An Application of the Universality Theorem for Tverberg Partitions

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Abstract

We show that, as a consequence of a new result of Pór on universal Tverberg partitions, any large-enough set P of points in $\mathbb{R}^d$ has a $(d+2)$-sized subset whose Radon point has half-space depth at least $c_d \cdot |P|$, where $c_d \in (0,1)$ depends only on d. We then give an application of this result to computing weak ϵ-nets by random sampling. We further show that given any set P of points in $\mathbb{R}^d$ and a parameter $\epsilon > 0$, there exists a set of $O\left(\frac{1}{\epsilon^{\lfloor \frac{d}{2} \rfloor + 1}}\right)$ $\lfloor \frac{d}{2} \rfloor$-dimensional simplices (ignoring polylogarithmic factors) spanned by points of P such that they form a transversal for all convex objects containing at least $\epsilon \cdot |P|$ points of P.

Keywords: Tverberg's theorem, Radon's lemma, weak ϵ-nets, half-space depth, transversals.

1 Introduction

Radon's lemma states that, given any set Q of $(d+2)$ points in $\mathbb{R}^d$, there always exists a partition of Q into two sets, say Q_1 and Q_2, such that $\text{conv}\, Q_1 \cap \text{conv}\, Q_2 \neq \emptyset$. Further, if Q is in general position, then a dimension argument implies that such a partition $\{Q_1, Q_2\}$ is unique—called the Radon partition of Q—and $\text{conv}\, Q_1 \cap \text{conv}\, Q_2$ consists of a single point, called the *Radon point* of Q and denoted by Radon Q.

In this paper we present an application of the following statement, which is one consequence of a recent theorem of Pór (see [1]).

Lemma 1 (Proof in Section 2)**.** *For every $d \in \mathbb{N}$ there is $f(d) \in \mathbb{N}$ such that every set $P \subset \mathbb{R}^d$ of $f(d)$ points in general position contains two disjoint sets $A, B \subset P$ with $|A| = d+2$, $|B| = d+1$ and the Radon point of A is contained in* conv B*. Furthermore, the Radon partition of A consists of two sets of sizes $\lfloor \frac{d}{2} \rfloor + 1$ and $\lceil \frac{d}{2} \rceil + 1$.*

We use Lemma 1 to prove the following theorem. Given a set P of points in $\mathbb{R}^d$, the *half-space depth* of a point $q \in \mathbb{R}^d$ with respect to P is defined to be the minimum number of points of P contained in any half-space containing q.

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Theorem 2 (Proof in Section 2)**.** *For every $d \in \mathbb{N}$ there is $h(d) \in \mathbb{N}$ such that every set P of at least $h(d)$ points in $\mathbb{R}^d$ in general position contains a set $P' \subseteq P$ of size $(d+2)$ with* Radon P' *being contained in at least $\frac{|P|}{h(d)}$ vertex-disjoint simplices spanned by the points of $P \setminus P'$. In particular,* Radon P' *has half-space depth at least $\frac{|P|}{h(d)}$.*

We expect that Theorem 2 will find further applications in discrete and combinatorial geometry. Here we give an application to the computation of a weak ϵ-net for a given set P of points in $\mathbb{R}^d$ (assumed to be in general position)

Definition 3. *Given a set P of points in $\mathbb{R}^d$ and a parameter $\epsilon > 0$, a set $N \subseteq \mathbb{R}^d$ is a weak ϵ-net with respect to convex sets for P if for every convex set K with $|K \cap P| \geq \epsilon \cdot |P|$, we have $K \cap N \neq \emptyset$.*

Consider the following simple algorithm to compute a weak ϵ-net for a given set P of points in $\mathbb{R}^d$.

Algorithm Weak-Nets (Input: a set of points P, parameter $\epsilon > 0$)

Let $R \subseteq P$ be a uniform random sample of size $\frac{g(d)}{\epsilon} \log \frac{1}{\epsilon}$, for a constant $g(d)$ depending only on d.

$$Q = \left\{ \text{Radon } R' \colon R' \in \binom{R}{d+2} \right\}.$$

return $Q \cup R$.

Our application of Theorem 2 is the following.

Theorem 4 (Proof in Section 3)**.** *Let P be a set of points in $\mathbb{R}^d$ in general position and $\epsilon > 0$ a given parameter. Then there is a $g(d) \in \mathbb{N}$ such that a uniform random sample $R \subseteq P$ of size $\frac{g(d)}{\epsilon} \log \frac{1}{\epsilon}$ satisfies the following properties with probability at least $\frac{9}{10}$.*

1. *$R \cup Q$ is a weak ϵ-net for P, where Q is the set of Radon points of all $(d+2)$-sized subsets of R. That is, Algorithm Weak-Nets returns a weak ϵ-net for P of size $O\left(\frac{1}{\epsilon^{d+2}}\right)$, and*
2. *each convex object containing at least $\epsilon|P|$ points of P intersects the convex hull of at least one $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$-sized subset of R.*

Remark. The first part of Theorem 4 gives a bound on the size of the ϵ-net that is weaker than the current best bound due to Matoušek and Wagner [5], which is of the order of $O\left(\frac{1}{\epsilon^d}\right)$ (ignoring polylogarithmic factors). Yet our construction of a weak ϵ-net is novel and interesting as it uses certain Radon points of the underlying set P.

2 Proof of Lemma 1 and Theorem 2.

We need some definitions. We set $m = (r-1)(d+1)+1$, and for $k \in [d+1]$ the block B_k is the set of integers $\{(r-1)(k-1)+1, (r-1)(k-1)+2, \ldots, (r-1)k+1\}$. The blocks are of size r each and they almost form a partition of $[m]$, only neighboring blocks have a common element, namely

$(r-1)k+1 \in B_k \cap B_{k+1}$ for all $k \in [d]$. Call an r-partition $\{I_1, \ldots, I_r\}$ of $[m]$ *special* if $|I_j \cap B_k| = 1$ for every $j \in [r]$ and every $k \in [d+1]$.

Pór's result is about sequences $S = (a_1, \ldots, a_N)$ of vectors in $\mathbb{R}^d$. A sequence $(b_1, \ldots, b_t)$ is a subsequence of length t of S if $b_j = a_{i_j}$ for all $j \in [t]$ where $1 \leq i_1 < i_2 < \ldots < i_t \leq N$. Given a sequence $S = (a_1, \ldots, a_m)$ where $m = (r-1)(d+1)+1$ and each $a_i \in \mathbb{R}^d$, an r-partition $\{S_1, S_2, \ldots, S_r\}$ of S is in one-to-one correspondence with an r-partition $\{I_i, \ldots, I_r\}$ of $[m]$ via $a_i \in S_j$ if and only if $i \in I_j$. An r-partition of S is called *special* if the corresponding r-partition of $[m]$ is special.

Tverberg's theorem states that given a set P of $(m-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of P into m sets whose convex-hulls contain a common point.

We can now state Pór's result [9].

Theorem A (Universality theorem for Tverberg partitions)**.** *Assume $d, r, t \in \mathbb{N}$, $r \geq 2$, and $m = (r-1)(d+1)+1 \leq t$. Then there exists $N = N(d,r,t) \in \mathbb{N}$ such that every sequence $S = (a_1, \ldots, a_N)$ of vectors (in general position) in $\mathbb{R}^d$ contains a subsequence $S' = (b_1, \ldots, b_t)$ (of length t) such that the Tverberg partitions of every subsequence of length m of S' are exactly the special partitions.*

Note that when the points of S (or P) come from the moment curve $\Gamma(x) = \{\gamma(x) : x \in \mathbb{R}^+\}$ where $\gamma(x) = (x, x^2, \ldots, x^d)$, then there is a natural ordering $S = (\gamma(x_1), \ldots, \gamma(x_n))$ with $x_1 < x_2 < \ldots < x_n$. Now let $0 < x_1 < \ldots < x_n$ a *rapidly increasing* sequence of real numbers, meaning that, for every $h \in [n-1]$, x_{h+1}/x_h is at least as large as some (large) constant $c_{d,r,h}$ depending only on d, r, h. It is not hard to check that in this case all Tverberg partitions of all $m = (r-1)(d+1)+1$ long subsequences of S are the special ones. This (and other examples as well) show that no other set of partitions can be universal.

We are going to apply the universality theorem in the special case $r = 3$ and $t = m = (r-1)(d+1)+1$. In this case $N(d,r,t)$ depends on d only, and thus we can set $f(d) = N(d,r,t) = N(d,3,2d+3)$.

Proof of Lemma 1. Order the elements of P arbitrarily to obtain a sequence $S = \big(p_1, \ldots, p_{f(d)}\big)$. Apply the universality theorem to S with $r = 3$, $t = m = 2d+3$. We get a subsequence S' of length m all of whose Tverberg 3-partitions are exactly the special ones. Define $I_1 = \{z \in [m] : z \equiv 1 \bmod 4\}$ and $I_2 = \{z \in [m] : z \equiv 3 \bmod 4\}$ and $I_3 = \{z \in [m] : z \text{ is even}\}$. Note that $|I_1| = \lceil \frac{d}{2} \rceil + 1$, $|I_2| = \lfloor \frac{d}{2} \rfloor + 1$ and $|I_3| = d+1$.

It is easy to see that $\{I_1, I_2, I_3\}$ is a special partition of $[m]$: every block contains exactly one element of I_1, I_2, I_3. Let the corresponding partition of S' be $\{S_1, S_2, S_3\}$. So $\bigcap_1^3 \operatorname{conv} S_i \neq \emptyset$. Set $A = S_1 \cup S_2$ and $B = S_3$. Then the Radon point of A, which is $\operatorname{conv} S_1 \cap \operatorname{conv} S_2$, is contained in $\operatorname{conv} B$. $\square$

Proof of Theorem 2. Consider a $(2d+3)$-uniform hypergraph $\mathcal{H} = (P, E)$ on the vertex set P, where $e \in E$ if and only if the $(2d+3)$ points of e can be partitioned into two sets $e = e_1 \cup e_2$ such that $|e_1| = d+2$, and Radon $e_1 \in \operatorname{conv} e_2$. We will call the set e_1 the *Radon-base* of the edge e. By the result of de Caen [3], any r-uniform hypergraph on n vertices and m edges contains an independent set of size at least

$$\frac{r-1}{r^{\frac{r}{r-1}}} \cdot \frac{n^{\frac{r}{r-1}}}{m^{\frac{1}{r-1}}}.$$

On the other hand, Lemma 1 implies that any set Q of $f(d)$ points of P must contain two disjoint sets—$A_Q \subseteq Q$ of size $(d+2)$ and $B_Q \subseteq Q$ of size $(d+1)$—such that Radon $A_Q \in \operatorname{conv} B_Q$. Then the $(2d+3)$ points $A_Q \cup B_Q$ form an edge in $\mathcal{H}$. This implies that no subset of P of size $f(d)$ can be independent in $\mathcal{H}$. Thus, with $r = 2d+3$, we have

$$\frac{2d+2}{(2d+3)^{\frac{2d+3}{2d+2}}} \cdot \frac{|P|^{\frac{2d+3}{2d+2}}}{|E|^{\frac{1}{2d+2}}} \;\leq\; \text{size of max. ind. set in } \mathcal{H} \;<\; f(d) \qquad \Longrightarrow \qquad |E| \geq \frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}.$$

By the pigeonhole principle, there exists a $(d+2)$-sized set $P' \subseteq P$ that is the Radon-base of a set E' of edges of E, where

$$|E'| \geq \frac{|E|}{\binom{|P|}{d+2}} \geq \frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}}.$$

The $(d+1)$-uniform hypergraph consisting of the sets $E'' = \{e' \setminus P' \colon e' \in E'\}$ has the property that the convex hull of the elements of each set contains Radon P'. It suffices to show that it contains a matching of size $\Omega(|P|)$—and this follows from known lower-bounds on matchings in uniform hypergraphs. For simplicity, we instead present a direct argument, though with worse constants.

Iteratively construct a matching by adding a $(d+1)$-sized set from E'' to the matching, and deleting all sets from E'' whose intersection with this added set is non-empty. Each set added to the matching can cause the deletion of at most $(d+1) \cdot \binom{|P|}{d}$ sets of E'', as a vertex of $P \setminus P'$ can belong to at most $\binom{n}{d}$ sets of E'' (each set in E'' has size $(d+1)$). The size of the final matching is the number of iterations, which, by the above discussion, is lower-bounded by

$$\frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}} \Bigg/ \binom{|P|}{d}(d+1).$$

A calculation then shows that

$$\frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}} \Bigg/ \binom{|P|}{d}(d+1) \geq \frac{|P|}{h(d)}, \quad \text{where } h(d) = \frac{2(2d+3)(d+1)f(d)^{2d+2}}{(d+2)!d!}.$$

□

3 Proof of Theorem 4

Definition 5. *Given positive integers d, p, q with $p \geq q > \lfloor \frac{d}{2} \rfloor$, let $\mathrm{CHS}(d,p,q)$ denote the smallest integer such that the following holds. For any compact convex object $K \subseteq \mathbb{R}^d$ and any set $P \subseteq \mathbb{R}^d \setminus K$ of points, if every subset of P of size p has a q-sized subset whose convex hull is disjoint from K, then P can be separated from K with $\mathrm{CHS}(d,p,q)$ half-spaces (that is, there exists a set $\mathcal{H}$ of $\mathrm{CHS}(d,p,q)$ half-spaces such that $K \subseteq \bigcap_{h \in \mathcal{H}} h$ and $\left(\bigcap_{h\in\mathcal{H}} h\right) \bigcap P = \emptyset$).*

Then the key statement is the following.

Lemma 6. *Let P be a set of n points in $\mathbb{R}^d$ and $\epsilon \in \left[0, \frac{1}{2}\right]$ a given parameter. Further let $q > \left\lfloor \frac{d}{2} \right\rfloor$ be an integer such that* $\mathrm{CHS}\left(d, q \cdot h\left(d\right), q\right)$ *is finite. Let R be a uniform random sample of P of size*

$$\frac{c_2 \cdot d \cdot \mathrm{CHS}\left(d, q \cdot h\left(d\right), q\right) \cdot \log \mathrm{CHS}\left(d, q \cdot h\left(d\right), q\right)}{\epsilon} \log \frac{1}{\epsilon},$$

where c_2 is a large-enough constant independent of d, ϵ and q. Then with probability at least $\frac{9}{10}$,

1. *$R \cup Q$ is a weak ϵ-net for P, where Q is the set of Radon points of all $(d+2)$-sized subsets of R, and*
2. *each convex object containing at least $\epsilon|P|$ points of P intersects the convex hull of at least one $\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-sized subset of R.*

Proof. The proof follows the method of Mustafa and Ray [6]; however they assumed a more restrictive case, so we present a proof modified appropriately to give a more general bound.

Set $p = q \cdot h\left(d\right)$.

Claim 7. *With probability at least $\frac{9}{10}$, R is an ϵ-net for the set system induced on P by the intersection of* $\mathrm{CHS}\left(d, p, q\right)$ *half-spaces in $\mathbb{R}^d$.*

Proof. The set system induced by the intersection of k half-spaces in $\mathbb{R}^d$ has VC-dimension $\Theta\left(d\, k \log k\right)$ [2]. Thus by the ϵ-net theorem, a uniform random sample of size

$$\Theta\left(\frac{dk \log k}{\epsilon} \log \frac{1}{\epsilon}\right) = \frac{c_2 \cdot d \cdot \mathrm{CHS}\left(d, p, q\right) \cdot \log \mathrm{CHS}\left(d, p, q\right)}{\epsilon} \log \frac{1}{\epsilon}$$

is an ϵ-net with probability at least $\frac{9}{10}$ (see [8]), where c_2 is a large-enough constant independent of d, ϵ and q. □

Assume that R is such an ϵ-net. Let K be any convex object containing at least $\epsilon|P|$ points of P.

Claim 8. *There exists $R_K \subseteq R$ of size p such that the convex hull of every subset of R_K of size q intersects K.*

Proof. If for every subset of R of size p there exists a q-sized subset whose convex hull is *disjoint* from K, then by the definition of $\mathrm{CHS}\left(d, p, q\right)$, all points of R can be separated from K by a set $\mathcal{H}$ of $\mathrm{CHS}\left(d, p, q\right)$ half-spaces. The common intersection of these half-spaces contains K and hence at least $\epsilon|P|$ points of P and no point of R, a contradiction to Claim 7. □

By Theorem 2, R_K has a $(d+2)$-sized subset, say R'_K, such that Radon $R'_K \in Q$ is contained in at least $\frac{|R_K|}{h(d)}$ vertex-disjoint simplices spanned by points of $R_K \setminus R'_K$. Now Radon R'_K must lie inside K: otherwise the half-space separating it from K must contain at least one point from each simplex containing Radon R'_K—namely it must contain at least $\frac{|R_K|}{h(d)} = \frac{p}{h(d)} = q$ points of R_K. But then the convex hull of these q points does not intersect K, a contradiction to Claim 8. Thus $R \cup Q$ is a weak ϵ-net for P, and further, the Radon partition of R'_K of size $\left\lfloor \frac{d}{2} \right\rfloor + 1$ must intersect K. This completes the proof. □

Proof of Theorem 4. It is known that $\mathrm{CHS}(d,p,q)$ is finite for large-enough values of q—this together with Lemma 6 implies the proof. In particular,

1. ([4]) For $p \geq q = d+1$ we have
$$\mathrm{CHS}(d,p,q) = O\left(p^{d^2} \log^{c' d^3 \log d} p\right),$$
where c' is an absolute constant.

2. ([7]) For any real $\beta > 0$ and $p \geq q = (1+\beta) \cdot \left\lfloor \frac{d}{2} \right\rfloor$ we have
$$\mathrm{CHS}(d,p,q) = O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right).$$

The second bound is stronger, but both of them imply, by setting $q = (d+1)$, $p = q \cdot h(d)$ and applying Lemma 6, the existence of a function

$$g(d) = O\Big(d \cdot \mathrm{CHS}\left(d, (d+1) \cdot h\left(d\right), (d+1)\right) \cdot \log \mathrm{CHS}\left(d, (d+1) \cdot h\left(d\right), (d+1)\right)\Big).$$

□

Acknowledgements. The first author was supported by the Hungarian National Research, Development and Innovation Office NKFIH Grants K 111827 and K 116769. The second author was supported by the grant ANR grant ADDS (ANR-19-CE48-0005).

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