Gain-Lever Lasers in Optical Subcarrier Transmission Systems

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Statement of Originality

The work presented in this thesis was carried out by the candidate. It has not been presented previously for any degree, nor is it at present under consideration by any other degree awarding body.

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Summary

This thesis is concerned with the gain-lever effect in quantum well semiconductor lasers with a view to their application in subcarrier multiplexed fibre optical systems.

The increased modulation efficiency and reduced noise figure of the gain-lever laser makes it very attractive for microwave optical transmission. However, their possible application to subcarrier multiplexed systems requires a full assessment of the effects of their intrinsic nonlinearity. With this in view, the first part of the study is dedicated to the modelling of the modulation and noise characteristics of the gain-lever laser based on the single-mode rate equations. These equations also provide an adequate basis for the analysis of the laser intrinsic dynamic nonlinearity. The Volterra series method of nonlinear system theory is then applied to develop analytical models that describe the nonlinear distortion.

Both intensity and frequency modulation are useful in gain-lever laser transmitters. Therefore analytical models were developed that account for distortion effects of both the intensity and frequency components of the laser light as well as of a nonlinear optical discriminator. The Volterra theory enables the calculation of both the harmonic and intermodulation distortion power. This measure is used to compare different gain-lever laser transmitters to the conventional DFB laser.

Finally, the analytic model is used to investigate both dynamic range constraints and the performance of a gain-lever laser transmitter in a fibre radio subcarrier multiplexed system.
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Chapter 1

Introduction

Gain-levered lasers exhibit a much larger modulation efficiency than their equivalent single-section counterparts. This characteristic makes them potentially attractive as microwave transmitters in fibre-optical links, because it entails considerable benefits such as a reduced link throughput loss and a low link noise figure.

However, the overall performance of analogue microwave links is also influenced by other transmitter characteristics such as its noise and linearity. Therefore, it is appropriate to investigate both the signal-to-noise ratio and the distortion characteristics of gain-levered lasers, in order to inform both link design and optimisation activities.

In this chapter the gain-lever effect in quantum well lasers is discussed and the earlier literature on the topic is reviewed. The possible role of the gain-lever laser in enhancing the performance of microwave fibre-optical links is put forward. Some possible application areas for these devices are suggested.

1.1 Background and motivation

Much interest has arisen in the gain-lever effect in quantum well (QW) semiconductor lasers [1]. The main reason for this interest stems from the fact that the gain-lever effect in quantum well lasers leads to a dramatically improved intensity modulation (IM) and frequency modulation (FM) response without a significant increase in both intensity or phase noise. To gain some understanding for this it is advisable to examine a simple qualitative model of the operation of a gain-lever laser.
1.1.1 The tandem-contact QW laser

The gain-lever effect arises most naturally in a two section QW semiconductor laser [1]. Figure 1.1 shows a schematic diagram of a two-section laser with its typical gain profile. Two electrically isolated contacts are made on top of the optical cavity allowing independent biasing of each section. This feature is crucial because it enables each section of the laser to operate on distinct points of the gain curve as shown in figure 1.1.

Optically the laser behaves as if it had one single cavity insofar as the photons inside the cavity are, to a first approximation, uniformly distributed along the axis of the laser. However, when each section of the laser is operated at distinct points on the gain versus carrier density curve, the dynamics of each section becomes different. To clarify this assume that a modulating signal is applied to section ‘a’ with the lower gain while section ‘b’, the slave section, is biased independently providing most of the gain. Normally, if the laser operates above threshold, the sum of the optical gains in each section must equal the threshold gain of the entire laser cavity. Furthermore, the sum of the optical gains of each section must stay clamped. When a
small signal current is applied to section 'a' the gain in this section is driven around its quiescent point and hence, due to the gain clamping condition, the gain in section 'b' goes through an identical variation along the gain curve but in the opposite direction.

The crucial point is that, due to the non-linearity of the gain curve, while the variations of gain in each section are of the same magnitude, the corresponding variations in the carrier density of each laser section are not. The larger the ratio of the slopes of the gain curve at the operating points, the larger is the ratio of the amplitudes of the corresponding change in carrier density. In summary, the gain-lever effect is due to the fact that a variation in carrier density around the bias point in the modulation section commands a much larger variation in the number of carriers in the slave section. This is a direct consequence of the sub-linear gain profile as shown in figure 1.1 which can also be expressed as $G'_a \gg G'_b$.

Above the laser threshold, any fluctuation in the number of carriers inside the laser's active cavity is efficiently converted into a variation in the internal photon density and thus in the emitted signal power. The leverage effect enables a small variation in the number of carriers in the signal section to command a significantly larger variation of carriers in the slave section which in turn translates into a large amplitude of emitted optical power. Similarly, the large carrier variation also implies a large variation in the refractive index of the optical cavity. Therefore, the centre wavelength of the emitted light also undergoes a large change.

The gain-lever effect thus provides a mechanism by which a small variation in the signal current can command a comparatively large variation both of optical emitted power and its wavelength. As a consequence both the IM and FM responses of gain-levered lasers are expected to have a much larger magnitude than that of their single-section counterparts. Unfortunately, the carrier lifetimes in both laser sections play a part in the saturation of the gain-lever effect decreasing its effectiveness at high optical power [1].

In some applications of semiconductor lasers to signal transmission over optical fibres it is required that the optical emitted waveform is a replica of the modulating signal. At this point one may ask if the non-linearity of the gain in QW lasers can impair
the transmission of signals with fidelity. It could be suggested that the non-linear gain characteristic in QW lasers may increase signal distortion.

1.1.2 Increased modulation efficiency

Above threshold, any current injected into the laser is almost entirely converted into photons which leads to a rather steep relationship between the light output power and the injected current. The derivative of this power-current relationship is defined as the slope or modulation efficiency of the laser. The simple model of a gain-lever laser presented in section 1.1.1 provides an understanding of the mechanisms responsible for the modulation efficiency enhancement in those devices. In the following, both earlier claims and experimental results on the modulation efficiency in gain-lever lasers are reviewed.

The “normal” gain-lever configuration

Two different interesting modes of operation of the tandem-contact laser have been used to enhance the magnitude of both their IM and FM responses. The “normal” gain-lever configuration arises when the signal section is biased with a small injection current, while the slave section is heavily biased providing most of the optical gain. In this mode of operation the modulating signal is applied to the lightly biased section as shown in figure 1.1.

Earlier researchers claim that, in theory, improvements of up to 40 dB in the intensity modulation efficiency can be achieved with a single quantum well (SQW) gain-lever laser operating at a moderately low power with a resonance frequency of 2 GHz [2]. The increase in the modulation efficiency itself was shown to decrease with increasing laser power. At a much higher resonance frequency of 10 GHz the same researchers predict an IM efficiency improvement of around 15 dB with a short 100 μm laser. Experimentally, however, the largest IM efficiency improvement observed was around 23 dB, obtained with a SQW GaAs 220 μm long device at 1.5 mW [2].

A British Telecom research group measured a much smaller increase in IM efficiency of 10 dB in a InGaAsP multi-quantum-well (MQW) gain-lever laser at a front facet output power of 5.7 mW [3]. As expected, the enhancement of IM efficiency is a strong
function of the bias current injected into the signal section and increases as the bias current is decreased. An increase of 2.5 dB in the relative intensity noise (RIN) of the gain-lever laser relative to the noise in a equivalent but uniformly biased device, was also measured at the point of largest modulation enhancement. Both these measurements prove that the enhancement in the modulation efficiency is not off-set by a correspondent increase in noise. A SNR improvement of 7.5 dB can be estimated from this data.

The measured IM efficiency improvements of 23 dB and 10 dB in different gain-lever lasers are not easily comparable since they relate to lasers with different cavity lengths and operating at different optical powers. The IM enhancement is dependent on the gain derivative at the threshold gain, and thus on the laser length, and is inversely proportional to the optical power \[G_b^2\].

It follows from the discussion of the qualitative model of the operation of a gain-lever laser presented in section 1.1.1 that the “normal” configuration must also engender a large frequency deviation of the emitted wavelength [4]. This has also been experimentally confirmed by Lau [5, 6] and also by McDonald [7]. However, an ideal FM transmitter should be free of residual IM. This is the main reason for investigating an alternative mode of operation to the “normal” gain-lever configuration.

**The “inverted” gain-lever configuration**

The “inverted” gain-lever configuration [5] arises when the arrangement shown in figure 1.1 is reversed with the signal section providing most of the gain while the gain of the slave section is only slightly positive.

The intensity modulation efficiency is actually suppressed in the inverted gain-lever mode [5]. This is because in this situation \(G_b^2 \gg G_a^2\), a symmetrical inversion of parameters in relation to the former case. On the contrary, the frequency modulation (FM) response is still kept at a high value resulting in a higher FM/IM ratio than in the “normal” gain-lever configuration [5].

Using a SQW GaAs laser biased in this configuration, Lau et. al. measured a mid-band FM efficiency of 15 GHz \(\cdot\) mA\(^{-1}\) at 3 mW output power per facet [5]. The laser had mirror coatings with reflectivity of about 0.7, and the signal and slave sections were
120 \mu m and 400 \mu m long, respectively. The same authors demonstrated experimentally that by performing an interferometric FM to IM conversion on the emitted light, an effective IM modulation efficiency of 24 mW \cdot mA^{-1} could be achieved, equivalent to an enhancement of 34 dB over a uniformly biased laser [6]. However, the measured improvement in signal to noise ratio (SNR) was only a modest 8 dB, relative to that of a uniformly pumped device.

McDonald et. al. measured the FM response enhancement in the same tandem-contact MQW InGaAs/InP laser [7] used by Westbrook and Seltzer [8],[3]. With the short section (60 \mu m) biased at a low injection current of 1 mA a maximum 6.4 dB enhancement in IM response was observed at 2 mW optical power. The frequency of the modulating signal supplied to the short electrode was 900 MHz. At the same time, with the same arrangement, a FM response of 2.58 GHz \cdot mA^{-1} was observed corresponding to an enhancement of 11.4 dB over uniform pumping. These results correspond to a "normal" gain-lever configuration.

When the bias current to the short section is increased to 12 mA the IM response is actually suppressed by 2.3 dB and the measured FM response is −295 MHz \cdot mA^{-1} which represents an enhancement of 2 dB over uniform pumping. Clearly in these circumstances the device is operating in the inverted gain-lever mode. This value of the frequency deviation is interesting because its magnitude is much smaller than the 15 GHz \cdot mA^{-1} obtained by Lau with a SQW GaAs laser.

1.1.3 RF transmission over optical fibre

A laser can be seen as an interface between two different media in the transmission of RF signals over optical fibres. In figure 1.2 the air side of the interface, radio or microwave frequencies are received and modulate the laser. On the optical side of the interface the properties of the laser beam change in a way that reflects the modulating signal. During the electro-optical conversion process some noise is also added to the light due to spontaneous events occurring inside the laser.

The changing properties of the laser beam are the intensity of the light and its phase or optical frequency. The modulation response is the product of the laser's intrinsic and parasitics responses, namely an RC circuit composed of the lasers series resistance
Figure 1.2: The laser as an interface between an air region with RF waves and a region supporting light waves.

and parallel capacitance. The intrinsic response can extend beyond 20 GHz for bulk lasers. Even higher frequencies are attainable with QW structures if the input circuit parasitics that limit the overall response are minimised.

Microwave transmission losses via copper cable are huge compared with optical transmission losses. On the other hand, the size and weight of optical fibres facilitate RF and microwave transmission. Moreover, they are more economical than copper cable, possess enormous bandwidth and are immune to electromagnetic interference. Clearly, there is a case here for optical technology being applied to microwave systems such as radar or earth-station antenna remoting [9, 10], links for mobile communications [11, 12, 13, 14], phased array antennas and microwave delay lines.

However, even such microwave fibre-optic systems are far from ideal. A significant limiting factor on the exploitation of RF or microwave optical links for carrying electrical signals is the high RF throughput loss and noise figure associated with directly modulated optical links [15, 16, 17].

The RF throughput loss, or link gain, is defined as the ratio of signal power received and the signal power required to modulate the laser. There is abundant literature showing that large losses around 20 dB or more are usually incurred rather than a gain [17, 18]. A brief estimate illustrates this point rather well. Take typical figures both
for the laser modulation efficiency of $0.4 \text{W} \cdot \text{A}^{-1}$ and for the photodiode response of $0.9 \text{W} \cdot \text{A}^{-1}$, and a coupling coefficient of 50% between laser-fibre-photodiode. The resulting RF loss is 15 dB.

The second problem is the high noise figure (NF) of such an optical link. The link noise figure is defined as the ratio of the output and input SNR, and represents a measure of its fidelity. The NF of the RF optical link is determined, primarily, both by the noise generated in the transmitter and by its modulation efficiency or transducer gain. The SNR at the output of the optical link depicted in figure 1.3 is lower than the SNR at the input because the transmitter adds some noise of its own to the thermal noise of the input resistance. Let’s say that the laser transducer gain is $G_{tx}$. According to the definition of noise figure we obtain

$$NF = \frac{N_a}{N_r} = 1 + \frac{N_{exc}}{G_{tx}^2 i_{th}^2}$$

(1.1)

where $N_{exc}$ is the laser noise and $i_{th}$ is the thermal r.m.s. noise current generated by the input resistance.

The detector also contributes thermal noise. However, assuming that the received optical power is large enough (> 0.5 mW for a 50 Ω receiver) thermal noise at the receiver can be neglected when compared to the laser’s generated noise. In such a case the link noise figure is dominated by the noise figure of the transmitter given above. Taking $N_{exc} = -150 \text{dB}$, $G_{tx} = 0.5 \text{mW} \cdot \text{mA}^{-1}$ and $50 i_{th}^2 = -174 \text{dB}$, the resulting noise
figure is approximately 26 dB. It is important to realise that both fibre and coupling losses after the transmitter have no influence in the link noise figure.

Such a high noise figure degrades the SNR, impairing the quality of service. Usually, to counter this degradation a considerable amount of electrical low noise amplification is employed prior to transmission over the optical link. The cost, bulkiness and added complexity of this solution depend on the frequency span being transmitted over the system.

Conversely, if both the throughput loss and noise figure of the optical link are significantly reduced, then RF transmission over fibres would be simpler and less expensive, greatly expanding their applications.

One, amongst many scenarios that illustrates the possibilities opened up by improved optical links, is within the field of mobile communication. Base stations perform a variety of processing functions that often have to be located at costly private premises. It would be desirable to relocate this complex equipment at the operator's premises to keep the simplicity and cost of the radio-interface to a minimum. Using optical links, base stations antennas could be remoted from the processing equipment which would be relocated at the network operator's own premises, achieving a considerable reduction in cost. This demands a compact and simple electro-optical interface that can send the signals from the base antennas to the operator's premises.

1.1.4 Subcarrier multiplexing in optical links

Most often than not RF/microwave optical links depicted in figure 1.3 are used to transmit several microwave subcarriers at the same time. Each of these subcarriers can be modulated with digital or analogue information waveforms before modulating the optical carrier transmitted through the link. This technique is known as subcarrier-multiplexing (SCM) and is used in a range of applications such as CATV [19, 20, 21, 22], radar or earth-station antenna remoting [9, 10], video distribution [23, 24, 25, 26], links for mobile communications [11, 12, 13, 14, 27] and broadband distribution [28, 29, 30].

In these applications, both the link throughput gain and noise figure are influential parameters in the system performance [31]. Moreover, the non-linear devices within the system also play an important role as they invariably are a source of undesirable
interference or noise. Amongst these devices, the laser is a major concern, although not
the only one [32, 33, 34, 35], because of its intrinsic non-linearity [31, 36, 37, 21, 38, 39,
40]. The noise or interference arising from the non-linear elements within the system is
usually known as harmonic distortion and intermodulation distortion and their relative
importance depends on the system’s frequency span.

Potential for improved performance

The simple analysis of the optical link depicted in figure 1.3 has shown that a laser
transmitter with a larger transducer gain, $G_{T_x}$, yields a link with both a lower through­
put loss and lower noise figure, according to eqn. 1.1, assuming that the other factors
are constant.

Theoretical and experimental evidence described above suggests that gain-levered
lasers have a much enhanced IM efficiency without their noise being significantly in­
creased. Therefore, such a laser is a strong candidate for the transmission of RF signals
over an optical link.

Gain-levered lasers have shown even larger enhancements of their FM efficiency,
which raised the case for RF-optical links employing both optical FM and an optical
discriminator as a FM to IM converter. Earlier research claims that an “inverted”
gain-lever laser followed by a Fabry-Pérot filter (FPF) could achieve a RF throughput
gain of 25 dB and a noise figure of 10 dB [15]. The net link gain of 25 dB assumes an
optical loss of 10% and a photodiode quantum efficiency of 80%. The authors argue
that when applying optimal impedance matching both to the transmitter and receiver
side a further enhancement of 25 dB in the link gain is possible in principle. The noise
figure is improved by matching on the transmitter side only which can bring a maximum
reduction of typically 11 dB. As the authors have noted, these are very attractive figures
for an optical link [15].

However, before gain-levered lasers can be employed as transmitters in fibre optical
links their full performance needs to be assessed in order to evaluate the suitability of
the viable architectures to a specific application. For example, the linearity of these
lasers must be modelled and its non-linear effects assessed. The gain-lever effect, in
particular, rests on the non-linearity of the laser’s gain profile, hence it would seem
plausible that this could result in a more distortive transmitter. A theoretical investigation of the intermodulation distortion performance of a MQW InGaAs laser operating at 1.3 \( \mu \)m wavelength has been carried out at carrier frequencies close to 1 GHz relevant to GSM and CT2 radio mobile systems [8]. The results suggested that the enhancement in the IM efficiency of gain-lever lasers is accompanied by a degradation of their intermodulation performance when compared with conventional DFB lasers.

The latter architecture employing optical FM and a discriminator, particularly, poses some modelling problems which are anticipated. Firstly, modelling of the non-linearity of gain-levered lasers under frequency modulation is necessary. Secondly, the FM to IM conversion efficiency, influencing the link RF gain, is proportional to the slope of the Fabry-Pérot discriminator, but the discriminator is also a non-linear device to an extent also dependent on its slope. Furthermore, the output noise also depends crucially on the slope of the discriminator.

These preliminary considerations show that the overall assessment of the system performance is a complex task requiring an effective method for modelling both the laser’s and the discriminator’s non-linearity. Theoretical methods particularly suited to systems with an arbitrary large number of modulating signals have been successfully applied to electronics since long [41], and were recently being used to capture the effects of the laser intrinsic non-linearity [36, 37, 38]. Moreover, this method of Volterra series expansion is suited to deal with complex systems employing both optical frequency modulation of diode lasers and non-linear discriminators [42, 43]. These analytical tools were applied in the study of gain-lever laser transmitters as well as the more conventional DFB laser.

1.2 Scope and organisation of the Thesis

In addition to the Introduction this Thesis is organised into eight chapters and appendices.

In Chapter 2 some concepts of laser action are introduced which are relevant to the modelling of semiconductor lasers, including the QW laser.

In Chapter 3 a set of rate equations for the gain-lever laser is used to study both
their intensity modulation and noise characteristics. A new expression for the SNR of gain-lever lasers is derived that indicates ways of maximising it [44].

In Chapter 4 both the frequency modulation and the phase noise in the gain-lever laser are studied and a new expression describing their FM response is derived. The new expression is cast in a form that shows that the FM response in gain-lever lasers is composed of two parts: a single-segment contribution and a two-segment contribution which is responsible for a large enhancement of the FM response. Furthermore, the FM efficiency improvement in the gain-lever laser, relative to the single-segment contribution, is defined and expressions are derived both for this quantity and for the SNR of gain-lever lasers under optical frequency modulation. These expressions are used to estimate both quantities and to suggest arrangements in which they are maximised.

In Chapter 5 a set of rate equations for a DFB laser is used as a model for the laser's intrinsic non-linearity. The method of Volterra series is introduced and used to obtain a solution for the rate equations. These methods are adapted to complex systems employing both optical frequency modulation of lasers [42] and optical discriminators [43].

In Chapter 6 the non-linear distortion in gain-lever lasers is studied considering the influence of the bias conditions. Their performance is assessed and compared with that in DFB and single-section QW lasers.

In Chapter 7 the tools developed in the previous chapters are applied to evaluate the performance of a SCM fibre-radio system employing a gain-lever transmitter.

Finally, Chapter 8 summarises the main conclusions of the research presented in this Thesis and identifies areas of further research.

1.3 Contributions

The main contributions of this work can be summarised as follows:

- A new analytical expression describing the theoretical SNR improvement in an intensity modulated gain-lever laser.
Chapter 1. Introduction

- Development of analytical models for the non-linear chirp of lasers based on the Volterra series expansion method.

- Extension of the previous models to include both the chirp and FM→IM non-linear conversion by an optical discriminator. Analysis of harmonic and intermodulation distortion of DFB lasers and optical discriminator suggesting the theoretical possibility of cancellation of distortion between FM and IM components.

- Study of non-linear distortion in the QW gain-lever laser and comparison with that of a DFB laser.

The contributions made during the course of this research have led to the following publications to date:


Chapter 2

Semiconductor laser modelling

2.1 Introduction

IMPORTANT concepts of the theory of laser action are reviewed in this chapter, which are relevant to the modelling of semiconductor lasers.

The theory of both the gain and the principal recombination mechanisms is outlined. In particular, some important characteristics and theoretical results of the gain in QW materials extant in the literature are presented and compared with experimental gain data in a GRIN-SCH-SQW laser also used to demonstrate the gain-lever effect. A phenomenological model of the laser dynamics is introduced in the form of a set of rate equations for the creation and annihilation of photons and carriers inside the laser's active cavity. The mechanisms behind gain compression are examined insofar as they play an important role in the laser dynamics. Finally, a waveguide mode analysis of the GRIN-SCH-SQW laser is presented which shows their mode structure and justifies the mode confinement factor.

2.2 Concepts of lasing action

The interaction between photons and atomic systems can take place in three distinct ways. Firstly, a photon can be absorbed and the atom raised to an excited state where an electron now occupies a higher energy level. Secondly, the excited atom can drop back to its original energy level by spontaneously emitting a photon with the appropriate energy. Finally, and perhaps more interestingly, an excited atom can also be induced by a photon to emit another photon thereby lowering its energy. The latter
process is called stimulated emission of photons and is the basis of laser action. It is very important to distinguish it from the process of spontaneous emission. Basically, the stimulated emitted photons are identical in frequency and phase to the photon that induces the emission, whereas the spontaneously emitted photons do not keep the same properties.

A laser system consists of a gain material enclosed between mirror interfaces which provide some energy feedback by trapping some of the light between them. Semiconductor lasers have a complex band structure. These bands form a continuum of states which are partially occupied depending on the temperature and on the rate of carrier injection. Spontaneous emission, absorption and stimulated emission take place between electron and hole states in the conduction and valence bands, respectively.

The rates of occurrence of these events are related by the Einstein relation which states that the probabilities of absorption and stimulated emission events are equal, and also depend on the probability of occupation of the conduction and valence bands by electrons and holes. In thermo-dynamical equilibrium, the band occupation probability is determined by the Fermi-Dirac distribution function.

2.2.1 Optical gain in semiconductor lasers

According to the Fermi-Dirac statistics the energy levels most likely to be occupied have energies below the quasi-Fermi energy, $E_{fc}$ and $E_{fv}$. A material provides gain, or amplification, when there is a large number of carriers in the conduction band, a condition designated by population inversion. This situation does not occur naturally because in semiconductors at room temperature there are not enough free electrons and holes to fill the bands significantly and consequently the material is normally absorptive.

A net gain arises when the quasi-Fermi energy of the conduction and valence band lies above the energy of the electron and hole band states at a given light wavelength, or

$$E_{fc} + E_{fv} > E_c + E_v$$  \hspace{1cm} (2.1)

where $E_c$ and $E_v$ are the energy of an electron or hole state, respectively [45]. The injection of carriers into the laser's active region is a common way of achieving the
necessary population inversion. Whenever, eqn. 2.1 is satisfied, the rate of stimulated emission exceeds the rate of absorption and, consequently, the intensity of the light travelling in the laser's active region experiences a sustained increase with distance described by

\[ I(z) = I_0 e^{G(z)z} \]

where \( G(z) \) represents the local net gain of the laser medium.

### 2.2.2 The ultra-low-threshold SQW laser

The first measurements of the modulation characteristics of gain-lever lasers were performed on a GaAs, graded index, (GRIN), separate confinement heterostructure, (SCH), single quantum well, (SQW), laser [2]. This particular type of laser was being pursued due to its extremely low-threshold current capabilities that stem from the reduced size of its active region formed by the quantum well [46]. Quantum well lasers can have threshold currents one order of magnitude less than conventional lasers fabricated from bulk materials [45, 47].

**Technology**

The technology required for the fabrication of these lasers is quite advanced in order to make the thin (= 0.01 \( \mu \)m) layers where the quantum confinement takes place. This particular laser structure was grown by using the technique of molecular beam epitaxy (MBE). The following layers were grown on a n-doped GaAs substrate [46]. First, a 0.5 \( \mu \)m n-doped GaAs buffer layer with a 0.1 \( \mu \)m super-lattice buffer layer on top. A n-doped \( \text{Al}_{0.5}\text{Ga}_{1-0.5}\text{As} \) cladding layer 1.5 \( \mu \)m thick. A 0.2 \( \mu \)m graded n-doped \( \text{Al}_x\text{Ga}_{1-x}\text{As} \) barrier layer \((x = 0.2 - 0.5)\), a 0.01 \( \mu \)m un-doped GaAs QW, a 0.2 \( \mu \)m graded un-doped \( \text{Al}_x\text{Ga}_{1-x}\text{As} \) layer \((x = 0.5 - 0.2)\), and a 1.5 \( \mu \)m p-doped \( \text{Al}_{0.5}\text{Ga}_{1-0.5}\text{As} \) cladding layer. A schematic diagram of the layers in a GRIN-SQW laser is shown in figure 2.1.

Before the second stage of the growth process could take place, mesas 2 \( \mu \)m wide were etched through the MBE layers down to the substrate. Finally, a buried heterostructure (BH) was re-grown by a liquid phase epitaxy (LPE) technique. This consisted of a 1 \( \mu \)m
p-doped $\text{Al}_{0.3}\text{Ga}_{0.7}\text{As}$ layer and a 3 $\mu$m n-doped $\text{Al}_{0.3}\text{Ga}_{0.7}\text{As}$ layer. A shallow zinc diffusion was performed to facilitate the ohmic contact on the p-side of the device.

The second stage of re-growth, where a material of larger band-gap is used, is necessary to provide both good current confinement into the active region and lateral optical confinement for the laser light. The finished devices had an active layer stripe width of 1 $\mu$m and were cleaved to different lengths.

**Main characteristics**

The GRIN structure used in the ultra-low-threshold laser helps confining the optical mode in the thin QW active region where optical gain takes place. The GRIN barrier region has a parabolically graded-index profile which supports Hermite-Gaussian modes in the direction perpendicular to the junction plane. By making the enclosed QW gain region very thin and locating it at the centre of the waveguide, the optical gain of the higher-order transverse modes are significantly smaller in these lasers than in regular (non-graded) SCH lasers. This provides additional strong discrimination against higher-order transverse modes in these GRIN-SCH lasers. As a result, all the odd-order transverse modes are completely suppressed irrespective of the width of the waveguide, while all the even modes are strongly discriminated because of the very much reduced mode gain [48].

Furthermore, it is also recognised that this GRIN region can act as a “funnel”, enhancing the capture of the injected carriers by the quantum well, and further reducing
the threshold current of the laser [49, 50]. It has been argued that the enhanced confinement of the carriers takes the form of a spatial distribution of electrons in which more of them will be localised in the vicinity of the quantum well layer [51].

**Intrinsic optical losses in the laser cavity**

The optical losses in a semiconductor laser can be represented by \[ \alpha = \Gamma \alpha_{fc} + (1 - \Gamma)\alpha_c + \alpha_s + \alpha_{cp} \] (2.3) where \( \alpha_{fc} \) denotes the free carrier absorption coefficient in the gain media, \( \alpha_c \) is the average absorption coefficient in the wave-guiding layers outside the gain media, \( \alpha_s \) is the optical scattering loss due to irregularities at the hetero-interfaces and lateral waveguide boundaries, and \( \alpha_{cp} \) is the coupling loss when the optical field spreads beyond the \( \text{Al}_x\text{Ga}_{1-x}\text{As} \) cladding layers which usually is negligible when the \( \text{Al}_x\text{Ga}_{1-x}\text{As} \) cladding layers are thick (2 \( \mu \text{m} \)).

Arguably, the thick cladding layers with high aluminium concentration, and the low doping of the waveguide layers, make it reasonable to assume that \( \alpha_c \) and \( \alpha_{cp} \) are each nearly zero in eqn. 2.3, [52].

The confinement factor in the GRIN-SCH with a well width of 0.01 \( \mu \text{m} \) is about 0.035 or one-tenth of that in a bulk laser. Since \( \Gamma \alpha_{fc} \) is proportional to the confinement factor and the free-carrier absorption loss of a conventional DH laser is typically 20 cm\(^{-1} \), the estimated free-carrier absorption loss \( \alpha_{fc} \) is 2 cm\(^{-1} \).

Furthermore, MBE-grown materials achieve better interface optical quality than that of a LPE-grown material which typically has a scattering loss of 10 cm\(^{-1} \) [52]. Therefore, the intrinsic loss of the QW laser is also expected to be low and this as been confirmed by Lau et. al. who measured an intrinsic loss of 3 – 9 cm\(^{-1} \) and a free-carrier absorption loss \( \Gamma \alpha_{fc} \) of 1 – 2 cm\(^{-1} \) in several GRIN-SQW lasers [45].

### 2.2.3 Gain in QW materials

Quantum well lasers have some distinctive characteristics when compared to bulk lasers. These peculiarities are mainly a consequence of the restriction of the movement of the carriers to the thin active QW region on the order of 0.01 \( \mu \text{m} \). Figure 2.2 illustrates
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0.01 μm

Figure 2.2: Energy level diagram of a semiconductor QW in the GaAs/AlGaAs material system. Confined energy levels are shown for electron (e), heavy-hole (hh) and light-hole (lh).

the concept of the quantum well energy-level problem in the GRIN-SCH laser. The movement of the carriers is restricted in one direction by the energy barriers due to the GaAs/AlGaAs material interfaces which causes the splitting of the band structure. The quantised sub-bands are separated by an energy equal to the difference between the energy levels of a particle in a potential well as depicted in figure 2.2.

Another distinctive feature of QW lasers is that the density of states in a sub-band is independent of the energy. Hence, the "stair-case" shape of the density of states as depicted in figure 2.3 in contrast to that in bulk materials. Consequently, the gain from a QW laser saturates as the carrier population in the conduction band increases, because of the limited number of available states in a given sub-band. Extra gain is still available from the next up energy sub-band, however, this implies a jump to a shorter emission wavelength.

The spectral gain, $G(E)$, of a QW laser material can be given by [53]

$$G(E) = \frac{\hbar e^2}{\epsilon_0 m^*_0 c n r} \int_{-\infty}^{\infty} d\varepsilon \sum_{n,j,l,h} \rho_{rj}(\varepsilon) M_j^2(\varepsilon) \frac{\hbar}{\pi r} \frac{f_{en}(\varepsilon) - f_{jn}(\varepsilon)}{(E - \varepsilon)^2 + \left(\frac{\hbar}{\pi r}\right)^2}$$

(2.4)

where $E$ is the photon energy, $\rho_{rj}$ is the reduced density of states, $M_j^2$ is the square of the dipole matrix element and the Fermi occupation probabilities are $f_{en}$ and $f_{jn}$. The
constants in eqn. 2.4 are the Planck’s constant $\hbar$, the electronic charge $e$, the speed of light $c$, the permittivity of free space $\varepsilon_0$, the electron rest mass $m_0$ and the effective refractive index of the medium $n_r$. The convolution integral in eqn. 2.4 accounts for the Lorentzian line shape broadening of the electron-hole transitions due to the intra-band electron collisions, with a time constant, $\tau$, typically around 0.1 ps [54].

In order to obtain $G(E)$ it is necessary to calculate the quasi-Fermi energy of both the electrons and holes, which determine the occupation factors, $f_{en}$ and $f_{jn}$. The quasi-Fermi energy of the conduction band can be found by inversion of the following expression [53, 55]

$$n = \left(\frac{4\pi k_B T}{h^2}\right) \sum_n m^*_e \ln \left[1 + \exp \left(\frac{E_{en} - E_n}{k_BT}\right)\right]$$

(2.5)

where $n$ represent the injected electron sheet density into the quantum well and $m^*_e$ is the carrier effective mass in the quantum well. A similar expression exists for holes. Normally, the QW remains absorptive until the rate of injection of carriers is sufficient to achieve a certain carrier density that makes it become transparent. The transparency carrier density, i.e. the carrier density at which the material exhibits zero gain, can be calculated from eqn. 2.1 and eqn. 2.5 [45].
2.2.4 Spontaneous recombination

In order to sustain gain in the laser material a constant current must be injected into the active region. In equilibrium conditions the injected current must balance the rate of carrier recombination due to both radiative and non-radiative processes.

Below the laser threshold, the radiative component of carrier recombination is found from the spectral spontaneous emission rate, averaged over all directions of emission in space. The total spontaneous emission rate is given by

\[ R_{sp} = \frac{8\pi^2 n^2 e^2 E_g}{m_0^2 h^2 c^3} \int_{E_g}^{\infty} |M|_{ave}^2 \sum_{n} \sum_{j=l,h} \rho_j(\epsilon) f_{cn}(\epsilon) f_{jn}(\epsilon) d\epsilon \]  

(2.6)

where \( E_g \) is the bandgap energy and \( |M|_{ave}^2 \) is the square of the dipole moment averaged with respect to all directions in space and all polarisations of the spontaneously emitted light. Once \( R_{sp} \) is calculated, a spontaneous recombination lifetime can be defined as

\[ \tau = \frac{n}{R_{sp}} \]  

(2.7)

When the injected carrier density is low, such that the Fermi-Dirac distribution can be safely approximated by the Boltzman distribution, reference [56] shows that \( R_{sp} \) is proportional to \( n^2 \), both for QW and bulk lasers. Consequently, it is customary to define an effective bimolecular recombination coefficient \( B \) in the following way

\[ R_{sp} = Bn^2 \]  

(2.8)

which shows that the spontaneous lifetime can also be given by

\[ \tau = \frac{1}{Bn} \]  

(2.9)

2.2.5 Non-radiative recombination

The non-radiative components of the carrier recombination are due to unconfined carriers in the barriers of the active region and to the Auger recombination process [53, 55]. The leakage current out of the SCH is relatively insignificant in comparison to the other contributions [53].

The Auger recombination processes involve four particle states (i.e. three electron and one hole states, two electron and two hole states, and so forth). Among the various
possibilities, the two electrons and one hole process (CHCC) and the one electron and
two hole (CHHS) process, are usually considered and each one leads to a different time
constant [53]. The combined lifetime for the Auger process may be written as

\[ \frac{1}{\tau_A} = C_1 np + C_2 p^2 \]  

(2.10)

where \( C_1 \) is the Auger recombination constant for the CHCC process, and \( C_2 \) is the
equivalent constant for the CHHS process. In GaAs these constants were calculated as
\( C_1 = 4.72 \cdot 10^{-30} \text{ cm}^6\text{s}^{-1} \) and \( C_2 = 0.64 \cdot 10^{-30} \text{ cm}^6\text{s}^{-1} \). The rate for the non-radiative
Auger recombination can be given by

\[ R_A = C_1 n^2 p + C_2 np^2 \]  

(2.11)

and if the impurity concentration is small compared to the injected carrier density, both
processes behave as \( C n^3 \) [57], since in this case \( p = n \).

2.2.6 Differential gain

The differential gain or the gain derivative with respect to the carrier density is a very
important quantity. It determines the frequency response of lasers, their modulation
bandwidth and pulse response. It also plays a central role in the theory of gain-lever
lasers.

It follows from the above theory that the gain derivative can be written as [58]

\[ G' = \frac{dG}{dn} = \frac{dG}{dE_{fc}} \frac{dE_{fc}}{dn} \]  

(2.12)

which highlights the importance of the rate of change of the Fermi-energy with the
injected carrier density in determining the gain derivative.

A theoretical calculation of both the gain and differential gain in the GRIN-SCH-
SQW laser is carried out in [54] where it is argued that the gain just above transparency
is identical to the value found for bulk double-heterostructure (DH) lasers and not larger
as had been suggested. Figure 2.4 plots the differential gain versus the modal gain both
for the GRIN-SQW and DH bulk laser as calculated in [54].

The cause of a lower than expected gain derivative is attributed to state-filling
[59], which consists of a significant occupation of electron states in the GRIN confining
region at room temperature. These electrons do not contribute to the gain because they lie much above the bandgap energy. However, both the differential gain and the transparency carrier density depend on the rate of change of the electron Fermi-level with the injected carrier density. In the presence of state-filling, the Fermi-level is less affected by these electrons than if they were only to populate the lower energy QW states, thus lowering the differential gain as indicated by eqn. 2.12. It is also shown in [59] that neglecting the state-filling of the GRIN sub-bands also leads to an underestimation of the transparency carrier density.

The abrupt increase in the differential gain in the QW, shown in figure 2.4 occurring around 80 cm$^{-1}$ is due to the onset of gain from the second quantised states of the QW as depicted in figure 2.3.

### 2.2.7 Optical gain measurement

One of the most widely used techniques to measure optical gain is the Hakki-Paoli technique [60]. In this measurement method the gain is obtained from the Fabry-Pérot resonances appearing in the laser emission spectrum below threshold. The output spectrum of the laser can be observed by scanning the light emerging from the laser facet. This light consists of spontaneously emitted photons coupled into the laser waveguide,
and undergoes successive partial reflections at the facet mirrors before being emitted. The transmission of the Fabry-Pérot cavity is given by

\[ T = \frac{(1 - R)^2 e^{\alpha L}}{(1 - R e^{\alpha L})^2 + 4 R e^{\alpha L} \sin^2(2\pi n r L / \lambda)} \]  

(2.13)

where \( g \) is the net gain experienced by the guided mode which takes into account internal optical losses and the fact that only a small fraction \( \Gamma \) of the light intensity travels in the active layer. The ratio, \( r \), of the maximum transmission at a crest over the minimum transmission at the neighbouring valley in the output spectrum, is given by

\[ r = \frac{T_{\text{max}}}{T_{\text{min}}} = \left[ \frac{1 + R e^{\alpha L}}{1 - R e^{\alpha L}} \right]^2 \]  

(2.14)

and hence the net gain \( g \) is

\[ g = \frac{1}{L} \left[ \ln \left( \frac{r - 1}{r + 1} \right) - \ln R \right] \]  

(2.15)

This theory assumes that the scanning of the observed spectrum is realized with an infinitely narrow passband. However, using monochromators at their best resolutions carries the inconvenience of reducing the experimental signal-to-noise ratio. On the contrary, opening up the monochromator slit increases the sensitivity, but at the expense of averaging errors in the apparent \( T_{\text{max}}/T_{\text{min}} \) ratio. This limitation in the sensitivity of the gain measurements is more serious when measuring small gain values.

### 2.2.8 Experimental SQW gain

The optical gain in GRIN-SCH-SQW lasers has been calculated from the emission spectrum below the lasing threshold by the method described in section 2.2.7 [45]. This measurement method gives the value of the net gain (eqn. 2.15), which is equal to the intrinsic gain of the laser material minus both the internal losses (eqn.2.3) and the mirror losses. Therefore, both these parameters had also to be measured independently by measuring the differential quantum efficiency of several lasers under different reflectivity coatings.

In the actual measurement, several lasers with cavity lengths of 120 \( \mu \)m, 200 \( \mu \)m, and 250 \( \mu \)m, with different mirror coatings were utilised. The data for the intrinsic gain
Figure 2.5: Measured (dots) intrinsic optical gain versus injected current density [45] and curve fit to the data (solid curve). The estimated gain derivative is plotted in the right-hand-side

as a function of the injected current for each device were obtained and, subsequently, the three sets of data were re-plotted to give the normalised gain curve for this type of laser.

Figure 2.5 shows the measured (dots) intrinsic optical gain as a function of the current density injected into the laser [45]. The solid curve represents an empirical model that was fitted to the data. Above the gain of 35 cm$^{-1}$ the curve follows a dependence given by $(\sqrt{j} - \sqrt{j_0})^{1/0.45}$, and below that value it is represented by a polynomial of third degree in the current.

The derivative of the gain versus the injected current is also shown in the figure (right) and was analytically calculated from the curve fit to the original data.

A significant feature of the measured gain-current curve is its highly sub-linear characteristic which is also noticed in the theoretical differential gain plot in figure 2.4. This feature is a direct consequence of the limited number of carrier states in any given quantum well energy level (figure 2.3), so that the available gain it can provide eventually saturates [45].

The dependence of the optical gain with the injected carrier density is also plotted in figure 2.6. This relationship was calculated from the original gain versus current data depicted in figure 2.5 by assuming a carrier bimolecular recombination mechanism of the type given by eqn. 2.8, which is typical of QW lasers [45, 61]; i.e. assuming that
the current density \( j = e L_z B n^2 \), where \( L_z \) is the thickness of the quantum well. With this model for the injected current versus carrier density, the gain dependence with carrier density, above \( 35 \) cm\(^{-1} \) can be represented approximately by \( G \propto \sqrt{n - n_0} \), in agreement with [62]. The gain derivative versus carrier density is also shown in the right-hand-side of figure 2.6.

The theoretical differential gain calculation whose results are depicted in figure 2.4 correspond to the same type of GRIN-SQW laser that was used to measure the gain depicted in figure 2.6 [54]. However, the theoretical differential gain calculation was carried out at peak gain [54], whereas the experimental gain (and thus its derivative) was measured at a constant wavelength. These two quantities are not exactly equivalent because the photon energy at which the maximum gain occurs shifts to higher energy with increasing injection.

The value for the differential gain at transparency (zero gain) obtained from figure 2.6 is \( G' = 1.4 \cdot 10^{-5} \text{ cm s}^{-1} \), which converted to the units used in figure 2.4 by dividing by the group velocity gives \( G' = 1.63 \cdot 10^{-15} \text{ cm}^2 \), a factor of 4 larger than the corresponding theoretical value read in figure 2.4. At a gain of \( 80 \) cm\(^{-1} \) the theoretical and experimental values for the differential gain are similar. This difference results from the different definitions of the differential gain used in the theoretical calculation and in the gain measurement as referred above. The present situation is similar to what occurs in bulk lasers. Here, the peak optical gain is commonly described by a linear
dependence in the injected carrier density, above the transparency point. However, this is not the case for a fixed emission wavelength for which the gain versus carrier density relationship becomes slightly sub-linear [63]. Clearly, the derivative of gain with carrier density is constant, if calculated at peak gain. However, if the gain derivative is calculated from the gain at a constant wavelength, its value depends on the carrier density. It follows that some degree of leverage may also be expected using bulk tandem-contact lasers [64], since it is the gain derivatives at the emitting wavelength in each section that are represented in figure 1.1.

2.3 Rate equation model

The dynamic response of lasers is often modelled by a pair of rate equations given by

\[
\frac{dp}{dt} = p \left( \Gamma G - \frac{1}{\tau_p} \right) + \beta R_{sp}
\]

\[
\frac{dn}{dt} = D\nabla^2 n + \frac{j}{ed} - R(n)
\]

which describe the time evolution of both the average photon density, \( p \), and carrier density, \( n \), in response to an injected current, \( j \). These equations provide a useful phenomenological model that is extensively used in this thesis.

Close examination of the rate equations reveals that they can be interpreted as a book-keeping of the supply, annihilation and creation of photons and charge carriers during their interactions. The first term inside the brackets in eqn. 2.16, given by \( Gp \) represents the rate of stimulated emission and \( \tau_p^{-1} \) is the photon decay rate due to both cavity and mirror losses

\[
\tau_p^{-1} = v_g \left[ \alpha + \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right) \right]
\]

where \( L \) is the cavity length and \( R_1 \) and \( R_2 \) are the facet reflectivities. The term, proportional to \( R_{sp} \) represents the photon rate of spontaneous emission. The parameter \( \beta \) is the spontaneous emission coupling factor which depends on the geometry of the laser cavity and guiding mechanisms. Its typical value for strongly guided lasers is around \( 10^{-4} \). The confinement factor \( \Gamma \) is the fraction of photons that travel in the active cavity.
The first term in eqn. 2.17 accounts for the carrier diffusion with a diffusion coefficient \( D \). The importance of this term depends on the particular geometry of the laser. In strongly guided lasers, such as the GRIN-SQW laser and BH laser structures in general, where the active region dimensions are small compared to the diffusion length, \( \sim 40 \mu m \), the diffusion term can be ignored thereby considerably simplifying the analysis. The second term describes the injection rate of carriers into the active region through, most often, external direct current injection. The last term \( R(n) \) encompasses the annihilation rate of carriers through several processes of both radiative and non-radiative recombination

\[
R(n) = An + Bn^2 + Cn^3 + Gp
\]  

(2.19)

where the constants \( A, B \) and \( C \) represent time constants associated with the various recombination processes. This model assumes that the time constants are independent of the pumping rate. However, at best this is only an approximation to the truth. For example, the spontaneous recombination in QW lasers follows the law \( Bn^2 \) but this is not expected to hold for very high injection levels [45, 56]. The last term in eqn. 2.19 is the rate of generation of photons by stimulated emission due to the stimulated recombinatoin of carriers. It is the product of the photon density and the rate of stimulated emission or gain, \( G \), given by eqn. 2.4. The rate of stimulated emission is also related to the gain coefficient by

\[
G = \frac{c}{n_g} g
\]  

(2.20)

where \( g \) is the gain per unit length as depicted in figure 2.6 and \( n_g \) is the group refractive index.

The rate equations have been cautioned in a number of occasions [65, 66, 67, 68]. One of the criticisms is that they describe the time evolution of the average number of photons and carriers, but in reality, those particles are not uniformly distributed inside the laser's cavity. However, for mirror reflectivities larger than 0.2 it was found that the spatial distribution is sufficiently uniform to justify the rate equation model [69]. The facets of semiconductor lasers have a reflectivity of around 0.3 and fall well within
this limit.

**Phase rate equation**

The rate equations describe the variation of both the carrier and photon density with time, as a result of a modulation of the current injected into the laser's active region. It also happens that the carrier density variations induce a variation in the optical gain which of necessity translates into a proportional variation in the refractive index of the cavity. The constant of proportionality between gain and index variations is known as the linewidth enhancement factor, \( \alpha \), defined as

\[
\alpha = -2k_0 \frac{\Delta G}{\Delta \mu}
\]

where \( k_0 \) is the light wavenumber and \( \mu \) is the refractive index. The linewidth enhancement factor is an important property of semiconductor lasers playing a large role in the theory of the width of their emission spectrum [70]. Moreover, a deviation in the cavity refractive index affects the optical frequency of the emitted light. It has been shown that this sequence of events linking the optical frequency deviation of the mode to the optical gain, is described by the following equation [63, 70]

\[
\Delta \phi = \frac{1}{2} \alpha \left( G - \frac{1}{\tau_p} \right)
\]

which can be interpreted as a rate equation for the frequency deviation of the longitudinal light mode. It is often used to describe the frequency modulation characteristics of lasers, in conjunction with the other rate equations that describe the gain variations caused by a modulating current [63].

### 2.3.1 Gain compression in lasers

The gain compression factor, \( \epsilon \), is an attempt to condense in one parameter the effects of a variety of physical phenomena that lead to the experimental observation of saturation of gain with power [71], which can be described by

\[
G_s = (1 - \epsilon p) G
\]
Chapter 2. Semiconductor laser modelling

The above mathematical expression of gain saturation was first derived by Tucker [72] by considering the effects of a lateral non-uniform carrier distribution inside a laser cavity with width \( w \). He argued that given the uneven profile of carrier distribution they must diffuse across the width of the laser and showed that this resulted in an effectively reduced electron lifetime. Mathematically, it followed that the effect of the lateral carrier diffusion and the effective electron lifetime was equivalent to a set of rate equations without the lateral spatial variation but with a non-linear gain expressed by eqn. 2.23, with \( \varepsilon \) given by

\[
\varepsilon = \frac{\gamma\tau_n}{2\left[1 + \left(\frac{2\pi L_{\text{eff}}}{w}\right)^2\right]} \tag{2.24}
\]

where \( L_{\text{eff}} \) is the effective carrier diffusion length.

The gain suppression due to both spectral hole-burning and spatial hole burning also contribute to the non-linear gain. Spectral hole-burning results from a depletion of carriers corresponding to the energy of the optical transition at the emitting wavelength. The rate of recombination of carriers fuelling the light emission increases at high optical power, depleting the carrier population. As a result a hole is burnt in the gain profile, effectively reducing the gain slightly at the emitting wavelength. Petermann [73] has shown that the effects of the gain suppression due to spectral hole-burning can be taken into account in the rate equations by using a non-linear gain, as given by eqn. 2.23, where \( \varepsilon \) has a larger value than the one required to account for lateral carrier diffusion.

The physical origins of the gain saturation have also been attributed to other different causes such as two-photon absorption, induced dielectric grating and carrier density dependent dispersion [74, 75]. However, none of these processes can alone explain the large gain saturation observed experimentally [74]. Gain saturation in lasers has been the subject of further investigation [76, 75]. Reference [74] points out that, in every model, the non-linear gain is a function of the local photon density. Since the photon density is not uniform along the laser cavity, the question arises as to what is the effective non-linear gain that should appear in the rate-equation. Reference [74] concludes that this large scale longitudinal photon density distribution has a negligible impact on the gain saturation of semiconductor lasers.
Figure 2.7: Large signal response of a laser to a current pulse showing damping of the relaxation oscillations. The pulse response is shown for two values of $\varepsilon = 1 \cdot 10^{-17} \text{ cm}^3$ (broken line) and $\varepsilon = 2.5 \cdot 10^{-17} \text{ cm}^3$ (solid line).

**Dynamic effects of the compression parameter**

Although the non-linear gain is quite a small effect it has important consequences for the dynamics of lasers as shown by a numerical solution of the rate equations. Figure 2.7 plots the response of a laser to a current pulse. The pulse response shows some oscillations which have a frequency equal to the resonance frequency of the laser. The height and damping of the resonance is controlled both by the spontaneous emission and compression parameters. Experimental results have shown that the coupling factor $\beta$ alone, can not account for the observed damping behaviour [71]. Above threshold the gain compression dominates the damping and thus spontaneous emission is often neglected. Gain compression also plays an important role in the modelling of the effects of the laser's intrinsic non-linearity as will be discussed in chapter 5 [42].

**Gain compression in QW lasers**

It has been suggested that gain compression in QW lasers might be enhanced by the carrier confinement in the QW [77] and also by a mechanism of well barrier hole-burning [78]. In an attempt to explain both the experimentally observed damping and
modulation speed in QW lasers, Rideout et. al. proposed a dynamic model for the laser with one extra rate equation [79]. Their model did not include a gain compression factor explicitly. However, the authors reduced the set of three equations to two by the inclusion of a gain compression term of the form given by eqn. 2.23. In the process of this reduction they derived an expression for $\epsilon$, and called this additional contribution to the overall $\epsilon$ due to well barrier hole-burning. It was predicted that the enhancement in $\epsilon$ would degrade the $K$ factor thus explaining why QW lasers fell short of attaining earlier expectations for their modulation speed: the $K$ factor is the proportionality constant between the damping rate $\gamma$ and the resonance frequency $f_r$ [77]. The maximum possible intrinsic modulation speed is determined by this factor $f_m = \sqrt{2\pi}/K$ and thus $K$ is often taken as a figure of merit for the high speed lasers.

Reference [80], using also an extra rate equation for the carriers crossing the barrier region of GRIN-SCH lasers, shows both theoretically and experimentally that carrier transport across the barriers degrades the $K$ factor via a reduction in the apparent differential gain, $G'$, and not by an enhancement in the gain compression factor, $\epsilon$, as was claimed in [79]. Furthermore, reference [78] argues that the degradation of modulation bandwidth in GRIN-SCH-SQW GaAs lasers is due to state filling rather than well barrier hole burning. State filling is the occupation of the lower energy sates of the barrier region. This means that a significant number of electrons are located in the barrier just outside the QW region. References [81, 59] provide further evidence of the connection between state filling and the reduction of modulation bandwidth.

Reference [82] demonstrates that a value of $\epsilon = 1.0 \cdot 10^{-17}$ cm$^3$ similar to that of bulk lasers, and a gain compression factor of the form given by eqn. 2.23, suffices to obtain good agreement with the experimentally measured modulation bandwidth of GRIN-SCH-MQW InGaAs lasers.

### 2.4 Waveguide modes in the GRIN-SQW laser

The GRIN barrier surrounding the QW active region acts as a “funnel” for the injected carriers enhancing the electrical properties of the laser. Similarly, the AlGaAs alloy making the buried heterostructure also plays an important role in reducing leakage
currents.

Both these structures provide also the physical mechanism guiding the optical modes propagating in the QW active region, thus determining important laser parameters such as the confinement factor, $\Gamma$, and the group velocity, $v_g$. The confinement factor is especially important because the optical gain, $G$, is obtained from the experimental modal gain divided by $\Gamma$.

The estimation of these laser parameters requires the propagating mode solutions for the two-dimensional waveguide problem. The exact solution to this problem is difficult. However, a good approximation to the exact solution can be found by assuming that the guiding in the orthogonal directions can be considered separately [83, 84]. This assumption is justified if the refractive index varies much more slowly along with the $x$ coordinate than with $y$ [85]. Under this approximation the two-dimensional waveguide problem can be reduced to two one-dimensional problems in the transverse and lateral directions relative to the propagation axis.

Since the laser is built as a buried heterostructure, there is a strong index guiding in the lateral direction provided by the index step of the AlGaAs alloy adjacent to the sides of the GaAs QW. This structure provides a mode guiding mechanism in the lateral direction identical to the dielectric slab problem [63].

The GRIN barrier is made of $\text{Al}_x\text{Ga}_{1-x}\text{As}$ where the percentage of Al in the alloy is progressively varied in order to obtain the desired refractive index variation. In the case of the ultra-low threshold GRIN-SQW laser a parabolic profile was chosen.

The laser structure can be viewed for the purpose of mode analysis as a core region surrounded by a dielectric barrier with a parabolic decreasing index in the transverse direction. The dimension of the QW is only a small fraction of the barrier dimension, 0.4 $\mu$m, and thus it is neglected. In the lateral direction the structure can be seen as a core region formed by the QW sandwiched between cladding regions with a lower index.

### 2.4.1 Mode solution

Here, a solution to this problem is proposed, with the intention to model both the confinement factor, $\Gamma$ and the group velocity, $v_g$, in the waveguide. First, a solution for the slab problem, along the $y$ coordinate, is found. Then, the complementary problem
of the GRIN barrier with a parabolic index profile is solved in the $x$ coordinate.

**Lateral modes**

The solution to the one-dimensional slab problem is obtained by solving the scalar Helmholtz equation, simultaneously, in the core region with a refractive index $n_2$, and in the cladding with a refractive index $n_1$, with $n_2 > n_1$.

$$\nabla_y^2 \phi(y) + (n_1^2 k_0^2 - k_e) \phi(y) = 0$$

$$\nabla_y^2 \phi(y) + (n_2^2 k_0^2 - k_e) \phi(y) = 0$$

(2.25)

where $\phi(y)$ is the electric field amplitude in the $y$ direction, $k$ and $\gamma$ are the wave-numbers in the core and cladding regions, respectively, and $k_e$ is the effective propagation constant. The solution of the “eigen-value” problem for the even TE modes is given by

$$\phi = A_e \cos(ky)$$

$$\phi = B_e e^{-\gamma |y - \frac{d}{2}|}$$

(2.26)

The requirement of continuity both of the electric field and its first derivative at one interface is sufficient to eliminate the constants $A_e$ and $B_e$ and gives the eigenvalue equation also known as the guiding condition

$$\gamma = k \tan(kd/2)$$

(2.27)

Furthermore, the wave-numbers $k$ (core) and $\gamma$ (cladding) must obey the following equations simultaneously, as found from substituting the solutions for $\phi(y)$ back in eqn. 2.25

$$k = k_0 \sqrt{\left(n_2^2 - n_e^2\right)}$$

$$\gamma = k_0 \sqrt{\left(n_e^2 - n_1^2\right)}$$

(2.28)

which results in a further condition

$$k^2 + \gamma^2 = k_0^2 \left(n_2^2 - n_1^2\right)$$

(2.29)
Combining both eqn. 2.27 and eqn. 2.29 in one equation, gives

$$\tan \frac{kd}{2} = \sqrt{\left(\frac{\Delta kd}{2} \right)^2 - \left(\frac{kd}{2} \right)^2}$$

(2.30)

where $\Delta k = k_0 \sqrt{(n_2^2 - n_1^2)}$. The solutions to this transcendental equation can be found by a graphical method. The right-hand side of eqn. 2.30 represents a circle with a radius $\Delta k$, while the left-hand side are the branches of the tangent as a function of $kd/2$. The intersections between both lines provide the solutions of the effective wave-number $k_e$ for all even TE modes.

The number of modes increases as $\Delta k$ increases since the circle becomes larger and intercepts more branches of the tangent function. Therefore the number of modes in the waveguide increases with a larger index gap between the core and the cladding regions.

**Transverse modes**

The solution of the one dimensional waveguide problem in the $x$ coordinate is obtained by solving the scalar Helmholtz equation with a parabolic index profile

$$\nabla_x^2 \psi(x) + \left( k_0^2 n_b^2 \left[ 1 - \left( \frac{x}{x_n} \right)^2 \right] - k^2 - \beta^2 \right) \psi(x) = 0$$

(2.31)

where $\psi(x)$ is the electric field amplitude in the $x$ direction, $k$ is determined by the eigenvalue equation for the slab problem and $\beta$ is the propagation constant along the waveguide. The core region is neglected since its thickness is negligible compared to the GRIN barrier. The validity of this approximation finds support in that the transverse confinement factor is found to be rather independent of the QW thickness [50].

The solution for the $x$ dependence of the electric field is given by the Hermite-Gauss functions [48, 49]

$$\Psi(x) = H_m \left( \frac{\sqrt{2x}}{w_0} \right) e^{-\left( \frac{x}{w_0} \right)^2}$$

(2.32)

where $H_m$ are the Hermite polynomials [50, 86] and $w_0$ is the beam radius

$$w_0 = \sqrt{\frac{2x_n}{n_b k_0}}$$

(2.33)

$$x_n = \frac{x_c}{\sqrt{1 - \left( \frac{w_1}{w_2} \right)^2}}$$

(2.34)
where \( x_c = 0.2 \, \text{\(\mu\)m} \) is the width of the GRIN barrier on either side of the active region, and \( n_b = 3.5 \) and \( n_c = 3.3 \) are, respectively, the maximum and minimum refractive indexes of the parabolic profile. With these values the parameter \( x_n = 0.6 \, \text{\(\mu\)m} \) and \( w_0 = 0.22 \, \text{\(\mu\)m} \).

The first Hermite polynomial is \( H_0 = 1 \), thus the fundamental mode of the electric field has a purely Gaussian profile in the \( x \) coordinate, as in eqn. 2.32.

**Propagation constant and group velocity**

The solution for the fundamental TE propagating mode in the core region of the waveguide in the \( z \) direction is given by

\[
E = \bar{E} \cos(ky) e^{-\left(\frac{x}{x_0}\right)^2} e^{-i\beta z}
\]

(2.35)

The propagation constant \( \beta \) is given by the dispersion relation

\[
\beta = \sqrt{\omega \mu - k^2 - (2m + 1)k_x}
\]

(2.36)

where \( \epsilon \) and \( \mu \) are parameters of the core region and \((2m + 1)k_x\) is the eigenvalue of eqn. 2.31, with \( k_x = \sqrt{2}/w_0 \), the group velocity being calculated from eqn. 2.36.

**Confinement factor**

The transverse confinement factor is defined by eqn. 2.37 which, for the fundamental transverse mode with a Gaussian profile is approximately equal to [50]

\[
\Gamma_T = \frac{\int_{-d/2}^{d/2} E^2(x)dx}{\int_{-\infty}^{\infty} E^2(x)dx} \approx \sqrt{\frac{2}{\pi}} \frac{d}{w_0}
\]

(2.37)

Using the calculated value for \( \omega_0 \) and the value \( d = 0.01 \, \text{\(\mu\)m} \) for the thickness of the QW active region, the transverse confinement factor \( \Gamma_T \) is 0.0369.

The lateral confinement factor for the fundamental transverse mode can be given by the simple expression

\[
\Gamma_L = \frac{\int_{-w/2}^{w/2} E^2(y)dy}{\int_{-\infty}^{\infty} E^2(y)dy} \approx \frac{D^2}{2 + D^2}
\]

(2.38)

which was found to be accurate to within 1.5% [63], where \( D \) is the normalised waveguide thickness. For a laser emitting at 0.85 \( \text{\(\mu\)m} \) wavelength and a stripe width of \( w \) of
1 \mu m, D \simeq 6.16 and \Gamma_L \simeq 0.95.

2.5 Summary

Important concepts of the theory of semiconductor lasers were introduced in this chapter, with an emphasis on QW lasers. Experimentally measured gain in GRIN-SQW lasers was presented and compared with theoretical gain calculations carried out for the same type of device. The principal recombination mechanisms in QW lasers were also outlined. A simple bimolecular carrier recombination model in a QW laser was adopted to transform the experimental gain-current data into a gain versus carrier density relationship which can be used to estimate the differential gain.

A set of rate equations was introduced as a model for the dynamical behaviour of the laser. The necessity of using a non-linear gain model in the rate equations was made clear. Finally, a simple analysis of the waveguide mode solutions in a GRIN-SQW laser was outlined that enabled the estimation of the mode confinement factor. The intrinsic gain used in the rate equations is obtained from the experimental modal gain by dividing by the confinement factor. All these issues are relevant to the modelling of lasers, which will be pursued in the subsequent chapters.
Chapter 3

Intensity modulation and noise in the gain-lever laser

3.1 Introduction

Earlier theoretical and experimental work on gain-lever lasers shows that these tandem-contact devices deliver significant improvements both in the intensity modulation efficiency and in the signal to noise ratio (SNR) compared to their single-section counterparts. This chapter builds on the earlier theory of the “gain-lever effect” and pursues it along two main vectors.

Firstly, the theoretical projections of the earlier researchers are analysed from this writer’s point of view. The intensity modulation efficiency improvement (IMEI) in the gain-lever laser is plotted in a realistic situation. The pattern of the relationship between the IMEI and the laser sections’ length emerges from these plots.

Secondly, the theory of the SNR in the gain-lever laser is developed by deducting a new expression for that quantity. The choice of the section’s relative length, $h$, that optimises both these quantities is made clear, at variance with earlier work.

3.2 Gain-lever laser dynamics

The intensity modulation properties of the gain-lever laser can be studied using the rate equation model governing the time evolution of both the photon and the carrier densities in both laser sections. An adequate set of equations for the tandem-contact SQW laser was proposed in reference [2] in order to model its modulation response.
This set of rate-equations is given by

\[ \frac{dp}{dt} = p\Gamma \left[ (1 - h)G_a + hG_b - \frac{1}{\Gamma \tau_p} \right] \] (3.1)

\[ \frac{dn_a}{dt} = \frac{j_a}{ed} - Bn_a^2 - G_ap \] (3.2)

\[ \frac{dn_b}{dt} = \frac{j_b}{ed} - Bn_b^2 - G_bp \] (3.3)

where \( p, n_a, \) and \( n_b, \) are the photon density inside the laser cavity and the carrier densities in each laser section, respectively. The \( j_a \) and \( j_b \) are the currents injected into the signal and slave sections, respectively, \( h \) is the fractional length of the slave section and \( \Gamma \) is a confinement factor. \( B \) is the bimolecular radiative coefficient, related to the spontaneous radiative recombination of carriers in the conduction and valence bands of the QW. The thickness of the active region is \( d \) and \( \tau_p \) is the photon lifetime. \( G_a \) and \( G_b \) represent the QW material gain in each section as a function of the carrier density and has been discussed in section 2.2.1.

It should be noticed that this set of rate equations does not include a gain compression factor. This suits the purpose of this chapter where, mainly, static characteristics of the laser are studied in which the phenomena of gain compression is unimportant. A non-linear gain will be introduced in these equations for the study of the non-linear characteristics of the laser in subsequent chapters. In the following the theory leading to the modulation efficiency of gain-lever lasers is outlined.

It follows from the steady-state solution of eqn. 3.1, in a laser operating above threshold, that the sum of the weighted gains in each section must equal the threshold gain \( G_0 \) of the entire optical cavity [2]

\[ (1 - h)G_{ao} + hG_{bo} = G_0 = \frac{1}{\Gamma \tau_p} \] (3.4)

This equation is known as the threshold condition.

To calculate the modulation response, a small-signal analysis (see section 5.4.2) of the rate equations is required. This analysis produces the following expression for the intensity modulation (IM) response in the gain-lever laser

\[ \frac{p}{j_a} = H_1(\omega) = \frac{p_0 \Gamma G_a(i\omega + \gamma_a)(1 - h)/ed}{-i\omega^3 - (\gamma_a + \gamma_b)\omega^2 + i\omega A_1 + A_2} \] (3.5)
Chapter 3. *Intensity modulation and noise in the gain-lever laser*

where $G'$ is the differential gain, $\omega$ is the modulation frequency and $A_1$ and $A_2$ are given by

$$A_1 = p_0 \Gamma [(1 - h)G'_a G_a + hG'_b G_b] + \gamma_a \gamma_b$$  \hspace{0.5cm} (3.6)$$

$$A_2 = p_0 \Gamma [(1 - h)G'_a G_a \gamma_b + hG'_b G_b \gamma_a]$$  \hspace{0.5cm} (3.7)$$

and the inverse stimulated lifetimes $\gamma_a$ and $\gamma_b$ are given by

$$\gamma_{a,b} = \frac{1}{\tau_{a,b}} + G'_{a,b} p_0$$  \hspace{0.5cm} (3.8)$$

where the second term is the stimulated part of the carrier lifetime and the first term is the inverse carrier lifetime

$$\frac{1}{\tau_{a,b}} = 2Bn_{a,b}$$  \hspace{0.5cm} (3.9)$$

The same analysis also produces the carrier density responses $F_a(\omega)$ and $F_b(\omega)$ in each section of the tandem-contact laser

$$F_a(\omega) = -\frac{G_a H_1(\omega) - j_a/ed}{i\omega + \gamma_a}$$  \hspace{0.5cm} (3.10)$$

$$F_b(\omega) = -\frac{G_b H_1(\omega)}{i\omega + \gamma_b}$$  \hspace{0.5cm} (3.11)$$

The carrier density response is the variation in the carrier density due to an injected sinusoidal current density with frequency $\omega$ and is used in the study of the laser chirp pursued in chapter 4.

When driven by a square pulse, the laser's output power exhibits damped periodic oscillations before settling down to its steady-state value. Such relaxation oscillations are due to an intrinsic resonance between the photons and the electrons. The angular frequency of this relaxation oscillation is given by

$$\omega_r^2 = p_0 \Gamma [(1 - h)G'_a G_a + hG'_b G_b]$$  \hspace{0.5cm} (3.12)$$

The IM response per unit current is obtained from 3.5 by multiplying the current density $j_a$ by the signal section area. The static IM response is calculated as

$$\frac{p}{j_a} = \frac{G'_a \gamma_b / eV}{(1 - h)G'_a G_a \gamma_b + hG'_b G_b \gamma_a}$$  \hspace{0.5cm} (3.13)$$

where $V$ is the active region volume. It will be necessary to compare some of the
quantities in the tandem-contact laser to the correspondent quantities in a physically equivalent single section device. The IM response of the single-section laser can be obtained from 3.5 by allowing both the signal and slave sections to become one single uniformly pumped optical cavity. In this case both the differential gain $G'_a$ and $G'_b$ and the lifetimes $\gamma_a$ and $\gamma_b$ are equal throughout the entire optical cavity and the gains $G_a$ and $G_b$ are equal to the threshold gain $G_0$. Consequently, the static IM response in the single-section laser is given by

$$\frac{p}{i} = \frac{1}{G_0 e V}$$

(3.14)

and its resonance frequency is

$$\omega_u^2 = \Gamma \frac{p_0 G_0 G'_0}{\tau_p}$$

(3.15)

### 3.3 IM efficiency improvement in gain-lever lasers

The intensity modulation efficiency improvement (IMEI) in the gain-lever laser is defined as the ratio of the IM response in the gain-lever laser and the IM response in the uniformly pumped laser. Using the expressions for the static modulation response, eqn. 3.5 and eqn. 3.14, the IMEI is given by (see appendix A)

$$\eta = \frac{p}{i} = \frac{\gamma_b}{(1 - h) \gamma_a R + \left(\frac{h \gamma_a}{g}\right) Q}$$

(3.16)

which agrees with Moore et. al. [2], where $R = G_a/G_0$, $Q = G_b/G_0$, and $g = G'_a/G'_b$. The quantities $R$ and $Q$ are themselves related via the equation

$$Q = [1 - (1 - h) R] / h$$

(3.17)

which follows from the threshold condition 3.4. The modulation efficiency $\eta$ is normalised to that of a uniformly pumped device: when $h = 0$, then $R = 1$ and $Q = 0$, resulting in $\eta = 1$.

To gain further insight into eqn. 3.16 consider cases near $h = 1$ when the slave section of the laser occupies the majority of the laser cavity. In this case, eqn. 3.16 can...
be simplified to

\[ \eta = \frac{\gamma_b G'_a}{\gamma_a G'_b} = \frac{1}{\tau_a} + \omega_r^2 \tau_p g \]  

(3.18)

For a laser operating at a small resonance frequency \( \omega_r \), which implies low-power operation according to 3.15 (below about 1 mW), the carrier spontaneous lifetimes dominate the ratio in 3.18 and the efficiency improvement becomes

\[ \eta = \frac{\tau_a G'_a}{\tau_b G'_b} \]  

(3.19)

However, when the laser is operated at a high resonance frequency, or high output power, the ratio in 3.18 becomes dominated by the terms containing \( \omega_r \) and expression 3.18 approaches a different limit, given by

\[ \lim_{\omega_r \to \infty} \frac{\gamma_a G'_a}{\gamma_b G'_b} = \frac{\omega_r^2 \tau_p}{g \omega_r^2 \tau_p} g = 1 \]  

(3.20)

and the IMEI is reduced to one (no improvement at all) [2].

Appendix A demonstrates that for any two given points on the gain curve in figure 1.1, the maximum IMEI is obtained by designing \( h \to 1 \). The IMEI is given by eqn. 3.18 and is directly proportional both to the ratio \( g = G'_a/G'_b \) and to the ratio of the inverse lifetimes. In order to maximise \( g \) the gain curve must exhibit a highly sub-linear shape. Furthermore, section ‘a’ should be biased at a low carrier density where the gain derivative \( G'_a \) is larger, whereas section ‘b’ should be biased towards the high carrier density region where the gain derivative \( G'_b \) is smaller. This also ensures a large ratio of carrier densities which in turn maximises the ratio \( \tau_a/\tau_b \), in 3.19. The penalty, compared to the maximum IMEI, incurred by making \( h < 1 \) is given by

\[ \eta_{Pen} = 20 \log \left[ \frac{(1 - h) + h \frac{G'_b}{G'_a}}{(1 - h) + h \frac{1}{\eta_{h=1}}} \right] \frac{1}{\eta_{h=1}} \]  

(3.21)

where \( \eta_{h=1} \) is the magnitude of the IMEI when \( h = 1 \) (appendix B).

Figure 3.1 is reproduced from [2] and plots \( \eta \), given by eqn. 3.16, against the slave’s section relative length for various values of the parameter \( g \) and of resonance frequency (or output power). It is assumed throughout the plot, that the signal section is biased at 10% of the laser’s threshold gain, \( G_0 \). Based on the plot in figure 3.1 it is claimed that, in theory, a short gain-lever laser could achieve a 40 dB enhancement of its modulation
efficiency [2]. However, it is difficult to accept this plot as conclusive justification for such a claim for the following reasons.

Firstly, eqn 3.16 is plotted in figure 3.1 under speculative conditions which may never be met in any real device. To grasp this, consider the plot where $h = 1$. When $h \to 1$, $G_b \to G_0$ (threshold gain) and thus $n_b$ becomes equal to the laser’s threshold carrier density. Furthermore, both from eqn. 3.19 and eqn. 3.9 it can be concluded that, for the curve marked $f_r = 2$ GHz and $g = 1$, $n_b/n_a \approx 10^{28/20}$ or $n_b \approx 25n_a$, at least. Moreover, because the curve refers to a situation where the optical power is nonzero, the carrier density $n_b$ must be even larger than $25n_a$, in order that $\eta$ may equal the indicated 28 dB. Since the signal section has to be biased above transparency which in a QW laser means that $n_a \approx 2 \cdot 10^{18}$ cm$^{-3}$, it becomes apparent that the plot is contemplating a laser with a threshold carrier density larger than $50 \cdot 10^{18}$ cm$^{-3}$. There can be little doubt that such a device will ever be fabricated because it wouldn’t be very useful. Similar criticism is applicable to the other curves in the plot including the one that indicates $\eta = 40$ dB where $h = 1$.

Secondly, the sole purpose of the plot in figure 3.1 is to show that $\eta$ can potentially be large especially if $g$ is large and the emitted power is relatively small. Otherwise, the pattern of the dependence of the IMEI ($\eta$) on the geometrical parameter $h$ for a laser

Figure 3.1: IM efficiency improvement versus $h$ in a gain-lever laser with 100 μm cavity length, at resonance frequencies of 2 GHz, 5 GHz, and 10 GHz. For each case three values of $g = 1, 5, 20$ are shown. (Reproduced from [2]).
with threshold $G_0$, can not be what is depicted in figure 3.1. It is worth clarifying this point before the appropriate plot is presented. Consider any given curve in figure 3.1 with constant $g$ and $G_a/G_0 = 0.1$. When $h$ is varied with $G_a$ also kept constant throughout the plot, $G_b$ has to change in order to satisfy the threshold condition, eqn. 3.4. But if $G_b$ changes, so, by necessity, does $G'_b$ and thus $g = G'_a/G'_b$. This shows that the curves in figure 3.1, for constant $g$ while varying $h$, are not compatible with a constant value of $G_0$. Therefore these curves can not represent data describing the real relationship between $\eta$ and $h$ for a particular laser with a given threshold $G_0$.

### 3.3.1 Variation of the IMEI versus $h$

Both sections of the tandem-contact laser can be independently biased by injecting current into each section separately. With the device operating above threshold and once the bias currents are settled, the quiescent value of the internal variables $G_a(n_a), G_b(n_b), p_0$ and the two bias currents are interrelated by the rate equations with the time derivatives equal to zero. The static version of eqn. 3.2 and eqn. 3.3 is given by

\begin{align*}
    j_a &= \left[ Bn_a^2 + G_a(n_a)p_0 \right] ed \\
    j_b &= \left[ Bn_b^2 + G_b(n_b)p_0 \right] ed
\end{align*}

and determine $p_0$ and one of the values of $G_a$ (or $G_b$). At the same time, the other gain $G_b$ (or $G_a$) is given by the threshold condition, eqn. 3.4, which follows from eqn. 3.1. Therefore, a valid bias situation for the laser can be devised by choosing both a specific $G_a$ for the signal section and any given photon density. The gain $G_b$ follows necessarily from the threshold condition and, finally, the bias currents $j_a$ and $j_b$ can be calculated from eqn. 3.22 and eqn. 3.23 (note that the gain $G(n)$ univocally determines $n$).

Figure 3.2 plots the magnitude of the IMEI as a function of the slave's section relative length $h$, for a laser with length 220 $\mu$m and threshold gain $G_0 = 55$ cm$^{-1}$. The signal section is biased at a fraction $R$ of the threshold gain, while $G_b$ changes to accommodate the threshold condition as $h$ is swept. The power $p_0$ is assumed constant throughout the plot. Note that the plot describes data about a particular laser with its electrical contact split at various length ratios. It is considered the same laser, despite
of the changing split ratio of its electrical contacts, because the threshold gain is kept constant. Moreover, the gain curve for the GRIN-SCH-SQW GaAs laser presented in figure 2.6 is used to calculate both the differential gains and carrier lifetimes in both sections of the laser.

![Diagram of intensity modulation efficiency improvement](image)

Figure 3.2: IM efficiency improvement versus $h$ in a gain-lever laser with 220 $\mu$m cavity length, at 1.5 mW and 4 mW for two values of $R = 0.1$ (-----) and $R = 0.2$ (------). The resonance frequency is 3.5 GHz and 5.7 GHz, respectively, for $h = 0.65$ and $R = 0.2$.

This plot, unlike the one in figure 3.1, clearly indicates that in order to maximise the IMEI the signal section must be lightly biased and the slave section should be designed as short as possible. It must be emphasised that this statement does not contradict an earlier one stating that for any two given points on the gain curve, $\eta$ is maximised when $h \rightarrow 1$. Failure to distinguish between these two situations seems to be the cause of error, at least in one occasion [3] which is cautioned both in [44] and also in section 3.5.

The trend shown in the curves in figure 3.2 is explained as follows. Given that $G_a$ is constant, the gain in the slave section, $G_b$, must increase as its relative length, $h$, is reduced, to ensure that the entire cavity reaches the threshold gain. At the same time, the ratio $g = G'_a/G'_b$ also increases due to the sub-linear shape of the gain curve, hence delivering an increasing IMEI as shown in the plot.

However, there are other factors playing against the design of a very short slave
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Figure 3.3: Static light-current characteristic in a gain-lever laser as a function of bias current $I_{b0}$ into the slave section, Left: for $h = 0.5$. The curves shown correspond to bias currents into the signal section from 1.5 mA to 10.5 mA in steps of 1 mA (from bottom to top). Right: for $h = 0.87$. The curves shown correspond to bias currents into the signal section $I_{a0}$ from 1 mA to 10 mA in steps of 1 mA (from bottom to top).

In the present arrangement, as the slave’s section length is reduced its gain must increase rapidly thus requiring an exceedingly large bias current. It is impossible to exceed a few kA/cm$^2$ of bias current without damaging the laser, and, accordingly, the plot is interrupted at $h = 0.5$.

The trend shown in figure 3.2 is indirectly confirmed by a plot of the static relationship between the injected current into the laser and its light output power. Figure 3.3 shows the static light-current plot of a gain-lever laser. These curves are related with the laser’s static IM response and confirm the trends observed in the plots of figure 3.2. The light-current plot shows the laser output power as a function of the bias current injected into the laser. Mathematically, the plots are obtained by solving the set of rate equations with the time varying terms equal to zero. An additional term has also been added to the right-hand side of eqn. 3.1 to account for spontaneous emission (otherwise negligible above threshold). The plots were obtained for a range of values of bias current injected into the laser’s signal section and the current going to the slave section was then ramped up. This was done for two different values of the parameter $h$.

When modulating the signal section one is effectively moving from one curve to another at constant $I_{b0}$. These curves are furthest apart just above threshold for low $I_{a0}$. Therefore, in this region of operation, the magnitude of the IMEI is greatest. On the contrary, for high optical power, or a larger $I_{a0}$, the plots are closer to one another.
indicating that the IMEI is reduced. It is also worth noting that the distances between plots on the vertical scale are uneven, indicative of signal distortion. These observations apply to both plots in figure 3.3 for $h = 0.5$ and $h = 0.87$. However, comparing both plots it is clear that for the case $h = 0.5$ the curves are more widely apart than the ones for the case $h = 0.86$. This clearly indicates that the IMEI is larger for $h = 0.5$ which confirms the situation depicted earlier in figure 3.2. The distortion in the gain-lever laser will be analysed in chapter 6.

### 3.4 Noise in the gain-lever laser

The gain-lever effect enhances the modulation efficiency of tandem-contact lasers. However, to be really significant, the enhancement in noise should not match the former signal enhancement. The intensity noise characteristics of gain-lever lasers have been theoretically and experimentally investigated [62, 3, 6]. The present research of this matter provides a different point of view that is both simpler and achieves more insight than the earlier one.

Spontaneous emission of photons causes fluctuations around the mean intensity of the laser mode. This fluctuation is known as relative intensity noise (RIN) and is defined as the auto-correlation of the photon density $\langle \delta p(t + \tau) \delta p(t) \rangle$. The random nature of the spontaneous emission process means that the photon output is a random variable. The evolution of this variable with time is a continuous stochastic process with spectral density given by the Fourier transform of the photon auto-correlation, given by

$$S_p(\omega) = \int_{-\infty}^{\infty} \langle \delta p(t + \tau) \delta p(t) \rangle e^{-i\omega \tau} \, d\tau$$  \hspace{1cm} (3.24)

The photon spontaneous emission process is usually modelled by a set of stochastic differential rate equations including a Langevin driving force $\Delta_p(t)$ representing the process of spontaneous emission. Here, this method is also used to study the noise characteristics in the tandem-contact laser. The rate equations describing the intensity noise are given by [62]

$$\frac{\delta p}{dt} = p \left[ (1 - h) \Gamma_Ga + h \Gamma_Gb - \frac{1}{\tau_p} \right] + \Delta_p(t)$$  \hspace{1cm} (3.25)
It should be recognised, by comparing the noise rate equations to eqns. 3.1–3.3, that the two sets of equations are similar. The main difference is that the driving force in the noise equations is $\Delta_p(t)$ whereas in the signal equations it is the input current signal. The present analysis of signal-to-noise ratio in the gain-lever laser is influenced by this similarity, unlike the analysis of reference [62]. Appendix C shows that, in the frequency domain, the photon density fluctuation is given by

$$\tilde{H}(\omega) = \frac{\Delta_p(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)}{i\omega^3 - (\gamma_a + \gamma_b)\omega^2 + i\omega A_1 + A_2}$$

and the carrier density fluctuations are

$$\tilde{F}_a(\omega) = -\frac{G_a\tilde{H}(\omega)}{i\omega + \gamma_a}$$

$$\tilde{F}_b(\omega) = -\frac{G_b\tilde{H}(\omega)}{i\omega + \gamma_b}$$

The RIN in the gain-lever laser is defined as

$$\text{RIN} = \frac{S_p(\omega)}{\tilde{p}_0^2}$$

and, as shown in appendix C, can be given by the following expression

$$\text{RIN} = \frac{R_{sp}}{\tilde{p}_0} \frac{(\gamma_a^2 + \omega^2)(\gamma_b^2 + \omega^2)}{\omega^2 [\omega^2 - (\omega_f^2 + \gamma_a\gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]}$$

This fresh derivation of the RIN produces an equation which is somewhat different from a similar one given by Gajic et. al. [62]. Their expression for the RIN differs from eqn. 3.32 in that the term $A_2$ (defined in eqn. 3.7) is replaced by an approximation

$$A_2 \approx \omega_f^2 ((1 - h)\gamma_b + h\gamma_a)$$

The right hand side of eqn. 3.33 is numerically identical to $A_2$ when the gain $G(n) = \sqrt{n - n_0}$, because in this case $G_aG'_a = G_bG'_b$ for any two points ‘a’ and ‘b’ on the gain curve. However, the experimental gain curve plotted in figure 2.5 deviates, if only slightly, from the square-root dependence, especially in the region just above trans-
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Moreover, by adhering to the accurate eqn. 3.32 it is possible to obtain a useful result [44] (section 3.5) which is lost once Gajic’s approximation is introduced.

3.5 SNR improvement in the gain-lever laser

Here, a simple expression that describes the SNR improvement in the gain-lever laser is derived, which leads to a different conclusion to that suggested in an earlier work [62]. An alternative equation to the static IMEI given by eqn. 3.16 is (see appendix A)

\[ \eta = \frac{G_a \gamma \omega^2}{G_u A_2} \]  

(3.34)

where \( \omega_u \) is the resonance frequency of the uniformly pumped laser as defined in eqn. 3.15. The subscript \( j = a, b, u \) stands for section ‘a’, section ‘b’ and an uniformly pumped laser respectively. \( A_2 \) is defined in eqn. 3.7 and its value does not contribute to this discussion.

The RIN of the gain-lever laser, normalised to that of a uniformly pumped device, was also calculated from the noise rate equations in section 3.4. The ratio \( N \) between the gain-lever and uniformly pumped laser RIN, at d.c., is given by

\[ N = \frac{R_{sp} \gamma_a \gamma_u^2 \omega^4}{R_{sp} \gamma_u^2 A_2^2} \]  

(3.35)

where the \( R_{sp} \) and \( R_{sp}^u \) are the rate of spontaneous emission in each device respectively.

As indicated in eqn. 3.35 the rate of spontaneous emission in the gain-lever laser, \( R_{sp} \), is different to that in an equivalent uniformly pumped device, \( R_{sp}^u \). This is due to the difference in the respective carrier densities.

The improvement in SNR (electrical) of the gain-lever laser, relative to the uniformly pumped laser, can be defined as

\[ SNR_i = \left( \frac{S}{N} \right)_g \left( \frac{N}{S} \right)_u \equiv \frac{\eta^2}{N} \]  

(3.36)

and, using eqn. 3.32 and eqn. 3.35, it is calculated as

\[ SNR_i = \frac{R_{sp}^u}{R_{sp}} \left( \frac{\gamma_a G_u^s}{\gamma_u G_u^s} \right)^2 \equiv A_r A_i \]  

(3.37)

where \( A_r = R_{sp}^u/R_{sp} \) and \( A_i = \left( \frac{\gamma_u G_u^s}{\gamma_a G_a^s} \right)^2 \).
The improvement in SNR resulting from the gain-lever effect is given by \( A_i \) alone. Its value is independent of the relative length \( h \) of the laser's sections. This apparently differs from earlier work by Gajic et al. [62]. The authors claim that the SNR improvement in a gain-lever laser is largest for \( h = 1 \) because the noise increase is minimised and \( \eta \) is maximised for \( h = 1 \). However, the last statement is only true for two fixed bias points in the gain curve (appendix D).

On the contrary, assuming that the gain of the signal section is fixed, then, as \( h \) varies, the gain of the slave section must change to obey the threshold condition. Figure 3.2 plots the modulation improvement \( \eta \) in such a scenario. In fact \( \eta \) increases with decreasing \( h \). As it happens the relative noise in the gain-lever laser also increases with decreasing \( h \). The combined result is that \( A_i \) is independent of \( h \) as shown in eqn. 3.37.

The factor \( A_r \) merely causes a reduction of the SNR improvement as discussed at this point. The rate of spontaneous emission \( R_{sp} \) into the lasing mode is given by [5, 6]

\[
R_{sp} = \beta [(1 - h)R_n(n_a) + hR_n(n_b)]
\] (3.38)

where \( \beta \) is the fraction of spontaneous emission coupling into the laser mode and \( R_n(n_{a,b}) \) are the spontaneous carrier recombination rates in each section. In a QW laser, they can be given by \( Bn_{a,b}^2 \) where \( B \) is a bimolecular recombination coefficient and \( n_{a,b} \) are the carrier densities.

Using the definition in eqn. 3.38 the \( R_{sp}^a/R_{sp} \) ratio, \( A_r \), is written as

\[
A_r = \frac{R_{sp}^a}{R_{sp}} = \frac{n_a^2}{(1 - h)n_a^2 + hn_b^2}
\] (3.39)

When \( h \to 1, n_b \to n_a \) and \( A_r \) approaches unity. When \( h \to 0, n_b \) has to increase so that eqn. 3.4 can be satisfied.

As a consequence, the rate of spontaneous emission \( R_{sp} \) of the gain-lever laser increases and \( A_r \) decreases, resulting in the SNR improvement decreasing with \( h \) as depicted in figure 3.4. It must be emphasised that this dependence on \( h \) has no connection with the gain-lever effect which is described by the factor \( A_i \) only.

The SNR improvement is plotted in figure 3.4 for two values of the photon density.
(proportional to the optical power). The signal section is biased at 10% and 20% of the laser's threshold gain. The maximum value of the SNR improvement (occurring at $h = 1$) is given by $A_i$. $A_i$ approaches unity with increasing photon density, because the stimulated part of the carrier inverse lifetimes $\gamma_a$ and $\gamma_b$ becomes dominant and cancels out in eqn. 3.37. Moreover, biasing the signal section at a low-gain level increases the gain derivative and therefore also the SNR improvement.

Figure 3.4: SNR improvement versus $h$ in a gain-lever laser with 220 $\mu$m cavity length, at 1.5 mW and 4 mW for two values of $R = 0.1$ (-----) and $R = 0.2$ (----). The resonance frequency is 3.5 GHz and 5.7 GHz, respectively, for $h = 0.65$ and $R = 0.2$.

3.5.1 Change of variables

Gajic et al. measured the SNR improvement in a GRIN-SCH-SQW GaAs gain-lever laser of the same type described in section 2.2.2 [62]. Unfortunately, neither the length nor the width of the laser used in this experiment are given. An increase in SNR of over 10 dB was measured at a power output of 1.5 mW. Since the area of the laser is not known it is impossible to relate the power to the internal photon density, which makes the comparison with theoretical results difficult because eqn. 3.37 is strongly dependent on the optical power.

A further hindrance, although of a different order, arises because eqn. 3.18 is depen-
dent both on the values of $G'(n)$ and of $\gamma(n)$ at specific bias points. Hence the accuracy of the plots in figure 3.2 and figure 3.4 depends on the accuracy of the gain versus carrier density data in figure 2.6. However, the experimentally measurable quantity is $G(J)$ as a function of the excitation current $J$ as given in figure 2.5. Although it is possible to measure the inverse lifetime $\gamma$ as a function of the injected current density this is not a simple task. Such a measurement could, in principle, be used to calculate the functional dependence of the carrier recombination rate, $R(n)$, on the carrier density, over a large range of currents [57]. Finally, this empirical model for $R(n)$ could be used to infer $G(n)$, and thus $G'(n)$, from the measured $G(J)$ data.

Lacking an empirical model for $R(n)$, the differential gain $G'(n)$ is estimated both from the measured data $G(J)$ and from the theoretical model for the recombination current $J = Bn^2ed$, where the value of the constant $B = 2 \cdot 10^{-10}$ s cm$^{-6}$ was picked from the literature [56]. However, this theoretical carrier recombination model, $R(n) = Bn^2$, although adequate for QW lasers, is not expected to hold over a large range of carrier densities such as one may encounter in gain-lever lasers. Therefore, it is worth to enquire if these approximations significantly affect the accuracy of both the IMEI and SNR improvement plots in figure 3.2 and figure 3.4. This is carried out, by finding valid alternative expressions both to eqn. 3.16 and to eqn. 3.37, for $\eta$ and $SNR_t$, respectively, that only depend on directly measurable quantities.

The relation between the injected current density $J$ and the carrier density $n$ inside the laser cavity, below threshold, is expressed by $J = edR(n)$ where $R(n)$ is the total rate of carrier recombination. Furthermore, the inverse of the differential lifetime $\tau$ is defined as

$$\frac{1}{\tau} = \frac{dR(n)}{dn} \quad (3.40)$$

Therefore, the gain derivative with respect to the carrier density can be given by

$$\frac{dG}{dn} = \frac{v_g dg_m dJ}{\Gamma} = \frac{v_g ed}{\Gamma} \frac{1}{\tau} \frac{dg_m}{dJ} \quad (3.41)$$

where $v_g$ is the group velocity and $g_m$ is the modal gain as a function of the injected current density. Using eqn. 3.41 the $SNR_t$ can be alternatively expressed as (see ap-
\[ SNR_i = A_r \left[ \frac{1 + K \frac{d\phi_s}{dJ}}{1 + K \frac{d\phi_s}{dJ}} \right]^2 \]  

(3.42)

where

\[ A_r = \frac{\beta_{sp} \eta_{isp} R(n_u)}{\beta_{sp} [(1-h)\eta_{isp} R(n_a) + h\eta_{isp} R(n_b)]} \approx \frac{J_u}{(1-h)J_a + hJ_b} \]  

(3.43)

\[ K = \frac{e}{\hbar \nu A \alpha_m} \]  

(3.44)

where \( \frac{dg}{dJ} \) are the derivatives of the modal gain with respect to the current density, calculated at given bias points compatible with eqn. 3.4. The area of the active cavity is \( A \), \( P_o \) is the output power in Watts, \( \alpha_m \) is the facet mirror loss and \( \nu \) is the emitted light wavelength. \( J_{a,b,u} \) are the corresponding current densities at those bias points as obtained from the measured gain versus injected current density curve, shown in figure 2.5. The expression for \( A_r \) implicitly contains the approximation that the internal spontaneous quantum efficiency, \( \eta_{isp} \), is similar in both sections of the gain-lever laser and in the uniformly pumped laser. This approximation is expected to hold reasonably well except for very high carrier densities where the Auger recombination may become significant in which case the value of \( A_r \) as given by eqn. 3.43 is overestimated (i.e. \( A_r \) is too small).

Likewise, the same procedure can be applied to eqn. 3.16, producing an alternative expression for the IMEI

\[ \eta = \left[ (1-h)R + hQ \left( 1 + K \frac{d\phi_s}{dJ} \right) \right]^{-1} \]  

(3.45)

and if \( h = 1 \)

\[ \eta = \frac{\left( 1 + K \frac{d\phi_s}{dJ} \right) \frac{d\phi_s}{dJ}}{\left( 1 + K \frac{d\phi_s}{dJ} \right) \frac{d\phi_s}{dJ}} \]  

(3.46)

Both eqn. 3.42 and eqn. 3.45 have a considerable advantage over their previous counterparts: both the derivatives \( \frac{dg_m}{dJ} \) and \( A_r \) can, in principle, be estimated from measured data for the modal gain versus injected current density (e.g. figure 2.5).

Figure 3.5 plots the IMEI using the model provided by eqn. 3.45. The estimated resonance frequencies for the 400 \( \mu \)m long laser are around 3.3 GHz and 5.3 GHz at
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Figure 3.5: IMEI versus $h$ at 1.5 mW and 4 mW for two values of $R = 0.1$ (---) and $R = 0.2$ (-----). A 220 μm cavity length on the left and a 400 μm cavity length on the right.

1.5 mW and 4 mW, respectively. Figure 3.6 plots the SNR improvement using the model provided by eqn. 3.42. Both curves match with the corresponding modelling results plotted in figure 3.4. This shows that eqn. 3.45 and eqn. 3.42 on the one hand,

Figure 3.6: SNR improvement versus $h$ at 1.5 mW and 4 mW for two values of $R = 0.1$ (---) and $R = 0.2$ (-----). The laser cavity length is 220 μm.

and eqn. 3.16 and eqn. 3.37 on the other hand, are two alternative and equivalent models for the IMEI and SNR improvement in gain-lever lasers. This conclusion is important because it establishes that the modelling of both $\eta$ and $SNR_i$ can be based on the experimentally measured $G(J)$ curve bypassing the need for a model of the
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carrier recombination $R(n)$ which enables translating between $G(J)$ and $G(n)$.

Since the alternative model is justified its predictions can be compared with experimental values of $\eta$. Figure 3.7 reproduces the measured IM response of two different gain-lever lasers relative to their uniformly pumped counterparts [2]. The experimental results show that an $\eta \approx 23$ dB was measured in a 220 $\mu$m laser with $h = 0.65$ at 1.5 mW, and an $\eta \approx 11$ dB was measured in a 400 $\mu$m laser with $h = 0.45$ equally at 1.5 mW. The theoretical results presented in figure 3.5 indicate a value of $\eta \approx 11$ dB for the 400 $\mu$m laser, but only $\eta \approx 13$ dB for the 220 $\mu$m laser in the same circumstances. Additionally, at 4 mW, a $\eta \approx 6$ dB is calculated for the 400 $\mu$m laser and only $\eta \approx 8$ dB for the 220 $\mu$m laser. These compare with the measured $\eta \approx 6$ dB and $\eta \approx 13.5$ dB, respectively.

The theoretical results depend on the accuracy of the values of the derivatives $g'(J)$, which are estimated from the fitting to the experimental data in figure 2.5. However, the data around the gain of 80 cm$^{-1}$ is sparse and may hinder the accuracy of the curve fit around that region. This is consistent with the observed discrepancy in the short 220 $\mu$m gain-lever laser, since when the signal section is biased at $G_a = 0.2G_0$ the slave section gain is $G_b \approx 80$ cm$^{-1}$. The theoretical and experimental IMEI values for the longer 400 $\mu$m laser agree very closely.

Table 3.1 shows in the last column the values of $g = g_a/g_b'$ required by eqn. 3.45 to achieve theoretical agreement with the measured values of $\eta$ in the 220 $\mu$m laser shown in

![Figure 3.7: Measured modulation response at output powers of 1.5 mW and 4 mW. Thin curves: uniformly pumped; thick curves: tandem pumped with $G_a/G_0 \approx 0.2$. Left: a 400 $\mu$m cavity with $h = 0.45$, right: a 220 $\mu$m cavity with $h = 0.65$. (Reproduced from [2]).](image)
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<th>—Measured—</th>
<th>—Required—</th>
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</tbody>
</table>

Table 3.1: Comparison between theoretical and experimental IMEI values in a 220 $\mu$m GRIN-SCH-SQW gain-lever laser.

Table 3.1 shows that in order to reach agreement between theoretical and experimental values of $\eta$, the value of $g$ has to be much larger than what is estimated from the gain curve. Moreover, the calculated resonance frequency (eqn. 3.12) becomes significantly smaller than the correspondent experimental value, due to the hypothetical smaller value of $G'_u$. Therefore, a full and final explanation of these difficulties has yet to be found. Notwithstanding, the relative close agreement between theoretical and experimental IMEI values in the 400 $\mu$m laser is encouraging.

### 3.5.2 SNR improvement versus power

Figure 3.8 plots the maximum $(h \rightarrow 1)$ SNR improvement as a function of the optical power. For each curve in the plot, the signal section is biased at a constant fraction $R = 0.1, 0.2, 0.3$ of the threshold gain. A similar calculation for an identical but shorter laser (120 $\mu$m long) indicates that a SNR improvement of 20 dB and 10 dB is obtained at 1 mW and 5 mW respectively with $R = 0.1$. This is because the threshold gain is larger for smaller lasers which causes $\gamma_u$ to be larger and $G'_u$ to be smaller.

#### Noise figure

The spot (1 Hz bandwidth) noise figure of a laser transmitter (figure 1.3) can be given by

$$NF = 10 \log \left[ 2 + \frac{RINP_0^2 R_i}{\eta^2 kT} \right]$$

(3.47)
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SNR improvement in gain-lever laser over homogeneous laser

Figure 3.8: The relative SNR improvement in the gain-lever laser versus output power for $R = 0.1, 0.2, 0.3$.

where $\eta$ is the modulation efficiency in the gain-lever laser in W/A and $k_B T$ is the available thermal noise current spectral density from the source resistance [16]. $R$ is equal to 50 $\Omega$ in the case of a resistive matched laser and 4 $\Omega$ for a perfectly matched input. Here the shot noise contribution from the photodetector is neglected to outline the effect of a large $\eta$ on the noise figure.

Using eqn. 3.32 and eqn. 3.13 for the static RIN ($h \rightarrow 1$) and modulation response $\eta$, respectively, the spot noise figure at frequencies approaching d.c. is given by

$$NF = 10 \log \left[ 2 + (eV)^2 \frac{R_{sp} \gamma_a^2}{p_0 (\Gamma G'_a)^2 kT} \frac{R}{R} \right]$$  \hspace{1cm} (3.48)$$

where $R_{sp}$ is the rate of spontaneous emission into the active cavity volume per unit volume. The value used for $R_{sp}$ is 1.28·10$^{12}$ s$^{-1}$, a typical value for GaAs lasers [63]. This value is divided by the laser active cavity volume $V$ and multiplied by the confinement factor $\Gamma$ to convert it to the units employed in eqn. 3.48.

The noise figure is optimised when the laser's signal section is biased close to transparency as the value of $G'_a$ is maximised. The noise figure is increased for values of $h < 1$ because the value of $R_{sp}$ slightly increases due to the increased carrier density in the slave section. Figure 3.9 plots the spot static noise figure in a gain-lever laser as a function of optical power for three values of the parameter $R$. The plot shows that as the optical power increases the NF also increases due to the reduction of $\eta_{gl}$ at high power.
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Figure 3.9: Spot noise figure in the gain-lever laser versus optical power for three values of the parameter $R = 0.1, 0.2, 0.3$.

At optical powers around 1 mW the noise figure is considerably reduced in relation to the noise figure of a single section laser, typically $\approx 26$ dB. This is due to the improvement in modulation efficiency $\eta_{gl}$ resulting from the gain-lever effect without the RIN offsetting it, as was shown. Also, an optimum operating point exists for which the NF is minimum. The noise figure can be further reduced if a better matching strategy is employed at the laser input. For a perfect lossless matching network, $R_i$ in eqn. 3.48 is typically $4 \Omega$. This matching would reduce the calculated noise figure shown in figure 3.9 by a factor of $4/50$ [15]. At 1 mW output power this laser could operate close to its theoretical optimum NF of 3 dB, assuming that the matching network does not degrade the laser noise performance.

### 3.5.3 Majorant

The SNR in a laser is a quantity that one would like to maximise. It is shown in appendix F that the SNR improvement in a gain-lever laser has a majorant given by

$$SNR_{i}^{Maj} = \left[ \frac{h\nu A\alpha_m}{e 2P_o g_{u}} + 1 \right]^2$$

Likewise, the IMEI also has a majorant which is given by

$$\eta_{Maj} = \frac{h\nu A\alpha_m}{e P_o g_{b}} + 1$$
A close look at eqn. 3.49 shows that low power operation increases both the \( SNR_t \) and IMEI. Furthermore, lasers operating at shorter wavelength or with larger area are also favoured, if everything else remains unchanged.

There is another important message contained in eqn. 3.49; while the \( SNR_t \) is proportional to the ratio \( g'_u/g'_u \), the limiting factor is \( g'_u \) rather then the ratio \( g'_u/g'_u \). In other words, a small value of \( g'_u \) gives a larger improvement than a large one, for a constant ratio \( g'_u/g'_u \). Given the similarity between eqn. 3.49 and eqn. 3.50 the same arguments apply also to the IMEI.

### 3.6 Summary

In this chapter both the intensity modulation and noise characteristics of the gain-lever laser were investigated, using an appropriate set of rate-equations as the starting point.

The theoretical conjectures of earlier researchers were interpreted and criticised. The IMEI in the GRIN-SCH-SQW gain-lever laser was plotted under realistic bias conditions, using the gain versus carrier density data derived from experimentation. A pattern of the relationship between the IMEI and the relative length of the laser sections emerged from these plots. To maximise the IMEI, the signal section should be biased just above transparency and the slave section length should be minimised, within the practical limits.

The noise in gain-lever lasers was also investigated. A new theoretical expression for the relative SNR improvement in a gain-lever laser was derived. This expression provides new insight at variance with earlier work. The gain-lever effect delivers a SNR improvement that depends on the bias of the tandem-contact laser but is essentially independent of the section's relative length. However, in practise, the SNR improvement in a gain-lever laser varies with \( h \) because of the dependence of the rate of spontaneous emission \( R_{sp} \) on this parameter. Reducing both the bias to the signal section and the optical power increase the SNR improvement. The relative length of the slave section should be maximised because this contributes to increase the SNR improvement by decreasing the parameter \( R_{sp} \).

Furthermore, the theoretical IMEI was compared with experimental values in two
different gain-lever lasers. Although the agreement between those values was satisfactory for the larger laser, for the shorter one the theoretical values are significantly smaller than the correspondent experimental values. The source of this apparent discrepancy was pursued by showing that both the theoretical IMEI and the SNR improvement can also be based on the experimental gain versus current data; the alternative model requires knowledge of the gain versus carrier density relationship which is derived from the experimental data assuming a carrier recombination law for QW lasers (e.g. $J = Bn^2ed$). The two alternative models were proved equivalent, thus effectively eliminating possible deviations to the carrier recombination law as the cause of the discrepancy between the experimental and theoretical values.

In lasers, IM is always accompanied by frequency modulation (FM), thus the next chapter will be devoted to the study of both the FM modulation and the phase noise in the gain-lever laser.
Chapter 4

Linear chirp and phase noise in the gain-lever laser

4.1 Introduction

The intensity modulation response in the gain-lever laser was studied in chapter 3. Here, it is shown that the gain-lever effect in SQW lasers also yields a large increase in the optical FM efficiency, compared to that in an equivalent single-section laser. A novel analytical expression for the relative FM efficiency in a gain-lever laser is derived and is related to the IMEI studied in the previous chapter.

The phase noise in gain-lever lasers is also studied. The SNR improvement in optical FM, relative to the SNR in an equivalent single-section laser is defined and a corresponding analytical expression is derived. Both the so called “normal” gain-lever and “inverted” gain-lever configurations are studied using the applicable theoretical expressions, and the merits of each configuration are discussed. The conversion of FM to IM is also addressed. Usually this can be performed by an optical discriminator such as a Mach-Zehnder interferometer or Fabry-Pérot étalon.

Finally, the gain-lever laser transmitter is appraised in respect of its possible advantages from a system standpoint. In subcarrier multiplexed applications where this laser may be used to transmit several channels over an optical fibre there is a need for further research that accesses the effects of both the laser and optical discriminator' non-linearities.
4.2 FM response in the gain-lever laser

If carrier clamping was perfect there would be no FM in a semiconductor laser at all. However, FM is employed in both single and multi-section lasers. In single section lasers, FM is due both to gain compression and spontaneous emission, whereas in multi-section lasers, a non-symmetry in both the linewidth enhancement factor and the differential gain between the sections also contributes to the overall FM response [87].

The FM response of the gain-lever laser can be obtained from an additional rate equation for the accumulated phase along the optical cavity

\[ \Delta \dot{\phi} = -\frac{\omega_0}{\mu_g} \Gamma [(1 - h)\Delta \mu_a + h\Delta \mu_b] \]

(4.1)

where \( \omega_0 \) is the optical mode frequency, \( \Delta \mu_a \) and \( \Delta \mu_b \) represent the refractive index change due to the carrier density dynamics in the laser’s active region. The linewidth enhancement factor is defined by the expression

\[ \alpha = -2k_0 v_g \mu' G' \]

(4.2)

where \( \mu' \) is the derivative of refractive index with respect to the carrier density and \( k_0 \) is the light wave-vector [63]. Using this definition, the linear chirp or frequency deviation, \( \Delta \dot{\phi} \), can be alternatively expressed as

\[ \Delta \dot{\phi} = \frac{\Gamma}{2} [(1 - h)\alpha_a G'_a n_a + h\alpha_b G'_b n_b] \]

(4.3)

where \( n_a \) and \( n_b \) are the small-signal carrier density deviations around their steady-state value [88]. When these are substituted in eqn. 4.3 by the corresponding carrier density frequency responses, \( F_a(\omega) \) and \( F_b(\omega) \), the following expression for the optical frequency deviation in the frequency domain is obtained

\[ \vartheta(\omega) = \frac{\Gamma}{4\pi} [(1 - h)\alpha_a G'_a F_a(\omega) + h\alpha_b G'_b F_b(\omega)] \]

(4.4)

where \( \omega \) is the frequency of the sinusoidal signal current, \( G' \) is the derivative of the gain \( G(n) \) without the gain compression parameter [42], contrary to what has been wrongly assumed, on some occasions, resulting in the total neglect of the contribution of gain compression from the final expression for the chirp [89]. To take proper account of the
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effect of gain compression on the overall FM response, the expressions for $F_a(\omega)$ and $F_b(\omega)$ must be calculated from a set of rate equations with a non-linear gain including the gain compression factor, e.g. $G(1 - \epsilon_0)$.

It is shown in appendix G that the FM response in the gain-lever laser can be given by

$$\frac{\vartheta(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G'_a \left[ \frac{(i\omega + \gamma_b)(i\omega + p\Gamma G_0 \epsilon) - E\omega^2}{-i\omega^3 - \omega^2(\gamma_a + \gamma_b + p_0\Gamma G_0 \epsilon) + i\omega A_1 + A_2} \right]$$

where $I$ is the amplitude of the injected signal current, the dimensionless factor $E$ and the frequency $\omega_b$ are defined by

$$E = \frac{\alpha_a}{\alpha_a - 1}$$

$$\omega_b^2 = (1 - 3\epsilon_0)p_0\Gamma G_b G'_b$$

and, in addition, the parameters $A_1$ and $A_2$ are given by

$$A_1 = \gamma_a \gamma_b + p_0\Gamma G_0 \epsilon (\gamma_a + \gamma_b) + (1 - 3\epsilon_0)p_0\Gamma [(1 - h)G_a G'_a + hG_b G'_b]$$

$$A_2 = (1 - 3\epsilon_0)p_0\Gamma [(1 - h)G_a G'_a \gamma_b + hG_b G'_b \gamma_a] + p_0\Gamma G_0 \epsilon \gamma_a \gamma_b$$

where $G_0$ is the threshold gain. These definitions of $A_1$ and $A_2$ differ slightly from their earlier definition in eqn. 3.6 and eqn. 3.7, because here the effect of the gain compression is taken into account.

Eqn. 4.5 for the linear chirp is similar, but not identical, to the one obtained by Kuznetsov et. al. for two-segment bulk lasers with different linewidth enhancement factors [90, 88]. In a QW laser, both the differential gain and the carrier lifetimes may also be different in each section bringing out the third-order polynomial in the denominator of eqn. 4.5, whereas in bulk lasers the denominator is of a second-order in $\omega$ [88].

The chirp in the gain-lever laser, given by eqn. 4.5, is also formally different from eqn. 5 of [5], derived by Lau to model the FM response in gain-lever lasers. Lau’s equation only applies where $h = 1$, and assumes both that $\epsilon = 0$, and that the linewidth parameter $\alpha$ in each section is inversely proportional to the corresponding $G'$, with the constant of inverse proportionality being $\mu'$. It is shown in appendix H that eqn. 4.5 can be transformed equivalently to Lau’s equation when $h = 1$, provided both that $\epsilon = 0$ and that $\alpha$ is inversely proportional to $G'$. 
The current equation is general, and comprises two clearly distinct terms; the first term proportional to the gain compression parameter is the single-segment contribution to the overall FM response. Its static magnitude is given by

$$\frac{\vartheta(0)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} a_\alpha e \eta$$

(4.10)

where \(\eta\) is the IMEI. This is formally similar to the static FM response in single-section lasers [63], given by

$$\frac{\vartheta(0)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} a_\alpha$$

(4.11)

Unlike the first, the second term in eqn. 4.5 is independent of the gain compression parameter, but is proportional to the dimensionless parameter \(E\). In [90, 88] this term is called the two-segment contribution to the FM response and in [5] only this term is associated with the "gain-lever effect" in the FM response. The static magnitude of the two-segment contribution is given by

$$\frac{\vartheta(0)}{I} = -\frac{1}{eV} \frac{\Gamma}{4\pi} a_\alpha G'_a \left[ \frac{g - 1}{g \left(1 - (1-h)R\right) \gamma_a \gamma_b + 1} \right]$$

(4.12)

where \(g = G'_a/G'_b\) and \(R = G_a/G_0\). Figure 4.1 plots the magnitude of the bracketed factor in eqn. 4.12 as a function of the parameter \(g\). When the signal section is lightly biased, \(R < 1\) and \(g > 1\), the bracketed factor can have a magnitude much larger than one as shown in the plot. If, however, the signal section is heavily biased such that
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$R > 1$ and $g < 1$, the bracketed factor will have a value between zero and minus one.

By using the approximation $G_aG'_a \approx G_bG'_b$, used earlier by Gajic in a different context [62] (see section 3.4), the FM response can also be given by

$$\frac{\vartheta(0)}{I} \approx \frac{1}{eV} \frac{\Gamma}{4\pi} \frac{\mu'}{\gamma} (R - 1) \eta$$

(4.13)

where $R = G_a/G_0$, showing that the magnitude of the two-segment term, being proportional to $\eta$, can also be greatly enhanced relatively to the magnitude of the FM response of a single-section laser. The single-element contribution is always blue shifted i.e. the optical frequency increases when the injected current increases. However, the two-segment contribution can be either blue-shifted or red-shifted depending on the value of $E$. The static two-section contribution when $h \rightarrow 1$, can also be given by

$$\frac{\dot{\vartheta}(0)}{I} = -\frac{1}{eV} \frac{\Gamma}{4\pi} \frac{G_a' (\alpha_b - \alpha_a)}{\gamma}$$

(4.14)

and, if $\mu'_a = \mu'_b$, can also be cast as

$$\frac{\dot{\vartheta}(0)}{I} = -\frac{1}{eV} \frac{\Gamma}{4\pi} \frac{\alpha_b}{\gamma} (G'_a - G'_b)$$

(4.15)

which coincides with earlier reported work [5]. It should be emphasised that eqn. 4.14 is more general than eqn. 4.15 which is valid only when $\mu'_a = \mu'_b$.

The bandwidth of the FM enhancement term is unlike the bandwidth of the IM response. The denominator of the IM response coincides with the denominator of eqn. 4.5. However, the two-segment term of the FM response misses the factor $i\omega + \gamma_b$ that appears in the numerator of the IM response. Effectively, the FM enhancement term is a low-pass filtered version of the IM response, with a bandwidth determined by the stimulated inverse carrier lifetime $\gamma_a$ in the signal section.

### 4.2.1 FM efficiency improvement

The relative improvement in the IM efficiency in a gain-lever laser has been defined as the ratio of the magnitude of the IM responses in the gain-lever laser and the magnitude of the FM response in its equivalent uniformly biased counterpart. The relative FM efficiency improvement in a gain-lever laser compared to the FM efficiency in a uniformly
biased device can be defined in a similar way, as

$$\eta_{FM} \equiv \frac{\left( \frac{\vartheta}{I} \right)_{h=0}}{\left( \frac{\varphi}{I} \right)_{h=0}} = \frac{G'_a}{G'_a} \eta - \frac{G'_0}{\gamma_b} h \left( 1 - \frac{G'_b}{G'_a} \right) \left( \frac{G_b}{G_0} \right) \eta$$  \hspace{1cm} (4.16)

which is obtained by dividing eqn. 4.5 by eqn. 4.11. This equation can be approximately given by

$$\eta_{FM} \approx -\frac{G'_0}{\gamma_b} h \left( 1 - \frac{G'_b}{G'_a} \right) \left( \frac{G_b}{G_0} \right) \eta$$  \hspace{1cm} (4.17)

when the two-segment contribution is dominant ($>> 1$) which occurs in the cases of practical interest. By using Gajic's approximation, $G_a G'_a \approx G_b G'_b$, one also gets

$$\eta_{FM} \approx \frac{G'_b}{\gamma_b} (R - 1) \eta$$  \hspace{1cm} (4.18)

Results for $\eta_{FM}$ will be given when discussing the "normal" and "inverted" gain-lever configurations.

### 4.3 Phase noise in the gain-lever laser

It was pointed out earlier that the process of spontaneous emission in a laser not only induces fluctuations in the internal photon density but also causes the phase of the light to change randomly. Moreover, the change in the carrier population and thus in optical gain also causes a refractive index change and consequently an additional variation in the phase of the light. This phenomena is responsible for an increase in the fundamental Schawlow-Townes laser linewidth due to the instantaneous phase change associated with the Langevin noise source $\Delta_{\phi}(t)$ in eqn. 3.25.

A similar analysis to that undertaken for the RIN in section 3.4 is also carried out here for the phase noise. The random phase fluctuation is modelled by a stochastic rate equation for the optical phase of the gain-lever laser including a Langevin noise force $\Delta_{\phi}(t)$ representing the random phase fluctuations due to spontaneously emitted photons

$$\dot{\phi} = -\frac{\omega_0}{\mu_q} \Gamma [(1-h)\Delta\mu_a + h\Delta\mu_b] + \Delta_{\phi}(t)$$  \hspace{1cm} (4.19)

Undertaking a small-signal analysis the following equation for the random frequency
fluctuation is obtained

$$\delta \phi = \frac{\Gamma}{2} [(1 - h)\alpha_a G_a' \delta n_a + h \alpha_b G_b' \delta n_b] + \Delta \phi(t)$$

(4.20)

It is shown in appendix I that the spectral density of the frequency fluctuations in the gain-lever laser can be given by

$$S_\phi(\omega) = \frac{R_{sp}}{4p_0} \times \left[ 1 + \Gamma^2 \frac{h^2 \alpha_a^2 G_b^2 G_a'^2 p_0^2 (\gamma_a^2 + \omega^2)}{\omega^2 [\omega^2 - (\omega_b^2 + \gamma_a \gamma_b)]^2 + [\omega^2 (\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$+ \frac{R_{sp}}{4p_0} \times \left[ \Gamma^2 \frac{(1 - h)^2 \alpha_a^2 G_a'^2 G_b'^2 p_0^2 (\gamma_b^2 + \omega^2)}{\omega^2 [\omega^2 - (\omega_b^2 + \gamma_a \gamma_b)]^2 + [\omega^2 (\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$+ \frac{R_{sp}}{4p_0} \times \left[ \Gamma^2 \frac{2h(1 - h)\alpha_a \alpha_b G_a G_b' G_b'^2 p_0^2 (\gamma_a \gamma_b + \omega^2)}{\omega^2 [\omega^2 - (\omega_b^2 + \gamma_a \gamma_b)]^2 + [\omega^2 (\gamma_a + \gamma_b) - A_2]^2} \right]$$

(4.21)

The d.c. noise spectral density when $h \rightarrow 1$ is simply given by

$$S_\phi(0) = \frac{R_{sp}}{4p_0} (1 + \alpha_b^2)$$

(4.22)

and is identical to the phase noise spectral density in a single-section laser [63].

### 4.3.1 SNR improvement

Here, the phase noise in the gain-lever laser is compared with the phase noise in an equivalent uniformly biased single-section laser. The increase in phase noise in the gain-lever laser relative to the phase noise in an uniformly biased device is defined by the ratio of the phase noise spectral densities in each device. At d.c. this ratio is given by

$$P \equiv \frac{S_{\phi}^l}{S_{\phi}^n} = \frac{R_{sp} \left( \frac{p_0 \Gamma}{A_2} \right)^2 [(1 - h)\alpha_a \gamma_b G_a G_a' + h \alpha_b \gamma_a G_b G_b']^2}{R_{sp}^2 1 + \alpha_a^2}$$

(4.23)

which can be simplified by realising that the second terms in both the numerator and denominator are usually much larger than 1. After some calculation one obtains

$$P \approx N \left[ (1 - h)R \left( \frac{\gamma_u}{\gamma_a} - \frac{\gamma_a}{\gamma_b} \right) + \frac{\gamma_u}{\gamma_b} \right]$$

(4.24)

where $N$ is the RIN increase in the levered laser, given by eqn. 3.35. It is reasonable to introduce yet another approximation that $\gamma_a \approx \gamma_b \approx \gamma_u$, also used in [87], which gives

$$P \approx N$$

(4.25)
showing that the phase noise increase in the gain-lever laser, relative to the phase noise in the equivalent single-section counterpart, is approximately equal to the RIN increase.

The increase in SNR in the gain-lever laser, relative to the SNR in an equivalent uniformly biased counterpart, as far as optical FM and phase noise are concerned, can be defined by

$$SNR_{i}^{FM} \equiv (\frac{S}{N})_{gl} \left( \frac{N}{S} \right)_{u} = \eta_{FM}^{2} \frac{P}{P}$$ \hspace{1cm} (4.26)

and upon substituting in both eqn. 4.17 for $\eta_{FM}$ and eqn. 4.23 for $P$ gives

$$SNR_{i}^{FM} \approx \frac{R_{sp}}{R_{sp}} \frac{\left( \frac{G_{a}}{G_{b}} \right)^{2}h^{2}(1 - \frac{G_{a}}{G_{b}})^{2}(\frac{G_{b}}{G_{a}})^{2}\eta^{2}}{1 + \left( \frac{R_{sp}}{R_{sp}} \right)^{2}(1 - h)^{2}(\gamma_{a}G_{a}G_{b} + \gamma_{a}\gamma_{b}G_{a}G_{b})^{2}}$$ \hspace{1cm} (4.27)

By introducing the approximation, $G_{a}G_{a} \approx G_{b}G_{b}$, a much simpler expression can be derived

$$SNR_{i}^{FM} \approx \frac{R_{sp}}{R_{sp}} \frac{\left( \frac{G_{b}}{G_{b}} \right)^{2}}{\gamma_{b}^{2}} \frac{(1 - R)^{2}}{(1 - h)R^{2}(\gamma_{b} - \gamma_{a}) + R\gamma_{a}]^{2}}$$ \hspace{1cm} (4.28)

and, assuming $\gamma_{a} \approx \gamma_{b}$, gives

$$SNR_{i}^{FM} \approx \frac{R_{sp}}{R_{sp}} \frac{\left( \frac{G_{b}}{\gamma_{b}^{2}} \right)^{2}}{(1 - R)^{2}}$$ \hspace{1cm} (4.29)

Furthermore, the SNR improvement in optical FM can be related to the SNR improvement in IM, by using both eqn. 4.18 and eqn. 4.25 to give

$$SNR_{i}^{FM} \approx \left( \frac{G_{0}}{\gamma_{b}^{2}} \right)^{2} (1 - R)^{2} SNR_{i}^{IM}$$ \hspace{1cm} (4.30)

showing that the “gain-lever effect” can deliver, in theory, a larger SNR increase in the optical frequency modulated component of the light than in the intensity modulated component.

### 4.4 FM response in the “normal” configuration

Fig. 4.2 plots both the FM and the phase responses in a GRIN-SCH-SQW GaAs gain-lever laser with 520 $\mu$m cavity length. The signal section is biased at a fraction $R = 0.286$ of the threshold gain $G_{0} = 35$ cm$^{-1}$, the ratio of differential gains in each section is $G_{a}'/G_{b}' = 3.65$ and thus $E > 0$ producing a large red-shifted FM response and
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Figure 4.2: Left: FM response in a “normal” gain-levered laser with 520 μm cavity length at 1 mW output power (solid curve). The signal section is biased at $R = 0.286$ with a current $I_a = 0.94$ mA and $I_b = 7.24$ mA. Other parameters: $h = 0.77$, $G'_a/G'_b = 3.65$, $\gamma_p = 3.24$ ps, $\gamma_a = 3 \times 10^8$ s$^{-1}$, $\gamma_b = 2.3 \times 10^8$ s$^{-1}$. The broken curves represent the single-segment contribution (curve A), and the two-segment contribution (curve B). Right: the phase of the FM response.

a rapidly varying phase. In these circumstances the laser is operating in the “normal” gain-lever configuration. The static FM response is $–30$ GHz · mA$^{-1}$, two orders of magnitude larger than the FM efficiency of an equivalent single-section laser, whose FM response is plotted in figure 4.3. The broken curves marked A and B in figure 4.2 represent, respectively, the single-segment and the enhancement contribution terms to the overall FM response in the gain-lever laser. The magnitude of the FM response in the gain-lever laser can be almost exclusively accounted for, by the enhancement contribution term.

Both the magnitude of the FM response and of the linewidth factor depend on the value of the refractive index derivative $\mu'$. A specific value for $\mu'$ in a 100 Å GRIN-SCH-SQW GaAs laser is not given in the literature. Alternatively, a value $\mu' = -2 \times 10^{-20}$ cm$^3$ is assumed which gives a plausible linewidth value $\alpha = 3$ in the SQW laser with 520 μm cavity length [91]. This value is also close to $\mu' = -8.5 \times 10^{-21}$ cm$^3$ measured in a laser with a 250 Å thick GaAs SQW [92]. The magnitude of the FM response in the single-section SQW laser is also proportional to the gain compression factor, assumed to have a value $\epsilon = 2.5 \times 10^{-17}$ cm$^3$ similar to that in other bulk lasers as suggested in section 2.3.1.

Unfortunately, the FM response in the levered laser has a small bandwidth with a
Figure 4.3: Left: FM response of a single-section SQW laser with 520 μm cavity length at an output power of 1 mW. Other parameters: $\tau_p = 3.24$ ps, $\gamma = 2.1 \cdot 10^9$ s$^{-1}$, $\varepsilon = 2.5 \cdot 10^{-17}$ cm$^3$. Right: the phase response.

frequency $f_{3dB} \approx 360$ MHz limited by the inverse carrier lifetime $\gamma_b$ in the slave section. Effectively, there is a trade-off between the magnitude of the static FM response in the gain-levered laser and its bandwidth. The magnitude of the two-segment contribution to the FM response increases as the ratio $G'_d/G'_b$ increases. However, biasing the slave section at a large carrier density where the differential gain $G'_b$ is smaller, reduces the stimulated part of the inverse lifetime $\gamma_b$ and hence the bandwidth of the response.

Figure 4.4 plots the relative increase in the static magnitude of the FM response in the gain-levered laser compared to the static magnitude of the FM response in an equivalent single-section laser, as a function of $h$. The relative FM efficiency is larger as $h$ decreases, a rather similar pattern to the $\eta$ versus $h$ plot depicted in figure 3.2. This similarity is consistent with eqn. 4.18 showing that the magnitude of the FM response is proportional to $\eta$.

4.4.1 Phase noise in the “normal” configuration

The phase noise increase in the gain-lever SQW laser, relative to the phase noise in an equivalent single-section laser, is plotted in the left-hand-side of figure 4.5 as a function of the geometrical parameter $h$, assuming that the signal section is biased at a constant fraction of the threshold gain. The phase noise increase is minimised for $h \rightarrow 1$, the same geometrical design that minimises the RIN increase, in conformity with eqn. 4.25.

The right-hand-side of figure 4.5 plots the relative SNR increase versus $h$ showing
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Figure 4.4: Relative increase of the FM efficiency in a “normal” levered SQW laser with 520 μm cavity length, as a function of \( h \), for two values of \( R = 0.1 \) and \( R = 0.286 \). The output power is 1 mW and the photon lifetime \( \tau_p = 3.24 \) ps.

Figure 4.5: Left: Relative increase in phase noise in a “normal” gain-lever laser with 520 μm cavity length, as a function of \( h \), for two values of \( R = 0.1 \) and \( R = 0.286 \). Right: SNR increase. Output power is 1 mW, \( \tau_p = 3.24 \) ps.
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Figure 4.7: Relative FM efficiency in “inverted” gain-lever with 520 μm cavity length, as a function of $h$, for two values of $R = 2$ and $R = 4$. Photon lifetime is $\tau_p = 5.67$ ps.

the large bandwidth is traded-off for a reduced magnitude of the FM response. The FM phase response, depicted in the right hand side of figure 4.6 has a much smoother variation than in the “normal” gain-lever and shows that the FM response is blueshifted.

Figure 4.7 plots the magnitude of the static FM response in the “inverted” gain-lever laser as a function of the geometrical parameter $h$, with the signal section biased at a constant high-gain given by $R = 2$ and $R = 4$. These curves show that in the “inverted” lever mode, the relative FM efficiency is maximised when $h \to 1$. When $h \to 0$, the relative FM efficiency decreases rapidly eventually approaching $\eta_{FM} = 0$ dB. This pattern is ever more accentuated for larger values of $R$. The dependence of $\eta_{FM}$ on $h$ in the inverted mode configuration is the inverse of that in the “normal” gain-lever, depicted in figure 4.4, where $\eta_{FM}$ increases when $h$ decreases. Given that the magnitude of $\eta_{FM}$ plotted in both instances is given by the same eqn. 4.17, some explanation is required to account for the differences between the plots. Furthermore, $\eta_{FM}$ is proportional to $\eta$, and $\eta$ itself is maximised when $h \to 0$, decreasing when $h \to 1$, in both the “normal” and “inverted” configurations. However, it must be realised that in the “inverted” gain-lever mode the slave section gain bias must decrease when its relative size decreases ($h \to 0$), according to the threshold condition, assuming a constant gain and $R > 1$ in the signal section. Indeed, for some value of $h < 1$, $G_h$ would eventually
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Figure 4.6: Left: FM response in a SQW “inverted” gain-lever laser with 520 μm cavity length at 3 mW output power. The signal section is biased at $R = 4$ with a current $I_a = 17.2$ mA, and $I_b = 3.85$ mA. Other parameters: $h = 0.77$, $G''/G' = 0.114$, $\tau_p = 5.67$ ps, $\gamma_a = 5.8 \cdot 10^9$ s$^{-1}$, $\gamma_b = 2 \cdot 10^{10}$ s$^{-1}$. The broken curves represent the single-segment contribution (curve A), and the two-segment contribution (curve B). Right: the phase response.

that for optical FM, it is much larger than the correspondent SNR increase for IM (compare with figure 3.4), as also suggested by eqn. 4.30. However, the magnitude increase in the FM case compared to the IM case is traded-off for a reduced bandwidth.

4.5 FM response in the “inverted” configuration

The “inverted” gain-lever laser is physically the same device as a “normal” gain-lever laser, with the bias points of the gain and signal sections interchanged [5]. In the “inverted” gain-lever configuration the slave section is biased with a small gain while the signal section provides most of the gain, hence $G''/G' < 1$. Consequently, the IM efficiency in the “inverted” mode configuration is actually suppressed because $\eta < 1$, hence the “inverted” mode has a large ratio of FM to IM efficiency [5], more suitable for an ideal FM transmitter [7].

Fig. 4.6 plots both the FM and phase responses in an “inverted” gain-lever laser of the same type used in the normal configuration except that in this case the facets have been mirror coated to a 0.7 reflectivity. The coating is required because the “inverted” lever mode is easier to achieve with a low threshold laser, since the shortest section must also supply the majority of the necessary gain. A low threshold laser has also a high internal photon density, thus increasing the bandwidth of the FM response. However,
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Figure 4.8: Left: Relative phase noise increase versus $h$ in “inverted” gain-lever laser with 520 $\mu$m cavity length, for two values of $R = 2$ and $R = 4$. Right: SNR improvement versus $h$ for $R = 2$ and $R = 4$. Output power is 3 mW, $\tau_p = 5.67$ ps.

be zero at which point $\eta = 1$ and the enhancement term contribution becomes zero according to eqn. 4.17, thus explaining the trend of the curves plotted in figure 4.7. It is also clear from figure 4.7 that the largest value of $R$ is not necessarily the most desirable although $\eta_{FM}$ increases with $R$ at $h \to 1$. However, because it is impossible to build a device with $h = 1$, it is best to bias with an $R$ value which maximises $\eta_{FM}$ at the designed $h$.

Nevertheless, the magnitude of the static FM response in the “inverted” gain-lever laser still exhibits a large increase compared to the FM efficiency in an equivalent single section laser as depicted in the plot in figure 4.7.

4.5.1 Phase noise in the “inverted” configuration

Figure 4.8 depicts the relative phase noise increase in the “inverted” lever laser as a function of the relative size of the slave section, $h$, assuming that the signal section is biased with a constant large gain given by $R = 2$ or $R = 4$. When $h \to 1$, $G_b \to G_0$, and thus the phase noise in the levered laser is, in the limit, equal to the phase noise in the equivalent single-section laser, as shown in the figure. When the signal section length increases ($h \to 0$) the phase noise increases rapidly because this section is biased much above the threshold gain and the number of spontaneous events is directly proportional to the gain.

This pattern has much in common with that in the “normal” gain-lever where both
the relative phase noise (figure 4.5) and RIN are also minimised when $h \to 1$ and both noises also increase as $h \to 0$. However, in the “normal” mode the noise increases slower than in the “inverted” configuration because the gain in the slave section only rises progressively above threshold as its size decreases.

Due to the rapid increase in noise depicted in figure 4.8, when the signal section is biased at high gain, the noise performance in the “inverted” lever laser can be quickly degraded when $h \to 0$. Notwithstanding, the “inverted” gain-lever is often also a low-threshold laser, for practical reasons, and thus a low-noise device.

The relative SNR for optical frequency modulation is plotted in the right hand side of figure 4.8 as a function of $h$, assuming a constant gain in the signal section. Due both to the increasing phase noise with decreasing $h$ and to a rapidly decreasing FM efficiency with decreasing $h$, especially for large values of $R$, the relative SNR increase is clearly maximised for $h \to 1$, in the “inverted” mode. When practical values of $h \approx 0.8$ are used, the best bias arrangement that maximises the SNR is the one with only a moderately large value of $1 < R < 2$, rather than a much larger value of $R$.

4.5.2 Appraisal of the theoretical FM response

The theoretical FM responses in the gain-lever laser shown in figure 4.2 and figure 4.6 can be compared with the experimental FM responses depicted in figure 4.9 [5]. The
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shapes of both the theoretical and experimental FM responses curves are similar which supports the validity of the theoretical model in eqn. 4.5. However, the static magnitudes of both the theoretical FM response (≈ 2.1 GHz/mA) and the experimentally measured FM response for the “inverted” mode (≈ 12.1 GHz/mA) shown in figure 4.9 differ by a factor of 6.

This discrepancy is significant enough and was investigated. Using the routine ATTGOAL of the MATLAB optimisation toolbox, a computer program was run to optimise a number of the laser parameters thus fitting the theoretical FM response (eqn. 4.5) to the experimental FM response subject to given constraints, e.g. bias currents, resonance frequency, section size and output power. The optimisation program produced an approximate solution as depicted in figure 4.10 whilst complying with the constraints to a large extent. The optimised values found by the computer program for the inverse lifetime, the derivative of the refractive index and the linewidth enhancement factor were, respectively, $\gamma_a = 0.1 \text{ns}^{-1}$, $\mu' = -1.5 \times 10^{-19} \text{cm}^3$, $\alpha_a = 67$ and $\alpha_b = 22$. The important point is that this value of $\mu'$ required to fit the model to the target experimental FM response is approximately one order of magnitude larger than the values of $\mu'$ measured in other SQW GaAs lasers [92, 93]. Moreover, the values of the linewidth factor $\alpha$ are also much larger than the values measured in actual GRIN-SCH-SQW GaAs lasers, typically between 2.9 to 3.9 [91].

In addition, it can also be argued theoretically that a relatively large value of $\mu'$ is indeed necessary to fit the theoretical FM response to the experimentally measured

![Figure 4.10: Model fit to the target experimental FM response curve.](image)
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FM response. An estimated value of $\mu'$ can be obtained both from eqn. 4.12 giving the magnitude of the static FM response as $\Gamma_0 \mu'/e V \lambda \gamma_a$ and from an estimate of $\gamma_a$ based on the experimental FM response curve in figure 4.9. The FM response is low-pass filtered and its 3dB frequency point is controlled by $\gamma_a$. The magnitude of the experimental FM response curve only starts decreasing slightly at around 2 GHz, thus enabling an estimate of $\gamma_a \approx 4\pi$ GHz or $\gamma_a \approx 10^{10}$ s$^{-1}$. Plugging in the above expression this estimate of $\gamma_a$, a cavity width of 2 $\mu$m established by the laser parameter optimisation and a magnitude of the FM response to approach 12.1 GHz/mA, it results that in order for the, $\mu'$ must be larger than $7.5 \times 10^{-20}$ cm$^3$.

The refractive index derivative has not, to our knowledge, been directly measured in GRIN SCH SQW GaAs lasers. However, it is possible to estimate its value from experimental measurements of the linewidth enhancement factor in a GRIN laser with a 70 Å SQW and an identical structure to the devices in which the measurements of the FM response have been performed. Derry et.al. reported a value of $\alpha = 2.9$ in a 250 $\mu$m laser with mirror coatings [91]. Using their estimate of the differential gain $G' = 6.2 \times 10^{-6}$ cm$^3$s$^{-1}$ the value of the refractive index derivative is estimated using eqn. 4.2 at $\mu' = -1.4 \times 10^{-20}$ cm$^3$.

The point has been made earlier in [5] that, in theory, the magnitude of the FM response in the “inverted” mode can be maintained at a high-level close to the magnitude of the FM response in the “normal” configuration. However, it is worth noticing, the argument is flawed, as is explained in the following.

The author in [5] considers a “normal” gain-lever laser with its section ‘a’ biased at a gain of 16 cm$^{-1}$ and its section ‘b’ biased at 65 cm$^{-1}$. Additionally, he also considers a similar laser, but with mirror coatings, biased in the “inverted” configuration where section ‘a’ is biased at 65 cm$^{-1}$ and section ‘b’ is biased at 16 cm$^{-1}$, i.e. a symmetric inversion of the parameters of the gain and modulation sections between the two configurations [5].

Using estimated values for the relevant quantities in eqn. 4.15 the author calculates the static FM efficiency in the “normal” configuration which is given by $\frac{\alpha_2}{\gamma_a} (G'_b - G'_a)$ and obtains a value of $-22$ GHz mA$^{-1}$ which is close to the experimental value. At the
same time, the author also calculates the laser's relative IM efficiency given by $\frac{G_a \gamma_b}{G_b \gamma_a}$ and obtains a value of 4.81 at 1 mW.

Since the FM efficiency is given by the same expression in both configurations, and given the symmetrical inversion of the bias points between the configurations, it is clear that the magnitude of the FM efficiency in the "inverted" configuration is given by $\frac{G_a \gamma_b}{G_b \gamma_a} (G'_b - G'_a)$.

Consequently, the absolute value of the ratio of the magnitudes of the two FM efficiencies is also given by $\frac{G'_a \gamma_b}{G'_b \gamma_a}$, which equals 4.81 at 1 mW. Hence, the FM efficiency in the "inverted" configuration would be 4.81 smaller than that in the "normal" configuration at the same power of 1 mW. Moreover, the emitted power in the "inverted" configuration is really 2 mW which corresponds to a much higher internal photon density than in the case of the "normal" configuration at 1 mW, because of the high reflectivity coatings. Since the FM efficiency is inversely proportional to the photon density through the value of $\gamma_a$ it follows that its magnitude must be even smaller than 4.6 GHz mA$^{-1}$ at 2 mW. The author in [5] wrongly estimates the FM response in this theoretical example of the "inverted" mode operation to have a magnitude of 16 GHz mA$^{-1}$, thus theoretically supporting the 12.1 GHz mA$^{-1}$ experimentally observed.

Extant measurements of the FM efficiency in both the "normal" and "inverted" configurations carried out by others in a MQW InGaAsP gain-lever laser show a large difference in the magnitudes of the respective FM responses, respectively, 2.58 GHz mA$^{-1}$ and −295 MHz mA$^{-1}$ [7]. These measurements are significant because they differ from those in [5] in two important aspects.

Firstly, the magnitudes of the FM response in both the "normal" and the "inverted" modes are much smaller in the MQW InGaAsP gain-lever laser than the correspondent values measured by Lau et. al. in the GRIN SCH GaAs laser. Partially, this could originate from the gain profile in MQW lasers such as the one used in these measurements by McDonald et. al.. MQW laser structures exhibit a reduced gain saturation compared with that occurring at large carrier densities in SQW devices. As the gain-lever effect depends on a highly sub-linear gain profile, the relatively smaller gain saturation in MQW contributes towards reducing the leverage and consequently the magnitude of
the FM response.

Secondly, in the MQW InGaAsP device the static FM response in the “normal” and in the “inverted” mode is respectively, red-shifted (negative) and blue-shifted (positive). In the GRIN GaAs laser, these roles are reversed; The FM response is blue-shifted in the “normal” gain-lever and it is red-shifted in the “inverted” mode. This reversal in the signs of the FM response between the two different types of device is indicative of their different physics. From eqn. 4.5 at $\omega = 0$ and $\epsilon = 0$ it is clear that in a “normal” gain-lever, where $G'_a > G'_b$, the sign of the FM response can be either negative, if $\alpha_b > \alpha_a$, or it can be positive if $\alpha_b > \alpha_a$. The single-segment term proportional to $\epsilon$ is to small relatively to the two-segment contribution to make a change to the current aspect. Lau et. al. in their study of the optical FM in GRIN SQW gain-lever lasers use the approximation that the refractive index derivative is unchanged, i.e. $\mu'_a = \mu'_b$, when the laser sections are non-uniformly biased. Hence, the linewidth enhancement factor increases with increasing carrier density according to eqn. 4.2, [5, 87], and consequently the sign of the static FM response must be negative in the “normal” mode and positive in the “inverted” mode. This is confirmed by their measurements of the FM response.

On the contrary, both theoretical and experimental evidence in MQW lasers indicates that the linewidth enhancement factor decreases with increasing carrier concentration [94, 93], because in these lasers the refractive index derivative also decreases rapidly with increasing carrier concentration [93]. In such MQW gain-lever lasers, biased in the “normal” mode, $G'_a > G'_b$ but $\alpha_a > \alpha_b$ so the static FM response must be positive or blue-shifted. Conversely, in the “inverted” configuration, $G'_a < G'_b$ and $\alpha_a < \alpha_b$ so the static FM response must be negative or red-shifted. This is also confirmed by the measurements by McDonald et. al. of the FM response in a gain-levered MQW InGaAsP laser [7].

Regardless of the bias mode, when the FM response is blue-shifted (positive), as in the case illustrated by figure 4.6, the single-segment contribution always subtracts from the two-segment contribution at the resonance frequency. Consequently, given that the magnitude of both resonance peaks is comparable (contrasting with their static magnitudes) the overall FM response is smaller than it would be if they were to add.
In the MQW InGaAsP gain-lever laser it is the "normal" gain-lever bias mode that produces a blue-shifted FM response. Hence, the magnitude of the FM response in the MQW InGaAsP (2.58 GHz mA\(^{-1}\)) is relatively small compared to the magnitude of the FM response in the GRIN SQW GaAs gain-lever laser (\(-22\) GHz mA\(^{-1}\)).

4.6 Addition of amplitude and phase noise

In optical communication systems using optical frequency modulation usually employ an optical discriminator to convert the light phase variation into an amplitude signal, prior to detection by a high-speed photo-diode. The RIN and the phase noise in gain-lever lasers have been analysed. Here, the effect of performing an interferometric conversion of phase noise into amplitude noise is also analysed. In system applications it is the total converted noise that sets the noise floor.

4.6.1 Signal and noise correlation

The FM and IM modulation responses both result from the same current stimulus. These carriers, injected at a varying rate, when converted to photons, make up the IM response and also change the refractive index which leads to the frequency response. It is, therefore, not surprising that both responses are correlated as shown in eqn. 4.13.

The spontaneously emitted photons inside the laser cavity cause the average number of photons to fluctuate thus originating the RIN. Furthermore, whenever the number of photons change the instantaneous laser gain also changes. By the very physical laws of the laser medium, this change in gain is automatically accompanied by an instantaneous change in refractive index which in turn causes a fluctuation in phase. This explains why the RIN and the phase noise in lasers are also partially correlated.

From the set of rate equations C.4 to C.6 for both the photon and carrier density fluctuation, by considering the case of\( h = 1\) and frequencies much lower than\( \gamma_b\), one obtains the following equation for the field amplitude fluctuation\( \delta \rho\)

\[
\frac{\delta \rho}{dt} = -\frac{p_0 G_b G_k}{\gamma_b} \delta \rho + \frac{\Delta_p(t)}{p_0}
\]

(4.31)

where\( \delta \rho \equiv \delta p/p_0\). Similarly, from both eqn. 4.20 for the phase noise and eqn. C.6 for
the carrier density fluctuation, emerges the following equation linking both phase and field amplitude fluctuations

\[ \delta \phi = -\frac{1}{2} \alpha_b \frac{p_0 \Gamma G_b G_b'}{\gamma_b} \delta \rho + \Delta \phi(t) + \Delta_0(t) \]  

(4.32)

This equation clearly demonstrates the correlation between field fluctuations, \( \delta \rho \), and phase fluctuations, \( \delta \phi \). Furthermore, an extra noise term \( \Delta_0 \) has been added to eqn. 4.32 in order to describe the measured extra non-Schawlow-Townes linewidth component [95]. The value of the fluctuation term \( \Delta_0 \) is given by \( \beta \Delta \phi \), where \( \beta = \Delta \omega_0 / \Delta \omega_{ST} \), where \( \Delta \omega_0 \) is the extra linewidth component and \( \Delta \omega_{ST} \) is the Schawlow-Townes linewidth.

### 4.6.2 Noise addition and de-correlation

The effect of performing an interferometric conversion of phase to amplitude noise is now analysed. The output phase noise is the sum of two distinct noise contributions: a ‘direct’ part giving the amplitude transmittance through the interferometer of the laser amplitude noise given by eqn. 4.31, and an ‘indirect’ part depending on the instantaneous phase fluctuation of the laser field relative to the lasing line centre given by eqn. 4.32. This is expressed by the following equation [95]

\[ \delta \rho_0(t) = T \delta \rho_I(t) + T' \delta \phi(t) \]  

(4.33)

where \( T \) and \( T' \) are, respectively, the interferometer transmittance and its derivative with respect to the angular frequency of the light. Substituting eqn. 4.31 and eqn. 4.32 for the phase and intensity fluctuations in eqn. 4.33, gives

\[ \delta \rho_0(t) = \left( T - \frac{1}{2} \frac{\alpha_b T' \omega_b^2}{\gamma_b} \right) \delta \rho_I(t) + T' (\Delta \phi(t) + \Delta_0(t)) \]  

(4.34)

Since \( \alpha_b \) is a positive quantity, for some positive value of the interferometer slope \( T' \) the term involving \( \delta \rho(t) \) in eqn. 4.34 is zero. In this case only the uncorrelated phase noise term remains, which will produce less intensity noise than what existed in the original laser beam [95]. In appendix J the following expression is obtained relating the output...
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Phase noise and RIN addition

Laser power = 1 mW Laser power = 3 mW

^ - 1 4 0

10

-170

-160

-150

-140

-130

-120

-110

-100

-90

-80

-70

-60

-50

-40

-30

-20

-10

0

10

Discriminator slope, h

Discriminator slope, h

Figure 4.11: Left: Output RIN in a SQW “normal” gain-lever laser with a 520 µm cavity length as a function of the discriminator’ slope. The output power is 1 mW, $h = 0.77$, and the signal section is biased at $R = 0.286$ as in figure 4.2. Right: Output RIN in a SQW “inverted” gain-lever laser with a 520 µm cavity length. The output power is 3 mW, $h = 0.77$, and the signal section is biased at $R = 4$ as in figure 4.6.

RIN, after the FM to IM conversion, with the original laser RIN

$$
\langle \delta \rho^2 \rangle_O = \left[ \left( 1 - \frac{1}{2} \frac{T'}{T} \frac{\alpha \omega_b^2}{\gamma_b} \right)^2 + \left( \frac{1}{2} \frac{T'}{T} \frac{\omega_b^2}{\gamma_b} \right)^2 (1 + \beta) \right] \langle \delta \rho^2 \rangle_I
$$

(4.35)

It is possible to achieve a reduction of the output intensity noise by minimising eqn. 4.35 with respect to $T'$ [96, 97]. The value of the optical discriminator’ slope that achieves this optimum noise is

$$
T' = -\frac{\alpha}{1 + \alpha^2 + \beta} \frac{T \gamma_b}{\omega_b^2}
$$

(4.36)

Figure 4.11 plots the output RIN in a gain-lever laser after the FM to IM conversion by an optical discriminator with slope $T'$. The dip in the output RIN noise is a consequence of the partial de-correlation. effected by the optical discriminator [15].

4.7 Noise figure

Both configurations of the gain-lever laser can deliver a significant increase in their FM signal to noise ratio relatively to its single-section counterpart. However, this increase is a relative quantity and thus it is more informative to examine the noise figure of these devices.

As far as both FM and phase noise are concerned the noise figure, for frequencies
approaching 0 Hz can be approximately given by

\[
NF \approx 10 \log \left[ 1 + \left( \frac{eV}{\Gamma} \right)^2 \frac{R_{sp} \gamma_a^2}{p_0 (G_a - G_b')^2} \frac{R_I}{kT} \frac{g_{1-(1-h)R} \gamma_a + 1}{g - 1} \right] \tag{4.37}
\]

which assumes \( \alpha_b \gg 1 \) so that \( \frac{1 + \alpha_b^2}{\alpha_b^2} \approx 1 \). The FM noise figure in the equivalent single-section laser can be given by

\[
NF \approx 10 \log \left[ 1 + \left( \frac{eV}{\Gamma} \right)^2 \frac{R_{sp}}{p_0 \epsilon^2} \frac{R_I}{kT} \right] \tag{4.38}
\]

which gives 51 dB at 1 mW, and 55 dB at 3 mW in the low-threshold device. For this calculation the value of the spontaneous emission factor \( R_{sp} \) has been estimated from eqn. 4.22 assuming that the linewidth of the equivalent single-section devices is, respectively, 5 MHz and 2 MHz for the low-threshold device. These are best case values amid measurements of the spectral linewidth in similar GRIN SCH SQW lasers [91, 87].

Figure 4.12 plots the FM noise figure both for the “normal” and “inverted” configuration. The noise figure in the “normal” gain-lever laser is 11 dB at 1 mW, a 40 dB improvement over the FM noise figure of an equivalent single-section laser. It is also much smaller than the noise figure in intensity modulated bulk DFB lasers.

The noise figure in the “inverted” gain-lever laser is 38 dB at 3 mW. This is far from impressive when compared to the noise figure in the “normal” configuration. These
values differ due to the following two main factors. Firstly, as shown in figure 4.1, the value of the factor in eqn. 4.37 inside the inner \[ \cdot \] is approximately 5 times larger in the “inverted” bias mode than in the “normal” bias mode. Secondly, the inverse lifetime \( \gamma_a \) is roughly an order of magnitude larger in the “inverted” lever laser because it has a larger resonance frequency than the “normal” gain-lever. The relatively large value of \( \gamma_a \) is essential to produce a flat FM response curve from the low frequency region of the FM response up to the resonance frequency.

The noise figure here calculated only takes into account both the FM component of the laser light and the phase noise. However, in order to be measured by a photodetector the FM signal must be converted to an IM signal by an optical discriminator. Hence, the total system noise results precisely from the addition of the laser RIN and the phase noise, according to the theory in section 4.6.

Figure 4.13 and figure 4.14 plot both the system noise figure and system gain as a function of the slope of the discriminator. The region of the plots corresponding to larger values of the slope, \( T' \), exhibit large gain because it is proportional to the slope. At the same time, the phase noise component is also multiplied by the slope and hence the system noise figure becomes equal to that for the FM component of the light, plotted earlier. These plots demonstrate that a large gain can be obtained by frequency modulating the light emission of the gain-lever laser and using a discriminator with slope between \( 10^{-10} \) s and \( 10^{-9} \) s. Figure 4.14, plots the noise figure in the “inverted” gain-lever. In the region of large gain the FM noise figure remains unattractive. The dip in the noise figure occurring at \( T' = 6 \cdot 10^{-11} \) s stems from the partial cancellation between the correlated parts of both the phase and intensity noise.

The noise figure in an optical link employing an “inverted” gain-lever laser has been estimated by earlier researchers [15]. The laser is frequency modulated as described in [6, 5] with an average output power of 3 mW/facet. A low-Q Fabry-Pérot discriminator was inserted between the laser and a high-speed photodiode whose output was displayed on a microwave spectrum analyser. The slope of the FP was about \( T' = 10^{-11} \) s. The RF input drive into the laser was varied and the effective IM modulation depth at the output of the FP was recorded. It was estimated that the RF drive current into the
laser needed for 100% modulation depth is −42 dBm or 0.035 mA. The noise figure for this “inverted” transmitter can be calculated using the following equation

\[ NF = 10 \log \left[ 1 + 2 \cdot RIN \cdot \left( \frac{i_{\text{full}}}{i_{\text{thermal}}} \right)^2 \right] \]  

(4.39)

where \( i_{\text{full}} \) is the root mean square current input for 100% modulation and \( i_{\text{thermal}}^2 \cdot 50 \Omega = -174 \text{ dBm/Hz} \) [15]. The measured RIN for the transmitter was −124 dB/Hz giving a noise figure of 11 dB [15], which is much smaller than that predicted in figure 4.14. However, this estimate of the noise figure remains troublesome because of the following difficulty with the underlying values. The signal output power for 100% modulation was estimated at −17 dBm which is equivalent to 0.63 mA into 50Ω. On the other hand, the signal intensity at the output of the FP can be given by \( P_s = i_s \cdot \eta_{FM} \cdot T' \cdot P_0 \) where \( i_s \) is the input current drive into the laser, \( P_0 \) is the average power from the laser and \( \eta_{FM} = 16 \text{ GHz/mA} \) is the frequency modulation efficiency in this particular “inverted” gain-lever laser [5]. This gives an optical signal output power of 0.018 mW and assuming a 80% quantum efficiency photodiode, translates into a photo-current of 0.015 mA which falls very short of the 0.63 mA projected for 100% modulation depth.

There is a region of interest in figure 4.13 for intensity modulation in the “normal gain-lever” configuration corresponding to small \( T' \) where the device exhibits both low noise figure and a relatively large modulation efficiency or gain. Possible applications
of the gain-lever laser have been suggested both by Lau [15] and by Westbrook and Seltzer [3, 8]. The latter researchers at British Telecomm studied a similar gain-lever laser operating at 1.5 μm with the objective of predicting the device's intermodulation distortion within an intensity modulation scheme, with a view in mobile radio applications [8]. The next section discusses how both an improved modulation efficiency and noise figure in laser transmitters might be exploited in certain applications utilising subcarrier multiplexing techniques.

4.8 Gain-lever: useful for multichannel transmission?

The increases in both the IM and FM efficiencies in gain-lever lasers as well as their improved SNR have been studied in detail in both chapter 3 and the current chapter. It has been pointed out in the introductory chapter that these improvements may be advantageously exploited in some optical-RF fibre transmission systems.

Say that a laser A has a larger SNR compared to that in a laser B. How can such an improvement in the SNR translate into a system advantage? In numerous transmission applications there is a strict requirement to meet a specified CNR prior to demodulation, in order to warrant a given signal quality or bit error rate (BER). This requirement can be translated back to a required modulation index $m$ at the transmitter, according to
the expression
\[
CNR_{TX} = \frac{1/2m^2}{RIN \cdot B}
\]  
(4.40)
where \(B\) is the channel bandwidth [98]. If laser A has a larger SNR than laser B, it means that either laser A is less noisy than laser B, or laser A has a larger modulation efficiency compared to laser B. In the first case, if laser A is preferred as a transmitter instead of laser B, a smaller modulation index is required in order to meet a given CNR requirement at the output, for the same signal power. In the second case, the same modulation index is required of both lasers, but because laser A has a larger modulation efficiency, it will need a smaller input signal power to produce it. In both cases a smaller CNR is required at the transmitter’s input by using the laser with the best SNR or noise figure.

It was shown both in chapter 3 and in the current chapter that a gain-levered laser can provide significant increases in both their IM and FM efficiencies as well as SNR. Consequently, employing such a laser in an optical-RF link could be a serious proposition giving superior noise figure and a larger link gain compared to a link employing a conventional bulk laser.

However, before such a transmitter can be envisaged in applications that require simultaneous transmission of several subcarriers (subcarrier multiplexing) an assessment of the distortion performance of these lasers is required.

In particular, the enhanced FM efficiency in an “inverted” gain-lever laser has raised the case for transmission systems employing both optical FM and an optical discriminator (Fabry-Perot filter) to convert the FM component of the light back to IM [15]. Here, an assessment of both the FM distortion performance of the laser as well as of the non-linear FPF is clearly necessary to evaluate the merits of such a transmitter for SCM applications. The full assessment of the transmitter characteristics enables a more accurate estimate of the required output CNR, which can be given by [98]
\[
CNR_{TX} = \frac{1/2m^2}{RIN \cdot B + m^2C_2 + m^6C_3}
\]  
(4.41)
where \(C_2\) and \(C_3\) are transmitter coefficients representing second and third order non-linear effects, and their value will be calculated in the subsequent chapters.
Chapter 4. *Linear chirp and phase noise in the gain-lever laser*

Without anticipating the results of the subsequent chapters here are some issues prompted by the gain-lever laser transmitter.

- Both the laser and discriminator non-linearities must be modelled because both are likely to be significant in determining the system distortion [21].

- Similarly, the distortion in both the FM and IM components of the laser light needs to be assessed because it is not clear if any of the two is insignificant compared to the other.

- Trade-offs are likely to occur because while the efficiency (gain) of the FM to IM conversion is proportional to the slope of the discriminator, both the system noise and the discriminator's distortion also increase with increasing slope [21].

These preliminary considerations show that the overall assessment of the system performance is a complex task, and thus an accurate yet tractable method for modelling both the laser's and the discriminator's non-linearity is required. This will be carried out using the Volterra series method [41] which has recently been applied in the study of performance assessment of optical SCM systems using intensity modulated DFB laser transmitters [38].

### 4.9 Summary

In this chapter some characteristics of the optical FM in gain-lever lasers were studied based on an extended set of rate equations including one equation for the accumulated phase of the light. The phase noise in the gain-lever laser was also studied using a similar method.

A novel analytical expression for the optical FM response in the gain-lever was derived, which is similar but not identical to an earlier expression for two-section bulk lasers but with identical differential gains in each section. The plots of the theoretical FM response compare very well in shape with the respective experimental plots.

The overall FM response in gain-lever lasers is composed of a single-segment contribution term identical to the usual FM response in single-section lasers with a magnitude
proportional to the gain compression factor; and, in addition, of an enhancement term contribution which does not depend on $\epsilon$, but is proportional to a dimensionless parameter $E$ related to the ratio of the differential gains in each section. This term describes the "gain-lever effect" in the FM response.

Two novel expressions were derived, one giving the static FM efficiency improvement in a gain-lever laser relative to an equivalent single-section laser. The other expression gives the relative SNR improvement in a gain-lever laser. Both the "normal" gain-lever and the "inverted" gain-lever configurations were studied by plotting these expressions. It was found that the "normal" gain-lever configuration provides the largest increases in both FM efficiency and SNR. Biasing the signal section at a low gain maximises both these improvements; furthermore, reducing $h$ also helps to maximise the FM efficiency, whereas the SNR improvement only varies slightly with $h$ in this configuration.

On the other hand, the "inverted" lever configuration also provides large increases in both the FM efficiency and the SNR, smaller, however, than the increases afforded in the "normal" configuration. As a trade-off, the bandwidth of the FM response, over which these improvements can be measured, is larger in the "inverted" configuration than in the "normal" configuration. More importantly, it was found that both the FM efficiency and the SNR in the "inverted" mode are maximised when $h \to 1$. To the same end it is best to bias the signal section at a moderate $R = 2$ rather than at a larger value of $R$, because the former value may give larger improvements for practical values of $h \approx 0.8$.

These results may be relevant for the design of two-section gain-lever lasers. It was suggested that gain-lever lasers in either configuration can be advantageously employed to transmit microwave signals down optical fibre links in a number of applications. Both the throughput electrical gain and noise figure of the link should be significantly improved compared to links employing conventional bulk lasers. However, in SCM applications involving multichannel transmission, it is also necessary to access the effects of both the laser and optical discriminator non-linearities before the suitability of gain-lever transmitter architectures can be established. This is the subject of the next chapter.
Chapter 5

Non-linear distortion in DFB lasers

5.1 Introduction

The study of both the IM and FM modulation properties in gain-lever lasers has shown great increases in their efficiency which suggests their use as optical transmitters in microwave optical links could be advantageous.

However, a full assessment of their suitability as transmitters requires the evaluation of their distortion levels. Here, the source of the non-linearity in lasers is presented and a suitable method of analysis is introduced. This method relies on the rate equation model of the laser and is introduced for a DFB laser first. The reasons for this precedence are twofold. The application of the Volterra series method to the analysis of the dynamic laser non-linearity is a relatively new topic in the literature. It was found that there was scope for the development of analytical models that could be applied to the assessment of FM distortion in DFB lasers whilst encompassing the non-linearity of optical discriminators. A review of the extant literature on the topic reveals that earlier models resort to approximations.

The second reason is that the study of distortion in DFB lasers is essential to gauge the non-linear distortion in gain-lever lasers.

5.2 What is distortion?

Systems comprising devices that exhibit some non-linear property often produce distortion effects that are peculiar and require special analysis techniques. A non-linear system is one that creates different frequency components to the ones present in the
Chapter 5. Non-linear distortion in DFB lasers

input signal. An ideal implementation of a multiplier is a simple example of a non-linear device. If such a device is presented with two sinusoidal signals of different frequencies, say \( f_1 \) and \( f_2 \), the output signal will contain different frequencies to those at the input. Simple trigonometry shows that there will also be sinusoidal components both at the sum frequency \( f_1 + f_2 \) and at the difference frequency \( f_1 - f_2 \).

This type of non-linear behaviour must be clearly distinguished from linear distortion. Systems affected by linear distortion do not create new frequencies at the output, rather they change the amplitude and phase of the original frequency components so that the shape of the output signal no longer resembles the input signal. Furthermore, linear distortion is a nuisance whereas the former can be quite useful in many cases.

A richer example of a non-linear system is the ideal diode, which performs a squaring operation on the input signal. Reverting to the same scenario as in the multiplier case, and again invoking basic trigonometry, one finds that the output signal of such a diode will also contain signal components at twice the frequency of the original signals, \( 2f_1 \) and \( 2f_2 \), and at d.c. in addition to the original frequencies and the sum and difference frequency components. Moreover, real diodes or systems are never ideal, so in general their non-linearity is best described by an infinite power series around some quiescent operating point. The output of such a system often contains a very rich mixture of frequency components. It is the task of non-linear analysis to predict the amplitudes of these components.

One way of accomplishing this task is best understood by considering a simple non-linear resistor with a terminal current-voltage given approximately by a third-order polynomial

\[
I = g_1 V(t) + g_2 V^2(t) + g_3 V^3(t) + ... \tag{5.1}
\]

Assume an input probing signal of the form \( \exp(i\omega_1 t) \). The first-order response of the resistor is given by \( I_1(\omega_1) = g_1 \exp(i\omega_1 t) \). The coefficient \( g_1 \) is called the first-order transfer function and is identical to the frequency response in linear system's theory. Since the resistor is a non-linear device, the linear frequency response is not sufficient to fully describe the system's response. To obtain higher-order components the input must be probed by an input signal of the form \( \exp(i\omega_1 t) + \exp(i\omega_2 t) \). Substituting this
on the second term of eqn. 5.1 one obtains the second-order response of the non-linear resistor, given by

\[ I_2 = g_2 \left[ e^{i2\omega_1 t} + 2e^{i(\omega_1 + \omega_2)t} + e^{i2\omega_2 t} \right] = \sum_f \frac{2!}{m_1!m_2!} g_2 e^{i(m_1\omega_1 + m_2\omega_2)t} \]

(5.2)

The second-order response embraces a number \( f \) of all possible combinations of two complex exponentials satisfying \( \sum_{r=1}^2 m_r \), where \( m_r \) represents the number of times a particular frequency appears in any given term of eqn. 5.2. The coefficient \( g_2 \) is called the second-order transfer function of the non-linear resistor. Any current component in eqn. 5.2 can be given by the second-order transfer function multiplied by a factor \( 2! / m_1!m_2! \) giving the number of ways in which a particular frequency component can be obtained. For instance, the complex exponential at frequency \( \omega_1 + \omega_2 \) can be obtained in two different ways eg. \( \omega_1 + \omega_2 \) and \( \omega_2 + \omega_1 \). The order of the transfer function reflects the fact that it describes the frequency components obtained by combining two of the input frequencies.

The analysis proceeds by probing with a signal of the form \( \exp(i\omega_1 t) + \exp(i\omega_2 t) + \exp(i\omega_3 t) \). Substituting this signal in the cubic term of eqn. 5.1 one obtains the third-order response of the non-linear resistor, given by

\[ I_3 = g_3 \left[ 6e^{i(\omega_1 + \omega_2 + \omega_3)t} + e^{i3\omega_1 t} + e^{i3\omega_2 t} + e^{i3\omega_3 t} + 3e^{i(2\omega_1 + \omega_2)t} \right. \\
+ 3e^{i(2\omega_1 + \omega_3)t} + 3e^{i(\omega_1 + 2\omega_2)t} + 3e^{i(\omega_1 + 2\omega_3)t} + 3e^{i(\omega_2 + 2\omega_3)t} + 3e^{i(2\omega_2 + \omega_3)t} \]  \\
\left. = \sum_f \frac{3!}{m_1!m_2!m_3!} g_3 e^{i(m_1\omega_1 + m_2\omega_2 + m_3\omega_3)t} \right] 

(5.3)

The coefficient \( g_3 \) is the third-order transfer function of the non-linear resistor and all the output current terms in eqn. 5.3 can be expressed by it. It should be noted that these transfer functions are frequency independent because the non-linear resistor has no memory, i.e. the output signal \( I(t) \) does not depend on the input values of \( V(t) \) for times \( t_0 < t \).

Alternatively, consider a simple model of a non-linear capacitor

\[ Q = C_1V(t) + C_2 V^2(t) + C_3 V^3(t) + \ldots \]  

(5.4)
with a terminal current-voltage transfer characteristic given by

\[ I = C_1 \frac{dV(t)}{dt} + 2C_2 V(t) \frac{dV(t)}{dt} + 3C_3 V^2(t) \frac{dV(t)}{dt} + \ldots \]  

(5.5)

By probing with \( \exp(i\omega_1 t) + \exp(i\omega_2 t) \) one obtains the second-order response of the capacitor given by

\[ I_2 = C_2 \left[ \omega_1 e^{i\omega_1 t} + i(\omega_1 + \omega_2)e^{i(\omega_1 + \omega_2)t} + i\omega_2 e^{i\omega_2 t} \right] \]

\[ = \sum_{m_1,m_2} \frac{2!}{m_1!m_2!} iC_2 (m_1 \omega_1 + m_2 \omega_2) e^{i(m_1 \omega_1 + m_2 \omega_2)t} \]

(5.6)

The second-order transfer function is given by \( iC_2(\omega_1 + \omega_2) \) and depends on the frequency of the input signals. This is a consequence of the memory of the capacitor whose output \( I(t) \) depends on past values of the input voltage.

An important point to retain here is that the amplitude of any type of second-order component in eqn. 5.6 is proportional to the same second-order transfer function calculated at the frequency of that component. Likewise, all third-order components are proportional to the third-order transfer function, in the same manner as the linear frequency response is described by the first-order transfer function so common in linear system’s analysis.

### 5.3 Effects of non-linear distortion in SCM systems

In SCM systems a laser is used to transmit several electrical microwave sub-carriers modulated on an optical carrier. The sub-carriers can modulate either the light intensity of the laser (IM) or its optical frequency (FM). This method of transmission is called sub-carrier multiplexing (SCM). The actual information can be amplitude modulated (AM) or frequency modulated (FM) onto the sub-carrier. AM systems employ low frequency sub-carriers around a couple of hundred MHz where the dynamic laser distortion is lowest. FM systems tend to use microwave sub-carriers and operate in a region where the dynamic distortion prevails because it is closer to the resonance frequency.

Figure 5.1 depicts a typical channel allocation scheme for a SCM system comprising three sub-carriers. Some harmonics and intermodulation products (IMP’s) of different
types that arise due system non-linearities are also shown in their relative positions. It is clear that the third-order IMP’s of both type $2f_{k_1} - f_{k_2}$ and type $f_{k_1} + f_{k_2} - f_{k_3}$ fall nearer to the transmission band whereas second-order terms fall further away from this band.

5.3.1 Distribution of intermodulation products

The number of intermodulation products that fall inside the transmission band depends on the type of IMP being considered.

When the system’s transmission band occupies less than one octave, the second-order intermodulation products of the type $f_{k_1} + f_{k_2}$ fall outside the transmission band and need not to be taken into account.

In such cases, only the third-order IMPs need to be considered. It can be shown that, for systems where the channel allocation plan consists of a sequence of equally spaced carriers, the total number of products of the type $2f_{k_1} - f_{k_2}$ whose frequency coincides with the frequency $f_k$ of a particular channel $r$, is given by the formula

$$IMP_{2f_{k_1} - f_{k_2}}^r = \frac{1}{2} \left[ N - 2 - \frac{1}{2} \left(1 - (-1)^N\right)(-1)^r \right]$$  \hspace{1cm} (5.7)\]

where $N$ is the number of channels [99]. Furthermore, the total number of products of the type $f_{k_1} + f_{k_2} - f_{k_3}$ whose frequency coincides with the frequency $f_k$ of a particular
Chapter 5. Non-linear distortion in DFB lasers

channel \( r \), is given by the formula

\[
IMP_{f_{k_1} + f_{k_2} - f_{k_3}}^r = \frac{r}{2} [N - r + 1] + \frac{1}{4} [(N - 3)^2 - 5] - \frac{1}{8} [1 - (-1)^N](-1)^{N+r} \quad (5.8)
\]

It results from eqn. 5.8 that the total number of IMPs of the type \( f_{k_1} + f_{k_2} - f_{k_3} \) grows proportionally to \( N^2 \) whereas the number of IMP's of type \( 2f_{k_1} - f_{k_2} \) only grows proportionally to \( N \). Therefore, as \( N \) increases the IMP's of the former type become predominant. Furthermore, as \( N \) increases the distribution function, eqn. 5.8, asymptotically approaches the value \( 3N^2/8 \) for the centre channel, and the value \( N^2/4 \) for a channel at the extremes of the frequency plan.

5.4 Origins of non-linear distortion in lasers

Semiconductor lasers possess non-linearities of different origins. A major logical division is to consider static non-linearities on the one hand and dynamic non-linearities on the other hand.

Static non-linear effects arise because the laser's current-power \( (I - P) \) characteristic is not perfectly linear. The \( I - P \) curve describes the static (i.e. measured at d.c.) relationship between the current injected into the laser and its output optical power. Often this type of non-linearity is due to leakage currents around the laser's active region. Since the leakage current is usually proportional to the injected current, this results in a non-linear relationship between the input current and the output power.

In many applications a laser is biased at a given point in the \( I - P \) curve. The effects of the static non-linearity can be analysed by expressing the output power as a power series around the current bias point. For small signals, keeping only the first three terms in the series is usually an acceptable approximation. In this way the laser static \( I - P \) non-linearity becomes quite similar to the non-linear resistor discussed in eqn. 5.1. The effects of this type of non-linearity are most significant at low frequency, below a few hundreds of MHz and is an important issue for laser applications in CATV systems.

At higher frequencies, non-linear distortion in lasers is dominated by the dynamic non-linearity. Fundamentally, the injected carriers into a laser cavity are converted
into photons which stimulate further recombination of carriers thus producing yet more photons. This interaction produces a resonance in the photon density response that gets stronger when the frequency of the injected current approaches the resonance frequency. This phenomena causes non-linear distortion in lasers and has been studied by several researchers using the rate equations [40].

Spatial hole burning (SHB) results from a non-uniform gain profile along the axis of the laser cavity. A standing-wave pattern arises in the cavity and the gain is more saturated at the points which coincide with the crests of the wave than at the points that coincide with the valleys. The effects of SHB are usually included in the rate equations by introducing a phenomenological parameter called the gain compression factor as discussed in section 2.3.1.

### 5.4.1 DFB laser model

Many aspects of the dynamics of DFB lasers can be described by the single-mode rate equations given by

\[
\frac{dn}{dt} = \frac{j}{ed} - \gamma_e n - (1 - \varepsilon_p) G p
\]

\[
\frac{dp}{dt} = p \left[ \Gamma(1 - \varepsilon_p) G - \frac{1}{\tau_p} \right] + \beta R_{sp}
\]

\[
\frac{d\phi}{dt} = \frac{a \Gamma}{2} G' n
\]

where \( \varepsilon \) is the gain compression factor which describes a first-order gain reduction due to spectral hole burning and other gain non-linearities, \( \beta \) is the spontaneous emission factor and \( \gamma_e \) is the inverse carrier lifetime.

The optical frequency (or wavelength) shift of the lasing mode is related to the phase variation \( \phi \) through

\[
\Delta \nu(t) = \frac{1}{2\pi} \frac{d\phi}{dt} = \frac{a \Gamma}{4\pi} G' n
\]

where \( \alpha \) is the linewidth enhancement factor with a typical value of \( \alpha \approx 5 \).
5.4.2 Perturbation analysis

The laser rate equations are a non-linear set of equations for which an exact analytical solution can not be found. However, they can be numerically integrated. Alternatively, in the small-signal regime a perturbation technique is available which enables approximate analytical solutions to be found [100].

The perturbation technique requires that the signal applied to the laser can be considered a small perturbation compared to the bias input current. It then assumes that the perturbing signal does not affect the steady-state solution and therefore that the response to the perturbation can be found independently from the steady-state solution. In mathematical terms, the input current density is written as

\[ j = j_0 + j_1 \]  

where \( j_0 \) represents the steady-state current which is time-independent and \( j_1 \) is the small signal perturbation. The laser output variables, the carrier density, the photon density and the emission frequency variation, are also written as a sum of contributions up to a third-order

\[ p = p_0 + p_1 + p_2 + p_3 \]
\[ n = n_0 + n_1 + n_2 + n_3 \]
\[ \nu = \nu_0 + \nu_1 + \nu_2 + \nu_3 \]  

Substituting these expressions in the rate equations, and separating out the terms of different order, one obtains the following steady-state equations

\[ 0 = p_0 [\Gamma (1 - \epsilon p_0) G_0 (n_0) n_0 - 1/t_p)] + \beta R_{sp} \]
\[ 0 = \frac{j_0}{ed} - \gamma_c n_0 - (1 - \epsilon p_0) G_0 (n_0) p_0 \]
\[ 0 = \nu_0 - \frac{\alpha \Gamma}{4 \pi} G'(n_0) n_0 \]  

The equations for the higher order perturbations can be written compactly as

\[ \frac{d^m p}{dt^m} + p_n \frac{\beta R_{sp}}{p_0} - p_0 \Gamma ((1 - \epsilon p_0) G_0' n_n + e G_0 p_n) - \beta \frac{n_n}{\tau} = C_n \]
\[
\frac{d^n n_n}{dt^n} + \gamma n_n + (1 - 2\epsilon \rho_0)G_0 p_n = -D_n \\
\nu_n = \frac{\alpha \Gamma}{4\pi} G' n_n = 0
\]  

(5.16)

where $\gamma$ is the inverse stimulated lifetime, $D_n$ and $C_n$ represent the driving forces. These driving forces are products of lower order $n's$ and $p's$ which are known quantities. For example, when solving for the second-order equations, the driving forces will be products of lower order quantities which are known solutions both of the steady-state and first-order equations. The reward of the perturbation method is that both eqn. 5.15 and eqn. 5.16 are linear, in contrast to the initial rate equations.

### 5.4.3 Volterra series expansion

A powerful technique for the analysis of non-linear systems with multi-tone inputs was developed by Volterra and Wiener as an extension to linear systems theory [38, 36]. This technique is introduced and is applied to the analysis and prediction of the dynamic distortion effects in lasers.

Consider first a linear operation $T_1$ on a time-dependent function $x(t)$ representing a physical input signal, to produce an output signal

\[ y(t) = T_1[x(t)] \]  

(5.17)

Assume that the signal $x(t)$ is some linear combination of base functions $x_n(t)$ which forms a representation of the signal. With this representation eqn. 5.17 is now written as

\[ y(t) = T_1 \left[ \sum_{n=1}^{N} c_n x_n(t) \right] \]  

(5.18)

The assumed linearity of the operator $T_1$ implies that

\[ y(t) = \sum_{n=1}^{N} c_n T_1[x_n(t)] \]  

(5.19)

so it also follows that

\[ y(t) = \sum_{n=1}^{N} c_n y_n(t) \]  

(5.20)

This shows that the response of a linear operation on a linear combination of signals is the same linear combination of individual responses to each input base signal.
Take a particular representation for any input signal $x(t)$ built by choosing the shape of a base function $x_n(t)$ to be a thin rectangle with time width $\delta$ and amplitude given by $1/\delta$ so that the elementary rectangle has unit area. Using this type of base function, the signal $x(t)$ can be represented by a linear combination with appropriate coefficients of an infinite sum of elementary rectangles displaced in time by an amount equal to their width $\delta$. This representation in pictorial terms consists of a stair-case approximation to the function $x(t)$ where the coefficients are the amplitudes of the stair at any given time. In mathematical terms this representation can be given by

$$x(t) = \sum_{n=\infty}^{\infty} \delta x(n\delta) \text{rect}_\delta(t - n\delta) \quad (5.21)$$

and the linear operation upon this signal is given by

$$y(t) = \sum_{n=-\infty}^{\infty} \delta x(n\delta) T_1[\text{rect}_\delta(t - n\delta)] \quad (5.22)$$

where the linearity of the operator was used. Assume also that the operator $T_1$ besides being linear is also a time-invariant (LTI) operator. Then the action of $T_1$ on the rectangle function does not depend on its position in time but only on its shape. The finite sum in eqn. 5.22 can be made to converge to an integral by taking the limit when the width of the rectangle goes to zero. In that case the approximation to the output signal $y(t)$ converges to $y(t)$ and the rectangle converges to an impulse distribution. In the limit the output signal is represented by

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_1(\tau - t) d\tau \quad (5.23)$$

where $h_1(t)$ is the impulse response of the system which results from eqn. 5.22

$$h_1(t) = \lim_{\delta \to 0} T_1[\text{rect}_\delta(t - n\delta)] = H_1[\delta(t)] \quad (5.24)$$

where $H_1$ is the LTI first-order operator.

Consider now an input signal $x(t)$ given by a bi-linear combination of base functions $x_n(t), x_m(t)$. Consider also of a second-order bi-linear operator $T_2$ acting on an input
signal $x(t)$ given by a bi-linear combination of base functions

$$y(t) = \sum_{n=1}^{N} \sum_{m=1}^{N} T_2[(x_n(t), x_m(t))]$$  \hspace{1cm} (5.25)$$

Note that this equation explicitly takes into account that the operator is bi-linear by stating that the output signal is a bi-linear combination of the responses of the operator on each product of base functions.

The functional form for the operator $T_2$ can be obtained by choosing the same base function representation as before, consisting of an infinite series of thin rectangles with width $\delta$ so as to make a stair-case approximation to the input signal. The difference is that in the present case this approximation has to be two-dimensional and can be written

$$y_\delta(t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta x(n\delta) \delta x(m\delta) T_2[\text{rect}_\delta(t-n\delta) \text{rect}_\delta(t-m\delta)]$$  \hspace{1cm} (5.26)$$

Assuming the second-order operator is also LTI, its action on the rectangle functions does not depend on their position on time but only on their shape. In the limit where $\delta \to 0$ the stair-case perfectly approximates the input function and the output is given by

$$y(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x(\tau_1) x(\tau_2) h_2(\tau_1-t, \tau_2-t) \delta \tau_1 \delta \tau_1$$  \hspace{1cm} (5.27)$$

where $h_2(t)$ is the generalised two-dimensional impulse response of the system which results from

$$h_2(t, t') = \lim_{\delta \to 0} T_2[\text{rect}_\delta(t-n\delta), \text{rect}_\delta(t'-m\delta)] = H_2[\delta(t), \delta(t')]$$  \hspace{1cm} (5.28)$$

A similar procedure could be followed to find the functional form for the third-order Volterra operator and indeed, for any superior order. The final result is the Volterra functional series expansion given by

$$y(t) = \int_{-\infty}^{\infty} h_1(\tau_1) j(t-\tau_1) d\tau_1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) j(t-\tau_1) j(t-\tau_2) d\tau_1 d\tau_2$$

$$+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{r=1}^{n} j(t-\tau_r) d\tau_r + \cdots$$  \hspace{1cm} (5.29)$$

where the $h_n(\tau_1, \ldots, \tau_n)$ are the generalised impulse responses. Their multi-dimensional
Fourier transform is the $n$th-order transfer function given by

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{r=1}^{n} e^{-i\omega_r \tau_r} d\tau_r \quad (5.30)$$

### 5.4.4 Volterra kernels for the DFB laser

The validity of the perturbation analysis of the rate equations carried out in section 5.4.2 rests on the assumption that the output variables can be expressed as a Taylor's series around the bias point. This is acceptable if the solution to the equation does not depart significantly from the bias point as results from the convergence properties of power series. The solutions of the linear equations give each and every term of the Taylor's series expansion.

The linearised rate equations also allow the calculation of the kernel transforms (eqn. 5.30). The Volterra expansion eqn. 5.29 is the analogue of a Taylor's series expansion for the functional $y(t)$. Each successive order of the perturbational solution is a linear, bi-linear and tri-linear operation on time functions similar to the Volterra $n$th-order operators. Therefore, the $n$th-order kernel transforms can be obtained by taking the $n$th-dimensional Fourier transform of the $n$th-order perturbational solution of the rate equations. The $n$th-order kernels of the photon density and carrier density are given by

$$H_n(\omega_1, \ldots, \omega_n) = \frac{-\frac{1}{n!} \left( p_0 \Gamma G_0' + \frac{\beta}{\gamma} \right) D_n(\omega_1, \ldots, \omega_n) + \frac{1}{n!} C_n(i\Omega + \gamma)}{-\Omega^2 - i\Omega (\gamma + p_0 \Gamma G_0 + \frac{\beta R_{sp}}{p_0}) + A_1}$$

$$F_n(\omega_1, \ldots, \omega_n) = \frac{-\left( 1 - 2\epsilon p_0 \right) G_0 H_n(\omega_1, \ldots, \omega_n) + \frac{1}{n!} D_n(\omega_1, \ldots, \omega_n)}{i\Omega + \gamma} \quad (5.31)$$

where $\Omega = \omega_1 + \cdots + \omega_n$, $G_0$ is the laser's threshold gain, $\tau$ is the spontaneous carrier lifetime and $A_1$ is given by

$$A_1 = \gamma \left( p_0 \Gamma G_0 \right) + \frac{\beta R_{sp}}{p_0} + \left( 1 - 3\epsilon p_0 \right) p_0 \Gamma G_0 G_0' + \left( 1 - 2\epsilon p_0 \right) \frac{\Gamma \beta G_0}{\tau} \quad (5.32)$$

The driving forces $D_n$ are functions of $\Omega$ and are given by

$$D_1 = \frac{j_1}{ed} \quad (5.33)$$

$$D_2 = \left( 1 - 2\epsilon p_0 \right) G_0 [H_1(\omega_1) F_1(\omega_2) + H_1(\omega_2) F_1(\omega_1)] - 2\epsilon G H_1(\omega_1) H_1(\omega_2) \quad (5.34)$$
\[ D_3 = 2[(1 - 2e\rho_0)G']_0 [H_1(\omega_1)F_2(\omega_2, \omega_3) + H_1(\omega_2)F_2(\omega_1, \omega_3) + H_1(\omega_3)F_2(\omega_1, \omega_2)] + 2[(1 - 2e\rho_0)G']_0 [H_2(\omega_1, \omega_2)F_1(\omega_3) + H_1(\omega_1)F_1(\omega_2) + \epsilon G'H_1(\omega_1)H_1(\omega_2)F_1(\omega_1)] - 2\epsilon G'H_1(\omega_1)F_1(\omega_2) + H_1(\omega_1)H_1(\omega_2)F_1(\omega_1)] - 4\epsilon G'H_1(\omega_1)H_2(\omega_2, \omega_3) + H_1(\omega_2)H_2(\omega_1, \omega_3) + H_1(\omega_3)H_2(\omega_1, \omega_2)] \] (5.35)

and \( C_n = \Gamma \times D_n \), except for \( C_0 = 0 \).

Finally, the \( n \)-th-order kernel of the frequency variation of the laser light is simply given by

\[ \vartheta_n(f_1, \ldots, f_n) = \frac{\alpha}{4\pi} \Gamma F_n(f_1, \ldots, f_n) \] (5.36)

### 5.4.5 Amplitudes of harmonics and intermodulation products

As illustrated by figure 5.1 non-linear distortion consists in the generation of new frequency components which were not present at the input. The amplitudes of the several types of harmonics and IMPs can be calculated using the theory of the Volterra functional series.

Substituting the \( n \)-th-order impulse response in eqn. 5.29 by its Fourier transform pair (eqn. 5.30) one obtains

\[ p_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(\omega_1, \ldots, \omega_n) \prod_{r=1}^{n} J(\omega_r) e^{i\omega_r t} d\omega_r \] (5.37)

where \( J(\omega) \) is the spectrum of the input signal. When the input signal is a sum of complex exponentials, \( j(t) = \sum e^{i\omega_n t} \), its input spectrum is \( J(\omega) = \sum_{r=1}^{n} \delta(\omega - \omega_r) \), and it follows from the above expression that

\[ p_n(t) = \sum_{r=1}^{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(\omega_1, \ldots, \omega_n) \prod_{r=1}^{n} \delta(w - \omega_r) d\omega_r \] (5.38)

and becomes

\[ p_n(t) = \sum_{f} \frac{n!}{m_1! \cdots m_n!} H_n(m_1[\omega_1], \ldots, m_n[\omega_n]) e^{i(m_1 \omega_1 + \cdots + m_n \omega_n) t} \] (5.39)

where the sum embraces all possible combinations of \( n \) frequencies out of the \( n \) complex exponentials present at the input, satisfying \( \sum_{r=1}^{n} m_r = n \) and the different \( m_r \) represent the number of times a particular frequency \( \omega_r \) appears in any given term of eqn. 5.39. If
Table 5.1: Amplitudes of some second and third-order harmonics and IMP’s when two sinusoids are present at the input.

\[
\begin{array}{cccccc}
 n & m_{-1} & m_{-2} & m_1 & m_2 & \text{IMP} & \text{Amplitude} \\
2 & 0 & 0 & 2 & 0 & 2\omega_1 & 1/2j_1H_2(\omega_1, \omega_1) \\
2 & 0 & 0 & 1 & 1 & \omega_1 + \omega_2 & j_1j_2H_2(\omega_1, \omega_2) \\
3 & 1 & 0 & 2 & 0 & 2\omega_1 - \omega_2 & 3/4j_1^2j_2^2H_3(\omega_1, \omega_1, -\omega_2) \\
3 & 0 & 1 & 0 & 2 & 2\omega_2 - \omega_1 & 3/4j_1j_2^2H_3(\omega_2, \omega_2, -\omega_1) \\
\end{array}
\]

the input consist of real sinusoids the amplitudes of the resulting harmonics and IMP’s are given by

\[
A(\omega_1, \ldots, \omega_n) = \frac{n!j_{k_1}, \ldots, j_{k_n}}{2^{n-1}m_1!, \ldots, m_n!}H_n(\omega_1, \ldots, \omega_n)
\]  

where the \( j_k \) represent the amplitudes of the sinusoidal inputs. Table 5.1 lists the amplitudes of various harmonics and IMP’s in the case of a two-tone input.

### 5.5 Non-linear chirp in a DFB laser

Much work has been presented in the literature on the use of semiconductor lasers coupled with sub-carrier multiplexing (SCM) techniques as a means of transmitting analogue and/or digital signals over optical fibres. SCM applications range from broadband video distribution systems \[29, 25\] including CATV \[19\], bidirectional networks \[28\], satellite systems \[10\] and fibre radio applications \[27\]. These systems have mainly concentrated on the intensity modulation (IM) of the optical power by either varying the laser injected current or by the use of external modulators \[16\].

Recently, a technique has been proposed \[101\] that takes advantage of the additional frequency modulation (FM) modulation of the optical carrier when the laser is directly modulated by FM/IM conversion using an interferometer which can either be of the Fabry-Pérot or Mach-Zehnder type \[21\]. As in the IM case intermodulation distortion, which occurs when several channels modulate the laser diode, can be an important limiting factor. A theoretical study of the non-linear FM response of the semiconductor laser is therefore of relevance for these type of applications if an accurate assessment of system performance is to be carried out.
Preliminary results of the non-linear chirp have been given [89] but which neglect a term proportional to the gain compression factor in the frequency chirping formula (eqn. 8 of [89]). This structure dependent term is sometimes called "adiabatic chirp" [102]. It is found that these results deserve some important criticism. By using the Volterra kernels for the laser, all the intermodulation products up to third-order are included in the analysis and we show that "adiabatic" chirp is an important factor in determining the FM response: gain compression is responsible for a reduction of more than 10 dB in second and third-order FM distortion in the region where the laser is to operate, that is well above threshold and below resonance.

5.5.1 Background

The analytic approach taken by Le Bihan and Yabre in [89] is based on the Bessel function method which is found to be rather more involved than the perturbation technique [103]. Moreover, it involved an approximation to the chirping formula which has significant impact and needs amendment, as is explained below.

The theoretical analysis of [89] starts from the single-mode rate equations for the electron density, photon density and optical phase (see eqn. 5.9 to 5.11). A frequency chirping formula is then obtained that relates the time-dependent frequency deviation in terms of the optical waveform (eqn. 8 of [89]) which is used to evaluate the FM non-linearity. This equation may be written as

\[ \Delta \nu \approx \frac{\alpha}{4\pi} \frac{1}{p} \frac{dp}{dt} \]  \hspace{1cm} (5.41)

Comparing this result with the previous work by Koch and Link [102] one realises that it neglects a term involving the gain compression factor \( \varepsilon \)—the "adiabatic chirp". This discrepancy occurs due to the inclusion of the factor \((1 - \varepsilon p)\) in the rate equation for the optical phase (eq. 5.11) with the result that it cancels out when calculating the chirping formula (section 4).

Frequency chirp is a result of the dependence of the refractive index with the carrier density, which is assumed to be linear, so that the photon density does not explicitly appear in eqn. 5.4 [73]. Since the FM non-linear distortion of [89] is obtained from eqn. 5.41 the intermodulation results reported do not include the effect of adiabatic
Chapter 5. Non-linear distortion in DFB lasers

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$: Volume</td>
<td>$0.45 \cdot 10^{-10}$</td>
<td>cm$^{-3}$</td>
</tr>
<tr>
<td>$\Gamma$: Optical confinement factor</td>
<td>0.25</td>
<td>—</td>
</tr>
<tr>
<td>$G'_0$: Gain slope</td>
<td>$2.9 \cdot 10^{-6}$</td>
<td>s$^{-1}$cm$^3$</td>
</tr>
<tr>
<td>$\epsilon$: Gain compression factor</td>
<td>$2.5 \cdot 10^{-17}$</td>
<td>cm$^3$</td>
</tr>
<tr>
<td>$\beta$: Spontaneous emission factor</td>
<td>$1 \cdot 10^{-4}$</td>
<td>—</td>
</tr>
<tr>
<td>$\tau_p$: Photon lifetime</td>
<td>1</td>
<td>ps</td>
</tr>
<tr>
<td>$\tau$: Carrier lifetime</td>
<td>1</td>
<td>ns</td>
</tr>
<tr>
<td>$I_{thr}$: Threshold current</td>
<td>17.1</td>
<td>mA</td>
</tr>
<tr>
<td>$n_0$: Transparency carrier density</td>
<td>$1 \cdot 10^{18}$</td>
<td>cm$^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.2: Parameter values for a DFB-BH laser.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D'_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{p_0} \left{ \frac{\alpha}{4n \tau_p} H_1(\omega_1)H_1(\omega_2) - \frac{1}{3} [\vartheta(\omega_1)H_1(\omega_2) + \vartheta(\omega_2)H_1(\omega_1)] \right}$</td>
</tr>
</tbody>
</table>
| 3   | $\frac{1}{p_0} \left\{ \frac{\alpha}{6 \tau_p} [H_1(\omega_1)H_2(\omega_2, \omega_3) + H_1(\omega_2)H_2(\omega_1, \omega_3) + H_1(\omega_3)H_2(\omega_1, \omega_2)] 
- \frac{1}{3} [\vartheta(\omega_1)H_2(\omega_2, \omega_3) + \vartheta(\omega_2)H_2(\omega_1, \omega_3) + \vartheta(\omega_3)H_2(\omega_1, \omega_2) 
+ H_1(\omega_1)\vartheta_2(\omega_2, \omega_3) + H_1(\omega_2)\vartheta_2(\omega_1, \omega_3) + H_1(\omega_3)\vartheta_2(\omega_1, \omega_2)] \right\}$ |

Table 5.3: Coefficient $D'_n$ of equation (5.43).

chirp. The ensuing consequences are discussed in what follows.

5.5.2 Non-linear chirp distortion

The Volterra series analysis which was discussed in some detail requires the calculation of the kernels of the system, also known as non-linear transfer functions. In section 5.4.4 the transfer functions $F_n(\omega)$ and $H_n(\omega)$, (with $\omega = \omega_1, ..., \omega_n$) associated with the carrier density and photon density, respectively, were calculated analytically. $H_n(\omega)$ determines the IM non-linear response of order $n$. The corresponding transfer function $\vartheta_n(\omega)$ associated with the laser FM response is directly proportional to the transfer function $F_n$ for the electron density and is given by eqn. 5.36. The results of table 5.1 may then be directly applied to the evaluation of the non-linear chirp once the linewidth
enhancement is specified.

Alternatively, and for comparison purposes, it is advisable to relate $\vartheta_n$ with $H_n$. This is achieved starting from eqn. 5.10 and neglecting spontaneous emission. Solving eqn. 5.10 for $G' n$ and substituting the result in eqn. 5.12 yields

$$
\Delta \nu \simeq \frac{\alpha}{4\pi} \left[ \frac{d}{dt} \ln(p) + \frac{\varepsilon}{\tau_p} \right]
$$

(5.42)

This is essentially the same equation given by Koch and Link in [102].

Proceeding with the calculation of the transfer functions and upon expanding $\ln(p)$ in a Taylor series around the bias point, for which the photon and carrier densities are $p_0$ and $n_0$, the final result for the $n$th-order transfer function is

$$
\vartheta_n(\omega_1, \ldots, \omega_n) = \frac{\alpha}{4\pi p_0} \left[ i(\omega_1 + \cdots + \omega_n) + \frac{\varepsilon}{\tau_p} p_0 \right] 
$$

$$
\times H_n(\omega_1, \ldots, \omega_n) + D'_n
$$

(5.43)

where the term $D'_n$ is given in table 5.3.

One is now in a position to evaluate the effect of gain compression on the FM intermodulation distortion. From eqn. 5.43 to a first approximation one expects the adiabatic chirp to become important at frequencies much lower than $p_0/\tau_p$ and at high bias currents.

It is appropriate to note that in 5.41 the gain compression term, $\varepsilon/\tau_p p_0$, was neglected. This has the effect that at low frequencies the modulation efficiency for FM appears to be very much reduced accentuating the relative influence of intermodulation distortion. It is found, however, that once the gain compression term is appropriately included the FM efficiency does not fall off at low frequencies and so the relative level of intermodulation distortion to signal is much improved. This is appreciated most easily by reference to specific illustrative results.

### 5.5.3 Results

Here it is considered, as an illustrative case, a distributed feedback buried-heterostructure (DFB-BH) laser with parameters taken from [104] and given in table 5.2. The package and laser chip parasitics are modelled as a RC network with $R = 5\ \Omega$ and
C = 8\,\text{pF} corresponding to a $-3\,$dB cutoff frequency of 4\,GHz.

The laser IM and FM responses to a sum of three frequencies are plotted in figures 5.2 to 5.4. The normalised FM and IM first-order transfer functions, also known as modulation response, are plotted in figure 5.2 (solid lines) for frequencies $\geq 100\,$MHz. These theoretical curves are identical to the ones obtained by Vodhanel et al. [104] through the use of a small-signal analysis of the rate equations, which he has shown to match closely the measured FM and IM responses for frequencies above 45\,MHz. Over this frequency range these results also agree in general with the reported experimental and theoretical results of Kobayashi [105]. The FM response of the DFB-BH laser used exhibits a characteristic dip in magnitude at approximately 300\,kHz [104] (not shown), typical of BH lasers, which is due to thermal effects [105], that can be neglected over the frequency range of interest ($> 50\,$MHz) and is therefore not included in this analysis.

Figures 5.3 and 5.4 show the second and third-order distortion relative to the carrier at $f_1$, of type $f_1 + f_2$ and $f_3 + f_2 - f_1$, respectively. The results were obtained for a bias current equal to twice the threshold current $I_0 = 2I_{th}$, corresponding to a resonance frequency of $f_0 = 6.5\,$GHz, $f_2 = f_1 + 0.2\,$GHz, $f_3 = f_2 + 0.5\,$GHz and for a constant optical modulation depth (OMI) of 20\%. As in [89] it is seen that both IM and FM second-order intermodulation distortion exhibit a maximum at approximately half the resonance frequency. However, no peak is seen at $f_0$ because our results are plotted

![Figure 5.2: IM and FM modulation response of a DFB-BH semiconductor laser biased at $2I_{th}$; (——) $\varepsilon = 8.62 \times 10^{-3}$, (-----) $\varepsilon = 0$ in chirp formula (5.42).](image-url)
Figure 5.3: Second-order FM and IM intermodulation distortion of type \( f_1 + f_2 \): \( I_0 = 2 I_{th}, \ f_2 = f_1 + 0.2 \ \text{GHz} \) and modulation depth \( m = 0.2; \ (---) \ \varepsilon = 8.62 \times 10^{-3}, \ (-\cdots) \ \varepsilon = 0 \) in chirp formula (5.42).

for a constant OMI (as in [106]) whereas in [89] the results are for a constant input signal current. For the latter case the OMI is higher at resonance thereby increasing the levels of distortion. The third-order intermodulation show maxima at \( f_0 \) and \( f_0/2 \). For the same reason given above the peak at \( f_0 \) is reduced, assuming constant OMI, when compared with the results of [89].

More important is the effect of the gain compression factor on the FM non-linear distortion. This effect can also be seen in these figures which show the modelled response when gain compression is neglected (dashed line), that is, taking \( \varepsilon = 0 \) in the chirp formula (5.42). This effect which is responsible for the reduction of the non-linear distortion at low frequencies becomes quite significant below \( f_0/2 \) where the laser is to be operated. At 1 GHz the reduction on the second and third-order distortion is 11 dB and 15 dB, respectively. It can be seen that in this region the FM second-order distortion of type \( f_1 + f_2 \) is higher than the corresponding IM distortion, yet the cross-over point occurs at lower frequencies than predicted when neglecting the term \( \frac{\varepsilon}{\tau_p} p \). On the other hand, the FM third-order distortion is lower than the IM distortion. At low frequencies the relative amplitude of the FM intermodulation product (\( \approx -60 \text{ dB} \)) shows only a slight increase with frequency until it reaches the relaxation oscillation frequency after which it decays rapidly. Furthermore, since for a large number of channels the distortion
Figure 5.4: Third-order FM and IM intermodulation distortion of type $f_3 + f_2 - f_1$: $I_0 = 2I_{th}, f_2 = f_1 + 0.2\, \text{GHz}, f_3 = f_2 + 0.5\, \text{GHz}$ and modulation depth $m = 0.2$; (-----) $\varepsilon = 8.62 \times 10^{-3}$, (-----) $\varepsilon = 0$ in chirp formula (5.42).

of type $f_3 + f_2 - f_1$ is the most important it can be concluded that FM modulation of the laser frequency is advantageous leading to better performance with respect to the laser non-linearity. This, in contrast to the conclusion of Le Bihan and Yabre [89], is in agreement with the general view given in Reference [21]: for FM modulation the laser requires lower signal amplitudes for the same OMI thereby behaving in a more linear fashion.

The "adiabatic chirp" is an important parameter in reducing the levels of distortion at frequencies below half the resonance of the laser. The results show that in SCM applications FM modulation of the laser followed by FM to IM interferometric conversion may place less stringent requirements on the laser non-linearity.

To actually compare IM and FM modulation in a system such as those described [21], an overall assessment of the laser and interferometer non-linearity and their combined effect is required. This is the subject of the next section.

5.6 Non-linear distortion in a DFB laser and FPF

Direct FM of diode lasers has been proposed for signal transmission in analogue systems because this method requires less modulation current than the IM methods making it likely for the laser to behave in a more linear fashion. This in turn would relax the
Chapter 5. Non-linear distortion in DFB lasers

stringent linearity criteria normally required from lasers [21].

Systems using optical FM must also employ an optical discriminator to convert the frequency deviation back into an amplitude signal. Systems employing a Fabry-Pérot (FP) optical discriminator to select one of various optical wavelengths in a WDM-SCM system, followed by direct detection of each optical carrier have also been studied [101, 107, 108]. In such systems the signal can be either optically frequency modulated [101, 107] or intensity modulated [108] to a variable extent. In any case, both IM and FM are present when the laser is modulated. The output signal is the result of the interaction of the transmission characteristic of the discriminator with the laser chirp and with the accompanying intensity modulated component.

Moreover, both the IM and FM components of the output signal are accompanied by intrinsic distortion due to the laser. It is by no means obvious if any of these contributions can be neglected. Therefore, in order to accurately model the distortion in these systems it may not be sufficient to take into account only the IM and FM laser responses. Both the intrinsic IM and FM distortion and their interaction with the non-linear transmission of the discriminator must be included in the model.

Previous studies of the non-linear distortion in these systems have not dealt with the problem in such a general way. In some cases the laser IM has been completely neglected and only the discriminator non-linearity has been taken into account while also neglecting the intrinsic laser's IM non-linearity, e.g. [21]. In other cases, both the IM and FM responses were taken into consideration along with the non-linear transmission of the optical discriminator. However, neither the FM nor the IM laser's non-linearities were taken into consideration [107, 108].

Here, the theory of Volterra kernels developed in the following section is applied to capture the non-linear distortion effects in a system where a laser is directly modulated and its light is passed through an optical discriminator prior to direct detection by a high speed diode. The model presented encompasses the intrinsic distortion of the laser, reflected both in the FM and IM components of the laser field, and its interaction with the non-linear transmission characteristic of the discriminator.
5.6.1 Kernels for IM and FM distortion plus FPF

The photon density at the output of the interferometer, $P_0$, can be obtained by referring to the common definition $P_0 = T(\nu)p$ where $T$ is the transmission characteristic of the discriminator as a function of the optical frequency $\nu$ and $p$ is the photon density at the output of the laser. This equation should be a good approximation since the response of the optical filter is rapid enough to follow the change in the wavelength [108]. Moreover, if the bandwidth of the microwave signal for each channel is small compared with the optical bandwidth of the discriminator, the output of the FPF is simply the input weighted by the filter response at the appropriate frequency [107]. The transmission $T$ is expanded in a Taylor's series up to a third-order in the optical frequency. Expressing the frequency deviation and photon density as a sum of zero, first, second and third-order contributions, $\Delta \nu = \nu_1 + \nu_2 + \nu_3$, $p = p_0 + p_1 + p_2 + p_3$, and separating out terms of equal order, one obtains

$$P_0 = Tp_0$$

$$P_1 = Tp_1 + T'\nu_1 p_0$$

$$P_2 = Tp_2 + T'\nu_1 p_1 + T''\nu_2 p_0 + 1/2T'''\nu^2_1 p_0$$

$$P_3 = Tp_3 + T'\nu_1 p_2 + T'\nu_2 p_1 + 1/2T'''\nu^2_1 p_1$$

$$+ T'\nu_3 p_0 + T''\nu_1 \nu_2 p_0 + 1/6T'''\nu^3_1 p_0$$

where $T'$, $T''$ and $T'''$ are the first three derivatives of the discriminator’s transmission function. These equations express the $n$th-order component of the photon density at the output of the interferometer as a sum of $n$th-order terms involving both the optical frequency deviation and the laser’s output photon density variation. Taking the Fourier transform of eqns. 5.45, 5.46 and 5.47 using the result in eqn. 5.39, the first three transfer functions for the photon density at the output of the discriminator are obtained

$$P_1(\omega_1) = TH_1(\omega_1) + T'\vartheta_1(\omega_1)p_0$$

$$P_2(\omega_1, \omega_2) = TH_2(\omega_1, \omega_2)$$

$$+ 1/2T' [H_1(\omega_1)\vartheta_1(\omega_2) + H_1(\omega_2)\vartheta_1(\omega_1)]$$

$$+ T'\vartheta_2(\omega_1, \omega_2)p_0 + 1/2T''\vartheta_1(\omega_1)\vartheta_1(\omega_2)p_0$$

(5.49)
Chapter 5. Non-linear distortion in DFB lasers

\begin{align}
P_3(\omega_1, \omega_2, \omega_3) &= TH_3(\omega_1, \omega_2, \omega_3) \\
&\quad + \frac{1}{3} T' [H_1(\omega_1) \vartheta_2(\omega_2, \omega_3) + H_1(\omega_2) \vartheta_2(\omega_1, \omega_3) + H_1(\omega_3) \vartheta_2(\omega_1, \omega_2)] \\
&\quad + \frac{1}{3} T' [\vartheta_1(\omega_1) H_2(\omega_2, \omega_3) + \vartheta_1(\omega_2) H_2(\omega_1, \omega_3) + \vartheta_1(\omega_3) H_2(\omega_1, \omega_2)] \\
&\quad + \frac{1}{6} T'' [H_1(\omega_1) \vartheta_1(\omega_2) \vartheta_1(\omega_3) + H_1(\omega_2) \vartheta_1(\omega_1) \vartheta_1(\omega_3) + H_1(\omega_3) \vartheta_1(\omega_1) \vartheta_1(\omega_2)] \\
&\quad + T'' \vartheta_3(\omega_1, \omega_2, \omega_3) p_0 \\
&\quad + \frac{1}{6} T''' \vartheta_1(\omega_1) \vartheta_1(\omega_2) \vartheta_1(\omega_3) p_0 \\
&\quad + \frac{1}{3} T''' [\vartheta_1(\omega_1) \vartheta_2(\omega_2, \omega_3) + \vartheta_1(\omega_2) \vartheta_2(\omega_1, \omega_3) + \vartheta_1(\omega_3) \vartheta_2(\omega_1, \omega_2)] p_0 \quad (5.50)
\end{align}

These transfer functions enable the calculation of the magnitude of any second and third-order harmonics or IMPs present at the output of the discriminator as a result of any number of sinusoidal signals applied to the laser provided that the above transfer functions replace the ones shown in table 5.1.

### 5.6.2 Discussion and results

The system studied comprises a DFB buried heterostructure laser emitting at 1.53 µm, followed by a Fabry-Pérot optical filter (FPF). The laser is biased at an optical power of 1.37 mW corresponding to a resonance frequency of 5 GHz. The laser parameters used in this simulation are taken from reference [104] and are given in table 5.1. The package and laser chip parasitics are modelled as a RC network with \( R = 5 \Omega \) and \( C = 8 \text{ pF} \) corresponding to a \(-3 \text{ dB}\) cutoff of 4 GHz.

As a check on the analytically developed model we compare plots (figure 5.5) of the transfer functions given in eqns. 5.48, 5.49 and 5.50 (lines) with the same transfer functions obtained by numerical integration of the laser’s rate equations coupled with the FPF’s transmission function (open circles). The numerical integration of the laser rate equations is carried out in the time domain. This is followed by a fast Fourier transform of the temporal response, which gives the total amplitude of any spectral line generated by the laser [31]. This method is only practical for a few (two or three) input frequencies and consumes a lot more processing power than the series approach.

Since the Volterra approach to the calculation of distortion is based on a convergent series, it cannot give exact results once the series is truncated. In the case of two input
modulating carriers at frequencies $f_1$ and $f_2$ one may be interested in the total distortion power appearing at frequency $2f_1 - f_2$. There is only one third-order term in the series that falls at that particular frequency. Its amplitude is given in table 5.1. However, there are also two other fifth-order terms that fall at the frequency of interest. These are terms involving the different combinations of the input frequencies such as $3f_1 - f_2 - f_1$ and also $2f_1 - 2f_2 + f_2$. But to consider contributions arising from all orders would render the series method totally impractical. Fortunately, the series is usually rapidly convergent which means that higher order distortion products have insignificantly lower amplitude. The number of terms in the series required to calculate the total amount of distortion within a given accuracy at a particular frequency depends on the amplitude of modulation and the strength of the laser's non-linearity.

Figure 5.5: Plot of the first, second and third-order transfer functions in eqns. 5.48, 5.49 and 5.50, respectively. The lines correspond to the Volterra series method. The open circles correspond to the numerical simulation using the intensity model for the FPF response. The crosses (+) correspond to the numerical simulation using the electrical field model. FPF bias: $T = 0.75$ and $T' = 8 \cdot 10^{-11}$ s. (A) $P_1(\omega_1)$, (B) $P_2(\omega_1, \omega_2)$ and (C) $P_3(\omega_1, \omega_1, -\omega_2)$, with $f_2 = f_1 + 0.2$ GHz.

As seen in figure 5.5, the analytical transfer functions are in good agreement with the numerical simulation using the equation $P_O = T(\nu)p$. This shows that the analytical model is justified whenever the above equation also holds.

However, this equation is in itself an approximation that facilitated the derivation
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of eqns. 5.48, 5.49 and 5.50. The accurate equation for the FPF response is given by

\[ E_0(\nu) = H_{FP}(\nu) \times E_i(\nu) \] \hspace{1cm} (5.51)

where \( E(\nu) \) represents the frequency spectrum of the laser field and \( H_{FP}(\nu) \) is the filter's frequency response, given by

\[ H_{FP} = \frac{1 - R}{1 - R \exp(i2\nu \tau)} \] \hspace{1cm} (5.52)

where \( \tau \) is the single-pass delay time. Therefore, the transfer functions of the system were also numerically calculated using the electrical field model and the results are also plotted in figure 5.5 (crosses). Comparison of the results of this simulation with those obtained using the approximate equation shows good agreement between the models where the bandwidth of the FPF is much larger than the maximum frequency present in the spectrum of the input signals. Therefore, the analytical model is justified provided that those conditions are met. For FM modulated signals, from Carson's rule, the bandwidth of the FPF should exceed at least twice the largest microwave frequency.

In what follows a simple study of a typical IMP2 at frequency 2\( f_1 \) and a third-order IMP3 at frequency \( f_1 + f_2 - f_3 \) will be carried out which shows important characteristics of the system performance. Furthermore, the relative contribution of each term in eqns. 5.49 and 5.50 to the overall system distortion will be examined.

Second-order distortion

When the discriminator is biased at the point of maximum slope, where \( T'' = 0 \), the fourth term in eqn. 5.49 is zero. In this case the second term is dominant but the third term also becomes significant at higher frequencies above approximately 1 GHz. The second term is determined by both the magnitudes and relative phase of the IM and FM laser responses, which shows that they are crucial to the modelling of system distortion [107, 108].

The first term in eqn. 5.49 only becomes significant when the IM component is larger than the FM component (i.e. \( T'' < 1.4 \cdot 10^{-10} \) s with \( T = 0.5 \)).

If \( T'' \neq 0 \), the discriminator (fourth term) can contribute significantly to the system's second-order distortion. When the ratio \( |T''/T'| \geq 1.6 \cdot 10^{-10} \) s, the magnitude of the
fourth term is comparable or larger than the magnitudes of the terms proportional to $T'$. The relative distortion of the discriminator, $IMP2_{\text{discr}}$, can be defined as the ratio of the fourth term and the FM signal power $T'\vartheta(\omega_1)p_0$. Its value, for a constant modulation index per channel $m$, is given by

$$IMP2_{\text{discr}} = \frac{\alpha_2^2}{16} T'^2 \left( \frac{T''^2}{T'^2} \right) m^2$$

(5.53)

where $\alpha_2$ depends on which IMP2 is being considered ($\alpha_2$ is at worst 1), and agrees with [21].

Figure 5.6: Relative power of harmonic distortion, located at frequency $2f_1$ as a function of $f_1$. The OMI is constant and equal to 4% for curve D. The output carrier power is the same for all curves.

Curve A) $T'\nu_1p_0 = 0.35T'p_1$ with $T' = 9 \times 10^{-11}$ s, $T = 0.75$ and $T'' = 0$.

Curve B) same conditions as for curve A but excluding the third term in eqn. 5.49.

Curve C) $T' = 9 \times 10^{-11}$ s, $T = 0.935$ and $T'' = -5 \times 10^{-20}$ s$^2$.

Curve D) pure IM (without discriminator).

The second-order harmonic distortion, relative to the carrier power, is plotted in figure 5.6 as a function of frequency $f_1$. The discriminator is a FPF with a bandwidth $\approx 10$ GHz and the FM index is $\approx 0.28$ at 1 GHz. Curve A plots the harmonic distortion when $T'' = 0$. Note that increasing $T'$, while maintaining a constant optical modulation index (OMI), decreases the relative distortion since this is proportional to the square of the amplitude of the modulating current which decreases with increasing slope for a
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constant OMI.

Curve B is plotted in the same conditions as curve A but the third term in eqn. 5.49, involving the laser’s FM distortion, was omitted. A comparison of curve A with curve B shows that the laser’s FM intrinsic distortion is also critical to model the system’s distortion, especially at higher frequencies. This term has not been included in previous analysis [21, 108, 107, 101]. However, the omission of FM distortion in reference [108] is a good approximation because the sub-carriers are located between 91.25 MHz and 403.25 MHz where the third term is negligible.

Curve C shows that a large cancellation of the second-harmonic distortion can be obtained at low frequency by off-setting the discriminator’s bias point from its maximum slope at \( T = 0.75 \). The origin of this cancellation can be understood from eqn. 5.49. The phase difference of the FM and IM responses in a DFB laser is zero at d.c.. Therefore, at low frequency, the second and fourth terms are roughly opposed in phase, because when \( T' > 0 \) and \( T > 0.75 \), then \( T'' < 0 \). Furthermore, the FPF can be biased in such a way that the magnitudes of those two terms also become equal, by changing \( T'' \), thereby causing the observed cancellation.

Curve D plots the harmonic distortion for the case of pure IM (without discriminator) for comparison purposes. The output carrier power is the same in all curves. A comparison between curve A and curve D shows that the system’s harmonic distortion performance is markedly better (curve A) than in the case of pure IM (curve D) at high frequency. At low frequency the system’s distortion performance can be very significantly improved by optimising the bias point of the FPF (curve C).

**Third-order distortion**

The contribution of each of the seven terms of eqn. 5.50 to the total third-order distortion will now be discussed.

When \( T'' = 0 \), the sixth term in eqn. 5.50 is largely dominant for \( |T''/T'| \geq 2 \cdot 10^{-18} \text{s}^2 \). If the equality holds, the magnitude of the terms proportional to \( T' \) is less than 1% of the magnitude of the sixth. The magnitude of the first term only becomes comparable to the magnitude of the terms proportional to \( T' \) when the IM component of the signal is larger than the FM component.
Figure 5.7: Relative power of IMP3 as a function of \( f_1 \), located at frequency \( f_1 + f_2 - f_3 \). The OMI is constant and equal to 4\% per channel for curve D. The output carrier power is the same for all curves.

Curve A) \( T'\nu_1p_0 \approx 0.38T\nu_1p_0 \) with \( T = 0.75 \) and \( T' = 8 \cdot 10^{-11} \text{s}, T'' = 0 \) and \( T''' = -5.4 \cdot 10^{-30} \text{s}^3 \).

Curve B) Same conditions as for curve A excluding all the terms in eqn. 5.50 but the sixth.

Curve C) \( T'\nu_1p_0 \approx 0.43T\nu_1p_0 \) with \( T = 0.67 \) and \( T' = 8 \cdot 10^{-11} \text{s}, T'' = 4.7 \cdot 10^{-21} \text{s}^2 \) and \( T''' = -3.6 \cdot 10^{-30} \text{s}^3 \).

Curve D) Same conditions as for curve C, excluding all terms but the ones proportional to \( T'' \) and \( T''' \) in eqn. 5.50.

Curve E) Same conditions as for curve C, excluding the seventh term in eqn. 5.50.

Curve F) pure IM (without discriminator).

The dominant sixth term accounts for the discriminator’s IMP3 performance. The relative distortion of the discriminator, \( \text{IMP3}_{\text{discr}} \), can be defined as the magnitude ratio of the sixth term and the FM signal power \( T'\varphi(\omega_1)p_0 \). For a constant modulation index per channel \( m \), its value is given by

\[
\text{IMP3}_{\text{discr}} = \frac{\alpha_3}{576} T^4 \left( \frac{T''}{T'^3} \right)^2 m^4
\]

(5.54)

where \( \alpha_3 \) depends on which IMP3 is being considered, and agrees with [21]. For a typical IMP3 at frequency \( f_1 + f_2 - f_3 \), \( \alpha_3 = 1/6 \).

When \( |T''/T'''| \geq 1.1 \cdot 10^9 \text{s} \), the magnitudes of the terms proportional to \( T'' \) become
Figure 5.8: Second and third order derivatives, \( T'' \) and \( T''' \), of both the FPF's transmission (left) and of the Mach-Zehnder's transmission (right), as a function of transmission for a constant value of \( T' = 8 \cdot 10^{-11} \) s.

Comparable or larger than the magnitude of the sixth term.

Figure 5.7 plots the system's IMPS for three tones at frequencies \( f_1, f_2 = f_1 + 0.2 \text{ GHz} \) and \( f_3 = f_1 + 0.4 \text{ GHz} \), falling at the frequency \( f_1 + f_2 - f_3 \), with the frequency \( f_1 \) being swept. The FPF has a bandwidth of \( \approx 16 \text{ GHz} \). The optical FM index is about 0.25 at 2 GHz.

Curve A plots the system's IMPS with the FPF biased at \( T = 0.75 \), where \( T'' = 0 \) [21]. A comparison of curves A and B shows that the system's distortion is dominated by the term proportional to \( T''' \), which is due to the FPF non-linearity.

Biasing the FPF at a lower \( T \) while keeping \( T' \) constant, increases \( T'' \) but decreases \( T''' \) as shown in figure 5.8 which plots both \( T'' \) and \( T''' \) as a function of the transmission \( T \), for a constant value of the slope \( T' \). The interesting features in this plot are that between \( 0.5 < T < 0.75 \), \( T'' \) and \( T''' \) have opposite signs. Furthermore, \( T''' \) as a zero at \( T = 0.5 \). It is also worth noting that these characteristics are not shared by the Mach-Zenhnder discriminator as can be observed in the plots on the right hand side of figure 5.8. In what follows, it will be demonstrated that the FPF has advantages when compared to the Mach-Zenhder.

Curve C plots the system's IMPS for \( T = 0.67 \), showing lower distortion than in curve A, due in part to the reduced value of \( T''' \). A comparison of curves C and D shows that the system's distortion is dominated both by the terms proportional to \( T'' \) and \( T''' \), except around both 2.7 GHz and d.c. Furthermore, a comparison of curves C and
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E, in particular, shows that the seventh term in eqn. 5.50, involving the laser's intrinsic FM distortion, is crucial to model the system's IMP3 accurately when the FPF is off-set from the point where $T'' = 0$. Curve F plots the system's IMP3 for the case of pure IM (without discriminator).

Quite interestingly, in a FPF $T''' = 0$ when $T \approx 0.5$, which is easily proved using the symbolic capabilities of the software Mathematica to find the zeros of the third derivative of the FPF transmission. Unfortunately, $T''$ is large when $T = 0.5$, so this is not the best bias point for the FPF if one is concerned about the IMD3 performance.

Figure 5.9 plots the system's IMP3 with the FPF biased at $T = 0.52$. At the chosen bias, the dominant contribution to the IMP3 comes from the terms proportional to $T''$ as shown by comparing curves A and B. Moreover, there is a large cancellation of IMP3 around 1.3 GHz. The origin of this cancellation is clarified in figure 5.10 which plots

![Figure 5.9: Relative power of IMP3 as a function of $f_1$, located at frequency $2f_1 - f_2$. The OMI is constant and equal to 4%.
Curve A) $T''p_0 \approx 0.47Tp_1$ with $T = 0.52$ and $T' = 6.88 \cdot 10^{-11}$ s, $T'' = 8.7 \cdot 10^{-21}$ s$^2$ and $T''' = -3.0 \cdot 10^{-31}$ s$^3$;
Curve B) Same conditions as for curve A but only the contribution from the terms proportional to $T''$ in eqn. 5.50 is included;
Curve C) pure IM (without discriminator).](image)

both the phase difference and the ratio of the magnitudes of the terms proportional to $T''$, for the same bias conditions used in curve A of figure 5.9. The cancellation is largely due both to the 180 degrees phase difference and approximately equal magnitudes of
these terms around 1.3 GHz. The two terms proportional to $T''$ are in phase opposition to each other at d.c., which is due to $\vartheta_2(0,0)$ being negative (red-shifted) while $\vartheta_1(0)$ is positive (blue-shifted). The evolution of the phase plotted in figure 5.10 is determined by the phase relation between $H_1(\omega)$ and $\vartheta_1(\omega)$, since $\vartheta_2(\omega_1, \omega_2)$ could be expressed as a bi-linear combination of terms of the form $H_1(\omega_1)\vartheta_1(\omega_2)$. Near the frequency where the cancellation occurs the phase difference between the terms proportional to $T''$ becomes 180 degrees while their magnitude is identical.

![Graph](image)

Figure 5.10: Magnitude ratio (solid curve) and phase difference (dotted curve) of terms proportional to $T''$ in eqn. 5.50.

**Effect of power on the cancellations of both IMP2 and IMP3**

The cancellations of both the IMP2 observed in figure 5.6 and the IMP3 observed in figure 5.9, are due to the phase opposition between certain dominant terms contributing to the system's non-linear response given by eqn. 5.49 and eqn. 5.50.

As far as the IMP3 is concerned, in the case shown in figure 5.9, the observed cancellation is due to the terms proportional to $T''$ having opposite phase.

The phase relation between each term of eqn. 5.49 and of eqn. 5.50 is ultimately dependent on the phase relation between the IM and FM laser responses as given by the transfer functions $H_1$ and $\vartheta_1$. These responses also determine the phase of the second and third-order transfer functions of both the IM and FM, since these higher-order
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the phase difference of the FM and IM responses is obtained from eqn. 5.43 and can be casted as

\[ \psi = \arctan\left(\frac{2\pi f}{\gamma}\right) \]  

where \( \gamma \) is the non-linear damping coefficient \( \gamma = \epsilon \rho_0 / \tau_p \). The damping coefficient depends on the cavity photon density and therefore the phase relation between all the laser transfer functions is also power dependent. The implication is that the frequency at which the cancellations occur depends on the power emitted by the laser. Figure 5.11 plots the phase difference of the FM and IM responses for the InAlGaP DFB laser as a function of the emitted power.

The cancellation of IMP3 shown in figure 5.9, between the terms proportional to \( T'' \), occurs at a frequency around 1.3 GHz for a laser power of 1.4 mW. From figure 5.11 this corresponds to a phase difference between the FM and IM responses of around 40 degrees. Consequently, one expects that the same cancellation will occur at a higher modulating frequency when the laser power is increased, since, as shown in figure 5.11, the same phase difference of 40 degrees arises at an increasing frequency for increasing

Figure 5.11: Phase difference between the FM and IM responses. The curves refer to different laser powers of 1.4 mW, 2.0 mW and 2.8 mW.
power.

Figure 5.12: Relative power of IMP3 as a function of $f_1$, located at frequency $f_1 + f_2 - f_3$. The OMI is constant and equal to 4%. On the left: Curve A) $T' \nu_1 p_0 \approx 0.54 T_1 p_1$ with $T = 0.567$ and $T' = 4.3 \cdot 10^{-11}$ s, $T'' = 2.7 \cdot 10^{-21}$ s$^2$ and $T''' = -2.3 \cdot 10^{-31}$ s$^3$. Curve B) pure IM (without discriminator).

Figure 5.12 plots the same IMP3 as shown in figure 5.9, for a power of 2.8 mW at a resonance frequency of 7 GHz. The bandwidth of the FPF is approximately 26 GHz. This plot shows that the cancellation of IMP3 can still be achieved for higher laser powers. A large photon density inside the laser cavity is also advantageous because the laser noise is reduced.

5.7 Summary

This chapter introduces both the subject of the laser dynamic non-linearity and its non-linear distortion effects in the context of SCM applications. The theory of Volterra kernels is also introduced and applied to the analysis of non-linear distortion in laser transmitters. The Volterra kernels in a DFB laser were used to capture the non-linear response of the device both in its IM and FM components of the light. Furthermore, the application of the theory was extended to incorporate also the non-linearity of an optical discriminator.
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The FM non-linear distortion of the semiconductor DFB laser when directly modulated was assessed and it was shown that for FM modulation the laser requires lower signal amplitudes for the same OMI thereby behaving in a more linear fashion. Moreover, the "adiabatic chirp" is an important parameter in reducing the levels of distortion at frequencies below half the resonance of the laser. The results show that in SCM applications FM modulation of the laser followed by FM to IM interferometric conversion may place less stringent requirements on the laser non-linearity.

The non-linear distortion of a complex optical systems employing both frequency and intensity modulation of a DFB laser and a FPF as a frequency discriminator was also assessed. The FPF has both a zero of $T''$ and of $T'''$ at $T \approx 0.5$ and $T \approx 0.75$, respectively. In between $T''$ is positive and $T'''$ is negative, if $T'$ is positive. However, optimising the system’s IMP2 or IMP3 performance is not just a matter of choosing the point where $T'' = 0$ or $T''' = 0$, respectively. It is also not good enough just to choose the right filter bandwidth according to Carson’s rule.

For instance, the IMP2 performance can be largely improved by off-setting the FPF bias point from $T'' = 0$ to a negative value, thereby largely cancelling IMP2 altogether at low frequency.

As far as the IMP3 performance is concerned the results show that the best strategy is neither to choose the point where $T'' = 0$ nor $T''' = 0$. By biasing the FPF between these points we found again that the relative IMP3 power is reduced and a large cancellation of IMP3 power can be achieved in some cases.

Furthermore, our results show that the laser’s FM distortion, in particular, can contribute significantly to both the IMP2 and IMP3 system performance. Therefore, in order to model accurately this system’s distortion it is essential to include also the contribution of the laser’s FM intrinsic non-linearity. Previous modelling work in this type of system tended to neglect this point.

The comparison of this system’s IMP2 and IMP3 performance with that of a purely IM laser without discriminator, perhaps favours the latter, overall, which should be largely ascribed to the FPF non-linear characteristic.
Chapter 6

Non-linear distortion in the gain-lever laser

6.1 Introduction

The methods developed in the previous chapter are used here to assess the non-linear distortion in gain-lever lasers. The intermodulation power resulting from both intensity and frequency modulation in gain-lever lasers is studied with attention paid to the influence of the bias conditions. Their performance is also compared to that of both an equivalent single-section laser and of a DFB laser. This both enables trade-offs to be identified and assesses the suitability of gain-lever lasers for optical subcarrier multiplexed applications.

6.2 Volterra kernels for IM in the gain-lever laser

In chapter 5, Volterra kernels were developed for a DFB laser capturing the effects of its dynamic non-linearity. Briefly, the nth-order kernels were obtained by taking the nth-dimensional Fourier transform of the nth-order perturbed solution of the rate equations. These kernels were used to assess the non-linear distortion in both intensity and frequency modulated DFB lasers.

A set of rate equations describing the dynamics of the photon and carrier density in the gain-lever laser was given in eqn. 3.1. A similar procedure is applied to these rate equations to obtain the Volterra kernels for intensity modulated gain-lever lasers,
which are given by

\[ H_n(\omega_1, \ldots, \omega_n) = \]

\[ = \frac{1}{n!} (1 - h) \left( p_G G'_a + \frac{\partial}{\partial G} \right) D_n(\omega_1, \ldots, \omega_n)(i\Omega + \gamma_b) \]

\[ -i\Omega^3 - \Omega^2 [(\gamma_a + \gamma_b) + p_G G'_b + \frac{\partial R_{sp}}{p_0}] + i\Omega A_1 + A_2 \]

\[ -i\Omega^3 - \Omega^2 [(\gamma_a + \gamma_b) + p_G G'_b + \frac{\partial R_{sp}}{p_0}] + i\Omega A_1 + A_2 \]

\[ + \frac{1}{n!} C_n(i\Omega + \gamma_b)(i\Omega + \gamma_a) \]

\[ F_{a_n}(\omega_1, \ldots, \omega_n) = \frac{(1 - 2\epsilon p_0)G_{a_0}H_n(\omega_1, \ldots, \omega_n) + \frac{1}{n!} D_{a_n}}{i\Omega + \gamma_a} \]

\[ F_{b_n}(\omega_1, \ldots, \omega_n) = \frac{(1 - 2\epsilon p_0)G_{b_0}H_n(\omega_1, \ldots, \omega_n) + \frac{1}{n!} D_{b_n}}{i\Omega + \gamma_b} \]

where \( \Omega = \omega_1 + \cdots + \omega_n \) and \( A_1 \) and \( A_2 \) are given by eqn. 4.8 and eqn. 4.9, respectively.

Both the gain saturation and the rate of spontaneous emission have been taken into account by writing the saturated gain as \( G(1 - \epsilon p_0) \) and adding an extra term to the photon density rate equation given by \( \beta R_{sp} \). The driving forces \( D_{a_n}(\omega_1, \ldots, \omega_n) \), \( D_{b_n}(\omega_1, \ldots, \omega_n) \) and \( C_n(\omega_1, \ldots, \omega_n) \) are given in appendix K. The driving forces include a number of terms proportional to the second and third gain derivatives of the gain with respect to the carrier density which account for the sub-linear gain profile of QW lasers. The value of these derivatives was obtained from the experimental gain curve in figure 2.5 as discussed in section 2.2.8.

The first-order transfer function corresponds to the laser’s linear intensity response to a sinusoidal modulation and was considered in chapter 3. The analytical transfer functions in an intensity modulated gain-lever laser are in good agreement with the correspondent transfer functions calculated by the numerical integration method as depicted in figure 6.1.

In the following the transfer functions are used to assess the non-linear distortion power generated by multi-tone modulation in a gain-lever laser.
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Figure 6.1: Plot of the first, second and third-order transfer functions in a 220 μm SQW gain-lever laser at 4 mW output power. The signal section bias is $R = 0.2$ and $h = 0.5$. The open circles correspond to the numerical integration method and the lines correspond to the Volterra series method. (A) $H_1(\omega_1)$, (B) $H_2(\omega_1, \omega_2)$ and (C) $H_3(\omega_1, \omega_1, -\omega_2)$, with $f_2 = f_1 + 0.2$ GHz.

6.2.1 IM distortion in the “normal” gain-lever laser

It was shown in chapter 3 that the intensity modulation efficiency in gain-lever lasers can be greatly increased by biasing the signal section close to transparency at a fraction of the threshold gain. This was attributed to the non-linear gain characteristic in QW lasers. In this study of intermodulation distortion in gain-lever lasers, similar bias conditions to those producing this IMEI are assumed in order to establish if the relative intermodulation power generated in these devices is significantly increased at the same time.

In particular, the influence of both the biasing conditions and the laser geometry on the relative power of both the IMP2 and the IMP3 intermodulation products is investigated. The laser geometry is varied by changing the relative length of the laser sections, while, at the same time, the gain in the slave section complies with the threshold condition. Furthermore, the laser output power is kept constant throughout.

Two sets of plots with different bias applied to the laser’s signal section are presented both for the second harmonic and a IMP3 at frequency $f_1 + f_2 - f_3$. The modulation index of the carriers is kept constant at 4% in all plots.
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Figure 6.2: Relative power in IMP2 at frequency $2f_1$ in a 220 $\mu$m gain-lever laser. Both the signal section bias, $R = 0.1$, and the power are kept constant. Three values of $h = 0.5, 0.7, 0.9$ are shown. On the left: 1.5 mW output power. On the right: 4 mW output power.

\textbf{Second harmonic distortion}

Both figure 6.2 and figure 6.3 plot the power in the second harmonic relative to the carrier power, whose frequency $f_1$ is swept. In the left hand side of figure 6.2 the steady-state laser power is 1.5 mW and the resonance frequency is approximately 2.1 GHz for the curve corresponding to $h = 0.5$ and 2.5 GHz for the curve corresponding to $h = 0.9$. On the right hand side of figure 6.2, the power is increased to 4 mW and the resonance frequency is approximately 3.5 GHz for the curve corresponding to $h = 0.5$ and 4 GHz for the curve corresponding to $h = 0.9$. The signal section gain is fixed at $R = 0.1$ of the threshold gain.

Figure 6.3 presents similar plots to those in figure 6.2 for a constant signal section bias of $R = 0.2$. The resonance frequency is approximately 2.5 GHz for the curves at 1.5 mW and approximately 4 GHz for the curves corresponding to 4 mW. The peak observed in the relative harmonic power occurs at a carrier frequency about one half of the laser's resonance frequency.

Both the plots in figure 6.2 and in figure 6.3 suggest that the relative harmonic power increases with increasing values of $h$, when both the laser power and the signal section bias are constant. This indicates that the relative harmonic power decreases when the intensity modulation efficiency increases as presented in figure 3.2.

The relative harmonic power in the gain-lever laser plotted in the left hand side of...
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Figure 6.3: Relative power in IMP2 at frequency $2f_1$ in a 220 $\mu$m gain-lever laser. Both the signal section bias, $R = 0.2$, and the power are kept constant. Three different values of $h = 0.5, 0.7, 0.9$ are shown. On the left: 1.5 mW output power. On the right: 4 mW output power.

Both figure 6.2 and figure 6.3 can be compared with the relative harmonic power in the DFB laser plotted in curve D of figure 5.6. The carrier modulation index is identical in both plots and the average power is approximately equal. Clearly, the relative harmonic power in the gain-lever laser is much larger than that in the DFB laser. Moreover, the harmonic power in the DFB laser is very small at low frequency increasing rapidly as the carrier frequency approaches the resonance. In contrast, the harmonic power in the gain-lever laser is large even at zero carrier frequency. The cause for this is examined in the following sections.

**Third-order distortion**

Both figure 6.4 and figure 6.5 plot the relative power in a IMP3 of the type $f_1 + f_2 - f_3$ at frequencies $f_1$, $f_2 = f_1 + 2$ MHz and $f_3 = f_1 + 4$ MHz with the frequency $f_1$ being swept.

Figure 6.4 plots the relative power in the IMP3 when the signal section is biased at a fraction $R = 0.1$ of the threshold gain, for two different laser powers, 1.5 mW and 4 mW. Several curves are plotted in each case for different laser geometries described by the value of $h$. Both the laser power and the signal section bias are kept constant throughout every single curve. Figure 6.5 also plots the relative power in the IMP3 in similar conditions to the ones in figure 6.4 but for the signal section bias which is a fraction $R = 0.2$ of the threshold gain.
Figure 6.4: Relative power in IMP3 at frequency $f_1 + f_2 - f_3$ in a 220 μm gain-lever laser. Both the signal section bias, $R = 0.1$ and the power are kept constant. Three values of $h = 0.9, 0.7, 0.5$ are shown. On the left: 1.5 mW output power. On the right: 4 mW output power.

Figure 6.5: Relative power in IMP3 at frequency $f_1 + f_2 - f_3$ in a 220 μm gain-lever laser. Both the signal section bias, $R = 0.2$ and the power are kept constant. Three values of $h = 0.9, 0.7, 0.5$ are shown. On the left: 1.5 mW output power. On the right: 4 mW output power.
As shown in these plots the relative IMP3 power decreases when both the output power and the geometrical parameter $h$ decrease. These are the same conditions that maximise the intensity modulation efficiency as shown in figure 3.2. Therefore, the bias conditions that increase the modulation efficiency in a gain-lever laser also entail the reduction of both the relative IMP3 and IMP2 power. This can be explained because the relative third-order intermodulation power is proportional to the square of the input signal current. For a constant modulation index, this current is reduced when the modulation efficiency is largest, hence reducing the relative intermodulation power at the same time.

A comparison of the relative IMP3 power in the gain-lever laser with that plotted in curve F of figure 5.7 for the DFB laser shows that the IMP3 power is also much larger in the gain-lever especially at the lower frequencies.

**Distortion in the uniformly biased laser**

The question arises as to what is the cause of the large relative intermodulation power in gain-lever lasers when compared with that in DFB lasers. The large modulation efficiency in gain-lever lasers is due to the sub-linear gain profile typical of QW lasers. Could this sub-linear characteristic also originate the high-levels of non-linear distortion in gain-lever lasers?

It helps the matter to consider the non-linear transfer functions in an uniformly biased SQW laser. Figure 6.6 plots the first, second and third-order transfer functions in a single-section SQW laser equivalent to the gain-lever laser. The unbroken curves plot the transfer functions assuming the non-linear gain model for the laser which is reflected in the large number of driving terms (given in appendix K) proportional to the gain derivatives $G''$ and $G'''$. The broken curves plot the transfer functions assuming a linear gain model, i.e. the gain is considered linear around the bias point, hence the derivatives $G''$ and $G'''$ are equal to zero.

As shown in the plot the effect of the driving terms proportional to $G''$ and $G'''$ is insignificant. Both the second and third-order transfer functions are identical in both models because the values of the gain derivatives $G''$ and $G'''$ are too small to make a difference. Therefore, it is increasingly clear that the non-linear gain in QW lasers can
Figure 6.6: Comparison of transfer functions in a SQW laser with a sub-linear gain model (-----) or a linear gain model (-----).

not be the cause of the relatively large non-linear distortion in these devices.

Figure 6.7 plots both the IMP2 and IMP3 in a uniformly biased tandem-contact laser emitting an output power of 1.5 mW. Effectively, there is no leverage in this situation because \( G'_a = G'_b \). With the exception of the two curves for \( h = 0 \), the other curves refer to a tandem-contact laser where the modulated section occupies a fraction, \( 1 - h \), of the laser's cavity. The other part of the optical cavity whilst providing gain remains un-modulated. The two curves corresponding to \( h = 0 \), refer to a laser where the signal section occupies the entire cavity, i.e. it is equivalent to a uniformly biased single-section laser. All the curves in the plot correspond to a constant power of 1.5 mW. By comparing the curves for \( h = 0 \) with the other curves, it is clear that both the relative IMP2 and IMP3 power in the tandem-contact laser are much larger than that in its single-section counterpart at frequencies approximately below one half of the resonance frequency.

Above this frequency both the relative IMP2 and IMP3 power are identical in both devices. In particular, in the tandem-contact laser both the relative IMP2 and IMP3 power increase as the relative length of the un-modulated (slave) section gets larger, at low frequency.
Chapter 6. Non-linear distortion in the gain-lever laser

Figure 6.7: Relative IMP2 (-----) and IMP3 (-----) power in a uniformly biased, $R = 1$, tandem-contact SQW laser. The two curves corresponding to $h = 0$ refer to an equivalent single-section device.

Figure 6.8: Relative IMP2 (-----) and IMP3 (-----) power in a levered, $R = 0.1$, SQW laser. This figure reproduces the plots on the left hand side of both figures 6.2 and 6.4.

Figure 6.7 reproduces the left hand side of both figures 6.2 and 6.4 plotting both the relative IMP2 and IMP3 power with the signal section biased at a fraction $R = 0.1$ of the threshold gain. In this bias situation the laser's IM efficiency is much larger than in the plot in figure 6.7 due to the gain-lever effect. The relative IMP2 power in the levered laser (figure 6.8) is only marginally larger than that in the uniformly biased laser. The relative IMP3 power in the levered laser is in some cases smaller and in some cases larger than that in the uniformly biased (non-levered) laser. This shows that both the relatively large IMP2 and IMP3 power in the gain-levered laser is mainly due to the presence of a un-modulated section. The sub-linear gain characteristic of QW devices is not the cause of the high levels of non-linear distortion found in these devices.

To emphasise this point, figure 6.9 compares both the relative IMP2 and IMP3 power in the DFB laser and in the single-section SQW laser. The DFB laser is the same device studied in chapter 5 operating at a resonance frequency of 4.2 GHz. To facilitate the comparison, both the modulation index and resonance frequency are similar in both devices. The plots show that the relative distortion power in the SQW device is similar to that in DFB lasers.
6.3 Kernels for FM in the gain-lever laser

The transfer functions of both the photon and carrier density were obtained from the gain-lever laser rate equations and are given in eqn. 6.1, eqn. 6.2 and eqn. 6.3. A phase rate equation is given in eqn. 4.1 and was used to derive a frequency chirping formula, eqn. 4.5.

The Volterra kernels describing the non-linear chirp derived from the phase rate equation can be given by

\[ \vartheta_n(\omega_1, \ldots, \omega_n) = -\frac{\omega_0}{\mu_g} [(1 - h)\mu'_a F_{an}(\omega_1, \ldots, \omega_n) + h\mu'_b F_{bn}(\omega_1, \ldots, \omega_n)] + G_n(\omega_1, \ldots, \omega_n) \]  

(6.4)

where the driving forces \( G_n(\omega_1, \ldots, \omega_n) \) are given by

\[ G_2 = k/2 [(1 - h)\mu''_a F_{a1}(\omega_1)F_{a2}(\omega_2) + h\mu''_b F_{b1}(\omega_1)F_{b2}(\omega_2)] \]  

(6.5)

\[ G_3 = k/6[(1 - h)\mu''_a F_{a1}(\omega_1)F_{a2}(\omega_2)F_{a3}(\omega_3) + h\mu''_b F_{b1}(\omega_1)F_{b2}(\omega_2)F_{b3}(\omega_3)] + k/3[(1 - h)\mu''_a F_{a1}(\omega_1)F_{a2}(\omega_2) + F_{a1}(\omega_1)F_{a2}(\omega_2, \omega_3) + F_{a1}(\omega_2)F_{a2}(\omega_1, \omega_3)] + k/3[h\mu''_b F_{b1}(\omega_1)F_{b2}(\omega_1, \omega_2) + F_{b1}(\omega_1)F_{b2}(\omega_2, \omega_3) + F_{b1}(\omega_2)F_{b2}(\omega_1, \omega_3)] \]  

(6.6)

where \( k = -\frac{\omega_0 G_1}{\mu_g} \), \( G_1 = 0 \) and \( \mu', \mu'' \) and \( \mu''' \) are the derivatives of the refractive index with respect to the carrier density.

The driving forces depend on the derivatives of the refraction index up to a third-
order. In general, this could be appropriate in QW lasers since the optical gain and its derivatives also vary with the carrier density. However, such a detailed experimental model for the refractive index variation does not exist in the extant literature of the GRIN-SCH-SQW laser. Moreover, the first derivative of the refractive index has been found to remain essentially constant with varying carrier density for the purpose of modelling the linear FM response in the GRIN-SCH-SQW gain-lever laser [87, 5]. Therefore, it can be assumed that both $\mu''$ and $\mu'''$ are very small, similarly to $G''$ and $G'''$, and thus the effect of the driving forces $G_2$ and $G_3$ can be neglected.

6.3.1 Distortion in the "normal" gain-lever laser and FPF

It was shown in chapter 4 that gain-lever lasers exhibit both a large FM efficiency and a much improved noise figure. These are desirable characteristics for a laser transmitter. Moreover, it has also been shown that frequency modulated lasers can behave in a more linear fashion than intensity modulated devices because a smaller signal current is required for the same modulation index [21, 42].

A gain-lever laser could be employed as a FM transmitter by converting the frequency modulated component of the light to an intensity variation with a FPF or other type of optical discriminator, prior to detection by a photodiode. Here, the relative power in both a IMP2 and a IMP3 generated in such a transmitter are analysed. The non-linear transfer functions of the transmitter can be given by both eqn. 5.49 and eqn. 5.50, if $H_n(\omega_1, \ldots, \omega_n)$ and $\vartheta_n(\omega_1, \ldots, \omega_n)$ for the gain-lever laser are substituted in.

The analytical transfer functions plotted in figure 6.10 (lines) are in good agreement with the numerical simulation of the transfer functions using the equation $P_0 = T(\nu)p_0$ for the the FPF response (circles). The results of the simulation of the transfer functions using the electric field model, eqn. 5.51, for the filter response are also shown in the figure (crosses). This simulation also agrees well with the analytical transfer functions provided that the signal frequency is much smaller than the filter’s bandwidth as discussed in section 5.6.2.

In what follows, the analytical transfer functions are used to analyse the intermodulation distortion in a gain-lever laser and FPF.
Figure 6.10: Plot of the first, second and third-order transfer functions in a 520 μm gain-lever laser and FPF. The lines correspond to the Volterra series method. The open circles correspond to the numerical simulation using the intensity model for the FPF response. The crosses (+) correspond to the numerical simulation using the electrical field model. FPF bias: $T = 0.5$ and $T' = -5 \cdot 10^{-11}$ s. (A) $P_1(\omega_1)$, (B) $P_2(\omega_1, \omega_2)$ and (C) $P_3(\omega_1, \omega_1, -\omega_2)$, with $f_2 = f_1 + 0.2$ GHz.

Second-order distortion

Figure 6.11 plots the relative harmonic power at frequency $2f_1$ in the frequency modulated gain-lever laser. The signal section is biased at a fraction $R = 0.29$ of the threshold gain and is shown in figure 4.18 to display a very large FM efficiency of about 30 GHz/mA at a relatively low power of 1 mW. The resonance frequency is 2 GHz.

The FM to IM conversion is performed by a FPF prior to the photo-detection. Curve A plots the harmonic distortion with the FPF biased at $T = 0.75$ where $T'' = 0$ achieving a much lower distortion than that shown in curve B where the FPF is biased at $T = 0.45$ where $T''$ is much larger.

To explain this it is helpful to refer back to eqn. 5.49 which gives the second-order transfer function for a frequency modulated laser followed by a FM to IM converter such as a FPF. It turns out that in a "normal" gain-lever laser the second and third terms in eqn. 5.49 are opposed in phase at d.c. regardless of the sign of $T'$, while their magnitude is comparable. Hence, these two terms cancel each other out to some extent at low frequency. Moreover, the contribution of the fourth term can be simply cancelled out by biasing the FPF at a transmission of $T = 0.75$ where $T'' = 0$. 
Chapter 6. Non-linear distortion in the gain-lever laser

Figure 6.11: Relative IMP2 power at frequency $2f_1$ in a 520 μm "normal" gain-lever laser and FPF. Laser bias: $R = 0.2857$ and 1 mW output power. Curve A: the FPF bias is $T = 0.75$, $T' = -6 \cdot 10^{-11}$ s, 22 GHz bandwidth. Curve B: the FPF bias is $T = 0.45$, $T' = -6 \cdot 10^{-11}$ s, $T'' = 8.7 \cdot 10^{-21}$ s$^2$, 15 GHz bandwidth.

The relative harmonic power in the frequency modulated laser can be compared with the relative harmonic power in the intensity modulated gain-lever laser plotted in figure 6.2. The comparison is justified because both the modulation index and signal power are approximately equal in both cases. The comparison shows that the harmonic power in the frequency modulated gain-lever laser can be considerably lower than in the intensity modulated device.

Third-order distortion

Figure 6.12 plots the relative IMP3 power of type $f_1 + f_2 - f_3$ in the frequency modulated gain-lever laser. The laser bias conditions are identical to those described earlier.

The strategy for biasing the FPF derives from eqn. 5.50 giving the third-order transfer function for this system. In this laser with a red-shifted FM response, the two terms in eqn. 5.50 proportional to $T''$ are opposed in phase at d.c. although their magnitude differs by approximately 13 dB. Hence the contribution of these terms partially cancels out regardless of the value of $T''$. This allows the choice of biasing the FPF closer to $T = 0.5$ where $T''' = 0$ thus also minimising the contribution of the term proportional to $T'''$, otherwise dominant, without increasing the value of the terms proportional to
However, this still leaves out the contribution of the terms proportional to $T''$ which is also significant.

As illustrated by curve A in figure 6.12 it is better to bias the FPF at $T = 0.44$ and with a negative slope, thereby cancelling out the remainder of the contributions of the terms proportional to $T''$ with that of the terms both proportional to $T'''$ and $T'$, at d.c.. In curve B the FPF is biased at $T = 0.6$, showing much larger distortion power at low frequency but exhibiting very low relative IMP3 power around 1.6 GHz near the laser's resonance frequency. The relative IMP3 power shown in curve A is considerably smaller than that obtained with the intensity modulated gain-lever laser for the same modulation index (see figure 6.4), at frequencies well below one half of the resonance frequency. On the other hand, the relative IMP3 power shown in curve B is slightly larger than in the intensity modulated gain-lever laser, except for a narrow frequency range near the resonance frequency.

In the context of section 4.8 a large transmitter gain (laser and FPF) is one of the main reasons that motivates the adoption of the gain-lever transmitters. For frequency modulated lasers the transmitter gain increases proportionally to the slope $T'$ of the FPF. For curve A the transmitter gain is about 10 dB at around 100 MHz, as indicated in figure 4.13. This gain is similar to the maximum gain achieved with a purely intensity modulated gain-lever laser, estimated at about 11 dB from figure 3.2 by assuming an efficiency in the non-levered laser of about 0.5 mW/ mA. For curve B the transmitter gain is down to 7 dB. As indicated in figure 4.13 the noise figure in this transmitter is about 18 dB and 21 dB for curve B and curve A, respectively, at frequencies approaching d.c..

6.3.2 Distortion in the “inverted” gain-lever laser and FPF

The non-linear distortion in the “inverted” gain-lever laser is now examined. The laser is biased in the “inverted” mode and is shown in chapter 4 to exhibit a large FM efficiency of around 2 GHz/ mA at an emitted optical power of 3 mW.
Chapter 6. Non-linear distortion in the gain-lever laser

Relative power in IMP3

\[ I = 0.29 \]

0.4

0.6

0.8

frequency \( f_1 \), GHz

-20

-40

-60

-80

-100

0

0.2

0.4

0.6

0.8

1

1.2

1.4

1.6

1.8

2

Relative power in IMP3

Figure 6.12: Relative IMP3 power at frequency \( f_1 + f_2 - f_3 \) in a 520 \( \mu \)m "normal" gain-lever laser and FPF. Laser bias: \( R = 0.29 \) and 1 mW power. Curve A: the FPF bias is \( T = 0.44 \), \( T' = -8 \cdot 10^{-11} \) s, \( T'' = 1.6 \cdot 10^{-29} \) s\(^2\), \( T''' = -1.7 \cdot 10^{-30} \) s\(^3\), 11 GHz bandwidth. Curve B: the FPF bias is \( T = 0.6 \), \( T' = -5 \cdot 10^{-11} \) s, \( T'' = 3.1 \cdot 10^{-21} \) s\(^2\), \( T''' = 5.1 \cdot 10^{-31} \) s\(^3\), 24 GHz bandwidth.

Second-order distortion

Figure 6.13 plots the relative harmonic power in the frequency modulated "inverted" gain-lever laser and FPF. The emitted power is 3 mW and the signal section is biased at a gain \( R = 4 \) times the threshold gain.

The following considerations guided the choice of the FPF bias point. In an "inverted" gain-lever laser the two terms proportional to \( T' \) in eqn. 5.49 are in phase opposition at d.c. and, up to the resonance frequency, their phase difference is always larger than 100 degrees. Although their magnitude differs by about 17 dB at d.c. and more with increasing frequency, they partially cancel each other out, regardless of the value of \( T' \). Furthermore, biasing the FPF close to \( T = 0.75 \) where \( T''' = 0 \) minimises the contribution of the fourth term in eqn. 5.49 which is proportional to \( T''' \).

For curve A in figure 6.13 the FPF is slightly off-set from the point \( T = 0.75 \) thereby achieving a cancellation of the harmonic power at d.c. due to the contribution of the term proportional to \( T'' \) cancelling out the net contribution of the dominant terms proportional to \( T' \). This is possible because these two contributions are in phase at d.c. but \( T''' \) and \( T' \) have opposite signs. Curve B exhibits relatively low harmonic distortion.
Figure 6.13: Relative harmonic power at frequency $2f_1$ in a 520 μm “inverted” gain-lever laser and FPF. Laser bias: R=4 and 3 mW output power. Curve A: The FPF bias is $T = 0.82, T' = 4 \times 10^{-11} \text{s}, T'' = -1.6 \times 10^{-21} s^2$, 32 GHz bandwidth. Curve B: The FPF bias is $T = 0.91, T' = 2 \times 10^{-11} \text{s}, T'' = -1.6 \times 10^{-21} s^2$, 52 GHz bandwidth.

at higher frequencies.

The relative harmonic power in this device can be compared with that in the “normal” configuration shown in figure 6.11. However, the signal power in this case is about three times larger than in the “normal” configuration, because here the emitted power is three times larger and the modulation index is identical in both cases. If the modulation index was reduced by a factor of $\sqrt{3}$ to restore equality of signal power, the relative distortion power would be reduced further by a factor of 3 or about 5 dB in this case. The result of the comparison is that the relative harmonic distortion is generally lower in the “inverted” mode than in the “normal” mode, for equal signal power. However, this is due to a reduced contribution to the distortion from the FPF.

**Third-order distortion**

Figure 6.14 plots the relative IMP3 power at frequency $f_1 + f_2 - f_3$. The laser bias conditions are identical to those described earlier.

In a frequency modulated “inverted” gain-lever laser a good strategy to reduce the relative IMP3 power is to bias the FPF at the point $T = 0.5$, where $T''' = 0$, thereby
Figure 6.14: Relative IMP3 power at frequency $f_1 + f_2 - f_3$ in a 520 μm "inverted" gain-lever laser. Laser bias: $R = 4$ and 3 mW emitted power. Curve A: the FPF bias is $T = 0.5$, $T' = 2 \cdot 10^{-11}$ s, $T'' = 7.9 \cdot 10^{-22}$ s$^2$, 50 GHz bandwidth. Curve B: the FPF bias is $T = 0.57$, $T' = 2 \cdot 10^{-11}$ s, $T'' = 6 \cdot 10^{-22}$ s$^2$, $T''' = -2.5 \cdot 10^{-32}$ s$^3$, 56 GHz bandwidth. Curve C: the FPF bias is $T = 0.67$, $T' = 1.5 \cdot 10^{-10}$ s, $T'' = 1.6 \cdot 10^{-20}$ s$^2$, $T''' = -2.3 \cdot 10^{-29}$ s$^3$, 8.4 GHz bandwidth.

eliminating the contribution of the sixth term in eqn. 5.50, proportional to $T'''$, which otherwise can make a large contribution. Moreover, when $T = 0.5$ the value of $T''$ is relatively large so the contribution of the two terms in eqn. 5.50 proportional to $T''$ would be large in a DFB laser. However, in a "inverted" gain-lever laser these terms have opposite phase and therefore cancel each other out partially at d.c. regardless of the value of $T'''$.

Curve A plots the relative IMP3 power with a 50 GHz bandwidth FPF biased at $T = 0.5$. Curve B corresponds to a 56 GHz bandwidth FPF with the same slope as in curve A but biased at $T = 0.57$, achieving a large cancellation in the relative IMP3 power in a narrow range of frequencies centred around 3.1 GHz where the distortion is very low.

The relative IMP3 power shown in both curves A and B is smaller than that in both the "normal" configuration plotted in figure 6.12 and the purely intensity modulated gain-lever laser plotted both in figure 6.4 and in figure 6.5, for the same modulation index. However, in both curves A and B the transmitter gain is only about −12 dB as
indicated in figure 4.14 although the laser's FM efficiency is large (2 GHz/mA).

Curve C plots the relative IMP3 power with a 8 GHz bandwidth FPF with a much larger slope of $T' = 1.5 \cdot 10^{-10}$ s for which the transmitter gain is about $-0.9$ dB. This gain is larger than in both curves A and B, but the relative IMP3 power, however, is also much larger than before and is now comparable to that in the earlier transmitters, for the same modulation index. An even larger gain can be obtained by further increasing the FPF slope, but this further increases the relative IMP3 power and reduces the bandwidth of the FPF.

6.4 Summary

This chapter presents Volterra kernels that capture the dynamic non-linearity in gain-lever lasers under both intensity and frequency modulation. These models were used to assess the intermodulation power generated in both the "normal" and "inverted" gain-lever lasers.

Contrary to expectations, the intermodulation power in single-section SQW lasers is not necessarily larger than that in DFB lasers, in spite of the sub-linear gain characteristic of QW devices. However, the relative intermodulation power in intensity modulated gain-lever lasers is considerably larger than that in single-section lasers, especially at frequencies below about half the resonance frequency. This stems from the partial modulation of the optical cavity in gain-lever lasers rather than from the sub-linear gain in a QW device. For this reason, the large gain and reduced noise figure in an intensity modulated gain-lever laser comes at the expense of a relatively large intermodulation power compared with DFB lasers. This trade-off is important to SCM applications which require very low intermodulation power.

On the other hand, the intermodulation power in frequency modulated gain-lever lasers followed by an optical discriminator, is found to be greatly influenced by the FPF bias point. Moreover, the relative intermodulation power can be lower in a transmitter comprising a frequency modulated gain-lever laser and a FPF, than in the purely intensity modulated gain-lever laser, especially in the "inverted" configuration where it is possible to obtain a large cancellation of IMP3 power by selecting the FPF bias point.
However, in the latter case the transmitter gain is relatively small compared with the gain obtained in the "normal" configuration, in spite of the large FM efficiency of the "inverted" lever transmitter. If the FPF slope is augmented to increase the gain, then the relative intermodulation power becomes comparable to that in the other transmitter configurations.

In the next chapter the performance of the "inverted" lever transmitter is evaluated in a mobile radio optical link, due to its low third-order distortion.
Chapter 7

System application assessment

7.1 Introduction

The principal motivation of this work has been to investigate if gain-lever lasers could be advantageously employed as transmitters in optical links in SCM applications.

In this chapter the performance of the “inverted” gain-lever laser transmitter is examined in the context of a SCM optical transmission link for a mobile radio system. The mobile transmitters can be scattered in a given area, and the effects of non-linear distortion are particularly aggravated in some spatial distributions. It is thus important that the optical transmission system can tolerate a large dynamic range of signal powers.

A worst case scenario is used to predict the system dynamic range which is shown to be limited by both the laser noise and by the intermodulation distortion.

7.2 System Carrier to Noise Ratio

The system carrier to noise ratio at the receiver for the channel $r$, at frequency $f_{kr}$, is given by

$$CNr = CNr_{rx}^{-1} + CNr_{rx}^{-1}$$

(7.1)

The carrier to noise ratio at the transmitter is given by

$$CNr_{tx}^{-1} = \frac{2RIN \cdot B}{m^2} + CIR_r^{-1}$$

(7.2)
where $CIR_r^{-1}$ is the distortion to carrier ratio. At the receiver, the carrier to noise ratio is given by

$$CNR_{RXr}^{-1} = \frac{2eI \cdot B + \langle i^2 \rangle B}{\frac{1}{2} m^2 I^2}$$

(7.3)

where $I$ is the average received current and $\langle i^2 \rangle$ is the input noise current of the receiver. Therefore, the output carrier to noise ratio at the end of the transmitter and receiver chain is given by

$$CNR_r = \frac{\frac{1}{2} m^2 I^2}{\langle i^2 \rangle B + I^2 RIN \cdot B + 2eI \cdot B + \frac{1}{2} m^2 I^2 CIR_r^{-1}}$$

(7.4)

### 7.2.1 Carrier to intermodulation ratio

In order to quantify the effect of the intermodulation interference (noise) on the system performance it is useful to define the CIR falling on the $r$-th channel located at a given frequency $\nu$, due to all other channels at frequencies $f_{k1}, \ldots, f_{kn}$ as

$$CIR_{n,r}(f_{k1}, \ldots, f_{kn}) = \frac{\int_{B} G_{1\nu}(f) df}{\int_{B} ||H_{BP}(f)||^2 G_{n\nu}(f) df}$$

(7.5)

where the integrals are evaluated over the receiver’s filter bandwidth.

The CIR due to all $n$-th order intermodulation products with energy falling within the bandwidth of a particular channel $r$ centred at frequency $\nu$ is given by

$$CIR_{n,r} = \frac{\int_{B} G_{1\nu}(f) df}{\sum_k \int_{B} ||H_{BP}(f)||^2 G_{n\nu}(f) df}$$

(7.6)

where the summation over $k$ includes all the distinct sets $k_1, \ldots, k_n$ subject to the condition

$$f_{k1} + \cdot \cdot + f_{kn} = m_{-K} f_{-K} + \cdot \cdot + m_K f_K = \nu$$

(7.7)

When the subcarriers are un-modulated, the intermodulation power is totally concentrated at the intermodulation frequency $\nu$ and is given by eqn. 5.40, so that

$$CIR_{n,r} = \frac{||j_{r} P_{1}(\nu)||^2}{\sum_k B_{nm}^2 ||j_{k1}, \ldots, j_{kn} P_{n}(f_{k1}, \ldots, f_{kn})||^2}$$

(7.8)

where

$$B_{nm} = \frac{n!}{m_{-K}! \ldots m_K !}$$

(7.9)
However, when the subcarriers are digitally modulated, the intermodulation power spectrum spreads out and is wider than the receiver's filter bandwidth. It is shown in appendix L that in this case, the carrier to interference ratio can be given by

\[
\text{CIR}_{\text{nr}} = \frac{\|j_rP_1(\nu)\|^2}{\sum_k B_{nm}^2 \alpha_n(m_{-K}, \ldots, m_K)\|j_{k_1} \ldots j_{k_n} P_n(f_{k_1}, \ldots, f_{k_n})\|^2}
\]

where the \( \alpha_n(m_{-K}, \ldots, m_K) \) denotes the fraction of the intermodulation power relative to the signal power that passes through the receiver's channel filter. In the case of a CPFSK digital modulation it has been estimated that \( \alpha_3(f_{k_1}, f_{k_2}, -f_{k_3}) = 0.8 \) and \( \alpha_3(f_{k_1}, f_{k_1}, -f_{k_2}) = 0.59 \), [109].

In general, only second and third-order IMP's need to be considered and accordingly eqn. 7.10 can be expanded into

\[
\text{CIR}_{r}^{-1} = \left( \frac{3}{2} \right)^2 \alpha_{11} \sum_{f_{k_1}+f_{k_2}-f_{k_3} = \nu} \frac{\|j_{k_1} j_{k_2} j_{k_3} P_3(f_{k_1}, f_{k_2}, -f_{k_3})\|^2}{\|j_r P_1(\nu)\|^2} \\
+ \left( \frac{3}{4} \right)^2 \alpha_{21} \sum_{2f_{k_1}-f_{k_2} = \nu} \frac{\|j_{k_1} j_{k_2} P_3(f_{k_1}, f_{k_1}, -f_{k_2})\|^2}{\|j_r P_1(\nu)\|^2} \\
+ \alpha_{11} \sum_{f_{k_1} \pm f_{k_2} = \nu} \frac{\|j_{k_1} j_{k_2} P_2(f_{k_1}, \pm f_{k_2})\|^2}{\|j_r P_1(\nu)\|^2} \\
+ \left( \frac{1}{2} \right)^2 \alpha_2 \sum_{2f_{k_1} = \nu} \frac{\|j_{k_1}^2 P_2(f_{k_1}, f_{k_1})\|^2}{\|j_r P_1(\nu)\|^2}
\]

This expression shows that the knowledge of the \( n \)-th order transfer functions is sufficient to predict at the design stage, both the CIR and the CNR in a non-linear system.

In what follows, eqn. 7.4 is applied to the analysis and optimisation of the dynamic range of a SCM optical transmission link.

### 7.3 Return link for mobile radio

A current and popular application of optical links lies in the developing field of mobile radio communications. As part of the RACE-II project MODAL-R2005 an optical link is used to convey the signals transmitted by mobile users from the antenna to the base station [110], as depicted in figure 7.1. Here, the combination of the SCM technique and the optical link enable the transmission of the signals over a considerable distance to the base station where the processing equipment can be concentrated. Figure 7.2
Chapter 7. System application assessment

Figure 7.1: Schematic diagram of the optical link for mobile radio.

gives the system frequency plan comprising 50 CPFSK modulated SCM channels with 1 MHz bandwidth.

This sort of application puts some stringent requirements on the system performance. The mobile users can emit from anywhere inside an area surrounding the receiving antenna. Consequently, the return link system has to accommodate a specified (40 dB) range of input signal power. At the same time, the CNR at the receiver end must exceed 10 dB to ensure a maximum BER of $10^{-3}$ in the demodulation of the CPFSK modulated information.

Figure 7.2: Frequency plan of the original SCM 50 channel CPFSK mobile radio system.

The dynamic range is one of the more demanding parameters and this study aims to find what is the best possible dynamic range achievable with the “inverted” transmitter. This will be carried out by consideration of the worst-case scenario [109].

Such a worst case scenario occurs when the remotest mobile from the antenna, is transmitting in the worst affected channel, and at the same time all the other mobile
users are transmitting in an area close to the antenna thereby maximising the intermodulation power falling in the channel of the weak transmitter.

The analysis of this worst case scenario enables the calculation of the dynamic range which is the maximum allowed range of input powers of the mobiles that still gives a CNR equal to the required minimum CNR.

7.3.1 Worst case analysis

Analysis of this problem begins with the identification of the channel that is the most affected by intermodulation distortion. This requires the calculation of the CIR for each channel and the results are plotted in figure 7.3. From this plot the worst affected channel can be identified as channel number 35.

Let $A_r$ be the amplitude of the farthest emitting channel and let $A_{\text{max}}$ be the amplitude of the interfering channels emitting nearest to the antenna. Define the dynamic range as $DR = 20 \log A_{\text{max}}/A_r$. If $m_r$ is the modulation index for the $r$th channel it follows that

$$A_{\text{max}} = \frac{m_r T p_0}{P_1(f_r)} \frac{10^{\frac{DR}{20}}}{\gamma}$$  \hspace{1cm} (7.12)
With these definitions the $CIR^{-1}$ of eqn. 7.11 can be written as

$$
CIR^{-1} = \left(\frac{3}{2}\right)^2 \left(10^{\frac{16R}{20}}\right)^6 p_0^4 m^4 T^4 \sum_{k=k_1\neq k_r} \frac{\|P_3(f_{k_1}, f_{k_2}, -f_{k_3})\|^2}{\|P_1(f_{k_r})\|^6} \tag{7.13}
$$

$$
+ \left(\frac{3}{4}\right)^2 \left(10^{\frac{16R}{20}}\right)^6 p_0^4 m^4 T^4 \sum_{k=k_1\neq k_r} \frac{\|P_3(f_{k_1}, f_{k_2}, -f_{k_3})\|^2}{\|P_1(f_{k_r})\|^6} \tag{7.14}
$$

where the summations only include the intermodulation products that do not include channel $r$. These terms have been neglected because they are proportional to $10^{\frac{16R}{20}}$ and are therefore far less significant than the terms retained in the above expression.

In the first summation the terms given by $f_{k_1} + f_{k_2} = f_{k_r}$ are not included. These terms obey $k_1 + k_2 = 2k_r$ and their number totals $N - k_r$ if $r > N_c$, or $r - 1$ if $r \leq N_c$, where $N_c$ is the centre channel number: that is $N_c = N/2$ if $N$ is even, otherwise $N_c = (N + 1)/2$ if $N$ is odd. In the present case there are $N = 50$ channels, and the number of terms that include channel 35 totals 12. They are far less than the remaining 819 intermodulation products, which do not include channel 35. The second summation contains all the intermodulation products of the form $2f_{k_1} - f_{k_2} = f_{k_r}$ and none of them involves channel $r$.

At this point it is convenient to define the distortion coefficients, given by

$$
D_{111} = \left(\frac{3}{2}\right)^2 \left(10^{\frac{16R}{20}}\right)^6 \frac{p_0^4}{N_{111}} T^4 \sum_{k=k_1\neq k_r} \frac{\|P_3(f_{k_1}, f_{k_2}, -f_{k_3})\|^2}{\|P_1(f_{k_r})\|^6} \tag{7.15}
$$

$$
D_{21} = \left(\frac{3}{4}\right)^2 \left(10^{\frac{16R}{20}}\right)^6 \frac{p_0^4}{N_{21}} T^4 \sum_{k=k_1\neq k_r} \frac{\|P_3(f_{k_1}, f_{k_2}, -f_{k_3})\|^2}{\|P_1(f_{k_r})\|^6} \tag{7.16}
$$

The distortion coefficients can be interpreted as an average value of the distortion per channel. With these definitions, the interference to carrier ratio $CIR^{-1}$ is given by

$$
CIR^{-1} = \left(10^{\frac{16R}{20}}\right)^6 m^4 (D_{111} N_{111} + D_{21} N_{21}) \tag{7.17}
$$

where $N_{111}$ and $N_{21}$ are the number of intermodulation terms in the summations of eqn 7.16 and eqn 7.16, respectively. The CNR at the link output, for the worst channel, is given by

$$
CNR_r = \frac{1}{2} m^2 \frac{1}{B} + P R I N \cdot B + 2eI \cdot B + \frac{1}{2} m^2 \cdot P^2 \cdot \alpha \cdot C_1 \tag{7.18}
$$

where $C_1 = D_{111} N_{111} + D_{21} N_{21}$ and $\alpha = 10^{\frac{16R}{20}}$. Now, $CNR$ is specified as the minimum required to achieve a BER of $10^{-3}$. Furthermore, the $RIN$, the bandwidth $B$, the
receiver noise are all given and $C_1$ is known.

Inspection of eqn. 7.18 shows that $m$ should increase from zero so that the required CNR can be achieved. However, if $m$ is too large the dynamic range will decrease to maintain the required CNR. Therefore, in between these extremes, there must be an optimum $m_{opt}$ that maximises the dynamic range for a given CNR.

The optimum value $m_{opt}$ that maximises the dynamic range can be obtained by solving the equation that results from equalising the derivative of $\alpha$ with respect to $m$ to zero. It is shown in appendix M that the required $m_{opt}$ for a prescribed value of $\alpha$ is given by

$$m_{opt}^4 = \frac{1}{3 \cdot CNR \cdot C_1 \cdot \alpha}$$

(7.19)

and the minimum received current required to achieve it is given by

$$I = \frac{-e - \sqrt{e^2 - W(i^2)}}{W}$$

(7.20)

where

$$W = RIN - \frac{\alpha \cdot C_1 \cdot m_{opt}^6}{B}$$

(7.21)

Alternatively, the $m_{opt}$ for the maximum dynamic range can also be given by

$$m_{opt} = \sqrt{\frac{3 \cdot CNR}{I^2} \cdot (I^2 RIN \cdot B + 2eI \cdot B + (i^2)B)}$$

(7.22)

The maximum dynamic range is obtained by substituting $m_{opt}$ back in eqn. 7.18, which gives

$$\alpha = \frac{I^4}{3^3 \cdot C_1 \cdot CNR^3 \cdot (I^2 RIN \cdot B + 2eI \cdot B + (i^2)B)^2}$$

(7.23)

If both the shot and thermal noise at the receiver are small in comparison with the transmitter RIN, the optimum modulation index and maximum dynamic range, reduce to the simpler expressions, given by

$$m_{opt} = \sqrt{3 \cdot CNR \cdot RIN \cdot B}$$

(7.24)

$$\alpha = \left(3^3 \cdot C_1 \cdot CNR^3 \cdot (RIN \cdot B)^2\right)^{-1}$$

(7.25)

These results are important because they show that the dynamic range and the link optical power budget can be theoretically determined beforehand, if the distortion coefficients are known.
Figure 7.4: The dynamic range (----) and the optimum modulation index (-----) versus the fibre length. The assumed receiver's input noise current is 10 pA \cdot Hz^{-1/2} and the output $RIN = -123$ dB/Hz.

The dynamic range is plotted in figure 7.4 as a function of the fibre length or received photo-current. Since the power emitted by the laser is a constant $3 \text{ mW}$, the received current depends on the fibre attenuation, $0.5 \text{ dB/Km}$, and other link losses, $10 \text{ dB}$. The CNR for each mobile channel is shown in figure 7.5, for the worst case scenario, where the modulation index for the remotest mobile is $m_{\text{opt}}$. Using eqn. 7.25 the optimum modulation index for the maximum dynamic range is 0.054%. The other mobiles transmit with a power $31 \text{ dB}$ larger, correspondent to the maximum calculated dynamic range for this system.

The dynamic range in the "inverted" gain-lever transmitter surpasses the system dynamic range specification of 40 dB with link distances up to 70 Km. On the other hand, it has been estimated that an intensity modulated DFB laser, biased at $50 \text{ mA}$, can also meet that specification [109]. By comparison with the DFB case study where the values of $D_{111} = 2.4 \cdot 10^{-4}$, $D_{21} = 6.2 \cdot 10^{-5}$ and $RIN = -148$ dB/Hz, it can be concluded that the relatively high value of the output RIN ($-123$ dB/Hz) in the present "inverted" gain-lever laser and FPF, is largely compensated for by its relatively small value of the distortion coefficients for the worst affected channel, $D_{111} = 4.3 \cdot 10^{-6}$ and $D_{21} = 6.8 \cdot 10^{-7}$. 
The RIN at the output of the FPF can be reduced by reducing $T'$ without changing the link noise figure. However, decreasing $T'$ does reduce the link throughput gain and, as shown in figure 7.6, it also causes both $D_{111}$ and $D_{21}$ to increase rapidly, thus adversely affecting the dynamic range. Note that the case where $T'$ becomes very small is effectively equivalent to a purely intensity modulated laser.

![Figure 7.6: The distortion coefficients and the maximum dynamic range versus the slope of the FPF (left) and versus the FPF transmission (right).](image)

Figure 7.6: The distortion coefficients and the maximum dynamic range versus the slope of the FPF (left) and versus the FPF transmission (right).

Figure 7.7 plots both the link NF and gain against the link bandwidth.

![Figure 7.7: Link NF (---) and gain (-----) across the link bandwidth. The laser power is 3 mW and the FPF slope is $T' = 2 \cdot 10^{-11}$ s.](image)

Figure 7.7: Link NF (---) and gain (-----) across the link bandwidth. The laser power is 3 mW and the FPF slope is $T' = 2 \cdot 10^{-11}$ s.

It can be concluded from the worst case analysis of the "inverted" transmitter that despite of the relatively large value of the output RIN, especially when the slope $T'$ is large (close to the minimum required bandwidth of the FPF), more than 40 dB in dynamic range can be achieved if the FPF bias point is carefully selected. However, as
indicated in figure 7.7 the throughput gain and noise figure are slightly worse than the corresponding figures obtained with both DFB lasers and with other configurations of the gain-lever laser.

7.4 Summary

The dynamic range of an optical SCM fibre transmission system has been analysed by considering a worst case situation.

The dynamic range is a very important parameter for mobile radio systems such as the one considered in this chapter. The "inverted" gain-lever transmitter can deliver a large dynamic range in this type of application due to a small value of the third-order distortion coefficients. This can be achieved by the combination of the "inverted" gain-lever laser and a FPF biased in such a way that minimises the relative IMP3 power generated by the transmitter. In such a case, it was found that the dynamic range achieved with the "inverted" transmitter is considerably larger than that obtained with an intensity modulated DFB laser.
Chapter 8

Conclusion

The main objective of the research presented in this thesis was to investigate the merits (trade-offs) of using gain-lever laser transmitters in optical subcarrier multiplexed applications.

Optical transmission of microwave signals in optical fibres is very advantageous because of both low fibre cost and low optical losses, making it possible to transmit signals over much larger distances than those attainable with coaxial cables.

Gain-lever lasers, in particular the “inverted” lever transmitter, have been proposed for optical link transmission in some application areas because they possess some very promising characteristics. These desirable characteristics result from the large modulation efficiency in gain-lever lasers when compared to conventional bulk lasers. Crucially to the point, this increased modulation efficiency is not off-set by an equally large increase in their spontaneous noise emission.

As a consequence of their increased modulation efficiency, gain-lever transmitters could increase the throughput RF gain in optical links whilst reducing their noise figure. Both these factors could boost the penetration of microwave optical transmission systems in many applications.

Optical subcarrier multiplexing is an important and simple technique for transmitting a large number of microwave subcarriers over an optical link. This technique, however, imposes strict limitations on the magnitude of cross-channel intermodulation noise generated by the transmitter. It was, therefore, essential to investigate the effects of the gain-lever laser non-linearity and assess the impact of the resulting intermodulation distortion in the system performance. To put it simply, does the large modulation
efficiency in gain-lever lasers lead to a large relative intermodulation power? Is there a particular configuration of the gain-lever laser best suited to subcarrier multiplexed applications? A theoretical methodology was adopted throughout this investigation to obtain some answers to these questions.

In both chapter 3 and chapter 4 a set of rate equations describing the dynamics of the gain-lever laser was used to model their modulation response and noise. Both the intensity and frequency modulation responses as well as their intensity and phase noise were investigated. This lead to an analytical expression giving the relative signal-to-noise ratio increase in these devices, which was used to find the laser bias conditions that maximise the increases in both their signal to noise ratio and modulation efficiency.

The theoretical intensity and frequency modulation responses in the gain-lever laser were compared to experimentally measured responses showing good agreement as far as the shape of the frequency response curves is concerned. However, the magnitudes of the theoretical responses fall short of the magnitude shown in some experimental curves.

Chapter 5 introduces the issue of the laser non-linearity. Frequency modulation in lasers is usually exploited by converting the frequency component of the laser light to an intensity modulated signal that can be detected by a photo-diode. Therefore, a Volterra series solution of the rate equations was obtained to assess the effects of the laser dynamic non-linearity in both the intensity and frequency modulated components of the laser light. Moreover, the Volterra series method was extended to capture the effects of the FPF non-linear transmission and its interaction with both components of the light. A study was carried out of both second and third-order intermodulation distortion in a DFB laser, which set a benchmark for gauging the corresponding performance in gain-lever lasers. It was found that both the intensity and frequency modulated components of the light can contribute significantly to the overall distortion in addition to the non-linearity of the FPF.

In chapter 6, the kernels developed in the previous chapter were used to study the intermodulation distortion in both the normal and “inverted” configurations of the gain-lever laser. It was found that the sub-linear profile of the material gain in QW
lasers does not lead to an increase in their intermodulation distortion when compared to that in conventional DFB lasers. On the other hand, an intensity modulated gain-lever laser exhibits a larger relative intermodulation distortion power, especially at low frequencies, as a result of the partial modulation of the laser cavity. Nevertheless, the frequency modulated “inverted” lever laser can exhibit a smaller relative intermodulation distortion power when compared to the intensity modulated gain-lever laser.

In chapter 7, the performance of the “inverted” lever transmitter in an optical link for a mobile radio application was assessed. The requirement for a link dynamic range of 40 dB is exceeded because of the small relative third-order intermodulation power in this transmitter.

The work carried out in this thesis suggests some areas where further investigation could be useful.

Measurement of both the modulation and noise responses of the gain-lever laser with a view to extract the values of laser parameters leading to accurate modelling of the device.

The laser chirp propagating along a dispersive fibre can originate additional non-linear distortion of the received signal [111, 112, 113, 114, 35]. Furthermore, spurious reflected light from connectors and other components can also produce additional distortion if re-admitted into the laser’s active cavity. Capturing these effects within the Volterra series framework could facilitate the overall assessment of system distortion.

The non-linear distortion analysis of an optical discriminator presented in this thesis uses a model of the discriminator based on its intensity transmission function. This greatly simplifies the model whilst including the effects of the laser intrinsic non-linearity in both the IM and FM components of the laser light. However, this model requires the bandwidth of the transmitted signals to be smaller than the discriminator bandwidth. A more general model should use the laser electrical field at the input to the discriminator. The challenge here is to use a representation of the electrical field that fully includes the effects of the laser intrinsic distortion whilst being simple enough to be handled analytically.
Appendix A

Derivation of expression for the IMEI

In this appendix, both the expression for the IMEI, eqn. 3.16, and its equivalent alternative, eqn. 3.34, are justified.

Dividing eqn. 3.13 by eqn. 3.14, one obtains

\[
\eta = \frac{p_0 \Gamma G_0 G'_0 G_a G'_b}{p_0 \Gamma G'_0 [(1 - h)G'_a G_a + hG'_b G'_a]} \tag{A.1}
\]

which, using eqn. 3.7, immediately reduces to

\[
\eta = \frac{G'_a G_b \omega_u^2}{G'_0 A_2} \tag{A.2}
\]

where \( \omega_u^2 = p_0 \Gamma G'_0 G_0 \). This is the alternative form for the IMEI. Furthermore, eqn. A.1 also gives

\[
\eta = \frac{G_0 G'_a G'_b}{(1 - h)G'_a G_a + hG'_b G'_a} \tag{A.3}
\]

and dividing both the numerator and the denominator by \( G_0 G'_a \), one also obtains

\[
\eta = \frac{G_b}{(1 - h)G'_b R + \frac{G'_a}{g} Q} \tag{A.4}
\]

where \( g = G'_a / G'_b \), \( R = G_a / G_0 \) and \( Q = G_b / G_0 \). This is the expression for the IMEI referred in the literature [2].
Appendix B

Penalty in the IMEI

In appendix D it is shown that the maximum IMEI achievable with a tandem-contact laser whose sections are biased at given points ‘a’ and ‘b’ in the gain curve, is given by eqn. 3.18 for $h = 1$. Here, the resulting penalty in relation to the maximum value of IMEI if the laser is biased at the same points ‘a’ and ‘b’, but with $h < 1$, is calculated. The eqn. 3.16 for the IMEI can be written, substituting in the definitions of $R$ and $Q$ and replacing $G_0$ by eqn. 3.17, as

$$
\eta = \frac{\gamma_b}{(1-h)G_a \gamma_b + h \sum_{k} G_k g (1-h)G_a + h G_b}
$$

(B.1)

After some algebraic manipulation eqn. B.1 transforms to

$$
\eta = \frac{(1-h) + h \frac{G_b}{G_a \eta_{h=1}}}{(1-h) + h \frac{G_b}{G_a \eta_{h=1}}}
$$

(B.2)

The maximum of eqn. B.2 occurs for $h = 1$ where it takes the value $\eta_{h=1}$ defined by eqn. 3.18. The penalty in dB compared to $\eta_{h=1}$ incurred when the tandem-contact laser is biased at the same points ‘a’ and ‘b’ but with $h < 1$ is obtained dividing eqn. B.2 by $\eta_{h=1}$

$$
\eta_{Pen} = 20 \log \left[ \frac{(1-h) + h \frac{G_b}{G_a \eta_{h=1}}} {1-h} \right]
$$

(B.3)
Appendix C

The RIN in the gain-lever laser

Under the Markovian condition where the correlation time of the noise sources is much shorter than the relaxation times of the laser system, the Langevin noise source $\Delta_p(t)$ satisfies the general relations

\[
\langle \Delta_p(t) \rangle = 0 \quad \text{(C.1)}
\]

\[
\langle \Delta_p(t) \Delta_p(t') \rangle = R_{sp} \delta(t - t') \quad \text{(C.2)}
\]

where the angle brackets denote an ensemble average [63]. The dual of eqn. C.2 in the frequency domain is [62]

\[
\langle \tilde{\Delta}_p(\omega) \tilde{\Delta}_p^*(\omega) \rangle = R_{sp} p_0 \quad \text{(C.3)}
\]

where $p_0$ is the steady-state average photon density and $R_{sp}$ is the rate of spontaneous emission.

A small-signal analysis of the stochastic rate equations is undertaken in a similar manner to that which leads to the modulation response (eqn. 3.5). A set of equations emerge for the photon and carrier densities fluctuations in each section of the laser

\[
\frac{\delta p}{dt} = p_0 \left[ (1 - h) \Gamma G'_a \delta n_a + h \Gamma G'_b \delta n_b \right] + \Delta_p(t) \quad \text{(C.4)}
\]

\[
\frac{\delta n_a}{dt} = -\gamma_a \delta n_a - G_a \delta p \quad \text{(C.5)}
\]

\[
\frac{\delta n_b}{dt} = -\gamma_b \delta n_b - G_b \delta p \quad \text{(C.6)}
\]

Fourier transforming these equations gives

\[
i \omega \tilde{H}(\omega) = p_0 \Gamma \left[ (1 - h) G'_a \tilde{F}_b(\omega) + h G'_b \tilde{F}_b(\omega) \right] + \tilde{\Delta}_p(\omega) \quad \text{(C.7)}
\]
Appendix C. The RIN in the gain-lever laser

\[ i\omega \tilde{F}_a(\omega) = -\gamma_a \tilde{F}_a(\omega) - G_a \tilde{H}(\omega) \]  
\[ i\omega \tilde{F}_b(\omega) = -\gamma_b \tilde{F}_b(\omega) - G_b \tilde{H}(\omega) \]  

from which one obtains

\[ \tilde{H}(\omega) = \frac{\Delta_p(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)}{i\omega^3 - (\gamma_a + \gamma_b)\omega^2 + i\omega A_1 + A_2} \]  
\[ \tilde{F}_a(\omega) = -\frac{G_a \tilde{H}(\omega)}{i\omega + \gamma_a} \]  
\[ \tilde{F}_b(\omega) = -\frac{G_b \tilde{H}(\omega)}{i\omega + \gamma_b} \]

Using the definition for the photon spectral density, eqn. 3.24 and Parseval's relation we can write

\[ S_p(\omega) = \langle |\tilde{H}(\omega)|^2 \rangle \]

and using eqn. C.3 the RIN is given by

\[ RIN = \frac{\langle |\tilde{H}(\omega)|^2 \rangle}{p_0^2} = \frac{R_{sp}}{p_0^2} \frac{(\gamma_a^2 + \omega^2)(\gamma_b^2 + \omega^2)}{\omega^2 - (\omega_a^2 + \gamma_a\gamma_b)^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]} \]
Appendix D

Maximum IMEI

Here, it is shown that for any two given points on the gain curve in figure 1.1, the maximum IMEI is obtained for $h \rightarrow 1$ and is given by eqn. 3.18. The ratio of eqn. 3.18 and the general expression for the IMEI given by eqn. 3.16 is given by

$$\frac{\eta_{h=1}}{\eta} = R(1 - h)\frac{\gamma_b}{\gamma_a}g + hQ$$  \hspace{1cm} (D.1)

which, using eqn. 3.17, transforms into

$$R(1 - h)\left(\frac{\gamma_b}{\gamma_a}g - 1\right) + 1$$  \hspace{1cm} (D.2)

For eqn. 3.18 to be the maximum IMEI, eqn. D.2 must be greater or equal to one. The expression $\frac{\gamma_b}{\gamma_a}g$ is identical to eqn. 3.18 (IMEI) which is always greater than one (see discussion following eqn. 3.18)

$$\frac{\gamma_b}{\gamma_a}g \geq 1$$  \hspace{1cm} (D.3)

Given this condition, eqn. D.2 is also larger or equal to one. Therefore, the maximum IMEI is obtained for $h \rightarrow 1$. 

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Appendix E

Alternative expression for the SNR improvement

Here, an alternative expression for the $SNR_i$ is obtained. Consider eqn. 3.37

$$SNR_i = A_r \left( \frac{\gamma_u G'_a}{\gamma_a G'_u} \right)^2$$

(E.1)

and substituting in, eqn. 3.41, one obtains

$$SNR_i = A_r \left[ \left( \frac{1 + \frac{g'_u \epsilon_0}{1 - \frac{1}{\gamma_a}} \frac{g'_u p_0}{\gamma_a}}{1 + \frac{g'_u \epsilon_0}{1 - \frac{1}{\gamma_a}} \frac{g'_u p_0}{\gamma_a}} \right)^2 \right]$$

(E.2)

from which it follows

$$SNR_i = A_r \left[ \left( 1 + \frac{g'_u \epsilon_0}{1 - \frac{1}{\gamma_a}} g'_u p_0 \right)^2 \right]$$

(E.3)

where $g'_i = d g_i / d J$. Using $P_o = p_0 \hbar v_s V \alpha_m / 2 \Gamma$ to transform from the photon density $p_0$ to the facet output power $P_o$, one finally obtains

$$SNR_i = A_r \left[ \left( 1 + \frac{2 P_o}{\hbar \nu \Lambda_{o_u} g'_u} \right)^2 \right]$$

(E.4)

where $A$ is the area of the active cavity and $\alpha_m$ is the mirror facet loss. The same procedure can be used to obtain an alternative expression for the IMEI from eqn. 3.16.
Appendix F

Existence of majorant

Here, it is shown that eqn. 3.42 has a majorant which is given by eqn. 3.49. Eqn. 3.42 can be written in a slightly different form as

\[ \text{SNR}_i = \left( \frac{(1 + k g'_u) g'_a}{(1 + k g'_u) g'_u} \right)^2 \]  

(F.1)

where \( k = \frac{e^2 P_0}{\hbar \nu A \alpha_m} \). Since \( g'_a = g g'_u \), one obtains

\[ \text{SNR}_i = \left( \frac{(g + k g g'_u)}{(1 + k g g'_u)} \right)^2 \]  

(F.2)

which can be further transformed to

\[ \text{SNR}_i = \left( \frac{g}{1 + k g g'_u} + \frac{k g g'_u}{1 + k g g'_u} \right)^2 \]  

(F.3)

and, it is necessarily true that

\[ \text{SNR}_i = \left[ \frac{g}{1 + k g g'_u} + \frac{k g g'_u}{1 + k g g'_u} \right]^2 < \left[ \frac{1}{k g'_u} + 1 \right]^2 \]  

(F.4)

Therefore, the majorant for the SNR improvement is given by

\[ \text{SNR}_i^{\text{Maj}} = \left[ \frac{\hbar \nu A \alpha_m}{e^2 P_0 g'_u} + 1 \right]^2 \]  

(F.5)

Eqn. 3.50 for the majorant of the IMEI can be obtained from eqn. 3.46 by following a similar procedure.
Appendix G

Derivation of the theoretical chirp in the gain-lever laser

Here, eqn. 4.5 for the linear chirp in the gain-lever laser is justified. By substituting equations 3.10 and 3.11 for the carrier density responses, $F_a(\omega)$ and $F_b(\omega)$, in eqn. 4.4 one obtains

$$\psi(\omega) = \frac{\Gamma}{4\pi} \left[ -r_a\alpha_a G'_a \frac{(1-2\epsilon_p)G_a H(\omega)}{\gamma_a + i\omega} + r_a\alpha_a G'_a \frac{j_a/ed}{\gamma_a + i\omega} - r_b\alpha_b G'_b \frac{(1-2\epsilon_p)G_b H(\omega)}{\gamma_b + i\omega} \right]$$ (G.1)

where $r_a = (1-h)$, $r_b = h$. This is transformed to

$$\psi(\omega) = \frac{\Gamma}{4\pi} \left[ -r_a\alpha_a G'_a G_a (1-2\epsilon_p)H(\omega) (\gamma_b + i\omega) \right. \\
\left. \frac{1}{(\gamma_a + i\omega)(\gamma_b + i\omega)} - \frac{r_a\alpha_b G'_b G_b (1-2\epsilon_p)H(\omega) (\gamma_a + i\omega)}{\gamma_a + i\omega} \right] + \frac{\Gamma}{4\pi} \left[ \frac{r_a\alpha_a G'_a j_a}{(\gamma_a + i\omega)} \right]$$ (G.2)

Using the fact that

$$r_a G'_a j_a/ed = \frac{H(\omega) D(\omega)}{(1-\epsilon_p)\eta_0 \Gamma (\gamma_b + i\omega)}$$ (G.3)

$$D(\omega) = (i\omega + \eta_0 \Gamma G_0) (i\omega + \gamma_a) (i\omega + \gamma_b) + i\omega \eta_0^2 + A$$ (G.4)

and substituting eqn. G.3 in eqn. G.2, results

$$\psi(\omega) = -\frac{\Gamma}{4\pi} r_a p_0 \Gamma (1-3\epsilon_p + 2(\epsilon_p)^2) G'_a \frac{j_a}{ed}$$

$$\times \left[ r_a\alpha_a G'_a G_a (\gamma_b + i\omega) + r_b\alpha_b G'_b G_b (\gamma_a + i\omega) \right] \left( \gamma_a + i\omega \right) D(\omega)$$

$$+ \frac{\Gamma}{4\pi} \left[ \frac{r_a\alpha_a G'_a j_a/ed D(\omega)}{(\gamma_a + i\omega) D(\omega)} \right]$$ (G.5)
and

\[ \theta(\omega) = \frac{r_a}{e d} \frac{j_a}{4\pi} \alpha_a G'_a \left[ \frac{(1-3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a G_a (\gamma_b + i\omega) + r_b \alpha_b G'_b G_b (\gamma_a + i\omega)]}{(\gamma_a + i\omega)D(\omega)} \right] + \frac{r_a}{e d} \frac{j_a}{4\pi} \alpha_a G'_a \left[ \frac{i\omega^2 + A + (i\omega + p_0 \Gamma G_0 \epsilon)(i\omega + \gamma_a)(i\omega + \gamma_b)}{(\gamma_a + i\omega)D(\omega)} \right] \] (G.6)

where

\[ \omega_r^2 = (1 - 3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a G_a (\gamma_b + i\omega) + r_b \alpha_b G'_b G_b (\gamma_a + i\omega)] \] (G.7)

\[ A = (1 - 3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a \gamma_b + r_b G_b \gamma_b \gamma_a] \] (G.8)

Furthermore, part of the numerator of eqn. G.6 can be transformed from

\[ \frac{(1-3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a G_a (\gamma_b + i\omega) + r_b \alpha_b G'_b G_b (\gamma_a + i\omega)]}{\alpha_a} + i\omega^2 + A \] (G.9)

to

\[ \frac{(1-3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a G_a (\gamma_b + i\omega) + r_b \alpha_b G'_b G_b (\gamma_a + i\omega)]}{\alpha_a} + \frac{(1-3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [r_a \alpha_a G'_a G_a (\gamma_b + i\omega) + r_b \alpha_b G'_b G_b (\gamma_a + i\omega)]}{\alpha_a} \] (G.10)

and to

\[-r_b \frac{\alpha_b - \alpha_a}{\alpha_a} \left( 1 - 3\epsilon p_0 + 2(\epsilon p_0)^2 \right)p_0 \Gamma G'_b G_b (\gamma_a + i\omega) \] (G.11)

Therefore

\[ \frac{\theta(\omega)}{r_a} = \frac{1}{e V_{ac} \frac{j_a}{4\pi} \alpha_a G'_a} \left[ \frac{(i\omega + p_0 \Gamma G_0 \epsilon)(\gamma_b + i\omega) - r_b \frac{\alpha_b - \alpha_a}{\alpha_a} \left( 1 - 3\epsilon p_0 + 2(\epsilon p_0)^2 \right)p_0 \Gamma G'_b G_b}{D(\omega)} \right] \] (G.12)

By defining both

\[ E = \frac{r_b (\alpha_b - \alpha_a)}{\alpha_a} = r_b \left( \frac{G'_a}{G'_b} - 1 \right) \] (G.13)

\[ \omega_b^2 = \left( 1 - 3\epsilon p_0 + 2(\epsilon p_0)^2 \right)p_0 \Gamma G'_b G_b \] (G.14)

which assumes that \( \mu'_a = \mu'_b \), eqn. G.12 transforms into

\[ \frac{\theta(\omega)}{I} = \frac{1}{e V_{ac} \frac{j_a}{4\pi} \alpha_a G'_a} \left[ \frac{(i\omega + p_0 \Gamma G_0 \epsilon)(\gamma_b + i\omega) - E \omega_b^2}{-i\omega^2 - \omega^2 (\gamma_a + \gamma_b + p_0 \Gamma G_0 \epsilon) + i\omega A_1 + A_2} \right] \] (G.15)
which is the expression for the linear chirp, with

\[ A_1 = \gamma_a \gamma_b + p_0 \Gamma G_0 \epsilon (\gamma_a + \gamma_b) \]
\[ + (1 - 3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [(1 - h)G_a G'_a + hG_b G'_b] \]  \hspace{1cm} (G.16)

\[ A_2 = (1 - 3\epsilon p_0 + 2(\epsilon p_0)^2)p_0 \Gamma [(1 - h)G_a G'_a \gamma_b + hG_b G'_b \gamma_a] + p_0 \Gamma G_0 \epsilon \gamma_a \gamma_b \]  \hspace{1cm} (G.17)
Appendix H

Derivation of Lau’s chirp equation

Here, it is shown that eqn. 4.5 for the FM response in a gain-lever laser is equivalent to equation 5 of [5].

For the case of $\epsilon = 0$, eqn. 4.5 gives

$$\frac{\dot{\theta}(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_G \left[ \frac{(i\omega + \gamma_b)i\omega - E\omega^2}{-i\omega^3 - \omega^2(\gamma_a + \gamma_b) + i\omega A_1 + A_2} \right]$$

(H.1)

The inverse of the normalised FM response is defined as

$$\frac{1}{f(\omega)} = H(\omega)/H(0) = H(0) \frac{D(\omega)}{N(\omega)}$$

(H.2)

where $N(\omega)$ and $D(\omega)$ are, respectively, the numerator and the denominator of the IM frequency response given elsewhere. With this definition it results that

$$D(\omega) = f(\omega) \frac{A_2}{\gamma_b} (i\omega + \gamma_b)$$

(H.3)

and consequently eqn. H.1 can be written as

$$\frac{\dot{\theta}(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_G \left[ \frac{(i\omega + \gamma_b)i\omega - h \left( \frac{\alpha_h}{\alpha_a} - 1 \right) \omega^2}{f(\omega)(i\omega + \gamma_b) \frac{A_2}{\gamma_b}} \right]$$

(H.4)

Multiplying both the numerator and the denominator of eqn. H.4 by $i\omega + \gamma_a$, one obtains

$$\frac{\dot{\theta}(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_G \left[ \frac{(i\omega + \gamma_a)(i\omega + \gamma_b)i\omega - h \left( \frac{\alpha_h}{\alpha_a} - 1 \right) \omega^2 (i\omega + \gamma_a)}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a) \frac{A_2}{\gamma_b}} \right]$$

(H.5)

It can also be shown that

$$i\omega(i\omega + \gamma_b)(i\omega + \gamma_a) = D(\omega) - i\omega \omega_a^2 - A_2$$

(H.6)
Therefore, eqn. H.5 transforms into
\[
\frac{\varphi(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G_a' \left[ \frac{D(\omega) \frac{\gamma_b}{\gamma_b} - i\omega \frac{\gamma_b}{\gamma_b} - \gamma_b - h \left( \frac{\alpha_b}{\alpha_a} - 1 \right) \frac{\gamma_b}{\gamma_a} (i\omega + \gamma_a)}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)} \right]
\]
(H.7)
and using eqn. H.2 for D(\omega) one obtains
\[
\frac{\varphi(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G_a' \left[ \frac{f(\omega)(i\omega + \gamma_b) - i\omega \frac{\gamma_b}{\gamma_b} - \gamma_b - h \left( \frac{\alpha_b}{\alpha_a} - 1 \right) \frac{\gamma_b}{\gamma_a} (i\omega + \gamma_a)}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)} \right]
\]
(H.8)
which is valid for any value of the parameter h. For h = 1 this equation gives
\[
\frac{\varphi(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G_a' \left[ \frac{f(\omega)(i\omega + \gamma_b) - \gamma_b + \left( 1 - \frac{\alpha_b}{\alpha_a} \right) \gamma_b - \frac{\alpha_b}{\alpha_a} \gamma_b i\omega}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)} \right]
\]
(H.9)
and, adding \(-i\omega + i\omega\) to the numerator, one obtains
\[
\frac{\varphi(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G_a' \left[ \frac{(f(\omega) - 1)(i\omega + \gamma_b) + \left( 1 - \frac{\alpha_b}{\alpha_a} \right) i\omega + \left( 1 - \frac{\alpha_b}{\alpha_a} \right) \gamma_b}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)} \right]
\]
(H.10)
Further, assuming as in [5], that the linewidth enhancement factor is inversely proportional the gain derivative one finally obtains
\[
\frac{\varphi(\omega)}{I} = \frac{1}{eV} \frac{\Gamma}{4\pi} \alpha_a G_a' \left[ \frac{(f(\omega) - 1)(i\omega + \gamma_b) + \left( 1 - \frac{\alpha_b}{\alpha_a} \right) i\omega + \left( 1 - \frac{\alpha_b}{\alpha_a} \right) \gamma_b}{f(\omega)(i\omega + \gamma_b)(i\omega + \gamma_a)} \right]
\]
(H.11)
which coincides with eqn. 5 of [5] apart from a sign disparity in the term \((f(\omega) - 1)(i\omega - \gamma_b)\) in [5]. However, this is more likely to be a printing mistake as a simplified form of the same equation appears correct in [87].
Appendix I

Phase noise in the gain-lever laser

Here, eqn. 4.21 for the phase noise in the gain-lever laser is justified. Starting with eqn. 4.20 and changing it to the frequency domain gives

\[
\frac{\delta\phi(\omega)}{dt} = \frac{\Gamma}{2} \left[ (1 - h)\alpha_a G_a^* F_a(\omega) + h\alpha_b G_b^* F_b(\omega) \right] + \tilde{\Delta}_\phi(\omega) \tag{I.1}
\]

Under similar conditions to those stated in appendix C the driving force \(\Delta_\phi(t)\) satisfies the relation

\[
\langle \Delta_\phi(t)\Delta_\phi(t') \rangle = \frac{R_{sp}}{4\rho_0} \delta(t - t') \tag{I.2}
\]

Substituting in eqn. 3.29 and eqn. 3.30 for \(F_a(\omega)\) and \(F_b(\omega)\), respectively, gives the following expression for the frequency deviation

\[
\frac{\delta\phi(\omega)}{dt} = \frac{\Gamma}{2} \left[ (1 - h)\alpha_a G_a^* \frac{H(\omega)}{\gamma_a + i\omega} + h\alpha_b G_b^* \frac{H(\omega)}{\gamma_b + i\omega} \right] + \tilde{\Delta}_\phi(\omega) \tag{I.3}
\]

The spectral density of the frequency noise is defined as

\[
S_\delta(\omega) = \langle ||\delta\phi(\omega)||^2 \rangle \tag{I.4}
\]

from which it follows that

\[
S_\delta(\omega) = \frac{\Gamma^2}{4} \frac{r_a \alpha_a G_a^* G_a H(\omega)(\gamma + i\omega) + r_b \alpha_b G_b^* G_b H(\omega)(\gamma + i\omega)}{(\gamma + i\omega)(\gamma_b + i\omega)} \| + \| \tilde{\Delta}_\phi(\omega) \|^2 \tag{I.5}
\]

which develops into

\[
S_\delta(\omega) = \frac{\Gamma^2}{4} \left[ \frac{r_a \alpha_a G_a^* G_a H(\omega)(\gamma + i\omega) + r_b \alpha_b G_b^* G_b H(\omega)(\gamma + i\omega)}{(\gamma + i\omega)(\gamma_b + i\omega)} \right] \times \left[ \frac{r_a \alpha_a G_a^* G_a H^*(\omega)(\gamma + i\omega)^* + r_b \alpha_b G_b^* G_b H^*(\omega)(\gamma + i\omega)^*}{(\gamma + i\omega)^*(\gamma_b + i\omega)^*} \right]
\]
where the asterisk indicates the complex conjugate. Substituting in $\tilde{H}(\omega)$ gives

$$S_\phi(\omega) = \frac{\Gamma^2}{4} \left[ \frac{r_a \alpha_a G' a G_a \Delta_p(\omega)(\gamma_b + i\omega) + r_b \alpha_b G' b G_b \Delta_p(\omega)(\gamma_a + i\omega)}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$\times \left[ \frac{r_a \alpha_a G' a G_a \Delta^*_p(\omega)(\gamma_b - i\omega) + r_b \alpha_b G' b G_b \Delta^*_p(\omega)(\gamma_a - i\omega)}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$+ \|\tilde{\Delta}_\phi(\omega)\|^2$$

(1.7)

Using $\|\tilde{\Delta}_p(\omega)\|^2 = R_{sp} p_0$ and $\|\tilde{\Delta}_\phi(\omega)\|^2 = R_{sp}/4p_0$, gives

$$S_\phi(\omega) = \frac{R_{sp}}{4p_0} \times \left[ \frac{1 + \Gamma^2 \frac{\alpha^2 G'^2 p_0^2 (\gamma_b^2 + \omega^2)}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2}}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$+ \frac{R_{sp}}{4p_0} \times \left[ \frac{\Gamma^2 \frac{(1-h)^2 \alpha^2 G'^2 p_0^2 (\gamma_b^2 + \omega^2)}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2}}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2} \right]$$

$$+ \frac{R_{sp}}{4p_0} \times \left[ \frac{\Gamma^2 \frac{2h(1-h)\alpha^2 \alpha_b G'_a G_a G'_b p_0^2 (\gamma_a \gamma_b + \omega^2)}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2}}{\omega^2 [\omega^2 - (\omega^2 + \gamma_a \gamma_b)]^2 + [\omega^2(\gamma_a + \gamma_b) - A_2]^2} \right]$$

(1.8)

which is the required expression for the phase noise spectral density in the gain-lever laser.
Appendix J

Phase and amplitude noise addition

Here, eqn. 4.35 describing the total output RIN after an interferometric conversion of phase to amplitude noise is justified. Using eqn. 4.34, the output field variance is calculated as

\[
\langle \delta \rho(t)^2 \rangle_O \equiv \langle \delta \rho_O(t) \delta \rho_O^*(t) \rangle \\
= T^2 \left[ \left( 1 - \frac{1}{2} \frac{T'}{T} \alpha_b \omega_b^2 \right)^2 + \left( \frac{T'}{T} \right)^2 \left( \frac{\langle \Delta_\phi^2 \rangle + \langle \Delta_\phi^2 \rangle}{\langle \delta \rho_I^2 \rangle} \right) \right] \langle \delta \rho_I^2 \rangle 
\]

(J.1)

substituting in

\[
\langle \delta \rho_I^2 \rangle = \frac{\gamma^2}{\omega_b^2} \langle \Delta_\rho^* \Delta_\rho \rangle 
\]

(J.2)

and also \( \langle \Delta_\phi^2 \rangle = R_{sp}/p_0 \), \( \langle \Delta_\phi^2 \rangle = R_{sp}/4p_0 \) and \( \langle \Delta_\phi^2 \rangle = \beta \langle \Delta_\phi^2 \rangle \), and dividing by \( T I_I = I_O \), gives

\[
\langle \delta \rho^2 \rangle_O = \left[ \left( 1 - \frac{1}{2} \frac{T'}{T} \alpha_b \omega_b^2 \right)^2 + \left( \frac{1}{2} \frac{T'}{T} \omega_b^2 \right)^2 (1 + \beta) \right] \langle \delta \rho^2 \rangle_I 
\]

(J.3)

This is the sought equation.
Appendix K

Driving forces

The expressions for the driving forces are given below

\[ D_{a1} = \frac{j_{a1}}{ed} \quad (K.1) \]

\[ D_{a2} = 2[B + 3Cn_{ao} \frac{1}{2} p_0 (1 - \epsilon p_0) G''_a] F_{a1}(\omega_1) F_{a1}(\omega_2) \]
\[- (1 - 2\epsilon p_0) G''_a [H_1(\omega_1) F_{a1}(\omega_2) + H_1(\omega_2) F_{a1}(\omega_1)] - \epsilon G_a 2 H_1(\omega_1) H_1(\omega_2) \quad (K.2) \]

\[ D_{a3} = [2B + 6Cn_{ao} + p_0 (1 - \epsilon p_0) G''_a] \]
\[ \times 2[F_{a1}(\omega_1) F_{a2}(\omega_2, \omega_3) + F_{a1}(\omega_2) F_{a2}(\omega_1, \omega_3) + F_{a1}(\omega_3) F_{a2}(\omega_1, \omega_2)] \]
\[ + [(1 - 2\epsilon p_0) G''_a] 2[H_1(\omega_1) F_{a2}(\omega_2, \omega_3) + H_1(\omega_2) F_{a2}(\omega_1, \omega_3) + H_1(\omega_3) F_{a2}(\omega_1, \omega_2)] \]
\[ + [(1 - 2\epsilon p_0) G''_a] 2[H_2(\omega_1, \omega_2) F_{a1}(\omega_3) + H_2(\omega_1, \omega_3) F_{a1}(\omega_2) + H_2(\omega_3, \omega_1) F_{a1}(\omega_1)] \]
\[- \epsilon G_a 2 [H_1(\omega_1) H_1(\omega_2) F_{a1}(\omega_3) + H_1(\omega_1) H_1(\omega_3) F_{a1}(\omega_2) + H_1(\omega_2) H_1(\omega_3) F_{a1}(\omega_1)] \]
\[ + [(1 - 2\epsilon p_0) G''_a] \]
\[ \times [H_1(\omega_1) F_{a1}(\omega_2) F_{a1}(\omega_3) + H_1(\omega_2) F_{a1}(\omega_1) F_{a1}(\omega_3) + H_1(\omega_3) F_{a1}(\omega_1) F_{a1}(\omega_2)] \]
\[- 2\epsilon G_a 2 [H_1(\omega_1) H_2(\omega_2, \omega_3) + H_1(\omega_2) H_2(\omega_1, \omega_3) + H_1(\omega_3) H_2(\omega_1, \omega_2)] \]
\[ + [C + \frac{1}{6} p_0 (1 - \epsilon p_0) G''_a] 6 F_{a1}(\omega_1) F_{a1}(\omega_2) F_{a1}(\omega_3) \quad (K.3) \]

The driving forces \( D_{b_n} \) are given by similar expressions with subscript \( a \) substituted by \( b \). The driving forces \( C_n \) are given by

\[ C_1 = 0 \quad (K.4) \]

\[ C_2 = (1 - \chi)[\beta b + \beta 3 Cn_{ao} + \frac{1}{2} p_0 \Gamma (1 - \epsilon p_0) G''_a] 2[F_{a1}(\omega_1) F_{a1}(\omega_2)] \]
\[ + \chi [\beta b + \beta 3 Cn_{bo} + \frac{1}{2} p_0 \Gamma (1 - \epsilon p_0) G''_b] [F_{b1}(\omega_1) F_{b1}(\omega_2)] \]
Appendix K. Driving forces

\[ C_3 = (1 - h)\beta_2 b + \beta_6 C n_{a0} + p_0 \Gamma (1 - \epsilon_0) G''_a \]
\[ \times 2[F_{a_1}(\omega_1)F_{a_2}(\omega_2, \omega_3) + F_{a_1}(\omega_2)F_{a_2}(\omega_1, \omega_3) + F_{a_1}(\omega_3)F_{a_2}(\omega_1, \omega_2)] \]
\[ + h[\beta_2 b + \beta_6 C n_{a0} + p_0 \Gamma (1 - \epsilon_0) G''_a] \]
\[ \times 2[F_{b_1}(\omega_1)F_{b_2}(\omega_2, \omega_3) + F_{b_1}(\omega_2)F_{b_2}(\omega_1, \omega_3) + F_{b_1}(\omega_3)F_{b_2}(\omega_1, \omega_2)] \]
\[ + (1 - h)\beta C + \frac{1}{6} p_0 \Gamma (1 - \epsilon_0) G''_a 6F_{a_1}(\omega_1)F_{a_1}(\omega_2)F_{a_1}(\omega_3) \]
\[ + h[\beta C + \frac{1}{6} p_0 \Gamma (1 - \epsilon_0) G''_a ] 6F_{b_1}(\omega_1)F_{b_1}(\omega_2)F_{b_1}(\omega_3) \]
\[ + (1 - h)\Gamma G'_a(1 - \epsilon_0) - p_0 \epsilon \]
\[ \times 2[H_1(\omega_1)F_{a_2}(\omega_2, \omega_3) + H_1(\omega_2)F_{a_2}(\omega_1, \omega_3) + H_1(\omega_3)F_{a_2}(\omega_1, \omega_2)] \]
\[ + (1 - h)\Gamma G'_a(1 - \epsilon_0) - p_0 \epsilon \]
\[ \times 2[H_1(\omega_1)H_2(\omega_2, \omega_3) + H_1(\omega_2)H_2(\omega_1, \omega_3) + H_1(\omega_3)H_2(\omega_1, \omega_2)] \]
\[ + h\Gamma G'_b(1 - \epsilon_0) - p_0 \epsilon \]
\[ \times 2[H_1(\omega_1)F_{b_2}(\omega_2, \omega_3) + H_1(\omega_2)F_{b_2}(\omega_1, \omega_3) + H_1(\omega_3)F_{b_2}(\omega_1, \omega_2)] \]
\[ + h\Gamma G''_b(1 - \epsilon_0) - p_0 \epsilon \]
\[ \times [H_1(\omega_1)F_{b_1}(\omega_2, \omega_3) + H_1(\omega_2)F_{b_1}(\omega_1, \omega_3) + H_1(\omega_3)F_{b_1}(\omega_1, \omega_2)] \]
\[ + h\Gamma G''_b(1 - \epsilon_0) - p_0 \epsilon \]
\[ \times [F_{a_1}(\omega_1)H_1(\omega_2)H_1(\omega_3) + F_{a_1}(\omega_2)H_1(\omega_1)H_1(\omega_3) + F_{a_1}(\omega_3)H_1(\omega_1)H_1(\omega_2)] \]
\[ - h2\Gamma e G'_a \]
\[ \times [F_{b_1}(\omega_1)H_1(\omega_2)H_1(\omega_3) + F_{b_1}(\omega_2)H_1(\omega_1)H_1(\omega_3) + F_{b_1}(\omega_3)H_1(\omega_1)H_1(\omega_2)] \]
\[ (K.6) \]
Appendix L

Carrier to intermodulation ratio for narrow-band modulation

In this appendix it is shown that the CIR in a non-linear system with several narrow-band input signals can be approximately given by eqn. 7.10. Moreover, for narrow-band modulation it is often possible to approximately estimate the factors $\alpha_n$ [115] using both both the Volterra theory of non-linear systems [41] and the theory of narrow-band digital stochastic processes [116].

The first part of the derivation of eqn. 7.10 consists in showing that the complex envelope of the $n$-th order response of a non-linear system to a sum of narrow-band input signals depends on the value of the $n$-th order transfer function only at the carrier frequencies.

Non-linear system response to a sum of narrow-band carriers

Consider a sum of narrow-band signals at the laser input centred at the frequencies $\nu_{k_1}, \ldots, \nu_{k_n}$, given by

$$j(t) = \sum_{k=1}^{N} \text{Re}\{z_k(t)e^{j2\pi\nu_k t}\} \quad \text{(L.1)}$$

where $z_k(t)$ is the complex envelope of the narrow-band signal. The input current can also be represented by

$$j(t) = \frac{1}{2} \sum_{k=-K}^{K} z_k(t)e^{j2\pi\nu_k t} \quad \text{(L.2)}$$

and its Fourier transform is

$$J(f) = \frac{1}{2} \sum_{k=-K}^{K} Z_k(f - \nu_k) \quad \text{(L.3)}$$
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where \( Z_k(f) \) is the Fourier transform of the complex envelope \( z_k(t) \). Recall that the photon density, \( p(t) \), at the system’s output can be represented by a Volterra series

\[
p(t) = \sum_{n=1}^{\infty} p_n(t)
\]  

where the photon density component of order \( n \) in the series is given by

\[
p_n(t) = \int \cdots \int P_n(f_1, \ldots, f_n) \prod_{r=1}^{n} J(f_r)e^{i2\pi f_r t} df_r
\]  

In the present case of \( N \) modulated carriers, \( J(f_r) \) is given by eqn. L.3 and reference [41] shows that the ensuing calculation results in

\[
p_n(t) = \sum_{k} \frac{n!2^{-n+1}}{m_{-K}! \cdots m_K!} \int \cdots \int P_n(f_1 + \nu_k, \ldots, f_n + \nu_k) \prod_{r=1}^{n} Z_{k_r}(f_r)e^{i2\pi f_r t} df_r
\]  

where \( k \) denotes that the sum is over all distinct sets of \( \{k_1, \ldots, k_n\} \) with \( k_r = -K, \ldots, K \) and \( m_k = 0, \ldots, n \) is the number of times each \( k_r \) occurs in this set, such that \( \sum_{k=-K}^{K} m_k = n \). The significance of this integral is that a non-linear system driven by a sum of narrow-band frequency components at the input, generates new narrow-band components centred at all the carrier intermodulation frequencies. For example, the photon density component, \( p_{\nu}(t) \), centred at frequency

\[
\nu = \sum_{r=1}^{n} \nu_k = \sum_{k=-K}^{K} m_k \nu_k
\]  

is generated by intermodulation of the input components at frequencies \( \nu_k, \ldots, \nu_{k_n} \). Denoting the complex envelope of \( p_{\nu}(t) \) by \( q_{\nu}(t) \) gives

\[
p_{\nu}(t) = \text{Re}\{q_{\nu}(t)e^{i2\pi \sum_{r=1}^{n} \nu_k t}\}
\]  

and from eqn. L.7 this complex envelope is given by

\[
q_{\nu}(t) = \frac{n!2^{-n+1}}{m_{-K}! \cdots m_K!} \int \cdots \int P_n(f_1 + \nu_k, \ldots, f_n + \nu_k) \prod_{r=1}^{n} Z_{k_r}(f_r)e^{i2\pi f_r t} df_r
\]
Appendix L. Carrier to intermodulation ratio for narrow-band modulation

When the input sinusoidal carriers are un-modulated, eqn. L.9 reduces to eqn. 5.40 showing that the system response is determined by the values of non-linear transfer functions at the input tone frequencies. On the contrary, if the carriers are modulated eqn. L.9 shows that, in principle, one needs to carry out a lengthy $n$-dimensional integration. However, if the system’s transfer function vary slowly within the narrow-band signal spectrum, the so called "Frequency Power Series Canonic Model" [41] achieves a considerable simplification by expanding the transfer function in a multivariate Taylor series about the carrier frequencies.

The first-order approximation in this model occurs when the transfer function can be approximately regarded as constant about the set of carrier frequencies $\nu = (\nu_{k_1}, \ldots, \nu_{k_n})$ [117]. Let this constant value be denoted by $P_n(\nu_{k_1}, \ldots, \nu_{k_n})$ which, for convenience, is abbreviated to $P_n(\nu)$. In this case the Fourier transform in eqn. L.9 can be easily carried out giving, [41]

$$q_{nu}(t) = \frac{n!2^{-n+1}}{m_{-K}! \cdots m_{K}!} P_n(\nu) \prod_{r=1}^{n} z_{k_r}(t)$$  \hspace{1cm} (L.10)

where the $n$-dimensional integral is replaced simply by a product of the $n$ complex envelopes of the intermodulation input signals. If a particular zone of frequencies about $\nu_k$ appears $m_k$ times in the intermodulation product then

$$\prod_{r=1}^{n} z_{k_r}(t) = \prod_{k=-K}^{K} z_{k}^{m_k}(t)$$  \hspace{1cm} (L.11)

where $m_k$ can be any integer from 0 to $n$ and the sum of all $m_k$ is $n$.

In the following, eqn. L.10 is used to calculate the complex envelope of an IMP originated by narrow-band digitally modulated carrier signals. Let us assume, for example, that the laser input current is a sum of $M$-ary CPFSK modulated signals centred at a frequency $\nu_{k_r}$. The low-pass equivalent of the current signal for channel $k_r$ can be given by

$$z_{k_r}(t) = \sum_{s=0}^{\infty} j_{k_r} e^{i\theta_{k_r}} e^{i2\pi f_d(t-sT)} g(t-sT)$$  \hspace{1cm} (L.12)

where $g(t-sT)$ represents a rectangular pulse shape of unit amplitude and duration $T$ seconds, $1/T$ is the symbol rate and $I_{sk_r}$ is a discrete random process representing the information symbol sequence of the channel centred at frequency $\nu_{k_r}$ [116]. Furthermore,
Appendix L. Carrier to intermodulation ratio for narrow-band modulation

\( f_d \) is the peak frequency deviation, \( \theta_k \) is a uniformly distributed initial phase and \( \phi_{sk} \), defined as

\[
\phi_{sk} = \sum_{i=0}^{s-1} I_{ik}
\]

(L.13)

represents the accumulation of all symbols up to time \((s-1)T\).

Therefore, the low-pass equivalent of the random process in eqn. L.11 reduces to the expression

\[
\prod_{r=1}^{n} z_{kr}(t) = \prod_{k=-K}^{K} [\sum_{s=0}^{\infty} j_k e^{j\theta_k} e^{j2\pi f_d \phi_{sk} + (t-s)I_{sk}} g(t-sT)]^m_k \]

(L.14)

where the \( r \)-th-channel data sequence \( I_{sk} \) appears \( m_k \) times and are assumed to be statistically independent. It is also assumed that both the pulse shape and bit rate are equal for all channels concerned. Further calculation gives

\[
\prod_{r=1}^{n} z_{kr}(t) = \prod_{k=-K}^{K} j_k^{m_k} e^{j \sum_{k=-K}^{K} m_k \theta_k \sum_{s=0}^{\infty} e^{j2\pi f_d \sum_{k=-K}^{K} m_k \phi_{sk} + (t-s)I_{sk}} g(t-sT)}
\]

(L.15)

At this point we define

\[
z_n(t) = e^{j \sum_{s=0}^{\infty} e^{j2\pi \phi_{sk} + (t-s)I_{sk}} g(t-sT)}
\]

(L.16)

where

\[
I_s = \sum_{k=-K}^{K} m_k I_{sk}
\]

(L.17)

\[
\phi_s = \sum_{k=-K}^{K} m_k \phi_{sk}
\]

(L.18)

\[
\theta = \sum_{k=-K}^{K} m_k \theta_k
\]

(L.19)

and using this definition, the low-pass equivalent of the \( n \)-th order intermodulation product in eqn. L.10 is given by

\[
q_{n\omega}(t) = \frac{n! 2^{-n+1}}{m_{-K}! \cdots m_K!} P_n(\nu) \left[ \prod_{k=-K}^{K} j_k^{m_k} \right] z_n(t)
\]

(L.20)

Intermodulation spectral density

Equations L.20 and L.16 facilitate a great deal the calculation of the spectral energy density of the random process in eqn. L.10 because the spectral density of a CPFSK
Appendix L. Carrier to intermodulation ratio for narrow-band modulation

$M$-ary digital signal with the form of eqn. L.16 is given by [116]

$$
\Phi_{zz}(f) = \frac{T}{4} \left[ \sum_{n=1}^{M} P_n A_n^2(f) + 2 \sum_{n=1}^{M} \sum_{m=1}^{M} P_n P_m B_{nm}(f) A_n(f) A_m(f) \right] \quad (L.21)
$$

where the $P_n$ represent the probabilities of occurrence of the $M$ different information symbols, and where

$$
A_n(f) = \text{sinc}[\pi T(f - (2n - 1 - M)f_d)] \\
B_{nm}(f) = \frac{\cos(2\pi f T - \alpha_{nm}) - \psi \cos(\alpha_{nm})}{1 + \psi^2 - 2\psi \cos(2\pi f T)} \\
\alpha_{nm} = 2\pi f_d T(m + n - 1 - M) \quad (L.22-24)
$$

where $\psi = \psi(j2\pi f_d T)$ is the characteristic function of the information symbols. This expression applies when $|\psi(j2\pi f_d T)| < 1$ which turns out to be the condition for which the spectrum contains no discrete frequency components. To complete the calculation of the spectral power of the $n$-th order IMP in eqn. L.20 one needs to calculate both the probabilities of the random symbols $I_s$ in eqn. L.16 and their characteristic function.

**Case of a IMP3 of the type $f_{k_1} + f_{k_2} - f_{k_3}$**

In the case of a three-tone IMP3, the random variable $I_s$ in eqn. L.17 can be given by

$$
I = a_{k_1} + a_{k_2} + a_{k_3} \quad (L.25)
$$

where the $a_k$ are the random processes carrying the information in channels $k_n$. Assuming each $a_k$ is a binary variable taking the values $\{-1, 1\}$, or $2k_n - 3$ for $k_n = 1, 2$, the random variable $I$ assumes values given by

$$
I = 2(k_1 + k_2 + k_3) - 9 \quad (L.26)
$$

which can also be written as

$$
I = 2k - 1 - M \quad (L.27) \\
k = 1, 2, \ldots, M
$$

where $M$ is the number of different values the variable $I$ can take. If each channel uses binary modulation, the $n$th-order channel can take $M = n + 1$ different values,
where \( n \) represents the number of interfering channels. However, the probabilities of the different values of \( I \) are generally not equal. If the symbols \( \{1,-1\} \) in each of the binary channels are equally probable, the probabilities for the variable \( I \) are given by

\[
P_k = \frac{C_{k-1}^{M-1}}{2^{M-1}}
\]

where \( C_{n-1}^{M-1} \) are the coefficients of the Newton binomial expansion with \( M \) terms and power \( M - 1 \). With these symbol probabilities, the characteristic function can be calculated as

\[
\psi(j2\pi f_d T) = \sum_{k=1}^{M} P_k e^{j2\pi f_d T(2k-1-M)}
\]

\[
= \cos^3(2\pi f_d T)
\]

(L.29)

**Case of a IMP3 of the type \( 2f_{k_1} - f_{k_2} \)**

In this case the random variable \( I \) can be given by

\[
I = 2a_{k_1} + a_{k_2}
\]

(L.30)

where the \( a_k \) are the binary random information in the two interfering channels located at frequencies \( f_{k_1} \) and \( f_{k_2} \). The random variable \( I \) can still be reduced to the same form as in eqn. L.26 showing that the values of \( I \) are the same as for the three-tone IMP3. However, in the two-tone case each symbol is equally probable. Consequently, the characteristic function is now given by

\[
\psi(j2\pi f_d T) = \sum_{k=1}^{M} P_k e^{j2\pi f_d T(2k-1-M)}
\]

\[
= \frac{1}{2} [\cos(6\pi f_d T) + \cos(2\pi f_d T)]
\]

(L.31)

**Carrier to intermodulation ratio**

Fig. L.1 compares the spectrum of a CPFSK digitally modulated signal with the spectrum of both a two-tone and a three-tone IMP3.

The spectrum of both the IMP3s spreads out when compared to the spectrum of the narrow-band CPFSK signal. This is because the IMP3s can assume one of four different values while the signal is binary. The spectrum of the three-tone IMP3 is
Figure L.1: Spectral density of narrow-band CPFSK modulation: the input signal (solid line), three-tone IMP3 (broken line), and two-tone IMP3 (broken-dotted line). The frequency deviation is $f_a T = 0.25$ (MSK).

more concentrated than that of the two-tone IMP3 because the symbols are equally probable in the latter case while for the former case, two of the symbols appear three times more frequently than the other two.

Using eqn. L.21, the spectral density of the $n$-th order IMP centred at frequency $\nu$ due to the modulated carriers centred at frequencies $f_{k_1}, \ldots, f_{k_n}$ is given by

$$G_{q_{n\nu}}(f) = B_{nm}^2 (j_{k_1}, \ldots, j_{k_n})^2 \Phi^{n}_s(f)$$ \hspace{1cm} (L.32)

where $B_{nm} = \frac{n!^2 (n+1)!^2}{m_{-K}!^2 \ldots m_{K}!^2}$ and $\Phi^{n}_s(f)$ is the spectral power of the random symbol sequence $I_s$ and is the same for all IMPs of a similar type.

The $n$-th order CIR for the $r$-th channel, with intermodulation power centred at the frequency $\nu$ due to input channels at frequencies $f_{k_1}, \ldots, f_{k_n}$ is defined as the ratio of the signal power to the power of the intermodulation product at the output of the channel filter. Consequently, the CIR is given by

$$CIR_{n_r}(f_{k_1}, \ldots, f_{k_n}) = \frac{\int_{-B/2}^{B/2} G_{q_{n\nu}}(f) df}{\int_{-B/2}^{B/2} G_{q_{n\nu}}(f) df}$$ \hspace{1cm} (L.33)

$$= \frac{\|P_1(\nu)\|^2}{B_{nm} \alpha_n(m_{-K}, \ldots, m_K) \|P_n(f_{k_1}, \ldots, f_{k_n})\|^2 (j_{k_1}, \ldots, j_{k_n})^2}$$
where the $\alpha(m_K, \ldots, m_k)$ denotes the fraction of the intermodulation power relative to the signal power that passes through the receiver's channel filter assumed to have unit gain across the channel bandwidth. Provided that the central frequency of the intermodulation product always coincide with the central frequency of the channel filter, the value of the $\alpha_n$ factors is the same for all intermodulation products of the same type regardless of their specific frequencies. In the case of the CPFSK digital modulation considered here, for the three-tone IMP’s $\alpha_3(1, 1, -1) = \alpha_{111} = 0.8$ and for the two-tone IMPs $\alpha_3(2, -1) = \alpha_{21} = 0.59$. The total $n$-th order CIR for channel $r$ due to all $n$-th order IMPs is obtained by the summation of the power in all the IMPs falling at frequency $\nu$ and is given by

$$ CIR_{n_r} = \frac{\|P_r(\nu)\|^2 j_{k_r}^2}{\sum_k B_{nm}^2 \alpha_n(m_{-K}, \ldots, m_K) \|P_r(f_{k_1}, \ldots, f_{k_n})\|^2 (j_{k_1}, \ldots, j_{k_n})^2} \quad (L.34) $$
Appendix M

Optimum modulation index

Here it is shown that the optimum modulation index that minimises the received current for a prescribed value of the dynamic range is given by eqn. 7.19. Equation 7.18 can be re-casted as

\[ I_P(f)B + P_R I N - B + 2eI'B \]

The derivative of \( \alpha \) with respect to the modulation index is calculated as

\[ \frac{d\alpha}{dm} = -\frac{2I^2}{CNR \cdot m^5 \cdot I^2 \cdot C_1} + \frac{6(i^2)B + I^2 RIN \cdot B + 2eI \cdot B}{\frac{1}{2}m^7 \cdot I^2 \cdot C_1} \]

and equating this equation to zero gives

\[ m_{opt}^6 = \frac{(i^2)B + I^2 RIN \cdot B + 2eI \cdot B}{I^2 \cdot C_1 \alpha} \]

The terms in the numerator of eqn. M.3 containing the current can be eliminated with the help of eqn. M.1, giving

\[ m_{opt}^4 = \frac{1}{3 \cdot CNR \cdot C_1 \cdot \alpha} \]

which is the required optimum modulation index, eqn. 7.19. The minimum received current can be obtained from eqn. M.3

\[ I^2 \cdot (RIN \cdot B - C_1 \cdot \alpha \cdot m_{opt}^6) + 2 \cdot e \cdot I \cdot B + (i^2)B = 0 \]

which has the solution

\[ I = \frac{-e - \sqrt{e^2 - W(i^2)}}{W} \]
where $W$ is given by

$$W = RIN - \frac{\alpha \cdot C_1 \cdot m_{opt}^5}{B}$$

(M.7)
Bibliography


[99] R. J. Westcott, "Investigation of multiple f.m./f.d.m. carriers through a satellite t.w.t. operating near to saturation," *Proc. IEEE*, vol. 114, no. 6, pp. 726–740, 1967.


