Geometrical Methods of Nonlinear Dynamics in Ship Capsize

Jesse d'Assunção Rebello de Souza, Junior

Thesis submitted for the degree of Doctor of Philosophy

Centre for Nonlinear Dynamics and Its Applications
UNIVERSITY COLLEGE LONDON
May 1995
ABSTRACT

After centuries of designing and building ships, understanding the dynamic behaviour of marine vessels in severe seas is a difficult problem that still challenges naval architects. Capsize in rough weather does occur regularly, perhaps because of this lack of understanding of dynamic stability. The many accomplishments in the field of mathematical modelling of large-amplitude ship motions still have to be matched by corresponding achievements in the understanding of the dynamics of those models.

We investigate in this work the essential features of current ship stability criteria, as well as the mathematical modelling of large-amplitude ship motions. Here we develop our own model of coupled heave-roll motions, in which both direct and internal parametric resonances are present. We then review the most relevant aspects of geometrical nonlinear dynamics with emphasis on some of the concepts and methods used to investigate the complex nonlinear phenomena related to ship capsize: attractor-following techniques, and bifurcation diagrams, transient and steady-state basin erosion phenomena, and integrity diagrams.

Finally, we show how this approach, based on theoretical and numerical studies, can lead to a simple yet robust method to evaluate the dynamic stability of ships. Here we base our results on key observations about the nature and features of the processes of erosion and loss of transient safe basins. Through the use of coarse grids of starting conditions the method allows the construction of boundaries of safe motion in the space of phase variables and parameters. The predictions of this method can be easily checked against the results of low-cost experiments with physical models.

With this work we hope to have contributed to the ongoing efforts to understand the complex nonlinear phenomena governing large-amplitude ship motions and capsize, and to have showed that such knowledge can be applied in the development of future practical methods of assessing ship stability.
ACKNOWLEDGEMENTS

I would like to thank the members of the Centre for Nonlinear Dynamics for providing a stimulating research atmosphere, and in particular to Dr Paul Holborn for his friendship during these last four years. Dr Steven Bishop has made significant contributions to this work, and I want to thank him for his help and guidance.

I am indebted to my supervisor Professor Michael Thompson for his help with this work and, more importantly, for his example of positive attitude towards serious research in engineering.

I also want to thank my colleagues at the Department of Naval Architecture and Ocean Engineering of the University of São Paulo for their support of my project of developing my doctoral studies in England.

This research programme was financed by the University of São Paulo (Brasil), the Brazilian Navy, and the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) to all of which I express my gratitude.

Finally, I want to thank my wife Denise who has now been for fourteen years a permanent source of love and inspiration for me, and is at this very moment nurturing inside her our first child.
TABLE OF CONTENTS

Abstract ........................................................................................................................ 2
Acknowledgements ................................................................................................... 3
Table of Contents ...................................................................................................... 4
List of Figures ............................................................................................................ 7
List of Tables ............................................................................................................ 10

1 INTRODUCTION

1.1 Scope of this work ................................................................................................ 11
1.2 Objectives of this work ......................................................................................... 17
1.3 Overview of the contents ..................................................................................... 18

2 CURRENT SHIP STABILITY CRITERIA

2.1 The need for better stability criteria: historical background ......................... 20
2.2 The GZ-Curve criterion: description ................................................................. 24
2.3 The GZ-Curve criterion: general comments ..................................................... 27
2.4 Brief survey of works on ship stability criteria ............................................... 31

3 MATHEMATICAL MODELLING OF SHIP MOTION

3.1 Objective of mathematical modelling ................................................................. 35
3.2 Basic ship capsize mechanisms in waves ......................................................... 36
   3.2.1 Pure loss of stability in waves ................................................................ 37
   3.2.2 Low-cycle (or parametric) resonance .................................................... 38
   3.2.3 Broaching ............................................................................................... 38
   3.2.4 Capsize in beam seas ............................................................................. 39
3.3 On the prediction of ship motions ..................................................................... 40
3.4 Basic formulation of the problem of ship motions .......................................... 41
3.5 The rôle of simplified mathematical models .................................................... 43
3.6 One-degree-of-freedom (1-DOF) models ......................................................... 43
3.6.1 Preliminaries ..............................................................................................43
3.6.2 The escape equation .................................................................................45
3.7 Two-degrees-of-freedom (2-DOF) models .......................................................48
   3.7.1 An archetypal 2-DOF model of ship motions ........................................49
   3.7.2 Equations of motion in non-dimensional form ......................................55
   3.7.3 Underlying potential system .................................................................56

4 GEOMETRICAL APPROACH TO NONLINEAR DYNAMICS

   4.1 Brief historical background ........................................................................60
   4.2 Preliminaries ...............................................................................................62
   4.3 Long-term dynamical behaviour: steady-state concepts ................................63
      4.3.1 A simple model of ship rolling .............................................................66
      4.3.2 Underlying potential system ...............................................................69
      4.3.3 Damped, unforced system .................................................................73
      4.3.4 Periodically forced system ................................................................76
   4.4 Elements of bifurcation theory ....................................................................79
      4.4.1 Bifurcations of periodic orbits .............................................................80
      4.4.2 Global bifurcations: tangling of invariant manifolds ............................84

5 DYNAMICS OF SHIP MOTIONS LEADING TO CAPSIZE

   5.1 Preliminaries: numerical methods in dynamics .........................................87
   5.2 Overview of steady-state response ...............................................................89
      5.2.1 Quasi-steady-state response in time series plots .................................89
      5.2.2 Steady-state response: Poincaré sections ...........................................92
   5.3 Bifurcation diagrams ..................................................................................96
      5.3.1 Attractor-following technique .............................................................96
      5.3.2 Bifurcations of the main sequence for the SIR model .......................98
   5.4 Basins of attraction ...................................................................................106
      5.4.1 Grids of starts ....................................................................................106
      5.4.2 Sample of basins of attraction for the SIR model ...............................107
   5.5 Transient dynamics and safe basins ..........................................................110
## Table of Contents

5.5.1 Transient *versus* steady-state dynamics .................................................. 111
5.5.2 Transient dynamics: preliminaries ............................................................. 112
5.5.3 Overview of transient escape in the SIR model .......................................... 118
5.5.4 Safe basins through grids-of-starts ......................................................... 122
5.6 Safe basins in control-parameter space for the SIR model ............................. 126
5.7 Evolution of safe basins .................................................................................. 133
  5.7.1 Loss of safe basin near resonance ........................................................... 133
  5.7.2 Processes of loss of safe basin ............................................................... 138
  5.7.3 Integrity diagrams ............................................................................... 147

### 6 THE METHOD OF A COARSE GRID-OF-STARTS

6.1 Preliminaries .................................................................................................. 152
6.2 Dynamics in potential wells ........................................................................... 153
  6.2.1 One degree-of-freedom systems ............................................................ 154
  6.2.2 Two-degrees-of-freedom systems .......................................................... 161
6.3 Heuristics and description of the method ...................................................... 163
6.4 Sensitivity study ............................................................................................ 168
  6.4.1 Selection of dynamical systems .............................................................. 168
  6.4.2 Results of sensitivity study ................................................................... 172
  6.4.3 Comparison with the literature ............................................................. 177
6.5 Boundaries of safe motion for the SIR model .............................................. 182

### 7 CONCLUSIONS ......................................................................................... 188

### APPENDIX A: POTENTIAL ENERGY FUNCTIONS ......................................... 194

A.1 Introduction ................................................................................................. 194
A.2 Expressions for potential energy ................................................................. 197
A.3 Results for homogeneous body of square cross-section ............................. 199
A.4 The SIR model assumption ......................................................................... 202

### REFERENCES ............................................................................................... 204
LIST OF FIGURES

2.1 - Hydrostatic forces acting upon a heeled ship .............................................. 25
2.2 - The IMO Weather Criterion ........................................................................... 26

3.1 - Motion of a particle in a potential well .......................................................... 47
3.2 - Restoring and potential functions for the escape equation .......................... 48
3.3 - Geometrical parameters for coupled heave-roll motion ............................... 50
3.4 - Potential well for heave-roll model ............................................................... 57

4.1 - Typical restoring moment function for a vessel .............................................. 67
4.2 - (a) Potential energy function for equation (4.13)
      (b) Outline of trajectories of (4.13) in phase space ...................................... 71
4.3 - Phase portrait for system (4.14) ..................................................................... 75

5.1 - Escape under slowly increasing amplitude of forcing .................................... 91
5.2 - Selection of attractors in the main sequence: Poincaré points ....................... 93
5.3 - Selection of attractors in the main sequence: trajectories ............................. 95
5.4 - Bifurcation diagrams for SIR equations ....................................................... 99
5.5 - Bifurcation diagrams for SIR equations - detail of region AA' .................... 101
5.6 - Bifurcation diagrams for SIR equations - detail of region BB' .................... 102
5.7 - Bifurcation diagrams for the SIR equations with $R=1.4$ ............................ 104
5.8 - Sketch of bifurcation curves for the SIR equations ........................................ 105
5.9 - Basin of attraction of the origin for the unforced SIR equations ................... 108
5.10 - Basins of attraction for the forced SIR equations .......................................... 109
5.11 - SIR model: values of $F$ required for escape within 8 cycles ...................... 119
5.12 - Contours of constant $F$-values required for escape .................................... 120
5.13 - Individual curves of minimum $F$ required for escape .............................. 121
5.14 - SIR model: safe basins in the control-parameter spaces ............................. 129
5.15 - 1-DOF approximation to SIR model: safe basins ..................................... 131
5.16 - Detail of safe basins in control-parameter spaces ......................................... 132
5.17 - Evolution of safe basins for the SIR equations with $\omega = 0.85$ ..........135
5.18 - Evolution of safe basins for the roll model (4.11) with $\omega = 0.85$ ..........136
5.19 - Evolution of safe basins for the escape equation with $\omega = 0.85$ ..........137
5.20 - Evolution of safe basins for the escape equation with $\omega = 0.55$ ..........142
5.21 - Evolution of safe basins for the roll model (4.11) with $\omega = 0.55$ ..........143
5.22 - Evolution of safe basins for the SIR equations with $\omega = 0.55$ ..........144
5.23 - Integrity diagrams for the escape equation (3.8).................................149
5.24 - Integrity diagrams for the SIR equations with varying $R$..........................150
5.25 - Integrity diagram for the SIR equations at $R = 1.7$...............................151

6.1 - Orbits around a centre.....................................................................................157
6.2 - Orbits around a sink........................................................................................158
6.3 - Particle in a two-dimensional potential well................................................162
6.4 - Example of a two-dimensional grid-of-starts.................................................167
6.5 - Potential well for equation (6.13).................................................................170
6.6 - Potential well for equation (6.14).................................................................170
6.7 - Potential well for equation (6.15).................................................................171
6.8 - Potential well for equation (6.16).................................................................171
6.9 - Potential well for equation (6.17).................................................................172
6.10 - Sensitivity study: equation (6.13).................................................................173
6.11 - Sensitivity study: equation (6.14).................................................................174
6.12 - Sensitivity study: equation (6.15).................................................................174
6.13 - Sensitivity study: equation (6.16).................................................................175
6.14 - Sensitivity study: equation (6.17), one-dimensional grids.........................175
6.15 - Sensitivity study: equation (6.17), two-dimensional grids........................176
6.16 - Boundaries of safe motion: equation (6.13)................................................179
6.17 - Boundaries of safe motion: equation (6.14)................................................179
6.18 - Boundaries of safe motion: equation (6.15)................................................180
6.19 - Boundaries of safe motion: equation (6.16)................................................180
6.20 - Boundaries of safe motion: equation (6.17)................................................181
6.21 - One- and two-dimensional grids for the SIR model....................................183
6.22 - Boundaries of safe motion for the SIR model: 1D and 2D grids.................184
6.23 - Boundaries of safe motion for the SIR model:
influence of grid size ................................................................. 185

6.24 - Boundaries of safe motion for the SIR model:
effect of grid density ................................................................. 187

A.1 - Main dimensions of body of rectangular cross-section .......... 195
A.2 - Basic coordinates for the floating square cross-section body ...... 196
A.3 - Relative positions between body and water ......................... 196
A.4 - Contours of constant potential energy ................................. 201
A.5 - Possible equilibrium floating positions ............................... 202
LIST OF TABLES

5.1 - Parameters for figure 5.17...............................................................................135
5.2 - Parameters for figure 5.18...............................................................................136
5.3 - Parameters for figure 5.19...............................................................................137
5.4 - Parameters for figure 5.20...............................................................................142
5.5 - Parameters for figure 5.21...............................................................................143
5.6 - Parameters for figure 5.22...............................................................................144
5.7 - Parameters for figure 5.23...............................................................................148
5.8 - Parameters for figure 5.24...............................................................................150
5.9 - Parameters for figure 5.25...............................................................................151

6.1 - Main features of selected systems..................................................................177

A.1 - Summary of results for the square cross-section body .........................203
Chapter One: Introduction

1. INTRODUCTION

1.1 SCOPE OF THIS WORK

This work is an application of the conceptual framework, and methods of nonlinear dynamics to the problem of ship capsize. Both subjects - ship dynamics and nonlinear dynamics - are today vast disciplines. No attempt can therefore be made here to provide a complete account of any of those areas. Far from that, we shall be just selecting a few tools available from the already considerable repertoire of nonlinear dynamics. As to ship dynamics, we shall try to define a problem in such a way that the resulting set of equations is at the same time amenable to analysis and significant from a practical point of view.

In any work of this kind many arbitrary choices are made, their number perhaps too large to allow comprehensive listing. The fact that they are arbitrary means that a different set of choices could equally be made, resulting in a different work. The only hope one may have is that the particular set of choices one makes proves itself useful.

Since we are concerned here with ship capsize, it is probably useful to start by explaining the general meaning of this phrase in the context of this work. We shall be referring to ships, boats, and vessels as synonyms throughout this work, without any particular sense of size necessarily attached to any of these terms. In general they will be referring to ship-like, sea-going vessels. Sometimes, statements may apply to a wider category of vessels, including offshore platforms, in which case we shall try to indicate it clearly.

From a naval architecture point of view, capsize can be defined in a few different ways. They all have in common that after capsizing the vessel is unable to recover its normal floating position without external aid. In this work we are particularly concerned with the loss of stability of the upright position (or more
often of acceptable oscillations around it) due to excessive roll motion - although
the effect of other motions, particularly heave, will be considered.

Before going into a more detailed discussion of ship capsize and nonlinear
dynamics, we would like to put the first of these two subjects into a broad
perspective. This might help us understanding how the modern approaches of
systems dynamics can contribute to real improvements in the knowledge of the
complex physics involved.

Capsize in severe weather is a long standing problem in Naval Architecture. Like
many other extreme responses of engineering systems, capsizing is not a common
event. Most vessels would not be expected to capsize during their operational
lives, either intact or in a damaged condition. Small vessels, like fishing boats
are, however, more at risk of intact capsizing, National Research Council (1991).
The number of capsizes still observed today is large enough to make professional
fishing at open sea the most dangerous occupation in the U.S.A., Jons et al
(1987). Capsizing of vessels usually represents large material losses, the risk of
human lives, and depending on the nature of the cargo, considerable
environmental damage.

For many centuries, sea-going vessels were designed and built solely based on
experience. In fact, successful use of sea-going vessels preceded any formal
knowledge of the underlying physics by thousands of years. In today's language
we may say that this relative success was due not only to the ingenuity of ancient
naval architects, but also to the relative lack of difficulty in building a stable
vessel. Trial-and-error allowed reasonably safe vessels to be built, stimulating
further developments in the field. At first thought, it is therefore somewhat
surprising that after so many years of experience and with all the present
knowledge of physics ships still capsize regularly.

Why can ships capsize at all? Why are they not built in such a way that capsizing,
or at least intact capsizing, is just impossible? Clearly the answer lies on the fact
that in ship design a compromise must be reached between many requirements,
Chapter One: Introduction

some of which conflict with ship stability. The vertical position of the centre of gravity is a typical case. A vessel can be built with such a low centre of gravity that the only stable floating position is upright. No matter how large the disturbance such vessel always returns to its original position. Unfortunately, however, such flexibility in positioning the centre of gravity of a vessel is not available when designing most merchant, naval or work vessels. The final position of the centre of gravity is the result of a large number of factors involved in the vessel's layout. As a consequence, for every one of those vessels there are indeed other stable floating positions apart from the desirable, upright one. The vessel is drawn towards them, and therefore towards capsize, by unacceptably large amplitude motions. This suggests that we turn our attention to the production of large amplitude motions, particularly roll motion.

There is a fundamental reason for the propensity of ships to display large amplitude roll motions. One main purpose of a vessel being to move forwards with minimum effort, the natural geometry is one in which the vessel's cross-section area transverse to the direction of motion is kept small. This will reduce hydrodynamic drag, especially if the shape of the vessel's bow is streamlined. Now, to maximise the displacement of the vessel (and consequently the amount of cargo it can carry) its length is increased. The resulting shape is that of a slender body. In the past, the slenderness of the vessel used to be limited by structural factors: amidships sections would have to be too heavy to withstand the mechanical loads imposed by a long body. More recently, the use of high tensile strength materials, particularly steel, has made common the construction of ships whose length can be more than ten times larger than their breadth or depth. As a consequence, typical sea-going vessels offer relatively small resistance to motions around their longitudinal axis - rolling motion - and therefore are likely to respond with large-amplitude roll motions to excitation in that direction.

The characteristic behaviour of ships as summarised above is obviously worsened by the presence of sea waves and other environmental agents. For instance, as an inevitable consequence of ship's slenderness, large hull areas will be exposed to
transverse waves. Considerable disturbances can be expected in such situations, even more so in severe weather. In fact, if large beam waves act upon ships, their pilots will normally change course, therefore avoiding prolonged exposure to that kind of excitation. There are, however, situations in which operational constraints restrict the pilot's freedom to alter course. In such cases prolonged exposure to beam waves may occur, and amplitudes of motion can build up unexpectedly.

Because large amplitude roll motion can occur quite easily, it has always been important to ensure that design procedures take relevant features into account in a suitable manner. As previously pointed out, empirical knowledge was, until very recently, the only source of information on how to design a ship. In the next chapter we shall be describing in some detail the presently internationally accepted International Maritime Organisation (IMO) Weather Criteria. It might be useful, however, to make a few preliminary comments regarding such criteria to put our approach to the problem of ship capsize into context. We note here that the IMO Weather Criteria still reflect the essentially empirical approach to the problem inherited from many generations of shipbuilders. Naval Architecture was regarded as an Art for many years; ship stability criteria certainly reflect it. This does not mean, however, that serious efforts to apply the laws of physics to this problem have not been produced. The extensive technical literature on the subject gives testimony to the continued effort of the research community to put the problem into a more rational framework.

An analysis of the IMO Criteria reveals however their empirical origin. They were derived from a study conducted in the 1930's by Rahola, who collected data on a number of ships that had capsized. He then defined, on a more or less arbitrary basis, certain minimum values for key parameters related to ship stability. The very definition and selection of those parameters were, of course, open to questioning - let alone the particular values he postulated for safe

---

1Examples of such operations are the deployment of subsea structures, excavation of trenches, and towed array duties of naval vessels.
operation. The strictly statistical nature of his calculations made it difficult to assess the extent to which his results might be suitable for ships of different categories. Rahola himself never claimed for his work the status of paradigm of ship stability criteria, but that is what it has actually become^.

The longevity of the paradigm embodied in Rahola's work has at least two reasons. First, the physics involved is very complex, making it difficult for more rational criteria to be developed. Second, the practical use of the various criteria developed upon Rahola's paradigm has been reasonably successful, particularly when applied to conventional ship forms (this is probably the main reason why it became a paradigm in the first place!). In spite of this, the naval architecture community have been unhappy about the internationally accepted ship stability criteria, see section 2.1. Some of the main reasons for that can be found in the scientific desire to develop a better criterion, and in the necessity to deal with unconventional vessels. In the next chapter we shall be describing some of the latest attempts to develop more rational stability criteria.

One major reason for the strong presence of empirical elements in ship stability criteria can be attributed to a lack of understanding of the physics involved in large amplitude ship motions. Such lack of understanding has little to do with ignorance of the equations of motion governing the phenomenon. With few exceptions every physical aspect of the phenomenon can be mathematically modelled with very good precision. However, we are here faced with an example of the knowledge of the governing equations being but a first step towards solving the problem. Large amplitude motions, of which capsize can be viewed as an extreme instance, are dominated by severe nonlinearities that give rise to a variety of complex dynamical phenomena. Analysing and understanding the behaviour of mathematical models of ship motions, and its relevance to ship stability in severe weather still poses a challenge.

^Science and engineering are full of similar cases where the author of a work sees his manifestly tentative ideas acquire a character of paradigm not at all originally intended. This obviously does not detract from the merits of their work. It is up to following generations of researchers to see the limitations of established views and to improve on them.
Chapter One: Introduction

It is here that recently developed geometrical methods of nonlinear dynamics can play a significant role. They are tailor-made to deal with strongly nonlinear systems, and with the aid of computer experiments can be used to build both qualitative and quantitative understandings of complex dynamical phenomena involved in problems such as ship capsize.

Research conducted since the late 80's at the University College London by J.M.T. Thompson and co-workers on the application of the conceptual framework of nonlinear dynamics to the problem of ship capsize has resulted in significant improvement of the basic understanding of the complex dynamics involved, revealing new phenomena and patterns of behaviour. Those results allowed for a more comprehensive picture of the problem and pointed to novel methods of evaluating ship stability, Thompson et al (1990), Rainey et al (1990), Rainey and Thompson (1991).

J. Falzarano, at the University of Michigan, drew on his experience on the mathematical modelling of ship motions in heavy seas to focus his research on the prediction of dangerous motions leading to capsize. His work differs from that carried out at UCL not only in the selection of tools of nonlinear dynamics he employed but also by the use of mathematical models directly extracted from the naval architecture practice. His work further illustrates the potentiality of geometrical methods of nonlinear dynamics as a way of tackling the problem of ship capsize, Falzarano (1990).

In Japan, research conducted by M. Kan and colleagues has approached many aspects of ship dynamics and capsize using the modern tools of nonlinear dynamics, supported by analytical, numerical and experimental investigations, Kan et al (1990), Kan and Taguchi (1991), Kan (1992). In the USA, at the Virginia Polytechnic Institute and State University, A. Nayfeh and his colleagues have been exploring modern concepts of nonlinear dynamics and their use in analysing ship dynamics and capsize. Their work include extensive analytical
investigations through the use of perturbation methods, and also numerical studies, see for example Nayfeh and Sanchez (1988) and Bikdash et al (1994).

1.2 Objectives of this work

From the point of view of the engineering problem that motivates this work, we believe that the ultimate goal is to achieve ship stability criteria based on sound knowledge of the complex dynamics involved. Presumably, such criteria should encompass the whole range of physical phenomena relevant to ship stability, including realistic modelling of ship dynamics and its interaction with the marine environment. Moreover, any resultant stability criterion should be of relatively straightforward application, if practical use is not to be jeopardised.

Here a difficult compromise must be reached. Even simple nonlinear models of ship motion can display bewildering wealth of dynamical phenomena. Mathematical modelling can, furthermore, be easily taken into depths of complexity and detail that may preclude a comprehensive study of dynamical features. As a result, stability criteria can easily be pushed to impractical degrees of complexity.

Having these points in mind, we make our main purpose in this work to contribute to the integration of recent methods of nonlinear dynamics into ship stability analysis. We intend to do that by approaching the problem from a geometrical viewpoint and revealing some fundamental properties of the dynamical processes involved in ship capsize. Throughout this work, a number of mathematical models of ship motion will be investigated; both their strong points and their limitations will hopefully be made clear.

It should perhaps be emphasised here that great attention is paid in this work to the possibility of use of these methods as a tool by the practising naval architect. We therefore avoid relying on massive computational power of still usually
Chapter One: Introduction

restricted access. We must however make clear that this work is intended as an investigation of the suitability of a certain approach to the problem of ship stability and capsize, and not as an effort to produce a finished stability criterion. We hope that the outlook we take in this work will be further pursued and eventually contribute to improvements on existing stability criteria.

1.3 Overview of the Contents

Chapter 2 reviews presently accepted ship stability criteria, with special emphasis on the International Maritime Organisation (IMO) Weather Criteria. We comment briefly on the origin of those criteria, and on the general assumptions behind them. The idea here is not to make an in-depth study of those criteria, but to describe them in enough detail to make possible a comparison between them and the approach we propose along the work.

In Chapter 3 we summarise the theory of ship motions as applied to large-amplitude motions and capsize. This distinction is a relevant one, since mathematical models of ship motion tend to be quite specialised according to their purpose. Thus, models designed, for instance, to help in the statistical analysis of sea-keeping, or in predicting manoeuvrability characteristics, are inadequate to deal with the problem of ship capsize.

After describing the fundamentals of ship motion mathematical modelling as they are traditionally approached in Naval Architecture, we turn our attention to the derivation of our own models. These will be used throughout the rest of the work, along with more conventional models.

Having described the engineering problem motivating this work, and having presented the class of mathematical models we shall be dealing with, we introduce in Chapter 4 the geometrical approach to nonlinear dynamical systems that will be the basis of our study of ship capsize. Many excellent texts are available that cover this topic with various degrees of depth and detail. It would
not be very helpful to try and present yet another general introduction to this subject. We shall instead take a different course and attempt to introduce the concepts and methods of nonlinear dynamics using the problem of ship motion and capsize as a guiding example. In doing that we will be at the same time making clear the context within which those mathematical ideas will be used in this work.

In Chapter 5 we present the results of our investigations about the complex nonlinear phenomena governing the failure of a system. We summarise some of the main features of both steady-state and transient dynamics of ship motions, with particular attention being paid to the behaviour of our 2-DOF model of coupled heave-roll motions. The last sections of Chapter 5 are dedicated to an investigation of the processes of erosion of safe basins of attraction. These results form the main input to the ideas we shall advance in Chapter 6.

In Chapter 6 we introduce our proposal for a method to assess ship stability against capsize, based on the results of previous chapters. We first describe the heuristics of the method, before we present some of the theoretical and numerical arguments that support the method.

Finally, we put forward the main conclusions of this work. We summarise here the main results of this work, and we also try to assess their potential for the development of future practical methods. We also make some suggestions on how future studies could extend this work both on the Nonlinear Dynamics and on the Naval Architecture fronts.
2. CURRENT SHIP STABILITY CRITERIA

The purpose of this chapter is to present a brief exposition of current ship stability criteria. The intent is not so much to analyse the various attempts produced along the years to tackle the problem of ship stability, as to present the wider background against which this work is set.

Instead of attempting to describe the large variety of existing criteria, we choose to concentrate on a single, yet representative example: the International Maritime Organisation (IMO) criteria for fishing vessels of 24 meters and over in length. The IMO criteria are probably the most widely accepted stability criteria, and appeared as a result of many years of discussions amongst international bodies and authorities. They are also representative of a whole trend in stability criteria, not only for fishing vessels, passenger and cargo ships, but also for the various types of offshore platforms and working vessels in operation around the world. This trend, or rather this approach, is characterised by the use of the vessel's righting arm curve as the main feature upon which stability analyses will be based. Many other stability criteria (particularly those adopted by Classification Societies such as Lloyd's Register of Shipping, American Bureau of Shipping, Det Norske Veritas, amongst others) employ the same general method, or approach we shall describe here.

The research on improved methods to assess ship stability is being continually carried on by various groups around the world. To give but a glimpse of this research effort, we briefly describe some of the works produced in the last two decades. We describe in somewhat more detail the work conducted at University College London, this being closely related to our own work.

2.1 THE NEED FOR BETTER STABILITY CRITERIA: HISTORICAL BACKGROUND

Before we proceed to comment on the main features of the IMO Weather Criteria, it might be in order to answer a question that should be at the root of
this work: what is the general feeling of the naval community towards the most widely accepted stability criteria. It might come as a surprise to learn that in spite of their wide acceptance and considerable practical success, the presently employed stability criteria do not seem to content the majority of the naval community.

Delegates to the First International Conference on Stability of Ships and Ocean Vehicles held in Glasgow in 1975 were asked to respond to questionnaires prepared by the organising committee in order to:

"establish the views of participants upon the existing methods of dealing with stability and the areas of the subject to which they believe future research should be directed." Kuo and Gordon (1975).

By collecting and classifying the answers to the questionnaires, several conclusions were outlined by Kuo and Gordon, among which we selected the most relevant to our study:

i. Existing Stability Criteria:

"Only 29% of respondents felt that the existing stability criteria based on the use of the righting arm curve met practical needs. Almost 50% felt that the criteria were unsatisfactory."

ii. When asked about how to improve existing methods of dealing with stability, respondents' opinions split as following:

1. Better stability criteria: 33%
2. Develop fresh treatment: 22%
3. Collect and analyse more practical data: 19%
4. Extensive training of masters and operators: 15%
5. Further refinement of the righting arm curve: 6.5%
6. Others: 4.5%

iii. When asked about the relative importance of various factors in the research for new improved methods, delegates' feelings suggested that the following factors were the most relevant (by order of importance):

1. Effects of waves.
2. Developing fresh methods.
3. Water on deck.
5. Scaled model experiments.
6. Wind effects.
7. Education, etc.

iv. Finally, when asked about the relative importance of factors affecting fresh stability criteria, ratings were as follows:

1. Simple to apply in practice: 27%
2. Taking into account as many influencing factors as possible: 25%
3. Incorporate effects of motion characteristics: 18%
4. Easy for design purposes: 17%
5. Be theoretically based: 13%

To summarise it, we could say that the prevalent opinion was that existing stability criteria should be replaced by new methods, probably fresh in their approach, that take into account as many influencing factors as possible, with emphasis on dynamic effects such as waves (only 7% of respondents felt existing criteria were acceptable for incorporating dynamic effects), and being at the same time simple enough not to preclude their practical use.
Almost two decades after that First Conference the same criteria about which the above comments were made are still in use. The reasons for the delay in adopting improved methods are not to be found in a lack of effort from the research community, as the following three International Conferences (1982, 1986, and 1990) on the subject demonstrate, but rather in the inherent complexity of the phenomena involved, allied to a certain number of practical constraints to the possible solutions, not the least significant of them being the requirement of simplicity of use. In fact, the simultaneous achievement of a comprehensive coverage of factors relevant to ship stability and simplicity of use seems to have posed too difficult a challenge. It is possible that only a redefinition of 'simplicity of use', on the basis of the present widespread availability of computers, will bring that objective closer to reality.

Regarding again the expectation that the research community apparently has about "a comprehensive coverage of factors relevant to ship stability" one point should perhaps be stressed here. It is our opinion that the concern with the large variety of practical factors that seem to play important roles in ship stability, such as loading conditions, accumulation of water and/or ice on deck, freeboard considerations, the effects of breaking waves, to name but a few, has obscured a much more basic fact: little is known about the dynamic behaviour of even simple models of ship motion. Obviously, we do not refer here to the linear theory of ship motions, which is relatively well understood. Linear models seem however to be utterly inappropriate to represent the large-amplitude motions involved in ship capsize, and we have to consider nonlinear models. It is the dynamics of those models that still lack fundamental understanding. It seems therefore suitable to investigate basic properties of nonlinear models in the hope of gaining important insight about extreme ship behaviour, and capsize.

---

1The Torremolinos International Convention for the Safety of Fishing Vessels, IMO (1977), proposed minimum stability requirements to be applied by the competent National Administrations concerning "new fishing vessels of 24 metres in length and over". The new recommendations followed the same principles already in use for passenger and cargo ships, viz., that of imposing certain conditions upon the righting arm curve. The Convention is not yet in force.
2.2 THE GZ-CURVE CRITERIA: DESCRIPTION

Most stability criteria are based on a hydrostatic model of ship motion. By *hydrostatic* we mean that all forces involved are supposed to be those calculated for the vessel standing still in a given position relative to the waves. Under this assumption, hydrodynamic forces such as added mass, damping, and some excitation terms must be either neglected or given by *ad hoc* formulations. In the IMO Weather Criteria simplification is carried a step further, and calculations are made for the vessel in still water. Figure 2.1 shows the basic forces involved here, together with some of the most common nomenclature.

The vessel's GZ-Curve is simply a plot of her static righting arm (or moment) for a range of angles of inclination. In its simplest form, the curve is defined for a vessel in calm waters, and can be determined in a variety of means including experiments with reduced physical models and numerical computation. Figure 2.2 depicts a typical GZ-Curve.

Calculations are usually performed under the assumption of equilibrium along the vertical direction, i.e., the vessel's weight $W$, acting along her centre of gravity $G$, is equalled by the weight of the displaced water $F$, which acts along the centre of buoyancy $B$. There is, however, a net moment around the centre of gravity. This moment is obviously given by $W \times GZ = W \times GM \times \sin(\theta)$. The metacentre $M$ is defined as the intersecting point between the vertical through $B$ and the vessel's centreplane (represented in figure 2.1 by its projection on the vertical plane). The metacentric height $GM$ is thus defined as the distance between $G$ and $M$. The righting arm $GZ$ is just the projection on the horizontal direction of the metacentric height $GM$, hence the importance attached to $GM$ as a measure of the vessel's stability. Clearly, since $B$ can change with the heeling angle $\theta$, $GM$ is itself a function of $\theta$. 
Figure 2.1 - Hydrostatic forces acting upon a heeled ship

The complete description of the criteria can be found in the *Guidance on a Method of Calculation of the Effect of Severe Wind and Rolling in Associated Sea Conditions* (Regulation 31), IMO (1977). We shall reproduce here only the essential aspects of the criteria.

"The ability of the vessel to withstand the effects of gusts and severe winds and rolling should be demonstrated using dynamic heeling moment taking into consideration the rolling angle due to waves. The criterion for adequate stability under these circumstances should show that the effect of the dynamic heeling moment $M_W$ (as indicated in figure 2.2) caused by wind pressure in the worst operating condition, taking into account the rolling angle, is equal to or less than the effect of the excess restoring moment (area "b" or the area under the corresponding excess restoring arm). This condition is considered to be fulfilled when the following condition is satisfied:
The ratio \( C_{wr} = \text{area "b" / area "a"} \) should not be less than unity.

![Figure 2.2 - The IMO Weather Criteria](image)

The wind heeling moments (steady and gust) should be calculated by the following expression:

\[
M_{w1,w2} = \frac{1}{2} \rho C_D K^2 \sum_{n=1}^{N} (V_n^2 A_n Z_n)
\]

Where:

- \( M_{w1} \) = heeling moment due to steady wind
- \( M_{w2} \) = heeling moment due to gust or severe wind
- \( \rho \) = air density
- \( C_D \) = appropriate non-dimensional drag coefficient
- \( K \) = wind speed factor
  - \( K = 1 \) for steady wind
  - \( K > 1 \) for gust wind
- \( V_n \) = wind speed at centroid of lateral area \( A_n \).
\[ A_n = \text{projected lateral profile area of element } n \]
\[ Z_n = \text{length of wind lever between centroid of } A_n \text{ and assumed point of action of the opposing forces} \]
\[ n = \text{integer} \]
\[ N = \text{number of elements of horizontal areas}. \]

The angles defined in figure 2.2 are:

\[ \theta_0 = \text{angle of heel under action of steady wind} \]
\[ \theta_i = \text{angle of roll to windward when rolling about } \theta_0 \]
\[ \theta_2 = \text{flooding angle, or angle specified by the Administration}. \]

The values of \( V_n, K, Z_n, \theta_0, \theta_i \) should be approved by the Administration.

2.3 **The GZ-Curve criteria: general comments**

Although simple in form, the GZ-Curve criteria lend themselves to more extensive analyses than it is possible to develop here. We shall therefore limit ourselves to some of the basic aspects and their possible connection with the study we perform here.

The present IMO criteria have their origins in the work of Rahola (1939), who collected statistics of 34 capsized vessels between 1871 and 1938. By considering the characteristics of those capsized vessels, Rahola proposed minimum GZ values at different angles of heeling such that adequate protection against capsize should be given. The whole method is therefore statistical and empirical. Later in the 1960s further statistical studies of capsized vessels were made within the same principle, Nadeinski and Jens (1968), Thomson and Tope (1970), and the resulting criteria remain almost unchanged to the present day. Also during the 1960s, the criteria were adapted to provide some cover for environmental effects, particularly wind, Sarchin and Goldberg (1962), later resulting in what is called the IMO Weather Criteria.
It seems important to emphasise that, in contrast with the IMO criteria for passenger and cargo ships, the various parameters and coefficients necessary for the assessment of the stability characteristics of a fishing vessel through the criteria shown in 2.2 are not given by IMO, but rather left to be supplied by each Administration. Therefore the criteria above could be better viewed as a guideline as to on which grounds to judge the stability of a given fishing vessel, i.e., as a statement that the GZ-Curve is the adequate instrument of assessing the stability of a vessel. It is useful to remember that the requirement of calculating and analysing the GZ-Curve was seen as a considerable advance when it was introduced by IMO. In fact, this requirement is still considered unrealistic for fishing boats under 24 meters in length. In the Voluntary Guidelines for the Design, Construction and Equipment of Small Fishing Vessels, IMO (1980), it is suggested that "where full stability information is not available", a formula for the minimum initial metacentric height (usually denoted by GM) can be used as a criterion for adequate stability\(^2\).

It should also be noted that vessels are required to fulfil the criteria "in the worst possible condition". This is intended to cover all operational conditions of a vessel, particularly with respect to loading conditions and consequent positions of its centre of gravity, thereby including the effects of changes in GM. Notwithstanding that, the criteria are often regarded as giving inadequate coverage of light loading conditions, Kuo and Gordon (1975).

In view of the above, it seems fair to say that the GZ-Curve criterion is more of a method than a criterion, and it would be convenient to start a critical analysis of this method by revealing its intrinsic features.

\(^2\)The formula is given in 4.2.3 of the Code of Safety for Fishermen and Fishing Vessels, IMCO (1975).
The GZ-Curve criteria are a *quasi-static energy method*. It is based on the idea that, from a pre-specified initial condition, the vessel should be able to resist a pre-specified quasi-static wind moment\(^3\).

The method is therefore based on what could be called a "standard capsize event", and the ability of a vessel to withstand that event without capsizing is regarded as a good measure of its stability qualities. The principle of defining appropriate "idealised events" and judging the performance of a system by its behaviour in such events is widespread in engineering. One reason for that is the almost infinite variety of situations in which a system will operate during its life. It is virtually impossible to test the response of a system to every combination of internal and external elements that the system may have to face. The engineering approach to this problem is to test the system under a few carefully chosen situations. There are often many procedures that can be used to define test situations, each one corresponding to a specific method or criterion.

A good example of the use of idealised events in engineering is the well-established use of standard tests to define the strength of materials. In the laboratory, specimens of pre-specified shapes are subjected to standard mechanical loads, under which their behaviour is measured and classified. As parts of real structural systems, materials will never be subjected to precisely the same loading with which specimens are tested. There are therefore criteria that relate the two situations, laboratory and real system: if a given material withstands a mechanical load \(X\) as part of a laboratory specimen, then it will withstand a mechanical load \(Y\) as part of a given structural system. As we could expect, the translation of results from laboratory to real structure is far from straightforward. Some degree of arbitrariness still remains, giving rise to a variety of different practical criteria.

---

\(^3\)The reference to a "dynamic wind heeling moment" made in the text of the regulation is probably related to the use of the term "dynamic stability diagram" to refer to the GZ-Curve, and is somewhat misleading since no dynamic effect is actually taken into account.
The case at hand - capsize - is no exception to the rule described above. Even if the dynamics of a ship in a realistic marine environment could be accurately modelled, the problem would persist of selecting situations under which to test the behaviour of a given vessel. Here again we will find that practical criteria are based on 'potential capsize' situations. If a given vessel withstands the potential capsize situation without capsizing, its stability is considered adequate. The IMO Criteria are based on a potential capsize situation in which a ship is heeled to a certain angle and then acted upon by a static wind moment of given magnitude. A quasi-static calculation is performed in which the work done by the wind moment has to be equalled by the work done by the vessel's restoring moment before a certain heel angle is achieved. The definition of this limit angle is rather arbitrary. It can be taken as the flooding angle (the angle at which the vessel starts to accumulate water on deck) or it may be otherwise defined by local Administration.

Perhaps a more important point to be made here is that the success of the approach of defining idealised events does not seem to be connected to how accurately the idealised events represent real conditions. In the above example, the potential capsize situation is clearly very far from actual capsize scenarios. Large waves, for instance, are a predominant feature of real capsize scenarios, and their action is not considered during the energy balance. One may conclude that the key element here is that the idealised events somehow capture the 'essential features' of the real phenomena involved. As far as statics go the IMO Criteria may have captured those essential features. But when trying to incorporate dynamic effects, a different approach must be devised. Here we believe that nonlinear dynamics may have an important role to play, for without studies on the dynamical mechanisms involved in large-amplitude motions and capsize there is little hope of achieving a better understanding of what the essential features of the problem are.

In this respect, some obvious shortcomings of the IMO Criteria should perhaps be mentioned. They are mainly motivated by the limited amount of information
the method is supposed to work upon, and by the requirement of keeping its use very simple (see Vassalos (1985)).

1. Strong winds are indeed a common feature in capsize scenarios, and the method rightly considers their action. But strong winds are frequently associated with large waves, whose effects are not given the same treatment in the method.

2. The waves that caused the ship to roll up to an angle \( \theta \), from which the gust wind is then supposed to act are not considered in the subsequent calculations.

3. Because no dynamic (time-related) effect is included, no consideration can be given to the oscillatory nature of some important capsize mechanisms.

4. Due to its empirical nature the method gives little or no information about "safety factors" eventually involved, which could be thought of as partially taking care of dynamic and waves effects.

These aspects are reflected in the general feeling among researchers that waves and dynamic effects are not adequately dealt with by the existing methods (see section 2.1). The question is clearly how to add dynamic effects to the analysis without making it too complicated to apply in practice. The answer seems to be made more difficult by the fact that, as it is now known, the behaviour of simple dynamical systems used to describe the most basic features of ship motion can be extremely complex. We hope, however, to show that the same methods that helped researchers to unveil the subtle complexities of dynamical models of ship motion can help them to synthesise new stability criteria of better quality, and simplicity compatible with present computational resources.

### 2.4 Brief Survey of Works on Ship Stability Criteria

The literature on ship stability and stability criteria is considerable, and the interest in the subject guarantees that it still grows. A variety of approaches
have been used in attempts to tackle the problem of producing simple yet comprehensive methods to assess the stability qualities of marine vessels.

There is a strong feeling among a number of researchers that a probabilistic approach must be used. This feeling seems to stem from the assertion that both the marine environment and ship operational conditions are too varied and unpredictable to allow the use of deterministic methods. Therefore a large number of works adopt a probabilistic approach, in which the capsizability of the vessel is measured as a probability of failure under a design excitation defined in terms of its statistical properties. See for instance Boroday and Nikolaev (1975), Huang (1990), Kaplan (1990), Kastner (1975), Kobylinski (1975), Tikka and Pauling (1990), Umeda et al (1990).

Research on improved stability criteria has been carried out for many years at Strathclyde University. As a result, new criteria have been proposed in which the effect of waves is explicitly taken into account, see for instance Martin et al (1982), Vassalos (1985), and Kuo et al (1986). In essence, this criterion is based on the idea of performing an energy balance between exciting and restoring forces along a carefully chosen critical cycle of oscillation. The main contribution of this method is to allow for the inclusion in the calculations of time-varying moments. Although a systematic procedure for the location of critical conditions is included in the method, a certain degree of arbitrariness remains. Furthermore, being a quasi-dynamic method, the Strathclyde criteria allow for limited inclusion of dynamic effects.

In a series of papers, L. Virgin has put forward a simplified criterion for stable ship roll motions in regular waves, Virgin (1987, 1989, 1990). His approach resembles our own in the selection of deterministic dynamics in phase space as the main tool for assessing ship stability. He proposes a criterion based on a comparison between the total mechanical energy of a limiting state of the ship (say, the ship heeled to the angle of vanishing stability) and the energy of trajectories. He argues that if the total energy of motion along a steady-state trajectory is kept well below the limiting value, the ship is safe. An empirical
safety factor defines exactly how much below the critical value the trajectories' energies should remain. The main limitation of Virgin's work may be its reliance on steady-state dynamics. As we shall see, the stability of steady-state trajectories can be a misleading parameter upon which to base ship stability analyses.

Work being carried out at the Centre for Nonlinear Dynamics of the University College London by J.M.T. Thompson and colleagues has employed an approach inspired in modern methods of nonlinear dynamics, where analytical and computational investigations combine to produce both qualitative and quantitative assessments of the dynamical behaviour of vessels in regular waves. These methods follow on a long tradition initiated by Henry Poincaré, who in the beginning of this century advocated the use of phase space representations as a valuable tool in the construction of our understanding of complex nonlinear dynamical systems, Poincaré (1899).

Following detailed studies of the behaviour of transient and steady-state motions of archetypal mathematical models of ship motion under the action of regular waves, Thompson and co-workers have proposed a remarkably simple method to assess ship stability against capsize. The method is based on the construction of a plot of critical amplitudes of forcing against the frequency of the forcing, the Transient Capsize Diagram, see for instance Rainey et al (1990), and Rainey and Thompson (1991).

The backbone of this method is the finding that, under quite general conditions, the loss of engineering integrity of a ship is dominated by a rapid erosion of its safe basin of attraction, Thompson et al (1990), Soliman and Thompson (1992). The authors propose that the values of forcing parameters at which such erosion starts can be approximately determined through a series of experiments (numerical or physical) in which the ship is excited from rest by short trains of regular waves. The objective of the experiments is to determine the minimum

---

4The term *rapid* is related to variations in forcing, not in time.
amplitude of excitation for escape (capsize) at each frequency within the range of interest. The domain of safe initial conditions (the safe basin of attraction) has been shown to be swiftly eroded by fractal incursions after a series of homoclinic events (bifurcations), making the particular choice of starting conditions relatively irrelevant, Soliman and Thompson (1989). Stable steady-state solutions can, however, persist within the safe basin for considerably larger levels of excitation. This makes analyses based on the stability of steady-state solutions dangerously non-conservative.

Our own work is part of the effort just described, and is intended to contribute to the research along similar lines tackling some of the points open to further investigation such as the inclusion of two-degree-of-freedom models.
3. MATHEMATICAL MODELLING OF SHIP MOTION

This chapter introduces mathematical models of ship motions. We give a brief overview of this vast subject, and proceed to describe in more detail the specific models we shall be using in this work.

3.1 OBJECTIVE OF MATHEMATICAL MODELLING

The purpose of a mathematical model is to represent interesting aspects of a phenomenon. Real problems are usually too complex to allow the definition of a mathematical model that represents all interesting aspects involved. Choices are therefore necessary, and the mathematical models we use are the result of a number of choices that privilege some factors whilst leaving others aside. The representation of the problem of a marine vessel moving in the sea through mathematical formulae is a prime example of that.

As a consequence of the complexity of the problem, mathematical models of ship motion tend to be rather specialised. We shall be dealing here with a small subset of mathematical models of ship motion directly concerned with ship capsize. Other classes of models exist which find use in different problems, including seakeeping, manoeuvrability, and structural design.

For our purposes the problem of ship motions can be phrased as follows. We imagine a marine vessel freely floating at sea under the action of wind and waves. We are solely concerned with the behaviour of the vessel as a rigid body. More specifically, we want to investigate under which conditions the vessel will display large-amplitude motions that can lead to capsize. We start with a brief description of the basic physical mechanisms known to be related to ship capsize.
3.2 Basic ship capsize mechanisms in waves

The physical mechanisms through which a vessel in waves can experience large-amplitude motions and eventual capsize form the basis of most efforts to model the problem mathematically. Simplified mathematical models in particular usually concentrate on one or two possible physical mechanisms. In what follows we present a well accepted classification of ship capsize mechanisms, and we establish a correlation between those mechanisms and typical mathematical models.

The U.S. Coast Guard mounted in the early 1970s a five-year program of trials with ship models in open water. The main goal was to achieve good understanding of the physical basis of ship stability in waves, and the then proposed classification of capsize mechanisms is still a landmark in the subject, see Paulling et al (1975). To their classification we add here a further mechanism, namely capsize in beam seas. This well known mechanism was not included in the work of Paulling and others probably based on the assumption that, in general, ship's heading can be adjusted to avoid prolonged exposure to beam waves.

Two reasons, however, make it desirable to take that unfavourable scenario into account. Firstly, the number of waves necessary to induce large roll angles or even capsize can be quite small, and is not known a priori. Numerous reports of real capsize events indicate that capsize can occur when the vessel is subjected to a short train of large waves, see for instance Kowaslky (1985) and National Research Council (1991). Thus, in practical situations the time for the crew to manoeuvre the ship and therefore avoid large amplitude motions can be short. Secondly, operational requirements sometimes make it difficult or undesirable for the ship to change course during considerable lengths of time. Although seakeeping considerations should determine the temporary disruption of normal

---

1 This is normally associated with operations such as the deployment of structures in the sea, the excavation of trenches, and others.
operation as soon as weather conditions become too rough (and therefore well before any real risk of capsize), the fact just mentioned above, namely the possibly short number of waves sufficient for capsize, makes these cautionary procedures less reliable.

The basic physical mechanisms through which intact ships have been observed to capsize are:

i. Pure loss of stability in waves;
ii. Low cycle (or parametric) resonance;
iii. Broaching;
iv. Capsize in beam seas.

It must be realised that this classification (like most attempts to classify real phenomena) serves the purpose of facilitating the study of a very complex problem. Real capsizes will generally involve more than one of the above mentioned capsize mechanisms, and it may be difficult to tell which one of them was predominant to the final outcome.

We now describe in more detail each one of the capsize mechanisms, and their possible connection with specific mathematical models.

3.2.1 Pure loss of stability in waves

This mode of capsize is most frequently observed when the ship travels at relatively high speeds, and experiences following waves whose effective lengths are not far from the ship's length. The ship may then find itself held in a wave crest for a considerable period of time. In such situation, ship static stability

\footnote{It should be noted that when we refer to a particular mode of capsize as "taking place" or "occurring" under certain situations, we mean that those situations are considered as necessary (or most favourable) for that particular mode. It does not mean that any ship will repeatedly capsize if subjected to the conditions described.}
features (basically its righting arm curve) are significantly affected. Maximum values of restoring moments, and the range of positive stability can be much reduced. Relatively small perturbations will produce large angles of roll and can easily induce capsize. In some cases, the initially stable upright position of the ship can become unstable (negative restoring moment), and therefore any tiny disturbance will make the ship to look for a new stable position at a larger heel angle. This is the capsize mode in which dynamical effects play the least important rôle. Static and quasi-static models are usually regarded as appropriate to investigate the vessel's robustness against this mode of capsize.

3.2.2 Low-cycle (or parametric) resonance

This mode of capsize takes place when ship's speed is such that its natural period is twice the waves' encounter period. In such situation the ship may find itself in a new wave crest at both port and starboard extremes of rolling motion, while oscillating at its natural roll period. The temporal variation of intrinsic restoring moments (the GZ-Curve itself) can induce autoparametrically excited oscillations. This mode of capsize is considered to be related to a well-known feature of parametrically excited systems: the Mathieu-type instability, see Jordan and Smith (1977). The simplest model used to investigate this mode of capsize is a one-degree-of-freedom model with time-varying restoring functions, see section 3.6.1.

3.2.3 Broaching

Broaching is the phenomenon of loss of directional stability experienced by a ship in waves. It is consequently related to the manoeuvring qualities of a ship, as well as to its conventional stability properties. In this mode of capsize, a

\[\text{The simple parametric resonance phenomenon is set in a deterministic scenario with regular excitation (waves).}\]

\[\text{For instance, rudder immersion and effectiveness.}\]
ship will usually be in following seas, when it loses directional stability and starts
to yaw quite heavily. The large rotational acceleration (in yaw), allied to
unbalanced roll moments causes the ship to roll violently and eventually capsize.
The loss of directional stability itself is primarily related to a dramatic reduction
in ship's "stiffness" to changes in course (yaw motion), observed under certain
circumstances. The broaching is made more serious when the ship has a
considerable forward velocity and is then "caught" on the front face of a wave,
so that it comes close to a "surfing" condition. The large unbalanced roll
moments generated by the turning motion are associated with considerable
relative (ship-water) lateral velocities and lever arms for lateral forces.

Broaching is a highly dynamical and nonlinear mode of capsize, and one that
requires sophisticated mathematical modelling. Works using mathematical
models to investigate this problem resort to considerable computational power,
see for instance de Kat and Paulling (1989).

3.2.4 CAPSIZE IN BEAM SEAS

In this scenario the ship is subjected to waves, and possibly wind, from a
lateral (relative) direction. It is observed that for certain values of waves' amplitude and period the ship is made to oscillate in large angles, and eventually capsize. Wind speed and the shipping of water on deck can also play an important rôle in this scenario, see Falzarano (1990). A more detailed description of the complex dynamic behaviour of a ship under these situations will be presented further ahead in this work. We note here that a large variety of dynamic patterns have been computationally observed, and documented in the literature, including periodic motions of different frequencies, as well as non-periodic (chaotic) motions, see for instance Thompson (1989). Because the most severe conditions for this mode of capsize tend to occur under waves of period close to the vessel's natural rolling period, capsize in beam seas is sometimes referred to as a main resonance phenomenon.
Mathematical models employed to study capsize in beam seas are often relatively simple. Rolling is the main dynamical variable, and models frequently reduce the problem to this single variable. We shall make extensive use of such models in this work.

3.3 On the prediction of ship motions

Before we proceed to introduce the general framework within which ship motions are mathematically modelled, a few comments on the rôle of the prediction of ship motions might be appropriate.

It seems reasonable to assume that the main purpose of trying to define mathematical models of ship motions is to have an instrument capable of reproducing actual ship behaviour. We can consider ship behaviour as given for example by the time evolution of its position and momentum as a rigid body in space. Previous efforts have taken one of two approaches here: to concentrate on individual time-histories, or to work with statistical properties. Note that the reason for adopting the latter has been just the evident difficulty in modelling real sea states by deterministic means, see for instance Hsieh et al (1994). Very often, when controlled experiments have been used to validate and/or calibrate mathematical models, a deterministic approach concentrating on individual trajectories (time-histories) has been adopted, see for instance de Kat and Paulling (1989) and Pawlowski (1991).

Possibly one of the major contributions of the theory of nonlinear dynamics to the field of ship stability has been so far to highlight the fact that individual time-histories may be unpredictable. This is strongly suggested by numerous studies that have shown that typical models of ship motions display sensitive
dependence on initial conditions\(^5\), see Soliman and Thompson (1991). Such phenomena are particularly relevant to the highly nonlinear regimes in which capsize can occur. These results indicate that, within the range of motions of interest to the researcher of ship capsize, the whole approach of validating sophisticated mathematical models with individual trajectories obtained from physical experiments may be flawed. It is likely that analyses of \textit{ensembles of trajectories} must be performed. But when one takes into account the present cost of running complex numerical packages developed to simulate in detail the behaviour of ships in waves one sees that a different approach must be envisaged. This is one of the main points of our work, and we shall explore it in more detail in the following chapters. First, however, we give more detailed information on the type of mathematical models we shall be working with.

\section*{3.4 Basic formulation of the problem of ship motions}

A complete description of the motion of a vessel as a solid in space requires the formulation and solution of the equations derived from Newton's Second Law:

\[ \frac{d}{dt} mv = f \] 

where \( \frac{d}{dt}(\cdot) \) denotes differentiation with respect to time \( t \), \( m \) is a 6 by 6 matrix of inertia, \( v \) is a 6 by 1 vector of linear and angular velocities, and \( f \) is a 6 by 1 vector of external forces and moments. Displacements, velocities and external excitation are defined in inertial reference frames.

In a wind-wave dominated scenario forces such as those generated by the propulsion and steering systems are often ignored. The remaining forces and

\(^5\)Note that \textit{initial conditions} can be understood here as comprising both phase variables (positions and momenta) and model parameters. Since none of those are known with great precision, individual trajectories would be for all purposes unpredictable.
moments are therefore those resulting from the interaction between the vessel and its surroundings. Let us concentrate on the hull-water interaction. An exact formulation of this problem requires writing down dynamical equations for the motion of the water around the vessel. For a Newtonian fluid of constant density, these are given by the Navier-Stokes equations (which express conservation of momentum) together with a continuity equation, see Newman (1977). The Navier-Stokes equations are a system of three nonlinear partial differential equations on which suitable boundary conditions must be imposed, reflecting the particular conditions of our problem. It is not necessary to say that, even in the absence of any vessel, the resulting equations of motion for real (viscous) sea waves are very complex. Indeed only approximate equations are usually dealt with in any practical sense. The most common approximations include reducing the problem to the ideal (non-viscous), linear case in which potential theory can be used, see Newman (1977) for a classical treatment of this problem.

The situation is further complicated by the real boundary conditions that must be applied in the ship-wave problem. Firstly, the vessel's hull has a complex geometry. Secondly, the very position and velocities of the vessel (necessary for the definition of the boundary condition) are the principal unknown variables of the problem. A number of methods have been envisaged along the years to try to solve this problem with various degrees of accuracy or sophistication. More recently, numerical methods have been combined with analytical tools to produce very powerful procedures applicable to large-amplitude motions, see for instance Miller et al (1986), and de Kat and Paulling (1989). Note that even these sophisticated packages make extensive use of empirical and semi-empirical formulations, particularly in the modelling of viscous effects. In their vast majority, these methods rely on the use of massive computational resources that render their operation very costly indeed.
3.5 The rôle of simplified mathematical models

In future, sophisticated numerical packages will probably be the norm in ship stability calculations. Technology is making increasing computational power widely and cheaply available. For the moment, however, it is still unrealistic to expect powerful numerical codes to be routinely employed, and simplified mathematical models are useful, and even necessary.

There is however a more fundamental reason for the interest in simple models of ship motion. As we said before in this work, there is still a considerable lack of basic knowledge about the dynamics of ship motion. Complex numerical models have far too many parameters to allow for systematic parametric studies that are necessary to advance our knowledge on the basic dynamical mechanisms responsible for large-amplitude motions and capsize. Many interesting phenomena can in fact be observed and investigated within less complicated contexts, such as those provided by one- and two-degrees-of-freedom models.

3.6 One-degree-of-freedom (1-DOF) models

3.6.1 Preliminaries

In 1-DOF models used in ship stability studies the problem is reduced to a single dynamic variable:

\[ \frac{d}{dt} I \dot{\phi} = M \]

where a dot denotes differentiation with respect to time \( t \), \( \phi \) is the vessel's displacement in roll, \( I \) is the vessel's inertia for roll motion, and \( M \) is the sum of relevant external moments. Various methods can be used to calculate external moments, varying from quite sophisticated and time-consuming procedures to simple lumped-parameter models. The work of William Froude records one of
the earliest attempts to present a mathematical description of the rolling motion of ships, in which the nonlinear nature of the problem was already highlighted, Froude (1874). In another early work, the instabilities ship rolling can experience were investigated by J.E. Kerwin in a study that concentrated on Mathieu-type instability caused by longitudinal waves, Kerwin (1955). More recently, many works have either focused on the development of 1-DOF roll motions or used them as their main working tool, see for instance Nayfeh and Khdeir (1986), Cardo et al (1984), Patel and Brown (1985), Kuo et al (1986), and Vassalos (1985). See also Witz et al (1989) for an application of 1-DOF models to the roll motion of semisubmersibles, and Grochowalski (1989) for an experimental work specifically designed to investigate physical mechanisms responsible for capsize, and their representation in 1-DOF models.

Simplified models of ship rolling can be defined by equations of the following type:

\[ I \ddot{\phi} + g(\dot{\phi}) + f(\phi) = M(t) \]

where \( g(\dot{\phi}) \) is a damping moment function, \( f(\phi) \) is the restoring moment function, and \( M(t) \) is the external moment function, which here is supposed to depend exclusively on time. Here again a variety of approaches can be used to calculate the above functions. One possible approach is to take \( f(\phi) \) as a (possibly polynomial) approximation of the vessel's restoring moment function in calm water \( f(t) = \sum_i a_i \phi_i \), \( M(t) \) as a simple harmonic function \( M(t) = F \sin(\alpha t + \varphi) \), and \( g(\dot{\phi}) \) as \emph{ad hoc} damping moment function such as \( g(\dot{\phi}) = \sum_j b_j \dot{\phi}^j \). The parameters \( a_i, b_i, \omega, \) and \( \varphi \) are either constants or vary with time in a pre-determined manner. The problem thus defined is then clearly reduced to a relatively simple lumped-parameter nonlinear oscillator. Although simple in definition this formulation of the problem of ship motion has been proved in a large number of works to be a valuable approximation to the real problem, see references in the previous sections.
The main limitations of the model (or class of models) described above can be summarised as follows:

i. Due to the absence of any other dynamic variable, i.e. other coupled modes of motion, interesting phenomena related to coupling between modes will be overlooked. Some of these phenomena are indeed known to play important rôles in some capsize mechanisms (see section 3.2). Perhaps the best known example is the parametric instability associated with the coupling between heave and roll. If already identified in higher dimensional models, some of these phenomena can be included in a 1-DOF model, albeit in approximate fashion. In the case just mentioned, one possible alternative is to model the parametric excitation induced by the heave motion as a time-varying (periodic) parameter affecting the vessel's restoring moment:

\[ I\ddot{\phi} + g(\dot{\phi}) + (1 + \varepsilon \cos(\alpha t))f(\phi) = F \sin(\alpha t) \]  

3.4

See Thompson et al (1992) for the rationale behind this approximation, and Kan (1992) for another recent example of its use.

ii. The approximation of the 'true' hydrostatic restoring moment by its calm-water counterpart places limitations on the valid ranges of wave parameters that can be investigated. In particular, only waves of length much larger than the vessel's breadth will have their action accurately modelled. Also, the 1-DOF approximation is best suited to represent beam, or near-beam wave situations.

3.6.2 THE ESCAPE EQUATION

The escape equation can be thought of as an archetypal model for the problem of the escape of a particle from a potential well. In a certain sense it represents the simplest possible model for the motion of a particle inside a potential well from which it can escape to infinity. This model was extensively
studied in the context of ship capsize by Michael Thompson and co-workers at University College London. A large variety of interesting nonlinear phenomena can be observed and investigated through this simple mathematical model; see Thompson (1989) for a comprehensive account of that.

The derivation of this well-known equation in the context of ship rolling motion is well documented elsewhere, see for instance Thompson et al (1990). Here we restrict ourselves to a brief analysis of its main features, with the intent of highlighting an approach that can be used to derive simple yet meaningful mathematical models of roll motion.

We can write down the equation of roll motion for this conservative system as:

\[ \dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 \]  

where \( \frac{\partial V(\phi)}{\partial \phi} \) denotes differentiation of the potential function \( V(\phi) \) with respect to the roll variable \( \phi \). The potential function \( V(\phi) \) is here given by the integration of the restoring moment function \( f(\phi) \):

\[ V(\phi) = \int_0^\phi f(x)dx \]

Ship rolling can, under these assumptions, be seen as the one-dimensional motion of a particle of mass \( m \) restricted to move along a line (whose geometry is totally defined by the potential function \( V(\phi) \)) under the action of a uniform gravitational field, see figure 3.1. Typically, the normal upright floating condition will be stable, and therefore will correspond to a local minimum of \( V(\phi) \). Likewise, the so-called angle of vanishing stability (at which the restoring moment is zero) will correspond to a local maximum of \( V(\phi) \). In this sense we can therefore see the capsize of a ship as the escape of a particle from a potential well. This is a viewpoint that we shall be employing throughout this work.
Now the simplest possible restoring moment function that comprises the possibility of escape is given by a polynomial of second order in $\phi$: $f(\phi) = \phi - \phi^2$ with corresponding potential function $V(\phi) = \frac{1}{2} \phi^2 - \frac{1}{3} \phi^3$, see figure 3.2.

Physically, this model can be related to a wind-wave scenario in which capsize against the wind is deemed impossible, see Thompson et al (1990).

After the addition of ad hoc damping and exciting moment functions, the resulting equation of motion can be non-dimensionalised through suitable scaling of time and displacement variables to give:

$$\ddot{\phi} + g(\dot{\phi}) + \phi - \phi^2 = M(t)$$

3.7
Particular choices of a linear function for the damping term, and of a simple sinusoidal function for the excitation yield the well-known escape equation:

\[ \ddot{\phi} + \beta \dot{\phi} + \phi - \phi^3 = F \sin(\omega t) \]  

where \( \beta \) is a constant damping coefficient, and \( F \) and \( \omega \) are the amplitude and frequency of the excitation, respectively.

It is clear from the development above that the approach used here is not restricted to a particular choice of restoring, damping or excitation functions. We shall in fact be using a variety of different damping, and restoring moment functions along this work to highlight the generality of our approach.

### 3.7 Two-degrees-of-freedom (2-DOF) models

A natural step towards more realistic mathematical modelling of ship motions is to consider the possible influence of internal coupling between roll
and a second dynamical variable. Here we find that depending on the relative
direction between wave propagation and the vessel's hull, some dynamical
variables will be naturally more relevant than others. This is a consequence of
the typical geometry of sea-going vessels, with their symmetry (or near-
symmetry) with respect to both vertical centreplanes. In particular, we note that
due to near-symmetry in a beam-wave scenario some dynamical variables are
likely to play less important rôles, noticeably pitch, yaw and surge. Two-
degrees-of-freedom models are not as common in the literature as their simpler,
1-DOF counterparts, but examples can be found in Bhattacharyya (1978).

3.7.1 AN ARCHETYPAL 2-DOF MODEL OF HEAVE AND ROLL MOTIONS

In collaboration with Prof. Michael Thompson we have developed for this
work a simple 2-DOF mathematical model of coupled heave and roll motions.
Our approach was to derive equations of motion from basic principles with the
explicit purpose of producing an archetypal model of the interaction between
heave and roll under regular wave forcing. We call this model the Symmetric
Internal Resonance (SIR for short) equations of motion.

We consider a typical (monohull) ship with a vertical plane of symmetry running
from bow to stern. The cross-section of the ship containing the centre of gravity,
G, is shown in figure 3.3. Other cross-sections will normally have a different
shape. We focus here on planar motions of the ship in which the cross-section
remains in a fixed vertical plane. The three degrees of freedom are then the
horizontal sway, $u$, the vertical heave, $v$, and the rotational roll, $\theta$. Within the
assumptions made here, the sway will not enter the analysis\(^6\) leaving us with just
two active degrees of freedom $(v, \theta)$. The ship floats in equilibrium on still water
in a reference state (R) with G at a distance $Y$ above the water surface A-A. Its
(reference-state) centre of buoyancy, B, is at a distance $S$ below the surface. The

---

\(^6\)The equation of motion in sway is trivial: $\ddot{u} = 0$. 
roll, $\theta$, and the upwards heave of $G$, denoted by $v$, are both taken to be zero in state $R$.

![Figure 3.3 - Geometrical parameters for coupled heave-roll motion](image)

To determine the type of heave-roll coupling that one might expect, we make here a careful analysis based on the assumption that the still water provides at all times just its simple hydrostatic buoyancy forces. The inertia of the water is therefore neglected, its surface and pressure field presumed to be undisturbed\(^7\), and there will be no resultant force in the horizontal sway direction. Damping forces are also neglected at this point. Under this assumption, the ship and water constitute a conservative mechanical system (see for example Thompson \textit{et al} (1992)), and we start by examining the form of the total potential energy function:

\(^7\text{This is a simplifying assumption known as the Froude-Krylov hypothesis, and is widely used in Naval Architecture, see Newman (1977).}\)
Chapter Three: Mathematical Modelling

$$V = V(v, \theta) \quad 3.9$$

By symmetry it is clear that $V$ will be an even function of $\theta$, satisfying the condition $V(v, \theta) = V(v, -\theta)$. The restoring force in heave, $H$, is equal to the partial derivative of $V$ with respect to $v$ which we write as $V_v$ giving:

$$H(v, \theta) = V_v(v, \theta) \quad 3.10$$

and the restoring moment in roll, $M$, is similarly:

$$M(v, \theta) = V_\theta(v, \theta) \quad 3.11$$

Now the conventional GZ curve gives the restoring moment as a function of $\theta$ under the condition of static balance in heave. This balance condition is:

$$H(v, \theta) = V_v(v, \theta) = 0 \quad 3.12$$

and we can in principle solve this equilibrium equation to obtain the slave relationship for the static variation of $v$ with $\theta$:

$$v = v_s(\theta) \quad 3.13$$

This satisfies the equilibrium condition $H = 0$ identically, so we have the defining identity:

$$V_v[v_s(\theta), \theta] = 0 \quad 3.14$$

Substituting the slave relationship back into $V$ gives us the potential energy under the condition of static balance, $\tilde{V}$, as:

$$\tilde{V}(\theta) = V[v_s(\theta), \theta] \quad 3.15$$
The derivative \( \ddot{V}_\theta \) is the conventional GZ restoring moment which we write as \( mgG_x(\theta) \), so we have:

\[
mgG_x(\theta) = \ddot{V}_\theta(\theta)
\]  

3.16

For our archetypal model, we take the simplest polynomial form for this moment, which for a symmetric ship is:

\[
mgG_x(\theta) = \ddot{V}_\theta(\theta) = c\theta \left[ 1 - (\theta/\theta_v) \right] \left[ 1 + (\theta/\theta_v) \right]
\]  

3.17

where \( c \) is the initial roll stiffness of the model and \( \theta_v \) is the angle of vanishing stability of the model corresponding to the unstable equilibrium state beyond which static capsize will occur. Although both \( c \) and \( \theta_v \) can be estimated for any real ship from its GZ curve, we should keep in mind that they represent parameters of the model, and therefore may differ from the actual values for the ship. Integrating this gives us:

\[
\ddot{V}(\theta) = \frac{1}{2} c \theta \left[ \left( \frac{\theta}{\theta_v} \right)^2 - \frac{1}{2} \left( \frac{\theta}{\theta_v} \right)^4 \right]
\]  

3.18

The problem now is how to devise a simple but realistic expression for the original energy function \( V(v, \theta) \) that will have (3.18) as its enslaved form. Ships are normally very linear in heave (see for example Faltinsen (1990)), so we can assume a constant value of heave stiffness, \( h \), valid for any departure from the enslaved state. This corresponds to assuming that away from \( v_s(\theta) \) the energy rises parabolically with \( (v - v_s) \), so we must add \( \frac{1}{2} h (v - v_s)^2 \) to the above \( \ddot{V} \) to give:

\[
V(v, \theta) = \ddot{V}(\theta) + \frac{1}{2} h (v - v_s(\theta))^2
\]  

3.19

To complete this exercise we assume a simple parabolic form for the symmetric function \( v_s(\theta) \) in terms of a single coefficient, \( \gamma \), as:
\begin{equation}
\nu_s(\theta) = \frac{1}{2} \gamma \theta^2
\end{equation} 

(3.20)

giving finally:

\begin{equation}
V(\nu, \theta) = \frac{1}{2} c \theta^2 \left[ (\theta/\theta_v)^2 - \frac{1}{2} (\theta/\theta_v)^4 \right] + \frac{1}{2} h \left( \nu - \frac{1}{2} \gamma \theta^2 \right)^2
\end{equation} 

(3.21)

The derivatives of (3.21) give us our restoring force and moment:

\begin{equation}
H(\nu, \theta) = V_\nu(\nu, \theta) = h \left( \nu - \frac{1}{2} \gamma \theta^2 \right)
\end{equation} 

(3.22)

\begin{equation}
M(\nu, \theta) = V_\theta(\nu, \theta) = c \theta \left[ 1 - (\theta/\theta_v) \right] \left[ 1 + (\theta/\theta_v) \right] - h \left( \nu - \frac{1}{2} \gamma \theta^2 \right) \gamma \theta
\end{equation} 

(3.23)

and we confirm that \( H = 0 \) gives \( \nu = \nu_s \), and substituting the latter gives the required \( M = mgG_r(\theta) \) and the required \( \tilde{V} \).

In terms of the total mass of the ship, \( m \), and its moment of inertia (in roll) about the centre of gravity, \( I \), the Lagrangian equations of motion are:

\begin{equation}
m \nu'' + h \left( \nu - \frac{1}{2} \gamma \theta^2 \right) = 0
\end{equation} 

(3.24)

\begin{equation}
I \theta'' + c \theta \left[ 1 - (\theta/\theta_v) \right] \left[ 1 + (\theta/\theta_v) \right] - h \left( \nu - \frac{1}{2} \gamma \theta^2 \right) \gamma \theta = -IAk \omega_f \sin(\omega_f \tau)
\end{equation} 

(3.25)

where a prime denotes differentiation with respect to the real time \( \tau \). Notice that we have added a forcing term to the right-hand side of the roll equation, which involves the moment of inertia \( I \), the wave-slope amplitude \( Ak \), and the wave encounter frequency \( \omega_f \). This expression was derived in Thompson et al (1992) for a ship in long beam waves: we retain the minus sign to maintain compatibility with this earlier study. The natural linear frequency in heave and roll are given by \( \omega^2_v = h/m \) and \( \omega^2_\theta = c/I \).
The enslavement coefficient $\gamma$ is not strictly a free parameter that we can adjust at will. When the ship is at its point of static capsize, namely at the saddle point, $C$, of the potential energy surface, $V(v, \theta)$, we have $\theta = 0$, $v = v_c = \frac{1}{2} \gamma \theta_c^2$ and $V_c = V(v_c, \theta_c) = \frac{1}{4} c \theta_c^2$. If we also evaluate $V_A = V(v, 0) = \frac{1}{2} h v_c^2$ we have:

$$\frac{V_A}{V_c} = \frac{1}{2} \frac{h}{c} (\gamma \theta_c)^2$$  \hspace{1cm} (3.26)

which gives us, in principle, one way of assigning a value to $\gamma$.

The potential energy surface of a ship at large angles of roll is never known with any precision: and different hull forms will give a wide range of surface characteristics. Some insight for the choice of a value for $\gamma$ may, however, be gained from the examination of a simple, geometric form such as a square cross-section ship. Detailed calculation of the exact potential energy function for this geometry reveals that $V_A$ and $V_c$ are of the same order of magnitude for a homogeneous ship (i.e. one with the centre of gravity at the geometric centre) within quite a wide range of average density values, see Appendix A. For these reasons we shall simply choose $\gamma$ to make $V_A = V_c$:

$$\gamma^2 = \frac{2c}{h \theta_c^2}$$  \hspace{1cm} (3.27)

This choice has the effect of eliminating the coefficient of $\theta^4$ in $V(v, \theta)$, but not of course the coefficient of $\theta^4$ in $\tilde{V}(\theta)$. We feel that the final equations with this $\gamma$ will represent an archetypal system, suitable for qualitative, phenomenological studies of heave-roll coupling. Detailed quantitative results, which would in any case vary dramatically from vessel to vessel, are quite beyond the scope of this type of basic, heuristic investigation.
3.7.2 EQUATIONS OF MOTION IN NON-DIMENSIONAL FORM

We now proceed to put these equations in a more compact form. We introduce the scaled variables:

\[ x = \frac{\theta}{\theta_v} \quad 3.28 \]
\[ y = \left( \frac{h}{2c\theta_v} \right)^{1/2} v \quad 3.29 \]
\[ t = \omega_\theta \tau \quad 3.30 \]

where the ratio of \( y \) to \( v \) has been chosen to make the coefficient of \( x^2 \) in the heave equation equal to unity. We further define the forcing magnitude and frequency ratios:

\[ F = \frac{A\omega^2}{\theta_v} \quad 3.31 \]
\[ \omega = \frac{\omega_\theta}{\omega} \quad 3.32 \]
\[ R = \frac{\omega_x}{\omega_\theta} \quad 3.33 \]

Substituting these and the adopted value of \( \gamma \) into the basic equations we have the non-dimensional potential energy function, \( U = \frac{V}{c\theta_v} \), given by:

\[ U(x,y) = \frac{1}{2} x^2 + y^2 - x^2 y \quad 3.34 \]

and the two coupled SIR equations of motion, suitable for studying symmetric internal resonance, are:

\[ \frac{2}{R^2} (\ddot{y} + 2\zeta R\dot{y}) + 2y = x^2 \quad 3.35 \]
\[ \ddot{x} + 2\zeta \dot{x} + x - 2xy - b = F \sin \alpha \]
where a dot denotes differentiation with respect to scaled time, \( t \). Here we have included \textit{ad hoc} linear viscous damping, with damping ratio \( \zeta \), in each equation of motion. We should remember that damping ratio is invariant under scaling, so that the damping ratio in the original non-scaled equations will also be \( \zeta \). We have also added an arbitrary bias, \( b \), that will allow us to examine the effect of breaking the symmetry in roll.

The condition of enslavement, valid for large values of the frequency ratio, \( R \), can be obtained by setting \( \ddot{y} = \dot{y} = 0 \) in the heave equation as:

\[
y_s = \frac{1}{2} x^2
\]

and substituting into the roll equation gives the conventional 1-DOF approximation (see section 4.3.1):

\[
\ddot{x} + 2\zeta \dot{x} + x - x^3 - b = F \sin \omega t
\]

Here, ignoring the added \( b \), the scaled GZ restoring moment is \( x - x^3 \) which is derivable from the non-dimensional potential function given by:

\[
\bar{U}(x) = \frac{\bar{V}}{c \bar{\theta}} = \frac{1}{2} x^2 - \frac{1}{4} x^4
\]

### 3.7.3 Underlying Potential System

As we have previously mentioned, the coupled heave-roll ship dynamics can also be seen as the motion of a mass point within a well, and it is interesting to rederive the SIR equations for such a system. Different well geometries correspond to different shapes of the ship's restoring moment curve (the so-called GZ curve), as given by different hull shapes and/or different loading
conditions. We therefore define suitably scaled variables \( x \) (roll) and \( y \) (heave), and we consider the following potential energy function.

\[
v = \frac{x^2}{2} + y^2 - x^3 y
\]  

3.39

Contours of \( v=k=\text{constant} \) are given by \( y = \pm \left[ k - x^2 / 2 + x^4 / 4 \right]^{1/2} + x^2 / 2 \), and a contour picture of the \( v=\text{constant} \) curves is shown in figure 3.4.

![Figure 3.4 - Potential well for heave-roll model](image)

To inspect the total potential energy function we write it in the form:

\[
v = \left[ y - \frac{x^2}{2} \right]^2 + \frac{x^2}{2} - \frac{x^4}{4}
\]  

3.40

and the corresponding partial derivatives as:

\[
\frac{\partial v}{\partial x} = -2x \left[ y - \frac{x^2}{2} \right] + x - x^3
\]  

3.41
the term in brackets vanishing when \( y = x^2 / 2 \). The significance of \( y_s \) will be made clear shortly. Saddles of the potential energy function are: \( x = \pm 1, \ y = 1/2, \ v = 1/4 \). Along the valley floor given by \( y_s(x) \) we see that \( v \) is just the integral of a one-degree-of-freedom curve. It rises away from the floor according to \( [y-y_s]^2 \), see figure 3.4. Equations of motion within the potential well defined by (3.39) are obtained by taking partial derivatives with respect to \( x \) and \( y \):

\[
\begin{align*}
\dot{y} + y &= 2xy \\
\frac{2}{R^2} \ddot{y} + 2y &= x^2
\end{align*}
\]

Here a dot denotes differentiation with respect to the scaled time \( t \). \( R \) is the ratio of the heave and roll natural frequencies. A large value for \( R \), which is typical of many ships, would induce motions following very closely the quasi-static heave values. Assuming static balance in the \( y \) direction by setting \( \frac{\partial V}{\partial y} = 0 \) gives us the slave relationship:

\[
y_s = \frac{x^2}{2}
\]

and substituting (3.45) into the \( x \) equation gives us the one-degree-of-freedom approximation, valid for large \( R \):

\[
\dot{x} + x - x^3 = 0
\]

To see heuristically the possibility of internal resonance we set \( R = 2 \). Assuming \( x = \sin t \), the right-hand side of the \( y \) equation becomes \((1 - \cos 2t) / 2\). A large oscillation of \( y \) at circular frequency 2 will thus be generated to feed back into the \( x \) equation. This multiplies \( x \), making it a parametric excitation: and being at
twice the $x$ natural frequency it induces the large principal Mathieu instability, generating a large response at the natural frequency of $x$. These conditions for internal resonance can be summarised as follows: $x$ stimulates a direct fundamental resonance of $y$, $y$ stimulates a principal parametric resonance of $x$.

Taking equations (3.43) and (3.44), and adding equal damping ratios, $\zeta$, a symmetry-breaking bias, $b$ and a direct sinusoidal external forcing on $x$, we reach the same equations (3.35), reproduced here:

$$\ddot{x} + 2\zeta \dot{x} + x - 2xy - b = F \sin \omega t$$

$$\frac{2}{R^2} (\dot{y} + 2\zeta \dot{y}) + 2y = x^2$$

3.47
4. THE GEOMETRICAL APPROACH TO NONLINEAR DYNAMICS

In this chapter we describe the geometrical approach to ship dynamics that constitutes the theoretical background against which the remainder of this work is developed. We start with a very brief review of the historical development of this field, and move on to a description of some basic elements of theoretical nonlinear dynamics.

Throughout our short 'theoretical tour' we try not to lose sight of the main problem that motivates this work: we are trying to understand the dynamical mechanisms associated with large-amplitude ship roll motions and capsize. We try therefore to produce a concise exposition in which only those topics directly related to our results are presented.

Our intent here will be that of introducing concepts and theoretical facts that give support to the results we shall describe later in this work. To achieve this goal we take, however, a slightly unconventional route: we attempt to outline the geometrical approach to nonlinear dynamics using the motions of ships at sea as a thread.

4.1 BRIEF HISTORICAL BACKGROUND

As with many other developments in theoretical dynamics, the origins of this approach are usually traced back to the field of Celestial Mechanics. The French mathematician, physicist, and philosopher Henry Poincaré is widely acknowledged as the first to propose the idea of describing the dynamical behaviour of systems by their geometrical properties, Poincaré (1899). He was partially motivated by insurmountable difficulties he encountered when applying perturbation techniques to approximate solutions to nonlinear equations of motion for celestial bodies. He was aware of the possibility of homoclinic tangling, and he reported particular bafflement before the seemingly infinite
complexity of some systems' behaviour. He is regarded as having had then a glimpse of chaotic dynamics.

Many researchers advanced the viewpoint suggested by Poincaré during the decades that followed, notably G.D. Birkhoff, G. Duffing, and B. Van der Pol, see for instance Birkhoff (1927), Duffing (1918), and Van der Pol (1927). Both Duffing and Van der Pol invested great efforts investigating what would become one of the main focal points of more recent research: low-dimensional, dissipative dynamics. In fact, many simplified one-degree-of-freedom mathematical models used nowadays in ship dynamics have been previously studied by Duffing.

Similar approaches developed in the 1940's found application in the study of systems of ordinary differential equations, Levinson (1949). The same decade saw the early work of E. Hopf on bifurcations in dynamical systems, see for example Hopf (1942). In the year of 1963 two events took place that had profound influence on the subsequent development of the field. Stephen Smale published a seminal work in which he proposed a mathematical framework for some of the fundamental dynamical mechanisms responsible for complex behaviour, Smale (1963). Concurrently, the American meteorologist Edward Lorenz was publishing in the Journal of Atmospheric Science a paper in which he reported numerical experiments with a simplified 3-dimensional model of atmospheric convection that seem to indicate that, in certain ranges of parameters, the system's solutions possessed an enormous sensitivity to initial conditions. Also, such solutions seemed to be non-periodic. The chaotic attractor he was then investigating became one of the archetypal models for chaotic dynamics, Lorenz (1963).

Lorenz's work remained largely undiscovered until the early 1970's. Today his work can be regarded as a precursor of a new era in the study of nonlinear dynamics, an era in which numerical experiments with computers would become at least as important as pen-and-paper theory. The years that followed witnessed
a blossoming of the field, with similar complex behaviour being identified in countless different systems. In parallel with that some theoretical strides were taken, none of them indicating, however, that theory by itself is close to producing general results for nonlinear systems with phase spaces of dimension 3 or greater. It is perhaps worth noting here that the description of ship motions even in a (forced) one-degree-of-freedom model requires a phase space of dimension 3.

4.2 Preliminaries

The geometrical approach to dynamics can be regarded as stemming from the key observation that, even if the details of dynamical behaviour of complex systems are too difficult (or even impossible) to grasp, valuable insight can be obtained if we focus our attention on some carefully constructed geometrical features of the dynamics. Central to this approach is the concept of phase space. Broadly speaking, a phase space is a topological space formed by taking each independent dynamical variable as a dimension. For our purposes, the regime of phase space relevant to ship motions will often be $\mathbb{R}^n \times \mathbb{R}$, where $n$ is the order of the associated system of first-order differential equations defining the dynamics, and the independent time-variable, say $t$, takes values in $\mathbb{R}$. Alternatively, we can work with an equivalent discrete formulation, or a map as we shall refer to it.

By adopting the idea of a phase space we shift our attention from individual time-histories, and try to visualise the dynamics as an ensemble of trajectories in phase space. We say that a mathematical model of ship motions given by a set of first-order differential equations:

$$\dot{x} = f(x, t)$$ 4.1
defines a vector field $f(x,t)$, where $f$ is supposed continuous with continuous derivatives (i.e. $f \in C^1$) at every point of the phase space. The collection of all trajectories (also referred to as orbits) of a system defines its associated flow. The main advantage of this outlook is that, as we said before, it is often possible to establish geometrical or topological properties of flows (and maps) without having to solve the equations of motion. We give in the next section an example of the type of qualitative information this approach yields. From a strictly engineering point of view, where quantitative information is always required at some point or another, the merit of a qualitative understanding of the dynamics of a given system is that such knowledge acts both as a conceptual framework within which the practitioner can reason about the problem, and as a guide as to how and where to concentrate quantitative study.

4.3 Long-term dynamical behaviour: steady-state concepts

Most of the theory of nonlinear dynamics concentrates on a rather idealised situation: we suppose that the system has been operating for an infinite time, so that all transient effects have died out. Obviously, no system operates for infinite time; moreover, a system's environment (whose influence is many times translated into the model as system's parameters) never remains stationary for long enough to allow transients to die away completely. Nevertheless, since taking transient effects aboard can complicate matters substantially (as we shall see later in this work), the approximation of a steady-state is widely used and, as we have said, permeates most of the theory. The main reason for that, apart from simplicity of treatment, is probably that steady-state concepts can act as an organising structure within which the dynamics can be understood. Let us illustrate that with a simple example in which some of the basic concepts can also be introduced.

We imagine the phase space of an unforced, damped 1-DOF model of roll motion:
We assume that \( g(x) \) and \( f(x) \) are \( C^1 \) real functions of the variables \( x \) and \( \dot{x} \), with \( g(\dot{x}) \) always opposed to the motion. The phase space here will be a 2-dimensional real space with roll displacement \( x_1 \) and roll velocity \( x_2 \) as coordinates. We try to visualise the dynamics as the collection of all possible trajectories in this space. We then see that, if we include transient trajectories in our analysis, a rather complicate picture emerges. The steady-state dynamics of this system is, however, extremely simple. In fact, all possible steady-state trajectories consist of the fixed points of the system (solutions of \( (\dot{x}_1, \dot{x}_2) = 0 \)) as defined exclusively by the zeroes of \( f(x) \). These can be classified into stable (local minima of \( V(x) \)) and unstable points (local maxima and local inflection points). Moreover, in physical systems only the stable solutions would be observed, which reduces the possibilities to stable fixed points: local minima of \( V(x) \). We see that, although the details of each possible trajectory of (4.3) may be difficult to obtain, the final fate of all of them is relatively simple to ascertain.

In our example the unforced ship would always be observed (after transients have significantly died out) at rest in one of its stable positions. The importance of the apparently trivial assertion that this theoretical conclusion is confirmed by direct observation should not, however, be underestimated.

The example just described of an unforced system, although useful to convey the idea of a simple phase space and some of its main organising features, gives little hope of improving our understanding of ship capsize in waves. For that we need to consider forced systems. Although many of the results we shall state here are

\[
\ddot{x} + g(\dot{x}) + f(x) = 0
\]

or, equivalently:

\[
\begin{align*}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{x}_2 &= \ddot{x} = -g(x_2) - f(x_1)
\end{align*}
\]
valid for quite general forcing functions\(^2\), our main concern throughout this work will be with *periodically forced* systems. More specifically, we consider systems defined by the following type of initial-value problem:

\[
\begin{align*}
\dot{x} + g(x) + f(x) &= F(t) \\
x(0) &= x_0
\end{align*}
\]

where \(x(t)\) is a vector of phase coordinates (typically non-dimensionalised positions and velocities), \(g(x)\) is called the damping function, \(f(x)\) is called the restoring (moment) function, and \(F(t)\) is the excitation or forcing function, periodic in the time variable \(t\) with period \(T\): \(F(t) = F(t + T)\).

Systems of the type shown in (4.4) define \((2n+1)\)-dimensional phase spaces, or in other words they are \((2n+1)\)-dimensional flows. We can effectively reduce the dimension of the system by 1 if we consider its associated *first return map*, also referred to as *Poincaré map*. Formally, given a system of the type defined by (4.4), we define its flow \(\phi_t(x_0) = \phi(t, x_0)\) as the collection of solutions of (4.4), where we allow \(x_0\) to take values in an subset \(S\) of \(\mathbb{R}^n\). The complete space in which \(\phi_t\) exists is clearly of dimension \((2n+1)\). In such space we take an \(2n\)-dimensional hyperplane \(\Sigma\) such that \(x_0 \in \Sigma\) and \(\Sigma\) is not tangent (i.e., is transversal) to \(\phi_t(x_0)\) at \(x_0\). Now, for any point \(x\) sufficiently close to \(x_0\) a periodic trajectory of (4.4) through \(x\) will cross \(\Sigma\) at a point \(P(x)\) near \(x_0\), see Perko (1991). The mapping \(x \mapsto P(x)\) is called the *first return map* or *Poincaré map* of \(\phi_t\).

The definition of \(\Sigma\) for a specific system is not a straightforward matter. Basically, this is due to the fact that the local direction of \(\phi_t\) is not immediately accessible; therefore it may be difficult to verify the transversality condition. Periodic non-autonomous systems like (4.4) lend themselves, however, to a simple solution to this problem, since one of the variables, namely time, is restricted to a one-dimensional circle. A hyperplane \(\Sigma\) can then be easily defined to be *globally* transversal to the flow. For that we take the period \(T\) and define

\(^2\)Particularly the numerical methods we shall introduce in the following chapter.
P: x(t) → x(t + T). Because the map is effectively sampling the system at regular intervals, this type of Poincaré map is also called a stroboscopic map.

One of the main advantages of using Poincaré maps instead of their associated flows is that, by the definition of a Poincaré map, periodic trajectories of the flow correspond to fixed points of the associated Poincaré map. Moreover, local stability properties of trajectories are preserved under the discretisation introduced by a Poincaré mapping, see for example Guckenheimer and Holmes (1990). The fact that Poincaré maps of nonlinear systems can rarely be defined in closed form does not really constitute a problem, since most theoretical results do not rely on such formulation, and because numerical computation of those maps can be easily implemented for most systems.

In the next sections we introduce some basic ideas from stability analysis. Local stability analysis deals with the problem of establishing conditions for the stability of singular points, and of classifying the possible types of behaviour in the vicinity of singular points. This topic is well documented elsewhere for both nonlinear flows and maps, see for example Thompson and Stewart (1986), or Guckenheimer and Holmes (1990). We select however in the next sections a few basic concepts and facts that we feel are necessary to the studies that will be described later in this work. Those concepts and facts are introduced, or rather exemplified, in the context of a simple model of roll motion. In doing so we progress from simpler to more complicated cases of the same model. We also use this example to present some important features of global dynamics, such as invariant manifolds.

4.3.1 A SIMPLE MODEL OF SHIP ROLLING

We continue our description of the geometrical approach to dynamics using a mathematical model of ship motion as an example. We derive this model
using a technique well known to Naval Architects, combined with a procedure that produces a convenient, non-dimensional equation.

Realistic restoring moment functions of ships in calm water, or equivalently their GZ-Curves, have the shape illustrated in figure 4.1.

Figure 4.1 - Typical restoring moment function for a vessel

A simple analytical function that reproduces the main features of the curve within the region depicted in figure 4.1 can be given by:

\[
f(\theta) = a_1 \theta - a_2 \theta^3
\]

where coefficients \(a_1\) and \(a_2\), \(a_1, a_2 > 0\) can be selected to approximate the curve. The damping effect of water and air surrounding the vessel can be approximated by a linear function of the roll velocity \(\theta'\), where a prime denotes differentiation with respect to real time \(t\):
and the excitation by regular beam waves can be modelled by a simple harmonic function of time:

\[ F(t) = A \sin(\Omega t + \Phi) \]

Under these assumptions, the equations of motion for a vessel of inertia \( I \) are:

\[ I \theta'' = -f(\theta) - g(\theta') + F(t) \]

or, substituting (4.5) to (4.7) into (4.8):

\[ I \theta'' + b \theta' + a_1 \theta - a_2 \theta^3 = A \sin(\Omega t + \Phi) \]

We make \( \Phi = 0 \), and divide throughout by \( I \). We also replace the original variables by suitably scaled variables to obtain a non-dimensional equation. For that we define scaled time \( t = \frac{\tau}{T} = \frac{T}{2\pi} \) and a new forcing frequency \( \omega = \frac{\Omega}{\Omega_n} \), such that the new linearised natural frequency is unity. We also define a scaled roll angle \( x = \frac{\theta x}{\theta_v} \), where \( \theta = \pm \theta_v \) are the angles at which the restoring moment drops to zero (the angles of vanishing stability). In terms of \( x \) the new angles of vanishing stability are therefore \( \pm 1 \).

Substituting the scaled variables into (4.9), and defining new damping and forcing parameters \( \beta \) and \( F \), respectively:

\[ \beta = \frac{b}{I \Omega_n} \quad F = \frac{A}{I \theta_v \Omega_n^2} \]

equation (4.9) can be rewritten as:
$$\ddot{x} + \beta \dot{x} + x - x^3 = F \sin(\omega t) \quad 4.11$$

4.3.2 UNDERLYING POTENTIAL SYSTEM

Equation (4.11) is a Duffing-type equation, on which much work has been published, see for example Nayfeh and Sanchez (1989) and references therein. From a theoretical point of view, system (4.11) is however slightly more delicate than its more usual double-well counterpart because it possesses unbounded trajectories. As we shall see later, this is not a serious problem as far as our studies are concerned.

To start understanding the steady-state behaviour of the symmetric Duffing model of equation (4.11) it is useful to consider first the underlying potential system obtained by taking the undamped, unforced case: $\beta = F = 0$:

$$\ddot{x} + x - x^3 = 0 \quad 4.12$$

Several useful concepts can be defined in this simpler scenario. We rewrite (4.12) as a system of first-order differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{f} = (f_1, f_2) = (x_2, -x_1 + x_1^3)$, giving:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3 \quad 4.13$$

Fixed points or singular points of (4.13) are the solutions of $\mathbf{f}(\mathbf{x}) = (\dot{x}_1, \dot{x}_2) = (0,0)$. For (4.13) there are therefore three singular points: $(x_1, x_2) = (0,0), (-1,0),$ and $(+1,0)$. Local stability analysis starts with the calculation of the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ at each of the singular points $\mathbf{x} = \mathbf{x}_*$.
Eigenvalues of $Df(x)$ are the characteristic exponents of a local expansion of the flow. If none of the eigenvalues of $Df(x)$ have zero real part at a singular point $x = x_*$ then $x_*$ is called a hyperbolic fixed point of the system. Near hyperbolic fixed points, the nonlinear system (4.13) is topologically equivalent to the linear system $\dot{x} = Ax$, where $A = Df(x)$. In particular, the stability of a hyperbolic fixed point $x = x_*$ is completely determined by the eigenvalues of $A$ calculated at $x_*$. The eigenvalues of this symmetric Duffing model are: (a) for the origin $(x_1, x_2) = (0, 0)$, $\lambda_{1,2} = \pm i$; (b) for $(x_1, x_2) = (\pm 1, 0)$, $\lambda_{1,2} = \pm \sqrt{2}$. We see therefore that the origin is a non-hyperbolic singular point of this system.

If this flow of a conservative system has no zero eigenvalues at a singular point $x_*$ then $x_*$ is called a non-degenerate singular point of the flow. Non-degenerate singular points of planar systems are either (topological) saddles or centres (see Perko (1991)). Centres are non-hyperbolic singular points at which eigenvalues of the flow are purely imaginary, like the origin in our example. Every trajectory sufficiently near to a centre is closed (periodic).

Looking at figure 4.2 we immediately recognise the three fixed points of this (potential) system: the origin $(x, \dot{x}) = (0, 0)$, and two hill-tops at $(x, \dot{x}) = (\pm 1, 0)$.

---

3Hyperbolic fixed points can be broadly classified into: (a) Sinks if all eigenvalues of $A$ have negative real parts; (b) Sources if all eigenvalues of $A$ have positive real parts; (c) Saddles if $A$ has at least two eigenvalues with real parts of opposite signs.

4This is the Hartman-Grobman Theorem: There exist neighbourhoods $S$ and $U$ of $x_*$ such that the flow of (4.13) in $S$ is homeomorphic to the flow of $\dot{x} = Ax$ in $U$, see Guckenheimer and Holmes (1990).
Figure 4.2 - (a) Potential energy function for the undamped, unforced symmetric Duffing equation, (b) Outline of trajectories in phase space
Chapter Four: Geometrical Approach to Nonlinear Dynamics

The origin is a centre, a stable fixed point in the (Lyapunov) sense that trajectories starting close to the origin remain close for all time. The interior of the well will be (densely) filled with periodic orbits of constant energy, see figure 4.2(b). Motions starting with zero velocity "outside" the well, $|x(0)|>1$, $\dot{x}(0)=0$ are also of constant energy but never visit the interior of the well: their potential energy tends to $-\infty (x \to \pm \infty)$ while their kinetic energy tends to $+\infty (\dot{x} \to \pm \infty)$.

Finally, we identify two special trajectories that are neither periodic nor wander to infinity. They are produced by motions that have a total energy equal to that of the hill-tops but have non-zero velocity. These two trajectories connect both hill-tops (obviously through different paths in phase space) taking, however, infinite time to do so. They are called heteroclinic trajectories, and the union of the two hill-tops and the two heteroclinic trajectories connecting them is called a homoclinic cycle.

The distinction between qualitative and quantitative information about the dynamics of a system may not always be clear-cut. It might therefore be interesting to use the definitions just presented to exemplify that distinction. The geometry (or topology) of the outline of trajectories shown in figure 4.2(b) gives us a wealth of information about the potential system underlying this symmetric Duffing model, equation (4.11): we can, for example, 'see' very clearly which trajectories remain close to the origin (or inside the well) for all time. In addition to that we can say that, typically, those trajectories are periodic. This, it could be said, is a qualitative information. However, if we were interested in the period of trajectories we would have to look further than just the sketch of figure 4.2(b). The period of trajectories is a common example of quantitative information. Although it may not be necessary to solve the equations of motion to obtain such information, it is certainly not directly available through inspection of figure 4.2(b).

A different example from the same system can, on the other hand, illustrate how precise, quantitative information can be drawn from simple geometrical analyses.
The outline shown in figure 4.2(b) indicates very clearly that, since the system is conservative\(^5\) (and therefore every trajectory is a constant energy curve in phase space), the maximum energy of a periodic trajectory is that of the hill-top fixed point. The (potential) energy associated with the saddle-type fixed points is a well-defined energy threshold beyond which no periodic trajectory of this system can exist.

4.3.3 DAMPED, UNFORCED SYSTEM

We progress in the analysis of our example by considering the inclusion of damping, here represented by a simple linear function of roll velocity. Our system then becomes:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 - x_1 + x_1^3
\end{align*}
\]

The concepts introduced in the previous section in the context of a conservative system are valid for more general cases. This is indeed highly desirable since the representation of a rolling ship by an undamped Newtonian model is qualitatively inaccurate. As we said before an unforced ship will always be found, after transients have died away, in one of its stable static positions. It is then of interest to understand how the addition of damping changes qualitatively the picture described in the previous section.

Fixed points of the system have not changed with the inclusion of damping. We note however that now the exact values of eigenvalues will obviously depend on the damping coefficient \(\beta\). They are now given by:

\[
\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - 4}
\]

\(^5\)See section 6.2.1 for a definition of conservative systems.
for the origin \((x, \dot{x}) = (0, 0)\), and by:

\[ \lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 + 8}}{2} \]

for the hill-tops at \((x, \dot{x}) = (\pm 1, 0)\).

Since \(\beta > 0\), the hill-tops are still saddle-type points with real eigenvalues of opposite signs and different magnitudes. The character of the origin depends on the exact value of \(\beta\). This situation mirrors the well-known classification of linear, damped systems into underdamped, overdamped, and critically damped cases. In our system the threshold between these cases is given by \(\beta = 2\), reflecting a (linearised) damping factor of 1, as we would expect. We are interested in small \(\beta\), characteristic of ship rolling, and at any rate smaller than 2. The origin will therefore be a stable focus with complex conjugate eigenvalues of negative real part.

As far as trajectories are concerned, the net effect of positive damping can be thought of as that of adding an inward (radial) component to every non-trivial trajectory of the flow. Thus, periodic orbits are transformed into spirals that converge to the origin. Such changes illustrate the important concept of structural stability, for which we now give a simple working definition. A vector field \( f(x) \in \mathbb{C}^1 \) is \textit{structurally stable} if it is topologically equivalent to all vector fields \( g(x) \in \mathbb{C}^1 \) sufficiently close to it. In our case, this means that if the undamped model (4.13) was structurally stable then the introduction of small damping in the form of a \( f \beta x_2 \) term should \textit{not} change the picture qualitatively. Since closed curves and spirals are qualitatively different trajectories, the picture is qualitatively changed, and the undamped system (4.13) is not structurally stable.

Obviously, not all trajectories of the damped, symmetric Duffing model of equation (4.14) converge to the origin. There are trajectories that escape from the attraction of the origin and become unbounded. The rôle of separatrices
between bounded and unbounded motions was played in the conservative case by
the heteroclinic trajectories. These trajectories are examples of an important
structure called an *invariant manifold*, which we shall define shortly. In
particular, some of the invariant manifolds of the hill-top saddles are frequently
the separatrices between regions of the phase space attracted to different
attractors. That is precisely what happens in this symmetric Duffing-type system.
We illustrate that in figure 4.3, which is a numerically computed portrait of
trajectories with the damping coefficient $\beta$ set to 0.1. Of special interest in figure
4.3 are the trajectories that converge to the unstable saddle-like fixed points.
They are numerical approximations of the stable manifolds of those points, and
form the separatrices between motions that converge to the stable origin and
motions that escape from it and become unbounded.

![Figure 4.3 - Phase portrait for the damped, symmetric system (4.14)](image-url)
4.3.4 Periodically forced system

We turn now our attention to the damped, forced symmetric system (4.11) in its complete form. The dynamics of autonomous planar flows is relatively well understood. Possible types of long-term behaviour in these systems are restricted to a few simple cases (the Poincaré-Bendixson Theorem, see Hale and Koçak (1991)). That is not, however, the case of flows in more than two dimensions. We would like for instance to know which types of long-term behaviour can be expected in the simple example of roll motion given by equation (4.11). Alternatively, we could ask how the picture in figure 4.2(b) changes if we introduce damping and periodic forcing? These questions do not have at present complete answers for arbitrary magnitudes of forcing. Some important features of systems like (4.11) are however known. To comment on the most relevant ones to our study we have to introduce a few more notions and facts.

An invariant set for this system is a subset \( S \) of the phase space of the flow such that trajectories containing a point of \( S \) remain in \( S \) for all time. Fixed points and periodic orbits are obvious examples of invariant sets. Other significant examples of invariant sets are the invariant manifolds of fixed points. Given a neighbourhood \( U \) of a fixed point \( x_* \), the local stable manifold of \( x_* \) is the set of points \( x \) of \( U \) such that: (a) the trajectory starting at \( x \) remains in \( U \) for all \( t \geq 0 \), and (b) the trajectory starting at \( x \) approaches \( x_* \) as \( t \to +\infty \). Analogously, the local unstable manifold of \( x_* \) is defined as the set of points \( x \) of \( U \) such that: (a) the trajectory starting at \( x \) remains in \( U \) for all \( t \leq 0 \), and (b) the trajectory starting at \( x \) approaches \( x_* \) as \( t \to -\infty \). Global stable and unstable manifolds of a fixed point are defined by taking unions of backward (for the stable manifold) and forward (for the unstable manifold) iterates of the local manifolds, see for example Guckenheimer and Holmes (1990) for formal definitions of these concepts.
An important theorem of Nonlinear Dynamics establishes the correspondence between invariant manifolds of hyperbolic fixed points and their linearised counterparts: the Stable Manifold Theorem, see Guckenheimer and Holmes (1990). We have already seen that invariant sets play a significant part in the organisation of the phase space. We shall see more examples of that in the next sections.

One of the central ideas in the dynamics of dissipative systems is that of an attractor and its basin of attraction. The \( \omega \)-limit set of a point \( x \) in the phase space \( S \) of a flow \( \phi(x,t) \) is the set of points \( y \) of \( S \) such that \( \phi(x,t) \rightarrow y \) as \( t \rightarrow +\infty \). Clearly, \( \omega \)-limit sets carry the asymptotic behaviour of the flow, and are therefore of primary importance in investigating steady-state dynamics. An attractor of a flow \( \phi(x,t) \) can be defined as an (indecomposable, closed) invariant set \( A \) such that for \( \infty \)-neighbourhood of \( A \) there exists a set \( B \) of points \( x \) in this neighbourhood for which \( A \) is their \( \omega \)-limit set, and \( \phi(x,t) \) is in \( B \) for \( t \geq 0 \). The basin of attraction or domain of attraction of the attractor \( A \) is the union of backward trajectories from \( B \). Every trajectory starting inside the basin of attraction of \( A \) will eventually get arbitrarily close to it. The basin of attraction of an attracting set is open and invariant under the flow.

We shall be frequently interested in the boundaries of basins of attraction. These are defined as the intersection between the closure\(^6\) of a basin of attraction and the closure of its complement. We see then that the basin boundary is also invariant under the flow. In fact, a basin boundary is formed by the union of one or more trajectories. When multiple disjoint attracting sets exist, their basins of attraction are non-intersecting sets (from uniqueness of solutions), and are separated by the stable manifolds of non-attracting (i.e. unstable) sets, see Guckenheimer and Holmes (1990) and Ott (1993). These sets are referred to as separatrices. We shall see in the following sections how the behaviour of those invariant sets is related to complicated dynamics and capsize.

---

\(^6\)The closure of a set \( B \) is the set of points \( x \) such that every open neighbourhood of \( x \) contains at least one point of \( B \).
A brief remark about multiple attractors should perhaps be made here. Their existence is made clear by the analyses we have been describing in this chapter. The practical implications of this simple fact may, however, be easily overlooked. Back in section 3.3 we made some comments about current procedures for validating sophisticated numerical codes for the prediction of ship motions. The potential flaw we then pointed out in those procedures can be understood as resulting from the disregard of this basic feature of nonlinear systems: typically, a set of trajectories can be attracted to more than one steady-state solution. More than that, it may be difficult, if not practically impossible, to ascertain in advance the fate of any particular trajectory or the number of possible solutions.

We note nevertheless that in a certain sense the existence of attractors often simplifies the steady-state dynamics of a system. Attractors are a prevalent feature of dissipative systems, which contrasts with their impossibility in conservative or Hamiltonian systems. They represent a large-scale contraction of the flow onto some sets, often of dimension much smaller than the phase space itself. In the example of system (4.3) we could rephrase our conclusion and say that a damped, unforced ship will always be observed in positions corresponding to local minima of the potential energy function because these are the only attractors for that system.

A periodically forced system like (4.11) obviously has no fixed points, since one of its phase variables (time) grows continually. The simplest possible invariant set of (4.11) is therefore a periodic orbit. In fact, the existence of periodic orbits near the origin of (4.11) for small periodic forcing is guaranteed by the Twist Theorem, see Guckenheimer and Holmes (1990). This theorem applies to the associated Poincaré map (which is area-preserving for (4.13)), and guarantees the persistence of fixed points of the map (periodic orbits of the flow) for small
(rigorously speaking, infinitesimal) damping and periodic forcing\footnote{This fact can also be demonstrated for flows using the Averaging Theorem, as we shall describe in Chapter 6, see also Guckenheimer and Holmes (1990).}. Note, however, that the possibility of other \( \omega \)-limit sets is not excluded: they may coexist with periodic orbits, and the basin boundaries separating domains of attraction of periodic and non-periodic solutions can be highly convoluted.

We can now turn our attention to the questions we asked in the beginning of this section, and perhaps rephrase them using the ideas just presented. System (4.13) displays small stable periodic orbits in the weakly damped and forced regime: a stable limit cycle. As we depart from small damping and forcing changes will take place in which the periodic attractor may be destroyed, and other attractors may be created. How do these changes occur? This is a natural next step in our analysis of the dynamics of nonlinear models of ship motions: to formalise the idea of \textit{qualitative change}. This leads us to a brief review of bifurcation theory.

\textbf{4.4 Elements of bifurcation theory}

Bifurcation theory is the study of qualitative (or topological) change in flows and maps. \textit{Local} bifurcation theory is concerned with changes that occur in small neighbourhoods of limit sets. Such changes can be usually characterised by the behaviour of the eigenvalues of the flow or map. To understand how some other bifurcations occur we have to consider the behaviour of the flow or map over larger subsets of phase space: the study of such bifurcations forms the focus of \textit{global} bifurcation theory. We shall see that both types of phenomena can play important rôles in ship capsize.

We have in a sense already defined bifurcations when we presented the idea of structural stability. The two concepts are connected by the idea of topological equivalence. In fact, given a system:
that depends on a vector of parameters $\mu \in \mathbb{R}^n$ we can define a \textit{bifurcation point} for the flow as a point $\mu = \mu^*$ such that 4.15 is structurally unstable.

In the example of our symmetric Duffing-type model of ship rolling, equation (4.11), if we write

\[
\mu = (\mu_1, \mu_2, \mu_3) = (\beta, F, \omega)
\]

we say that $p^* = (0, 0, p^*)$ is a \textit{bifurcation set} for (4.11), since for any $\beta \neq 0, F \neq 0$ the flow is qualitatively different from that at $\mu^* = (0, 0, \mu_3)$.

Local bifurcation theory is largely concerned with bifurcations of fixed points of flows and maps. Since fixed points of Poincaré maps represent periodic orbits of their associated flows, this covers also bifurcations of periodic orbits of flows. A classification of local bifurcations is outside the scope of this work, and is a topic rich with excellent references, see for example Chow and Hale (1982), Thompson and Stewart (1986), Hale and Koçak (1991), and Wiggins (1991). We shall however mention some basic bifurcations that have been shown in several studies to be directly related to ship capsize, see Thompson (1989). Also we shall be mainly concerned with codimension 1 bifurcations$^8$.

\section*{4.4.1 Bifurcations of periodic orbits}

As we have seen, a ship under small (not necessarily infinitesimal) forcing oscillates with the frequency of the forcing. It is interesting therefore to investigate the typical bifurcations that such a periodic orbit can undergo as we allow one parameter to vary smoothly. These bifurcations can be seen as a first

$^8$For our purposes here we can define codimension of a bifurcation as the number of parameters that have to be simultaneously controlled to produce the bifurcation generically. See Wiggins (1991) for a formal definition.
step in a ladder of phenomena of increasing complexity that will ultimately lead to capsize.

Bifurcations of periodic solutions of ship motion can be better studied through a Poincaré map. Given a system as the one shown in (4.15) we define a Poincaré map $\mathbf{P}(\mathbf{x})(t) \rightarrow \mathbf{x}(t + T)$ such that periodic solutions of the flow are represented by fixed points of $\mathbf{P}(\mathbf{x})$. Now given a fixed point $\mathbf{x}^*$ of $\mathbf{P}(\mathbf{x})$ we linearise the map around $\mathbf{x}^*$: $\mathbf{P}(\mathbf{x}_{i+1}) = \mathbf{J}_{\mathbf{x}^*}\mathbf{x}_i$. The position of the eigenvalues of the Jacobian matrix $\mathbf{J}$ in the complex plane are used to study the stability of the periodic orbit they represent. We assume $\mathbf{J}$ has distinct, non-zero eigenvalues. If all eigenvalues $\lambda_j$ are inside the unit circle $|\lambda_j| < 1$ then the periodic orbit is (asymptotically) stable.

Three basic ways in which the stability condition is violated give rise to three fundamental codimension 1 bifurcations, which we now proceed to describe briefly. We assume in our description that one control parameter is changed smoothly.

*The cyclic fold or saddle-node bifurcation*

Let $\mu$ be a control parameter (for instance, the magnitude or frequency of forcing, or perhaps the damping level), and let us assume that (4.15) has a stable periodic solution for $0 < \mu < \mu^*$. If for $\mu = \mu^*$ one of the eigenvalues of the associated Poincaré map equals $+1$ we say that $\mu = \mu^*$ is a cyclic fold or saddle-node bifurcation point of (4.15). More precisely we can say that, in the presence of typical nonlinear terms, the periodic orbit goes through a cyclic-fold or saddle-node bifurcation at $\mu = \mu^*$.

For $\mu > \mu^*$ the periodic orbit no longer exists. Which stable solutions, if any, replace the periodic orbit is a difficult question. Several cases are possible, depending solely on the details of the system. There are at least three cases that deserve our attention. In the simplest scenario a stable periodic orbit of same period exists after the bifurcation, and the system may settle onto a periodic
behaviour, typically of different amplitude. If we decrease \( \mu > \mu^* \) the new stable periodic solution may undergo itself a saddle-node bifurcation at a different parameter value \( \mu^* < \mu^* \) after which it may jump back onto the original periodic orbit. Thus the two saddle-node bifurcations define a \textit{hysteresis loop}.

It may also happen that the system jumps straight onto an attracting non-periodic (quasiperiodic or chaotic) solution. These non-periodic attractors may be the only attracting solution after the bifurcation, or they may compete with other attractors. The basin boundaries of competing attractors can be highly intertwined, and which of the stable attracting solutions will be observed depends exclusively on initial conditions. In other words, the precise manner in which the control parameter is changed through the bifurcation value affects the outcome. Finally, there may be no (bounded) attracting solutions for \( \mu > \mu^* \): the system diverges to infinity. Details of these and other scenarios can be found for example in Thompson (1989), Thompson \textit{et al} (1990), and Thompson and McRobie (1993).

Let us examine briefly the significance of the phenomena described above to ship stability and capsize. First, it is useful to remember that we are dealing here with steady-state dynamics, i.e. with long-term behaviour after transients have all but decayed. Perhaps the most intuitive way of depicting the above situation is to imagine that a control parameter, say amplitude of forcing, is allowed to vary \textit{slowly}, so that steady-state solutions are closely followed\(^9\). We then see that under increasing forcing the stable periodic solution can lose its stability and experience a variety of post-bifurcation behaviours. The system can settle onto a different periodic motion. We should note, however, that even in this simple scenario the amplitude of periodic motion \textit{can be excessive}, and may correspond to practical capsize. The same remark is valid for more complex attracting solutions after a fold. The amplitude of chaotic motion may be acceptable or, more often than not, it may involve large excursions from the origin that in

\(^9\)Obviously, the term \textit{slowly} must here be understood in comparison with intrinsic system time-scales as given, for instance, by the period of the forcing. This is the idea behind the attractor-following algorithm that we shall describe in chapter 5.
practice mean capsize, or at least danger. In the last and most dramatic scenario it is clear that solutions becoming unbounded indicate capsize. Finally it should be noted that saddle-node bifurcations are not necessarily preceded by a significant increase in amplitude of motion: despite being parabolic, they can occur suddenly with little early signal.

*The flip bifurcation*

When the largest eigenvalues are real and distinct bifurcations can occur either with one eigenvalue going through +1 or -1. The former case corresponds to the saddle-node bifurcation we have just described. With appropriate nonlinearities, the latter corresponds to a *flip bifurcation*. Here the previously stable periodic orbit is replaced by a different periodic solution of twice its period: this is the familiar period-doubling scenario. The term *replaced* means here that, if nonlinearities are of a certain sign, a smooth bifurcation path exists in the space of control parameters linking stable fixed points before and after the bifurcation, in which case we call this bifurcation a *super-critical flip*. A flip bifurcation is often followed by other bifurcations forming *cascades of period-doubling* that are widely recognised as a typical route to chaotic dynamics. Pioneering work by M. Feigenbaum in the late 1970's has demonstrated some universal features of such period-doubling cascades, such as the convergence of control-parameter intervals between successive bifurcations to a single (system-independent) ratio, Feigenbaum (1980).

From the point of view of ship capsize, the flip bifurcation can perhaps be seen as a possible early signal of impending disaster. It should be noted, however, that the whole cascade of period-doubling bifurcations typically occurs within a small range of parameter values. This means that at the point of first period-doubling the system can be already close to capsize. Unfortunately, the practical relevance of this process is somewhat limited by the fact that long sequences of regular
waves would have to reach the vessel to make it possible to observe a period-doubled motion.

The Neimark bifurcation

The two previous bifurcations we described occur when the eigenvalues are real. We consider now a codimension 1 bifurcation associated with a pair of complex eigenvalues going through the unit circle. This is the Neimark bifurcation, which is analogous to (although more complex than) the Hopf bifurcation of periodic orbits of flows. Here the typical outcome is a quasiperiodic motion on a 2-torus, see for example Hale and Koçak (1991).

In terms of practical relevance to ship stability, similar comments to those we made about the flip bifurcation apply. Depending on the sign of nonlinear coefficients, the change from periodic to quasiperiodic motion can be smooth, in the sense of a continuous path in control space. Therefore, the relative "size" in phase space of the invariant torus in which orbits are now contained can grow gradually (although it may be fast) as the control parameter is increased above the bifurcation value. Having said that, one should always keep in mind that the bifurcations described above are all local: no information can be extracted from these bifurcation about the large-scale behaviour of the system.

4.4.2 Global bifurcations: tangling of invariant manifolds

Many global (and local-global) bifurcations, or events as we also refer to them, can occur in nonlinear flows and maps. We shall concentrate here on a single type of global events that have been shown to play a very significant part in determining the engineering (as opposed to mathematical) stability of

\[10\] Under special conditions upon the eigenvalues strong resonances can occur, in which 3-T- or 4-T-periodic attractors are created, see Thompson and Stewart (1986).
trajectories in models of ship motion, see Soliman and Thompson (1992) and McRobie (1992). These are global bifurcations resulting from **tangencies of invariant manifolds** of unstable fixed points of the map.

Let us take the escape equation (3.8) as an example, and let us assume small damping, and $\omega$ close to resonance. We start by recalling that for small forcing a small, stable periodic oscillation takes place around the origin. We note also that small unstable oscillations build up around the hill-top saddle. It is in fact the behaviour of the invariant manifolds of this saddle cycle that will concern us here. Initially, these invariant manifolds do not form any loops: this fact is guaranteed by the topological equivalence of the flow in the vicinity of the saddle for the unforced and lightly forced cases, see Hale and Koçak (1991). Under increasing forcing there will be a point at which the stable and unstable manifolds of the hill-top saddle intersect (touch) tangentially. At this point a homoclinic orbit appears connecting the saddle to itself, thus creating what is called a homoclinic loop (or homoclinic connection). The dynamics near the saddle point (of the corresponding Poincaré map) following the creation of a homoclinic loop is very complex, and is often regarded as giving rise to chaotic motion. Practically every textbook in modern Nonlinear Dynamics includes detailed descriptions of the complex patterns of dynamical behaviour that accompany homoclinic (and heteroclinic) tangling, see for instance Guckenheimer and Holmes (1990), Arrowsmith and Place (1990), and Hilborn (1994). In particular, it has been shown that the dynamics of a system (or of its Poincaré map) that displays a homoclinic tangling is at least as complex as that of a horseshoe map\(^{11}\). We shall however restrict our attention here to the significance of those global events to the stability of bounded motions.

First, we note that the global bifurcation above occurs without any apparent connection with the local bifurcations we described in the previous section. In fact, it may be argued that the homoclinic tangency itself has no immediate effect on the low-energy trajectories, namely the stable periodic motion inside the

\(^{11}\text{This is the Smale-Birkoff theorem, see for example Guckenheimer and Holmes (1990).}\)
potential well. Numerical evidence gathered for a variety of simple nonlinear maps seems to indicate that there can be a considerable gap (in control space) between the first homoclinic tangency and the saddle-node and flip bifurcations, see for example Thompson (1989) and McRobie (1992). There are however reasons to regard homoclinic tangencies as an early signal of loss of stability. To explain that we have to introduce one of the central ideas of this work, namely that of *basin erosion*. This is done in the next chapter, where we describe in more detail the different processes that can result in loss of integrity of safe motions.
5. DYNAMICS OF SHIP MOTIONS LEADING TO CAPSIZE

In this chapter we present a collection of the results from our studies on the dynamics of ship motions associated with and leading to capsize. We shall apply methods of geometrical dynamics to the 2-DOF model of ship rolling that we introduced in sections 3.7.1 and 3.7.2. One-degree-of-freedom models were extensively investigated by Michael Thompson and co-workers at the Centre for Nonlinear Dynamics at UCL, leading to the proposal of a new method to assess ship stability: the Transient Capsize Diagram, see section 2.4.

Here, the methods are implemented in a number of computer programmes we developed for this work. We start with a useful approximation to steady-state dynamics, and proceed to examine steady-state response through Poincaré sections. We then present bifurcation diagrams generated with attractor-following techniques. We conclude our study of steady-state dynamics with an investigation into basins of attraction for 1- and 2-DOF models of ship motions.

The last two sections in this chapter deal with transient dynamics. This is a topic that deserves particular attention in our work, for reasons that will become clear shortly. We offer some preliminary considerations on transient versus steady-state dynamics, and present results on safe basins and basin erosion processes linked to capsize. These results will be some of the main arguments in favour of the approach we suggest for ship stability analysis in the last chapter of this work.

5.1 PRELIMINARIES: NUMERICAL METHODS IN DYNAMICS

Analytical methods can and do play an important part in investigations of nonlinear ship dynamics. A variety of perturbation techniques has been employed successfully in 1-DOF models of ship rolling, see Szemplinska-Stupnicka (1992) and Nayfeh and Sanchez (1990) for recent accounts on that. Analytical methods
can provide much insight into the basic mechanisms in operation, and are therefore very useful to the researcher of nonlinear dynamics. The Melnikov method, for instance, has been used to estimate parameter values at which the first homoclinic tangency occurs in simple models, see Thompson (1989).

The main limitation of analytical methods as an engineering tool is the type of mathematical models they can work with. Detailed models of ship motion, particularly in more than one degree-of-freedom, would be far too complicated to study with these techniques. Fortunately, numerical methods offer the possibility of working with models of great complexity, which allows detailed investigation of ‘realistic’ models of ship motion. This is however done at a cost. First, numerically generated results allow little scope for general statements or conclusions. Also they do not usually constitute rigorous proof of interesting facts: more often than not numerical results are used to suggest or reinforce arguments. This is very much in line with the engineering approach to problem-solving, in which a combination of theory and practice is regarded by many as the best overall strategy.

The results we shall present in this chapter were developed as part of an engineering effort to provide some insight into ship capsize. Numerical methods are in this sense adequate tools for this work, and they have been extensively used in our work. Being numerical in their nature, these results carry the two main drawbacks of almost every numerical study: strictly speaking, they are particular, and they do not offer rigorous proofs. This makes the choice of mathematical models ever so important, and that is one of the main reasons why we have opted for a relatively simple, almost archetypal, model. We should, by doing this, be avoiding exploiting specific features of sophisticated models, that might not be present in other mathematical representations of ship motion. Obviously, the use of simple models also represents considerable economy of computation time, allowing more detailed studies to be performed. Perhaps more importantly, we can be reasonably confident that results we present for the basic model of our study will in some form be present in more realistic representations.
Equally relevant is the fact that all these methods can be applied, often with no modification, to arbitrary mathematical models of large-amplitude ship motion.

5.2 Overview of steady-state response

5.2.1 Quasi-steady-state response in time-series plots

We consider the coupled heave-roll dynamics of a ship under regular forcing. Under the assumptions of section 3.7.1 equations of motion can be given by:

\[ \ddot{x} + 2\omega_0 \dot{x} + x - 2xy - b = F \sin(\omega t) \]

\[ \frac{2}{R^2} (\ddot{y} + 2\omega R \dot{y}) + 2y = x^2 \]

We shall refer to this model as the SIR (Symmetric Internal Resonance) equations. This model contemplates two basic resonance mechanisms: direct roll resonance, and internal resonance. An internal resonance parameter \( R \), which is just the ratio between natural frequencies in heave and roll, can be adjusted to reveal more of one or the other mechanism. For large \( R \) response follows heave (vertical) equilibrium values closely, and the model's behaviour effectively resembles that of a 1-DOF system. For moderate \( R \) internal resonance can be a dominant feature. To see that we set \( R = 2 \) and assume roll response sinusoidal at roll natural frequency, say \( x = \sin(t) \). The right-hand side of the heave equation will then be proportional to \( \cos(2t) \), inducing large heave oscillations at twice the natural frequency in roll. This oscillation feeds back into the roll equation, and because it multiplies \( x \) it constitutes a parametric excitation. Being at twice the natural frequency in roll it induces the principal Mathieu instability, Jordan and Smith (1977). It is worth noting that typical values of \( R \) for some classes of sea-going ships are not far from 2.

---

1 This would in fact be the 1-DOF system we presented in chapter 4, see equation (4.11).
Chapter Five: Dynamics of Ship Motions

The analysis of steady-state response is a time-consuming task, mainly due to the necessity of dissipating transients. A first overview of steady-state response can however be obtained without resorting to long simulations if we allow parameters to vary slowly, so that steady-state is closely followed. We call this *quasi-steady-state* response. We are primarily interested in following system's behaviour as we increase the amplitude of forcing (corresponding to larger waves) from ambient conditions until escape (capsize) occurs. For system (5.1)-(5.2) we are particularly interested in evaluating the effect of the internal resonance parameter $R$ upon the overall response.

Figure 5.1 is a collection of plots of amplitude of roll motion $x$ against time $t$. The amplitude of sinusoidal forcing is varied according to $F(t) = 2 \times 10^{-3} t$, so that the horizontal ordinate can also be viewed as the amplitude of forcing $F$. Vertical and horizontal scales correspond to $-1.1 < x < 1.1$ and $0 < F < 0.5$. The frequency of forcing is $\omega = 0.85$, corresponding to near-optimal escape for the associated 1-DOF model.

Several features can be drawn from the time-series plots of figure 5.1. Escape under minimum amplitude of forcing occurs near the point of internal resonance $R = 2 \omega = 1.7$. We can also see that a *jump to roll resonance*, accompanied by typical over-shoot, takes place early in the time-series for all but the smallest values of $R$. For $R$ equal to 1.7 and 1.8 the system exhibits considerable ranges of irregular, possibly chaotic, response. Finally, we note that as $R$ is increased, bringing the system closer to a 1-DOF, single-well Duffing oscillator, symmetry-breaking bifurcations seem to occur at high values of $F$.

We shall see in the next section that symmetry-breaking phenomena are also present at lower values of $R$ corresponding to fully coupled motion. We shall look at some of these features in more detail in the next sections. We shall

---

2This is the frequency for escape under minimum amplitude of forcing, see Lansbury and Thompson (1990).
Chapter Five: Dynamics of Ship Motions

The constant bias parameter $b$ is set to 0, and damping level is given by $\zeta = 0.05$.  

<table>
<thead>
<tr>
<th>$R$</th>
<th>Figure 5.1 - SIR equations: escape under slowly increasing amplitude of forcing.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>$R=1.6$</td>
</tr>
<tr>
<td>1.7</td>
<td>$R=1.7$</td>
</tr>
<tr>
<td>1.8</td>
<td>$R=1.8$</td>
</tr>
<tr>
<td>1.9</td>
<td>$R=1.9$</td>
</tr>
<tr>
<td>2.0</td>
<td>$R=2.0$</td>
</tr>
<tr>
<td>2.1</td>
<td>$R=2.1$</td>
</tr>
<tr>
<td>2.2</td>
<td>$R=2.2$</td>
</tr>
<tr>
<td>2.3</td>
<td>$R=2.3$</td>
</tr>
<tr>
<td>2.4</td>
<td>$R=2.4$</td>
</tr>
<tr>
<td>2.5</td>
<td>$R=2.5$</td>
</tr>
<tr>
<td>2.6</td>
<td>$R=2.6$</td>
</tr>
<tr>
<td>2.7</td>
<td>$R=2.7$</td>
</tr>
<tr>
<td>2.8</td>
<td>$R=2.8$</td>
</tr>
<tr>
<td>2.9</td>
<td>$R=2.9$</td>
</tr>
<tr>
<td>3.0</td>
<td>$R=3.0$</td>
</tr>
<tr>
<td>3.1</td>
<td>$R=3.1$</td>
</tr>
<tr>
<td>3.2</td>
<td>$R=3.2$</td>
</tr>
<tr>
<td>3.3</td>
<td>$R=3.3$</td>
</tr>
</tbody>
</table>
include the effects of different frequencies of excitation when we discuss the transient behaviour of the system.

5.2.2 Steady-state response: Poincaré sections

The behaviour of periodic flows is best studied through Poincaré mapping. Usually, we would sample trajectories at regular intervals of one period of the forcing, $T$, and project the Poincaré points (intersections of trajectories with Poincaré sections) onto a single, global Poincaré section. Here, we have opted for a double sampling rate (half the period of the forcing, $T/2$), to allow for visual inspection of symmetry-breaking phenomena. To help visualisation we also draw on this global Poincaré section contours of constant potential energy corresponding to the energy of hill-top saddles: for the hill-top saddles these contours are given by $y = \pm \sqrt{\frac{2}{2}} \left( \frac{x^2}{2} - \frac{1}{2} \right) + \frac{x^2}{2}$. Since we are interested in investigating the possible effects of internal resonance, we shall be taking $R = 1.7$, $\omega = 0.85$ as parameters values for our numerical studies, and we use a damping parameter $\zeta = 0.05$. Numerical integration of equations of motion is performed here and elsewhere in this work using a fixed-step 4th-order Runge-Kutta procedure, and unless indicated otherwise we shall be using a step size of $T/80$.

Figure 5.2 shows a sample of steady-state trajectories for the SIR equations. We are interested here in observing broad features of response as we increase the amplitude of forcing from rest. This means that we start with the system in equilibrium at the origin and increase $F$ in small steps. After transients have considerably dissipated (we allow 300 cycles of the forcing for dissipation of transients in this example) we project a certain number of Poincaré points onto our Poincaré section. In this example we have plotted a further 500 steady-state points.
Figure 5.2 - SIR equations selection of attractors in the main sequence: Poincaré points
It is important to note that we are in fact using an *attractor-following technique* here, see section 5.3. For each new \( F \) we start simulations from a point within the attractor obtained for the previous \( F \)-value. This procedure is envisaged as a means of following the evolution of the initial attractor (the ambient equilibrium) as we wind up the amplitude of forcing, thereby inspecting what we call the *main sequence* of attractors for this system, see 5.3.2. The same procedure can be applied to follow the evolution of any attractor of the system.

We turn now to the main features of figure 5.2. As we could expect, the system settles onto a small-amplitude period-1 response for small \( F \) (\( F \leq 0.13 \)). For moderate forcing (\( 0.14 \leq F \leq 0.17 \)), this periodic attractor is replaced by a quasi-periodic attracting set of larger amplitude of motions. Further increase in \( F \) reveals locking onto a period-21 response, \( F=0.18 \), at which value response still seems to be symmetric. In the final stages of motion (\( F > 0.18 \)) before escape response looks non-periodic (chaotic), and a symmetry-breaking bifurcation occurs. We shall look at these bifurcations in more detail in the next sections.

Figure 5.3 contains a sample of trajectories for the SIR equations similar to that of figure 5.2, with the actual trajectories superimposed on the Poincaré points. To avoid overloading the picture with curves we have kept only the first 40 Poincaré points and their associated trajectories (after 300 cycles of the forcing for dissipation of transients). The periodic and quasi-periodic modes are clearly visible for \( F \leq 0.16 \), as is the symmetry of the motion for these values of forcing. At \( F=0.17 \) the system seems to be close to a saddle-node bifurcation in which locking onto periodic motion will occur. Obviously, the trajectories at \( F=0.18 \) (period-21 response) are not very helpful because of the large period of the motion. Trajectories within the (possibly) chaotic regime, \( F > 0.18 \), show clearly that symmetry has already been lost for these larger values of \( F \).
Figure 5.3 - SIR equations selection of attractors in the main sequence: trajectories
5.3 BIFURCATION DIAGRAMS

5.3.1 ATTRACTOR-FOLLOWING TECHNIQUE

Bifurcation diagrams are graphical representations of the system's behaviour as one or more parameters of the model are changed. Usually, the behaviour of the system is represented on the diagram by a single phase variable. For the SIR equations we choose the roll angle of its associated Poincaré map as the variable to be plotted in the bifurcation diagrams. Bifurcation diagrams can be generated (one might say, approximated) by a variety of numerical schemes. These include algorithms based on continuation methods, such as efficient path-following algorithms (see Foale (1993)), some of which have already made their way into commercially available packages (Doedel (1986), Parker and Chua (1989)). For our own work, we choose to apply a simple yet very robust method of attractor-following, which we now proceed to describe briefly.

Let $x_{\mu_i}$ be an asymptotically stable fixed point of a Poincaré map $P: x(t) \rightarrow x(t + T)$. The map $P$ is a first return map for the system $\dot{x}(t) = f(x, \mu, t)$, $f$ is $T$-periodic. The system depends on a parameter $\mu$, and $x_{\mu_i}$ is a fixed point for $\mu = \mu_i$. We are interested in following the evolution of $x_{\mu_i}$ as we vary the parameter $\mu$. In other words, we want to construct a series of attracting sets $x_{\xi}$ for a series $\xi = \{\mu_1, \mu_2, \ldots, \mu_s\}$ of parameter values. Obviously, the problem just formulated does not admit a unique solution in general. That is because nonlinear systems usually possess many attracting sets for each point in the space of control parameters. Which attracting set will eventually capture a particular trajectory depends exclusively on initial conditions.

We are in fact interested in one particular series of attracting sets, namely the series a physical system would follow from $x_{\mu_i}$ if the control parameter $\mu$ was allowed to vary smoothly and slowly. The attractor-following algorithm we use in this work is a numerical procedure to estimate that particular series of attracting sets. It consists of solving a sequence of initial-value problems. For
that we take $x_{\mu_1}$ as the initial condition of a numerical integration of \( \dot{x}(t) = f(x, \mu, t) \), with $\mu = \mu_2$, and we perform this simulation for $0 \leq t \leq n_1 T$, where $n_1$ is an arbitrary integer, say 200. By doing that we intend to allow transients to die away. We then assume $x(n_1 T) \in x_{\mu_1}$, where $x_{\mu_2}$ is an attracting set for $\dot{x}(t) = f(x, \mu_2, t)$. Since $f \in C^1$, it varies smoothly with $\mu$, which means that if there is not a bifurcation point $\mu^*$, $\mu_1 \leq \mu^* \leq \mu_2$, $x_{\mu_2}$ will be approximately given by $x_{\mu_2} = x_{\mu_1} + (\mu_2 - \mu_1) \frac{\partial P}{\partial \mu}_{\mu_1}$. If there is a bifurcation point between $\mu_1$ and $\mu_2$, $x_{\mu_2}$ will be the attracting set whose basin of attraction includes $x_{\mu_1}$. We now extend the numerical integration of the system for a further $n_2$ cycles (say $n_2 = 100$) to display the attracting set $x_{\mu_2}$. The number $n_2$ is related to the maximum order of periodicity we want to be able to see in the bifurcation diagram\(^3\). The algorithm proceeds with $x_{\mu_1}$ replaced by $x_{\mu_2}$.

With or without bifurcations, the attractor-following procedure generates the desired series of attracting sets, namely the steady-state responses that would be observed in a physical system under slowly changing parameters. In particular, the procedure is suitable for studies of hysteresis loops, as we shall demonstrate shortly. Obviously, only periodic attracting sets can be fully displayed by this or any other numerical algorithm. That is one of the reasons why it is interesting to visualise Poincaré sections for a system, in which quasi-periodic and chaotic responses can be seen more easily.

\(^3\)From a practical point of view, the choice of $n_2$ is also influenced by the graphical resolution of the bifurcation diagram.
5.3.2 BIFURCATIONS OF THE MAIN SEQUENCE FOR THE SIR MODEL

For the 2-DOF model of ship motions, we define the main sequence as the sequence \( x_\xi, \xi = \{\mu_1, \mu_2, \ldots, \mu_n\} \) such that \( x_{\mu_i} \) is the stable equilibrium of the system at rest. In other words, for the SIR model, if \( x_\xi \) is the main sequence then \( x_{\mu_i} \) is the origin: \( x_{\mu_i} = 0 \). In our numerical examples we choose the amplitude of forcing \( F \) as the control parameter \( \mu \), so that the main sequence starts at \( \mu_i = 0 \). From that point, we initially investigate the response of the system as the amplitude of forcing is increased until capsize occurs.

Figure 5.4 shows a complete bifurcation diagram for the SIR equations with parameters \( \mu = 0.85, \zeta = 0.05, R = 1.7 \). Two hundred cycles of the forcing were allowed for dissipation of transients. The control parameter here is the amplitude of forcing \( F \), displayed as the horizontal ordinate. The vertical ordinate is the amplitude of the roll variable \( x \); the vertical scale is \(-1.2 \leq x \leq 0.3\). The figure shows the attracting sets (stroboscopically sampled) for \( 0.1 \leq F \leq 0.2 \). For \( 0 \leq F < 0.1 \) the response is limited to a small-amplitude period-1 motion. Escape occurs shortly after \( F = 0.2 \).

First we notice the general agreement between the attracting sets showed in figure 5.4 and the sample of Poincaré sections of figures 5.2 and 5.3. As we mentioned elsewhere in this section, the two types of graphical representation complement each other. The Poincaré sections of figures 5.2 and 5.3 are suitable for visualising the type of attracting set at each value of \( F \): periodic, quasiperiodic, chaotic. Obviously, if we allow \( F \) to be decreased as well as increased, the different steady-state responses of a hysteresis loop can be observed. The bifurcation diagram is, however, a more direct and complete visualisation of the evolution of steady-state responses as a parameter is varied.

From inspection of figures 5.2 to 5.4, the evolution of the main sequence for this set of parameters can be summarised as follows. For small \( F \) the system exhibits
Figure 5.4 - Bifurcation diagrams for SIR equations
periodic response with the period of the forcing. At around $F = 0.1380$ the system goes through a supercritical Neimark bifurcation\(^4\), settling onto a resonant quasi-periodic motion. Further increase in $F$ will increase the amplitude of quasi-periodic motions until, at around $F = 0.1385$, the system goes through the first of a series of saddle-node bifurcations. For $F$ in the range between 0.14 and 0.18, the amplitude of motion varies relatively little, but the system goes through a complex cascade of saddle-node bifurcations (map explosions) in which periodic windows are interspersed among regions of quasi-periodic response. Just before $F = 0.18$ a relatively long period-21 response settles in at a further saddle-node bifurcation, its Poincaré section clearly visible in figures 5.2 and 5.3. At around $F = 0.185$ a saddle-node bifurcation (intermittency explosion) destroys the stability of the periodic attractor, resulting in stable motions of apparently chaotic nature. The chaotic behaviour persists until shortly after $F = 0.2$, when escape occurs at a blue-sky catastrophe.

Figures 5.5 and 5.6 are detailed bifurcation diagrams of the regions marked by points AA' and BB' in figure 5.4, where $0.133 < F < 0.142$ and $0.1765 < F < 0.1860$, respectively. We explore these two regions in more detail to locate and identify the corresponding bifurcation points more accurately. In figure 5.5 we show a detail of the bifurcation diagram plus three examples of stroboscopically sampled trajectories in a $(x,y)$ plane. Each one of these phase portraits shows 500 Poincaré points after discarding the first 100 points as transient. At $F = 0.1375$ we can observe the critical slowing of trajectories converging to a node that is progressively weakly attracting. At $F = 0.1384$ (thus after the Neimark bifurcation) trajectories are attracted to a (initially weakly) attracting limit cycle. At $F = 0.1393$ the drift ring (quasiperiodic attractor) is clearly visible.

\(^4\)We employ here the nomenclature of Thompson et al (1994).
Figure 5.5 - Bifurcation diagrams for SIR equations - detail of region AA' - and sample of attractors.
Figure 5.6 - Bifurcation diagrams for SIR equations - detail of region BB'
Figure 5.6 explores in more detail the region of the bifurcation diagram between points B and B' (see figure 5.4). Again the top picture is for a forward sweep in $F$, and the bottom picture for a reverse sweep. Here a hysteresis loop is clearly visible between a saddle-node fold at $F = 0.1768$ and a catastrophic blue sky instability at $F = 0.1802$. In this region there are at least two competing attractors: one periodic, and one quasi-periodic. Their basins of attraction are shown for $F = 0.1800$ in figure 5.10. The saddle-node at $F = 0.1855$ is not accompanied by a hysteresis loop, indicating that it is an intermittency explosion.

Obviously, bifurcation sequences change as we vary parameters of the system. To see this in the SIR model we take a different value for the internal resonance parameter $R$, and we allow $\omega$ to vary. We then build a series of bifurcation diagrams, one for each value of $\omega$. By marking the position of the main bifurcations along the diagrams we can construct a preliminary picture of bifurcation curves for this model in the control-parameter plane spanned by $F$ and $\omega$. Figure 5.7 shows bifurcation diagrams for different $\omega$ values, and in figure 5.8 we show the bifurcation curves generated from these diagrams. For all these figures we take $R = 1.4$ and $\zeta = 0.05$. Like in the previous figures of this section, two hundred cycles of the forcing we allowed here for the dissipation of transients.

It can be seen from figure 5.7 that bifurcation sequences can indeed vary substantially for different values of the system parameters. For the lower values of $\omega$ bifurcation sequences are very simple, with the system escaping at the first fold bifurcation. As we increase the frequency of the forcing $\omega$, bifurcation sequences eventually become more complex. Above $\omega = 0.95$ the periodic response initially goes through a symmetry-breaking bifurcation. This is followed by a sequence of bifurcations possibly similar to that displayed for $R = 1.7$ (see figures 5.4 to 5.6) with the appearance of windows of periodic motion bounded by regions of quasi-periodic and/or chaotic response.
Figure 5.7 - Bifurcation diagrams for SIR equations with $R = 1.4$, and $\omega$ varying from 0.80 (top) to 1.10 (bottom) in equal steps of 0.05.
From the data showed in figure 5.7 we have built a preliminary picture of bifurcation curves for the SIR model in the plane of control-parameters $F$ and $\omega$. These schematic bifurcation curves, shown in figure 5.8, are only partial, covering the same range of frequencies $\omega$ of figure 5.7. They do, however, show the broad features of bifurcation sequences for this model and, in particular, the approximately "V-shaped" region of escaping values typical of resonant systems, Thompson (1989).

![Figure 5.8](image)

**Figure 5.8 - Sketch of bifurcation curves for the SIR equations. Curves were approximated from attractor-following data of figure 5.7.**
5.4 Basins of attraction

We conclude our investigation of steady-state features of the SIR equations with the topic of basins of attraction. As we have seen in the bifurcation diagrams of the previous section, the SIR equations possess in some regions of control space more than one bounded attractor. In fact, the existence of competing attractors is a prevalent feature of this or any other system where an attractor at infinity does exist. To see that we just have to realise that, for every point in control space, i.e., for every combination of model parameters (such as damping level, frequency and amplitude of forcing, static bias, and internal resonance parameter) there are solutions that leave the potential well and tend to infinity, effectively becoming unbounded. Clearly then, the periodic, quasi-periodic and chaotic responses featured in the bifurcation diagrams of the previous section must have basins of attraction that do not cover the entire phase space. In this section we investigate those basins of attraction with the help of numerical procedures.

5.4.1 Grids of starts

There are a few numerical procedures available for the approximation of basins of attraction. An efficient method is the cell-to-cell mapping technique, Hsu (1989), which essentially consists of dividing the phase space into a finite number of cells, and exploring this discretisation to accelerate the process of locating basins of attraction. This is obtained by 'marking' the cells visited by the trajectory that was attracted to a certain attractor, and later including all those cells in its basin of attraction. In two-dimensional phase spaces this procedure is very reliable and several times more efficient than simple grids-of-starts. As the dimension of the phase space increases, so do the complexities of implementing this technique. Moreover, the time-related advantage of this method over simple grids-of-starts is somewhat reduced by the need to calculate distances in higher dimension.
In our work we generate numerical estimates of basins of attraction with the use of a method of grids-of-starts. Following an argument similar to the one that led us to opt for an attractor-following technique for production of bifurcation diagrams, we chose a grid-of-starts method here based on its simplicity, reliability, and robustness. Possibly the only disadvantage of this method is the time requirement. Fine grids-of-starts of more than 300,000 points as the ones used to generate figure 5.10 can take several hours running on a desktop computer.

Briefly stated, the method of grids-of-starts consists of numerically integrating the equations of motion from a number of initial conditions given by a suitably placed grid of starting points in the phase space of the system. After transients have effectively died away, the response is analysed to reveal a steady-state behaviour, be it periodic, quasi-periodic or chaotic. Usually, points of the grid attracted to the same solution are colour-coded to produce a visual representation of the basin.

5.4.2 Sample of Basins of Attraction for the SIR Model

In the case of the SIR model with its five-dimensional phase space (or four-dimensional, if we refer to its Poincaré map) basins of attraction cannot be fully visualised on paper. We limit ourselves here to two-dimensional cross-sections of those four- or five-dimensional objects. Figure 5.9 is a two-dimensional view of the basins of attraction for the SIR model with zero forcing. Here initial velocities were taken to be zero, so that the grid-of-starts used is also two-dimensional. Parameters of the model are $R = 1.7$, $\zeta = 0.05$; the grid has 640 by 480 points. The basin of attraction of the stable fixed point at the origin is depicted in white, whereas points attracted to infinity are coloured in black. We have also superimposed contours of constant energy corresponding to the saddle-like points of the potential energy function. We observe that even for this unforced (but damped) case the boundaries of the basin of attraction of the
origin are quite complex in shape. This complexity, we must keep in mind, may be due to the fact that we are viewing a two-dimensional cross-section of a four-dimensional object.

Figure 5.9 - Basin of attraction of the origin for the unforced SIR equations

Figure 5.10 is a two-dimensional representation of the basins of attraction of two competing attractors that exist at $F = 0.18$, $\omega = 0.85$. We have allowed 100 cycles of the forcing to dissipate transients; all other parameters are the same as those of figure 5.9. We have coloured in black starting conditions leading to the period-21 response shown in figures 5.2 and 5.3, and we have coloured in white the initial points attracted to a competing non-periodic attractor. Points that diverge towards infinity are represented in grey. The complex geometrical pattern of basins of attraction for the SIR equations is here very clear. Although the interior of the basins of bounded attractors (periodic and non-periodic)
seems to be reasonably well-defined, the *boundaries* of these basins where they interface the basin of attraction of the unbounded solutions (attractor at infinity) have the appearance of a fractal. Also the two basins of bounded motions seem to be highly intertwined.

![Figure 5.10 - Basins of attraction for the forced SIR equations, \( F = 0.18, \omega = 0.85 \)](image)

**Some comments on the accuracy of numerically-determined basins of attraction**

We should perhaps remark here that a certain degree of uncertainty remains with respect to these numerical estimates of basins of attraction. These uncertainties have several causes. First there is always the question of dissipation of transients. The displaying of transients of arbitrary lengths is a well known feature of nonlinear systems, particularly when close to certain types of bifurcations. Therefore when we say that a given trajectory is attracted to a non-periodic attractor (possibly chaotic) we must bear in mind that it may just be that we have not allowed enough time for dissipation of transients. In some cases,
theory tells us that no amount of time would suffice to significantly dissipate transients of a system on the verge of a bifurcation in which a chaotic attractor is created.

A second cause of uncertainty in the numerical identification of basins of attraction is the escape criterion that must be applied to label trajectories attracted to infinity. Obviously one cannot follow these trajectories indefinitely. There are cases in which energy arguments may be used to identify trajectories that have gone too far away from the origin to be ever brought back near it. In these cases it may happen that a large-amplitude periodic, quasi-periodic or chaotic attractor will eventually capture some of these trajectories. In practical terms, one must also avoid numerical overflow that tends to occur when trajectories drift far away from the origin. For all these reasons there is some degree of uncertainty concerning the identification of trajectories that are attracted to infinity.

Finally, periodic and quasi-periodic attractors are also subjected to inaccuracy in their numerical determination. Periodic trajectories are always periodic within a given numerical precision. Quasi-periodic motion is not simple to detect by direct, automatic inspection of Poincaré points. Depending on numerical parameters used a quasi-periodic motion can easily be mistaken for periodic or chaotic. In our numerical studies we prefer using visualisation of sequences of Poincaré points to identify quasi-periodic motion.

\section{5.5 Transient dynamics and safe basins}

In this section we introduce some of the fundamental ideas behind the method that we present in the next chapter. Here, for the first time in this work, we depart from steady-state dynamics, and start considering the \textit{transient behaviour} of nonlinear models of ship motion.
5.5.1 Transient versus steady-state dynamics

As we pointed out in the beginning of chapter four, most of the theory of nonlinear dynamics deals with steady-state or long-term behaviour. There are, however, at least two reasons why we should investigate transient behaviour in models of ship motion. First, the steady-state approximation (that transients have all but died away) is not a particularly good one in the problem of vessels moving under the action of sea waves. Because of the constantly changing aspect of real marine environment (waves and wind, for example), one can say that vessels never experience conditions of near-steady-state. Above all, real capsize events frequently involve the vessel being acted upon by a short train of steep waves. Clearly under these conditions there is no time to achieve any significant steady-state.

A second reason for the interest in the transient dynamics of ships stems from results of previous investigations that showed that an analysis based on the stability of steady-state solutions can lead to non-conservative results. In other words, if we assess the stability properties of a vessel by its ability to sustain steady-state motion of acceptable magnitude, we may be neglecting important transient phenomena that will, in practice, result in much reduced stability qualities for the vessel. This is one of the most significant results of the work that has been carried out by Michael Thompson and co-workers at the Centre for Nonlinear Dynamics - UCL, see for example Thompson et al (1990) and references therein. In essence, these studies have shown that the erosion of transient basins by incursive fractal striations of escaping solutions can lead to reduction of engineering integrity of safe motions, and that such reduction of engineering integrity may occur well before the final loss of stability of steady-state motion. We shall return to these ideas shortly.

From the point of view of experimental verification, the study of transient dynamics presents considerable advantages. The generation of long, clean sequences of regular waves in a wave tank is troublesome. Furthermore, it is
often very difficult (and expensive) to test the capsize of models under steady-state excitation, even if irregular: it may take hours of costly operation of the tank to produce a few capsizes of doubtful statistical relevance. Finally, the same applies to numerical investigation. For example, it is much faster to integrate equations of motion for a few cycles of the excitation than it is to search for long-term solutions.

5.5.2 Transient Dynamics: Preliminaries

Because transient dynamics is a topic little mentioned in textbooks and survey papers, we introduce here some basic definitions that will be used in the remainder of this work.

We consider the system defined by an equation of the type:

\[ \dot{x}(t) = f(x, \mu, t) \]

with initial condition \( x(0) = x_0 \). The vector of phase variables \( x = (x_1, x_2, \ldots, x_n)^T \) takes values in a submanifold \( S \) of \( \mathbb{R}^n \). If the vector field \( f \) is periodic with period \( T \) we may substitute the system by its Poincaré map \( P: x(t) \rightarrow x(t + T) \) with similar initial conditions. Examples of systems that can be represented in this form are the escape equation (3.8), the symmetric single-well roll equation (4.11), and the SIR equations (5.1-5.2) of coupled heave-roll motion. As before, \( \mu \) represents a vector of control parameters, and \( t \) is time.

A \textit{escape criterion} \( \Pi \) is an inequality \( \Phi(x, Q) \geq 0 \), involving some or all phase variables \( x_i \), and possibly a set of real constants \( Q \). We shall see shortly examples of escape criteria. A point \( x \in S \) that satisfies \( \Pi \) is called an \textit{escaping point}. The \textit{escape region} \( \Sigma(\Pi) \) of the system (5.3) is the set of its escaping points. The escape region of a system is therefore the region of its phase space in which the escape criterion is verified. A \textit{transient trajectory} \( \Gamma(x_0, t_f) \) of (5.3) is any
solution of the initial-value problem $\dot{x}(t) = f(x, \mu, t)$, $x(0) = x_0$, where $t \in [0, t_f]$. We say that $\Gamma(x_0, t_f)$ is an escaping trajectory of (5.3) under the escape criterion $\Pi$ if there is a point $x \in \Gamma(x_0, t_f)$ such that $x$ is an escaping point. It is clear from these definitions that if the initial point $x_0$ of a trajectory $\Gamma(x_0, t_f)$ is in $\Sigma(\Pi)$ then $\Gamma(x_0, t_f)$ is an escaping trajectory. We call these trajectories trivial escaping trajectories. If an escaping trajectory is not trivial we say that it is a non-trivial escaping trajectory. We shall be mainly concerned with this latter type of escaping trajectories, although both trivial and non-trivial escaping trajectories can play a part in the numerical determination of safe basins.

We say that a transient trajectory is a safe trajectory $\Gamma_s(x_0, t_f)$ if the trajectory is not an escaping trajectory, i.e., if there is not a point $x \in \Gamma_s(x_0, t_f)$ such that $x \in \Sigma(\Pi)$. The global safe basin $G(\Pi, \mu, t_f)$ of system (5.3) is the union of all safe trajectories of (5.3) under escape criterion $\Pi$, with control parameters $\mu$, and transient time defined by $t \in [0, t_f]$:

$$G(\Pi, \mu, t_f) = \bigcup_{x_0 \in \mathbb{S}} \Gamma_s(x_0, t_f)$$

The intersection of $G(\Pi, \mu, t_f)$ with any hyperplane $t = c$, $c \in [0, t_f]$ defines a local safe basin of (5.3). We shall refer to the local safe basin defined by the choice of $c = 0$ as simply the safe basin of the system, and we shall denote it by $B$. The safe basin of system (5.3) is therefore the union of all initial conditions $x_0$ that lead to safe trajectories.

Given a system like (5.3) and an escape criterion $\Pi$ we would like to determine its safe basin $B$. In other words, we would like to determine the set of initial conditions $x_0$ such that $\Gamma(x_0, t_f)$ is a safe trajectory. Obviously, for a given system $B$ depends not only on the escape criterion $\Pi$, but also on the superior time limit $t_f$. Let us deal with these two factors separately, starting with the latter.

---

5For the sake of brevity we shall refer to transients trajectories as just trajectories when there is no possible confusion between the two terms.
We have made the above definitions in such a way that in the limit $t_f \to +\infty$ the transient concepts presented can be made to approach their steady-state counterparts. This means that for $t_f$ sufficiently large the safe basin of the system should tend to the set of points such that their steady-state trajectories are safe trajectories, i.e., none of their points belong to the escape region of (5.3). We shall, however, be more interested in relatively short transients, usually given by a few cycles of the forcing function. In such situations transient safe basins and their steady-state counterparts may differ considerably. This points leads to a brief discussion about the choice of $t_f$.

Comments on the choice of $t_f$

We note that if $t_1$ and $t_2$ are two time limits with $t_1 < t_2$ then $B_{t_1} \subset B_{t_2}$. To see this we observe that some trajectories that have not escaped for $0 \leq t \leq t_1$ may escape for $t_1 < t \leq t_2$. This fact would suggest that, in the interest of safety, we should choose the largest possible $t_f$, effectively approximating the steady-state result. This should give us the smallest $B$, corresponding to initial conditions that never escape. Here, however, we must remember that one of the reasons for our interest in transient dynamics was that marine vessels are never excited by an infinite number of regular waves. We may argue that from an engineering point of view, the steady-state safe basin is of little relevance. Pursuing this viewpoint we ask how long should we allow transients to be? We believe that the answer may come from a combination of two factors. First, we note that the probability of a sequence of more than ten steep near-regular waves is very small, see for instance Jasper (1956) or Ochi (1978) for calculations of ocean wave statistics. Second, sensitivity studies may allow us to recognise how much more the transient safe basin is reduced after a given number of cycles of the forcing have been taken into account. In fact, such studies were carried out by Thompson and colleagues for the escape equation (3.8), showing that the safe basin obtained with 8 cycles of the forcing was, for that system, a good approximation of the
steady-state safe basin, see for instance Soliman and Thompson (1989). Unless indicated otherwise we shall adopt in our numerical studies a maximum duration of transients given by 10 cycles of the forcing.

We return now to the first of the two factors that have been seen to influence the determination of the safe basin of a system: the escape criterion.

Comments on the choice of escape criteria

Escape criteria can be seen in at least two different ways. A $\bowtie$ escape criterion can be viewed as a numerical scheme, necessary to computations, whose main purpose is to approximately locate trajectories that are attracted to infinity. Of course, this viewpoint assumes that there is such an attractor in the system's dynamics, as it is the case for our models of ship motions. This is a more mathematical, and perhaps more rigorous, way of regarding escape criteria. This is also the way we should define escape criteria if we want to be able to approximate asymptotically steady-state definitions as $t_f \to +\infty$.

But the definition of $\bowtie$ escape criterion can be made on a different approach. First, we recognise the fact that attractors at infinity have no concrete counterpart in ship motions. In physical terms, trajectories that depart from the potential well around the origin will be attracted to other bounded motions. These motions will typically constitute situations of capsize. Our mathematical models, however, do not include representations for these or any other very-large-amplitude motions. In our mathematical modelling, such motions have been replaced by attractors at infinity. Therefore, we can say that $\bowtie$ escape criterion, in physical terms, should identify trajectories that are attracted to very-large-amplitude motions.

More importantly, perhaps, from a physical viewpoint the final fate of a trajectory is not its most relevant feature. For if a trajectory visits regions of
Chapter Five: Dynamics of Ship Motions

Phase space far away from the safe origin capsize has, physically speaking, already occurred. This is the approach we shall take when defining escape criteria. We feel that this approach leads to more relevant results, and it is consistent with our choice of giving priority to transient aspects of the dynamics.

According to the idea above, the escape region will be defined as the whole phase space with the exception of a certain neighbourhood of the origin (which we might call the safe region). There are several ways in which this neighbourhood can be defined. Virgin et al (1992) use an energy-based measure in which the total mechanical energy of the system, as given by the sum of kinetic and potential terms, is compared with the energy of some critical state. The safe region would then be defined as the set of points \( x \in S \) such that:

\[
E(x) = E_{\text{kinetic}} + E_{\text{potential}} \leq E_{\text{critical}}
\]

The exact geometry of the safe region would in this case depend on the details of the potential energy function for the system.

Another procedure is to adopt the usual Euclidean distance of a point of the trajectory to the origin\(^6\), and compare it with a critical distance. The safe region would then be a topological \( n \)-sphere:

\[
d(x) = \sqrt{\sum_{i=1}^{n} x_i^2} \leq d_{\text{critical}}
\]

In our own numerical studies of the SIR equations we choose to employ an even simpler escape criterion: we define the escape region using only the roll angle variable \( x_1 \). The safe region will be defined by the condition:

\[
d(x) = |x_1| < d_{\text{critical}} = d_{\text{esc}}.
\]

\(^6\)We assume here that the origin has been defined to coincide with the stable fixed point of the unforced system.
Although the above function is not a distance function for the SIR equations, we note that this choice of escape criterion has some advantages from the computational as well as the experimental points of view. It is faster to compute, and far easier to implement on physical experiments because it does not require continuous recording of all displacements and velocities. Also, it relates directly to limitations of our mathematical modelling: if the amplitude of roll motion becomes too large (perhaps beyond our $d_{\text{esc}}$) we can expect significant changes in the physics of the problem, such as the effect of water-on-deck. This would suggest that our choice of $d_{\text{esc}}$ should take the angle of vanishing stability as one of its parameters. As we have seen before in this work, the angle of vanishing stability is many times defined as the angle at which hydrostatic restoring moment drops to zero. This angle, of course, would be a local maximum or saddle of potential energy: it corresponds to the location of the unstable saddle-like fixed point of the unforced system.

Finally, in our numerical experiments with the SIR equations, we have observed that once the amplitude of roll motion has achieved a certain (large) magnitude, that trajectory will quickly diverge to infinity. Since our simple escape criterion can be seen as a particular case of the two other procedures described above, it would suggest that all the different alternatives presented above for the choice of escape criterion may lead to similar results, making it even more advantageous to use the simplest possible defining function. In other words, numerical experiments seem to show that criteria of the form $\Phi(x,Q) \geq 0$ can produce good approximations of regions of phase space that are attracted to infinity in our mathematical models.
5.5.3 Overview of transient escape in the SIR model

We illustrate the ideas presented in the preceding section with an overview of escape values for the SIR model of coupled heave-roll motion under transient conditions. We recall the SIR equations (3.35) as defined in chapter 3:

\[
\begin{align*}
\frac{2}{R^2}(\ddot{y} + 2\zeta \dot{y}) + 2y &= x^2 \\
\dot{x} + 2\zeta \dot{x} + x - 2xy - b &= F \sin \omega t
\end{align*}
\]

We keep here the same notation of chapter 3 in which \(x\) and \(y\) are the roll and heave displacement variables, respectively. The damping parameter \(\zeta\) is kept constant at a value of 0.05, and the bias in forcing, \(b\), is assumed to be zero. We are therefore basically interested in finding minimum values of excitation amplitude \(F\) for escape within a (small) number of forcing cycles for a range of values of excitation frequency \(\omega\) and internal resonance parameter \(R\). For this study we shall consider a maximum duration of transients as defined by eight periods of the forcing function.

The procedure we have followed to determine transient escape values consists of numerically integrating the system from rest, \((x,\dot{x},y,\dot{y}) = (0,0,0,0)\), for a given set of parameters \((F,\omega,R)\). Escape is here defined by \(|x(t)| \geq 1.27\) for \(t \leq \frac{2\pi}{\omega}m\), where \(m = 8\). If escape has not occurred we step up \(F\), and repeat the operation until escape occurs. We then record \(F\), and move on to a different combination of \(\omega\) and \(R\). For each new value of \(F\) we start the system at rest. We repeat the procedure for all combinations of \(\omega\) and \(R\) of interest.

Figure 5.11 is a three-dimensional representation of minimum values of \(F\) for escape. The most noticeable feature of this picture is the "ridge" whose projection onto the \((\omega,R)\)-plane develops roughly along the line \(R = 2\omega\) (see also figure 5.12).

Note that for the SIR model \(x=1\) and \(x=-1\) are the roll coordinates of the hill-top saddles of the potential energy function, see section 3.7.2.
Of course, $R = 2\omega$ is just the (theoretical) condition for internal resonance, see section 3.7.2. At first thought it seems contradictory that the system should require high values of $F$ for escape near conditions of internal resonance. We can perhaps understand this phenomenon if we think along the following lines. First we must remember that near internal resonance there is strong coupling between heave and roll motion, which means that there is significant flow of energy between the two "modes". In particular, since all external energy input (excitation) comes from the roll equation, there will be a significant flow of energy from roll to heave motion near internal resonance. Now, the system cannot escape in heave (the potential well is just parabolic in heave), therefore near internal resonance the system will be using a (relatively) large proportion of its energy to drive high-amplitude heave motion that does not lead to escape. Of course, there will be combinations of $\omega$ and $R$ that require even larger values of $F$ for escape, these corresponding to situations where no resonance of any type exists. But minimum values of $F$ for escape will not occur near internal resonance, a fact that at first would seem puzzling.
Figure 5.11 is useful to reveal that the surface generated by minimum values of $F$ for escape has a saddle-like shape. The minimum height of the surface along the ridge occurs around $R = 1.7$, a point that we feel may have special significance and in fact have already been investigating, see section 5.2.

In figure 5.12 we show contours of constant $F$ required for escape. In this figure the existence and position of the ridge is most clear. But to help visualising minimum and maximum values for escape for each $R$, we have split figure 5.11 in its 21 generating curves, and have displayed those points in three sets of 7 curves in figure 5.13.
Figure 5.13 - Individual curves of minimum $F$ required for escape with $R$ varying from 1.0 to 1.3 (top), 1.35 to 1.65 (middle), and 1.7 to 2.0 (bottom)
5.5.4 Safe basins through grids-of-starts

Before we turn our attention to the main focus of this section, the evolution of safe basins, we present a brief description of the method of grids-of-starts as it is employed in this work. This is a simple and robust method to generate numerical approximations to the safe basins of a system, as well as some interesting basin-related features. For the sake of clarity we choose to describe the method separately for phase space and control-parameter space.

Safe basins in phase space

A grid-of-starts in phase space is the product set of discrete vectors of starting coordinates:

\[ M_s = s_1 \otimes s_2 \otimes \ldots \otimes s_n \]

where \( s_i = \{ s_{i1}, s_{i2}, \ldots, s_{iN_i} \} \), \( i = 1, 2, \ldots, n \). Here \( n \) is the number of phase variables covered by the grid (the dimension of the system), and \( N_i \) is the number of starting coordinates in the grid for the phase variable \( x_i \). Of course, the vectors \( s_i \) do not have to have the same number of elements. In our example of section 5.4.2 we have \( n = 4 \), with \( N_1 = 640, N_2 = N_3 = 1, N_4 = 480 \): we used a grid of \( 640 \times 1 \times 480 \times 1 \) points. Each element of a grid-of-starts in phase space is then a point \( x_o \in M_s \) that will be used as the initial condition of an initial-value problem.

The method of grids-of-starts consists of solving numerically a series of initial-value problems:

\[ \dot{x}(t) = f(x, \mu, t), \quad t \in [0, t_f] \]

\[ x(0) = x_o, x_o \in M_s \]
to generate $N = N_1 \times N_2 \times \ldots \times N_n$ series of discrete points $\{x(k), k = 1, 2, \ldots m \times p\}$. The $\{x(k)\}$ are numerical approximations to transient trajectories, where $m$ is the number of cycles of forcing (of period $T$), and $p$ is the number of points per cycle of the forcing. The parameter $m$ defines the maximum duration of the transient, $t_f = mT$. Each point of $\{x(k)\}$ is then tested against the escape criterion, and classified as safe or escaping. If any point of $\{x(k)\}$ is classified as escaping, the trajectory it approximates is considered to be an escaping trajectory. The set of points $\mathbf{x}_0$ such that the corresponding $\{x(k)\}$ approximates a safe trajectory is our numerically-generated safe basin for the system.

Comments on the definition of practical grids-of-starts

This method can lead to very long computations even for 2-DOF systems and the relatively short duration of transients that we have used. This is because the number of points of a grid increases exponentially with the dimension of the system. For example, even a relatively modest grid with, say, 100 points in each dimension would have $10^8$ points. If each trajectory took one second to be integrated the complete grid would take more than three years to be computed! We must therefore exercise some judgement when defining grids-of-starts of practical use. The procedure we shall employ more often consists of effectively reducing the dimension of the grid by considering only some dimensions of the system.

Perhaps the most straightforward procedure to generate a grid of starting points is to define an $n$-dimensional window in the phase space of the system, and place a uniformly distributed grid of points covering the window. That is the procedure we shall use most in our work. Let us exemplify this procedure with the SIR equations. The generalisation for other finite-dimensional systems is trivial.

---

8In our numerical algorithms we test every intermediate point in the 4th-order Runge-Kutta integration scheme. This is necessary to avoid numerical overflow.

9If $a$ is the number of different ordinates for each dimension, then the total number of points in a grid of dimension $n$ is $N = a^n$. 

An \textit{n-dimensional window} is a region of the phase space $S$ of the system defined by:

$$x_{\text{inf}_i} \leq x_i \leq x_{\text{sup}_i}, \ i = 1, 2, 3, 4$$

where each one of the $x_i$ corresponds to one of the phase variables of the system. The \textit{limits of the grid} are $(x_{\text{inf}_i}, x_{\text{sup}_i})$. Now let $N_i$ be the number of distinct ordinates of the grid along dimension $x_i$. We say that $(x_{\text{inf}_i}, x_{\text{sup}_i}, N_i), i = 1, 2, 3, 4$ define $N = N_1 \times N_2 \times N_3 \times N_4$ \textit{n-dimensional cells} given by:

$$\left(x_{i, j}, x_{i, j+1}\right) = \left(x_{\text{inf}_i} + \frac{j}{N_i} (x_{\text{sup}_i} - x_{\text{inf}_i}), x_{\text{inf}_i} + \frac{j+1}{N_i} (x_{\text{sup}_i} - x_{\text{inf}_i})\right), \ j = 0, 1, 2, \ldots, N_i - 1, \ i = 1, 2, 3, 4$$

The starting points $x_0(j) = (x_{01}, x_{02}, x_{03}, x_{04})$ composing the grid $M$ are the \textit{geometric centres} of each of the cells defined above:

$$x_{0i}(j) = \left(\frac{x_{i, j} + x_{i, j+1}}{2}\right), \ j = 0, 1, 2, \ldots, N_i, \ i = 1, 2, 3, 4$$

As we have said before the $N_i$'s do not have to be the same for all dimensions. The procedure we employ more often in our work to reduce the computational problem to manageable proportions is to make one or two of the $N_i$'s equal to unity. Since our graphical representations of safe basins are always two-dimensional, one option is to make the $N_i$'s corresponding to dimensions not present in the picture unity. This will often correspond to taking cross-sections of the safe basins for zero velocities, $x_2 = x_4 = 0$.

One of the features of grids-of-starts defined by windows in phase space is that there may be points of the grid that belong to the escape region. These points would correspond to trivial escaping trajectories, see 5.5.2. In the case of our
escape criterion, this would only happen if the grid extends beyond $d_{sw}$ in the $x_1$ direction.

**Safe basins in control-parameter space**

As we shall see in the next section, safe basins in *phase space* are very useful as a means of observing the evolution of basins as we change a control parameter. More specifically, we shall see how numerically generated safe basins in phase space can reveal underlying processes of loss of safe basins and basin erosion as we increase the amplitude of the periodic forcing, $F$. They are not, however, very efficient in locating regions of control-parameter space where interesting dynamics may be occurring. For that, it is perhaps better to concentrate on a single (starting) point in phase space, say, the origin $x(0) = x_0 = 0$, and then determine which points (in control-parameter space) lead to escape. Of course, we shall be concerned here with *transient* escape within a given number of cycles of the forcing.

Let us then define a grid-of-starts in control-parameter space as the product set of discrete vectors of starting parameter values:

$$M_r = r_1 \otimes r_2 \otimes \ldots \otimes r_q$$

where $r_i = \{r_{i1}, r_{i2}, \ldots, r_{iq_i}\}$, $i = 1, 2, \ldots, q$. Here $q$ is the number of control parameters covered by the grid (the dimension of the vector $\mu$ of control variables), and $Q_i$ is the number of starting values in the grid for the control parameter $\mu_i$. If a control parameter $\mu_i$ will not be varied in a given grid we just say that its $Q_i$ equals 1, and keep it constant throughout the grid. In the study of the SIR model that follows, that is the case with the damping parameter $\zeta$ and the symmetry-breaking bias $b$, which we keep constant at 0.05 and 0, respectively. Each element of a grid-of-starts in control-parameter space will then be a point $\mu \in M_r$ that will be used to solve an initial-value problem:
Chapter Five: Dynamics of Ship Motions

\[ \dot{x}(t) = f(x, \mu, t), \ \mu \in M, \ t \in [0, t_f] \]

5.6

\[ x(0) = x_0 \]

to generate \( Q = Q_1 \times Q_2 \times \ldots \times Q_q \) series of discrete points \( \{x(k), k = 1, 2, \ldots, m \times p\} \).

Here, as in the case of grids in phase space, the \( \{x(k)\} \) are numerical approximations to transient trajectories, where \( m \) is the number of cycles of forcing (of period \( T \)), and \( p \) is the number of points per cycle of the forcing. We test each point \( x(k) \) for escape to classify it as safe or escaping. If all points \( x(k) \) of a (numerically approximated) trajectory are safe we say that the trajectory is safe. We then define the safe basin of the system in control-parameter space as the collection of points \( \mu \in M \) such that the trajectories they generate are safe.

5.6 Safe Basins in Control-Parameter Space for the SIR Model

We investigate here the behaviour of safe basins in control-parameter space for the SIR model. The main purposes of this study are to give us a broad view of transient escape regions, and to help identifying regions of control-parameter space where interesting dynamics may be happening, as for example with the "ridge" identified in section 5.5.3.

We shall be studying safe basins in two different control-parameter spaces. One is defined by the \((F, \omega)\) pair, and we shall refer to it as the oscillator control-parameter space. The other space we shall be concerned with here is defined by the \((F/\omega^2, \omega)\) pair, and we refer to it as the vessel control-parameter space. This is related to specific features of the ship capsize problem under regular waves as represented by the SIR model. To see why we use \( F/\omega^2 \) we recall from Thompson et al (1992) that the non-dimensionalised amplitude of periodic
forcing $F$ can be related to the parameters of the regular, incoming beam waves by:

$$F \equiv \frac{Ak\omega^2}{\theta_v}$$

where $A$ is the height of the wave ($2A$ from crest to trough), $k$ is the wave number ($k = 2\pi/\lambda$, $\lambda$ is the wave length), and $\theta_v$ is the angle of vanishing stability for the vessel, see 3.7.1. Hence $Ak$ is the wave slope, and $Ak/\theta_v$ is the wave slope relative to the angle of vanishing stability (i.e. normalised). But from equation (5.7) we have $Ak/\theta_v = F/\omega^2$, therefore if we want to examine escape regions in a space defined by (normalised wave slope, wave frequency) we have, in terms of $F$ and $\omega$, to work with the pair $(F/\omega^2, \omega)$.

The figures we show here were generated using the grid-of-starts method described in section 5.5.4. We have kept the damping parameter $\zeta$ constant at 0.05 throughout this study, and unless otherwise stated the symmetry-breaking bias $b$ is assumed to be zero. We have employed fine grids of more than 300,000 points (640 by 480 in the $(F, \omega)$ or $(F/\omega^2, \omega)$ planes) that allow us to visually inspect the fractal-like structure of basin boundaries.

Figure 5.14 shows a collection of control-parameter portraits of safe basins (in white) for a range of values of the internal resonance parameter $R$. Both the oscillator (left column) and the vessel (right column) control-parameter spaces of the SIR model are represented. Numerical simulations used to generate these pictures start at the origin of the phase space $(x, \dot{x}, y, \dot{y}) = (0, 0, 0, 0)$, and run for eight cycles of the forcing, after which if escape (defined by $|x(t)| \geq 1.2$) has not occurred the point is declared safe. We have kept the damping parameter $\zeta$ constant at 0.05, and the symmetry-breaking bias $b$ is assumed to be zero.

The two control-parameter spaces are, of course, qualitatively similar. The vessel control-parameter space can, for example, be viewed as a distorted version of the oscillator control-parameter space, in which the escape regions
(black regions in the pictures) are twisted and stretched in a clockwise direction. We have therefore extended the range of \( \omega \) upwards (i.e. into large values) to allow for the fact that escape should be expected for low values of \( F/\omega^2 \) (in the vessel control-parameter space) at large \( \omega \). This is a simple consequence of the fact that the \( F \)-equivalent to a given \( F/\omega^2 \) is multiplied by \( \omega^2 \). In other words, the vessel control-parameter space at large \( \omega \) is a representation of the oscillator control-parameter space at large \( F \). Conversely, escape regions in the oscillator control-parameter space for small \( \omega \) represent the escape behaviour of the vessel at large \( F/\omega^2 \). The two representations are therefore not just qualitatively equivalent but actually complement each other. Having said that, we should remark that most of the interesting behaviour, as well as the lower values of amplitude of forcing necessary for escape, seem to occur around \( \omega = 1 \), making largely irrelevant the choice of either control-parameter space for further studies. We shall return to this point shortly.

Figure 5.14 contains a wealth of information about the transient behaviour of the SIR model. We start by observing the highly convoluted shape of these basins, indicative of an underlying fractal structure that has, in fact, already been noted in our steady-state investigation of safe basins in phase space, see section 5.4. Also clear in the pictures of figure 5.14 are the already familiar "V"-shaped regions corresponding to escape under resonance (direct or parametric). As we have noted above these regions tend to concentrate around \( \omega = 1 \). However, differently from previously investigated one-degree-of-freedom systems (see for example Thompson (1989)), escape regions in control-parameter space for the SIR model are roughly split in two by a region of escape suppression. These escape suppression regions protrude from below, forming in the extended control-parameter space of \((F, R, \omega)\) a ridge that lines itself up following approximately the internal-resonance plane of \( R = 2\omega \).

\(^{10}\)As an apparent exception to this rule we note that for \( R=2.0 \) the lowest value for escape occurs somewhere between \( \omega = 2.5 \) and \( \omega = 3.0 \). However at such large values of \( \omega \) some fundamental hypotheses of our mathematical modelling, such as the long-wave assumption, are probably no longer valid.

\(^{11}\)In section 5.5.3 we had already identified this phenomenon and presented a preliminary explanation for its existence in terms of the energy coupling between heave and roll motions.
Figure 5-14 - SIR model: safe basins in the control-parameter spaces of \((F, \omega)\) (left column), and \((F/\omega^2, \omega)\) (right column) for a range of \(R\) values.
The behaviour of the system as an oscillator for large values of \( \omega \), say \( \omega \approx 2 \), is usually not investigated because at such conditions escape tends to occur only for large values of \( F \) (when compared to the minimum \( F \) necessary for escape). However, when we consider the ship capsize problem as modelled by the SIR model, we see that those large \( \omega \) values could, in principle, be of interest because now the relevant amplitude of forcing is proportional to \( F/\omega^2 \). There is no reason why the minimum values for escape in the oscillator and vessel problems should therefore be the same, and we have to investigate both. Inspection of figure 5.14 shows however that, with the exception of \( R = 2.0 \), minimum values for escape occur around \( \omega = 1 \) for both the oscillator and the vessel problems.

We can nevertheless detect a clear tendency for minimum escape values to occur for larger \( \omega \) values as we increase \( R \) in the vessel control-parameter space. To investigate this occurrence we consider the limit case of \( R = \infty \), which defines a one-degree-of-freedom problem with a symmetric potential well, see equation (4.11). Figure 5.15 shows that we have to extend our grid-of-starts almost up to \( \omega = 5 \) to see values of amplitude for escape that are lower than those of the \( \omega \approx 1 \) region. At such large values of frequency, however, our mathematical modelling of the ship motion problem is probably no longer valid, so that we can say that within the limits of validity of our mathematical models most of the interesting dynamics (in terms of the nonlinear oscillator and ship capsize) will occur around \( \omega = 1 \).
As far as minimum values for escape are concerned, we see, again from figure 5.14, that with transients running for up to eight cycles of the forcing no escape is observed below $F$ (or $F/\omega^2$) = 0.1. The absolute minimum escape value for the range of $R$ considered here seems to lie between $R = 1.2$ and $R = 1.6$. We have therefore taken this range of $R$, and looked at it more closely in figure 5.16. Here we have also increased the maximum duration of transients to sixteen cycles to produce a more detailed picture of the "lobes" of escaping conditions at the tips.
Figure 5.16 - Detail of safe basins in the oscillator (left column), and vessel (right column), control-parameter planes for the SIR model.
of the escaping regions. The system displayed considerable sensitivity to the maximum duration of transients, with minimum escape values falling to about 0.08. This is consistent with the "lobe-type", fractal structure of the boundaries of escape regions that can be observed in figure 5.16, see McRobie and Thompson (1991), and Soliman and Thompson (1992). Roughly speaking, each "lobe" represents escape under a specific period of the excitation, so that the longer we allow transients to go the more "lobes" we see, each subsequent "lobe" extending slightly further down from the previous one, i.e. to regions of lower $F$ values.

5.7 Evolution of Safe Basins

5.7.1 Loss of Safe Basin Near Resonance

In this section we present some examples of the application of the ideas above to the study of the evolution of safe basins as a parameter of the model is varied. We concentrate here on the loss of safe basin experienced by 1- and 2-DOF models of ship motion near resonance.

Figure 5.17 is a sample of numerically generated safe basins for the SIR equations at internal resonance with parameters shown in table 5.1. We recall that in the context of the SIR equations we write $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$. The figure shows the evolution of the safe basins (displayed in black) as the amplitude of periodic forcing $F$ is increased from 0.06 to 0.30. To give a better view of the four-dimensional safe basins of the SIR equations we present two different cross-sections. In figure 5.17(a), (c) and (e) we show cross-sections defined by the hyperplane $y = \dot{y} = 0$, with $x$ and $\dot{x}$ as the horizontal and vertical ordinates, respectively. In figure 5.17(b), (d) and (f) we see cross-sections of the same corresponding basins, defined by $\dot{x} = \dot{y} = 0$, with $x$ and $y$ as the horizontal and vertical ordinates, respectively.
We postpone until next section a more detailed analysis of the process of loss of safe basin in operation here, but we note that the shape of the safe basin can become very complex, possibly fractal, giving little hope of its identification by any means other than numerical investigation. We note however that the apparently isolated 'islands' of safe starting conditions clearly visible in figure 5.17 (c) and (d) are probably connected in the full four-dimensional phase space.

In figure 5.18 we can see the evolution of safe basins as we increase the amplitude of periodic forcing for the 1-DOF symmetric single-well model of ship roll (4.11), see parameters in table 5.2. We recall that this model can be seen as a limit case of the SIR equations for $R \to +\infty$. Where applicable we use the same parameter values of figure 5.17. Pictures have $x$ and $\dot{x}$ as horizontal and vertical ordinates, respectively. Safe basins are displayed in black.

Figure 5.18 illustrates one of the main dynamical mechanisms of loss of safe basin: the erosion of the safe basin by incursive fractal fingers. This mechanism has been extensively studied at the Centre for Nonlinear Dynamics -UCL, and its discovery was one of the main arguments for the proposal of a Transient Capsize Diagram as a method to assess the engineering integrity of motions of a vessel, Rainey and Thompson (1991). We shall comment on this and other mechanisms of loss of safe basin in the next section.

We conclude this brief exposition of safe basins and their evolution with an example of the behaviour of the escape equation (3.8). Again we concentrate on near-resonance conditions, and we follow safe basins as the amplitude of periodic forcing is increased for equation (3.8), see table 5.3 for the parameters of both these figures. The pictures in figure 5.19 show the safe basin being eroded by fractal incursions in a processes that resembles the one observed for the symmetric-well system.
Table 5.1 - Parameters for figure 5.17

<table>
<thead>
<tr>
<th>Parameters of the Model</th>
<th>$\omega = 0.85$, $\zeta = 0.05$, $R = 1.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the Grid</td>
<td>$(x_{ref}, x_{sup}, y_{ref}, y_{sup}, y_{rig}, y_{rig}) = (-1.1, 1.1, -1.0, 1.0, -0.6, 0.6, -0.5, 0.5)$</td>
</tr>
<tr>
<td></td>
<td>$(N_1, N_2, N_3, N_4) = (320, 1, 240, 1)$</td>
</tr>
<tr>
<td>Numerical Parameters</td>
<td>$p = \frac{T}{80}$, $m = 8$, $d_{esc} = 1.2$</td>
</tr>
</tbody>
</table>

Figure 5.17 - Evolution of safe basins for the SIR equations with $\omega = 0.85$
Table 5.2 - Parameters for figure 5.18

<table>
<thead>
<tr>
<th>Parameters of the model</th>
<th>$\omega = 0.85, \zeta = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the grid</td>
<td>$(x_{\text{inf}}, x_{\text{sup}}, \dot{x}<em>{\text{inf}}, \dot{x}</em>{\text{sup}}) = (-1.2, 1.2, -1.0, 1.0)$</td>
</tr>
<tr>
<td></td>
<td>$(N_x, N_y) = (640, 480)$</td>
</tr>
<tr>
<td>Numerical parameters</td>
<td>$p = \frac{T}{80}, m = 8, d_{\text{esc}} = 1.2$</td>
</tr>
</tbody>
</table>

Figure 5.18 - Evolution of safe basins for the roll model (4.11) with $\omega = 0.85$
Table 5.3 - Parameters for figure 5.19

<table>
<thead>
<tr>
<th>Parameters of the Model</th>
<th>$\zeta = 0.05, \omega = 0.85$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the Grid</td>
<td>$(x_{inf}, x_{sup}, \dot{x}<em>{inf}, \dot{x}</em>{sup}) = (-1.2, 1.2, -1.0, 1.0)$</td>
</tr>
<tr>
<td></td>
<td>$(N_1, N_2) = (200, 200)$</td>
</tr>
<tr>
<td>Numerical Parameters</td>
<td>$p = \frac{T}{80}, m = 6, d_{sec} = 1.2$</td>
</tr>
</tbody>
</table>

Figure 5.19 - Evolution of safe basins for the escape equation with $\omega = 0.85$
5.7.2 Processes of loss of safe basin

In this section we present some comments on the processes of loss or reduction of safe basin that can be responsible for the reduction of engineering integrity of transient ship motions. This topic was a subject of detailed studies in the context of the escape equation (3.8), see Soliman and Thompson (1989). After describing the basic mechanisms of loss of safe basin we produce a preliminary classification of processes motivated by the knowledge of those mechanisms and also by results of numerical investigations of 1-DOF and 2-DOF models.

Basic mechanisms of loss of safe basin

Several nonlinear dynamical phenomena can be associated with processes of reduction or loss of safe basins. These phenomena constitute the basic mechanisms of loss of safe basin that we now describe. As we shall see, these basic mechanisms are extracted from the realm of steady-state dynamics. They can therefore represent a link between steady-state and transient phenomena. For convenience of description we assume that the amplitude of periodic forcing $F$ is the parameter being varied. Variations of other parameters such as the frequency of forcing can yield similar result.

We identify three basic mechanisms of loss or reduction of safe basins that have been found to be relevant to our studies:

i. Jump to resonance

These are well-known phenomena often associated with a fold or saddle-node bifurcation, see 4.4.1. A typical scenario of loss of safe basin through a jump to resonance can be summarised as follows. At low or moderate values of $F$ most trajectories are attracted to a small-amplitude (in other words, safe)
solution. At a critical value of $F$, a large-amplitude\textsuperscript{12} attractor is created. This new and unsafe attractor may coexist with the small-amplitude solution for a certain range of $F$ (corresponding to hysteric behaviour). Beyond a certain value of $F$ the small-amplitude attractor is destroyed, possibly in another saddle-node bifurcation. Only a large-amplitude attractor remains, and all trajectories will be attracted to unsafe solutions.

We must remark that there may be more than one attractor for the system even at moderate values of $F$. Even more likely, there will be two or more large-amplitude attractors for higher values of $F$. Many of these will, however, have very small basins of attraction and may be irrelevant from a practical viewpoint. In any case, attractors will be either safe or non-safe (escaping), and the comments we made apply to each of those categories.

A second comment should be made with regard to the transient nature of trajectories used to define safe basins. It may happen that, due to this transient character, some trajectories remain safe when safe attractors no longer exist. This is particularly true in the aftermath of a saddle-node bifurcation or a blue-sky catastrophe event when transients can be very long, see Soliman and Thompson (1989). With suitable definition of maximum time of transients the underlying loss of safe solutions will eventually manifest itself on the behaviour of safe basins. Moreover our main interest is in transient escape, and therefore trajectories that escape only after long transients are to be considered safe.

ii. Gradual shift

This is a mechanism of loss of safe basin related to the shift or drifting of attractors and their basins in phase space. This is a consequence of the fact that, even in the absence of bifurcations, attracting sets do not remain stationary in

\textsuperscript{12}By a large-amplitude attractor we mean an attractor containing one or more points belonging to the escape region of the system.
Chapter Five: Dynamics of Ship Motions

Phase space as parameters of the system are varied. The effects of this phenomenon are noticeable in some of our numerically-determined safe basins, see for example figures 5.20 and 5.21. They have also been observed in other numerical studies, see for example Soliman and Thompson (1989).

iii. Fractal erosion

This mechanism of reduction or loss of safe basin originates from the complex tangling of invariant manifolds following homoclinic and/or heteroclinic tangencies of invariant sets of saddle-like singular points. These complex phenomena have been investigated in depth in several studies, Grebogi et al (1983), Grebogi et al (1986), Thompson (1989), McRobie (1992).

From the point of view of transient escape it is perhaps useful to distinguish between two different situations. The first case is typified by the fractal erosion of safe basins in near-resonance conditions as seen in figures 5.18 and 5.19\(^1\). Here, a number of fractal whiskers emanating from the vicinity of the hill-top saddle develop inside the former safe basin, rapidly sweeping across it. It should be stressed that these fractal structures appear and grow within a narrow range of parameter values, so that shortly after their first appearance the bulk of the safe basin is eroded by them. We shall see in the next chapter that these considerations can be crucial to the development of a practical method to assess the stability of transient motions.

There exists however a second situation in which the fractal structures created after transversal intersections of invariant sets do not seem to display the global behaviour described above. This case is exemplified in figures 5.20, 5.21 and 5.22, for the escape equation (3.8), single-well symmetric roll model (4.11), and SIR equations, respectively. Here, basin boundaries seem to acquire a fractal

\[^1\]Figure 5.17 probably fits in this category but fractal structures are not as easily observable due to the dimension of the system.
structure, but the fractal incursions are arranged in phase space in such a way as to be macroscopically perceived as a single tongue-like structure. In addition to that, this 'single' fractal incursion does not intrude the bulk of the safe basin, but remains close to its boundary while growing in length and width.

In terms of safe basins, the net effect of this kind of global behaviour is to give safe basins a macroscopic smoothness. As we can see in figures 5.20, 5.21 and 5.22 safe basins experience in this case a seemingly smooth and gradual reduction. In fact, Soliman and Thompson (1989) argue that safe basins should become smooth again before the final collapse of the system in a saddle-node bifurcation. It is equally important here to remark that the erosion of the safe basins in this case will occur gradually and slowly. Again we see in the next chapter how these considerations affect the definition of methods of evaluating the stability of transient motions.

A preliminary classification of processes of loss of safe basin

The basic dynamical mechanisms and the numerical results above, suggest that processes of loss of safe basin can be broadly classified into catastrophic, sudden, and gradual processes. Catastrophic processes are defined by the discontinuous change in shape, size and position of safe basins that result for instance from a jump to resonance. In extreme cases, the whole safe basin can vanish almost instantaneously as motions are attracted to a resonant attractor that happens to be outside the safe region. This is a process that occurs not only because of our definition of escape in which unsafe motions are not necessarily only those attracted to infinity but any motion that implies the system visiting

---

14The pictures in figure 5.22, being only two-dimensional cross-sections of four-dimensional objects are less clear than those of figures 5.20 and 5.21. The contrast with figure 5.17, which corresponds to a near-resonance condition, is however marked. Here again we have reason to believe that the seemingly isolated 'islands' are in fact connected in the full four-dimensional phase space.
Chapter Five: Dynamics of Ship Motions

Table 5.4 - Parameters for figure 5.20

<table>
<thead>
<tr>
<th>Parameters of the Model</th>
<th>( \xi = 0.05, \ \omega = 0.55 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the Grid</td>
<td>( (x_{inf}, x_{sup}, \dot{x}<em>{inf}, \dot{x}</em>{sup}) = (-1.2, 1.2, -1.0, 1.0) )</td>
</tr>
<tr>
<td>(( N_1, N_2 ))</td>
<td>(200, 200)</td>
</tr>
<tr>
<td>Numerical Parameters</td>
<td>( p = \frac{T}{80}, \ m = 6, \ d_{esc} = 1.2 )</td>
</tr>
</tbody>
</table>

Figure 5.20 - Evolution of safe basins for the escape equation with \( \omega = 0.55 \)
Table 5.5 - Parameters for figure 5.21

<table>
<thead>
<tr>
<th>Parameters of the Model</th>
<th>( \omega = 0.85, \zeta = 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the Grid</td>
<td>((x_{\text{inf}}, x_{\text{sup}}, \dot{x}<em>{\text{inf}}, \dot{x}</em>{\text{sup}}) = (-1.2, 1.2, -1.0, 1.0))</td>
</tr>
<tr>
<td></td>
<td>((N_1, N_2) = (640, 480))</td>
</tr>
<tr>
<td>Numerical Parameters</td>
<td>( p = \frac{T}{80}, m = 8, d_{\text{esc}} = 1.2 )</td>
</tr>
</tbody>
</table>

Figure 5.21 - Evolution of safe basins for the roll model (4.11) with \( \omega = 0.55 \)
Table 5.6 - Parameters for figure 5.22

<table>
<thead>
<tr>
<th>PARAMETERS OF THE MODEL</th>
<th>( \omega = 0.55, \zeta = 0.05, R = 1.7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PARAMETERS OF THE GRID</td>
<td>((x_{ref}, \dot{x}<em>{ref}, y</em>{ref}, \dot{y}<em>{ref}, z</em>{ref}, \dot{z}_{ref}) = (-1.1, 1.1, -1.0, 1.0, -0.6, 0.6, -0.5, 0.5))</td>
</tr>
<tr>
<td>((N_1, N_2, N_3, N_4))</td>
<td>((640, 1, 480, 1))</td>
</tr>
<tr>
<td>NUMERICAL PARAMETERS</td>
<td>(p = \frac{T}{80}, m = 8, d_{esc} = 1.2)</td>
</tr>
</tbody>
</table>

Figure 5.22- Evolution of safe basins for the SIR equations with \( \omega = 0.55 \)
regions of phase space outside the safe limits. The same process can occur when for example a system jumps straight from a low-amplitude non-resonant attractor to infinity.

**Sudden loss of safe basin** is associated with fractal erosion such as we have observed for our models in near-resonance conditions. The fact that fractal incursions sweep rapidly across the bulk of the safe basin as a parameter is varied means that the whole process of reduction and loss of safe basin happens within a narrow range of parameter values. This is the process that originates the sharp cliffs in *integrity diagrams*\(^{15}\) observed by Thompson and co-workers, and it is also associated with the proposal of a Transient Capsize Diagram as a tool in evaluating the stability qualities of a vessel, see Thompson *et al* (1990) for a description of integrity diagrams in the context of ship motions, see 2.4 and also Rainey and Thompson (1991) for definition and details of Transient Capsize Diagrams.

Finally, **gradual reduction and loss of safe basin** is the process resulting from the gradual shift and/or gradual erosion of safe basins such as illustrated by figures 5.20, 5.21, and 5.22. The latter mechanism is probably related to fractal erosion in the sense that basins seem to have fractal *boundaries* during most of the process. However, in contrast with the sudden fractal erosion of the previous case, loss of safe basin is here macroscopically smooth and, equally importantly, gradual.

It may be interesting to comment here on the possible connection between local and global bifurcations that the considerations above somehow suggest. It is reasonable to assume that the global features of transient trajectories, and therefore of transient basins, are linked to underlying steady-state structures such as invariant manifolds that form the boundaries between different attractors. For example it seems reasonable to ascribe the complex geometry of (transient) safe basins to underlying complexities of (absolute or steady-state) basins of

---

\(^{15}\)See section 5.7.3.
attraction. Rigorous facts substantiating such links remain, however, to be established. In spite of this lack of theoretical surety, results such as those shown in this section seem to give evidence against the belief that:

"This global bifurcation [the homoclinic tangency of invariant manifolds] occurs in addition to, and independently of, the local period doubling or jump (fold) bifurcations of fixed points referred to above."\(^{(16)}\)

In fact, one main difference between the evolution of steady-state dynamics for near-resonance and off-resonance situations is the different bifurcation sequences that the system goes through for different values of frequency. This has been established, for example, for the escape equation, see Soliman and Thompson (1989). It also remains to be shown if these different bifurcations of the main sequence have some bearing on the global organisation of invariant manifolds. What the numerical studies we have described in this section seem to indicate is that different bifurcation sequences correspond to different organisation of global structures, and more importantly, that such differences do have practical consequences as far as safe basins are concerned.

In the next chapter we present a method to assess the stability of transient ship motions that is based on the preliminary classification just described. We shall then see how the knowledge of these different processes can help us to define a simple and robust procedure to investigate the stability properties of vessels. Before we do that, however, we illustrate the concept of loss of safe basin, and the different processes just described with the aid of integrity diagrams.

\(^{(16)}\)See Guckenheimer and Holmes (1990). The authors make this comment in the context of a 1-DOF double-well Duffing equation not dissimilar to our system (4.11) or to the escape equation.
5.7.3 Integrity Diagrams

Integrity diagrams have been extensively used by Thompson and co-workers to quantify basin erosion. In Soliman and Thompson (1989) the authors propose and investigate the use of several different integrity measures. One of them, which we find particularly useful, is the direct ratio between safe and non-safe (escaping) transient trajectories defined by a suitable grid of starting points in the phase space of the system. Let us define this procedure in more detail.

We recall from 5.5.4 that a grid of starts \( M \) is defined as a collection of \( N \) starting points in the phase space \( S \) of the system. We have also shown how a grid is computed for a system, following the idea of selecting an \( n \)-dimensional window in \( S \). Let then \( N_s \) and \( N_e \) be the number of elements of \( M \) whose corresponding transient trajectories are safe, and escaping, respectively, under a given escape criterion. Following Soliman and Thompson we define a Global Integrity Measure \( G \) for the system as the ratio between \( N_s \) and \( N_e \):

\[
G = \frac{N_s}{N_s + N_e} = \frac{N_s}{N}
\]

We note that \( G \) is independent of the knowledge of fine details of the underlying steady-state dynamics, such as to which attractor each trajectory is attracted. On the other hand, \( G \) is clearly dependent of both the escape criterion (including the maximum duration of transients \( t_f \)) and the particular grid used to compute trajectories. We have already considered the effect of different choices of \( t_f \), see 5.5.4. With respect to the possible influence of particular choices of a grid of starts we make two comments here. First, the number of elements of the grid is restricted by practical considerations as we have seen in 5.5.4, but a reasonable number of starting conditions, say \( 10^4 \) for a four-dimensional uniform grid, should give a good indication of both quantitative and qualitative behaviour of \( G \) as we vary a parameter of the model. Second, with respect to the window defining the grid, we feel that a natural choice is to place the window around the safe potential well, extending its limits to include the nearest unstable fixed
There is not much point, however, in extending the window too far beyond the potential barrier, since by doing this we would only be increasing the proportion of trivial escaping trajectories.

For our purposes we define an Integrity Diagram as a plot of the integrity measure $G$ against a parameter of the model. More specifically, we will be interested in following values of $G$ as we increase the parameter representing the amplitude of excitation, $F$. We illustrate this useful representation of the engineering integrity of a system with examples taken from the escape equation \((3.8)\), and from the SIR equations.

Figure 5.17 shows three integrity diagrams for the escape equation \((3.8)\) with different frequencies of the excitation, $\omega = 0.55$, $\omega = 0.85$, and $\omega = 1.00$. Other parameters for this figure are given in table 5.7 below.

<table>
<thead>
<tr>
<th>Table 5.7 - Parameters for figure 5.23</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PARAMETERS OF THE MODEL</strong></td>
</tr>
<tr>
<td><strong>PARAMETERS OF THE GRID</strong></td>
</tr>
<tr>
<td>$(N_1, N_2) = (200, 200)$</td>
</tr>
<tr>
<td><strong>NUMERICAL PARAMETERS</strong></td>
</tr>
</tbody>
</table>

The most noticeable feature of figure 5.23 is the sudden and sharp fall of $G$ for $\omega = 0.85$ and $\omega = 1.00$, and the absence of such sharp reduction of integrity for the lower value of $\omega = 0.55$. This confirms our observations on the nature of the processes of safe basin loss for different values of frequency. The integrity diagram for the lower value of frequency would illustrate a case of gradual reduction and loss of safe basins, whereas the two other values correspond to situations where a sudden reduction is observed.
In figures 5.24 and 5.25 we can see integrity diagrams for the SIR equations. The first of these figures shows integrity diagrams for a variety of values of $R$, the internal resonance parameter, including $R = +\infty$, which is in fact the symmetric single-well model of roll motion (4.11). All the diagrams of this figure are for the near-resonance condition of $\omega = 0.85$. Other parameters can be seen in tables 5.8 and 5.9, for figure 5.24 and 5.25, respectively.

We can see from figure 5.24 that variations in $R$ do not seem to affect the character of the underlying process of reduction of safe basin: for all values of $R$ a sudden reduction of safe basin can be seen. This observation seems to indicate that the nature of the process of loss of safe basin is not so much related to internal resonance as it is related to the direct roll resonance (the resonance in roll is kept for all these integrity diagrams by our keeping of a constant $\omega$). Note that the shift of the integrity curve for $R = +\infty$ is only due to different grids being used for this 1-DOF system and the 2-DOF system with finite $R$ (the grid for the $R = +\infty$ case is the same as for the other values of $R$, apart from $(N_1, N_2) = (10, 10)$).
Chapter Five: Dynamics of Ship Motions

Table 5.8 - Parameters for figure 5.24

<table>
<thead>
<tr>
<th>Parameters of the Model</th>
<th>$\omega = 0.85, \zeta = 0.05, R$ as shown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters of the Grid</td>
<td>$(x_{\text{ref}}, x_{\text{sup}}, \dot{x}<em>{\text{ref}}, \dot{x}</em>{\text{sup}}, y_{\text{ref}}, y_{\text{sup}}, \dot{y}<em>{\text{ref}}, \dot{y}</em>{\text{sup}}) = (-1.1, 1.1, -1.0, 1.0, -0.6, 0.6, -0.5, 0.5)$</td>
</tr>
<tr>
<td>$\mathbf{N} = (N_1, N_2, N_3, N_4) = (10, 10, 10, 10)$</td>
<td></td>
</tr>
<tr>
<td>Numerical Parameters</td>
<td>$p = \frac{T}{80}, m = 8, d_{\text{sc}} = 1.2$</td>
</tr>
</tbody>
</table>

Figure 5.24 - Integrity diagrams for the SIR equations with varying $R$

Figure 5.25 shows two integrity diagrams for the SIR equations with $R = 1.7$. One of them is for the internal resonance case $\omega = 0.85$, the other being for an off-resonance condition of $\omega = 0.55$. We have used the same grid of points for the two conditions. As we would now expect, a sharp, sudden reduction of $G$ is observed for $\omega = 0.85$. Also shown along with this integrity curve are the three points, A, B, and C corresponding to the values of $F$ for which figure 5.17 shows cross-sections of safe basins, 0.06, 0.24, and 0.30, respectively. Once again we can notice the sudden reduction of $G$ being directly related to complex safe basins that are eroded by fractal-like incursive striations. The diagram for
$\omega = 0.55$ shows that for that frequency loss of safe basin starts earlier in terms of $F$ but the reduction is more gradual. We shall see in the next chapter how these differences can affect our choice of a method to assess the stability of roll motions.

<table>
<thead>
<tr>
<th>Table 5.9 - Parameters for figure 5.25</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PARAMETERS OF THE MODEL</strong></td>
</tr>
<tr>
<td><strong>PARAMETERS OF THE GRID</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>NUMERICAL PARAMETERS</strong></td>
</tr>
</tbody>
</table>

Figure 5.25 - Integrity diagram for the SIR equations with $R = 1.7$
6. THE METHOD OF A COARSE GRID-OF-STARTS

6.1 PRELIMINARIES

In this chapter we propose a method of assessing the dynamical stability of ship motions through the use of a grid of starting points suitably placed in the phase space of the system. We first investigated this method in Bishop and de Souza (1993), although the idea can be seen as a possible extension of the work by Thompson and colleagues on the Transient Capsize Diagram, see for example Rainey et al (1990). The main purpose of the method is to build boundaries of safe motion in the space of control parameters of a system. We illustrate the method with the construction of diagrams depicting maximum amplitudes of excitation for a range of frequencies of the forcing. The method can, of course, be used to build similar pictures involving other parameters of a system including damping factors, restoring function characteristics (GZ-Curves), and various forcing functions.

In the next section we review the general theoretical results that give support to the idea we propose. We try and frame this brief exposition of theory within the wider context of nonlinear mechanical oscillators whose dynamics are governed by underlying potential wells. Next, we offer a brief justification of the method in heuristic terms. Here we collect our arguments based not only on theoretical facts but also on the numerical studies of chapter five.

The last two sections of this chapter contain results of the use of the method of a coarse grid-of-starts to build boundaries of safe motion in the control space of the system. We present results for a variety of systems, including the escape equation (3.8), the symmetric single-well Duffing oscillator (4.11), and the SIR equations. We also perform a comparison of the method we propose with results of similar studies found in the literature.
We note that, although the method we propose here was originally envisaged as a means of determining boundaries of safe ship roll motion, its potential application in similar problems is obvious. This versatility is a result of theoretical knowledge of some basic facts concerning the dynamics in potential wells, but it is also suggested by numerical investigation carried out for various nonlinear systems, some of which are included here for comparison.

### 6.2 Dynamics in Potential Wells

In this section we outline the framework within which we shall study the loss of stability of transient ship motions. Here we consider the problem of ship motions from the viewpoint of general mechanical oscillators described by systems of the form of equation (6.1) below:

\[ \ddot{x} + g(\dot{x}) + f(x) = F(t) \]  

Here \( x \) is a \( n \)-dimensional vector of space coordinates describing the motion of the system, and a dot denotes differentiation with respect to time \( t \). The functions \( f(x) \) and \( g(\dot{x}) \) are the restoring and damping functions, respectively. Both \( f(x) \) and \( g(\dot{x}) \) are assumed to be \( n \)-dimensional vectors of \( C^1 \)-functions 
\[
\begin{align*}
    f(x) &= [f_1(x), f_2(x), \ldots, f_n(x)]^T \\
    g(\dot{x}) &= [g_1(\dot{x}), g_2(\dot{x}), \ldots, g_n(\dot{x})]^T.
\end{align*}
\]
We shall also be assuming that the restoring function has a finite number of zeroes \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \), and that \( f(x) \) has non-zero derivatives at each one of them i.e.:

\[
\left. \frac{\partial f(x)}{\partial x_j} \right|_{x=x^*} \neq 0 \quad i, j = 1, 2, \ldots, n
\]

The forcing term \( F(t) \) is a \( n \)-dimensional vector of smooth \( T \)-periodic functions of time \( t \): \( F(t) = F(t + T) \) for some \( T \) thus called the period of the forcing. More specifically, we consider \( F(t) \) as given by a simple harmonic function:
\[ F(t) = F \sin \left( \frac{2\pi}{T} t \right) = F \sin(\omega t) \]

6.2

We shall be working here with systems characterised by one or two space coordinates, i.e. \( n = 1 \) or \( n = 2 \). The number of space coordinates necessary to characterise the dynamics of the system is often referred to as the number of degrees of freedom of the system.

### 6.2.1 ONE DEGREE OF FREEDOM SYSTEMS

When \( x \) is a scalar, i.e. \( n = 1 \), equation (6.1) is equivalent to a set of two first order differential equations:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -g(\dot{x}) - f(x) + F(t)
\end{align*}
\]

6.3

In the absence of damping and forcing, i.e. \( g(\dot{x}) = F(t) = 0 \), the system defined by equation (6.1) reduces to

\[ \ddot{x} + f(x) = 0 \]

6.4

or equivalently:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -f(x)
\end{align*}
\]

6.5

Systems defined by equation (6.4) are completely integrable Hamiltonian systems. For such systems there is a Hamiltonian function

\[ H(x, y) = \frac{y^2}{2} + \int_{0}^{x} f(s) \, ds \]

such that equation (6.5) is equivalent to:

\[
\begin{align*}
\frac{dx}{dt} &= \dot{x} = \frac{\partial H}{\partial y} \\
\frac{dy}{dt} &= \dot{y} = -\frac{\partial H}{\partial x}
\end{align*}
\]

6.6
Clearly, the Hamiltonian \( H(x,y) \) defined above can be identified to the \textit{total mechanical energy} of the system given by the sum of kinetic and potential terms, 
\[
T(y) = \frac{y^2}{2}, \quad \text{and} \quad V(x) = \int_0^x f(s) \, ds,
\]
respectively:
\[
H(x,y) = T(y) + V(x) \quad 6.7
\]

We can also see from equation (6.6) that \( \frac{dH}{dt} = 0 \), and therefore solution curves of (6.5) are restricted to level curves defined by \( H(x,y) = \text{constant} \). These constraints place limitations on the variety of qualitatively different types of final motion the system can display. In particular, it is known that for any system described by equation (6.4) the only possible non-wandering sets are fixed points, closed orbits, and the unions of fixed points and the trajectories connecting them, Guckenheimer and Holmes (1983).

In addition to what we have seen above, systems given by equation (6.4) are \textit{conservative}, since their vector fields \(-f(x)\) can always be expressed as the gradient of a \( C^2 \) potential function \( V(x) \):
\[
\dot{x}(t) = -f(x) = -\frac{\partial V(x)}{\partial x} = -\frac{\partial}{\partial x} \left[ \int_0^x f(s) \, ds \right] \quad 6.8
\]

The dynamical properties of conservative planar systems have been extensively studied, and some general results regarding qualitative features of their solutions have been known for some time, see for example Andronov \textit{et al} (1966), and Hirsch and Smale (1974). For our present study, the important general results for conservative planar systems are the following (see Hale and Koçak (1991) for a proof):
Suppose that $\bar{x}$ is a critical point of a potential function $V(x)$ such that $\bar{x} = (\bar{x},0)$ is an equilibrium point of the conservative vector field given by equation (6.5). Then:

(i) $\bar{x}$ is a saddle point if $\frac{d^2V(x)}{dx^2} \bigg|_{x=\bar{x}} < 0$

(ii) $\bar{x}$ is a center if $\frac{d^2V(x)}{dx^2} \bigg|_{x=\bar{x}} > 0$

Note that the case $\frac{d^2V(x)}{dx^2} = 0$ is ruled out by the assumption that $f(x)$ has non-zero derivatives at any of its zeroes.

Moreover, if the potential function $V(x)$ is generic\(^1\), then there is a neighbourhood of $V(x)$ in the space of $C^2$ functions such that the conservative vector field defined by any potential function in this neighbourhood is topologically equivalent to the one defined by $V(x)$.

It is our primary interest to consider cases in which a non-degenerate minimum of $V(x)$ corresponds to a desirable condition for our system. We are therefore particularly interested in systems whose underlying conservative counterparts display a center at the origin. Under conditions of genericity of the governing potential function, the result above guarantees that if a given conservative planar system has a center at the origin, then systems governed by slightly different $C^2$ potential functions still have a center at the origin. This fact is of practical relevance since small modelling errors are unavoidable.

It is important to observe here that the set of points belonging to periodic orbits around the origin typically constitutes a set of non-zero measure in phase space,

\(^1\)A potential function $V(x)$ is said to be generic if it satisfies the following conditions:

(i) there are finitely many critical points of $V(x)$;

(ii) each critical point of $V(x)$ is nondegenerate, that is, $\frac{d^2V(x)}{dx^2} \bigg|_{x=\bar{x}} \neq 0$ for all critical points $\bar{x}$;

(iii) no two maximum values of $V(x)$ are equal;

(iv) $V(x)$ is unbounded for $x \to \pm \infty$. 
i.e., there is a region of finite size densely filled with periodic orbits around the origin. Indeed, such region will be circumscribed by the homoclinic orbit connecting the lowest adjacent maximum of $V(x)$ (also referred to as a hill-top saddle) to itself, see for example figure 6.1.

![Figure 6.1 - Orbits around a centre](image)

To consider the effects of damping we investigate the system given by:

$$\dot{x} + g(x) + f(x) = 0$$  \hspace{1cm} 6.9

If $g(x)$ satisfies the conditions specified earlier then the system is called a dissipative planar system if infinity is a source for the system: every bounded trajectory moves and stays away from infinity. Such systems possess unique global attractors, usually given by the union of (asymptotically) stable fixed points (sinks, minima of $V(x)$) and the unstable manifolds of saddle-type fixed points

---

2Note that the definition above excludes systems for which $V(x) \to -\infty$ as $x \to \pm \infty$. For such systems there are in fact unbounded trajectories, and these systems can be of practical interest. We remark that for these systems typical trajectories either tend to infinity or approach asymptotically a sink. The determination of basins of attraction of the unbounded solutions can however be difficult.
points (maxima of $V(x)$). This means that for such systems all non-trivial trajectories eventually approach a minimum of $V(x)$ and tend asymptotically to it.

Obviously, the final state of the system attracted to an arbitrary minimum of $V(x)$ may correspond to large amplitude (perhaps unacceptable) displacements. The relevant fact here is, however, that under the conditions stated above the origin (which we assume is a desirable minimum of $V(x)$) has a basin of attraction of non-zero measure in phase space. To see this we observe that points belonging to periodic orbits around the origin in the undamped case (figure 6.1) will, in the presence of positive damping, be necessarily attracted to the origin, now a sink, see figure 6.2.

![Homoclinic orbit](image)

Figure 6.2 - Orbits around a sink

We consider now the addition of a periodic forcing and we assume initially that the magnitude of the forcing is small:

$$F(t) = \varepsilon F \sin(\alpha t), \quad 0 < \varepsilon << 1$$

such that equation (6.1) could be written as:
\[ \dot{x} + g(x) + f(x) = \varepsilon F \sin(\omega t) \]  

In this case, the Averaging Theorem states that there is an autonomous averaged system whose solutions are locally diffeomorphic to the solutions of (6.10). Moreover, the theorem states that any hyperbolic fixed point of the averaged (unforced) system is close to a hyperbolic periodic orbit of (6.10), and that both are of the same stability type, see Guckenheimer and Holmes (1983). Note that the hyperbolicity of fixed points of (6.9) is a consequence of the assumption that \( f(x) \) has non-trivial derivatives at any one of the critical points of \( V(x) \). This means that under small periodic forcing the origin will no longer be a stable fixed point. The stable solutions now are small-amplitude periodic orbits whose period is in fact \( T = \frac{2\pi}{\omega} \). Likewise, the theorem asserts that unstable periodic orbits (of same period \( T \)) will develop around the previous unstable fixed points associated with maxima of \( V(x) \), the so called hill-top saddles. Indeed, the invariant manifolds of the lowest adjacent hill-top saddle will continue to play an important role. In particular, the stable manifold of the hill-top saddle will still be the separatrix between trajectories attracted to the small-amplitude solution and trajectories that will eventually settle onto some other attractor. Furthermore, given a point belonging to the stable manifold of a (hyperbolic) periodic orbit of (6.10), and a point belonging to the stable manifold of the corresponding fixed point of the autonomous averaged system, the Averaging Theorem guarantees that if these two points are close for \( t = 0 \) they will remain close for all time \( t \in [0, \infty) \). This means that the basin of attraction of the desirable sink of the unforced system will be essentially preserved as the basin of attraction of the small-amplitude solutions for the lightly driven oscillator, i.e. \( \varepsilon \) is a small parameter.

Finally we consider the case when the magnitude of the forcing is not small. Here the variety of possible behaviours is much larger, and there are few general results. Nevertheless, as we have just described, we can expect that stable fixed points of the unforced system—will be replaced by stable small-amplitude
oscillations for lightly forced systems, and that basins of attraction will be initially preserved. As the magnitude of the forcing is increased we can expect quantitative as well as qualitative changes both in the orbit structure of the system and in the organisation of basins of attraction.

Detailed studies with simple nonlinear mathematical models have demonstrated the existence of complex hierarchies of bifurcations leading the system into large-amplitude regimes, and eventually to escape, Thompson (1989), Thompson et al (1987). Among the most significant events related to the reduction of basin area of small-amplitude motions are global bifurcations associated with homoclinic and heteroclinic tangencies of stable and unstable manifolds of hill-top saddle solutions. These bifurcations affect the structure of basin boundaries which under the variation of a parameter become fractal. Soon after such qualitative changes or metamorphoses take place the safe basin of attraction is rapidly eroded by fractal incursions of the unsafe basin. A small basin may however persist around the stable fixed point\(^3\) for a range of parameter values, making results of traditional steady-state analysis unreliable, Soliman and Thompson (1992). In such cases, insight into escape mechanisms can also be achieved through the use of perturbation techniques, Szemplinska-Stupnicka (1992).

As we have just seen, the inclusion of finite damping and external forcing adds immense new complexity to the problem, \textit{even for a one degree of freedom system}. Nevertheless, in terms of visualising the dynamics, it can prove useful to retain the picture of the system as a small particle moving in a surface whose local height above a reference plane is given by \(V(x)\). Interesting systems will typically display at least one local minimum of \(V(x)\), corresponding to a stable fixed point of the unforced, undamped system. We assume that the vicinity of one of the minima of \(V(x)\) corresponds to the desirable situation for the operation of the system, so that trajectories that visit regions of the phase space far away from that minimum are to be considered unsafe.

\(^3\)This is the fixed point located along the main sequence.
6.2.2 Two degrees of freedom systems

We consider now systems composed of two coupled nonlinear mechanical oscillators. For that, we take equation (6.1) with \( n=2 \), so that \( x=(x_1,x_2) \):

\[
\begin{align*}
\ddot{x}_1 + g_1(x) + f_1(x) &= F_1 \sin(\alpha t) \\
\ddot{x}_2 + g_2(x) + f_2(x) &= F_2 \sin(\alpha t)
\end{align*}
\]

We assume that damping, restoring, and forcing functions possess the same properties listed for equation (6.1). In a similar manner as before, we start by considering the associated Hamiltonian system:

\[
\begin{align*}
\dot{x}_1 + f_1(x) &= 0 \\
\dot{x}_2 + f_2(x) &= 0
\end{align*}
\]

If a second conserved quantity functionally independent from the Hamiltonian can be identified for equation (6.12) the system is said to be completely integrable, in which case each trajectory is restricted to a two-dimensional surface in \( \mathbb{R}^4 \), and the dynamics may be relatively simple.

Most systems described by equation (6.12) are not, however, completely integrable, i.e., a second independent conserved quantity does not exist\(^4\). Of course, the Hamiltonian is a conserved quantity, and therefore trajectories are still confined to submanifolds of \( \mathbb{R}^4 \) in which \( H(x,\dot{x}) = \text{constant} \). These will typically be three-dimensional subsets of \( \mathbb{R}^4 \), and therefore the flow is in fact restricted to three dimensions. Flows in three dimensions can be very complex indeed, and a general classification of all possible types of dynamic behaviour in three-dimensional flows remains a challenge for researchers. In particular, we know that attracting sets in three dimensions include fixed points, periodic and quasi-periodic orbits, and strange attractors.

\(^4\)It can be difficult to prove the non-integrability of a system, but strong evidence may be gathered via numerical investigation.
The analogy with a particle moving in a potential well can be carried forward to higher dimensional spaces. Here we imagine the particle moving in a two-dimensional surface defined by the two space coordinates $x_1$ and $x_2$ as shown in figure 6.3.

![Figure 6.3 - Particle in a two-dimensional potential well](image)

We can still assume that the origin of our coordinates is some (non-degenerate) minimum of $V(x) = V(x_1, x_2)$ which we deem a desirable condition for the system. In the absence of any damping, the general (non-integrable) Hamiltonian system given by equation (6.12) can display very complex dynamics. In particular, small-amplitude periodic orbits do not always survive infinitesimal perturbations.

The addition of damping can simplify the dynamics considerably. For example, if damping is always positive there will be a monotonic contraction of volumes in phase space, so that final motion will be restricted to sets of zero volume, and the dynamics will typically give rise to attractors. In particular, we can easily see that for systems governed by potential wells and damping all non-trivial trajectories will be attracted to one of the local minima of $V(x)$. 
If we consider the inclusion of small periodic forcing, the Averaging Theorem guarantees that fixed points of the unforced system will be typically replaced by small-amplitude periodic oscillations around them in the forced case. The same remarks regarding invariant manifolds and their relation to basins of attraction are therefore also valid here. As a consequence, we can still expect basins of attraction of a desirable sink (usually the origin) to be essentially preserved as the basin of attraction of a small-amplitude, stable solution. Here again we keep in mind that this holds true as far as the forcing is small (rigorously speaking, infinitesimal).

Finally we consider the more general case in which the forcing is of finite magnitude. As with one-degree-of-freedom systems, the dynamics can be very complex. As the magnitude of the forcing is increased the system will usually undergo a series of quantitative and qualitative changes in behaviour that will eventually lead to trajectories escaping from the safe potential well.

One of the basic assumptions of the method we describe here is that detailed knowledge of the very complex dynamics that may precede escape is not needed for approximate determination of boundaries of safe motion in control space. The only assurance we need is that there is a finite safe basin of attraction around the desirable origin in which we can place a suitable grid of starting conditions. As we have just seen systems governed by underlying potential wells do, under quite non-restrictive conditions, fulfil this basic condition.

6.3 Heuristics and Description of the Method

We summarise here the basic ideas behind the proposed method of a coarse grids-of-starts for the identification of unsafe regions of operation in parameter space. We draw on existing knowledge of the details of nonlinear dynamical behaviour, much of it has been studied in the previous chapters, to
construct a sensible picture of the main phenomena governing large-amplitude motions and capsize.

We start by recalling that our purpose is to identify regions in the control space of a system where motions are safe. The term safe is meant here in the context of safe trajectories as defined in chapter five, i.e., transient trajectories that remain inside the safe region of the system.

The method is based upon the assumption, corroborated both by the theoretical arguments of the previous section and by the numerical studies of previous chapters, that safe basins are progressively reduced as we vary parameters of the system. In particular, we have seen that as we increase the magnitude of the forcing complex sequences of events determine the reduction or loss of safe basins. We have also seen that these processes of loss can be associated with specific underlying mechanisms such as jumps to resonance, gradual shift, fractal erosion, and gradual erosion.

Clearly, if computational effort is not an issue, all we have to do is to construct pictures of safe basins, or alternatively integrity diagrams, and follow the reduction of safe basins as we vary parameters. However, what we argue here is that, if our main concern is to distinguish safe combinations of parameter values from unsafe ones, we do not have to build complete pictures of safe basin evolution. Based on the knowledge of how safe basins are reduced and lost, we can argue that if we test a small number of points suitable spread over the initial safe basin of the system we should be able to identify, for example, maximum values of amplitude of forcing at which the system still retains acceptable engineering integrity. This small number of points comprise what we call a coarse grid-of-starts.

Let us see why a coarse grid-of-starts should identify situations of reduced engineering integrity. Again we employ the magnitude of forcing as an example. Let us assume that the origin of the phase space is made to coincide with a local
minimum of potential energy. The origin is a stable fixed point of the unforced system, and we are interested in following the evolution of safe basins of motions inside the potential well around the origin. We assume that for $F = F_0 = 0$ the system has a safe basin of size $V_0$, its initial safe basin. The size of the safe basin can be defined as the volume of the safe region, see chapter five for a definition of safe region. Note that we can see the integrity measure $G$ (also defined in the previous chapter) as a numerical approximation of $V$. Now as $F$ is increased from zero the size, shape, and character (fractal or smooth) of the safe basin may change. In particular, we can expect that as $F$ is increased fewer points will lead to safe trajectories, and therefore $V$ will eventually diminish. This is certainly true of large values of $F$.

Let us treat each process of safe basin reduction separately to see that a coarse grid-of-starts can detect dangerous loss of engineering integrity. See section 5.6.2 for a description of processes of reduction and loss of safe basin.

If the safe basin experiences a catastrophic reduction, any point inside the safe basin previous to the catastrophic loss will indicate loss of engineering integrity. The same applies to a sudden reduction of safe basin such as that associated with the mechanism of fractal erosion. Because the loss is sudden, and particularly because of the way in which fractal incursive structures sweep quickly across the bulk of the safe basin, any point inside the previously safe basin will indicate capsize within a very narrow range of $F$-values.

There may be instances when points located in different regions of the safe basin experience escape for $F$-values that are not necessarily close. This is the case of gradual reduction of safe basin associated with a gradual shift of the safe basin and/or with its gradual erosion. Here we need to test points belonging to each of these regions to be sure of detecting the loss of safe basin early. For example, loss of safe basin can occur from the periphery of the safe basin (near its initial boundary) towards its interior. This is the case of the escape equation below

---

5This decrease in $V$ is not necessarily monotonic, see Soliman and Thompson (1989).
resonance with $\omega = 0.55$, see figure 5.19. Here the interior of the safe basin is affected by incursive fractal striations well after points in the periphery of the basin have become unsafe. The same phenomenon can be observed for the SIR equations, see figure 5.17.

A straightforward procedure to place starting points in different regions of the initial safe basin is to lay a uniform grid of points over it, i.e. a grid of equally spaced points, see for example figure 6.4. A complete grid of points will have the same dimension of the safe basin or of the phase space of the system. This, as we have seen before in this work, can lead to time-consuming simulations of a large number of points. If the dimension of a grid is smaller than that of the phase space of the system we say the grid is reduced.

We propose a simplified procedure in which a reduced grid is placed spanning only the displacement directions (roll angle, and heave displacement, for example, for the SIR equations). We assume that it is unlikely that the reduction of safe basin will occur only along privileged directions of phase space. In other words, we assume that if the size of the safe basin is decreasing then its cross-section with the plane formed by the displacement variables will typically decrease as well. Of course, we could choose any other direction in phase space based on this argument. Our choice of the displacement variables is dictated exclusively by practical considerations of future experiments that may be used to validate these results. In experiments with physical models of ships or other marine vessels adjusting simultaneously the initial position and velocity of the model can be quite a laborious task, and a major source of experimental error.

The formal definition of a coarse grid-of-starts is similar to the definition of a grid that we have introduced in section 5.5.3, with the difference that the only vectors used are those associated with displacement variables (or, more generally, variables associated with the potential energy function). Therefore, a

---

6We shall see that good results can be obtained with grids even further reduced to include only the roll variable direction.
coarse grid-of-starts in this work is always a reduced grid, i.e., a grid of dimension inferior to that of the phase space. Another important defining feature of a coarse grid-of-starts is the small number of elements comprising the grid. We shall see that a very small number is sufficient for the cases we consider here.

Once a coarse grid-of-starts has been placed for the system the generation of boundaries of safe transient motion consists of sweeping a range of parameter values while checking if escape has occurred. Again taking the example of frequency and amplitude of excitation (as we shall be doing in the remainder of this study) we fix one of the control parameters, the frequency of excitation $\omega$, and increase $F$, the amplitude of excitation, until escape occurs within a pre-specified number of cycles of the forcing (usually eight or ten). For each set of values of the control parameters we start the system at each of the points comprising the grid, and if escape occurs for any of the points we mark the value of $(F, \omega)$ as belonging to the boundary of safe motion for the system.
Clearly, the precision with which we are able to estimate the values of \((F, \omega)\) comprising the boundaries of safe motion is related to the numerical steps we take when sweeping those control parameters. One procedure that speeds up computations considerably is to employ a Newton-Raphson-type algorithm in which we start with a fairly coarse step, and progressively halve it until we achieve a desired precision. Unfortunately the fractal character of basin boundaries (see for example figure 5.14) makes this procedure rather unreliable unless we use a fine starting step, in which case the advantage of the halving procedure is largely reduced. Nevertheless, in terms of computation times the halving procedure still retains some comparative advantage over a fixed-step method, and for this reason we have chosen to employ it, adopting a fine initial step.

6.4 SENSITIVITY STUDY

The influence of the main parameters and features defining a coarse grid-of-starts can only be assessed through a sensitivity study, which must necessarily involve specific models. For this first sensitivity study we have selected four different 1-DOF models, and one 2-DOF model.

6.4.1 SELECTION OF DYNAMICAL SYSTEMS

With the purpose of demonstrating the applicability of the method we propose here, we have selected from the recent international literature four different one-degree-of-freedom systems, and a two-degree-of-freedom system. They are all periodically-driven nonlinear oscillators in which failure can be viewed as the escape from a potential well. The basic criterion used in the selection of these specific systems was the availability of diagrams in the control plane depicting safe (non-escaping) and unsafe regions of operation. These
systems will be used for both the sensitivity study, and for the comparison with other methods that we shall perform in the next section.

The dynamical systems considered here are:

The escape equation (3.8):

\[ \ddot{x} + \beta \dot{x} + x - x^2 = F \sin(\omega t + \delta) \] 6.13

The symmetric single well roll model (4.11) (a Duffing equation):

\[ \ddot{x} + \beta \dot{x} + x - x^3 = F \sin(\omega t + \delta) \] 6.14

An asymmetric double-well model with quadratic and cubic nonlinearities from Virgin et al (1992):

\[ \ddot{x} + \beta \dot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_4 x^3 = F \sin(\omega t + \delta) \] 6.15

A symmetric single-well roll model with cubic and fifth order nonlinearities from Soliman and Thompson (1991):

\[ \ddot{x} + \beta_1 \dot{x} + \beta_2 |\dot{x}| \dot{x} + c_1 x + c_3 x^3 + c_5 x^5 = F \sin(\omega t) \] 6.16

A symmetric two-dimensional potential well (2-DOF system) from Virgin et al (1992):

\[ \ddot{x}_1 + \beta \dot{x}_1 + x_1 - 4x_1 x_2^2 = F \sin(\alpha x) \] 6.17

\[ \ddot{x}_2 + \beta \dot{x}_2 + 2.25x_2 - 4x_1^2 x_2 = F \sin(\alpha x) \]

Figures 6.5-6.9 depict the potential wells associated with the systems above.
Chapter Six: Coarse Grids-of-Starts

Figure 6.5 - Potential well for equation (6.13)

Figure 6.6 - Potential well for equation (6.14)
Figure 6.7 - Potential well for equation (6.15)

Figure 6.8 - Potential well for equation (6.16)
6.4.2 Results of sensitivity study

We start with a sensitivity study involving the main parameters defining a grid-of-starts: grid length, and grid size. For each one of the systems represented above by equations (6.13)-(6.17) we perform the numerical determination of boundaries of safe motion under transient conditions, using three different grids. The influence of grid size is assessed by including a one-point grid as well as two two-point grids; the influence of grid length is evaluated by taking different lengths for the two two-point grids.

Results for this sensitivity study are shown in figures 6.10-6.13, in which boundaries of safe motion for the one-degree-of-freedom models are depicted. Figures 6.14 and 6.15 aim at comparing also the effect of employing one- or two-dimensional grids for the study of two-degree-of-freedom models.
Figure 6.10 - Sensitivity study: equation (6.13)

As far as absolute minima of excitation amplitude for escape\(^7\) are concerned little distinction can be made between the simplest, fastest one-point grids and the more demanding three-point grids. In fact, with the exception of the escape equation (6.13), the minimum values of \(F_{esc}\) predicted by the three grids investigated is essentially the same. Invariably, such minima correspond to near-resonance conditions. This conclusion is supported by detailed investigations on the process of basin erosion that characterises the loss of engineering integrity in near-resonance conditions. Basically, those studies show that, following global events in which basin boundaries become fractal, the bulk of the safe region of phase space is *swiftly* eroded by fractal incursions. This would indicate that one-point and many-points grids should give similar estimates for the amplitude of excitation for escape, provided that all starting points are situated inside the initial safe region.

\(^7\)The condition of escape under minimum amplitude of the excitation is sometimes referred to as *optimal escape*. 
Figure 6.11 - Sensitivity study: equation (6.14)

Figure 6.12 - Sensitivity study: equation (6.15)
The same, however, does not hold true in far-from-resonance conditions. Figures 6.10-6.13 show that longer grids yield considerably lower estimates for escape.
values. As expected, the longer the grids used are, the lower are the estimated escape values. In some cases, as with equations (6.13) and (6.15), escape values predicted by longer grids can be less than half of those predicted by a one-point grid. This signals for the necessity of longer grids if an accurate global picture of safe boundaries is to be achieved through the method proposed here.

The comparison between one- and two-dimensional grids for equation (6.17) reveals a somewhat similar pattern. Good agreement is observed between all grids near resonance. But at far-from-resonance parameter values two-dimensional grids can indicate significantly lower escape values.

In view of the results described above, we have selected the longer grids for comparison with the international literature. For equation (6.17) we have selected the longer, two-dimensional grid.
6.4.3 COMPARISON WITH THE LITERATURE

Table 6.1 summarises the main numerical features of the data we are using here for comparison. In our own simulations we have employed the same parameter values, but we have used different starting points, maximum transient duration, and escape criteria.

<table>
<thead>
<tr>
<th>FEATURE → EQUATION</th>
<th>REFERENCE</th>
<th>PARAMETERS</th>
<th>STARTING POINTS</th>
<th>MAXIMUM TRANSIENT DURATION</th>
<th>ESCAPE CRITERION</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6.13)</td>
<td>Thompson (1989a)</td>
<td>β = 0.1</td>
<td>x = 0</td>
<td>32 T</td>
<td>x ≥ 20</td>
</tr>
<tr>
<td>(6.13)</td>
<td>Virgin et al (1992)</td>
<td>β = 0.1</td>
<td>x = 0</td>
<td>30 T</td>
<td>x &gt; 1</td>
</tr>
<tr>
<td>(6.14)</td>
<td>Virgin et al (1992)</td>
<td>β = 0.1</td>
<td>x = 0</td>
<td>30 T</td>
<td></td>
</tr>
<tr>
<td>(6.14)</td>
<td>Kan (1992)</td>
<td>β = 0.0455</td>
<td>x = 0</td>
<td>20 T</td>
<td></td>
</tr>
<tr>
<td>(6.15)</td>
<td>Virgin et al (1992)</td>
<td>β = 0.1, ω₀ = 139.93, α₂ = 4132.7, α₃ = 194.82</td>
<td>x = 0</td>
<td>30 T</td>
<td>x &lt; -7.143</td>
</tr>
<tr>
<td>(6.16)</td>
<td>Soliman and Thompson (1991)</td>
<td>β₁ = 0.0555, β₂ = 0.1659, c₁ = 0.2227, c₂ = -0.0694, c₃ = -0.0131</td>
<td>x = 0</td>
<td>16 T</td>
<td></td>
</tr>
<tr>
<td>(6.17)</td>
<td>Virgin et al (1992)</td>
<td>β = 0.1, x₁ = x₂ = 0</td>
<td>30 T</td>
<td>√x₁² + x₂² &gt; 0.901</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1 - Main features of selected systems

We have used a maximum transient duration of 10 cycles of the forcing in all computations. Our escape criterion is based on the distance to the nearest local maximum of potential energy: the system is deemed to have escaped if the magnitude of the displacement variable \(x\) exceeds 1.2 of that distance. For asymmetric potential wells, namely equations (6.13) and (6.15), only displacements in the direction of escape are considered. In the case of equation (6.17) with its two-dimensional potential well, we have used a similar criterion:
escape is assumed to have occurred if the distance in the phase space of
displacement variables between a point in the system's trajectory and the origin
\[ d(x) = \sqrt{x_1^2 + x_2^2} \] exceeds 1.2 of the distance between the origin and the nearest
unstable fixed point of the unforced system. Incidentally, equation (6.17)
displays a symmetric potential well, and the distance between the origin and each
of the four saddle-like unstable fixed points is the same.

With respect to starting points we have used three-point grids symmetrically
placed around the origin. The length of the grids is always 0.65 of the distance
between the origin and the nearest unstable fixed point of the unforced system.
For one-dimensional systems, that point is just the lowest local maximum of
potential energy adjacent to the origin. For equation (6.17) the length of the grid
in each direction is taken as 0.65 of the distance \textit{along that direction} between
the origin and the nearest saddle-like fixed point of the unforced system.

It should be noted that actual starting points are placed at the centre of each
rectangle defined by the grid, which means for example that extreme starting
points are placed at less than half the total length of the grid in each direction.
For a three-point grid extreme points are actually placed at 1/3 of that distance,
see figure 6.4.

Figures 6.16-6.20 show boundaries of safe motion obtained from the recent
international literature, together with results from our own simulations for the
same systems. The first point to be noted is the general agreement between those
results. Boundaries follow approximately similar patterns, with the possible
exception of equation (6.15). Again, \textit{global} minima of amplitude of excitation
for escape are remarkably similar, especially considering the differences in
starting points, maximum transient duration, and escape criteria between our
studies and the ones selected for comparison. Once again, this can be interpreted
as evidence for the rapid erosion of safe basins of attraction that seems to
dominate the loss of integrity under increasing amplitude of forcing \textit{in near­}
resonance conditions.
But as far as a complete picture of safe motion is concerned, i.e. the whole boundary of safe motion covering the entire range of frequencies, we see once
more that major differences can be obtained with the use of longer, larger grids. The results from one-point grids extracted from the literature tend to

![Graph 1](image1.png)

Figure 6.18 - Boundaries of safe motion: equation (6.15)

![Graph 2](image2.png)

Figure 6.19 - Boundaries of safe motion: equation (6.16)

overestimate amplitudes of excitation necessary for escape. This fact is reflected on higher boundaries of safe motion, and is particularly noticeable at the
extremes of frequencies covered by the numerical experiments. In most cases amplitudes of excitation leading to escape from one-point grids can be as much as twice the value predicted by our three-point grids, see figures 6.16-6.20. It is interesting to note that our maximum duration of transients is shorter than in any of the other experiments. Our escape criterion is also more stringent than those used in the other experiments, with the exception of the results from Kan (1992) for equation (6.14) and Thompson (1989a) for equation (6.13).

The fact that significantly lower values of amplitude for escape can be obtained if we use starting points other than the minimum of the potential well signals to a possibly different process of safe basin erosion that may be operating at far-from-resonance conditions. This is a consequence of the fact that if a swift basin erosion process is in operation little difference is to found between escape predictions from any starting point inside the original (i.e. unforced) safe basin.
6.5 BOUNDARIES OF SAFE MOTION FOR THE SIR MODEL

We conclude this chapter on the application of the method of a coarse grid-of-starts with an investigation of boundaries of safe motion for the SIR model introduced in chapter three, see equation (3.35). This system offers a further example of the use of this method in 2-DOF oscillators, but we feel that some peculiarities of a 2-DOF model of coupled heave-roll motions deserve separate treatment.

We recall that we are considering here coarse grids-of-starts in which the initial velocity of the model is always set to zero. Furthermore, we note that experiments with reduced-scale physical models of vessels would be much facilitated if the starting points to be tested were all located along the static balance curve of the vessel. We recall from chapter three that this curve is just the solution of the \( \frac{\partial v}{\partial y} = 0 \) equation, where \( v \) is the potential energy function, see equation (3.45). In other words, we would like to test initial conditions of static balance in the heave direction. This would reduce the preparation of the model's initial position to just setting the appropriate roll angle, leaving the model free to find its natural floating position. For the SIR equations, the static balance curve is given by \( y = \frac{x^2}{2} \), where we retain the nomenclature of chapter three in which \( x \) is the roll angle, and \( y \) is the heave displacement.

A grid of starting points located on the static balance curve of the vessel is still a one-dimensional grid, although both \( x \) and \( y \) vary in the grid. We would like to assess the possible effect on boundaries of safe motion of using a two-dimensional grid spread over the \( x-y \) plane. For that we consider uniform two-dimensional grids whose extreme points are located on the same level curve of potential energy as the (extreme points of the) corresponding one-dimensional grid. Clearly, there is an infinity of such grids; we arbitrarily choose grids such that their maximum \( x \)-value is half that of the corresponding one-dimensional grid. Since escape can occur towards either positive or negative \( x \) we employ
grids symmetric with respect to the $y$ axis. Note that these two-dimensional grids not necessarily include the origin as one of their points.

Figure 6.21 illustrates the ideas just described, where because of symmetry we show only half of the grids. The dotted curve is the vertical static balance curve on which we place starting points (represented by black dots) for one-dimensional grids. The closed curve passing through the extreme points of the grids is a level curve of potential energy. Note that the maximum potential energy of points in both one- and two-dimensional grids is therefore the same.

Figure 6.21 - One- and two-dimensional grids for the SIR model

One- and two-dimensional grids

Figure 6.22 shows boundaries of safe motion for the SIR model derived from four different grids. We have used two one-dimensional (1D), and two two-dimensional (2D) grids. For each type of grid we see results of two different
sizes of grid: 0.15 and 0.60. Here we prefer to measure the size of the grid by the potential energy of its extreme points, and we express this number as a fraction of the potential energy of the two hill-top saddles.

The 1D grids are each made of fifteen points uniformly distributed (in the x-axis) along the vertical static balance curve for this system. The 2D grids are made of 225 points (15 by 15) equally spaced in both x and y ordinates. As we have described above, the corresponding (i.e. same size) grids are such that the maximum potential energy of their points is the same. The maximum x ordinate (in modulus) of 2D grids is half that of the corresponding 1D grid.

![Figure 6.22 - Boundaries of safe motion for the SIR model: 1D and 2D grids](image)

Inspection of figure 6.22 reveals that boundaries of safe motion estimated with 1D grids placed along the vertical static balance curve are very close to corresponding boundaries generated from 2D grids. It is interesting to observe that close to the two resonance regions (around \( \omega = 0.85 \) we have the main
internal resonance, and around \( \omega = 1.0 \) we have the main direct roll resonance) all grids, even the smaller ones, give essentially the same results. This confirms our conjectures about the character of the process of basin erosion (sudden fractal erosion) that operates in these conditions, see chapter five. On the other hand, in off-resonance conditions, and perhaps particularly below resonance, significant differences can be seen between the results of different grids. Even here, however, differences between 1D grids and their 2D counterparts are small. Differences due to different sizes of grids, however, are significant and warrant further study.

*Effect of grid size*

Having established that 1D grids produce results similar to those of more demanding 2D grids, we try to evaluate the influence of grid size. We examine grids of six different sizes, varying from zero (origin only) to 0.75 in regular steps of 0.15. Results are shown in figure 6.23.
The curves in figure 6.23 show once more that within resonance regions all grids yield similar results. In particular, the results from a small grid of size 0.15 are almost indiscernible from results of the largest grid near resonance. But here again it is the off-resonance conditions that seem to require larger grids for their adequate inspection. It may be useful to compare the behaviour of boundaries of safe motion for the SIR equations in the low-frequency range, say $0.55 < \omega < 0.70$, with the corresponding low-frequency range results for the escape equation, see figure 6.10. In both of these cases pictures of safe basin erosion indicate a gradual loss of safe basin volume, see figures 5.20 and 5.22. This similarity of behaviour reinforces the ideas put forward in chapter five that macroscopically different processes or mechanisms of loss of safe basin operate here.

Based on the results discussed above, we select a grid size of 0.60 for further studies on the influence of grid density.

**Effect of grid density**

To assess the possible effect of grid density we compare grids comprised of 1, 3, 5, and 15 points, where we retain the zero-sized, 1-point grid for reference. With the obvious exception of the 1-point grid, all other grids have the same size of 0.60. The results shown in figure 6.24 leave little doubt that the density of the grid has a very small effect on the resulting boundaries of safe motion. In fact, with the possible exception of a narrow range at the largest frequencies in figure 6.24, the curves for 3, 5, and 15 points are practically the same.

This is a positive result as far as the practical use of the method we investigate here is concerned. First, because coarser grids mean less demanding computations and, more importantly perhaps, easier and cheaper experiments
Chapter Six: Coarse Grids-of-Starts

with physical models. Secondly, the virtual coincidence of boundaries estimated with 3- or 15-point grids is well supported by our arguments concerning the way in which safe basins can be eroded. Following those ideas, it should be sufficient to inspect the safety of a small number of points, as long as they belong to well-separated regions of the original safe basin. The results of this sensitivity study seem to confirm this idea.

Figure 6.24 - Boundaries of safe motion for the SIR model: effect of grid density
7 CONCLUSIONS

The problem of large-amplitude ship motions and their relevance to ship dynamic stability is still a challenging one. The theory of ship motions historically has already achieved a good level of development, and has recently received considerable thrust from additional numerical methods. As far as the dynamics of ship motions are concerned, however, a similar degree of understanding is perhaps still lacking. We believe that the integration of modern geometrical methods of nonlinear dynamics can prove useful in bridging the present gap between the knowledge of the equations of motion and an understanding of how the system behaves.

In this work we have made an attempt to further our knowledge of the dynamics of large-amplitude ship motions and their connection with ship capsize. We have initially surveyed the theory of ship motions in the context of ship stability and capsize, and proceeded to describe briefly the currently accepted international stability criteria, as well as some of the more recent research efforts in the field of ship stability. Our purpose during this brief exposition of stability criteria and mathematical modelling of ship motions has been not so much of contributing to this highly specialised field, but mainly to put our own research effort into proper perspective. We have, nevertheless, derived our own mathematical model of coupled heave-roll motions in regular waves, in which both internal and direct roll resonances can be present.

Having established mathematical models for our investigation, we have then turned our attention to the theory of nonlinear dynamics. Here we have tried to produce a concise yet useful description of the main topics of interest carefully selected from the already vast domain of this young field of applied mathematics. We have covered the main results of steady-state dynamics of one- and two-degree-of-freedom oscillators considering scenarios of increasing complexity from small-amplitude, unforced, undamped motions to large-amplitude periodically driven oscillations. Still within the realm of steady-state dynamics,
we have summarised some of the main bifurcational phenomena associated with the passage from small-amplitude motions to large-amplitude ones, and eventually capsize.

Apart from describing some of the relevant theoretical results of nonlinear dynamics regarding models of ship motion, we have here attempted to familiarise ourselves with the analytical-numerical approach to nonlinear dynamics that characterises the application of recent geometrical methods to this field. In this context we have investigated a variety of numerical procedures that can be used to build a sensible picture of the main dynamical phenomena governing the behaviour of a system. These methods have included building quasi-steady-state time series of key dynamical variables, steady-state Poincaré sections, and an attractor-following technique to generate bifurcation diagrams. Special attention has been paid to basins of attraction. These objects are of fundamental relevance to the engineering integrity of a system, since they give an indication of how much the system can be disturbed, and still retain the same pattern of dynamical behaviour.

In the last part of our investigation on analytical and numerical methods of nonlinear dynamics we have concentrated on transient dynamics. We believe that this is an area in which considerable theoretical development is perhaps still needed. The problem of ship capsize is a good example of the potential for applications of transient dynamics. A mathematical model of ship motions in which the effect of sea waves is represented by periodic excitation can only be seen, from a practical point of view, as a short-term representation. More importantly perhaps, ship capsize is an essentially transient phenomenon. The tools for investigation here are basically numerical methods, and indeed rigorous theoretical results linking transient and steady-state features of a system's behaviour are few and far between. Our numerical experiments have, however, allowed us to identify an interesting phenomenon of escape suppression under conditions of internal resonance for the SIR model. A preliminary explanation of
this phenomenon was offered in terms of the energy exchange between heave and roll motions.

The approach and specific methods described above have then been used to construct detailed pictures of dynamical behaviour for various mathematical models of ship motion including Thompson's escape equation, a symmetric single-well Duffing-type model of ship roll, and our 2-DOF model of coupled heave-roll motions, the SIR model. Both steady-state and transient dynamical features have been investigated with the exclusive aid of a variety of computer programs we have developed during this work. Our main findings here related to two topics. First, we have produced a study of the bifurcational behaviour of the SIR model, in which typical routes to large-amplitude motions were analysed. Secondly, transient and steady-state basins of attraction of the SIR model have been studied, with particular attention to the various processes of loss of safe basin (basin erosion). This study, coupled with similar results for other systems (for example the escape equation, and various Duffing-type oscillators) investigated both in this work and in other works suggested the formulation of a new procedure to assess the stability of ship motions.

The description and justification of the procedure mentioned above comprises the last part of our work. The purpose of what we called the method of a coarse grid-of-starts is to build contours of safe transient motion for a system in the control plane (or space) spanned by any of its parameters. We have illustrated the use of the method to construct boundaries of safe operation in the control plane defined by the excitation parameters of frequency and amplitude. In essence the method is based on the assumption that safe basins are progressively eroded following one or more basic mechanisms studied in chapter five. These comprise catastrophic, sudden, and gradual reductions of safe basin volume. If the safe basin of a system is reduced through any combination of the above mechanisms then a grid of starting points suitably located in phase space should give a good indication of safe limits of operation. We have investigate this claim through detailed sensitivity studies and comparison with results published in the
research literature. In these studies we have employed a variety of different mechanical oscillators, including the escape equation, symmetric and asymmetric Duffing-type models, a model of roll motion for a specific vessel, the Gaul, and one 2-DOF model of coupled mechanical oscillators. Main parameters for suitable grids-of-starts have then been defined.

We conclude the investigation of the method of a coarse grid-of-starts with an application to the SIR model. We have performed a sensitivity study involving the main features of a grid: its dimension, size in phase space, and density (the number of points comprising the grid). We have concluded that one-dimensional grids comprised of just three points located on the vertical static balance curve of a vessel (its 'natural' floating position) can give good estimates of safe boundaries.

We see the method we propose here simply as an initial attempt at applying the findings of geometrical nonlinear dynamics to the problem of ship stability against capsize. We would like to think that the approach we use here can be used to develop future practical procedures that can lead to better dynamic stability criteria.

Future studies

One of the few certainties about any research work is that it can be extended and improved upon. We see at least two fronts on which this work could be carried further. On the nonlinear dynamics front many interesting points were just touched upon. The bifurcational behaviour of the SIR model needs to be further investigated firstly to the point of producing a complete picture of bifurcations of the main sequence and then extending the study to cover bifurcations of coexisting attractors of practical relevance.
Another point we have hinted at in our work is the possible connection between specific cascades of local bifurcations and global events such as the development of homoclinic and heteroclinic tangles. The fact remains that the macroscopic features of fractal basin erosion are different in qualitatively different regions of the bifurcation diagram, such as for frequencies below or above 'optimal escape' (which itself can be defined by a codimension-two local bifurcation). The macroscopic differences in basin erosion are certainly relevant from a practical point-of-view, since they seem to imply different geometry of fractal incursions, and different rates of erosion. It should be interesting to understand the reasons for these differences. Some insight might be gathered from the application of concepts of lobe dynamics.

Another theoretical blank that should be interesting to fill is the relation between basins of attraction (steady-state) and safe basins (transient). There is strong numerical evidence, some of it can be seen in this work, that safe basins somehow 'follow' their underlying steady-state counterparts. For example, when (steady-state) basin boundaries become fractal, their transient counterparts also seem to acquire a fractal character. As far as we know this or other links between transient and steady-state basins have not yet been rigorously established.

Also it would be desirable to strengthen and extend the argument behind the method of a coarse grid-of-starts. We feel that the heuristics of the method are quite strong, particularly when applied to ship capsize. The method can, of course, be regarded as a more general procedure to determine boundaries of safe motion for mechanical driven oscillators, in which case the argument behind the method should be further investigated.

On the Naval Architecture front there is much work to be done if this approach is to result in new, improved stability criteria. One of the main advantages of the method of a coarse grid-of-starts over traditional stability analysis is the speed with which parametric studies, a necessity in this field, can be carried out. This
allows for extensive yet cost-effective investigations on the influence of each and every parameter of a mathematical model, and even of completely different mathematical models. These would include more sophisticated models in which different aspects of practical interest, such as water-on-deck, cargo shift, and ice accretion, could be incorporated. Also from a practical point of view sensitivity studies involving the level of damping would be quite important. If our results are found to hold true for arbitrary models, very fast computer simulations will let designers assess the stability qualities of various engineering solutions, and perhaps help them to reach a better overall result interactively at the design stage.

Finally, particular care has been taken throughout the development of our method of a coarse grid-of-starts to facilitate the comparison of its predictions with those from physical experiments with hull forms. Experiments necessary for confirmation and/or calibration of our method consist of relatively straightforward runs with reduced-scale models in regular waves. Wave steepness (or amplitude) would be increased in small steps, and the model excited by a short train of, say, eight waves for each amplitude. For each value of wave steepness the model would simply have to be held in three different angles of roll at the start of each simulation, being allowed to find its natural floating position for each new angle. The experiment is repeated until capsize occurs for any of the initial conditions. Wave frequency would then be varied, and a plot of minimum wave steepness versus wave frequency would be quickly generated. We feel that this kind of verification would be highly desirable, and would perhaps suggest how the method should be improved, see for example MacMaster and Thompson (1994).
APPENDIX A

POTENTIAL ENERGY FUNCTION OF A RECTANGULAR CROSS SECTION BODY

A.1 INTRODUCTION

In this appendix we develop explicit and exact functions for the total potential energy of prismatic bodies of rectangular cross-sections of arbitrary aspect ratio. These simple geometric shapes can be thought of as first approximations to ship-like floating structures. The expressions presented here can then be useful in verifying some basic hydrostatic properties of interest to us (see section 3.7.1), as well as allowing, under certain assumptions, the development of equations of motion valid for large-amplitude regimes.

We consider a body of breadth $E$, depth $dE$, length $b$, and mass $m$. Such a body has three planes of symmetry: two vertical (longitudinal and transversal) and one horizontal. We assume that the centre of gravity $G$ is located along the line defined by the intersection of the two vertical planes of symmetry, and at a distance $a$ from the baseline, see figure A.1.

Figure A.2 defines basic coordinates and geometrical parameters. Vertical displacements (heave motions) are defined by the distance between $G$ and the water surface (positive upwards), and denoted by $v$. Angular displacements (roll motions) are defined as the angle between the body and the water surface (zero for the upright position, and positive clockwise), and denoted by $\theta$.

The gravitational potential energy function $V(v, \theta)$ is given by the sum of body and displaced water individual energies:
Appendix A: Potential Energy Functions

Figure A.1 - Main dimensions of prismatic body of rectangular cross-section

\[ V(v, \theta) = V_{\text{body}}(v) + V_{\text{water}}(v, \theta) \]  

We adopt a zero potential energy for the body when its centre of gravity is at the water surface, and we assume that the water displaced by the body forms an infinitely thin film on the water surface, so that the individual potential energies can be given by 

\[ V_{\text{body}}(v) = mgv \]  
\[ V_{\text{water}}(v, \theta) = \rho g h A v_g \]

where \( \rho \) is the water density, \( g \) is the gravitational acceleration, \( A \) is the submerged area (of the cross-section), and \( v_g \) is the depth of the buoyancy centre as shown in figure A.2.

Expressions for \( V_{\text{water}} \) can be more easily determined if we consider three different cases as shown in figure A.3. Cases 1, 2, and 3 are defined in terms of the relative position between the body and the water surface: in Case 1 the water surface crosses both vertical sides of the body, in Case 2 one vertical side of the body is crossed by the water surface, along with the bottom of the body, and finally in Case 3 one vertical side is crossed along with the top of the body. If we define three parameters \( v_1, v_2, \) and \( v_c \) as:
Figure A.2 - Basic coordinates and geometric parameters for the floating square cross-section body

Figure A.3 - Different relative positions between body and water plane
\[ v_1 = \frac{E}{2} (\lambda \cos \theta - \sin \theta) \tag{A.2} \]
\[ v_2 = \frac{E}{2} (\lambda \cos \theta + \sin \theta) \tag{A.3} \]
\[ v_c = v - \left( a - \frac{\lambda E}{2} \right) \cos \theta \tag{A.4} \]

Then the following relations define Cases 1 to 3:

Case 1: \(-v_1 \leq v_c \leq v_1 \) \tag{A.5}

Case 2: \(v_1 \leq v_c \leq v_2 \) \tag{A.6}

Case 3: \(-v_2 \leq v_c \leq -v_1 \) \tag{A.7}

Note that we assume \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), so that values of \(v_c\) larger than \(v_2\) in modulus correspond to situations in which the body is either completely emerged (\(v_2 > 0\)) or completely submerged (\(v_2 < 0\)). These situations will not concern us here.

### A.2 Expressions for Potential Energy

Expressions for \(V(v, \theta)\) in terms of its two independent variables can be quite long. We therefore recall that the total potential energy is:

\[ V(v, \theta) = mgv + \rho g b A(v, \theta) v_b(v, \theta) \tag{A.8} \]

It will then suffice to present expressions for the submerged cross-sectional area \(A(v, \theta)\) and for the position of the centre of buoyancy \(v_b(v, \theta)\). To keep expressions even shorter we choose to introduce a few auxiliary parameters, and we make notation shorter by dropping the explicit dependence on \(v\) and \(\theta\). A key
geometrical parameter for the following calculations is given by $D$, the water surface elevation measured along the body vertical centre line (see figure A.2):

$$D = a - \frac{v}{\cos \theta}$$  \hspace{1cm} (A.9)

**Case 1**

The expressions for $A$ and $v_b$ for Case 1 are given by:

$$A = ED$$  \hspace{1cm} (A.10)

$$v_b = \left(\frac{D}{2} + \frac{E^2}{24D} \tan^2 \theta\right) \cos \theta$$  \hspace{1cm} (A.11)

**Case 2**

The expressions for $A$ and $v_b$ for Case 2 are given by:

$$A = \frac{1}{2 \tan \theta} \left(\frac{E}{\tan \theta + D}\right)^2$$  \hspace{1cm} (A.12)

$$v_b = \frac{1}{3} \left(D \cos \theta + \frac{E}{2} \sin \theta\right)$$  \hspace{1cm} (A.13)

**Case 3**

The submerged cross-sectional area $A$ can be given by:

$$A = \lambda E^2 - A_i$$  \hspace{1cm} (A.14)

Where the cross-sectional area above the water surface $A_i$ is given by:
Appendix A: Potential Energy Functions

\[ A_r = \frac{1}{2} \left( \frac{E}{2} + \frac{(\lambda E - D)}{\tan \theta} \right)^2 \tan \theta \quad A.15 \]

The depth of the centre of buoyancy can be given by:

\[ v_b = d \cos \theta \quad A.16 \]

Where the auxiliary parameter \( d \) and further auxiliary parameters are given by:

\[ d = D - \delta_v + \delta_h \tan \theta \quad A.17 \]

\[ \delta_v = \frac{\lambda^3 E^3}{2} - \frac{A_l}{A} \]

\[ \delta_h = \frac{A_l l_h}{A} \quad A.19 \]

\[ l_v = \lambda E - \frac{1}{3} \left( \frac{E}{2} \tan \theta + \lambda E - D \right) \]

\[ l_h = \frac{E}{2} - \frac{1}{3} \left( \frac{E}{2} + \frac{\lambda E - D}{\tan \theta} \right) \quad A.21 \]

A.3 RESULTS FOR A HOMOGENEOUS BODY OF SQUARE CROSS-SECTION

In order to get some feel for the shape of potential energy surfaces as we vary both heave and roll displacements we consider the specific case of a homogeneous body of square cross-section. Here, of course, the centre of gravity \( G \) will remain fixed at the geometric centre of the body. We, however, allow the average density of the body, \( \rho \), to vary, and we then inspect the resulting hydrostatic properties, such as stable and unstable floating positions, as well as potential energy surfaces. By doing so we will at the same time be checking on the feasibility of one of our assumptions (see equation 3.27) regarding the development of the SIR equations.
Appendix A: Potential Energy Functions

We therefore consider a floating homogeneous prismatic body of square cross-section of side $E$, length $b$, and average density relative to the water given by $ho = \frac{\rho_{\text{body}}}{\rho_{\text{water}}}$. We want to inspect (total, i.e. body plus water) potential energy surfaces for this system, and some of the basic hydrostatic properties that can be extracted from such examination, such as stable and unstable floating equilibrium positions.

Figure A.4 summarises the evolution of potential energy surfaces as we vary the relative average density of the body, $\rho$, between 0.1 and 0.9. The left column of the picture contains plots of contours of constant potential energy, $V$. Contour curves are equally spaced in steps of $0.1 \times (V_{\text{unstable}} - V_{\text{stable}})$, where $V_{\text{stable}}$ is the potential energy value for the stable floating equilibria (local minima of $V$), and $V_{\text{unstable}}$ is the potential energy value for the unstable floating equilibria (saddle-like points of $V$). The right column of figure A.4 shows corresponding three-dimensional views of potential energy surfaces meant to help visualising the shape of $V(\nu, \theta)$. Inspection of figure A.4 reveals the existence of at least two distinct stable floating positions: a square-shape position in which (in a front view) the body floats with two of its sides parallel to the water plane, and a diamond-shape position in which (again in a front view) the body has a diagonal parallel to the water plane, see figure A.5 (a) and (b), respectively. The stability of these floating positions is affected by the density $\rho$, such that for very light ($\rho = 0.1$ and 0.2) or very heavy ($\rho = 0.8$ and 0.9) conditions the square-shape position is stable, whereas for intermediate values of density ($\rho = 0.3$ to 0.7) it is the diamond-shape position that is stable.
Figure A.4 - Contours of constant potential energy (left column), and three-dimensional views of potential wells for the homogeneous square cross-section body. Relative average density $\rho$ goes from 0.1 (top pictures) to 0.9 (bottom pictures) in equal steps of 0.1.
Appendix A: Potential Energy Functions

Figure A.5 - Two possible equilibrium floating positions for the homogeneous body of square cross-section, (a) square-shape, and (b) diamond-shape

A.4 The SIR Model Assumption

Numerical calculation with this example also allows us to assess the validity of an assumption we made in the development of the SIR equations. We recall the notation of section 3.7.1, in which a subscript \( v \) indicates properties related to the angle of vanishing stability, which of course correspond to unstable equilibria here. To that notation we add a subscript \( s \) indicating properties related to stable equilibria. In a similar fashion to section 3.7.1 we then define \( V_A = V(v, \theta_v) \) and \( V_c = V(v, \theta_c) \). The assumption we made in section 3.7.1 was that the shape of the potential well for the SIR model was such that \( \frac{V_A}{V_c} = 1 \): here we can compare this assumption with values for the homogeneous square cross-section body. Inspection of the contours of constant potential energy of figure A.4 reveals that shapes of potential wells here will not be such that the condition \( \frac{V_A}{V_c} = 1 \) is verified exactly. We are, however, more interested in
Appendix A: Potential Energy Functions

checking to what extent, if at all, this *ad hoc* assumption can be justified on the basis of exact results for a simple (yet significant) geometric shape.

To circumvent any problems related to particular choices of potential energy datum we compare $V_A$ and $V_C$ using the energy of the local minimum, $V_*$, as a datum, and we denote this ratio simply by $\frac{V_A}{V_C}$. Table A.1 shows a summary of results for the homogeneous square cross-section body. We can see that values for the ratio $\frac{V_A}{V_C}$ usually stay within the order of magnitude of 1, giving some support to the assumption we made in section 3.7.1. The shape of level curves of potential energy around the stable equilibria for the square cross-section body also resemble, for some values of $\rho$, that of the SIR equations (compare for instance figure 3.4 with pictures for $\rho=0.3$ and 0.8 in figure A.4).

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\nu_v/E$</th>
<th>$\theta_v$</th>
<th>$V_*$</th>
<th>$\nu_v/E$</th>
<th>$\theta_v$</th>
<th>$V_*=V_C$</th>
<th>$V_A$</th>
<th>$V_A/V_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.400</td>
<td>0</td>
<td>441.4500</td>
<td>0.391</td>
<td>$\pm \pi/4$</td>
<td>486.8588</td>
<td>441.8473</td>
<td>0.0087</td>
</tr>
<tr>
<td>0.2</td>
<td>0.300</td>
<td>0</td>
<td>784.8000</td>
<td>0.260</td>
<td>$\pm \pi/4$</td>
<td>802.3882</td>
<td>792.6480</td>
<td>0.4462</td>
</tr>
<tr>
<td>0.3</td>
<td>0.159</td>
<td>$\pm \pi/4$</td>
<td>1006.3844</td>
<td>0.200</td>
<td>0</td>
<td>1030.0500</td>
<td>1015.0283</td>
<td>0.3653</td>
</tr>
<tr>
<td>0.4</td>
<td>0.075</td>
<td>$\pm \pi/4$</td>
<td>1120.1841</td>
<td>0.100</td>
<td>0</td>
<td>1177.2000</td>
<td>1124.1167</td>
<td>0.0690</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000</td>
<td>$\pm \pi/4$</td>
<td>1156.1196</td>
<td>0.000</td>
<td>0</td>
<td>1226.2500</td>
<td>1156.1196</td>
<td>0</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.075</td>
<td>$\pm \pi/4$</td>
<td>1120.1841</td>
<td>-0.100</td>
<td>0</td>
<td>1177.2000</td>
<td>1124.1167</td>
<td>0.0690</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.159</td>
<td>$\pm \pi/4$</td>
<td>1006.3844</td>
<td>-0.200</td>
<td>0</td>
<td>1030.0500</td>
<td>1015.0283</td>
<td>0.3653</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.300</td>
<td>0</td>
<td>784.8000</td>
<td>-0.260</td>
<td>$\pm \pi/4$</td>
<td>802.3882</td>
<td>792.6480</td>
<td>0.4462</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.400</td>
<td>0</td>
<td>441.4500</td>
<td>-0.391</td>
<td>$\pm \pi/4$</td>
<td>486.8588</td>
<td>441.8473</td>
<td>0.0087</td>
</tr>
</tbody>
</table>

Table A.1 - Summary of results for the homogeneous body of square cross-section with $E=b=1$ and $g=9.81$
REFERENCES


Rahola, J. (1939). "The judging of the stability of ships and the determination of the minimum amount of stability," The University of Finland, Helsinki, Finland.


