# A characterisation of ordered abstract probabilities 

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#### Abstract

In computer science, especially when dealing with quantum computing or other non-standard models of computation, basic notions in probability theory like "a predicate" vary wildly. There seems to be one constant: the only useful example of an algebra of probabilities is the real unit interval. In this paper we try to explain this phenomenon. We will show that the structure of the real unit interval naturally arises from a few reasonable assumptions. We do this by studying effect monoids, an abstraction of the algebraic structure of the real unit interval: it has an addition $x+y$ which is only defined when $x+y \leq 1$ and an involution $x \mapsto 1-x$ which make it an effect algebra, in combination with an associative (possibly non-commutative) multiplication. Examples include the unit intervals of ordered rings and Boolean algebras.

We present a structure theory for effect monoids that are $\omega$ complete, i.e. where every increasing sequence has a supremum.

We show that any $\omega$-complete effect monoid embeds into the direct sum of a Boolean algebra and the unit interval of a commutative unital $C^{*}$-algebra. This gives us from first principles a dichotomy between sharp logic, represented by the Boolean algebra part of the effect monoid, and probabilistic logic, represented by the commutative $\mathbf{C}^{*}$-algebra. Some consequences of this characterisation are that the multiplication must always be commutative, and that the unique $\omega$-complete effect monoid without zero divisors and more than 2 elements must be the real unit interval. Our results give an algebraic characterisation and motivation for why any physical or logical theory would represent probabilities by real numbers.


## I. Introduction

Probability theory in the quantum realm is different in important ways from that of the classical world. Nevertheless, they both crucially rely on real numbers to represent probabilities of events. This makes sense as observations of quantum systems must still be interpreted trough classical means. However, in principle one can imagine a world governed by different physical laws where even the standard notion of a probability is different, or wish to study probabilistic models where one does not care about the specifics of their probabilities; such an approach can for instance be found in categorical quantum mechanics [1], [2], [3]. In this paper we study a reasonable class of alternatives to the real unit interval as the set of allowed probabilities. We will establish that this quite general seeming class actually only contains (continuous products of) the real unit interval. This shows that any 'reasonable' enough physical theory must necessarily be based on probabilities represented by real numbers.

In order to determine the right set of alternatives to the real unit interval we must first find out what structure is crucial for abstract probabilities. There are a variety of operations
on the real unit interval that are used in their interpretation as probabilities. First of all, to be able to talk about coarsegraining the probabilities of mutually exclusive events, we must be able to take the sum $x+y$ of two probabilities $x, y \in[0,1]$ as long as $x+y \leq 1$. Second, in order to represent the complement of an event we require the involution given by $x^{\perp} \equiv 1-x$. The probability $x^{\perp}$ is the unique number such that $x+x^{\perp}=1$. Axiomatising this structure of a partially defined addition combined with an involution defines an effect algebra [4]. The unit interval of course also has a multiplication $x \cdot y$. This operation is needed in order to talk about, for instance, joint distributions. An effect monoid is an effect algebra with an associative distributive (possibly non-commutative) multiplication, and hence axiomatises these three interacting algebraic structures (addition, involution and multiplication) present in the unit interval.

In order to define an analogue to Bayes' theorem we would also need the division operation that is available in the unit interval: when $x \leq y$, then there is a probability $z$ such that $y \cdot z=x$ (namely $z=x / y$ ). We actually will not require the existence of such a division operation, as it turns out to follow (non-trivially) from our final requirement:

A property that sets the unit interval $[0,1]$ apart from, for instance, the rational numbers between 0 and 1 , is that $[0,1]$ is closed under taking limits. In particular, each ascending chain of probabilities $x_{1} \leq x_{2} \leq \ldots$ has a supremum. In other words: the unit interval is $\omega$-complete. We have then arrived at our candidate for an abstract notion of the set of probabilities: an $\omega$-complete effect monoid.

Further motivation for the use of this structure as a natural candidate for the set of probabilities is its prevalence in effectus theory. This is a recent approach to categorical logic [5] and a general framework to deal with notions such as states, predicates, measurement and probability in deterministic, (classical) probabilistic and quantum settings [6], [7]. The set of probabilities in an effectus have the structure of an effect monoid. Examples of effectuses include any generalised probabilistic theory [8], where the probabilities are the unit interval, but also any topos (and in fact any extensive category with final object), where the probabilities are the Boolean values $\{0,1\}$ [6].

An effectus defines the sum of some morphisms. In a $\omega$ effectus, this is strengthened to the existence of some countable sums (making it a partially additive category [9]). In such an effectus the probabilities form an $\omega$-complete effect monoid [6].

Effect monoids are of broader interest than only to study
effectuses: examples of effect monoids include all Boolean algebras and unit intervals of partially ordered rings. Furthermore any effect monoid can be used to define a generalised notion of convex set and convex effect algebra (by replacing the usual unit interval by elements of the effect monoid, see [7] $179 \& 192]$ or [10]).

This now raises the question of how close a probability theory based on an $\omega$-complete effect monoid is to regular probability theory.

Our main result is that $\omega$-complete effect monoids can always be embedded into a direct sum of an $\omega$-complete Boolean algebra and the unit interval of a bounded- $\omega$-complete commutative unital $\mathrm{C}^{*}$-algebra. The latter is isomorphic to $C(X)$ for some basically disconnected compact Hausdorff space $X$. If the effect monoid is directed complete, such thay any directed set has a supremum, then it is even isomorphic to the direct sum of a complete Boolean algebra and the unit interval of a monotone complete commutative $\mathrm{C}^{*}$-algebra.

This result basically states that any $\omega$-complete effect monoid can be split up into a sharp part (the Boolean algebra), and a convex probabilistic part (the commutative $\mathrm{C}^{*}$-algebra). This then gives us from basic algebraic and order-theoretic considerations a dichotomy between sharp and fuzzy logic.

As part of the proof of this embedding theorem we find an assortment of additional structure present in $\omega$-complete effect monoids: it has a partially defined division operation, it is a lattice, and multiplication must necessarily be normal (i.e. preserve suprema).

The classification also has some further non-trivial consequences. In particular, it shows that any $\omega$-complete effect monoid must necessarily be commutative.

Finally, we use the classification to show that an $\omega$-complete effect monoid without zero divisors must either be trivial, $\{0\}$, the two-element Boolean algebra, $\{0,1\}$, or the unit interval, $[0,1]$. This gives a new characterisation of the real unit interval as the unique $\omega$-complete effect monoid without zero divisors and more than two elements, and could be seen as a generalisation of the well-known result that the set of real numbers is the unique Dedekind-complete Archimedian ordered field.

In so far as the structure of an $\omega$-complete effect monoid is required for common actions involving probabilities, (coarsegraining, negations, joint distributions, limits) our results motivate the usage of real numbers in any hypothetical alternative physical theory.

## II. Preliminaries

Before we state the main results of this paper technically, we recall the definitions of the structures involved.
Definition 1. An effect algebra (EA) $\left(E, \otimes, 0,()^{\perp}\right)$ is a set $E$ with distinguished element $0 \in E$, partial binary operation $\varnothing$ (called sum) and (total) unary operation $x \mapsto x^{\perp}$ (called complement), satisfying the following axioms, writing $x \perp y$ whenever $x \oslash y$ is defined and $1 \equiv 0^{\perp}$.

- Commutativity: if $x \perp y$, then $y \perp x$ and $x \oslash y=y \boxtimes x$.
- Zero: $x \perp 0$ and $x \oslash 0=x$.
- Associativity: if $x \perp y$ and $(x \boxtimes y) \perp z$, then $y \perp z$, $x \perp(y \boxtimes z)$, and $(x \boxtimes y) \boxtimes z=x \boxtimes(y \boxtimes z)$.
- For any $x \in E$, the complement $x^{\perp}$ is the unique element with $x \boxtimes x^{\perp}=1$.
- If $x \perp 1$ for some $x \in E$, then $x=0$.

For $x, y \in E$ we write $x \leq y$ whenever there is a $z \in E$ with $x \oslash z=y$. This turns $E$ into a poset with minimum 0 and maximum 1. The map $x \mapsto x^{\perp}$ is an order anti-isomorphism. Furthermore $x \perp y$ if and only if $x \leq y^{\perp}$. If $x \leq y$, then the element $z$ with $x \boxtimes z=y$ is unique and is denoted by $y \ominus x$ [4].

A morphism $f: E \rightarrow F$ between effect algebras is a map such that $f(1)=1$ and $f(x) \perp f(y)$ whenever $x \perp y$, and then $f(x \otimes y)=f(x) \otimes f(y)$. A morphism necessarily preserves the complement, $f\left(x^{\perp}\right)=f(x)^{\perp}$, and the order: $x \leq y \Longrightarrow f(x) \leq f(y)$. A morphism is an embedding when it is also order reflecting: if $f(x) \leq f(y)$ then $x \leq y$. Observe that an embedding is automatically injective. We say $E$ and $F$ are isomorphic and write $E \cong F$ when there exists an isomorphism (i.e. a bijective morphism whose inverse is a morphism too) from $E$ to $F$. Note that an isomorphism is the same as a surjective embedding.
Example 2. Let $\left(B, 0,1, \wedge, \vee,()^{\perp}\right)$ be an orthomodular lattice. Then $B$ is an effect algebra with the partial addition defined by $x \perp y \Longleftrightarrow x \wedge y=0$ and in that case $x \boxtimes y=x \vee y$. The complement, ()$^{\perp}$, is given by the orthocomplement, ()$^{\perp}$. The lattice order coincides with the effect algebra order (defined above). See e.g. [11, Prop. 27].

Example 3. Let $G$ be an ordered abelian group (such as the self-adjoint part of a $\mathrm{C}^{*}$-algebra). Then any interval $[0, u]_{G} \equiv$ $\{a \in G ; 0 \leq a \leq u\}$ where $u$ is a positive element of $G$ forms an effect algebra, with addition given by $a \perp b \Longleftrightarrow a+b \leq u$ and in that case $a \otimes b=a+b$. The complement is defined by $a^{\perp}=u-a$. The effect algebra order on $[0, u]_{G}$ coincides with the regular order on $G$.

In particular, the set of effects $[0,1]_{C}$ of a unital $\mathrm{C}^{*}$ algebra $C$ forms an effect algebra with $a \perp b \Longleftrightarrow a+b \leq 1$, and $a^{\perp}=1-a$.

Effect algebras have been studied extensively (to name a few: [12], [13], [14], [15], [16], [17], [18]) and even found surprising applications in quantum contextuality [19], [20] and the study of Lebesque integration [21]. The following remark gives some categorical motivation to the definition of effect algebras.

Remark 4. An effect algebra is a bounded poset: a partially ordered set with a minimal and maximal element. In [22] it is shown that any bounded poset $P$ can be embedded into an orthomodular poset $K(P)$. This is known as the Kalmbach extension [23]. This extends to a functor from the category of bounded posets to the category of orthomodular posets, and this functor is in fact left adjoint to the forgetful functor going in the opposite direction [24]. This adjunction gives rise to the Kalmbach monad on the category of bounded posets. The Eilenberg-Moore category for the Kalmbach monad is
isomorphic to the category of effect algebras, and hence effect algebras are in fact algebras over bounded posets [25].

The category of effect algebras is both complete and cocomplete. There is also an algebraic tensor product of effect algebras that makes the category of effect algebras symmetric monoidal [26]. The monoids in the category of effect algebras resulting from this tensor product are called effect monoids, and they can be explicitly defined as follows:

Definition 5. An effect monoid (EM) is an effect algebra $\left(M, \otimes, 0,^{\perp}, \cdot\right)$ with an additional (total) binary operation $\cdot$, such that the following conditions hold for all $a, b, c \in M$.

- Unit: $a \cdot 1=a=1 \cdot a$.
- Distributivity: if $b \perp c$, then $a \cdot b \perp a \cdot c, \quad b \cdot a \perp c \cdot a$, $a \cdot(b \boxtimes c)=(a \cdot b) \boxtimes(a \cdot c)$, and $(b \boxtimes c) \cdot a=(b \cdot a) \boxtimes(c \cdot a)$.

Or, in other words, the operation • is bi-additive.

- Associativity: $a \cdot(b \cdot c)=(a \cdot b) \cdot c$.

We call an effect monoid $M$ commutative if $a \cdot b=b \cdot a$ for all $a, b \in M$; an element $p$ of $M$ idempotent whenever $p^{2} \equiv$ $p \cdot p=p$; elements $a, b$ of $M$ orthogonal when $a \cdot b=b \cdot a=0$; and we denote the set of idempotents of $M$ by $P(M)$.

Example 6. Any Boolean algebra $\left(B, 0,1, \wedge, \vee,()^{\perp}\right)$, being an orthomodular lattice, is an effect algebra by Example 2 and, moreover, a commutative effect monoid with multiplication defined by $x \cdot y=x \wedge y$. Conversely, any orthomodular lattice for which $\wedge$ distributes over $\varnothing$ (and thus $\vee$ ) is a Boolean algebra.

Example 7. The unit interval $[0,1]_{R}$ of any (partially) ordered unital ring $R$ (in which the sum $a+b$ and product $a \cdot b$ of positive elements $a$ and $b$ are again positive) is an effect monoid.

Let, for example, $X$ be a compact Hausdorff space. We denote its space of continuous functions into the complex numbers by $C(X) \equiv\{f: X \rightarrow \mathbb{C} ; f$ continuous $\}$. This is a commutative unital $\mathrm{C}^{*}$-algebra (and conversely by the Gel'fand theorem, any commutative $\mathrm{C}^{*}$-algebra with unit is of this form) and hence its unit interval $[0,1]_{C(X)}=\{f$ : $X \rightarrow[0,1]\}$ is a commutative effect monoid.

In [6, Ex. 4.3.9] and [11, Cor. 51] two different noncommutative effect monoids are constructed.

Definition 8. Let $M$ and $N$ be effect monoids. A morphism from $M$ to $N$ is a morphism of effect algebras with the added condition that $f(a \cdot b)=f(a) \cdot f(b)$ for all $a, b \in M$. Similar to the case of effect algebras, an embedding $M \rightarrow N$ is a morphism that is order reflecting. Also here an isomorphism of effect monoids is the same thing as a surjective embedding of effect monoids.

Example 9. It is well-known that any Boolean algebra $B$ is isomorphic to the set of clopens of its Stone space $X_{B}$. This yields an effect monoid embedding from $B$ into $[0,1]_{C\left(X_{B}\right)}$.
Remark 10. A physical or logical theory which has probabilities of the form $[0,1]_{C(X)}$ can be seen as a theory with
a natural notion of space, where probabilities are allowed to vary continuously over the space $X$. Such a spatial theory is considered in for instance Ref. [27].

Example 11. Given two effect algebras/monoids $E_{1}$ and $E_{2}$ we define their direct sum $E_{1} \oplus E_{2}$ as the Cartesian product with pointwise operations. This is again an effect algebra/monoid.

Example 12. Let $M$ be an effect monoid and let $p \in M$ be some idempotent. The subset $p M \equiv\{p \cdot a ; a \in M\}$ is called the left corner by $p$ and is an effect monoid with $(p \cdot a)^{\perp} \equiv$ $p \cdot a^{\perp}$ and all other operations inherited from $M$. Later we will see that $a \mapsto\left(p \cdot a, p^{\perp} \cdot a\right)$ is an isomorphism $M \cong p M \oplus p^{\perp} M$. Analogous facts hold for the right corner $M p \equiv\{a \cdot p ; a \in$ $M\}$.

Definition 13. Let $E$ be an effect algebra. A directed set $S \subseteq E$ is a non-empty set such that for all $a, b \in S$ there exists a $c \in S$ such that $a, b \leq c$. $E$ is directed complete when for any directed set $S$ there is a supremum $\bigvee S$. It is $\omega$-complete if directed suprema of countable sets exist, or equivalently if any increasing sequence $a_{1} \leq a_{2} \leq \ldots$ in $E$ has a supremum.

Remark 14. A directed complete partially ordered set is often referred to by the shorthand dcpo. These structures lie at the basis of domain theory and are often encountered when studying denotational semantics of programming languages as they allow for a natural way to talk about fix points of recursion. Note that being $\omega$-complete is strictly weaker. For effect algebras we could have equivalently defined directed completeness with respect to downwards directed sets, as the complement is an order anti-isomorphism.

Example 15. Let $B$ be a $\omega$-complete Boolean algebra. Then $B$ is a $\omega$-complete effect monoid. If $B$ is complete as a Boolean algebra, then $B$ is directed-complete as effect monoid.

Example 16. Let $X$ be an extremally disconnected compact Hausdorff space, i.e. where the closure of every open set is open. Then $[0,1]_{C(X)}$ is a directed-complete effect monoid. If $X$ is a basically disconnected [28, 1H] compact Hausdorff space, i.e. where every cozero set has open closure, then $[0,1]_{C(X)}$ is an $\omega$-complete effect monoid [28, 3N.5].

## III. OvERVIEW

The main results of the paper are the following theorems:
Theorem. Let $M$ be an $\omega$-complete effect monoid. Then $M$ embeds into $M_{1} \oplus M_{2}$, where $M_{1}$ is an $\omega$-complete Boolean algebra, and $M_{2}=[0,1]_{C(X)}$, where $X$ is a basically disconnected compact Hausdorff space (see Theorem 68).

Theorem. Let $M$ be a directed-complete effect monoid. Then $M \cong M_{1} \oplus M_{2}$ where $M_{1}$ is a complete Boolean algebra and $M_{2}=[0,1]_{C(X)}$ for some extremally-disconnected compact Hausdorff space $X$ (see Theorem 69).

By Example 9, the Boolean algebra $M_{1}$ also embeds into a $[0,1]_{C\left(X_{M_{1}}\right)}$, and hence we could 'coarse-grain' the direct
sums above and say that any $\omega$-complete effect monoid embeds into the unit interval of $C\left(X_{M_{1}}+X\right)$, where + is the disjoint union of the topological spaces. This observation suggests a Stone-type duality that we discuss in more detail in the conclusion.

Other results for an $\omega$-complete effect monoid $M$ that either follow directly from the above theorems, or are proven along the way are the following:

- $M$ is a lattice.
- $M$ is an effect divisoid [7].
- The multiplication in $M$ is normal: $a \cdot \bigvee S=\bigvee a \cdot S$.
- If $M$ is convex (as an effect algebra), then scalar multiplication is homogeneous: $\lambda(a \cdot b)=(\lambda a) \cdot b=a \cdot(\lambda b)$ for any $\lambda \in[0,1]$ and $a, b \in M$.
- $M$ is commutative.
- If $M$ has no non-trivial zero-divisors (i.e. $a \cdot b=0$, implies $a=0$ or $b=0$ ), then $M$ is isomorphic to $[0,1],\{0,1\}$ or $\{0\}$.
It should be noted that the scalars in a $\omega$-effectus satisfying normalisation have no non-trivial zero-divisors [6] and hence using the last point above, we have completely characterised the scalars in such $\omega$-effectuses, splitting them up into trivial, Boolean and convex effectuses.

The paper is structured as follows. In Section IV we recover and prove some basic results regarding effect algebras/monoids. Then in Section $\bar{\square}$ we will show that in any $\omega$-complete effect monoid $M$, we can define a kind of partial division operation which turns it into a effect divisoid. Using this division we show that the multiplication must be normal. Then in Section VI we study idempotents that are either Boolean, meaning that all elements below $p$ must also be idempotents, or halvable, meaning that there is an $a \in M$ such that $p=a \oslash a$. We establish that an $\omega$-complete effect monoid where 1 is Boolean must be a Boolean algebra, while if 1 is halvable then it must be convex. In Section VIIwe show that a maximal collection of orthogonal idempotents of $M$ can be found that consists of a mix of halvable and Boolean idempotents. The corner $p M$ associated to such an idempotent will either be convex (if $p$ is halvable) or Boolean (if $p$ is Boolean). Using normality of multiplication we show that $M$ embeds into the direct sum of the corners associated to these idempotents. Letting $M_{1}$ be the direct sum of the Boolean corners, and $M_{2}$ be the direct sum of the convex corners, we see that $M$ embeds into $M_{1} \oplus M_{2}$, where $M_{1}$ is Boolean and $M_{2}$ is convex. In Section VIII we recall some results regarding order unit spaces and use Yosida's representation theorem to show that a convex $\omega$-complete effect monoid must be isomorphic to the unit interval of a $C(X)$. Then in Section IX we collect all the results and prove our main theorems. Finally in Section $X$ we conclude and discuss some future work and open questions.

## IV. BASIC RESULTS

We do not assume any commutativity of the product in an effect monoid. Nevertheless, some commutativity comes for free.

Lemma 17. For any $a \in M$ in an effect monoid $M$, we have $a \cdot a^{\perp}=a^{\perp} \cdot a$.
Proof. $a^{2} \otimes\left(a^{\perp} \cdot a\right)=\left(a \oslash a^{\perp}\right) \cdot a=1 \cdot a=a=a \cdot 1=$ $a \cdot\left(a \oslash a^{\perp}\right)=a^{2} \otimes\left(a \cdot a^{\perp}\right)$. Cancelling $a^{2}$ on both sides gives the desired equality.

Lemma 18. An element $p \in M$ is an idempotent if and only if $p \cdot p^{\perp}=0$.
Proof. $p=p \cdot 1=p \cdot\left(p \otimes p^{\perp}\right)=p^{2} \otimes p \cdot p^{\perp}$. Hence $p=p^{2}$ if and only if $p \cdot p^{\perp}=0$.
Lemma 19. For $a, p \in M$ with $p^{2}=p$, we have

$$
p \cdot a=a \quad \Longleftrightarrow \quad a \cdot p=a \quad \Longleftrightarrow \quad a \leq p
$$

Proof. Assume $a \leq p$. Then $a \cdot p^{\perp} \leq p \cdot p^{\perp}=0$, so that $a \cdot p^{\perp}=$ 0 . Similarly $p^{\perp} \cdot a=0$. Hence $a=a \cdot 1=a \cdot\left(p \otimes p^{\perp}\right)=$ $a \cdot p \boxtimes a \cdot p^{\perp}=a \cdot p$. Similarly $p \cdot a=a$.

Now assume $p \cdot a=a$. Then immediately $a=p \cdot a \leq p \cdot 1=$ $p$. The final implication (that $a \cdot p=a \Longrightarrow a \leq p$ ) is proven similarly.

Lemma 20. Let $M$ be an effect monoid with idempotent $p \in$ $M$. Then $p \cdot a=a \cdot p$ for any $a \in M$.

Proof. Clearly $p \cdot a \leq p \cdot 1=p$ and so by Lemma $19 p$. $a \cdot p=a \cdot p$. Similarly $a \cdot p \leq p$ and so $p \cdot a \cdot p=p \cdot a$. Thus $p \cdot a=p \cdot a \cdot p=a \cdot p$, as desired.
Corollary 21. Let $M$ be an effect monoid with idempotent $p \in M$. The map $e \mapsto\left(p \cdot e, p^{\perp} \cdot e\right)$ is an isomorphism $M \cong$ $p M \oplus p^{\perp} M$.

The following two lemmas are simple observations that will be used several times.

Lemma 22. Let $a \leq b$ be elements of an effect algebra $E$. If $b \otimes b^{\prime} \leq a \otimes a^{\prime}$ for some $a^{\prime} \leq b^{\prime}$ from $E$, then $a=b$ (and $a^{\prime}=b^{\prime}$ ).

Proof. Since $a \leq a^{\prime}$ and $b \leq b^{\prime}$, we have $a \oslash a^{\prime} \leq b \oslash b^{\prime}$, and so $a \otimes a^{\prime}=b \otimes b^{\prime}$. Then $0=\left(b \otimes b^{\prime}\right) \ominus\left(a \otimes a^{\prime}\right)=$ $(b \ominus a) \otimes\left(b^{\prime} \ominus a^{\prime}\right)$, yielding $b \ominus a=0$ and $b^{\prime} \ominus a^{\prime}=0$, so $b=a$ and $b^{\prime}=a^{\prime}$.

Lemma 23. Let $p$ be an idempotent from an effect monoid $M$, and let $a, b \leq p$ be elements below $p$. If $a \otimes b$ exists, then $a \boxtimes$ $b \leq p$.
Proof. Since $a \leq p$, we have $a \cdot p^{\perp}=0$, and similarly, $b \cdot p^{\perp}=$ 0 . But then $(a \otimes b) \cdot p^{\perp}=0$, and hence $(a \otimes b) \cdot p=a \otimes b$. By Lemma 19 we then have $a \otimes b \leq p$.

We defined directed set to mean upwards directed. Using the fact that $a \mapsto a^{\perp}$ is an order anti-isomorphism, a directedcomplete effect algebra also has all infima of downwards directed (or 'filtered') sets (and similarly for countable infima in a $\omega$-complete effect algebra).

Recall that given an element $a$ of an ordered group $G$ a subset $S$ of $G$ has a supremum $\bigvee S$ in $G$ if and only if $\bigvee_{s \in S} a+s$ exists, which follows immediately from the
observation that $a+(): G \rightarrow G$ is an order isomorphism. For effect algebras the situation is a bit more complicated, and we only have the implications mentioned in the lemma below. We will see in Corollary 39 that the situation improves somewhat for $\omega$-complete effect monoids.
Lemma 24. Let $x$ be an element and $S$ a non-empty subset of an effect algebra $E$. If $S \subseteq\left[0, x^{\perp}\right]_{E}$, then

$$
\begin{aligned}
\bigvee S \text { exists } & \Longrightarrow \quad x \otimes \bigvee S=\bigvee x \otimes S, \text { and } \\
\bigwedge x \otimes S \text { exists } & \Longrightarrow \quad x \otimes \bigwedge S=\bigwedge x \otimes S
\end{aligned}
$$

Here "=" means also that the sums, suprema and infima on either side exist. Similarly, if $S \subseteq[x, 1]_{E}$, then

$$
\begin{aligned}
\bigwedge_{s \in S} s \ominus x \text { exists } & \Longrightarrow(\bigvee S) \ominus x=\bigvee_{s \in S} s \ominus x, \text { and } \\
\bigwedge S \text { exists } & \Longrightarrow \quad(\bigwedge S) \ominus x=\bigwedge_{s \in S} s \ominus x
\end{aligned}
$$

Moreover, if $S \subseteq[0, x]_{E}$, then

$$
\begin{array}{rll}
\bigvee S \text { exists } & \Longrightarrow & x \ominus \bigvee S \\
\bigvee x \ominus S \text { exists } & \Longrightarrow & x \ominus \bigwedge S \\
\bigvee \ominus \bigvee, \text { and } \\
\bigvee x \ominus S
\end{array}
$$

Proof. Note that $a \mapsto x \bowtie a$ gives an order isomorphism $\left[0, x^{\perp}\right]_{E} \rightarrow[x, 1]_{E}$ with inverse $a \mapsto a \ominus x$. Whence $x \otimes()$ preserves and reflects all infima and suprema restricted to $\left[0, x^{\perp}\right]_{E}$ and $[x, 1]_{E}$. Surely, given elements $a \leq b$ from $E$, and a subset $S$ of the interval $[a, b]_{E}$, it is clear that any supremum (infimum) of $S$ in $E$ will be the supremum (infimum) of $S$ in $[a, b]_{E}$ too (using here that $S$ is nonempty). The converse does not always hold, but when $S$ has a supremum in $[a, 1]_{E}$, then this is the supremum in $E$ too (and when $S$ has an infimum in $[0, b]_{E}$, then this is the infimum in $E$ too). These considerations yield the first four equations. For the latter two we just add the observation that $x \ominus()$ gives an order reversing isomorphism $[0, x]_{E} \rightarrow[0, x]_{E}$.

We can now prove a few basic yet useful facts of $\omega$-complete effect monoids. These lemmas deal with elements that are summable with themselves: elements $a$ such that $a \perp a$ which means that $a \boxtimes a$ is defined. For $n \in \mathbb{N}$ we will use the notation $n a=a \otimes \ldots \boxtimes a$ for the $n$-fold sum of $a$ with itself (when it is defined). We study these self-summable elements to be able to define a " $\frac{1}{2}$ " in some effect monoids later on.

Lemma 25. For any $a \in M$ in some effect monoid $M$, the element $a \cdot a^{\perp}$ is summable with itself.

Proof. Since $1=1 \cdot 1=\left(a \oslash a^{\perp}\right) \cdot\left(a \oslash a^{\perp}\right)=a \cdot a \oslash a \cdot a^{\perp} \oslash$ $a^{\perp} \cdot a \oslash a^{\perp} \cdot a^{\perp}$, and $a \cdot a^{\perp}=a^{\perp} \cdot a$ by Lemma 17) we see that $a \cdot a^{\perp} \otimes a \cdot a^{\perp}$ indeed exists.

Lemma 26. Let $a$ be an element of an $\omega$-complete effect monoid $M$.

1) If $n a$ exists for all $n$ then $a=0$.
2) If $a^{2}=0$ then $a=0$.
3) If $a \perp a$ then $\bigwedge_{n} a^{n}=0$.

Proof. For point 1, we have $a \oslash \bigvee_{n} n a=\bigvee_{n} a \oslash n a=$ $\bigvee_{n}(n+1) a=\bigvee_{n} n a$, and so $a=0$.

For point 2, since $a^{2}=0$ we have $a=a \cdot 1=a \cdot\left(a \oslash a^{\perp}\right)=$ $a \cdot a^{\perp}$, and hence (because of Lemma 25) $a$ is summable with itself. But furthermore $(a \otimes a)^{2}=4 a^{2}=0$, and so $(a \otimes a)^{2}=0$. Continuing in this fashion, we see that $2^{n} a$ exists for every $n \in \mathbb{N}$ and $\left(2^{n} a\right)^{2}=0$. Hence, for any $m \in \mathbb{N}$ the sum ma exists so that by the previous point $a=0$.

For point 3 , write $b \equiv \bigwedge_{n} a^{n}$. As $(2 a)^{n}=2^{n} a^{n}$ and $b \leq a^{n}$ we see that $2^{n} b$ is defined. But this is true for all $n$, and so again by the point $1, b=0$.

## V. Floors, CEILINGS AND DIVISION

In this section we will see that any $\omega$-complete effect monoid has floors and ceilings. These are respectively the largest idempotent below an element and the smallest idempotent above an element. We will also construct a "division": for $a \leq b$ we will find an element $a / b$ such that $(a / b) \cdot b=a$.

Then using ceilings and this division we will show that multiplication in a $\omega$-complete effect monoid is always normal, i.e. that $b \cdot \bigvee S=\bigvee b \cdot S$ for non-empty $S$ for which $\bigvee S$ exists. This technical result will be frequently used in the remaining sections.

Definition 27. Let $\left(x_{i}\right)_{i \in I}$ be a (potentially infinite) family of elements from an effect algebra $E$. We say that the sum $\bigotimes_{i \in I} x_{i}$ exists if for every finite subset $S \subseteq I$ the $\operatorname{sum} \bigotimes_{i \in S} x_{i}$ exists and the supremum $\bigvee_{\text {finite } S \subseteq I} \boxtimes_{i \in S} x_{i}$ exists as well. In that case we write $\bar{\emptyset}_{i \in I} x_{i} \equiv$ $\bigvee_{\text {finite } S \subseteq I} \boxtimes_{i \in S} x_{i}$.
Lemma 28. Given $a \in M$ for an effect monoid $M$, we have

$$
\left(a^{N}\right)^{\perp}=a^{\perp} \otimes a^{\perp} \cdot a \otimes a^{\perp} \cdot a^{2} \otimes \cdots \otimes a^{\perp} \cdot a^{N-1}
$$

for every natural number $N$.
Proof. From the computation

$$
\begin{aligned}
& 1=a^{\perp} \oslash a \\
& =a^{\perp} \otimes\left(a^{\perp} \otimes a\right) \cdot a \\
& \equiv a^{\perp} \otimes a^{\perp} \cdot a \otimes a^{2} \\
& =a^{\perp} \otimes a^{\perp} \cdot a \otimes\left(a^{\perp} \otimes a\right) \cdot a^{2} \\
& \equiv a^{\perp} \otimes a^{\perp} \cdot a \oslash a^{\perp} \cdot a^{2} \otimes a^{3} \\
& \vdots \\
& =\left(\bigotimes_{n=0}^{N-1} a^{\perp} \cdot a^{n}\right) \otimes a^{N}
\end{aligned}
$$

the result follows immediately.
Corollary 29. The sum $\bigotimes_{n=0}^{\infty} a^{\perp} \cdot a^{n}$ exists for any element $a$ from an $\omega$-complete effect monoid $M$.
Definition 30. Given an element $a$ of an $\omega$-complete effect monoid $M$

$$
\lceil a\rceil \equiv \bigotimes_{n=0}^{\infty} a \cdot\left(a^{\perp}\right)^{n} \quad \text { and } \quad\lfloor a\rfloor \equiv \bigwedge_{n=0}^{\infty} a^{n}
$$

are called the ceiling of a and the floor of $a$, respectively.
We list some basic properties of $\lceil a\rceil$ and $\lfloor a\rfloor$ in Proposition 35, after we have made the observations necessary to establish them.

Lemma 31. Given an element $a$ of an $\omega$-complete effect monoid $M$, we have $\bigwedge_{n} a^{\perp} \cdot a^{n}=0$.
Proof. Write $b \equiv \bigwedge_{n} a^{\perp} \cdot a^{n}$. Since $a$ and $a^{\perp}$ commute by Lemma 17 we compute

$$
1=1^{n}=\left(a^{\perp} \oslash a\right)^{n}=\bigotimes_{k=0}^{n}\binom{n}{k}\left(\left(a^{\perp}\right)^{k} \cdot a^{n-k}\right)
$$

and in particular see that the sum $\binom{n}{1}\left(a^{\perp} \cdot a^{n-1}\right) \equiv n\left(a^{\perp}\right.$. $a^{n-1}$ ) exists. Because $b \leq a^{\perp} \cdot a^{n-1}$, the $n$-fold sum $n b$ exists too and hence $b=0$ by Lemma 26

Lemma 32. We have $\lfloor a\rfloor=\lfloor a\rfloor \cdot a=a \cdot\lfloor a\rfloor$ for any element $a$ of an $\omega$-complete effect monoid $M$.

Proof. Using Lemmas 17 and 31 we compute $\lfloor a\rfloor \cdot a^{\perp}=$ $\left(\bigwedge_{n} a^{n}\right) \cdot a^{\perp} \leq \bigwedge_{n} a^{n} \cdot a^{\perp}=\bigwedge_{n} a^{\perp} \cdot a^{n}=0$, and so $\lfloor a\rfloor \cdot a=$ $\lfloor a\rfloor$. The other identity has a similar proof.

Lemma 33. Given elements $a, b_{1}, b_{2}, \ldots$ of a $\omega$-complete effect monoid $M$ such that $\bigotimes_{n} b_{n}$ exists, and $a \cdot b_{n}=0$ for all $n \in \mathbb{N}$, we have $a \cdot \emptyset_{n} b_{n}=0$.
Proof. Writing $s_{N} \equiv \bigotimes_{n=1}^{N} b_{n}$, we have $s_{1} \leq s_{2} \leq \cdots$ and $a \cdot s_{n}=0$ for all $n$. Since $s_{n}=\left(a \otimes a^{\perp}\right) \cdot s_{n}=a \cdot s_{n} \oslash$ $a^{\perp} \cdot s_{n}=a^{\perp} \cdot s_{n}$ for all $n \in \mathbb{N}$, we have

$$
\bigvee_{n} s_{n}=\bigvee_{n} a^{\perp} \cdot s_{n} \leq a^{\perp} \cdot \bigvee_{n} s_{n} \leq \bigvee_{n} s_{n}
$$

which implies that $a^{\perp} \cdot \bigvee_{n} s_{n}=\bigvee_{n} s_{n}$, and thus $a \cdot \bigvee_{n} b_{n} \equiv$ $a \cdot \bigvee_{n} s_{n}=0$.
Proposition 34. Given elements $a$ and $b$ of an $\omega$-complete effect monoid $M$,

$$
a \cdot b=0 \quad \Longrightarrow \quad a \cdot\lceil b\rceil=0 .
$$

Proof. If $a \cdot b=0$, then also $a \cdot b \cdot\left(b^{\perp}\right)^{n}=0$ for all $n$. Hence by Lemma $33 a \cdot\lceil b\rceil \equiv a \cdot \boxtimes_{n=1}^{\infty} b \cdot\left(b^{\perp}\right)^{n}=0$.
Proposition 35. Let $a$ be an element of an $\omega$-complete effect monoid $M$.

1) The floor $\lfloor a\rfloor$ of $a$ is an idempotent with $\lfloor a\rfloor \leq a$. In fact, $\lfloor a\rfloor$ is the greatest idempotent below $a$.
2) The ceiling $\lceil a\rceil$ of $a$ is the least idempotent above $a$.
3) We have $\lceil a\rceil^{\perp}=\left\lfloor a^{\perp}\right\rfloor$ and $\lfloor a\rfloor^{\perp}=\left\lceil a^{\perp}\right\rceil$.

Proof. Point 3 follows from Lemma 28 . Concerning point 1 . Since $\lfloor a\rfloor \cdot a^{\perp}=0$ (by Lemma 32) we have $\lfloor a\rfloor \cdot\left\lceil a^{\perp}\right\rceil=0$ by Proposition 34, and so $\lfloor a\rfloor \cdot\lfloor a\rfloor^{\perp}=0$ because $\lfloor a\rfloor^{\perp}=\left\lceil a^{\perp}\right\rceil$ by point 3 . Hence $\lfloor a\rfloor$ is an idempotent. Also, since $\lfloor a\rfloor=\bigwedge_{n} a^{n}$, we clearly have $\lfloor a\rfloor \leq a$. Now, if $s$ is an idempotent in $M$ with $s \leq a$, then $s=s^{n} \leq a^{n}$, and so $s \leq \bigwedge_{n} a^{n} \equiv\lfloor a\rfloor$. Whence $\lfloor a\rfloor$ is the greatest idempotent below $a$. Point 2 now follows easily from 1 since $\lceil\cdot\rceil$ is the dual of $\lfloor\cdot\rfloor$ under the order anti-isomorphism $(\cdot)^{\perp}$.

Lemma 36. $\lceil a \otimes b\rceil=\lceil a\rceil \vee\lceil b\rceil$ for all summable elements $a$ and $b$ of an $\omega$-complete effect monoid $M$ (that is, $\lceil a \oslash b\rceil$ is the supremum of $\lceil a\rceil$ and $\lceil b\rceil$ ).

Proof. Since $\lceil a \otimes b\rceil \geq a \otimes b \geq a$, we have $\lceil a \otimes b\rceil \geq\lceil a\rceil$, and similarly, $\lceil a \otimes b\rceil \geq\lceil b\rceil$. Let $u$ be an upper bound of $\lceil a\rceil$ and $\lceil b\rceil$; we claim that $\lceil a \otimes b\rceil \leq u$. Since $\lceil a\rceil \leq u$ and $\lceil b\rceil \leq$ $u$, we have $a \leq\lceil a\rceil \leq\lfloor u\rfloor$ and $b \leq\lceil b\rceil \leq\lfloor u\rfloor$, and so $a \otimes b \leq$ $\lfloor u\rfloor$ by Lemma [23. Whence $\lceil a \otimes b\rceil \leq\lfloor u\rfloor \leq u$.

Any $\omega$-complete effect monoid is a lattice effect algebra [29]:

Theorem 37. Any pair of elements $a$ and $b$ from an $\omega$ complete effect monoid $M$ has an infimum, $a \wedge b$, given by

$$
a \wedge b=\bigotimes_{n=1}^{\infty} a_{n} \cdot b_{n} \text { where }\left[\begin{array}{ll}
a_{1}=a & a_{n+1}=a_{n} \cdot b_{n}^{\perp} \\
b_{1}=b & b_{n+1}=a_{n}^{\perp} \cdot b_{n}
\end{array} .\right.
$$

Consequently, any pair also has a supremum given by $a \vee b=$ $\left(a^{\perp} \wedge b^{\perp}\right)^{\perp}$.

Proof. First order of business is showing that the sum $\emptyset_{n=1}^{N} a_{n} \cdot b_{n}$ exists for every $N$. In fact, we'll show that $a \ominus \bigotimes_{n=1}^{N} a_{n} \cdot b_{n}=a_{N+1}$ for all $N$, by induction. Indeed, for $N=1$, we have $a \ominus a \cdot b=a \cdot b^{\perp}=a_{2}$, and if $a \ominus \emptyset_{n=1}^{N} a_{n} \cdot b_{n}=a_{N+1}$ for some $N$, then $a_{N+2}=a_{N+1} \cdot b_{N+1}^{\perp}=a_{N+1} \ominus a_{N+1} \cdot b_{N+1}=$ $\left(a \ominus \bigotimes_{n=1}^{N} a_{n} \cdot b_{n}\right) \ominus a_{N+1} \cdot b_{N+1}=a \ominus \bigotimes_{n=1}^{N+1} a_{n} \cdot b_{n}$. In particular, $\boxtimes_{n=1}^{\infty} a_{n} \cdot b_{n}$ exists, and, moreover,

$$
a=\bigwedge_{m=1}^{\infty} a_{m} \otimes \bigotimes_{n=1}^{\infty} a_{n} \cdot b_{n} .
$$

By a similar reasoning, we get

$$
b=\bigwedge_{m=1}^{\infty} b_{m} \otimes \bigotimes_{n=1}^{\infty} a_{n} \cdot b_{n}
$$

Already writing $a \wedge b \equiv \emptyset_{n=1}^{\infty} a_{n} \cdot b_{n}$, we know at this point that $a \wedge b \leq a$ and $a \wedge b \leq b$. It remains to be shown that $a \wedge b$ defined above is the greatest lower bound of $a$ and $b$. So let $\ell \in$ $M$ with $\ell \leq a$ and $\ell \leq b$ be given; we must show that $\ell \leq a \wedge b$.

As an intermezzo, we observe that $\left(\bigwedge_{n} a_{n}\right) \cdot\left(\bigwedge_{m} b_{m}\right)=$ 0 . Indeed, we have $\left(\bigwedge_{n} a_{n}\right) \cdot\left(\bigwedge_{m} b_{m}\right) \leq \bigwedge_{n} a_{n} \cdot b_{n}$, and $\bigwedge_{n} a_{n} \cdot b_{n}=0$, because $\mathbb{Q}_{n=1}^{\infty} a_{n} \cdot b_{n}$ exists (see Lemma 26). By Proposition 34 it follows that $\left(\bigwedge_{n} a_{n}\right) \cdot\left\lceil\bigwedge_{m} b_{m}\right\rceil=0$. Whence writing $p \equiv\left\lceil\bigwedge_{m} b_{m}\right\rceil$, we have $p \cdot \bigwedge_{n} a_{n} \equiv\left(\bigwedge_{n} a_{n}\right)$. $p=0$ using Lemma 20. Observing that $\bigwedge_{n} b_{n} \leq p$ and using Lemma 19 we also have $p^{\perp} \cdot \bigwedge_{n} b_{n}=0$. We then calculate $p \cdot a=p \cdot\left(\bigwedge_{n} a_{n} \otimes a \wedge b\right)=p \cdot(a \wedge b)$ and similarly $p^{\perp} \cdot b=p^{\perp} \cdot(a \wedge b)$.

Returning to the problem of whether $\ell \leq a \wedge b$, we have

$$
\begin{aligned}
& \ell=p \cdot \ell \oslash p^{\perp} \cdot \ell \leq p \cdot a \oslash p^{\perp} \cdot b \\
& \quad=p \cdot(a \wedge b) \oslash p^{\perp} \cdot(a \wedge b)=a \wedge b
\end{aligned}
$$

Whence $a \wedge b$ is the infimum of $a$ and $b$.

The presence of finite infima and suprema in $\omega$-complete effect monoids prevents certain subtleties around the existence of arbitrary suprema and infima.

Corollary 38. Let $a \leq b$ be elements of an $\omega$-complete effect monoid $M$, and let $S$ be a non-empty subset of $[a, b]_{M}$.

Then $S$ has a supremum (infimum) in $M$ if and only if $S$ has a supremum (infimum) in $[a, b]_{M}$, and these suprema (infima) coincide.

Proof. It is clear that if $S$ has a supremum in $M$, then this is also the supremum in $[a, b]_{M}$. For the converse, suppose that $S$ has a supremum $\bigvee S$ in $[a, b]_{M}$, and let $u$ be an upper bound for $S$ in $M$; in order to show that $\bigvee S$ is the supremum of $S$ in $M$ too, we must prove that $\bigvee S \leq u$. Note that $b \wedge u$ is an upper bound for $S$. Indeed, given $s \in S \subseteq[a, b]_{M}$ we have $s \leq b$, and $s \leq u$, so $s \leq b \wedge u$. Moreover, one easily sees that $b \wedge u \in[a, b]_{M}$ using the fact that $S$ is non-empty. Whence $b \wedge u$ is an upper bound for $S$ in $[a, b]_{M}$, and so $\bigvee S \leq b \wedge u \leq u$, making $\bigvee S$ the supremum of $S$ in $M$. Similar reasoning applies to infima of $S$.

Corollary 39. Given an element $a$ and a non-empty subset $S$ of an $\omega$-complete effect monoid $M$ such that $a \boxtimes s$ exists for all $s \in S$,

1) the supremum $\bigvee S$ exists iff $\bigvee a \oslash S$ exists, and in that case $a \oslash \bigvee S=\bigvee a \oslash S$;
2) the infimum $\bigwedge S$ exists iff $\bigwedge a \otimes S$ exists, and in that case $a \otimes \bigwedge S=\bigwedge a \boxtimes S$.

Proof. The map $a \otimes():\left[0, a^{\perp}\right]_{M} \rightarrow[a, 1]_{M}$, being an order isomorphism, preserves and reflects suprema and infima. Now apply Corollary 38

Now that we know more about the existence of suprema and infima, we set our sights on proving that multiplication interacts with suprema and infima as desired, namely that it preserves them. To do this we introduce a partial division operation.

Definition 40. Given elements $a \leq b$ of an $\omega$-complete effect monoid, set

$$
a / b \equiv \bigotimes_{n=0}^{\infty} a \cdot\left(b^{\perp}\right)^{n}
$$

Note that the sum exists, because $\bigotimes_{n=0}^{N} a \cdot\left(b^{\perp}\right)^{n} \leq$ $\emptyset_{n=0}^{\infty} b \cdot\left(b^{\perp}\right)^{n} \equiv\lceil b\rceil$ for all $N$.

Lemma 41. Let $b$ be an element of an $\omega$-complete effect monoid $M$.

1) $b / b=\lceil b\rceil$.
2) $\left(a_{1} \otimes a_{2}\right) / b=a_{1} / b \boxtimes a_{2} / b$ for all summable $a_{1}, a_{2} \in M$ with $a_{1} \boxtimes a_{2} \leq b$.
3) $(a \cdot b) / b=a \cdot\lceil b\rceil$ for all $a \in M$.
4) $(a / b) \cdot b=a$ for all $a \in M$ with $a \leq b$.
5) $\{a \cdot b ; a \in M\} \equiv M b=[0, b]_{M} \equiv\{a ; a \in M ; a \leq b\}$.
6) The maps $a \mapsto a \cdot b, b \cdot a: M\lceil b\rceil \rightarrow M b$ are order isomorphisms.

Proof. Points 1 and 2 are easy, and left to the reader. Concerning 3, first note that
$(a \cdot b) / b \equiv \emptyset_{n=0}^{\infty} a \cdot b \cdot\left(b^{\perp}\right)^{n} \leq a \cdot \bigotimes_{n=0}^{\infty} b \cdot\left(b^{\perp}\right)^{n}=a \cdot\lceil b\rceil$.
Thus $(a \cdot b) / b \leq a \cdot\lceil b\rceil$. Since similarly $\left(a^{\perp} \cdot b\right) / b \leq a^{\perp} \cdot\lceil b\rceil$, we get, using 2

$$
\lceil b\rceil=b / b=(a \cdot b) / b \otimes\left(a^{\perp} \cdot b\right) / b \leq a \cdot\lceil b\rceil \otimes a^{\perp} \cdot\lceil b\rceil=\lceil b\rceil,
$$

forcing $(a \cdot b) / b=a \cdot\lceil b\rceil$ (see Lemma 22). For point 4 note that given $a, b \in M$ with $a \leq b$ we have $a=a \cdot\lceil b\rceil$ (by Lemma 19, since $a \leq b \leq\lceil b\rceil$,) and so

$$
\begin{array}{rlr}
a=a \cdot\lceil b\rceil & =(a \cdot b) / b & \text { by point } 3 \\
& \equiv \bigotimes_{n=0}^{\infty} a \cdot b \cdot\left(b^{\perp}\right)^{n} & \\
& =\bigotimes_{n=0}^{\infty} a \cdot\left(b^{\perp}\right)^{n} \cdot b & \text { by Lemma } 17 \\
& \leq\left(\bigvee_{n=0}^{\infty} a \cdot\left(b^{\perp}\right)^{n}\right) \cdot b & \\
& =(a / b) \cdot b .
\end{array}
$$

Since similarly $b \ominus a \leq((b \ominus a) / b) \cdot b$, we get

$$
\begin{aligned}
b=a \oslash(b \ominus a) \leq(a / b) \cdot b & ((b \ominus a) / b) \cdot b \\
& =(b / b) \cdot b=\lceil b\rceil \cdot b=b,
\end{aligned}
$$

which forces $a=(a / b) \cdot b$. For point 5 note that $M b, b M \subseteq$ $[0, b]_{M}$ since $b \cdot a, a \cdot b \leq b$ for all $a \in M$, and $[0, b]_{M} \subseteq M b$, because $a=(a / b) \cdot b$ for any $a \in[0, b]_{M}$ by point 4 .

Finally, concerning point 6, the maps $a \mapsto a \cdot b: M\lceil b\rceil \rightarrow$ $M b$ and $a \mapsto a / b: M b \rightarrow M\lceil b\rceil$ are clearly order preserving, and each other's inverse by points 3 and 4 and thus order isomorphisms. The proof that $a \mapsto b \cdot a: M\lceil b\rceil \rightarrow M b$ is an order isomorphism follows along entirely similar lines, but involves $b \backslash a$ defined by $b \backslash a \equiv \bigotimes_{n}\left(b^{\perp}\right)^{n} \cdot a$ and uses the fact that $b M=[0, b]_{M}=M b$.

Remark 42. From the previous lemma it follows that any $\omega$ complete effect monoid is a so called effect divisoid [7, §195]. The converse is false: later on we will show that any $\omega$ complete effect monoid is commutative, but there exists a noncommutative effect divisoid. [6, Ex. 4.3.9].

Finally, we can prove that multiplication is indeed normal:
Theorem 43. Let $b$ and $b^{\prime}$ be elements of an $\omega$-complete effect monoid $M$, and let $S \subseteq M$ be any (potentially uncountable or non-directed) non-empty subset.

1) If $\bigvee S$ exists, then so does $\bigvee_{s \in S} b \cdot s \cdot b^{\prime}$, and $b \cdot(\bigvee S) \cdot b^{\prime}=$ $\bigvee_{s \in S} b \cdot s \cdot b^{\prime}$
2) If $\bigwedge S$ exists, then so does $\bigwedge_{s \in S} b \cdot s \cdot b^{\prime}$, and $b \cdot(\bigwedge S) \cdot b^{\prime}=$ $\bigwedge_{s \in S} b \cdot s \cdot b^{\prime}$.
[^0]Proof. Suppose that $\bigvee S$ exists. We will prove that $b \cdot \bigvee S=$ $\bigvee_{s \in S} b \cdot s$, and leave the remainder to the reader. Note that $b \cdot():[0,[b\rceil]_{M} \rightarrow[0, b]_{M}$, being an order isomorphism by Lemma 416, preserves suprema and infima. The set $S$ need, however, not be part of $[0,\lceil b\rceil]_{M}$, so we consider instead of $b$ the element $b^{\prime} \equiv b \otimes\lceil b\rceil^{\perp}$, for which $\left\lceil b^{\prime}\right\rceil=$ $\lceil b\rceil \vee\lceil b\rceil^{\perp}=1$ by Lemma 36. We then get an order isomorphism $b^{\prime} \cdot(): M \rightarrow\left[0, b^{\prime}\right]_{M}$, which preserves suprema, so that $b^{\prime} \cdot \bigvee S$ is the supremum of $b^{\prime} \cdot S$ in $\left[0, b^{\prime}\right]_{M}$, and hence in $M$, by Corollary 38. Then

$$
\begin{aligned}
\left(b \boxtimes\lceil b\rceil^{\perp}\right) \cdot \bigvee S & =\bigvee_{s \in S}\left(b \boxtimes\lceil b\rceil^{\perp}\right) \cdot s \\
& \leq \bigvee_{s \in S} b \cdot s \otimes \bigvee^{\prime}\lceil b\rceil^{\perp} \cdot s^{\prime} \\
& \leq b \cdot \bigvee S \boxtimes\lceil b\rceil^{\perp} \cdot \bigvee S \\
& =\left(b \boxtimes\lceil b\rceil^{\perp}\right) \cdot \bigvee S
\end{aligned}
$$

forces $\bigvee_{s \in S} b \cdot s=b \cdot \bigvee S$ (see Lemma 22).

## VI. Boolean algebras, halves and convexity

We are ready to study the two important types of idempotents in an effect monoid: those that are Boolean and those that are halvable.

Definition 44. We say that an element $a$ of an effect monoid $M$ is Boolean when each $b \leq a$ is idempotent. We say an effect monoid is Boolean when 1 is Boolean.
Proposition 45. The set of idempotents $P(M)$ of an effect monoid $M$ is a Boolean algebra. Thus an effect monoid is Boolean iff it is a Boolean algebra.

Proof. First we will show that in fact $p \cdot q=p \wedge q$ for $p, q \in$ $P(M)$, where the infimum $\wedge$ is taken in $M$. Using Lemma 20 , we see $(p \cdot q)^{2}=p \cdot q \cdot p \cdot q=p \cdot p \cdot q \cdot q=p \cdot q$ and so $p \cdot q$ is an idempotent. Let $r \leq p, q$. Then $r \cdot p=r$ and $r \cdot q=r$ so that $r \cdot p \cdot q=r$, and hence $r \leq p \cdot q$ by Lemma 19, which shows $p \cdot q=p \wedge q$. As the complement is an order antiisomorphism, we find $p \vee q=\left(p^{\perp} \wedge q^{\perp}\right)^{\perp}$ and hence $P(M)$ is a complemented lattice. It remains to show that it satisfies distributivity: $p \wedge(q \vee r)=(p \wedge q) \vee(p \wedge r)$. By uniqueness of complements, it is easily shown that $p \vee q=p \boxtimes\left(p^{\perp} \cdot q\right)=$ $q \otimes\left(p \cdot q^{\perp}\right)$. The remainder is a straightforward exercise in writing out the expressions $p \wedge(q \vee r)$ and $(p \wedge q) \vee(p \wedge r)$ and noting that they are equal.

Proposition 46. If $M$ is $\omega$-complete, then the Boolean algebra of projections $P(M)$ is $\omega$-complete.

Proof. Let $A \subseteq P(M)$ be a countable subset. Pick an enumeration of its elements $p_{1}, p_{2}, \ldots$ Let $q_{n}$ be iteratively defined as $q_{1} \equiv p_{1}$ and $q_{n} \equiv q_{n-1} \vee p_{n}$. Then the $q_{n}$ form an increasing sequence and hence it has a supremum $q$. We claim that $q$ is also the supremum of $A$. Of course $q \geq q_{n} \geq p_{n}$ and hence $q$ is an upper bound. Suppose that $r \geq p_{n}$ for all $n$. Then $r \geq q_{1}$, and hence by induction if $r \geq q_{n}$ then $r \geq q_{n} \vee p_{n}=q_{n+1}$. Hence also $r \geq q$.

Proposition 47. Let $M$ be an $\omega$-complete Boolean effect monoid. Then $M$ is an $\omega$-complete Boolean algebra.
Proof. By Propositions 45 and $46 P(M)$ is an $\omega$-complete Boolean algebra. But by assumption every element of $M$ is an idempotent, and hence $M=P(M)$.

The counterpart to the Boolean effect monoids, are the halvable effect monoids

Definition 48. We say that an element $a$ of an effect algebra $E$ is halvable when $a=b \otimes b$ for some $b \in E$. We say an effect algebra is halvable when 1 is halvable.

A halvable effect monoid actually has much more structure then might be apparent:
Definition 49. Let $E$ be an effect algebra. We say $E$ is convex if there exists an action $\cdot[0,1] \times E \rightarrow E$, where $[0,1]$ is the standard real unit interval, satisfying the following axioms for all $a, b \in E$ and $\lambda, \mu \in[0,1]$ :

- $\lambda \cdot(\mu \cdot a)=(\lambda \mu) \cdot a$.
- If $\lambda+\mu \leq 1$, then $\lambda \cdot a \perp \mu \cdot a$ and $\lambda \cdot a \boxtimes \mu \cdot a=(\lambda+\mu) \cdot a$.
- If $a \perp b$, then $\lambda \cdot a \perp \lambda \cdot b$ and $\lambda \cdot(a \otimes b)=\lambda \cdot a \oplus \lambda \cdot b$. - $1 \cdot a=a$.

In a convex effect monoid we will usually write the convex action without any symbol in order to distinguish it from the multiplication coming from the monoid structure. So if $\lambda \in$ $[0,1]$ is a real number and $a, b \in M$ is a convex effect monoid, then we write $\lambda(a \cdot b)$. Note that a priori it is not clear whether $\lambda(a \cdot b)=(\lambda a) \cdot b=a \cdot(\lambda b)$.

Proposition 50. Let $M$ be a halvable $\omega$-complete effect monoid. Then $M$ is convex.

Proof. Pick any $a \in M$ with $a \oslash a=1$. Let $q=\frac{m}{2^{n}}$ be a dyadic rational number with $0 \leq m \leq 2^{n}$. We define a corresponding element $\bar{q} \in M$ by $\bar{q}=m a^{n}$, which is easily seen to be independent of the choice of $m$ and $n$. This yield an action $(q, s) \mapsto \bar{q} \cdot s$ that satisfies all axioms of Definition 49 restricted to dyadic rationals.

Assume $\lambda \in(0,1]$. Pick a strictly increasing sequence $0 \leq q_{1}<q_{2}<\ldots$ of dyadic rationals with $\sup q_{i}=\lambda$.

We will define $\bar{\lambda} \in M$ by $\bigvee_{i} \overline{q_{i}}$, but first we have to show that it is independent of the choice of the sequence and that it coincides with the definition just given for dyadic rationals. So assume $0 \leq p_{1}<p_{2}<\ldots$ is any other sequence of dyadic rationals with $\sup p_{i}=\lambda$. For any $p_{i}$ we can find a $q_{j}$ with $p_{i} \leq q_{j}$, so $\overline{p_{i}} \leq \overline{q_{j}}$, hence $\bigvee_{i} \overline{p_{i}} \leq \bigvee_{j} \overline{q_{j}}$. As the situation is symmetric between the sequences, we also have $\bigvee_{j} \overline{q_{j}} \leq \bigvee_{i} \overline{p_{i}}$ and so $\bigvee_{j} \overline{q_{j}}=\bigvee_{i} \overline{p_{i}}$. Hence $\bar{\lambda}$ is independent of the choice of sequence. Next, assume $\lambda \equiv q$ is a non-zero dyadic rational. Pick $m$ with $2^{-m} \leq q$. Then $q_{n} \equiv q-2^{-(m+n)}$ is a sequence of dyadic rationals with $\sup q_{n}=q$. We have $\bigvee_{n} \overline{q-2^{-(m+n)}}=\bar{q} \ominus$ $\bigwedge_{n} \overline{2^{-(m+n)}}=\bar{q} \ominus \bigwedge_{n} a^{m+n}=\bar{q}$, (where in the last step we used Lemma 26) so both definitions of $\bar{q}$ coincide. As a result we are indeed justified to define $\bar{\lambda}=\bigvee_{i} \overline{q_{i}}$. We can then define
an action by $(\lambda, s) \mapsto \bar{\lambda} \cdot s$. As both addition and multiplication preserve suprema by Theorem 43, it is straightforward to show that this action indeed satisfies all the axioms of a convex action.

## VII. Embedding theorems

We now have what we need to show that any $\omega$-complete effect monoid embeds into a direct sum of a Boolean effect monoid and a halvable one, which lies at the heart of our results.

Lemma 51. Let $M$ be an $\omega$-complete effect monoid, and let $a$ be a halvable element. Then the ceiling $\lceil a\rceil$ is halvable as well.

Proof. Write $a \equiv b \otimes b$. We compute

$$
\begin{aligned}
&\lceil a\rceil \equiv \bigotimes_{n} a \cdot\left(a^{\perp}\right)^{n}=\bigotimes_{n}(b \otimes b) \cdot\left(a^{\perp}\right)^{n} \\
&=\left(\bigotimes_{n} b \cdot\left(a^{\perp}\right)^{n}\right) \otimes\left(\bigotimes_{n} b \cdot\left(a^{\perp}\right)^{n}\right)
\end{aligned}
$$

and hence it is indeed halvable.
Proposition 52. Each $\omega$-complete effect monoid $M$ has a subset $E \subseteq M$

1) that is a maximal collection of non-zero orthogonal idempotents, and
2) such that each element of $E$ is either halvable or Boolean.

Proof. Let $H$ be a maximal collection of non-zero orthogonal halvable idempotents of $M$, and let $E$ be a maximal collection of non-zero orthogonal idempotents of $M$ that extends $H$. (Such sets $E$ and $H$ exist by Zorn's Lemma). By definition, $E$ is a maximal collection of non-zero orthogonal idempotents of $M$, so the only thing to prove is that each $e \in E \backslash H$ is Boolean. Hence, let $a$ be an element of $M$ below some $e \in$ $E \backslash H$; we must show that $a$ is an idempotent. Note that $2\left(a^{\perp}\right.$. $a) \equiv a^{\perp} \cdot a \boxtimes a^{\perp} \cdot a$ (which exists by e.g. Lemma 25) is halvable, and $2\left(a^{\perp} \cdot a\right) \leq e$, because $2\left(a^{\perp} \cdot a\right) \cdot e=2\left(a^{\perp} \cdot a \cdot e\right)=2\left(a^{\perp} \cdot a\right)$. Then the idempotent $\left\lceil 2 a \cdot a^{\perp}\right\rceil \leq e$, which is halvable by Lemma 51, is orthogonal to all $h \in H$ (since it is below $e$ ) and must therefore be zero lest it contradict the maximality of $H$. In particular, $a^{\perp} \cdot a=0$ since $a^{\perp} \cdot a \leq\left\lceil 2 a^{\perp} \cdot a\right\rceil=0$, and so $a$ is an idempotent by Lemma 18 Whence $e$ is Boolean.

Note that the only idempotent that is both Boolean and halvable is zero, and hence each element in the above set is either Boolean or halvable.

Proposition 53. Given a maximal orthogonal collection of non-zero idempotents $E$ of an $\omega$-complete effect monoid $M$, the map

$$
a \mapsto(a \cdot e)_{e}: M \longrightarrow \bigoplus_{e \in E} M e
$$

is an embedding of effect monoids.
Proof. The map obviously maps 1 to 1 , and preserves addition. Hence it also preserves the complement and the order. By Lemma20we have $(a \cdot e) \cdot(b \cdot e)=(a \cdot b) \cdot(e \cdot e)=(a \cdot b) \cdot e$, and so the map also preserves the multiplication. It remains to show
that the map is order reflecting. Note that if we had $\emptyset E=1$, then for any $a$ by Theorem $43 a=a \cdot 1=a \cdot \bigotimes_{e \in E} e=$ $\bigotimes_{e \in E} a \cdot e$, and hence if $a \cdot e \leq b \cdot e$ for all $e \in E$ we have $a=\bigotimes_{e \in E} a \cdot e \leq \rrbracket_{e \in E} b \cdot e=b$, which proves that it is indeed order reflecting.

So let us prove that $\boxtimes E=1$. Suppose $u$ is an upper bound for $E$; we must show that $u=1$. Note that $\lfloor u\rfloor$ is an upper bound for $E$ too, since $E$ contains only idempotents. It follows that the idempotent $\lfloor u\rfloor^{\perp}=\left\lceil u^{\perp}\right\rceil$ is orthogonal to all $e \in E$, which is impossible (by maximality of $E$ ) unless $\left\lceil u^{\perp}\right\rceil=0$. Hence $\left\lceil u^{\perp}\right\rceil=0$, and thus $u^{\perp}=0$.

Theorem 54. Let $M$ be an $\omega$-complete effect monoid. Then there exist $\omega$-complete effect monoids $M_{1}$ and $M_{2}$ where $M_{1}$ is convex, and $M_{2}$ is an $\omega$-complete Boolean algebra such that $M$ embeds into $M_{1} \oplus M_{2}$.
Proof. Let $E=H \cup B$ be a maximal collection of nonzero orthogonal idempotents of Proposition 52 such that the idempotents $p \in H$ are halvable, while the $q \in B$ are Boolean.

Let $M_{1} \equiv \bigoplus_{p \in H} p M$ and $M_{2} \equiv \bigoplus_{q \in B} q M$. It is easy to see that $M_{1}$ is then again halvable and $M_{2}$ is Boolean. By Propositions 50 and $47 M_{1}$ is convex while $M_{2}$ is an $\omega$-complete Boolean algebra. By the previous proposition $M$ embeds into $\bigoplus_{p \in E} p M \cong M_{1} \oplus M_{2}$.

One might be tempted to think that the above result could be strengthened to an isomorphism. The following example shows that this is not the case:

Example 55. Let $X_{1}$ and $X_{2}$ be uncountably infinite sets, and let $A$ be the set of all pairs of functions

$$
A \equiv\left\{\left(f_{1}: X_{1} \rightarrow[0,1], f_{2}: X_{2} \rightarrow\{0,1\}\right)\right\}
$$

Let $S_{0}, S_{1} \subseteq A$ be subsets where both functions are unequal to 0 respectively 1 only at a countable number of spots:
$S_{0} \equiv\left\{\left(f_{1}, f_{2}\right) ;\right.$ both $\left[\begin{array}{l}\left\{x_{1} \in X_{1} ; f_{1}\left(x_{1}\right) \neq 0\right\} \\ \left\{x_{2} \in X_{2} ; f_{2}\left(x_{2}\right) \neq 0\right\}\end{array}\right.$ countable $\}$
$S_{1} \equiv\left\{\left(f_{1}, f_{2}\right) ;\right.$ both $\left[\begin{array}{l}\left\{x_{1} \in X_{1} ; f_{1}\left(x_{1}\right) \neq 1\right\} \\ \left\{x_{2} \in X_{2} ; f_{2}\left(x_{2}\right) \neq 1\right\}\end{array}\right.$ countable $\}$
Finally, define $M=S_{0} \cup S_{1}$. It is then straightforward to check that $M$ is a $\omega$-complete effect monoid. It is easy to see that $M$ has no maximal halvable idempotent, and hence for any $M_{1}$ halvable and $M_{2}$ Boolean, necessarily $M \neq M_{1} \oplus M_{2}$.

Though the embedding is not an isomorphism for $\omega$ complete effect monoids, when we assume full directedcompleteness, we can derive a stronger result.

Lemma 56. Let $M$ be a directed-complete effect monoid. Then there is a maximal element that is summable with itself. That is: there is an $a \in M$ such that $a \perp a$ and for any other $b \geq a$ such that $b \perp b$, we have $b=a$. Furthermore,

1) $a \oslash a$ is an idempotent and
2) if $s \leq(a \boxtimes a)^{\perp}$, then $s$ is idempotent for any $s \in M$.

Proof. Write $A \equiv\{s \in M ; s \perp s\}$. We will show that $A$ has a maximal element using Zorn's Lemma. To this end, suppose $D \subseteq A$ is a chain. We have to show that it has an upper bound in $A$. If $D$ is empty, then $0 \in A$ is clearly an upper bound, so we may assume that $D$ is not empty. Define $s \equiv \bigvee D$. It is sufficient to show $s \perp s$ as then $s \in A$.

Assume $d, d^{\prime} \in D$. We claim $d \perp d^{\prime}$. Indeed, assuming without loss of generality that $d^{\prime} \leq d$, we see $d^{\prime} \leq d \leq d^{\perp}$ and so $d \perp d^{\prime}$.

So $\bigvee_{d^{\prime} \in D} d \otimes d^{\prime}$ exists. As addition preserves suprema, we have $\bigvee_{d^{\prime} \in D} d \boxtimes d^{\prime}=d \boxtimes \bigvee_{d^{\prime} \in D} d^{\prime}=d \boxtimes s$. Hence $\bigvee_{d \in D} d \boxtimes s$ exists and $\bigvee_{d \in D} d \boxtimes s=\left(\bigvee_{d \in D} d\right) \boxtimes s=s \boxtimes s$, so indeed $s \in$ $A$. By Zorn's Lemma we know there is a maximal element of $A$. Pick such a maximal element $a \in A$.

Next we will show that $a \oslash a$ is idempotent. Define $b \equiv$ $(a \otimes a)^{\perp}$. Note that $b \cdot b^{\perp}$ is summable with itself. As a result $b=b \cdot 1 \geq b \cdot\left(b \cdot b^{\perp} \otimes b \cdot b^{\perp}\right)=b^{2} \cdot b^{\perp} \otimes b^{2} \cdot b^{\perp}$. Now note that $1=a \oslash a \oslash(a \oslash a)^{\perp}=a \oslash a \oslash b \geq a \oslash a \oslash b^{2} \cdot b^{\perp} \otimes b^{2} \cdot b^{\perp}$, and hence $a \oslash b^{2} \cdot b^{\perp}$ is summable with itself. Since $a$ is maximal with this property we must have $b^{2} \cdot b^{\perp}=0$. But then $\left(b \cdot b^{\perp}\right)^{2}=b^{2} \cdot b^{\perp} b^{\perp}=0$ and so $b \cdot b^{\perp}=0$ by Lemma,26, Hence $b^{\perp}=a \oslash a$ is indeed idempotent.

Now let $s \leq(a \oslash a)^{\perp} \equiv b$. We have to show that $s$ is idempotent. By Lemma $25 s \cdot s^{\perp}$ is summable with itself and so $s \cdot s^{\perp} \oslash s \cdot s^{\perp} \leq b \equiv(a \oslash a)^{\perp}$ by Lemma 23. Thus $a \oslash a \perp$ $s \cdot s^{\perp} \boxtimes s \cdot s^{\perp}$ and so $a \oslash s \cdot s^{\perp}$ is summable with itself. By maximality of $a$ we must have $s \cdot s^{\perp}=0$, which shows that $s$ is indeed an idempotent.

Theorem 57. Let $M$ be a directed-complete effect monoid. Then there exists a convex directed-complete effect monoid $M_{1}$, and a complete Boolean algebra $M_{2}$ such that $M \cong M_{1} \oplus$ $M_{2}$.

Proof. Let $a$ be a maximal element that is summable to itself from Lemma 56 It was shown that $p=(a \otimes a)^{\perp}$ is an idempotent such that any $s \leq p$ is also an idempotent. Hence by an adaptation of Proposition 47 to directed-complete effect monoids, $p M$ is a complete Boolean algebra.

But of course $p^{\perp} M$ is halvable and hence by Proposition 50 $p^{\perp} M$ is convex. Hence letting $M=p^{\perp} M \oplus p M$ gives the desired result.

## VIII. ORDER UNIT SPACES

In this section we will see that a convex $\omega$-complete effect monoid $M$ is isomorphic to the set of continuous functions of some basically disconnected compact Hausdorff space $X$ to the real unit interval, or equivalently by Gel'fand duality, isomorphic to the unit interval of an $\omega$-complete commutative $\mathrm{C}^{*}$-algebra. We do this by using a known representation of convex effect algebras as unit intervals of order unit spaces, and then apply Yosida's representation theorem for latticeordered vector spaces.
Definition 58. An order unit space (OUS) is an ordered vector space together with distinguished element $1 \in V$, such that for every $v \in V$, there is a $n \in \mathbb{N}$ such that $-n \cdot 1 \leq v \leq$
$n \cdot 1$. An OUS is called Archimedean provided that $v \leq \frac{1}{n} \cdot 1$ for all $n \in \mathbb{N}$, implies $v \leq 0$.
Definition 59. Given an OUS $V$, its unit interval $[0,1]_{V} \equiv$ $\{v \in V ; 0 \leq v \leq 1\}$ is a convex effect algebra with $a \perp b$ iff $a+b \leq 1$ and then $a \otimes b=a+b ; a^{\perp}=1-a$; and the convex structure being given by the obvious scalar multiplication. We say $V$ is $\omega$-complete/directed complete whenever $[0,1]_{V}$ is.
Lemma 60. Let $V$ be an order unit space. It is directed complete if and only if it is bounded directed complete; that is if every directed subset of $V$ with upper bound has a supremum. Similarly $V$ is $\omega$-complete iff it is bounded $\omega$ complete.

Proof. Since any subset of the unit interval is obviously bounded, any bounded $\omega$-complete/directed-complete order unit space is $\omega$-complete/directed complete. For the other direction, assume $S \subseteq V$ is some (countable) directed subset with upper bound $b \in V$. Pick any $a \in S$ and define $S^{\prime} \equiv$ $\{v ; v \in S ; a \leq v\}$. It is sufficient to show that $S^{\prime}$ has a supremum. Pick any $n \in \mathbb{N}, n \neq 0$ with $-n \cdot 1 \leq a, b \leq n \cdot 1$. Then clearly $\left\{\frac{1}{n}(s-a) ; s \in S^{\prime}\right\} \subseteq[0,1]_{V}$ has a supremum and so does $S^{\prime}$ as $v \mapsto n v+a$ is an order isomorphism.
Lemma 61. A bounded $\omega$-complete OUS is Archimedean.
Proof. Assume $V$ is a bounded $\omega$-complete OUS. Let $v \in V$ be given with $v \leq \frac{1}{n} 1$ for all $n \in \mathbb{N}$. As $a \mapsto-a$ is an order anti-isomorphism, all bounded directed subsets of $V$ have an infimum too, so $v \leq \bigwedge_{n} \frac{1}{n} 1=\left(\inf _{n} \frac{1}{n}\right) 1=0$ as desired.

Recall that the unit interval of an OUS is a convex effect algebra. In fact, every convex effect algebra is of this form.

Theorem 62. Let $M$ be a convex effect algebra. Then there exists an order unit space $V$ such that $M \cong[0,1]_{V}$ [30].

Proposition 63. Let $M$ be an $\omega$-complete convex effect monoid. Then the multiplication is "bilinear": $\lambda(a \cdot b)=$ $(\lambda a) \cdot b=a \cdot(\lambda b)$ for any $a, b \in M$ and $\lambda \in[0,1]$.
Proof. Clearly $n\left(a \cdot\left(\frac{1}{n} b\right)\right)=a \cdot b=n \frac{1}{n}(a \cdot b)$ and so $a \cdot\left(\frac{1}{n} b\right)=$ $\frac{1}{n}(a \cdot b)$. Hence $m\left(a \cdot\left(\frac{1}{n} b\right)\right)=a \cdot\left(\frac{m}{n} b\right)=\frac{m}{n}(a \cdot b)$ for any $m \leq$ $n$. We have shown the second equality for rational $\lambda$. With a similar argument one shows the first equality holds as well for rational $\lambda$. To prove the general case, let $a, b \in M$ and $\lambda \in$ $[0,1]$ be given. Pick a sequence $0 \leq q_{1}<q_{2}<\ldots$ of rationals with $\sup _{n} q_{n}=\lambda$. Let $V$ be an OUS with $M \cong[0,1]_{V}$ (as a convex effect algebra), which exists due to Theorem 62, Note that the multiplication of $M$ is only defined on $[0,1]_{V}$. For $a \in[0,1]_{V}$ we have $a \leq\|a\| 1$ and so $\|a \cdot b\| \leq\|a\|\|b\|$ for any $b \in[0,1]_{V}$. Hence

$$
\begin{aligned}
& \|\lambda(a \cdot b)-a \cdot(\lambda b)\| \\
& \quad=\left\|\left(\lambda-q_{i}\right)(a \cdot b)+a \cdot\left(q_{i} b\right)-a \cdot(\lambda b)\right\| \\
& \quad=\left\|\left(\lambda-q_{i}\right)(a \cdot b)-a \cdot\left(\left(\lambda-q_{i}\right) b\right)\right\| \\
& \quad \leq\left(\lambda-q_{i}\right)\|a \cdot b\|+\left(\lambda-q_{i}\right)\|a\|\|b\| .
\end{aligned}
$$

The right-hand side vanishes as $i \rightarrow \infty$. Thus $\| \lambda(a \cdot b)-$ $a \cdot(\lambda b)=0 \|$. Since $V$ is Archimedean by Lemma 61, \|•\|
is a proper norm so that then $\lambda(a \cdot b)-a \cdot(\lambda b)=0$. That $\lambda(a \cdot b)=(\lambda a) \cdot b$ follows similarly.
Theorem 64 (Yosida [31], cf. [32]). Let $V$ be a normcomplete lattice-ordered Archimedean OUS. Denote by $\Phi$ the Yosida spectrum of $V$ : the compact Hausdorff space of real-valued finite-suprema-preserving unital linear functionals on $V$ with the induced pointwise topology of $\mathbb{R}^{V}$. Then the map $\vartheta: V \rightarrow C(\Phi)$ given by $\vartheta(v)(\varphi)=\varphi(v)$ is an order isomorphism.
Remark 65. In fact the category of compact Hausdorff spaces with continuous maps between them is dually equivalent to the category of lattice-ordered norm-complete Archimedean order unit spaces with linear unital finite-supremum-preserving maps between them. [32]

Theorem 66. Let $M$ be a convex $\omega$-complete effect monoid. Then $M \cong[0,1]_{C(X)}$ for some basically disconnected Hausdorff space $X$. If $M$ is even directed complete, then $X$ is extremally disconnected.
Proof. The effect monoid $M$ is a lattice by Theorem 37. By Theorem 62 there is an OUS $V$ such that $M \cong[0,1]_{V}$ as a convex effect algebra. It is easy to see that $V$ is a lattice as well as any supremum reduces to one in the interval $[0,1]_{V} . V$ is bounded $\omega$-complete by Lemma 60 and hence Archimedean by Lemma 61, and norm complete by [33, Lemma 1.2]. Then by Theorem 64 we have $V \cong C(\Phi)$, where $\Phi$ is the Yosida spectrum of $V$. Note $C(\Phi)$ is bounded $\omega$-complete iff $\Phi$ is basically disconnected and $C(\Phi)$ is bounded directed complete iff $\Phi$ is extremally disconnected. The multiplication of $M$ induces a multiplication on $[0,1]_{C(\Phi)}$, which we will denote by $f * g$ to distinguish it from the pointwise multiplication $f \cdot g$. To prove the Theorem, it remains to be shown that $f * g=f \cdot g$.

Pick any $\varphi \in \Phi$. It is sufficient to show that $\varphi(f * g)=$ $\varphi(f) \varphi(g)$ —indeed then $(f * g)(\varphi)=\varphi(f * g)=\varphi(f) \varphi(g)=$ $f(\varphi) g(\varphi)=(f \cdot g)(\varphi)$. The remainder of the proof is based on [34, Lemma 5.26]. Write $d(a, b) \equiv a \vee b-a \wedge b$ for either $a, b \in[0,1]$ or $a, b \in[0,1]_{C(\Phi)}$. Note $f *(g \vee h) \leq$ $(f * g) \vee(f * h)$ and $f *(g \wedge h) \geq(f * g) \wedge(f * h)$, hence $d(f * g, f * h) \leq f * d(g, h)$ for $f, g, h \in[0,1]_{C(\Phi)}$. So in particular $d(f * g, f *(\varphi(g) 1)) \leq f * d(g, \varphi(g) 1) \leq$ $d(g, \varphi(g) 1)$. The multiplication $*$ is "bilinear" by Proposition 63, so $f *(\varphi(g) 1)=\varphi(g) f$. By definition $\varphi$ preserves infima, suprema and 1 and so $d(\varphi(f), \varphi(g))=\varphi(d(f, g))$ for any $f, g \in[0,1]_{C(\Phi)}$. Combining the last three facts, we see

$$
\begin{aligned}
d(\varphi(f * g), \varphi(f) \varphi(g)) & =\varphi(d(f * g, f *(\varphi(g) 1))) \\
& \leq \varphi(d(g, \varphi(g) 1)) \\
& =d(\varphi(g), \varphi(g))=0
\end{aligned}
$$

So indeed $\varphi(f * g)=\varphi(f) \varphi(g)$, hence $f * g=f \cdot g$.
Remark 67. The previous theorem can also be proven using Kadison's representation Theorem [35], which states that any norm-complete Archimedean OUS with a bilinear and positive product is isomorphic to $C(X)$. This requires one to show that the product on $[0,1]_{V}$ extends to a bilinear and positive
product on the entirety of $V$, which can be done, but is a bit tedious (cf. [11, Theorem 46]).

## IX. Main theorems

We now collect our previous results and prove our main theorems.

Theorem 68. Let $M$ be an $\omega$-complete effect monoid. Then there exists a basically disconnected compact Hausdorff space $X$, and an $\omega$-complete Boolean algebra $B$ such that $M$ embeds into $[0,1]_{C(X)} \oplus B$.
Proof. By Theorem 54 there exist $\omega$-complete effect monoids $M_{1}$ and $M_{2}$ such that $M$ embeds into $M_{1} \oplus M_{2}$, where $M_{1}$ is convex and $M_{2}$ is an $\omega$-complete Boolean algebra. By Theorem $66 M_{1}=[0,1]_{C(X)}$ for a basically disconnected compact Hausdorff space $X$.
Theorem 69. Let $M$ be a directed-complete effect monoid. Then there exists an extremally-disconnected compact Hausdorff space $X$, and an complete Boolean algebra $B$ such that $M \cong[0,1]_{C(X)} \oplus B$.
Proof. Same as previous theorem but using Theorem 57
Corollary 70. Let $M$ be an $\omega$-complete effect monoid. Then $M$ is commutative.

Proof. $M$ embeds into $[0,1]_{C(X)} \oplus B$ where $B$ is a Boolean algebra and $X$ is a basically disconnected compact Hausdorff space. The multiplication in $B$ is given by the join and hence is commutative, while the multiplication in $C(X)$ is given by pointwise multiplication in the real numbers and hence is also commutative. The multiplication of $M$ is then necessarily also commutative.

Theorem 71. Let $M$ be an $\omega$-complete effect monoid with no non-trivial zero divisors. Then either $M=\{0\}, M=\{0,1\}$ or $M \cong[0,1]$.

Proof. Assume that $M \neq\{0,1\}$ and $M \neq\{0\}$. We remark first that for any idempotent $p \in M$ we have $p \cdot p^{\perp}=0$, and hence by the lack of non-trivial zero divisors we must have $p=0$ or $p=1$. Since $M \neq\{0,1\}$, there is an $s \in M$ such that $s \neq 0,1$, and hence we must have $s \cdot s^{\perp} \neq 0$. By Lemma 25 we then have an element $2\left(s \cdot s^{\perp}\right)$ that is halvable. Hence by Lemma $51\left\lceil 2 s \cdot s^{\perp}\right\rceil$ is also halvable. As this ceiling is an idempotent it must be equal to 1 or to 0 . If it were zero then $2 s \cdot s^{\perp} \leq\left\lceil 2 s \cdot s^{\perp}\right\rceil=0$, which contradicts $s \cdot s^{\perp} \neq 0$. So $1=\left\lceil 2 s \cdot s^{\perp}\right\rceil$ is halvable. By Proposition 50, $M$ is then convex. Hence, by Theorem $66 M=[0,1]_{C(X)}$ for some basically disconnected $X$. We will show that $X$ has a single element, which will complete the proof.

As idempotents of $[0,1]_{C(X)}$ correspond to clopens of $X$, there are only two clopens in $X$, namely $X$ and $\emptyset$. Reasoning towards contradiction, assume there are $x, y \in X$ with $x \neq y$. By Urysohn's lemma we can find $f \in C(X)$ with $0 \leq f \leq 1, f(x)=0$ and $f(y)=1 . U_{x} \equiv f^{-1}\left(\left[0, \frac{1}{3}\right)\right.$ and $U_{y} \equiv f^{-1}\left(\left(\frac{2}{3}, 1\right]\right)$ are two open sets with disjoint closure. Using Urysohn's lemma again, we can find $g \in C(X)$
with $g\left(\overline{U_{x}}\right)=\{0\}$ and $g\left(\overline{U_{y}}\right)=\{1\}$. As $X$ is basically disconnected, we know $\overline{\operatorname{supp} g}$ is clopen. We cannot have $\overline{\operatorname{supp} g}=\emptyset$ as $y \in U_{y} \subseteq \overline{\operatorname{supp} g}$. Hence $\overline{\operatorname{supp} g}=X$. But then $x \in U_{x} \subseteq X \backslash \overline{\operatorname{supp} g}=\emptyset$. Contradiction. Apparently $X$ has only one point and so $M \cong[0,1]$.

Remark 72. In [6] it is shown that any $\omega$-effectus satisfying a natural property called normalisation (basically, that any substate can be normalised to a state) must have scalars without zero divisors. We have hence also completely classified the allowable scalars in a $\omega$-effectus satisfying normalisation.

Remark 73. As noted in Example 7, the unit interval of any partially ordered ring where the sum and product of positive elements is again positive forms an effect monoid. Hence the previous results can also be seen as, to our knowledge, new results for $\sigma$-Dedekind-complete ordered rings. Although it is not clear how our results for the unit interval can be extended to results concerning the entire ring, our results might eventually lead to a version of the characterisation of the real numbers as the unique Dedekind-complete ordered field, where the condition of being a field can be weakened to being a domain (i.e. a ring without non-trivial zero-divisors).

## X. CONCLUSION AND OUTLOOK

We have shown that any $\omega$-complete effect monoid embeds into a direct sum of a Boolean algebra and the set of continuous functions from a given Hausdorff space $X$ into the real unit interval $[0,1]$. As a result, the only $\omega$-complete effect monoids without any zero divisors are either degenerate ( $\{0\}$ ), the Booleans $(\{0,1\})$, or the real unit interval $([0,1])$, resulting in a dichotomy of sharp logic and fuzzy probabilistic logic.

The structure of an $\omega$-complete effect monoid; partial addition, involution, multiplication and directed limits; captures in a sense the structure present in the real unit interval and hence these results give a foundational underpinning to why the unit interval should be the designated structure of the scalars in any non-sharp physical or logical theory.

An interesting follow-up question is to consider whether the $\omega$-completeness can be weakened somehow. There exist pathological (non-commutative) effect monoids ([6, Ex. 4.3.9] and [11, Cor. 51]), but all the known ones are nonArchimedean (i.e. they have infinitesimal elements). So perhaps it is possible to embed any Archimedean effect monoid into a directed-complete effect monoid, similar to the Dedekind-MacNeille completion of a poset into a complete lattice. If this is the case, then we can essentially get rid of the assumption of $\omega$-completeness.

Another matter that deserves further investigation is the categorical aspect of our results. It can be shown that Theorem 69 has the following consequence: the category of directedcomplete effect monoids with effect monoid homomorphisms between them (which need not preserve suprema) is dually equivalent to the category which has as objects pairs $(X, S)$, where $S \subseteq X$ is a clopen subset of a extremally disconnected compact Hausdorff space $X$, and morphisms $(X, S) \rightarrow(Y, T)$ are continuous maps $f: X \rightarrow Y$ satisfying $f(S) \subseteq T$.

Obtaining a similar duality for $\omega$-complete effect monoids would in our estimation be rather non-trivial, but as it involves $\omega$-complete algebras, it can perhaps use results analogous to those of Ref. [36].
We also aim to use our results regarding $\omega$-complete effect monoids to study $\omega$-effectuses [6]. It seems possible to use this framework in combination with our results to make a "reconstruction of quantum theory" [37], [38] that does not a priori refer to the structure of the real unit interval, ${ }^{2}$ for instance by adopting the effectus-based axioms of Ref. [39].

Finally, when the associative bilinear multiplication of an effect monoid is replaced by a binary operation satisfying axioms related to the Lüders product $(a, b) \mapsto \sqrt{a} b \sqrt{a}$ for positive $a$ and $b$ in a $\mathrm{C}^{*}$-algebra, one gets a sequential effect algebra. In forthcoming work [40] we use the techniques and results of this paper to show that a spectral theorem holds in any normal sequential effect algebra [41] and furthermore that any such algebra is isomorphic to the direct sum of a Boolean, convex and a "almost convex" normal sequential effect algebra.
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## References

[1] S. Abramsky and B. Coecke, "A categorical semantics of quantum protocols," in Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004. IEEE, 2004, pp. 415-425.
[2] S. Gogioso, "Fantastic quantum theories and where to find them," 2017. [Online]. Available: https://arxiv.org/abs/1703.10576
[3] S. Tull, "A categorical reconstruction of quantum theory," 2016. [Online]. Available: https://arxiv.org/abs/1804.02265
[4] D. J. Foulis and M. K. Bennett, "Effect algebras and unsharp quantum logics," Foundations of physics, vol. 24, no. 10, pp. 1331-1352, 1994.
[5] K. Cho, B. Jacobs, B. E. Westerbaan, and A. A. Westerbaan, "An introduction to effectus theory," 2015. [Online]. Available: https://arxiv.org/abs/1512.05813
[6] K. Cho, "Effectuses in categorical quantum foundations," Ph.D. dissertation, Radboud Universiteit Nijmegen, 2019. [Online]. Available: https://arxiv.org/abs/1910.12198
[7] B. E. Westerbaan, "Dagger and dilations in the category of von Neumann algebras," Ph.D. dissertation, Radboud Universiteit Nijmegen, 2019. [Online]. Available: https://arxiv.org/abs/1803.01911
[8] J. Barrett, "Information processing in generalized probabilistic theories," Physical Review A, vol. 75, no. 3, p. 032304, 2007.
[9] M. A. Arbib and E. G. Manes, "Partially additive categories and flowdiagram semantics," Journal of Algebra, vol. 62, no. 1, pp. 203-227, 1980.
[10] B. Jacobs, "Probabilities, distribution monads, and convex categories," Theoretical Computer Science, vol. 412, no. 28, pp. 3323-3336, 2011.
[11] B. E. Westerbaan, "Sequential product on effect logics," Master's thesis, Radboud University Nijmegen, 2013. [Online]. Available: https://www.ru.nl/publish/pages/813276/masterscriptie_bas_westerbaan.pdf
[12] I. Chajda, R. Halaš, and J. Kühr, "Every effect algebra can be made into a total algebra," Algebra universalis, vol. 61, no. 2, p. 139, 2009.
[13] K. Ravindran, "On a structure theory of effect algebras," Ph.D. dissertation, Kansas State University, 1996.
[14] S. Gudder, "Examples, problems, and results in effect algebras," International Journal of Theoretical Physics, vol. 35, no. 11, pp. 2365-2376, 1996.
[15] S. P. Gudder and R. Greechie, "Effect algebra counterexamples," Mathematica Slovaca, vol. 46, no. 4, pp. 317-325, 1996.

[^1][16] G. Jenča, "Blocks of homogeneous effect algebras," Bulletin of the Australian Mathematical Society, vol. 64, no. 1, pp. 81-98, 2001.
[17] A. Dvurečenskij, "Every state on interval effect algebra is integral," Journal of Mathematical Physics, vol. 51, no. 8, p. 083508, 2010.
[18] D. J. Foulis and S. Pulmannová, "Type-decomposition of an effect algebra," Foundations of Physics, vol. 40, no. 9-10, pp. 1543-1565, 2010.
[19] S. Staton and S. Uijlen, "Effect algebras, presheaves, non-locality and contextuality," Information and Computation, vol. 261, pp. 336-354, 2018.
[20] F. Roumen, "Cohomology of effect algebras," in QPL 2016, ser. EPTCS, vol. 236, 2016, pp. 174-202. [Online]. Available: https://arxiv.org/abs/1602.00567
[21] B. Jacobs and A. A. Westerbaan, "An effect-theoretic account of Lebesgue integration," Electronic Notes in Theoretical Computer Science, vol. 319, pp. 239-253, 2015.
[22] R. Mayet and M. Navara, "Classes of logics representable as kernels of measures," Contributions to General Algebra, vol. 9, pp. 241-248, 1995.
[23] G. Kalmbach, "Orthomodular lattices do not satisfy any special lattice equation," Archiv der Mathematik, vol. 28, no. 1, pp. 7-8, 1977.
[24] J. Harding, "Remarks on concrete orthomodular lattices," International Journal of Theoretical Physics, vol. 43, no. 10, pp. 2149-2168, 2004.
[25] G. Jenča, "Effect algebras are the Eilenberg-Moore category for the Kalmbach monad," Order, vol. 32, no. 3, pp. 439-448, 2015.
[26] B. Jacobs and J. Mandemaker, "Coreflections in algebraic quantum logic," Foundations of physics, vol. 42, no. 7, pp. 932-958, 2012.
[27] P. Enrique Moliner, C. Heunen, and S. Tull, "Space in monoidal categories," in Proceedings 14th International Conference on Quantum Physics and Logic, Nijmegen, The Netherlands, 3-7 July 2017, ser. Electronic Proceedings in Theoretical Computer Science, B. Coecke and A. Kissinger, Eds., vol. 266. Open Publishing Association, 2018, pp. 399-410.
[28] L. Gillman and M. Jerison, Rings of continuous functions. Springer, 2013.
[29] Z. Riečanová, "Generalization of blocks for D-lattices and latticeordered effect algebras," International Journal of Theoretical Physics, vol. 39, no. 2, pp. 231-237, 2000.
[30] S. Gudder and S. Pulmannová, "Representation theorem for convex effect algebras," Commentationes Mathematicae Universitatis Carolinae, vol. 39, no. 4, pp. 645-660, 1998.
[31] K. Yosida, "On the representation of the vector lattice," Proceedings of the Imperial Academy, vol. 18, no. 7, pp. 339-342, 1942.
[32] B. E. Westerbaan, "Yosida duality" 2016. [Online]. Available: https://arxiv.org/abs/1612.03327
[33] J. M. Wright, "Measures with values in a partially ordered vector space," Proceedings of the London Mathematical Society, vol. 3, no. 4, pp. 675688, 1972.
[34] A. C. M. van Rooij, Lecture notes on Riesz Spaces, 2011. [Online]. Available: https://github.com/awesterb/riesz-spaces
[35] R. V. Kadison, A representation theory for commutative topological algebra. American Mathematical Society, 1951, no. 7.
[36] R. Furber, D. Kozen, K. Larsen, R. Mardare, and P. Panangaden, "Unrestricted stone duality for Markov processes," in 2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). IEEE, 2017, pp. 1-9.
[37] L. Hardy, "Quantum theory from five reasonable axioms," 2001. [Online]. Available: https://arxiv.org/abs/quant-ph/0101012
[38] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Informational derivation of quantum theory," Physical Review A, vol. 84, no. 1, p. 012311, 2011.
[39] J. van de Wetering, "An effect-theoretic reconstruction of quantum theory," 2018. [Online]. Available: https://arxiv.org/abs/1801.05798
[40] A. A. Westerbaan, B. E. Westerbaan, and J. van de Wetering, "The three types of normal sequential effect algebras," 2019, forthcoming.
[41] S. Gudder and R. Greechie, "Sequential products on effect algebras," Reports on Mathematical Physics, vol. 49, no. 1, pp. 87-111, 2002.


[^0]:    ${ }^{1}$ Cho shows that there is a non-commutative division effect monoid. Any division effect monoid is an effect divisoid as well.

[^1]:    ${ }^{2}$ In the reconstruction of Ref. [3], almost all the work is done without any assumptions on the scalars, but in the end, the structure of the unit interval is necessary to get to standard quantum theory, which is something which would not be necessary with our approach.

