Yarymovych 1960, Leven et al. 1985], which give qualitative agreement to the numerical results. A physical demonstration of the parametrically excited pendulum is shown in figure 10.1. The pendulum consists of a thin metal rod with a plastic disk on the end to aid visualisation. The rod is pivoted to allow motion in the vertical plane, and the support is driven by a scotch yoke mechanism by a small variable speed d.c. motor. The equipment is crude with no means of measuring angular displacement, and the displacement of the pivot point is of fixed amplitude. Despite these obvious limitations, the experimental setup allows some of the predicted motions to be observed. No quantitative measurements were taken since there are too many unknown parameters such as damping, natural frequency of the pendulum, exact form of excitation, and so on although in principal these quantities could be estimated with a little extra work.

10.2 Physically Observed Solutions

The hanging solution is easily observed at low frequency, and can be shown to be stable by slightly perturbing the pendulum and watching it return to the hanging
state. By increasing the forcing frequency and perturbing the pendulum away from the hanging solution by a significant angular displacement, after the transient behaviour is allowed to die away, the symmetric period-2 oscillation can be seen. By increasing the forcing frequency further, we expect the symmetry breaking bifurcation to occur. In practice, this is difficult to observe for two reasons. Firstly, the symmetry breaking bifurcation is very close to the final event which sees the disappearance of all the major non-rotating solutions. Secondly, the physical system is not completely symmetric, and so the symmetry breaking bifurcation will be replaced by a sudden sharp rise in amplitude of the pendulum. The experimental setup does not afford the fine control necessary to distinguish the sequence of bifurcations leading up to the final event, and in practice, what is observed is a complicated transient behaviour where the pendulum tumbles back and forth with varying amplitude before falling on to a coexisting rotating solution.

At high frequency, the pendulum can be made to stabilise in the inverted state. Decreasing the frequency slightly produces small inverted oscillations which appear to be asymmetric; a fact also noted by an earlier experimentalist [Stephenson 1908]. This is due to the pitchfork bifurcation predicted in chapter 4. Increasing the frequency causes the inverted pendulum to topple over as the stability is lost.

Some rotating solutions with \( n = r \) are easily observed, but distinguishing even (1,1) and (2,2) solutions is difficult with the limited precision available. Modifying the equipment to measure angular displacement or velocity is highly desirable if these solutions are to be distinguished. No solutions with \( n \neq r \) were observed as expected since the basins of attraction in the mathematical model were very small.

The only tumbling solution that can be realistically observed is the tumbling
chaos. Indeed, it was to show this behaviour that the demonstration was originally constructed. The tumbling chaos is stable over a wide range of the forcing frequency, and the random left-right sequence of tumbles may be regarded as the typical behaviour of the system. Often whilst observing the chaotic tumbling there are large portions of time in which the behaviour appears to be periodic, with the sequence of tumbling appearing very regular. This is to be expected since the ergodic properties of the chaotic attractor dictate that there will strongly recurrent portions of time history around the unstable periodic orbits. This is an important feature when distinguishing the chaotic nature of the pendulum from that of a purely random process.
Figure 10.1: Experimental demonstration of the parametrically excited pendulum. The thin metal rod is driven by a small d.c. motor, and the plastic disk is to aid visualisation.
For real engineering systems operating in noisy environments, all that may be required is a description of which orbits exist under various values of the control parameters. This is given in figure 11.1, which is a combination of the results of the previous chapters. The complexity of this figure itself underlines the need for the individual consideration of the various types of possible motion in terms of the stability of solutions, and the locus of bifurcations. For the non-rotating solutions, H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of point c. S is the locus of a symmetry breaking bifurcation, and F represents the end of a period-doubling cascade although only the initial period-doubling is shown since the cascade is very rapid. The period-2 fold line A associated with the sub-critical pitchfork bifurcation is also shown, along with the secondary unstable zone H'. For the rotating solutions, J represents the creation of a (1,1) rotating orbit, which period-doubles at line B, and again at line G, which is close to the final bifurcation to tumbling chaos. The period-1 rotating solution restabilises at line I. Two period-3 subharmonic rotating orbits are also shown in windows (3,1) and (3,3). The pendulum is stable in the inverted position in the 'inverted zone'. For more detailed information it is necessary to refer to the individual chapters concerned with the various aspects of the bifurcational behaviour.

For higher dimensional systems, the qualitative features described may be applied to predict failure, or to estimate global dynamics from the simple model. The importance of parametric excitation in terms of instability of equilibria (see chapter 2) must also be appreciated if the stability of a structure is to be considered.
Chapter 11: Conclusions and Discussion

We have presented a detailed analysis of the bifurcational behaviour of the parametrically excited pendulum, from the relatively simple hanging behaviour, to the more complex oscillatory motions, inverted solutions, rotating orbits, and tumbling chaos. Two approaches have been used; a numerical attack based on cell-mapping, path following, and bifurcation following, and a variety of analytical techniques from harmonic balance, the method of strained parameters, to the more unusual concepts from braid and knot theory. Both approaches revealed details that the other missed, but there was also a great deal of agreement between the two which served to reinforce the results. Some physical experimentation was carried out, and the findings gave qualitative agreement with some of the numerical results.

The main points revealed by this investigation regarding non-rotating orbits centred around the approach of reducing the system to an escape from a potential well problem. As we saw in chapter 3, the parametrically excited pendulum is a generic example of a system which permits escape from a symmetric potential well under parametric excitation, and this allowed much of the bifurcational behaviour to be predicted. By considering the horseshoe formed by the invariant manifolds of the hilltop saddles, odd periodic orbits were located numerically, and found to be stable in the escape zone. These solutions were ignored by earlier investigations [Koch & Leven 1985, Bryant & Miles 1990c]. The importance of this result is that it is possible to achieve an odd periodic non-rotating solution for the parametrically excited pendulum, but caution must be exercised since the solutions are only stable
Figure 11.1: Bifurcation diagram for the parametrically excited pendulum. H is where the equilibrium loses stability at a pitchfork bifurcation which is sub-critical to the left of point c. S is the locus of a symmetry breaking bifurcation, and F represents the end of a period-doubling cascade although only the initial period-doubling is shown since the cascade is very rapid. The period-2 fold line A associated with the sub-critical pitchfork bifurcation is also shown, along with the secondary unstable zone H'. J represents the creation of a (1,1) rotating orbit, which period-doubles at line B, and again at line G, which is close to the final bifurcation to tumbling chaos. The period-1 solution restabilises at line I. Two period-3 subharmonic orbits are also shown in windows (3,1) and (3,3). The pendulum is stable in the inverted position in the 'inverted zone'.
over small parameter regimes. Braid and knot theory allowed pseudo-Anosov orbits to be located, which imply that the parametrically excited pendulum has chaotic dynamics, and by developing a crossings algorithm some bifurcational precedence relationships were obtained which were not immediately obvious from the numerical results.

For rotating orbits, there is still much work to be done, but the results in chapter 9 represent an important start in classifying rotating orbits without resorting to further braid and knot theory, and a wide range of subharmonic solutions have been located numerically.

There are many possible extensions to this work, which could include the effect of damping on the system, a more complete description of the rotating orbits based on braid and knot theory, or on the more practical side, the physical realisation of more of the numerically located orbits could be attempted together with some nonlinear control techniques to stabilise the subharmonic orbits. However, this would involve considerable modifications to be made to the experimental apparatus by way of added measuring sensors and finer control of the forcing frequency. There is also scope to apply the results obtained here to some physical systems which can be modelled by the parametrically excited pendulum, and to put to use the expertise gained in predicting the failure of physical systems subject to parametric excitation.
Bibliography and References


Mechanics 22 89-98.


Tritton, D.J. [1986]. "Ordered and chaotic motion of a forced spherical pendulum,"


