To my parents, without whom I could not have achieved half of what I have.

And to my brother, Gurjit who helped support me throughout this thesis.
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The dynamics of friction oscillators

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Abstract

The aim of this thesis is to investigate the nonsmooth dynamical behaviour of a mechanical system, which undergoes self-sustained oscillations and chaotic motion induced by dry friction. Dry friction itself is not very easy to model and so a number of different models are investigated numerically. An experimental study is undertaken with a mechanical system which closely matches the mathematical system, to allow the formulation of the most physically realistic friction model for this system.

It is shown that the complexity of the system can be increased by an appropriate increase in the dimension of the phase space of the system. The nonsmooth nature of the system and of its dynamical behaviour is further complicated by the fact that the phase space dimension varies. Thus classical methods of analysis are not especially applicable. To analyse the system new numerical techniques for determining Poincaré maps and Lyapunov exponents are presented.

The exploration of the multi-dimensional phase space of such a system would be difficult and computationally expensive. Therefore a method of reducing the complexity and dimension of the system to a lower dimensional map is presented. The different attractors of the system are found and bifurcation’s of the system identified by use of the lower dimensional map.

The significance of damping and external forcing on the system are investigated, as well as discussing the engineering considerations that they bring. Bifurcational behaviour of the system is then investigated using classical methods of analysis along with the computer program AUTO.

A number of methods which prevent the stick-slip behaviour of the system are investigated with a view of eventually controlling the type of behaviour the system exhibits. A number of control methods are discussed and the application of an appropriate control mechanism for the system is presented.
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Chapter 1

Introduction

Many early civilisations have tried to master friction and its effects, some succeeding and others failing in varying degrees. The one early civilisation that seems to have overcome many of the problems of friction and used it to their own advantage was the Egyptian civilisation, where the transportation of heavy stone statues and building blocks were used in the construction of the pyramids and palaces for the Pharaohs.

Some of the statues and blocks transported from the quarries to their resting places weighed up to 50 tons, with some of the largest statues and obelisks varying in height from 19.5m to 32m with weights in the range 120 tons to about 500 tons. The Egyptian method of moving large stone statues is depicted in the well known painting from a grotto at El-Bersheh dated about 1880BC shown in figure 1.1. The colossus is secured to a sledge, but the figure features no rollers or levers to help the sledge along, as suggested by many scholars.

The most interesting feature of this painting is that it shows an Egyptian official standing in front of the pedestal pouring lubricant from a jar on the ground immediately in front of the sledge.

Three other men are each shown to be carrying two jars at the side of the sledge,
presumably to supply more lubricant to the official on the pedestal while a similar number of men carry a wooden plank on which the sledge was probably towed. The wooden plank has a rough jagged surface on one side of it, which was probably placed on the ground, whilst the other side had a smooth surface upon which the sledge was towed. Thus the plank did not move when the sledge was towed on top of it. Even if water had been used as the lubricant on the planks it could have led to a significant reduction in frictional resistance.

There is some merit in considering the frictional implications of the painting and of the movement of large stone statues on the assumption that they are accurate representations of the process. It would be both interesting and instructive to attempt to quantify the coefficient of friction for the situation depicted in the paintings. The weight of the colossus \( W \) shown in figure 1.1 has been estimated at about 60 tons (600 kN). The painting shows 172 men pulling on all ropes. It has been estimated that each man can exert a pull in the range 534N to about 1068N. It is doubtful that those toiling under the Egyptian sun could have consistently achieved the higher range. Therefore the mean tractive effort of 800N per man will be used. On this basis the total effort, which must at least equal the friction
force \( F \) becomes:

\[ F = 172 \times 800N \]

(1.1)

Thus the coefficient of friction \( \mu \) is given by:

\[ \mu = \frac{F}{W} = 0.23 \]

(1.2)

It is interesting to note that Bowden and Tabor (Bowden & Tabor 1956) quote coefficients of friction for wood-on-wood as follows:

- *wet*: 0.2
- *clean(dry)*: 0.25 - 0.5

A comparison of the calculated coefficient of friction for the Egyptian sledge with the values quoted by Bowden and Tabor suggests that the sledge could in all probability have been sliding over lubricated planks of wood.

During the Renaissance 1450 - 1600 AD, the study of the mechanical sciences flourished. As expected this era also had a significant impact upon the study of friction and its effects. In the first place the subject during this period benefited from the inspiration of a man whose genius is almost beyond comprehension; Leonardo da Vinci. Leonardo’s direct contribution to the Renaissance movement undoubtedly lies in the arts, but his contributions to the mechanical sciences and tribology, the study of friction, in particular cannot be overlooked.

In assessing the significance of Leonardo’s studies of friction it is important to realise that although the force of friction had been recognised by Aristotle some 2000 years earlier, and numerous attempts had been made to minimise its effects in many of the earlier civilisations, his work represented the first recorded qualitative study of the subject.

Figures 1.2 and 1.3 show sketches from the *Codex Atlanticus* and *Codex Aryndel* made by Leonardo.
Figure 1.2: Sketches from the Codex Atlanticus and the Codex Arundel showing experiments to determine: (a) the friction force between horizontal and inclined planes; (b) the influence of apparent contact area upon the friction force (From MacCurdy E. (1938)).
Figure 1.3: Sketches from the Codex Atlanticus and the Codex Arundel showing experiments to determine: (c) the friction force on a horizontal plane by means of a pulley; (d) the friction torque on a roller and a half bearing (From MacCurdy E. (1938)).
These sketches demonstrate that Leonardo measured the force of friction between objects on both horizontal and inclined planes. Figure 1.2 is very interesting since it clearly relates to Leonardo's studies of the influence of apparent contact area upon frictional resistance. Of the apparent area of contact and the normal force upon friction Leonardo wrote:

"The friction made by the same weight will be of equal resistance at the beginning of its movement although the contact may be of different breadths and lengths."

And:

"Friction produces double the amount of effort if the weight be doubled."

The above observations are entirely in accord with statements of the first two laws of friction, namely:

- The force of friction is independent of the apparent area of contact.
- The force of friction is directly proportional to the applied load.

Leonardo introduced for the first time the concept of the coefficient of friction as the ratio of the force of friction to the normal load. He correlated the friction to surface roughness by stating:

"The friction is of different strength for different bodies as much as there are differences in their slipperiness. Bodies whose surface is smoother have a smaller friction. With the assumption of a smooth place and polished surfaces each body resists with the fourth part of its weight."

Leonardo also related friction to his interest in the "music of the spheres", indicating a concept of friction induced vibration. To this regard he noted that the heavenly boundaries are either smooth or full of lumps and roughs, and that friction would have rubbed away the boundaries of each and there would not be any more friction anymore, and the sound would cease, and the dancers would stop.
Figure 1.4: Amontons' illustration of a beam (AB) being drawn up a frictionless plane (CD) (From Dowson D. (1979)).

It is apparent that Leonardo's studies of friction were formed on the application of scientific methods to contemporary problems. He appears to have conceived his experiments, reached conclusions and recorded findings, which on the whole have stood the test of time.

Leonardo's ideas on friction were forgotten for quite some time, during which Sir Issac Newton (1642 - 1727) published his *Principia*. Newton's contribution included his first and second laws of motion of a body. Meanwhile towards the end of the seventeenth century work on friction was initiated in France when Leonardo's dry friction laws were rediscovered independently by Guillaume Amontons. His experiments and interpretations of his results were discussed in a now classical paper presented to the Académie Royale on 19th December 1699 (Amontons 1699).

Figures 1.4, 1.5, 1.6 and 1.7 show sketches of the various experiments carried out by Amontons. Figure 1.4 is an illustration of a beam (AB) being drawn up a plane (CD). Amontons argued that it was of little value to know that by classical mechanics the force required to move the beam AB up an inclined plane CD is equal to the weight of the beam multiplied by the sine of the angle of inclination.
Figure 1.5: Amontons' illustration of his apparatus for friction experiments. Test materials: A-A, B-B. Spring providing normal loading: C-C-C. Spring balance with scale for friction measurements: D (From Dowson D. (1979)).

when

"...the resistance incurred in the rubbing of this beam against the earth may not simply be equal to this force, but may even exceed it by a considerable number of times."

The apparatus used by Amontons in his "Experiment concerning the rubbing of various materials one against the other" is shown in figure 1.5. Test specimens like AA and BB were loaded together by various springs depicted by CCC, while the force required to overcome friction and initiate sliding was measured on the spring balance D.

Amontons' main findings were:

- That the resistance caused by rubbing only increases or diminishes in proportion to greater or lesser pressure (load) and not according to the greater or lesser extent of the surfaces.
Figure 1.6: Amontons' representation of friction between multiple surfaces (From Dowson D. (1979)).

Figure 1.7: Amontons' representation of elastic asperities by springs (A) (From Dowson D. (1979)).
• That the resistance caused by rubbing is more or less the same for iron, lead, copper and wood in any combination if the surfaces are coated with fat.

• That this resistance is more or less equal to one-third of the pressure (load).

The first observation embodies the first and second laws of friction, namely:

• The force of friction is directly proportional to the applied load.

• The force of friction is independent of the apparent area of contact.

The same conclusions were made by Leonardo da Vinci’s work but during Amontons’ period these facts were not generally known. However Amontons still anticipated that these statements, particularly that regarding the null effect upon friction of the size of the rubbing bodies, might not meet with ready and universal acceptance. He clearly thought of the fundamental cause of friction in terms of surface roughness and the force required to lift interlocking asperities over each other in sliding motion. He wrote:

“...It is impossible that these irregularities shall not be partly convex and partly concave, and when the former enter upon the latter they shall produce a certain resistance when there is an attempt to move them, since in order to do this they will have to raise that which presses them against each other, and the action of these unevenness or else the effect which these can produce is the same as that of inclined planes which are used in raising loads, it follows that the greater pressure (load) the greater the resistance to movement; furthermore as in the case under consideration one must assume that the pressure is equally distributed over the whole area of the surfaces: it follows again that where you have several surfaces of differing areas loaded with equal weights each part forming the larger one carries less weight than each part of the same areas which form the smaller ones, this following the proportion these surfaces have between themselves.... It follows
Further that the resistance caused by the rubbing of surfaces of differing areas is always the same when they are loaded with equal weights..."

In a very short space of time Amontons had succeeded in formulating the basic laws of friction, which in due course became known as Amontons' laws. However much of his work failed to attract the attention of other investigators and thus he failed to achieve the recognition he really deserved.

Any review of the important contributions to the understanding of friction would not be complete without discussing the work of Leonhard Euler. In 1748 while working in Berlin, Euler submitted papers on friction to the Academy of Sciences. The most important contribution of his work was the first clear distinction drawn between static and dynamic friction. His analysis of sliding motion down an inclined plane with friction between the mass and plane can be summarised by looking at figure 1.8. Euler argued that if the plane was inclined at the limiting slope for static equilibrium and the angle was then increased slightly, the mass would move with an exceedingly small velocity.

However, he found by experiment that this was not the case and that the body moved very quickly once the equilibrium was disturbed. He concluded in his
paper that dynamic friction must be smaller than static friction. Euler’s work is important as it developed a clear analytical approach to friction, it introduced the well known symbol \( \mu \) for coefficient of friction and it marked the transition from studies of static to dynamic friction.

Some eighty years after Amontons had presented his paper another French engineer called Charles Augustine Coulomb was formulating a theory that would cause a revolution in the understanding of friction.

In 1785 Coulomb submitted his now classical friction paper (Coulomb 1785) (see figure 1.9) to the Paris Academy of Sciences. His paper detailed the most comprehensive study of friction undertaken up to that period. In the introduction of his friction memoir he outlines the current state of knowledge with references to the work by Amontons which Coulomb had rediscovered.

![Coulomb's classical paper](image)

Figure 1.9: Coulomb’s classical paper.

His apparatus for the study of sliding friction between plane surfaces is shown
in figure 1.10. Une table très solide of substantial proportions was carefully prepared to form the lower surface of the sliding pair. Sledges of various lengths were connected either directly to a weight or indirectly via a lever to a weight by means of a rope which passed over a pulley mounted on a shaft. Strips of various widths were nailed to the underside of the sledge as shown in Coulomb's fig.2 and weights of various magnitudes were added to the sledges as shown in his fig.3 to investigate the apparent contact area and load. Many materials were used in various combinations in both dry and lubricated conditions. Smooth and rough surfaces were investigated along with maximum speeds and pressures of 4.8m/s and 30.4MN/m² adequately covering the range of practical conditions. Periods of repose ranging from 0.5s to 4 days provided data of considerable importance in the development of Coulomb's theory of friction.

A wide range of friction experiments were reported and the results recorded in great detail. On the whole Coulomb found that in most cases friction was proportional to the load and independent of the size of the contacting surfaces. The second important observation that is discussed is the relationship between dynamic and static friction. In general dynamic friction was found to be smaller than static friction, the difference being quite large for fibrous materials but almost imperceptible for metals.

Towards the end of his paper Coulomb begins by summarising the four principal features of his experimental findings which for the foundations for his theory. They are:

- For wood sliding on wood under dry conditions the friction rises initially but soon reaches a maximum. Thereafter the force of friction is essentially proportional to the load.

- For wood sliding on wood the force of friction is essentially proportional to load at any speed, but dynamic friction is much lower than the static friction related to long periods of repose.
Figure 1.10: Coulomb's apparatus for the study of sliding friction (From Dowson D. (1979)).
• For metals sliding on metals in dry conditions the force of friction is essentially proportional to load and there is no difference between static and dynamic friction.

• For metals on wood under dry conditions the static friction rises very slowly with time of repose and might take four, five or even more days to reach its limit. With metal-on-metal the limit is reached almost immediately and with wood-on-wood it takes only one or two minutes. For wood-on-wood or metal-on-metal under dry conditions speed has very little effect on dynamic friction, but in the case of wood-on-metal the dynamic friction increases with speed.

From his experimental results Coulomb formulated the third law of friction:

Dynamic friction is independent of the sliding velocity.

In this forthright, brief but ingenious and simple way, Coulomb explained the nature of friction. His results provided a workable theory for generations of engineers and scientists.

Coulomb's results and findings were confirmed by another series of more careful experiments carried out by General A.J. Morrin, the results of which were published in 1830. On the basis of their experiments Coulomb and Morrin concluded that the coefficients of friction are constant for any pair of materials. Their experiments were carried out under conditions of very small loads and relatively low speeds. Consequently, the results obtained by them hold true only for moderate, not extreme conditions.

Subsequent investigations by Captain Douglas Galton on the friction between brake-blocks and railway wheels showed that the coefficients of friction did not remain constant. Galton's preliminary findings were first presented to a meeting of the Institution of Mechanical Engineers in Paris and subsequently published
CHAPTER 1. INTRODUCTION

in *Engineering* on the 14th of June 1878. The major conclusions that Galton formulated from his findings were:

- The coefficient of dynamic friction between the brake-blocks and the wheels decreased as the sliding speed increased.

- The coefficient of dynamic friction between the brake-blocks and the wheels decreased with time at a fixed speed.

Thus Galton had shown that Coulomb's simple theory was inadequate in explaining the friction between two bodies under a variety of conditions.

During this period the investigations of Lord Rayleigh are of particular interest. Lord Rayleigh in his major work *The Theory of Sound* (Rayleigh 1945) first published in 1877, discussed the the wine goblet rubbed with a finger and self-sustained oscillations of violin strings caused by dry friction. He introduced the simplest mechanical model, shown in figure 1.11, which modelled the dynamical behaviour of a violin string being bowed. During this same period Hemholtz also observed similar behaviour in bowed instruments.

![Figure 1.11: Schematic of the simplest model exhibiting self-sustained oscillations.](image-url)
He noticed that the string was carried along by the bow by friction, then suddenly detached itself and moved rapidly backwards until it was caught by the moving bow again.

Recently there has been increasing interest in the type of behaviour observed by Rayleigh and Hemholtz, so called self-sustained oscillation or stick-slip motion. This kind of dynamical behaviour generally occurs when elastic surfaces are driven against each other. In the presence of friction, intermittent behaviour is observed in which a period of rest follows a slip of one surface with respect to the other, which in turn is followed by a period of rest.

Stick-slip behaviour is also observed in the frictional sliding of rocks. This led Burridge and Knopoff (Burridge & Knopoff 1967) to propose it as a mechanism for earthquakes. A long period of rest usually follows a short period of tectonic plate movement, the kind of behaviour which is seen in systems which exhibit stick-slip oscillations. They introduced two mechanical models for earthquakes, which consisted of masses connected by springs and driven at a particular velocity on a surface with friction.

This was further investigated by Brace and Byerlee (Brace & Byerlee 1966) who investigated the instabilities of pre and post seismic slip by using the models proposed by Burridge and Knopff (Burridge & Knopoff 1967).

Nussbaum and Ruina (Nussbaum & Ruina 1987) expanded on these ideas and introduced a two degree of freedom earthquake model, shown in figure 1.12, with static and dynamic friction. Drawing from previous works from Byerlee (Byerlee 1970; Byerlee 1978) Rice and Tse (Rice & Tse 1986) and Cao and Aki (Cao & Aki 1986), Nussbaum and Ruina shifted emphasis towards the patterns of events after many slip cycles have occurred over long periods of time. They also showed that the configuration of the two degree of freedom system can be characterised by a single variable, the difference in the stretch of the driving springs at the end of an event.
Huang and Turcotte (Huang & Turcotte 1990) in their investigations into the two degree of freedom model showed that if an asymmetry is introduced in the friction forces, then the two degree of freedom system, introduced by Nussbaum and Ruina can present chaotic behaviour. More specifically they considered the asymmetrical case in which the friction force on one mass is different from the friction force on the other mass. They then compared their results with the motion of tectonic plates in California and Japan and observed that such tectonic plate motion seems to have the same kind of chaotic dynamics as observed in the two degree of freedom model.

Galvanetto and Bishop (Galvanetto, Bishop & Briseghella 1993) extended the system presented by Huang and Turcotte by introducing a specific dynamic friction curve whose friction force diminishes with increasing slip velocity. The friction curve also possessed a discontinuity which ultimately gave the system a non-smooth nature. The non-smooth nature of this system and of its dynamics do not especially endear themselves to classical methods of analysis. Therefore Galvanetto and Bishop presented new techniques for determining Poincaré maps and Lyapunov exponents. The dynamical behaviour of such systems becomes even more complicated as they belong to a class of non-smooth systems with variable phase space dimension, where the number of degrees of freedom alters through time.
1.1 Thesis overview

The proposed course of research will be divided into a number of distinct areas. The first main area of research is to determine whether any of the dynamic friction characteristics used in other investigations are physically realistic.

Numerical studies of the single degree of freedom model, introduced by Lord Rayleigh, will be carried out using a variety of different dynamic friction characteristics.

Extensive experimental studies of a mechanical model, which closely matches the mathematical model will be carried out.

Once these two initial investigations are completed, a direct qualitative comparison between numerical and experimental data will be made, to determine a physically realistic dynamic friction characteristic for this system.

Once a realistic dynamic friction curve has be obtained, the emphasis will now shift towards a two degree of freedom model. The behaviour of the two degree of freedom system will be analysed by using techniques introduced by Galvanetto and Bishop.

The construction of a three dimensional Poincaré section and the reduction of the four dimensional phase space to a one dimensional map will allow the identification of any bifurcational behaviour of the system. Using the definition of the one dimensional map introduced by Galvanetto and Bishop, the computation of the most significant Lyapunov exponent for the system will be possible.

The role of damping and external forcing on the system are investigated, as well as discussing the engineering considerations that they bring. The bifurcational behaviour of these systems will be investigated using the computer program AUTO as well as with classical methods of analysis.
Since in the majority of engineering applications, stick-slip oscillations are highly undesirable and should be avoided at all costs since they diminish the precision of motion and safety of operation. They also cause noise and promote wear and tear. Therefore it would be intriguing to evaluate the effectiveness of a number of methods which would allow the reduction or prevention of stick-slip behaviour in the system presented here.

These methods are investigated with a view of eventually controlling the type of behaviour the system exhibits. A number of control methods are discussed and the application of an appropriate control mechanism for the system will be investigated.
Chapter 2

The Single Degree of Freedom Model

The single degree of freedom model shown in 2.1 was developed by Lord Rayleigh (Rayleigh 1945) during his investigations into the oscillations of violin strings as mentioned in chapter 1. The model was introduced as being the simplest mechanical model which exhibited stick-slip or self-sustained oscillations induced by dry friction. Dry friction forces provide the excitation mechanism for the large-amplitude oscillations of the violin string when it is being bowed.

The generally accepted theory of stick-slip oscillations is that if two-bodies are in contact with each other, and when the two bodies slide relative to one another, they are said to be in a state of *slipping*, where dry friction then acts as a resistance against this relative motion. When the two bodies do not slide relative to one another they are said to be in a state of *sticking*, where any impending motion of the bodies is resisted by dry friction.

Sticking and slipping can occur successively in a system capable of vibration and the system exhibits a periodic motion. Such vibrations may often be observed indirectly by the sound they radiate into the surrounding air, as in the case of the
violin string being bowed or a wine goblet being rubbed by a finger and producing a note.

When describing the dynamics of a model which are affected by dry friction, such as in figure 2.1, the models' dynamics depend upon a number of parameters such as the relative velocity of the bodies, normal contact force, surface properties and material properties. A specific property of dry friction was addressed by people such as Euler, Coulomb and Galton, relates to the changing character of the friction force, adding further complexity to the nature of dry friction. Experimental investigations carried out by Coulomb and Morrin both showed that the friction force developed between two surfaces at relative rest, that is when they are in a state of sticking, is greater than the friction force that exists between the same surfaces while slipping is in progress. Thus the friction force at rest can be greater than the friction force during motion.

To distinguish between these two cases the friction developed between two bodies which are motionless is referred to as the static friction force and the friction force between sliding surfaces as dynamic friction. In terms of the single degree
of freedom model shown in 2.1, static friction is the friction force that is present between the mass and the belt, and provides resistance to any impending motion of the system from a position of equilibrium, i.e. rest. Dynamic friction is a force which always opposes the motion of the system so as to return it to a position of equilibrium.

Static friction characterises the sticking mode of behaviour, dynamic friction on the other hand characterises the slipping mode and together they both contribute to stick-slip motion.

If the static friction is greater than the elastic force of the spring, then the mass is sticking and it rides along the belt at a uniform velocity equal to that of the belt. If the static friction force is exceeded by the elastic force of the spring then the mass slips, where upon the dynamic friction opposes its dynamical motion.

2.1 The mathematical model

Before the mathematical model of the system is specified a number of assumptions are made regarding the system, with reference to many of ideas that have been made in the introduction to this section.

Assumption 1: During a stick phase the mass will ride upon the belt at a constant velocity, and have zero acceleration.

Assumption 2: The mass cannot attain a velocity greater than that of the belt.

With these two assumptions in mind the general equation of motion for the single degree of freedom system during the slip phase can be defined as:

\[ m\ddot{x} + kx = F_d(\dot{x} - v_{dr}) \]  

(2.1)
Where \( m, k, x \) and \( v_{dr} \) are the mass, spring stiffness, displacement and belt velocity, respectively. The first and second derivatives of the displacement, with respect to time, are given as \( \dot{x} \) and \( \ddot{x} \). The dynamic friction force \( F_d(\dot{x} - v_{dr}) \) is typically considered to be a function of the relative velocity of the mass.

If equation 2.1 is now written as a system of first order ordinary differential equations:

\[
\dot{x} = v, \quad \dot{v} = \frac{-kx + F_d(v - v_{dr})}{m}
\] (2.2)

This system can now be easily solved by using a Runge-Kutta integration scheme to solve the equations. During a stick phase the equations of motion for the system which satisfy assumption 1 are:

\[
\dot{x} = v_{dr}, \quad \dot{v} = 0
\] (2.3)

The relationship between the elastic forces of the spring and the static friction \( F_s \), provide the conditions for impending slip i.e. the transition from a stick phase to a slip phase. During the stick phase the static friction is greater than the elastic forces of the spring:

\[
F_s > kx
\] (2.4)

Once the transition point is reached where the change in behaviour of the system from a stick phase to slip phase is impending, then the static friction is balanced by the elastic forces of the spring:

\[
F_s = kx
\] (2.5)
 CHAPTER 2. THE SINGLE DEGREE OF FREEDOM MODEL

Figure 2.2: The steady state behaviour of the limit cycle, showing the position of impending slip (shown as a black dot).

Hence the position of the stick to slip transition point can be determined. Figure 2.2 shows the steady state behaviour of the limit cycle and relationship between the static friction and the elastic forces of the spring.

For the duration of this investigation, the value of the maximum static friction force will be $F_s = 1$. The system possesses a fixed point located at:

$$ x = \frac{F_d(-v_{dr})}{k}, \quad \dot{x} = 0 $$

(2.6)

The stability of such a configuration can be established by choosing a point in the immediate neighbourhood of the fixed point. In this case we consider a point which is slightly deviated from the fixed point by a displacement of $\xi$:

$$ x = \xi + \frac{F_d(-v_{dr})}{k}, \quad \dot{x} = \dot{\xi} $$

(2.7)

and by considering the linearised equation:
\[ \ddot{\xi} + \frac{k}{m} \dot{\xi} - \frac{\dot{\xi}}{m} \left[ \frac{dF_d(\dot{\xi} - v_{dr})}{d\dot{\xi}} \right] = 0 \] (2.8)

Equation 2.8 is a linear differential equation with a damping coefficient equal to:

\[ - \left[ \frac{dF_d(\dot{\xi} - v_{dr})}{d\dot{\xi}} \right] \] (2.9)

Therefore if such a coefficient is positive, the equilibrium point is stable whereas in this case the equilibrium point is unstable due to a negative damping coefficient.

Now that a fully defined mathematical model of the single degree of freedom system has been constructed, investigations into the dynamical behaviour of the model can be initiated.

By using different dynamic friction characteristics in the model, that have various functional dependencies upon the relative velocity, paramount importance is placed upon identifying features of a dynamic friction characteristic are crucial for the occurrence of stick-slip oscillations.

2.2 Computational and Numerical Techniques

In order to make the numerical routines as efficient as possible, a number of algorithms are used which make use of a few system properties. These numerical and computational tools will considerably reduce the time and effort taken to carry out the analysis of the system, and increase the accuracy of the numerical simulation as a whole.
2.2.1 Integration of the equations of motion

The equations of motion for the single degree of freedom system are second order differential equations as shown in equation 2.1. Such problems that involve higher order differential equations can always be reduced to the study of first order differential equations, as shown in equation 2.2.

Here $x$ is the displacement of the system, $k$ the stiffness of the spring and $F_d$ the dynamic friction force, which is taken to be a function of the relative velocity $\dot{x} - \dot{v}_{dr}$ and where $\dot{v}$ is the new variable. Now the numerical integration routine is applied to each first order equation during each interval.

The main underlying idea of any routine for solving these types of differential equations is always: rewrite the $\delta x'$s and $\delta v'$s as finite steps, $\Delta x$ and $\Delta v$, and multiply the equations by $\Delta t$. This gives algebraic formulas for the change in functions when the independent variable $t$ is stepped by one stepsize $\Delta t$. In the limit of making the stepsize very small, a good approximation to the underlying differential equation is achieved. The literal implementation of this procedure results in Euler's method, which is not recommended for any practical use. However Euler's method is conceptually important, but all practical methods come down to the same basic idea: add small increments to your functions corresponding to the derivatives multiplied by stepsizes.

2.2.2 The Runge Kutta algorithm

The most regularly used algorithm to numerically integrate differential equations, is the classical fourth-order Runge-Kutta method. The Runge-Kutta family of algorithms has its origins in the idea of approximating a function by its Taylor series.

The forth-order Runge-Kutta method, that is presented here, requires the compu-
...tion of four auxiliary quantities \( k_1, k_2, k_3 \) and \( k_4 \) per time step \( \Delta t \). If we rewrite equation 2.2 as a set of equations which are expressed as functions of the form:

\[
\begin{align*}
    x_n & = f(v_n) \\
    v_n & = f(x_n, v_n)
\end{align*}
\]

Then the steps in the calculating the auxiliary quantities are:

\[
\begin{align*}
    x_{k1} & = \Delta t f(v_n) \\
    v_{k1} & = \Delta t f(x_n, v_n) \\
    x_{k2} & = \Delta t f(v_n + \frac{vk_1}{2}) \\
    v_{k2} & = \Delta t f(x_n + \frac{xk_1}{2}, v_n + \frac{vk_1}{2}) \\
    x_{k3} & = \Delta t f(v_n + \frac{vk_2}{2}) \\
    v_{k3} & = \Delta t f(x_n + \frac{xk_2}{2}, v_n + \frac{vk_2}{2}) \\
    x_{k4} & = \Delta t f(v_n + vk_3) \\
    v_{k4} & = \Delta t f(x_n + xk_3, v_n + vk_3)
\end{align*}
\]

\( x_{n+1} \) and \( v_{n+1} \) are then evaluated as a weighted averages over the \( k_n \) values as:

\[
\begin{align*}
    x_{n+1} & = x_n + \frac{1}{6} (x_{k1} + 2x_{k2} + 2x_{k3} + x_{k4}) \\
    v_{n+1} & = v_n + \frac{1}{6} (v_{k1} + 2v_{k2} + 2v_{k3} + v_{k4})
\end{align*}
\]
Figure 2.3: The Fourth-order Runge-Kutta method. In each step the derivative is evaluated four times: once at the initial starting point, twice at trial midpoints, and once at a trial endpoint. From these derivatives the final function solution (shown as a filled dot) is calculated. (From *Numerical Recipes in 'C'*).

Figure 2.3 shows how the auxiliary quantities are used by the Fourth-order Runge-Kutta method to calculate the solution for a particular differential equation.

The Runge-Kutta method detailed above can be used for nearly all conditions, however a good ODE integrator should exert some adaptive control over its own progress, making frequent changes in its stepsize. The purpose of this adaptive stepsize control is to achieve some predetermined accuracy in the solution with minimum computational effort. Many small steps should tiptoe through difficult areas of the system whilst a few great strides should speed through smooth and uncomplicated areas. The resulting gains in efficiency are not mere tens of percents or factors of two; they can sometimes be factors of ten, a hundred, or more.
2.3 Numerical algorithms used for the Stick-Slip system

A number of algorithms are used in the iteration of the stick-slip system. They are used to make the integration process more simpler as well as making it far more accurate than would otherwise be obtained be possible with brute force methods over a similar time period.

2.3.1 During a stick phase

During a stick phase the equations of motion are very simple are shown in equation 2.3. The mass rides upon the belt at the same velocity as the belt i.e. \( \dot{x} = v_{dr} \). This is probably the simplest form that the equations of motion take during the stick-slip cycle and as such are very easily integrated.

2.3.2 Initial slipping from a global stick phase

By choosing an initial condition where a global stick phase is obtained for the system, presents the problem of how to detect the point at which the mass will start upon a slip phase. This can be done quite easily by solving the impending slip condition in equation 2.5 where the static friction \( F_s \) is balanced by the elastic forces of the spring. However in practice it is actually easier to consider the impending slip condition as an inequality.

\[
F_s > kx
\]  

(2.13)

If the maximum static friction \( F_s \) is greater than the elastic forces of the spring, \( kx \) then the mass sticks. When \( F_s \) is less than \( kx \) then the mass slips, hence the
Figure 2.4: The point where the transition from sticking to slipping behaviour is shown by the black dot. The impending slip inequality is shown around various areas of the limit cycle.

exact position of the point of slipping can be determined. Figure 2.4 shows the premise of this idea more clearly.

2.3.3 During a slip phase

During a slip phase the following equation of motion is numerically integrated using the Runge-Kutta method.

\[
\begin{align*}
\dot{x} & = v \\
\dot{v} & = -kx \pm F_d(\dot{x} - v_{dr})/m \\
\end{align*}
\] (2.14)

As mentioned before the sign for the dynamic friction \(F_d\), is determined by the current relative velocity of the mass in question. In all the cases in this chapter it has been assumed that if the relative velocity is less that 0, i.e. \(\dot{x} - v_{dr} < 0\), then the dynamic friction takes on the form \(+F_d\), and vice versa.
2.3.4 End of a slip phase

Once the system has started upon a slip phase, then the dynamics of the system are integrated by using a suitable Runge-Kutta routine. During the slip phase the numerical routine also has to check if the slipping mass has started to stick, or if in the case of a system where more than one mass is involved, if any other masses have changed their behaviour from slipping to sticking during another mass's slip phase.

The numerical routine carries out such a check when the function $f(x, v_{dr}) = \dot{x} - v_{dr}$ changes sign. Because of the assumption that the mass cannot have a velocity greater than the belt velocity $v_{dr}$, the function $f(x, v_{dr})$ can never be positive.

Therefore in order to find the exact location of this point where a mass starts to stick following a slip phase, the numerical routine has to find the intersection of the flow of the dynamical system and the subset of the phase space defined by $f(x, v_{dr}) = 0$.

Following a particular article by Hénon (Hénon 1982), in which the computation of Poincaré maps is discussed, the integration routine can be arranged in such a way that one integration point exactly lies on such a subset. That is, the integration routine can be arranged in such a way that one integration point lies exactly on the subset where $f(x, v_{dr}) = 0$, and hence determine the exact instant in which a mass finishes slipping. The only requirement is that the integration routine should allow independent time steps, i.e. each new time step can be computed independently of the preceding time steps.

Hénons' method of computing Poincaré maps will be developed further and used to determine the point where $f(\dot{x}, v) = 0$. The numerical routine continuously checks the value of the function $f(\dot{x}, v)$. Equation 2.15 describes the system during the slip phase.
\[ \begin{align*}
\dot{x} &= v = f_1 \\
\dot{v} &= \frac{-kx \pm F_d(\dot{x} - v_d)}{m} = f_2 
\end{align*} \] (2.15)

If a change in the sign of the function \( f(\dot{x}, v) \) is detected, the numerical routine returns to the previous step and rearranges its own equations by dividing all but the last equations in 2.15 by the last one, and inverting the last equation i.e.

\[ \frac{\delta x}{\delta v} = \frac{f_1}{f_2}, \frac{\delta t}{\delta v} = \frac{1}{f_2} \] (2.16)

Now \( t \) is the dependent variable, \( v \) being independent. The system described by equations 2.16 is integrated by the Runge-Kutta method by using a *velocity time step* of \( \Delta v = -f(v, v_d) \), which in one *time step* gives the exact locations of the point at which \( f(\dot{x}, v) = 0 \) i.e. where \( v = v_d \). After this point has been established the Runge-Kutta integrating routine reverts back to the system described by equations 2.15.

Another aspect of the above algorithm that should be addressed is the accuracy and computational efficiency of the algorithm. The fact that the algorithm gives the exact location of the end of the slip phase for a particular motion of the mass means that there are *no errors* being carried forwards to the next integration step.

If a brute force method had been used to calculate the location of the end of the slip phase only *approximate* answers would have been achieved. The Runge-Kutta routine would have checked for the change in sign of the function \( f(\dot{x}, v) \), and once a change had been detected then it would return back to its previous integration point. At this previous point the time step interval would be reduced by a factor of say ten, and the integration process would start again. The change in the function \( f(\dot{x}, v) \) would be looked for again and the system would revert back to the previous integration point. The point at which the change in sign for the function \( f(\dot{x}, v) \) would be compared with the one previously obtained. If the difference between...
the two values was within a previously specified accuracy then that point would be taken to be a reasonably accurate estimate of the location of the end of the slip phase. The integration process would now continue with the system in a stick phase. Hence only an approximate location of the end of the slip phase would be obtained as well as being extremely computationally expensive.

2.4 The Classic Coulomb Friction Model

The classic Coulomb friction model is probably the most widely used friction law used in the study of systems which involve friction. There are many underlying reasons for this but the main one seems to be, that it is extremely simple, can be solved by classical means and is straightforward to implement. In fact before the availability of personal computers nearly all research on friction based systems used the Coulomb model, since at least some limited analysis was available. Den Hartog (Den Hartog 1931) solved such systems for periodic orbits, and more recently Shaw (Shaw 1986) determined the stability of such orbits.

With the classic Coulomb friction model it is assumed that the dynamic friction force remains constant during a period of slipping in a particular direction. A discontinuity is present where the mass is in a state of sticking and has a uniform velocity which is equal to that of the belt, i.e. $\dot{x} = v_{dr}$, with the relative velocity equal to zero, i.e. $\dot{x} - v_{dr} = 0$. Figure 2.5 shows a schematic diagram of the classic Coulomb friction model.

The general equation of motion for the system during the slip phase is the same as used in equation 2.1.

During the slip phase the general conditions for determining the values of the dynamic friction force $F_d$ based on a classic Coulomb model are given by:
The dynamic friction force $F_d$ during sticking i.e. when $\dot{x} = v_{dr}$ is as mentioned before, not uniquely defined, as it lies upon the discontinuity where the relative velocity is equal to 0 i.e. $\dot{x} - v_{dr} = 0$.

The results of the numerical simulation of the system using the classic Coulomb friction model are shown in figures 2.6(a) to 2.6(c).

Figure 2.6(a) shows the displacement time history for the single degree of freedom model. It shows that from initial starting conditions of $x = 0$ and $\dot{x} = 0$ the displacement of the mass increases gradually to a maximum of approximately $x = 1.5$. From this point it loses displacement until it attains a displacement of $x = 0.5$ at which point it starts to increase its displacement until it again reaches a maximum displacement of $x = 1.5$. The displacement of the mass then follows the same sinusoidal pattern for the rest of the simulation.
Figure 2.6: Time histories for the single degree of freedom model using the classic Coulomb dynamic friction curve.
Figure 2.6(b) shows the corresponding time history for velocity. It shows that from initial starting conditions of $x = 0$ and $\dot{x} = 0$ the mass is initially rapidly accelerated towards the belt velocity $v_{dr}$. The mass achieves the belt velocity and prepares to stick.

The mass has now entered the stick region of the friction curve and it rides along the belt at a uniform velocity of $\dot{x} = v_{dr}$ for about 2.5 seconds. During this period of sticking the static friction force $F_s$ is greater than the elastic force of the spring i.e. $F_s > kx$. The dynamic friction force is not uniquely defined during sticking, and is therefore not plotted.

At approximately time= 3.0s the mass begins to slip, its velocity decreases to a minimum of $-0.5$ but then recovers and increases until it reaches the belt velocity $v_{dr}$. It is interesting to note that during the initial period of slipping the displacement of the mass is still increasing, as shown in figure 2.6(a). This is because after the stick phase the mass is still being accelerated, the displacement reaches its maximum value only when the velocity of the mass has reached zero. After the mass has attained the belt velocity it then immediately starts to slip again and eventually oscillates with a constant period $2\pi$ for the rest of the numerical simulation, which corresponds to a oscillation at the natural frequency.

Figure 2.6(c) shows the time history for the dynamic friction force for this particular case.

At approximately time=1.0 when the mass begins to attain the belt velocity $v_{dr}$ and stick, the dynamic friction force is undefinable as the relative velocity of the mass is equal to zero and lies upon the discontinuity $\dot{x} - v_{dr} = 0$.

After the stick phase has passed, at approximately time=3.0, the friction force remains constant at its maximum value of $+F_d$ for the rest of the simulation. This is because the mass never attains a velocity greater than the belt velocity and always remains in an area of the dynamic friction function where $\dot{x} - v_{dr} < 0$, which has a corresponding constant dynamic friction value of $+F_d$. 
Figure 2.7: The phase portrait for the single degree of freedom model using the classic Coulomb dynamic friction curve.

Figure 2.7 shows the corresponding phase portrait for this system.

From the phase portrait it can be observed that the mass is accelerated from the initial conditions \( x = 0 \) and \( \dot{x} = 0 \) towards the belt velocity \( v_{dr} \), as with the velocity time history. Once achieving the belt velocity the mass enters the stick region of the dynamic friction curve, and rides along the belt with a uniform velocity of \( \dot{x} = v_{dr} \). This stick phase can be seen on the phase portrait as the straight horizontal line which touches the limit cycle. Eventually the mass begins to slip as the elastic forces of the spring overcome the static friction \( F_s \). The point at which this occurs at has co-ordinates \( x = 1.0, v = 0.5 \). At first it loses velocity and displacement due to the elastic forces of the spring pulling it back and slowing the mass down. The mass continues to slip with increasing negative velocity as the spring exerts its elastic forces upon the mass, until it reaches a point where the elastic forces of the spring are actually less then the static friction. At this point the mass then begins to recover and increases its velocity until it reaches the belt velocity \( v_{dr} \), at which point it immediately starts to slip again. The motion of the system follows the same pattern for the rest of the simulation.

Further investigation of the classic Coulomb friction model was required to verify
an important aspect of the results obtained from the numerical simulation. After
the initial stick phase there does not seem to be another stick phase for the rest of
the simulation, although it seems from the phase portrait and time histories that
the mass appears to achieve the belt velocity. Another aspect that could be seen
from the phase portrait, figure 2.7, is that the position where the mass slips is the
same as, or is extremely close to the position where the mass sticks. In this case
the mass could be slipping continuously, never being able to stick thus stick-slip
motion was not occurring.

For these questions to be adequately answered it was necessary to rerun the simu­
lation with an extremely small time step, in this case \( \delta t = 10^{-6} \), in order to make
it very accurate and check if a stick phase occurs after the initial stick phase. The
results of the accurate rerun of the simulation were:

- The mass reaches the belt velocity on a number of occasions, after the initial
  stick phase.
- The mass actually enters the stick phase of the friction curve.
- The total duration for a stick phase, after the initial stick phase is 0.0088
  seconds, and the actual velocity change before and after the stick phase
  occurs is 0.00445 m/s.

It was confirmed that the mass does stick for an extremely small period of time,
and immediately starts to slip again. The period of time that the mass was sticking
was so small that it would have been totally missed if a very small time step was
not chosen. Thus stick-slip motion occurs throughout the simulation, even if the
period of sticking is small when compared to the overall dynamic time.
2.5 The Continuous Coulomb Friction Model

The continuous Coulomb friction model is basically the same as the classic Coulomb model, but it differs in the fact that it does not possess a discontinuity at $\dot{x} - v_{dr} = 0$. Instead the sticking region of the friction curve has a slight negative slope to it, instead of the vertical discontinuity as in the classic Coulomb case.

The major difference of having a stick section with a gradient is that it is now assumed that there is a small zone of velocities at which sticking can occur, instead of the single sticking velocity $\dot{x} = v_{dr}$ at the discontinuity as assumed in the classic Coulomb model.

The consequence of this assumption is that three velocities have to be specified. Velocities $v_{dr1}$ and $v_{dr2}$ are the velocities which specify the transition between sticking and slipping behaviour, while $v_{dr}$ specifies the actual belt velocity $v_{dr}$. Together they also govern the actual slope of the sticking region of the friction curve. Figure 2.8 shows a schematic diagram of the continuous Coulomb friction curve.

With the continuous Coulomb friction model is assumed that the dynamic friction force remains constant as the mass is slipping in a particular direction, as in the classic Coulomb friction model. In this case the dynamic friction force can be defined during the sticking phase, unlike the classic Coulomb friction model.

The general equation of motion for the system during the slip phase is the same as in equation 2.1.

During the slip phase the general condition for determining the dynamic friction force is given by:

\[
F_d(\dot{x} - v_{dr}) = \begin{cases} 
1, & \dot{x} < v_{dr1} \\
-1, & \dot{x} > v_{dr2}
\end{cases}
\]  

(2.18)
Figure 2.8: The continuous Coulomb dynamic friction curve. Velocities $v_{dr1}$ to $v_{dr2}$ signify the range of velocities that sticking can occur, whilst $v_{dr}$ is the actual belt velocity.

$v_{dr1}$ to $v_{dr2}$ are velocities that signify the range of within which sticking can occur, whilst $v_{dr}$ is the actual belt velocity.

During the stick phase of the general condition for determining the dynamic friction force is given by:

$$F_d(\dot{x} - v_{dr}) = G_{\text{stick}} \ast (\dot{x} - v_{dr}), \quad v_{dr1} < \dot{x} < v_{dr2}$$

In equation 2.19 the parameter $G_{\text{stick}}$ refers to the gradient of the stick section on the friction curve.

During the period where the onset of sticking occurs the relative velocity of the mass $\dot{x} - v_{dr}$ is not equal to zero. The consequence of this is that the mass does not immediately start to stick, as in the classic Coulomb model but begins a gradual path along the sloping stick region of the friction curve towards sticking and trying to achieve a relative velocity equal to zero.
The results obtained from the continuous Coulomb friction model are shown in figures 2.9(a) to figure 2.9(c).

Figure 2.9(a) shows the displacement time history for the single degree of freedom model. From the time histories it can be seen that initially the mass is rapidly accelerated from the initial conditions $x = 0$ and $\dot{x} = 0$ until it attains a maximum velocity just below the belt velocity $v_{dr}$. The dynamic friction force during this period of rapid acceleration decreases rapidly from its maximum possible value of $+F_d$ to a minimum of approximately 0.27.

Once the mass attains its maximum velocity it is now within the sticking region of the friction curve, as it has already attained a velocity greater than $v_{dr1}$, the velocity which specifies the onset of sticking. The mass is now in a stick phase with a velocity somewhere between $v_{dr1}$ and $v_{dr}$, and it gradually loses velocity at a constant rate, until a velocity slightly lower than $v_{dr1}$ is achieved at which point the mass begins to slip.

During this stick phase the dynamic friction force steadily increases until it reaches its maximum possible value of $+F_d$, which coincides with the onset of slipping for the mass.

The mass has now entered the slip section of the friction curve and its velocity decreases to a minimum of approximately $-0.3$ as the slip phase continues, sending the mass into an area of the friction curve where the mass has a relative velocity which is negative, $\dot{x} - v_d < 0$. The dynamic friction force during this period remains constant at $+F_d$.

The velocity of the mass now begins to increase until it reaches a velocity of $v_{dr1}$, the velocity which indicates the onset of the sticking phase. The mass however immediately slips and eventually seems to oscillate with a constant period $2\pi$, corresponding to an oscillation at the natural frequency of the system. The dynamic friction force during this period remains constant at $+F_d$. 
Figure 2.9: Time histories for the single degree of freedom model using the continuous Coulomb dynamic friction curve.
After the initial starting interval, the velocity amplitude reached for the mass is \( v_{dr1} \), which is the velocity at which the onset of sticking occurs. The mass never attains a velocity higher than \( v_{dr1} \), so the sticking part of the dynamic friction curve is never breached so a constant force is exerted on the mass.

Figure 2.10 shows the phase portrait for this system. It can again be observed that the mass is accelerated from rest towards the belt velocity.

The mass never actually reaches the belt velocity, but achieves a velocity slightly lower than the belt velocity \( v_{dr} \). The mass has now entered the stick section of the friction curve as it has already attained a velocity of \( v_{dr1} \), the velocity which specifies the onset of sticking. The mass is now in a stick phase with a velocity somewhere between \( v_{dr1} \) and \( v_{dr} \), and it gradually loses velocity during the stick phase until it attains a velocity of \( v_{dr1} \), at which point the mass begins to slip. This can be seen as a straight line with a slight negative slope to it which connects to the limit cycle.

Eventually the mass slips, at first it loses velocity and displacement. The mass then begins to recover and increases its velocity until it reaches the velocity \( v_{dr1} \), at
which point it immediately starts to slip again. The motion of the system follows the same pattern for the rest of the simulation.

Apart from the initial stick phase the system never achieves a high enough velocity to undergo another stick phase. The system seems to be in a perpetual cycle of continuous slipping rather reminiscent of the long term behaviour of the Classic Coulomb system.

The only difference being that the amplitude of the oscillation in the continuous Coulomb case is less than that attained by the mass in the classic Coulomb case. This is attributable to the fact that the maximum static friction force is attained at a lower velocity of $v_{d1}$ and not at the belt velocity. Thus the mass will start upon its stick phase at the lower velocity, experience less displacement, and in turn have a lower amplitude.

### 2.6 The Discontinuous friction model with a positively sloping slip section

In the two friction models that have been previously discussed, it has been assumed that the dynamic friction force remains constant during a period of slipping in a particular direction.

The discontinuous friction model with a positively sloping slip section assumes otherwise. Drawing from conclusions by Glaton, after many experimental studies, it is now assumed that the dynamic friction force is now a decreasing function of the relative velocity.

Figure 2.11 shows a schematic diagram of the friction curve for this model. It basically resembles the classic Coulomb friction model as it possesses a vertical discontinuity at $\dot{x} = v_{dr}$, but with the slip sections of the dynamic friction characteristic having a slight positive slope to them.
The general equation of motion for the system during the slip phase is the same as in equation 2.1.

The general conditions for the determining the friction force during the slip phase are given by:

\[
F_d (\dot{x} - v_{dr}) = G_{slip} * (\dot{x} - v_{dr}) + F_s, \quad \dot{x} < v_{dr}
\]

\[
F_d (\dot{x} - v_{dr}) = G_{slip} * (\dot{x} - v_{dr}) - F_s, \quad \dot{x} > v_{dr}
\]

(2.20)

In equation 2.20 the parameter $G_{slip}$ refers to the gradient of the slip section of the dynamic friction curve.

During the period where sticking occurs the relative velocity $\dot{x} - v_{dr}$ is equal to zero, therefore the mass has a motion which is uniform with a constant velocity of $\dot{x} = v_{dr}$. The sticking region lies on the discontinuity and thus the dynamic friction force in this region cannot be determined.
As the slip sections of this dynamic friction curve have a small positive slope to them, an intersection of the slip sections with the relative velocity axis would correspond to a force reversal, which is physically unrealistic. Therefore only a velocity range around the belt velocity $v_{dr}$, sufficiently far from any intersections with the relative velocity axis is considered as valid.

The results obtained for the discontinuous friction model with a positively sloping slip sections, are shown in figure 2.12(a) to figure 2.12(c).

Figure 2.12(a) shows the displacement time history for the single degree of freedom model. It shows that from initial starting conditions of $x = 0$ and $\dot{x} = 0$ the mass' displacement increases gradually to a maximum of approximately $x = 1.5$. From this point it loses displacement until it attains a displacement of $x = 0.4$ at which point it starts to increase its displacement until it again reaches a maximum displacement of $x = 1.5$. The displacement of the mass then follows the same sinusoidal pattern for the rest of the simulation.

From the velocity time history shown in figure 2.12(b) it can be observed that the mass is rapidly accelerated from rest until it reaches the belt velocity $v_{dr}$. The friction force during this period of rapid acceleration increases slightly to achieve its maximum possible value of $F_a$. The mass is now in a stick phase there follows a significant period where sticking occurs, after which the mass then begins to slip. The dynamic friction force during the period of sticking is multi-valued and is therefore undefinable as it lies upon the discontinuity $\dot{x} - v_{dr} = 0$.

The mass has now entered the slip section of the friction curve, and its velocity decreases as the slip phase continues, sending the mass into an area of the friction curve where the mass has a relative velocity which is negative, i.e. $\dot{x} - v_{dr} < 0$. Eventually the velocity of the mass begins to increase until it again attains the belt velocity $v_{dr}$. The dynamic friction force during this period decreases slightly reaching a minimum value of approximately 0.9475 which coincides with the minimum velocity of $-0.55$ attained by the mass, but then begins to increase
Figure 2.12: Time histories obtained for the single degree of freedom model using the discontinuous dynamic friction curve with positively sloping slip sections.
as the velocity of the mass increases and eventually reaches its maximum possible value of $+F_d$ when the belt velocity is achieved. The mass is now in a stick phase, but slips after a short period. This period of sticking is significantly shorter than the initial sticking period, and the motion of the system then follows the same pattern of behaviour of slipping and sticking for short periods for the rest of the numerical simulation.

This has been the first simulation where there has been a sustained stick-slip motion throughout the simulation. All previous simulations using the classic Coulomb friction model and the continuous Coulomb friction model have only exhibited stick-slip motion in the beginning of their numerical simulations.

The degree of sticking in this case increases the period of the oscillation and this in turn leads to a lowering of the fundamental frequency of the oscillation. The friction force oscillates almost sinusoidally and remains close to $+F_d$, fluctuating only a small amount.

Figure 2.13 shows the phase portrait of the system; again it shows the mass being accelerated to the belt velocity and then sticking for a relatively long period of time. The mass then reaches its stick to slip transition point and slips. After undergoing a slip phase the mass again sticks at the belt velocity $v_d$. The motion of the system then follows this same pattern for the duration of the simulation.

Unlike the other dynamic friction models looked at so far, the limit cycle obtained in this case has the transition points clearly separated. As a result transitions from the onset of sticking to the onset of slipping are separated by relatively long periods of sticking.
2.7 Continuous Friction model with positively sloping slip sections

The continuous friction model with positively sloping slip sections is based upon the continuous Coulomb friction model. It differs from the other models' in that both the stick section and the slip section of the dynamic friction curve have a slight gradient to them. The dynamic friction force during a slip phase decreases with increasing relative velocity at a linear rate.

Figure 2.14 shows a graphical representation of this type of friction curve.

As mentioned, the major aspect of this friction model is that as the stick section has a slight gradient, and it is now assumed that there are now a small zone of velocities at which sticking can occur, instead of the single sticking velocity usually assumed with the discontinuous models. This is the same assumption that is made with the other continuous friction models.

The consequence of this assumption is that three velocities have to specified. Ve-
Figure 2.14: Schematic of the continuous friction model with positively sloping slip sections. Velocities $v_{dr1}$ to $v_{dr2}$ signify the range of velocities where sticking can occur, whilst $v_{dr}$ is the actual belt velocity.

Velocities $v_{dr1}$ and $v_{dr2}$ are the velocities which specify the transition between sticking and slipping behaviour, while $v_{dr}$ specifies the actual belt velocity $v_{dr}$. Together they also govern the actual slope of the sticking region of the friction curve.

As the slip sections of this friction model have a small positive slope to them, an intersection of the slip sections with the relative velocity axis would correspond to a force reversal which is physically unrealistic. Therefore only a velocity range around the belt velocity $v_{dr}$, sufficiently far from any intersections with the relative velocity axis is considered as valid.

The general equation for the system during the slip phase is the same as equation 2.1, which was expressed in the introduction to this chapter.

The general conditions for determining the friction force are given by:
\[ F(\dot{x} - v_{dr}) = G_{\text{stick}} \ast (\dot{x} - v_{dr}) + 1.01, \quad \dot{x} < v_{dr1} \]
\[ F(\dot{x} - v_{dr}) = G_{\text{stick}} \ast (\dot{x} - v_{dr}), \quad v_{dr1} < \dot{x} < v_{dr2} \]  
\[ F(\dot{x} - v_{dr}) = G_{\text{slip}} \ast (\dot{x} - v_{dr}) - 1.01, \quad \dot{x} > v_{dr2} \]

In equation 2.21 the parameter \( G_{\text{stick}} \) refers to the gradient of the stick section of the dynamic friction curve, whilst the parameter \( G_{\text{slip}} \) refers to the gradient of the slip section of the friction curve.

During the period where the onset of sticking is occurring, the relative velocity of the mass \( \dot{x} - v_{dr} \) is not equal to zero. The consequence of this is that the mass begins a gradual path towards sticking instead of the sudden slipping to sticking behaviour experienced with the various discontinuous models.

The results obtained from the continuous friction model with positively sloping slip sections are shown in figure 2.15(a) to figure 2.15(c).

Figure 2.15(a) shows the displacement time history for the single degree of freedom model. It shows that from initial starting conditions of \( x = 0 \) and \( \dot{x} = 0 \) the mass’s displacement increases gradually to a maximum of approximately \( x = 1.3 \). From this point it loses displacement until it attains a displacement of \( x = 0.65 \) at which point it starts to increase its displacement until it again reaches a maximum displacement of \( x = 1.3 \). The displacement of the mass then follows the same sinusoidal pattern for the rest of the simulation.

From the velocity time history show in figure 2.15(b) it can be seen that the mass is initially rapidly accelerated from rest towards the belt velocity \( v_{dr} \). However the mass never attains the same velocity as the belt but achieves a velocity which is slightly less.

The dynamic friction force during this initial period starts to decline rapidly as the mass accelerates towards the belt velocity. The lowest value for the dynamic friction force of approximately 0.3 is obtained as the mass achieves its maximum
Figure 2.15: Time histories obtained for the single degree of freedom model using the continuous dynamic friction curve with positively sloping slip sections.
velocity of approximately 0.45, a fraction below the actual belt velocity $v_{dr}$.

The mass is now in the sticking region of the friction curve as it has already attained a velocity greater than $v_{dr1}$, the velocity which specifies the onset of the sticking phase. The mass is now in a stick phase with a velocity somewhere between $v_{dr1}$ and $v_{dr}$, and it gradually loses velocity during the stick phase until it reaches a velocity just below $v_{dr1}$, a velocity at which the mass begins to slip.

During the stick phase the dynamic friction force steadily increases until it reaches its maximum possible value of $+F_d$ (+1 in this case).

The mass has now entered the slip section of the friction curve and its velocity decreases as the slip phase continues, sending the mass into an area of the friction curve where the mass has a relative velocity which is negative, i.e. $\dot{x} - v_{dr} < 0$. The dynamic friction force during the slip phase is reduced slightly, but not by a significant amount. The mass attains a minimum velocity of approximately $-0.45$ whilst the dynamic friction force attains a value of approximately 0.95 during this period.

The mass’s velocity now begins to increase until it reaches a velocity of $v_{dr1}$, the velocity which indicates the onset of the sticking phase. The dynamic friction force at this point attains its maximum value of $+F_d$.

The mass begins to stick, but the velocity of the mass is still increasing, a fact that is confirmed by the dynamic friction force decreasing during this period (remember that it is now a decreasing function of increasing relative velocity). The mass eventually attains a maximum velocity just a little higher than $v_{dr1}$, and then begins to lose velocity during the rest of the stick phase. The dynamic friction force during this period begins to increase and reaches its maximum possible value of $+F_d$ towards the end of the stick phase. The mass now attains a velocity slightly lower than $v_{dr1}$, at which point it begins to slip. The mass's motion follows the same pattern of behaviour for the rest of the numerical simulation.
Figure 2.16 shows the phase portrait for this particular case. It can be observed that the mass is accelerated from rest to a velocity that is slightly lower than the belt velocity $v_{dr}$. It has now entered the stick region of the friction curve and it gradually loses velocity along the stick path until it reaches a velocity of $v_{dr1}$, the velocity which specifies the onset of the slipping phase. The mass slips and eventually sticks again at a velocity of $v_{dr1}$, after completing its slipping phase. The mass then undergoes a very quick stick phase and slips again. The masses motion now follows the same pattern of behaviour for the rest of the numerical simulation.

The motion produces a limit cycle, the only possible asymptotic motion for this type of system. The stick phases in this system, if the first initial stick phase is ignored, are extremely short and quick with the velocity of the mass during the stick phase fluctuating between $v_{dr1}$ and $v_{dr}$. 
2.8 Discontinuous friction model with positively sloping non-linear slip sections

In all the friction models that have been discussed so far the dynamic friction force has been assumed to be constant or a linear function of the relative velocity of the mass. In this case the discontinuous friction model with positively sloping non-linear slip sections is radically different. Figure 2.17 shows a schematic diagram of this friction model. The slip sections of this friction curve have a positive slope to them, but are also asymptotic to the relative velocity axis so they do not intersect with it.

This particular friction model possesses a discontinuity during a stick phase, where the mass has the same velocity as the belt, $\dot{x} - v_{dr} = 0$, thus at this point it is not uniquely defined.

The general equation of motion for the system during the slip phase is the same
as in equation 2.1, which was expressed at in the introduction to this chapter.

The general conditions for determining the dynamic friction force during the slip phase are:

\[
F(\dot{x} - v_{dr}) = \begin{cases} 
\pm \frac{F_d}{1 + \gamma |\dot{x} - v_{dr}|}, & \dot{x} < v_{dr} \\
-\frac{F_d}{1 + \gamma |\dot{x} - v_{dr}|}, & \dot{x} > v_{dr}
\end{cases}
\]  

(2.22)

During the period where sticking is occurring the relative velocity is equal to zero, \( \dot{x} - v_{dr} = 0 \), and the mass has a motion which is uniform with a constant velocity of \( \dot{x} = v_{dr} \).

The results obtained from the discontinuous friction model with positively sloping non-linear slip sections are shown in figure 2.18(a) to figure 2.18(c). Figure 2.18(a) shows the displacement time history for the single degree of freedom model. It shows that from initial starting conditions of \( x = 0 \) and \( \dot{x} = 0 \) the mass's displacement increases gradually to a maximum of approximately \( x = 1.4 \). From this point it loses displacement until it attains a displacement of \( x = -0.97 \) at which point it starts to increase its displacement until it again reaches a maximum displacement of \( x = 1.4 \). The displacement of the mass then follows the same sinusoidal pattern for the rest of the simulation. The displacement time history also shows another feature that should be addressed. The actual shape of the time history takes on a saw tooth function shape which corresponds to periods of sticking being greater that periods of slipping.

From the velocity time history it can be seen that the mass is initially rapidly accelerated from rest toward the belt velocity, \( v_{dr} \). The mass achieves the belt velocity and begins to stick. The friction force during this period increases as the mass reaches the belt velocity reaching a value just short of its maximum possible value, \( F_d \). The mass has now entered the stick region of the friction curve and it travels along the belt at a uniform velocity of \( \dot{x} = v_{dr} \), for approximately 3.0
Figure 2.18: Time histories obtained for the single degree of freedom model using the discontinuous dynamic friction curve with positively sloping non-linear slip sections.
seconds. The dynamic friction force during the stick phase is not uniquely defined as the relative velocity of the mass lies upon the discontinuity at $\dot{x} - v_{dr} = 0$. The mass then begins to slip and follows a slip path. During the slip path the velocity of the mass decreases to a minimum of approximately $-1.15$, but then recovers and increases until it reaches the belt velocity $v_{dr}$ at which point it begins to stick again. The friction force during this period also decreases in value as the mass follows its slip path, reaching its lowest value of $0.15$, at the point where the mass has its lowest velocity, and then increases to just short of its maximum possible value, $+F_{fr}$. The mass has now again entered the stick region of the friction force and rides along the belt at a uniform velocity $\dot{x} = v_{dr}$. This time the period of sticking is about 5.0 seconds which is even longer than the first initial sticking period of 3.0 seconds. After this extensive period of sticking the mass eventually begins to slip. The velocity time history of the mass then follows the pattern of behaviour as previously described for the rest of the numerical simulation.

Figure 2.19 shows the phase portrait for this system. It can be seen that the mass is accelerated from rest to a the belt velocity very rapidly. Once achieving the belt velocity the mass enters the stick region of the friction curve, and rides along the belt with a uniform velocity of $\dot{x} = v_{dr}$. This stick phase can be seen as the flat top of the limit cycle. Eventually the mass begins to slip, and follows a slip path where the mass at first loses velocity and displacement. The mass then recovers and increases its velocity until it reaches the belt velocity $v_{dr}$, where it sticks again. The mass now sticks for a long period and follows the path of the flat top of the limit cycle and eventually slip again. The motion for the rest of the simulation follows the same pattern.

There is one aspect about this particular friction model which sets it aside from the various other friction models that have been discussed; the stick phase seems to last a considerable period of time. This so far happens to be the only dynamic friction model which has exhibited this prolonged sticking phase.

There seems to be a distinct stick-slip behaviour in this case and not just a very
quick and short lived stick phase, as experienced with some other friction models.

2.9 Continuous friction model with positively sloping non-linear slip sections

The continuous friction model with positively sloping non-linear slip sections is based upon the discontinuous friction model with positively sloping slip sections. It differs from the discontinuous friction model in that it does not possess a discontinuity at \( \dot{x} = v_{dr} \) but the slip section of the dynamic friction curve has a slight gradient to it.

With this continuous friction model as with the other continuous models it is assumed that there are now a small zone of velocities at which sticking can occur, instead of the single sticking velocity usually assumed with a discontinuous model.

The consequence of this assumption is that three velocities have to be specified. Velocities \( v_{dr1} \) and \( v_{dr2} \) are the velocities which specify the transition between sticking
and slipping behaviour, while \( v_{dr} \) specifies the actual belt velocity \( v_{dr} \). Together they also govern the actual slope of the sticking region of the friction curve. Figure 2.20 shows a schematic diagram of this type of friction curve.

The slip sections of this friction curve have a slight positive slope to them, but they are asymptotic to the relative velocity axis and so they do not intersect with it.

The general equation of motion for the system during the slip phase is the same as in equation 2.1, which was expressed at in the introduction to this chapter.

The general conditions for determining the dynamic friction force are:

\[
F(\dot{x} - v_{dr}) = \frac{F_s}{1 + \gamma |\dot{x} - v_{dr}|}, \quad \dot{x} < v_{dr1}, \\
F(\dot{x} - v_{dr}) = G_{stick} \ast (\dot{x} - v_{dr}), \quad v_{dr1} < \dot{x} < v_{dr2}, \\
F(\dot{x} - v_{dr}) = \frac{-F_s}{1 + \gamma |\dot{x} - v_{dr2}|}, \quad \dot{x} > v_{dr2}
\] (2.23)
In equation 2.23 the parameter $G_{stick}$ corresponds to the gradient of the stick section of the dynamic friction curve.

During the period where the onset of sticking occurs, the relative velocity of the mass $\dot{x} - v_{dr}$ is not equal to zero. The consequence of this is that the mass begins a gradual path towards sticking at the belt velocity $v_{dr}$ instead of the sudden slipping to sticking behaviour experienced with the various discontinuous models.

The results obtained from the continuous friction model with positively sloping non-linear slip sections are shown in figure 2.21(a) to figure 2.21(c).

Figure 2.21(a) shows the displacement time history for the single degree of freedom model. It shows that from initial starting conditions of $x = 0$ and $\dot{x} = 0$ the mass's displacement increases gradually to a maximum of approximately $x = 1.2$. From this point it loses displacement until it attains a displacement of $x = -0.7$ at which point it starts to increase its displacement until it again reaches a maximum displacement of $x = 1.2$. The displacement of the mass then follows the same pattern for the rest of the simulation. Again as in the discontinuous model the displacement time history takes on a saw tooth function shape which corresponds to periods of sticking being greater that periods of slipping.

From the velocity time history it can be seen that the mass accelerated from rest towards the belt velocity $v_{dr}$. However the mass never actually attains the same velocity as the belt but achieves a velocity of $v = 0.45$ which is slightly below $v_{dr}$.

The friction force during this period increases to a maximum of $+F_d$ as the mass accelerates towards $v_{dr1}$. Once the mass begins a path towards sticking, the friction force drops dramatically.

The mass begins to stick, but the velocity of the mass is still increasing, reducing the friction force even further. The friction reaches its lowest value of 0.25 when the mass attains its maximum velocity of $v = 0.45$, a little below the belt velocity $v_{dr}$. 
Figure 2.21: Time histories obtained for the single degree of freedom model using the continuous dynamic friction curve with positively sloping non-linear slip sections.
Once the mass attains its maximum velocity it is now within the sticking region of
the friction curve as it has already attained a velocity of \( v_{dr1} \), the velocity which
specifies the onset of the sticking phase. The mass is now in a stick phase with
a velocity somewhere between \( v_{dr1} \) and \( v_{dr} \), and it gradually loses velocity during
the stick phase until it reaches a velocity of \( v_{dr1} \), at which point the mass begins
to slip.

During this stick phase the dynamic friction force steadily increases until it reaches
its maximum possible value of \( +F_d \) (+1 in this case), which coincides with the onset
of slipping for the mass.

The mass has now entered the slip section of the friction curve and its velocity
decreases as the slip phase continues, sending the mass into an area of the friction
curve where the mass has a relative velocity which is negative, \( \dot{x} - v_{dr} < 0 \).

The friction force decreases during this period of the slip phase, to a point where
it reaches a minimum value of 0.2 when the mass reaches its minimum velocity of
\( v = -0.95 \).

The velocity of the mass now begins to increase until it reaches a velocity of \( v_{dr1} \),
the velocity which indicates the onset of the sticking phase. The mass begins to
stick, but the velocity of the mass is still increasing, a fact that is confirmed by
the dynamic friction force decreasing during this period (remember that it is a
decreasing function of increasing relative velocity). The mass eventually attains a
velocity of \( V = 0.58 \) which is higher than the belt velocity \( v_{dr} \), and then proceeds
to lose velocity for the rest of the stick phase, until it reaches a velocity of \( v_{dr1} \), at
which point the mass begins to slip.

The friction force during this period begins to increase from its lowest value of
0.2, at the minimum velocity of \( v = -0.95 \) for the mass during the slip phase. It
reaches a peak of \( +F_d \) (+1 in this case) when the mass attains a velocity of \( v_{dr1} \),
after which it decreases dramatically to a value of \(-0.4 \) when the mass attains its
maximum velocity of \( v = 0.58 \) during the start of the stick phase.
As the mass proceeds to lose velocity during the rest of the stick phase the friction force increases to its maximum possible value of $-F_s$ when the mass attains a velocity of $v_{d1}$ and begins to slip.

The motion of the mass follows the same pattern of behaviour for the rest of the numerical simulation.

Figure 2.22 shows the phase portrait for the discontinuous friction model with positively sloping non-linear slip sections. It can be seen that the mass is accelerated from rest to a velocity just below the belt velocity $v_{dr}$. Once achieving this velocity the mass enters the stick region of the friction curve and sticks. The stick phase is the straight line at a gradient that makes up the top of the limit cycle.

During the stick phase the mass loses velocity until it reaches a velocity of $v_{d1}$, at which point the mass begins to slip. The mass then follows a slip path during which the mass at first loses velocity and displacement. The mass then recovers increasing its velocity until it reaches a velocity a little higher than the belt velocity $v_{dr}$, where it sticks again. The motion for the rest of the simulation follows the same pattern of behaviour.
As with the discontinuous friction model with positively sloping non-linear slip sections this friction model has stick phases which tend to last for a considerable period of time. The limit cycle that is produced for this system is very similar to the discontinuous model but as the stick phase path has a slight gradient to it. Thus during the stick phase the velocity of the mass fluctuates between $v_{dr1}$ and $v_{dr}$.

2.10 Summary

This chapter has dealt with the various features of the dynamic friction characteristic, in order to determine their effect upon the behaviour of the single degree of freedom system.

The gradient of the slip section of the friction characteristic was found to be a crucial parameter. Only a non-zero positive slope would lead to a distinct stick-slip oscillation of constant amplitude. The friction characteristics which were related to a Coulomb friction type of curve, where the dynamic friction force is constant during the slip phase, did not allow for the possibility of distinct stick-slip oscillations possible. Therefore these types of friction characteristics are deemed as being physically unrealistic models for dry friction, for this particular system.

Experimental investigations in the next chapter will determine which if any of the above friction characteristics will be deemed the most realistic for the single degree of freedom system presented here.
Chapter 3

Experimental Investigations

The single degree of freedom model for friction induced stick-slip vibrations was first introduced by Lord Rayleigh (Rayleigh 1945) in his investigations into the oscillations of violins strings, although Hemholtz also carried out similar investigations around about the same time.

Since the system is one of the simplest mechanical systems which exhibits the phenomenon of stick-slip of self sustained oscillations induced by dry friction, it has been investigated by various research groups around the world. Researchers such as Feeny and Moon, Perrin and Heslot, Popp and Wiercigroch are but a few who are active contributors in the study of these systems. Many of these groups have carried out experimental investigations of systems designed to mimic the behaviour of the single degree of freedom model, and have used their findings to substantiate many of their theoretical and numerical hypotheses, and to distinguish between numerical and physical effects.

Thus the purpose of this chapter is to examine the experimental results from a mechanical model, which closely matches the single degree of freedom model, with numerical simulations that have been carried out in the previous chapter, and determine which of the various dynamic friction curves gives the most physically
realistic results.

The previous work in this thesis dealt more with determining which parts of the dynamic friction model had the most profound effect upon the stick-slip behaviour of the single degree of freedom model. However, in order to understand the nature of the system the most physically realistic dynamic friction model should be used. It would be wrong to suggest that only an exact model should be used, as dry friction is a far more complicated problem for that to be possible, but a reasonably similar qualitative model would be acceptable.

3.1 The experiment

A schematic representation of the single degree of freedom system is shown in figure 3.1 where a mass is connected by a spring to a fixed support and is placed upon a moving conveyor belt.

Figure 3.2 shows a photograph of the actual experiment used in the experimental investigations. The experimental rig was constructed by Mr Malcolm Sayatch of UCL’s Civil Engineering technical department.

The experimental rig consists of a sturdy frame, which holds two rollers apart. The distance between the rollers can be altered by moving the axle of a roller along a threaded section of the frame and tightening up nuts either side of the axle.

This allows belts to be put over the rollers and then tensioned by moving one of the rollers slightly outwards. The mass, if placed directly upon the tensioned conveyor belt, would cause the belt to sag and affect the motion of the mass.

To prevent the belt from sagging, a piece of aluminium sheeting a little wider than the belt and shorter than the distance between the rollers was fabricated. The sheeting was then placed between the rollers at a height a little lower than the top
of the rollers and fixed to the frame of the rig.

The belt, when put over the rollers, also went over the sheeting and cleared it by a fraction of a millimetre. To minimise any effects of friction between the underside of the belt and the aluminium sheet, the sheet was polished and high gloss wax was applied to it.

To drive the conveyor belt a constant torque motor was used, so that irrespective of the load upon the belt, the speed of the motor would not alter. The motor used a rubber belt to drive one of the rollers, so that any vibrations from the motor to the experiment would be minimised. The motor was controlled by a speed controller, which allowed the conveyor belt to operate at a minimum speed of 0.01 m/s up to a maximum speed of 0.5 m/s.

The belt itself was approximately 2 metres long, and when placed over the rollers gave a total length of approximately 800mm for the mass to ride upon. The belt was made by the technical staff at UCL, of a tough synthetic PVC material which had a cloth underside. This material was felt to be the best medium for the belt as it possessed strength and the cloth underside did not allow excessive elasticity, something which could have caused the belt to slip off the rollers if stretched.

Upon the belt, sand which had been sieved to the required grade, was stuck on using glue. A number of belts were constructed using different grades of sand particles from very coarse to very fine sand. So using different belts could allow different friction characteristics to be investigated.

The mass was made of wood as other denser materials like metal caused the spring to deform when it elongated due to the weight of the mass. A number of different shapes were tried for the mass. Initially a cube was used, but this did not work very well as it was constantly tipping over on its corners. So finally a disc shape was used for the mass, at the suggestion of Professor Bishop, who happened to have watched ice hockey that weekend!
A number of masses of diameter 100mm were made with different grades of sand glued to their underside's. So now, both the underside of the mass and the conveyor belt were rough enough so that the system had sufficient static friction for a stick phase to take place.

Observing the experiment it was noted that it was quite easy to detect a distinct stick and slip phase. However due to the continual rubbing of the two contact surfaces, the roughness of the underside of the mass and the belt diminished as the surfaces became polished and continuous slipping of the mass was subsequently observed.

During this period of continuous slipping the mass remains at a reasonably constant displacement with the spring elongation also remaining constant. This gives the impression that the mass was in fact stationary.

This polishing effect was due to sand, which was stuck to both contact surfaces, becoming dislodged due to the physical forces applied to the individual grains of sand. Eventually one of the surfaces, usually the underside of the mass, would no longer have sand on its surface. The surface of the mass would become smoother as it rubbed with the rough surface of the conveyor belt. Saw dust and residue from the glue would stick between the sand particles on the belt giving a smooth polished feel to some areas of the belt. The mass would now begin to slip continuously as sufficient static friction was not present between the two surfaces for a stick phase to take place. This ruled out any long-term observations, so the time each experimental run was limited to a minute or so.

Observing the experimental model it was quite clear how stick-slip oscillations occurred within such a system and how sticking and slipping can occur successively in such a system and produce a periodic or near periodic motion.

However such observations could not discriminate the nature of the dynamic friction that was present in such a system. To do this direct and accurate readings of the displacement and velocity of the system would be required.
CHAPTER 3. EXPERIMENTAL INVESTIGATIONS

Figure 3.1: Schematic of the single degree of freedom model.

Figure 3.2: Photograph of the experimental rig. The threaded supports upon which the axles are fixed can be seen just below the rollers. The speed controller for the motor (not shown) is in the foreground.
Many researchers who are active in the experimental side of non-linear dynamics have access to large funds of money and sometimes are funded by large corporations who may have a commercial interest in the results of such experiments.

In this case, experimental work was not originally envisaged but during the numerical investigations of the single degree of freedom model, it was though appropriate to get a physical feel and understanding for any phenomena that occur.

As a result the experimental rig was conceived which could provide an overview of the dynamics, but neither the funds or relevant expertise were available to create a precise experimental set up on par with other experimental researchers (Bogaz & Ryczek 1997; Feeny & Moon 1994).

Thus the experimental investigation was simply going to be limited to obtaining qualitative results, therefore extremely accurate measurements would not be required, as would be required if a quantitative analysis was required. The easiest way in which the displacement of the mass could be determined would have been to use a laser Doppler device which would have given very fast, if not real-time results. The other advantage of a laser Doppler system is that it is a non-contacting method of measuring displacements and would not have affected the motion of the mass in any way. However laser Doppler's were not available so another method had to be found.

### 3.2 The first experimental trial

The first attempt to measure the displacement of the mass during an experiment was carried out by using a high-speed video camera. The video camera was set to record at a speed of 400 frames per second so that it could capture all the possible behaviour of the system, especially the transition of behaviour from sticking to slipping. Before the experiment was set up and the mass placed onto the conveyor belt, a few changes had to made so that the mass would be more visible when
The mass was painted black and a thin black rod was fixed to its centre pointing vertically upwards. Directly behind the conveyor belt a large white board was placed, onto which a metre rule with a black scale was stuck. Figure 3.3 shows a schematic representation of the experimental set up. The premise of this was as the mass oscillated backwards and forwards along the conveyor belt the black rod, which was attached to the mass would move along in front of the metre rule and give the appropriate displacement of the mass. The time was recorded by the video camera, as it had a built in digital timer which was observable on the videotape. After filming an experiment, the video film was played back on a special video player at a much slower speed.

The displacements of the mass were obtained by lining up the black rod sticking out of the mass with the scale behind it and recorded by hand along with the time. Once a reasonable amount of data was collected the displacements were entered into a computer. The appropriate velocities were calculated by using the finite difference method: displacement between two particular points and dividing by
the time taken between the points. The results were then shown graphically as a time series of displacement and velocity and a phase portrait, typical traces of which are shown in figures 3.4, 3.5 and 3.6.

Figure 3.4 shows the time series for displacement for the mass. The mass starts from an initial positive displacement of approximately 390mm but was stationary i.e. the initial velocity was equal to zero. The belt was started and the masses displacement increases to a maximum of approximately 410mm from which point it begins to decrease rapidly, indicating a slip phase. At approximately time(s) = 0.45 the mass attains a minimum displacement of 50mm, upon which the mass stops slipping and starts upon a stick phase. The displacement of the mass increases steadily until it reaches a maximum displacement of 390mm. It should be noted that the previous maximum displacement of 410mm has not been achieved again by the mass.

The displacement of the mass then starts to decrease again at a rapid rate, indicating another slip phase, until a minimum of 70mm is achieved at approximately Time(s) = 1.45. After attaining this minimum displacement, the mass embarks
upon another stick phase which takes its final displacement, before the experiment ends, to approximately 320mm. The displacement time series shows a reasonably smooth time series for the displacement of the mass and also shows the *saw tooth* function shape that was exhibited by the displacement time series obtained from many of the numerical simulations.

Figure 3.5 show the velocity time series for the mass. The velocity of the mass, as mentioned before, is calculated by using the time taken for the mass to cover a specific distance between two points. The velocity time series shows that from relative rest the velocity of the mass starts to increase rapidly to a value of approximately 100mm/s. However the velocity of the mass fluctuates and stays within a range of 50mm/s - 100mm/s. The velocity of the mass then starts to decrease rapidly indicating a slip phase, to a minimum of approximately -410mm/s, after which it begins to increase to a peak value of 300mm/s for a moment, but then drops to approximately 150mm/s. Again the velocity fluctuates, but now tends to stays within a narrow range of 140mm/s - 160mm/s for about 0.5 seconds. The mass during this period, where its velocity fluctuates within this narrow range, indicates
that it is in a stick phase. After the stick phase ends the velocity of the mass again
decreases to a minimum of approximately -400mm/s at Time(s) = 1.3, however
this is an extremely approximate value as the velocity is fluctuating around this
minimum. The velocity then increases rapidly reaching a maximum of 200mm/s
and then stays within the range of 140mm/s - 160mm/s, indicating the stick phase
of the mass, which lasts for approximately 0.4 seconds and then the velocity drops
to zero as the experiment ends.

The velocity time series, if fluctuations are ignored, has distinct areas in which
the mass is slipping and in which it is sticking. Areas where the velocity stays
within the narrow range indicates the stick phase of the system, and areas where
the velocity is rapidly changing indicates the slipping phase.

The results obtained are similar to some of the velocity time series obtained in the
numerical simulations, but unlike them the velocity time series here is very noisy
with fluctuations in the calculated velocity giving a very disjointed trajectory for
the mass, instead of a time series without these fluctuations.

This is primarily down to the algorithm which is used to calculate the velocity.
This assumption can be made as the initial displacement time series, from which
the velocity time series is derived is free of any sudden fluctuations in displacement.
The velocity is actually varying smoothly but the algorithm produces fluctuations
in the calculated velocity.

Figure 3.6 shows the phase portrait obtained for the experimental system. From
a starting displacement of approximately 390mm and with zero velocity the phase
space trajectory begins by increasing both displacement and velocity until a max-
imum velocity of approximately 150mm/s is attained with a maximum displace-
ment of 415mm. The trajectory then starts to lose both velocity and displacement
but starts to increase its velocity after reaching a minimum velocity of -420mm/s,
with the displacement of the trajectory still decreasing. The displacement of
the trajectory attains a minimum of approximately 50mm where the velocity in-
increases suddenly to a value of 300mm/s but then falls within a range of 140mm/s - 160mm/s, indicating the beginning of a stick phase, and stays within this range for a short period of time. Once the trajectory reaches a displacement of 370mm the velocity decreases with the displacement still increasing until the velocity of the trajectory falls to zero. The trajectory then starts to lose both displacement and velocity, indicating the slip phase of the system, but then starts to increase its velocity after reaching a minimum velocity of approximately -400mm/s with the displacement still decreasing. The trajectory reaches a minimum displacement of 75mm when the velocity increases to a velocity of 200mm/s and for a short period of time continues to fluctuate within a range of 140mm/s -160mm/s, indicating the stick phase of the system, but then decreases to zero as the experiment ends.

The phase portrait obtained for this experiment showed a rather noisy trajectory for the mass. The velocity and displacement of the mass fluctuate along areas of the stick and slip phase, something that was not observed in any of the numerical simulations.

Again this can be directly attributable to the algorithm used to calculate the
velocity producing fluctuations in its calculations. In fact pseudo velocities were actually being calculated, something which will be discussed later in this chapter.

The time histories and the phase portrait show that for successive stick and slip phases the mass did not follow the same path, as was the case in the numerical simulations. This was primarily due to the friction characteristics of the system changing as the experiment progressed, both the mass and belt became smoother and thus the same displacements or velocities could not be achieved. However the overall results obtained were qualitatively similar to the numerical simulations.

The disadvantage of this method of measurement is that it was very time consuming. The experiment needs to be filmed and replayed through a video player at a slower speed and the appropriate displacements taken down by hand. Approximately 2 seconds worth of data took nearly 2 hours to collect! Although the calculations for the velocity can be carried out by computer, entering the appropriate displacements into the computer by hand is very time consuming and prone to many errors.

The advantage of this method was that it was readily available in the UCL Civil Engineering labs and did not require the purchase of any extra equipment.

The experiment produced reasonable results, in the sense that it gave an indication that some of the numerical simulations were producing qualitatively similar behaviour.

The most important aspect about the experimental results was that, for all their imperfections they gave the reassuring fact that the results did not give something that was unexpected. They merely reconfirmed that the methods and hypotheses used in the numerical simulations were reasonably accurate. However it was felt that the time taken to collect the data and analyse it was far too long for such a short experiment. Another method would have to be found.
3.3 The second experimental trial

An alternative method of measuring the displacement of the mass involved the use of a rotary encoder. These encoders are normally used in mechanical devices where the angle or displacement of a moving part needs to be calculated accurately. The encoder used in this experiment is usually found in computerised lathes, where the exact depth the cutting tool is being deployed at, is required.

This encoder, when its shaft was rotated 1 revolution gave 4096 square wave pulses. A photograph of the actual rotary encoder used for this experiment is shown in figure 3.7.

The first problem with using the encoder was that a method of translating the horizontal displacements of the mass into the rotational movements of the encoder's shaft had to be found. Once this had been done, whenever the mass moved, it would be a matter of counting the number of pulses given as the encoder rotated. As the diameter of the encoder's shaft was known, the displacement of the mass could be calculated. Theoretically it should be possible to get measurements to a few hundredths of a millimetre.

However another pressing problem arose which had to be addressed before any measurements could be taken. The encoder gave square wave pulses when it was rotated, 4096 in one revolution of the shaft to be exact. However the encoder gave square wave pulses whichever direction it was rotated. When the mass moves backwards the shaft of the encoder is rotated anticlockwise and when it moves forwards the shaft is rotated clockwise. Figure 3.8 explains the premise of this idea more clearly. Therefore it would be necessary to distinguish between anticlockwise rotations which corresponded to backward motion of the mass and clockwise rotations which corresponded to forward motion of the mass.

This proved to be quite a simple problem, which was solved by noticing that the square waves given when the encoder's shaft was rotated clockwise were out
Figure 3.7: A photograph of the rotary encoder.
Rotary encoder

Clockwise pulses are out of phase by +90° from anticlockwise pulses.

Figure 3.8: Square wave pulses given by the rotary encoder as it was rotated. Anticlockwise rotations produce pulses, which are out of phase by 90 degrees with clockwise pulses.
of phase by 90 degrees from the square waves given when the shaft was rotated anticlockwise.

A special circuit called a J-K flip-flop circuit was built by Mr Leslie Wade at the UCL Civil Engineering Labs, to distinguish clockwise rotations from anticlockwise rotations.

The circuit had output channels for connection to a counter timer or a computer. The J-K flip-flop circuit allowed the number of anticlockwise pulses on one of the output channels and the number of clockwise pulses on another output channel to be counted. The number of clockwise and anticlockwise pulses from the encoder when combined give the number of pulses from the starting datum point, and hence the displacement can be calculated. When these values are used in conjunction with an accurate timer, the velocity can be calculated.

Since these pulses had to be counted, stored and used in calculations to obtain the displacement and velocity of the mass, a computer data acquisition board with an built in counter timer would be used. The LabPC16+ counter timer board made by Amplicon Plc, was thought to be the best for this purpose as it had 4 input channels for data acquisition and allowed upto $2^{32}$ pulses per channel to be counted. Also, software routines were provided with the counter timer board which allowed our own software to access the data from the board very easily.

The first experiment using the encoder to measure the displacement of the mass is shown in figure 3.9. The mass has a piece of waxed string attached to it which is drawn back over the shaft of the encoder, which is attached to a fixed support. The idea was that as the mass moves it pulls the string which in turn rotates the shaft around which it is wrapped.

To keep the string under constant tension a small weight is placed at the end of it. However it was soon found that this configuration was unsatisfactory as the waxed string was slipping around the shaft and many of the readings were not registering as the encoder's shaft was not rotating and so the final displacement
was incorrect. Also it was found that the weight on the end of the string adversely affected the motion of the mass in a way that was wholly unacceptable. It was actually impeding the sticking process to such an extent that on many of the experimental trials a constant slipping phase was observed.

The second experiment using the encoder is shown in figure 3.10.

The string connected to the mass has been replaced by an extremely light balsa wood beam. It was connected to the mass by the use of a universal joint which allows movement in all directions.

The other end of the balsa wood beam sits on top of the encoder’s shaft which has a piece of rubber pipe covering it. The idea here is that as the mass oscillates backwards and forwards, the balsa wood beam moves along with it which in turn causes the encoder’s shaft to rotate. This system worked extremely well and gave some good results.

However it failed to give accurate readings when the mass came backwards quickly.
When the mass was pulled backwards quickly by the spring it caused the balsa wood to lose contact with the encoder's shaft as it bounced upwards. Thus some readings were going unrecorded by the computer and so eventually the overall displacement of the mass was incorrect. Another factor which caused wrong readings to be recorded was that the balsa wood beam was not travelling over the surface of the shaft in a perfectly transverse and horizontal fashion. In some instances it actually fell off the surface of the shaft, causing the experimental run to be abandoned. Another cause for concern was the fact that the balsa wood beam was slipping slightly over the encoder's shaft and not rotating it, causing errors to creep into the total displacement of the mass.

Despite the difficulties mentioned the second experiment detailed above gave very good results until either the balsa wood beam bounced up and lost contact with the encoder's shaft or when the beam fell off the shaft of the encoder.

The third experiment using the encoder is shown in figures 3.11 and 3.12 dealt with all of the above problems successfully.

The shaft of the encoder had a cog put onto it, whilst the balsa wood beam had a track stuck to its underside which fitted the cog perfectly. This was to make sure that the shaft and the balsa wood beam could not slip over each other, as they
Figure 3.11: Schematic diagram of the third experiment using the rotary encoder.

Figure 3.12: Photograph of the third experiment using the rotary encoder.
were now mechanically forced to stay in contact. Figure 3.13 shows a magnified region of figure 3.11 which illustrates the balsa wood beam with its track and encoder shaft with the cog fitted.

A small feather spring was attached to a fixed support just above the point where the shaft of the encoder and the beam were in contact with each other. The spring pressed very lightly onto the top of the beam ensuring that the beam and the shaft were always in contact with one another, this stopped the beam bouncing which had been described earlier. The last problem of the beam falling off the encoder's shaft was solved by placing two rods coated with PTFE, which made them very slippery, vertically on either side of the beam. Figure 3.14 shows this more clearly. This ensured that the beam only moved only a very small amount transversely across the top of the shaft. The rods caused no real impediment to the motion of the mass because as they were coated with PTFE they caused very little friction to occur, thus motion was not adversely affected.

Typical results obtained from the third encoder experiment are shown in figures
Figure 3.14: Schematic diagram of the third experiment using rods coated with PTFE to guide the balsa wood beam.

3.15, 3.16 and 3.17. Figure 3.15 shows the displacement time history for the mass for this experiment. Distinct areas of the time series highlight slipping and sticking of the mass. From the start of the shown time series the mass is in a stick phase for a short period of time. The mass then starts to slip and reaches a minimum displacement of $X \approx -12\text{cm}$ when it starts to stick again. The motion of the mass repeats this pattern for the duration. The time series also has the saw tooth function shape that was exhibited by the displacement time series obtained from some of the numerical simulations.

One important aspect of the time series that needs to be addressed is the fact that the mass does not slip or stick at the same displacements, as was indicated by the numerical simulations. The experiment seems to gives results which are inherently noisy.

This shifting of the positions where sticking and slipping takes place is primarily due to the friction characteristics of the belt and the contact surface of the mass altering with time. The mass is becoming more smoother as time progresses as it is polished by the belt, which is also getting smoother as particles of saw dust get embedded into its surface, so altering the distribution of contact points.
between the two surfaces. Also during the slipping process, the temperature of the contacting surfaces can vary, which in due course can induce a variation in the friction properties.

Another contributing factor which may explain the differing paths of the trajectories is the joint in the conveyor belt. In the numerical simulations it is assumed that the belt is seamless and perfect. The slight raising of the belt surface which is present at the joint causes the mass to momentarily lose contact with the belt, when the experiment is in progress. This would obviously effect the overall dynamics of the system, as the jumping of the mass when it came into contact with the joint would be similar to a perturbation being applied at periodic intervals to the mass in the single degree of freedom system.

These phenomena may contribute to hidden, unseen state variables in the modelling of friction. The noise many in fact be deterministic chaos involving such hidden variables.

Figure 3.16 shows the velocity time series for the experiment. Distinct areas of
sticking and slipping are visible from the time series. From the time series the mass starts off with an initial velocity of $V \approx 22\text{cm/s}$, but seems to immediately start slipping. The mass attains a minimum velocity $V \approx -80\text{cm/s}$ at which point it begins to increase its velocity until it reaches a velocity of $V \approx 25\text{cm/s}$ at which point it begins to stick. The mass sticks for a short while, approximately 1.5 seconds. During this period of sticking the velocity of the mass does fluctuate, but remains within a small range of the belt velocity. After the end of the stick phase, the mass then begins to slip and the motion of the mass repeats this pattern for the duration of the experiment. Observing the time series it can be seen that the mass does not slip or stick at exactly the same velocity as indicated by the numerical simulations. Also the minimum slip velocity that the mass attains is also different each time. Again this is explained by the fact that the friction characteristics of the mass and belt are changing with time, but the overall features of the system remain the same, that stick-slip motion is still observed.

Another aspect of the velocity time series is the apparent fluctuation of the velocity of the mass during its stick phase. The velocity of the mass should remain constant during the stick phase, as indicated by the numerical simulations.

This fluctuation at first seems to have been caused by some external vibration having entered the system, such as the vibration of the motor driving the conveyor belt. However this premise is not correct as the displacement time series was not affected by fluctuations. So again the algorithm used to calculate the velocity seems to be in question.

Figure 3.17 shows the phase portrait obtained for this experiment. It shows a phase portrait that was more or less expected from the experiment, as predicted by the numerical simulations. The phase portrait shows that the trajectory started from the co-ordinates $X \approx 12\text{cm}$ and $V \approx 22\text{cm/s}$. The trajectory then almost immediately starts to follow a slip phase where it loses both velocity and displacement until it reaches a minimum velocity of $V \approx -80\text{cm/s}$. From there it starts to gain velocity, but still lose displacement until it attains a velocity of $V \approx 22\text{cm/s}$
at which point the trajectory follows a stick phase. During the stick phase the velocity fluctuates a small amount but stays within a small range. The displacement during this period steadily increases until $X \approx 15\text{cm}$ is reached at which point the trajectory starts to slip and the whole process repeats itself.

The experimental phase portrait and time histories look similar to some of the numerical data, however when directly compared with the results obtained from the numerical simulations, the experimental results seem to have too many fluctuations in the calculated velocity of the mass.

Where the phase portrait trajectory for the numerical simulation seems to be well defined without fluctuations along the slip and stick phases and it follows the exact same path each time, the experimental phase portrait has a trajectory, which is constantly fluctuating and does not follow the same path for each stick and slip phase.

The velocity of the system is calculated by a simple algorithm which calculates the velocity between data points. The algorithm uses the first order approximation:
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Displacement (cm)

Velocity (cm/s)

Figure 3.17: Phase portrait obtained from the third experiment.

\[ f'(x) \approx \frac{f(x + h) - f(x)}{h} \]  \hspace{1cm} (3.1)

Where the \( f(x) \) is the displacement and is defined implicitly by the system and the derivative of the system is \( f'(x) \), as the distance between two points \( h \rightarrow 0 \). However upon closer investigation of the above algorithm two sources of error become apparent, the truncation error and roundoff error (Press, Teukolsky, Vetterling & Flannery 1992). The truncation error comes from the higher terms in the Taylor series expansion:

\[ f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f'''(x) + \ldots \]  \hspace{1cm} (3.2)

Where

\[ \frac{f(x + h) - f(x)}{h} = f' + \frac{1}{2}hf'' + \ldots \]  \hspace{1cm} (3.3)
Hence the higher terms are ignored and not accounted for, resulting in a truncation error. The roundoff error occurs in the stepsize \( h \) in the following manner. Suppose a point \( x = 10.3 \) is chosen and a small stepsize of \( h = 0.0001 \) is also chosen. Neither \( x = 10.3 \) nor \( x + h = 10.3001 \) is a number with an exact representation in binary, each is therefore represented with some fractional error characteristic of the computer's floating point format. This immediately implies at least the same large fractional error in the calculated derivative. Therefore the algorithm seems to be calculating *pseudo-velocities* of the mass rather than the actual velocity.

However it should be pointed out that the algorithm used in 3.1 in this instance, is fundamentally flawed for calculating velocities from the displacements, as the stepsize \( h \) in this case is not small enough. In the actual implementation of this method, called the finite difference method, the velocity of the mass is the gradient between two points on the displacement time history. However the gradient between the two points may not actually be the gradient of the time history if the distance between the two points, \( h \) does not approach zero.

This produces a large error that is unacceptable. Figure 3.18 shows this idea clearly, how *pseudo-velocities* can be inadvertently calculated using this method.

The errors, as discussed, could be reduced by decreasing the stepsize i.e. \( t_2 - t_1 \) between the two points \( p_1 \) and \( p_2 \). However this would have a direct effect upon the sampling rate and hence dramatically increasing the size of the data file. This would not be very practical as the stepsize would have to be reduced to a very small value for any real benefit, the amount of space that the data file would occupy would be enormous and the calculations carried out would be very computationally expensive.

Michael Davies (Davies 1997), outlined a number of methods for numerically estimating the derivatives of a system from its time series in the presence of noise. In the paper, Davies shows how a group of non-recursive filters (also known as Polynomial Least Squares filters) such as a Savitsky-Golay (SG) filter can be used
(a) A reasonably accurate approximation to the gradient at $p_1$.

(b) A very inaccurate approximation to the gradient at $p_1$.

Figure 3.18: The approximation to the gradient is made with the line L. The gradient of the tangent T gives the correct gradient for $p_1$. In (a) there is a slight difference in the gradients of T and L giving a reasonably accurate approximation to the gradient at $p_1$. In (b) there is a considerable difference of the gradients of T and L giving an inaccurate approximation to the gradient at $p_1$. 
to fit a fixed order polynomial to the data within some data window. The effect of such a filter is to smooth any noisy data that may be present whilst still preserving the characteristics of the data. The premise of data smoothing is that a variable is being measured that is both slowly varying and being corrupted by random noise. Once the polynomial has been fitted the derivative within the data window can be estimated by polynomial interpolation.

To investigate whether these numerical methods would correctly calculate the derivatives of the system, displacement time series data from the experiment was used, so that a direct comparison can be made.

A velocity time series from the experiment, obtained from displacements, using the finite difference method is shown in figure 3.19.

The velocity time series produced from the same experimental displacement time series, but which had undergone data smoothing using the SG filter is shown in figure 3.20.
The results are quite remarkable. The smoothed data has not lost any of the characteristics of the non-smoothed data as all the heights and widths have been preserved.

The phase portrait calculated with the finite difference algorithm is shown in figure 3.21, whilst the phase portrait of the system after the data has been processed by the SG filter is shown in figure 3.22.

The smoothed phase portrait does not show any fluctuations or scattered data around the stick and slip paths. It shows a well defined trajectory where a distinct change in behaviour from sticking to slipping can be observed. It also shows the characteristic flat top during the stick phase. Qualitatively the phase portrait obtained from the experiment looks very similar to the phase portrait obtained from the discontinuous dynamic friction model with non-linear slip sections. This friction model will be chosen as being the most physically realistic for the single degree of freedom system.
Figure 3.21: Phase portrait obtained from the third experiment without using *data smoothing* techniques.

Figure 3.22: Phase obtained from the third experiment with *data smoothing* using the SG filter.
3.4 Summary

Qualitatively the results produced from the experiment did resemble some of the numerical simulations. There are some differences between numerical and experimental results but that is to be expected. The numerical model epitomises perfect laboratory conditions where everything remains constant, conditions which we all know are not achievable in the real world.

The phase portrait obtained from the experiment shows that not all trajectories follow the same path, even if they show the same behaviour. The numerical simulations on the other hand suggest that all trajectories are exactly the same and follow the same path.

The reason for this type of behaviour on the experimental system is that the friction characteristics of the system are constantly changing whilst in the numerical system they remain the same. The friction characteristics of the system change as the mass is constantly being sanded away by the action of the conveyor belt. It is becoming smoother on its contact surface with the belt, whilst the belt is also getting smoother as particles of sawdust get embedded into its surface.

These factors would influence the static friction force which would alter the position of the transition points for the change in behaviour from slipping to sticking and vice versa.

From these experimental investigations it has been established that any realistic friction model, for a system such as this, should include variability in the static friction between the two surfaces. This variability could be dependent upon many factors, however further investigations into the overall characteristics of the friction model will be pursued without this variability.

From the experimental results is has been concluded that the discontinuous dynamic friction model with non-linear slip sections is the most physically realistic
friction model for the single degree of freedom system that is being investigated.
Chapter 4

The Two degrees of freedom system

Typically, in physical processes a single degree of freedom model is an oversimplification and although the same overall configuration may apply, further degrees of freedom must be considered in order to simulate a more realistic physical system. For this reason the single degree of freedom model has been extended so as to achieve a higher dimensional system. The method that has been chosen to increase the dimensionality of the system in this case was to add another mass. There were others options available to increase the dimensionality of the system such as considering lateral or vertical displacements of a single mass, but it was felt that in this way a smoother progression of ideas was achieved. This allowed many of the principles considered in the single degree of freedom system to be implemented in the higher dimension system without having to alter many of the core fundamental ideas, such as equations of motion.

The two degrees of freedom system which now operates within a four dimensional phase space is shown in figure 4.1. As the phase space of the system is four-dimensional it is now possible for the system to exhibit chaotic behaviour, which was not possible in the single degree of freedom system which operated within a
CHAPTER 4. THE TWO DEGREES OF FREEDOM SYSTEM

Figure 4.1: Schematic diagram of the two degrees of freedom model

two dimensional phase space.

Two masses, $m_1$ and $m_2$ respectively, are connected by a linear spring with an elastic constant $k_c$ and both supported by conveyer belt, moving with velocity $v_{dr}$. The contact surface of the masses is rough so that two friction forces $F_1$ and $F_2$ operate on the masses. The two masses are additionally connected to a fixed support by two exterior springs whose elastic constants are respectively $k_1$ and $k_2$.

The basic type of motion for the two degrees of freedom system is similar to the single degree of freedom system. If a driving velocity $v_{dr}$ is applied, starting from some initial condition in which both masses start to move at the same velocity as the belt, there exists an initial stick-phase during which both the masses ride on the moving belt without deformation of the coupling spring but with stretching and contraction of the two exterior springs, a so called global stick phase.

After a time the elastic force applied by one of the exterior springs to one of the masses exceeds the static friction for that mass so that the mass starts to slip. This accelerated motion increases the elastic force upon the coupling spring which may in turn cause the second mass to slip. After this motion has ceased, a stick-phase may again exist where both masses again ride along the belt at the belt velocity until one of the masses starts to slip again. However there may be cases where both masses never ride along the belt at the belt velocity at the same time, one
mass may stick whilst the other slips, and vice versa. These different states for the masses in the two degree of freedom system are discussed below.

**Both masses slipping**

The equations of motion where both masses are in a state of slipping are:

\[
m_1 \ddot{x}_1 + (k_1 + k_c)x_1 - k_c x_2 = F_1(\dot{x}_1 - v_{dr}) \quad (a)
\]
\[
m_2 \ddot{x}_2 + (k_2 + k_c)x_2 - k_c x_1 = F_2(\dot{x}_2 - v_{dr}) \quad (b)
\]

**Both masses sticking**

During the state where both masses are sticking, called a global stick phase, the motion of both masses is uniform with the belt velocity, i.e.

\[
\dot{x}_1 = v_{dr} \quad (a)
\]
\[
\dot{x}_2 = v_{dr} \quad (b)
\]

**1 mass sticking, 1 mass slipping**

This state is described by a combination of equations where one mass is slipping and the other is sticking i.e. equation 4.1(a) and 4.2(b) or equation 4.1(b) and 4.2(a).

The relationship between the elastic forces of the springs and the static friction forces, provide the conditions for impending slip, the transition from sticking to slipping:

\[
\pm F_{s1} = (k_1 + k_c)x_1 - k_c x_2
\]
\[
\pm F_{s2} = (k_2 + k_c)x_2 - k_c x_1
\]

(4.3)
The parameters $F_{s1}$ and $F_{s2}$ represent the maximum static frictional forces acting upon the contact surfaces of the masses $m_1$ and $m_2$ and the conveyor belt respectively.

From the experimental results and the numerical results from the undamped single degree of freedom system it has been established that the most realistic dynamic friction force for this particular type of system is one where the dynamic friction force is assumed to be a decreasing function of the relative velocity of the form:

$$F_i = \frac{F_{si}}{1 + |\dot{x}_i - \dot{v}_{dr}|/v_f} \text{sgn}(\dot{x}_i - \dot{v}_{dr})$$  \hspace{1cm} (4.4)

Where $v_f$ is a reference velocity introduced in equation 4.4 for dimensional reasons.

To reduce the number of parameters in the system, initially a number of hypotheses are assumed:

$$m_1 = m_2 = m$$
$$k_1 = k_2 = k$$  \hspace{1cm} (4.5)

The coupling spring $k_c$ is also defined in terms of $k$: $k_c = \alpha k$. The equations of motion for the system shown in equation 4.1 are simplified by a change of variables based upon the above assumptions. A number of dimensionless ratios are introduced to rescale the equations of motion for the system:

$$\alpha = \frac{k_c}{k}$$
$$\beta = \frac{F_{s2}}{F_{s1}}$$
$$\tau = \sqrt{\frac{k}{m}}$$
$$X_i = \frac{x_i}{k_{s1}}$$  \hspace{1cm} (4.6)

The parameter $\tau$ is the dimensionless time and $X_i$ are the dimensionless displacements. If the dimensionless ratios in equation 4.6 are substituted into the equa-
tions of motion in 4.1 and into the impending slip conditions in equations 4.3 the following dimensionless rescaled equations are obtained:

\[\begin{align*}
\dot{X}_1 + X_1 + \alpha(X_1 - X_2) &= \frac{\pm 1}{1 + \gamma |\dot{X}_1 - V_{dr}|} \\
\dot{X}_2 + X_2 + \alpha(X_2 - X_1) &= \frac{\pm \beta}{1 + \gamma |\dot{X}_2 - V_{dr}|}
\end{align*}\] (4.7)

The parameter \(\gamma = F_s1/(v_f\sqrt{km})\) which is present in equations 4.7 is a measure of the rate at which the dynamic friction force decreases as the rescaled relative velocity \(\dot{X}_i - V_{dr}\) increases, where \(V_{dr}\) is the rescaled belt velocity given by \(V_{dr} = \sqrt{kmv_{dr}/F_s1}\).

For the impending slip conditions the equations become:

\[\begin{align*}
X_1 + \alpha(X_1 - X_2) &= \pm 1 \quad (a) \\
X_2 + \alpha(X_2 - X_1) &= \pm \beta \quad (b)
\end{align*}\] (4.8)

In each of the rescaled equations the "+" sign is adopted if the relative slipping velocity \((\dot{X}_1 - V_{dr})\) is negative and the "-" sign adopted if the relative slipping velocity is positive.

Equations 4.7 can be rewritten as a system of first order ordinary differential equations in the standard form \(\dot{x} = f(x)\) where \(x \in \mathbb{R}^4\). For the slip phase for each of the masses the equations of motions are:

\[\begin{align*}
\dot{X}_1 &= V_1 \\
\dot{V}_1 &= -X_1 - \alpha(X_1 - X_2) \pm \frac{1}{1 + \gamma |V_1 - V_{dr}|} \\
\dot{X}_2 &= V_2 \\
\dot{V}_2 &= -X_2 - \alpha(X_2 - X_1) \pm \frac{\beta}{1 + \gamma |V_2 - V_{dr}|}
\end{align*}\] (4.9)
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4.1 System parameters

The system equations incorporate three parameters $\alpha, \beta, \gamma$. The parameter $\alpha$, as mentioned before, is the ratio of the stiffness of the coupling spring $k_c$ and the stiffness of the exterior springs. The parameter $\alpha$ can vary between 0 and $+\infty$ and governs the degree to which the masses are coupled. The condition $\alpha = 0$ effectively decouples the masses and the system is then constituted by two single degree of freedom systems.

The condition $\alpha \to \infty$ forces the masses to follow the same dynamics. This is because a large value for $\alpha$ would signify a proportionally larger stiffness for the coupling spring than for the other springs, a condition which would cause the coupling spring to act more as a solid connector between the masses forcing them to follow each other's behaviour. From other research carried out on similar systems (Nussbaum & Ruina 1987; Huang & Turcotte 1990; Galvanetto, Bishop & Briseghella 1993; Galvanetto & Bishop 1994) a value of $\alpha = 1.2$ has been chosen and is used throughout all the numerical simulations.

The parameter $\beta$ corresponds to the ratio between the maximum static friction force acting on the second mass, $F_{s2}$, and the maximum static friction force acting on the first mass, $F_{s1}$, and can also be varied between 0 and $+\infty$.

If $\beta = 0$ the second mass is not subjected to a stick-slip force. This is more clearly seen if the equation of motion for the system during the slip phase is examined in 4.7 (b). If $\beta = 0$ then the right hand side of equation 4.7 (b) is equal to zero, indicating that dynamic friction is not present thus stick-slip oscillations cannot occur.

If the impending slipping conditions for the second mass are examined in equation 4.8 (b) we see that $\beta$ also governs the maximum value for the static friction for the mass. If $\beta = 0$ then the static friction for the mass is also zero, therefore it does not stick, thus when under this form of regime the second mass is constantly
slipping. Figure 4.2 shows this type of behaviour more clearly. It shows that from initial starting conditions of $X_2 = -1.10, V_2 = 0.08075$, and with a belt velocity of $V_{dr} = 0.08075$ the second mass should have been in a position to immediately start sticking, but instead the blocks velocity and displacement start to increase. The velocity of the mass reaches a maximum of approximately $V_2 = 1.7$ then starts to decrease to zero where the mass attains a maximum displacement of $X_2 = 1.05$. The velocity and displacement of the mass then starts to decrease as the elastic forces of the springs pull the mass back. However as the mass starts to increase its velocity the mass again attains the belt velocity and surpasses it without slipping. The mass remains in a state of constant slipping with the motion being purely oscillatory, without any form of sticking. As a comparison, figure 4.3 shows the motion of the first mass with $\beta = 0$ and with the initial starting conditions of $X_1 = 0.2, V_1 = 0.08075$ and with the same belt velocity of $V_{dr} = 0.08075$. It shows the classic type of motion that is expected of stick-slip behaviour.

If $\beta \to \infty$ the exact opposite of the behaviour for $\beta = 0$ occurs; that is the second mass would move along the conveyor belt without slipping. If the second mass did continue to move along the conveyor belt without slipping then this
would insinuate an infinitely long conveyor belt. However it should be pointed out that this situation would be extremely unrealistic as it would also signify a phenomenal maximum static friction force present between the contact surface of the second mass and conveyor belt which neither the coupling spring or exterior springs could overcome. If $\beta \rightarrow \infty$ then the second mass affects the behaviour of the first mass by constantly shifting its position of impending slip and impending stick to the right, as it is exerting a constant force upon the coupling spring pulling the first mass along with it. Figure 4.4 shows this behaviour more clearly. The value $\beta = 1$ makes the equations of motion, equations 4.9 and the impending slip conditions, equations 4.8 symmetric with respect to the couples of variables $X_1, V_1$ and $X_2, V_2$. With spatial heterogeneity, that is, with the ratio of frictional forces $\beta$ not equal to one, the dynamical behaviour of the two degree of freedom system is generally more chaotic. Therefore for the time being only the asymmetric case will be investigated, and drawing from other investigations which have also used the parameter $\beta$ in the same fashion, (Huang & Turcotte 1990; Galvanetto, Bishop & Briseghella 1993; Galvanetto & Bishop 1994) a value of $\beta = 1.3$ is chosen and used in all numerical simulations of the asymmetric case.
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Figure 4.4: Behaviour of the second mass when $\beta \rightarrow \infty$.

Figure 4.5: Behaviour of the first mass when $\beta \rightarrow \infty$. 
The parameter $\gamma$ as mentioned before, measures the rate at which the dynamic friction force decreases with increasing relative velocity. $\gamma$ can in principle vary between $-\infty$ and $+\infty$. From previous research carried out on similar systems which also have used the parameter $\gamma$ (Huang & Turcotte 1990; Galvanetto, Bishop & Briseghella 1993; Galvanetto & Bishop 1994), a value of $\gamma = 3.0$ is chosen and used in all numerical simulations. The dynamic friction curve for this value of $\gamma$ is shown in figure 4.6 along with other curves with different values of $\gamma$.

Figures 4.7 and figure 4.8 show typical time histories for the velocity and displacements for the first mass 1 while figure 4.9 shows typical phase portraits obtained. It can be deduced by looking at these figures that for these parameter values, the motion of mass 1 is aperiodic. The same can be said of the motion of the second mass 2, whose time histories for velocity and displacement are shown in figures 4.10 and figure 4.11 while its phase portrait is shown in figure 4.12.

The phase portraits of both masses show something different from that obtained with the single degree of freedom model. The phase portrait for single degree of freedom system showed the classical limit cycle, which was an isolated periodic
solution for that system (Guckenheimer & Holmes 1983).

However the two degrees of freedom system, which now has a higher dimension and is now capable of chaotic behaviour, shows phase portraits for both masses where series of trajectories for the stick-slip behaviour of the masses, which never follow exactly the same path during each stick-slip cycle.

The position of the transition point where the change in behaviour from sticking to slipping occurs also differs after each stick-slip cycle, as does the position of the point where sticking occurs at the end of a slip phase.

It can also be observed that sometimes a particular trajectory seems to be preparing for a stick phase, but is then forced to follow another path and now is again in a slip phase. This type of motion is more evident when observing the phase portrait of the second mass. This is primarily due to the effect of the first mass pulling the second mass via the coupling spring out of a potential stick phase. This behaviour is shown in figure 4.11
Figure 4.8: Time history showing displacement for the first mass $m_1$.

Figure 4.9: Phase portrait for the first mass $m_1$. 
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Figure 4.10: Time history showing velocity for the second mass $m_2$.

Figure 4.11: Time history showing displacement for the second mass $m_2$. 
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Figure 4.12: Phase portrait for the second mass $m_2$.

Figure 4.13: Behaviour of the second mass as it is pulled out of a potential stick phase by the action of the first mass. The shaded area highlights the area where this occurs.
4.2 Analytical fixed points

The slip phases of the system are described by equation 4.9 which posses an apparent fixed point where the equations have a relative velocity of $-V_{dr}$.

The only physically meaningful fixed points are obtained by selecting the "+" sign in equations 4.9. Solving the algebraic homogeneous system obtained from equations 4.9 it is possible to find the co-ordinates of the fixed point as follows:

\[
\begin{align*}
X_1 &= \frac{1 + \alpha + \alpha \beta}{(1 + 2 \alpha)(1 + \gamma V_{dr})} \\
V_1 &= 0 \\
X_2 &= \frac{\beta + \alpha + \alpha \beta}{(1 + 2 \alpha)(1 + \gamma V_{dr})} \\
V_2 &= 0
\end{align*}
\]

(4.10)

This point corresponds to the position in which the elastic forces acting on the masses are balanced by the dynamic friction force corresponding to the relative slipping velocity $V_{rel} = -V_{dr}$. Any discussion of the stability of the fixed point is complex due to the four dimensional nature of the system. The calculation of the Jacobian matrix of equation 4.9 and the calculation of its determinant give rise to the characteristic equation:

\[
H_0 \lambda^4 + H_1 \lambda^3 + H_2 \lambda^2 + H_3 \lambda + H_4 = 0
\]

(4.11)

where
\[ H_0 = 1 \]
\[ H_1 = -\gamma \frac{1 + \beta}{\chi^2} \]
\[ H_2 = \frac{\beta \gamma^2}{\chi^4} + 2 + 2\alpha \]
\[ H_3 = -\gamma \frac{(1 + \alpha)(1 + \beta)}{\chi^2} \]
\[ H_4 = 1 + 2\alpha \]  

(4.12)

where \( \chi \) is given by \( \chi = (1 + \gamma V_{dr}) \). The eigenvalues of the characteristic equation 4.11 indicate the stability of the fixed point, shown in equation 4.10. If all the roots, \( \lambda \), of equation 4.11 have negative real parts the fixed point is asymptotically stable. The general discussion of the solutions of equations 4.11 may be performed by means of the Routh-Hurwitz criterion (Leipholz 1970), remembering that in the present system \( \alpha > 0, \beta > 0, \gamma \in \mathbb{R}, V_{dr} > 0 \). According to the Routh-Hurwitz criterion the solution of equation 4.11 have negative real parts if:

\[ H_1 > 0 \]
\[ D_2 = H_1H_2 - H_0H_3 > 0 \]
\[ D_3 = D_2H_3 - H_1^2H_4 > 0 \]
\[ H_4 > 0 \]

(4.13)

The first condition imposes that \( \gamma < 0 \), the second and third conditions become:

\[-\gamma \frac{1 + \beta}{\chi^2} \left[ 1 + \alpha + \frac{\beta}{\chi^4}\gamma^2 \right] > 0 \]
\[ \frac{\beta \gamma^2}{\chi^4}(1 + \alpha) + \alpha^2 > 0 \]

(4.14)

which are satisfied if \( \gamma < 0 \) while the fourth condition is always satisfied. Observing the condition \( \gamma < 0 \), the only one that guarantees the stability of the fixed point, given in equation 4.10, imposes a positive slope to the dynamic friction function so that this condition coincides with the stability condition of the single degree of freedom system.
In particular it is possible to evaluate the solutions for a parameter set. For example if $\alpha = 1.2, \beta = 1.3, \gamma = 3.0, V_{dr} = 0.1$, then equation 4.11 has the four solutions:

$$
\begin{align*}
\lambda_1 &= 0.483167789 \\
\lambda_2 &= 3.036888037 \\
\lambda_{3,4} &= 0.281392203 \pm 1.495980660
\end{align*}
$$

(4.15)

from which it is evident that the fixed point is unstable.

### 4.3 Definition of the failure locus

Since the two degrees of freedom system is four dimensional, graphical representation of the whole system dynamics are not possible. However it is possible to view the response of the system via projections of the system dynamics onto two-dimensional planes. The obvious ones would be to view individual phase portraits of the masses with the $(X_1, V_1)$ and $(X_2, V_2)$ plane projections.

Another representation that will be useful in revealing useful aspects of behaviour is the phase space projection on the plane $(X_1, X_2)$. This particular projection is shown on a plane with a failure locus, which is defined by the impending slip conditions shown in equation 4.8. The principle of using such a failure locus has in the past been adopted by researchers such as (Nussbaum & Ruina 1987) and (Huang & Turcotte 1990) in their investigations of similar systems. However an important feature that should be pointed out about the failure locus used in this investigation is that it allows slipping in both directions, that is slipping both backwards and forwards. The investigations carried out by Nussbaum and Ruina (Nussbaum & Ruina 1987) and Huang and Turcotte (Huang & Turcotte 1990) allowed slipping to occur in one direction only. This is mainly due to the fact that
much of their work was based upon modelling the interactions of tectonic plates 
in a geophysical fault, which only tend to slip in only one direction.

Figure 4.14 shows this failure locus in the plane \((X_1, X_2)\) shown with a particular 
motion of the system. A global stick phase, where both masses are simultaneously 
riding on the belt, is represented on the failure locus as the straight line \(X_2 - X_1 = \text{const.}\).

Each masses failure line on the failure locus is indicated by its number. As the 
motion progresses slipping eventually occurs, at the point where a trajectory intersects the failure locus, hence if a trajectory intersected a failure line annotated with "1" then mass 1 would slip.
4.4 Types of events

Following the definition introduced by Nussbaum and Ruina in 1987 in the seismological field an event is defined as a system undergoing a cycle which begins at relative rest, through some form of motion of one or more of the masses, until the point where it comes to rest once again. Thus an event is characterised by a motion of the system in which a stick phase occurs. By using this definition and integrating the equations of motion for the system different events can be observed in the \((X_1, X_2)\) plane.

4.4.1 One mass events

These events are the simplest that can occur where only one mass slips and undergoes some form of motion, and then comes to rest (i.e. sticks) after a period of time during which the second mass remains sticking to the moving belt. Two separate one mass events are shown in figure 4.15. The system accumulates elastic energy along the straight line \(X_2 - X_1 = \text{const}\) from \(P_0\) to \(P_1\). As the trajectory reaches the failure locus at \(P_1\), mass 1 starts to slip. The slip phase of mass 1 finishes at \(P_2\) at which point another global stick phase begins until point \(P_3\). The point \(P_3\) marks the onset of the slip phase for mass 2, which continues to slip until point \(P_4\) is reached.

4.4.2 Two mass events

Two mass events occur when the second mass begins to slip while the first mass is also slipping. An example of this is shown in figure 4.16. It is possible to recognise this type of motion because the trajectory, started from \(P_0\), intersects the mass 2 failure locus at \(P_1\) and then continues to intersect the mass 1 failure locus at \(P_2\) without the system attaining a stick phase between the two loci intersections. The
consequent motion appears as a curved trajectory as shown in figure 4.14. The trajectory eventually attains a *global* stick phase at point \( P_3 \).

### 4.4.3 Multiple mass events

Multiple mass events are similar to two mass events but are characterised by the presence of more than two intersections with the failure locus before a stick phase occurs again. Figure 4.17 shows such a multiple mass event. The trajectory starts from \( P_0 \) with a global stick phase, until it intersects the mass 2 failure line at \( P_1 \) where mass 2 begins to slip. The trajectory then intersects the mass 1 failure line at \( P_2 \) at which point mass 1 begins to slip, however the trajectory continues to intersect the mass 2 failure line again at \( P_3 \) only attaining a global stick phase at \( P_4 \).
Figure 4.16: The failure locus of the two degree of freedom system showing a two mass event.

Figure 4.17: The failure locus of the two degree of freedom system showing a multiple mass event.
4.5 The Poincaré map

Generally a Poincaré map is used to investigate the system dynamics (Guckenheimer & Holmes 1983; Thompson & Stewart 1986; Devaney 1992), but in this case it is not immediately possible to define a stroboscopic section due to the fact that the system is autonomous without periodic external forcing.

In a periodically forced system the Poincaré section could be defined relatively easily. The standard technique (Thompson & Stewart 1986) in dealing with the phase space of the periodically forced system would be to inspect the projection \((x, \dot{x})\) whenever time \(t\) is an integer multiple of the forcing period \(2\pi\). Observing the phase projection only at the specific times, \(t = mt(m = 0, 1, 2, \ldots)\) would yield a structure comprising of a sequence of dots, representing the Poincaré section. Transient motions will appear as rather scattered dots, and the emergence of a stable fundamental solution would be seen as the eventual repetition of just one fixed point. However the emergence of a stable sub-harmonic of order \(n\) would be seen as a systematic jumping between \(n\) fixed points.

In a system such as the one presented by the two degree of freedom system, the forcing in this case is constant. Therefore a new definition for a Poincaré section for this system has to be developed.

A Poincaré section is defined as an \(n - 1\) dimensional hypersurface in the state space that is transverse to the flow in the \(n\) dimensional space spanned by \(x\) (Foale & Thompson 1991; Thompson & Stewart 1986; Nayfeh & Balachandran 1995). Examining the equations for the two degree of freedom system, it can be shown that the four dimensional flow is transverse to the three dimensional volume at \(V_1 = 0\), which immediately suggests that this may be chosen as a Poincaré section. The derivative

\[
\frac{dX_1}{dV_1} = \frac{dX_1/dt}{dV_1/dt} = \frac{V_1}{-X_1 - \alpha(X_1 - X_2) + \frac{1}{1 + \eta|V_1 - V_0|}} = 0
\] (4.16)
and thus the function $X_1(V_1)$ is perpendicular to $V_1$ satisfying the transversality conditions for a Poincaré section. The Poincaré map is three dimensional and consists of the co-ordinates $(X_1, X_2, V_2)$ assumed by the system when $V_1 = 0$ passes from negative to positive values.

The motions of this fourth order system can be classified according to the Poincaré map defined in this way so that if the map has an attracting fixed point then this corresponds to an attracting limit cycle for the global system and the time between two intersections with the Poincaré section is the period of the cycle. However it should be noted that the period between consecutive cycles does not necessarily remain constant, due to the system being autonomous.

This definition of the Poincaré map allows the rigorous investigation of the dynamics of the system even if events are not occurring, i.e. even if the motion of the system does not include a stick phase.

### 4.6 Attractors of the system

Figure 4.18 show the type of bifurcation diagram which is obtained when the belt velocity, $V_{dr}$ is plotted against the first co-ordinate of the Poincaré point, $X_{1p}$. The bifurcation diagram obtained for the two degrees of freedom system shows that two attractors are present. A number of bifurcation's can be seen including period doubling, reverse period doubling, chaotic explosions and jumps to remote attractors as well as a chaotic zone with periodic windows. It is interesting to note that the majority of the behaviour which could be termed bifurcational occurs at relatively low belt velocities.

Each of the attractors is labelled in figure 4.18. Attractor 1 seems to be the main one which is attracting the majority of the motion, both chaotic and periodic. This would seem to imply that it possesses the larger basin of attraction, whilst attractor 2 exists in a very narrow range of initial starting co-ordinates.
The location of the attractors, their basins of attraction and boundaries was carried out by using a grid of initial starting conditions and iterating the system forward until a steady state behaviour was achieved. However this becomes computationally expensive if the phase space is extensive and a large number of attractors are present. A method of cell mapping (Hsu 1987) for the location of attractors and their basins gives a considerable improvement in efficiency over the grid of starts method.

### 4.7 The One Dimensional Map

For those motions which incorporate events where one or both of the masses undergoes a stick phase, the behaviour of the system can be characterised by monitoring the stretch difference between the masses \(X_2 - X_1\) (Nussbaum & Ruina 1987). The four dimensional flow of the system is effectively reduced to a one-dimensional map, which would allow easier and straightforward identification of the bifurca-
tional behaviour of the system.

The one dimensional map for the four dimensional flow is derived by monitoring the stretch difference between the masses \( d = X_2 - X_1 \), specifically noting its value during a global stick phase. A stick phase is characterised by a constant stretch difference

\[
d = X_2 - X_1
\] (4.17)

During the evolution of a motion, the system generates in a natural way a sequence of values \( d_0, d_1, d_2, \ldots, d_n \), which can be interpreted as a one dimensional map of the type

\[
d_{n+1} = f(d_n)
\] (4.18)

which describes the state of the system. In this case \( f \) is implicitly defined and it is computed via the numerical integration of the system dynamics. A periodic motion composed of various events will lead to a sequence of stretch values which repeat themselves providing simple graphical presentations of complex behaviour.

For a given map \( f \) (Devaney 1989) a **fixed point** is defined to be any \( d \) such that \( f(d) = d \). It appears on the iteration map i.e. one dimensional map as an intersection of the curve \( f(d) \) with the 45° line \( d_n = d_{n+1} \). A point \( d \) and its iterates comprise a **periodic orbit of period** \( n \) if \( f^n(d) = d \). A fixed point can correspond to a **limit cycle**, which is a periodic orbit of period 1 that is unique in its own neighbourhood.

The slope of a function \( f \), such as the one dimensional map determines the stability of its fixed points (Devaney 1989; Ott 1994). A fixed point is unstable if the slope of the map at that point has a magnitude greater than 1, and is stable if the slope of the map at that point is less than 1. If the slope of the map equals
±1, higher derivatives are necessary to determine the stability at that point. An attracting fixed point $d_0$, has the property that for $d$ sufficiently close to $d_0$ i.e. inside the region of attraction, $f^{n+1}(d)$ is closer to $d_0$ than $f^n(d)$. A repelling fixed point has an analogous property. A fixed point with a slope of ±1 is called non-hyperbolic, otherwise it is called hyperbolic. Non-hyperbolic fixed points have the property of being structurally unstable, as small change in the system can result in a qualitative change in the sequence of events.

An exploration of the full four dimensional phase space would be computationally expensive but a significant reduction in time can be achieved by using this definition for the one-dimensional map. This will allow attractors for the system to be located and followed in control space and determine their basins of attraction. The application of this method is particularly suitable for systems whose transient motions are characterised by global stick-phases.

Extensive numerical simulations suggest that motions which are characterised by global stick-phases, are only possible for this system if $V_{dr} < 0.20$, with the system parameters being $\alpha = 1.2, \beta = 1.3, \gamma = 3.0$. If the driving velocity is within this bound then any initial condition will be the beginning of a transient motion in which, at least for some of the time, both masses undergo a global stick-phase. Thus the ability to represent all stick-phase motions coincides with the ability to characterise all transient motions. In the case of transient motions characterised by a global stick-phase, all the motions with the same initial stretch difference $(X_2 - X_1)$ and the same driving velocity $V_{dr}$ have the same temporal evolution. This means that in the plane $(X_2, X_1)$ the initial conditions aligned with a straight line $X_2 - X_1 = const$, into the failure locus give rise to different temporal parameterisation of the same motion.

Consequently the attractors of these motions can be studied from a one-dimensional grid of starts aligned with a line which is transverse to the direction of the bisection line into the failure locus. Since this line represents all the possible global stick motions for the system, thus the basins of attraction, or at least their projections
are plotted.

The search for all the attractors via the one-dimensional map can be made more efficient from a computational point of view by using a cell-mapping type of algorithm (Hsu 1987; Foale & Thompson 1991).

The essential assumption made when using a cell-mapping algorithm is that whole cells map to whole cells. A line of initial starting conditions can be divided into small sub-intervals or cells. Thus given a starting cell $C_1$, the centre of which maps under the action of the flow to somewhere in $C_2$, it can be assumed that the process can be continued by mapping the centre of $C_2$ to a point in cell $C_3$, where again recentering takes place before continuing. The process is repeated in this particular fashion until the line of cells settle onto an attractor.

This method implies that all the cells of the line of cells that settle onto the attractor, all the points within in these cells will themselves converge to that same attractor.

### 4.7.1 Bifurcations of the one dimensional map

From the previous section it has been established that the two degrees of freedom system possesses 2 attractors. Using the one dimensional map the path of one of these attractors will be followed to establish the nature of any bifurcation's that occur. The attractor whose bifurcational behaviour will be investigated is attractor 2 in figure 4.18.

Attractor 2 is shown in more detail in figure 4.19. The figure is generated by plotting the first co-ordinate of the three dimensional Poincaré map, $X_{1P}$, against the belt velocity, $V_{dr}$. Following an earlier definition using equation 4.16, a three dimensional Poincaré section of the four dimensional phase space is detected by the condition $V_1 = 0$ and the Poincaré map is constituted by the successive values
of the variables \((X_1, X_2, V_1)\) when \(V_1\) is equal to zero and passes from negative to positive values.

The attractor itself exits approximately in the range \(0.0806 < V_{dr} < 0.170\).

**Collision with the basin boundaries**

The introduction of the one dimensional map helps to explain the sudden crisis that occurs to the chaotic attractor. The evolution of the one dimensional map as the belt velocity parameter \(V_{dr}\) decreases is shown in figures 4.20 (a)-(d). As \(V_{dr}\) decreases the permanent branches of the one dimensional map, which are shown in bold, spread towards the basin boundaries of the attractor. The attractor itself disappears at approximately \(V_{dr} \approx 0.08059\), a value at which the permanent branches of the one dimensional map collide with its basin boundaries.

Consequently it is assumed that a similar type of collision takes place in the four dimensional phase space, between the chaotic attractor and a stable manifold.
The term crisis was first introduced by Grebogi, Ott and Yorke (Grebogi, Ott & Yorke 1983) to describe certain sudden qualitative changes in the chaotic dynamics of dynamical systems as a control parameter is varied. A crisis occurs when a chaotic attractor comes into contact with an unstable periodic solution. It should be noted that when a chaotic attractor comes into contact with the stable manifold of an unstable periodic solution, due to the nature of the stable manifold, it eventually comes into contact with the unstable periodic motion. At this point the chaotic attractor is suddenly destroyed, and in the post-bifurcational state the motion is transiently chaotic before settling down to a bounded solution, which in this case is a periodic solution.

More specifically this crisis can be further categorised as a boundary crisis, an example of what is called a blue sky catastrophe. This catastrophe refers to the disappearance of an attractor from the state space of a system (Thompson & Stewart 1986).

Reverse Period Doubling

In the range $0.0820 < V_{dr} < 0.0842$ the system dynamics seem to undergo an apparent reverse period doubling bifurcation. This type of bifurcation can easily be described and confirmed by use of the one-dimensional map. In figure 4.21 (a) the second iterate of the one-dimensional map is shown at a driving velocity of $V_{dr} = 0.0838$. It can be clearly seen that the second iterated map has two stable fixed points which are separated by one unstable fixed point. Figure 4.21 (b) shows the second iterated map as the driving velocity is increased to $V_{dr} = 0.0842$. It can be clearly seen that the shape of the map has changed quite dramatically, as there only seems to be one stable attractor (fixed point?). The shape of the map has changed in a way that the three fixed points have collided and have collapsed in such a way as to give rise to a unique stable fixed point, as shown in figure 4.19 (b).
Figure 4.20: The one-dimensional map showing the collision of the attractor with its basin boundaries.
Figure 4.21: The one dimensional map showing the reverse period doubling bifurcation. 
u=unstable fixed point, s=stable fixed point.
Fold Bifurcation

From figure 4.19 it can be quite clearly seen that the attractor disappears at approximately $V_{dr} \approx 0.170$. The second iterated map for $V_{dr} = 0.165$ is shown in figure 4.22 (a), and shows two stable fixed points separated by an unstable fixed point.

As the driving velocity is increased to $V_{dr} = 0.175$, the shape of the second iterated map changes, as shown in figure 4.22 (b). The intersections of the map with the bisection line disappear, along with the two stable fixed points with just the unstable fixed point remaining.

4.8 Lyapunov exponents

The Lyapunov exponent is a convenient indicator of the sensitivity to small perturbations characteristic of chaotic systems. The Lyapunov exponent measures the exponential rates of divergence or convergence of nearby trajectories of an attractor in the state space.

Methods for the calculation of Lyapunov exponents for dynamic systems with discontinuities have been proposed (Oestreich, Hinrichs & Popp 1996; Muller 1995). The method used in this instance (Oestreich, Hinrichs & Popp 1996) seems to be the ideal method for this investigation as it not only allows the calculation of Lyapunov exponents but also utilises the one-dimensional map.

As described in the initial description of the one dimensional map, the non-smooth four-dimensional system of the stick-slip system can be reduced to a one-dimensional map, $f$. By means of this system reduction three Lyapunov exponents have been lost. Since the phase space dimension of the system varies between two and four, depending whether the system is in a stick phase or slip phase, two Lyapunov exponents can be considered as $\lambda_4 \rightarrow -\infty, \lambda_3 \rightarrow -\infty$ because of the
Figure 4.22: The one dimensional map showing the fold bifurcation. u=unstable fixed point, s=stable fixed point.
The measure of average local stability, the Lyapunov exponent, \( \lambda \), can be given in the case of the one-dimensional map \( f \) (Oestreich, Hinrichs & Popp 1996; Hagedorn 1988) as:

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(d_n) |
\]

Thus the last Lyapunov exponent can be determined by evaluating the slope, \( f'(d_n) \), of the one dimensional map at points crossed by the orbit. Figure 4.23 shows the premise of this idea more clearly.

For the two degree of freedom system the one dimensional map is implicitly defined, therefore the function \( f'(d_n) \) will be evaluated in terms of \( d_n \). Using figure 4.23 it can be ascertained that

\[
f'(d_n) = \frac{d_{n+1} + \delta - d_{n+1}}{d_n + \delta - d_n}
\]

This can be simplified by using the following known property

\[
f(d_n + \delta) = d_{n+1} + \delta
\]

\[
f(d_n) = d_{n+1}
\]
Therefore

\[ f'(d_n) = \frac{f(d_{n+1}) - f(d_n)}{\delta} \]  \hspace{1cm} (4.22)

Thus in terms of the map, the Lyapunov exponent \( \lambda \) of the one dimensional map \( f \) gives the average exponential rate of divergence or convergence of infinitesimally nearby initial conditions. That is, on average, the separation between two infinitesimally displaced initial points \( d_n \) and \( d_n + \delta \) where \( \delta \rightarrow 0 \), typically grows exponentially as the two points are evolved by the map \( f \).

The Lyapunov exponent can be interpreted in the following way:

(i) \( \lambda < 0 \), \hspace{0.5cm} the orbit is stable and periodic,

(ii) \( \lambda = 0 \), \hspace{0.5cm} the orbit is neutrally stable and quasiperiodic,

(iii) \( \lambda > 0 \), \hspace{0.5cm} the orbit is chaotic.

Figure 4.24 (a) shows the computed Lyapunov exponent for different values of the driving velocity \( V_{dr} \) along with the bifurcation diagram for attractor 2. Figure
4.24 (b) shows the good agreement between the values obtained for the Lyapunov exponents and the bifurcation diagram.

4.9 Summary

The two degree of freedom system was a logical extension to the single degree of freedom system, by way of providing a higher dimensional system and allowing a smoother progression of ideas to take place from one model to the other. The two degree of freedom system exhibits rich and complex dynamical behaviour, and since the phase space of the system is four dimensional a three dimensional Poincaré section has been defined to describe the system.

The ability to classify certain events of the system allows the introduction of a one-dimensional map to classify any bifurcation's of the system that may occur. The reduction of the four dimensional phase space of the system to the one-dimensional map allows the extensive exploration of the dynamical behaviour of the system without being computationally intensive.

The same one-dimensional map can also be utilised to calculate of the Lyapunov exponent of the system using methods used in other investigations (Oestreich, Hinrichs & Popp 1996; Hagedorn 1988).
Figure 4.24: Diagram (a) A plot of attractor 2. (b) A plot of the Lyapunov exponents for 3 fixed belt velocities from figure (a).
Chapter 5

The damped single degree of freedom system

Friction and damping affect most mechanical systems, and thus plays a huge role in engineering dynamics. For a mathematical model to fully simulate the behaviour of single degree of freedom system the mechanical system some form of structural damping must be incorporated. The inclusion of damping is a natural extension of the single degree of freedom model towards a more physically realistic model. In some cases, damping is left out of numerical simulations of mechanical systems for mathematical convenience, so that functions can be solved more easily or be more amenable to classical methods of analysis.

The single degree of freedom mechanical system presented in chapter 2 does not incorporate any form of damping, and thus by our own definition given above is physically unrealistic.

As described in chapter 1 and 2 of this thesis the single degree of freedom stick-slip model was proposed by Burridge and Knopoff (Burridge & Knopoff 1967) as a mechanism for earthquakes. A long period of rest usually follow a short period of tectonic plate movement, the kind of behaviour which is seen in systems which
exhibit stick-slip oscillations. The single degree of freedom system is a low order approximation to the behaviour of the earth's crust and the occurrence of faults. The mass is analogous to a tectonic plate, and the force responsible for the motion of the plates is mantle convection, which in geophysical circles is typically assumed to be constant. Hence the constant velocity of the belt.

The inclusion of damping does not produce a more accurate model of tectonic plate movement, or any other system which exhibits stick-slip oscillations, but rather a better understanding of the dynamic processes involved. As with many real world problems, in our modelling process we are trying to produce mechanisms from observations.

Although much work has been conducted into the single degree of freedom system, both undamped and damped (McMillan 1997; Feeny & Moon 1994; Feeny & Moon 1993; Hinrichs, Oestreich & Popp 1997; Popp & Stelter 1990b; Popp & Stelter 1990a) much of it has concentrated upon the behaviour of such a system under a few limited damping values. A full investigation of the damped single degree of freedom system would focus upon the behaviour of the system as the structural damping is varied, and the occurrence of any bifurcation's in the solution paths of the system.

As mentioned before in chapter 2, the single degree of freedom system being investigated is one of the simplest models which undergoes self-sustained oscillations. A schematic diagram of the damped version of this system is shown in figure 5.1.

The system consists of a mass $m$ which is connected by a spring with stiffness $k$, and a linear dashpot, to a fixed support and placed upon a moving conveyor belt with velocity $v_{dr}$.

Drawing from the earlier work in chapter 2 on the undamped system the equation of motion for the damped system during the slip phase is:
Figure 5.1: Schematic of the damped single degree of freedom model

\[ m \ddot{x} + c \dot{x} + kx = F_d(\dot{x} - v_{dr}) \]  

(5.1)

Where \( c \) is the coefficient of linear viscous damping and \( F_d \) is the dynamic friction. Conclusions from chapter 3, which dealt with the experimental investigations of the single degree of freedom system, established that the most physically realistic model for the dynamic friction in this case is a decreasing function of the relative velocity of the form:

\[ F_d = \frac{F_s}{1 + \gamma |(\dot{x} - v_{dr})|} \]  

(5.2)

The parameter \( \gamma \) in equation 5.2 is a measure of the rate at which the friction force decreases as the relative velocity \( |\dot{x} - v_{dr}| \) increases. The shape coefficient of the dynamic friction \( \gamma \) can, in principle, vary between \(-\infty\) and \(+\infty\). For the purpose of this chapter the value \( \gamma = 3.0 \) is chosen, as suggested by Huang and Turcotte (Huang & Turcotte 1990).

A schematic diagram of the dynamic friction curve is shown in figure 5.2 along
with dynamic friction curves with different values of $\gamma$.

As discussed in chapter 2 the relationship between the elastic forces of the spring and the static friction force $F_s$, provide the conditions for impending slip, the transition from sticking to slipping.

During sticking the static friction is greater than the elastic force of the spring i.e.:

$$F_s > kx \quad (5.3)$$

Once the transition point is reached, the static friction is balanced by the elastic force of the spring.

$$F_s = kx \quad (5.4)$$

Hence the position of the transition point can be determined. Since the transition point gives the exact position of impending slip, and the mass is still sticking at that instant, then the velocity of the mass is equal to the belt velocity $v_{dr}$. 

Figure 5.2: Schematic diagram of the dynamic friction curve with different values for $\gamma$. 
Therefore, as the mass at impending slip, has zero relative velocity the acceleration of the mass at this point must also be equal to zero. For the undamped system this can be demonstrated by using the equation below.

\[ m\ddot{x} + kx = F_d(\dot{x} - v_{dr}) \quad (5.5) \]

If the impending slip conditions are used, 5.4, and it is assumed that \( F_s = 1 \) and \( k = 1 \) then the position of the transition point is also \( x = 1 \).

Therefore equation 5.5 using the definition for the dynamic friction in equation 5.2, can now be written as:

\[ m\ddot{x} + 1 = \frac{1}{1 + \gamma |(\dot{x} - v_{dr})|} \quad (5.6) \]

And since \( \dot{x} = v_{dr} \) it follows that equation 5.6 becomes:

\[
\begin{align*}
  m\ddot{x} + 1 & = 1 \\
  \downarrow & \\
  \ddot{x} & = 0 
\end{align*}
\quad (5.7)
\]

Which follows from the assumptions that have been made about the system.

However if we use the same assumptions and same impending slip conditions, equation 5.4, for the damped system, we attain the equation below:

\[ m\ddot{x} + cv_{dr} + 1 = 1 \quad (5.8) \]

which does not suggest that \( \ddot{x} = 0 \), as is required at impending slip. For the damped system to acquire zero acceleration at impending slip a new impending
slip condition must be applied. For the new impending slip condition to correctly describe the behaviour at impending slip the energy dissipation caused by the damping must be taken into account. The new condition takes the form:

\[ F_s - cv_{dr} = kx \]  \hspace{1cm} (5.9)

With the presence of damping, the elastic force exerted by the spring has less work to do as some of the energy in the system has been dissipated. Incorporating this condition, equation 5.8 can now be written as:

\[
\begin{align*}
mx + c\ddot{x} + 1 - cv_{dr} &= 1 \\
\downarrow \\
m\ddot{x} + cv_{dr} + 1 - cv_{dr} &= 1 \\
\downarrow \\
\ddot{x} &= 0
\end{align*}
\]  \hspace{1cm} (5.10)

which now correctly describes the damped system at impending slip.

From previous chapters it has been established that the only possible steady state motion for this type of system when undamped is a limit cycle similar to that shown in figure 5.3. The limit cycle represents the stable periodic solution of the stick-slip system, and all trajectories are attracted onto the stable periodic solution.

Figure 5.3 also shows transient motions from within the limit cycle being repelled by an unstable fixed point within the limit cycle and quickly converging onto the limit cycle.
Chapter 5. The Damped Single Degree of Freedom System

5.1 Behaviour of the damped system

Figure 5.4 shows phase portraits for the damped system with increasing damping coefficient values.

When the system is undamped a limit cycle is obtained, as shown in figure 5.4(a). The limit cycle as mentioned before represents the stable periodic solution of the system. Transient motions outside of the limit cycle are attracted very quickly onto the limit cycle. Likewise all transient motions within the limit cycle are repelled towards the limit cycle, indicating that the fixed point within the limit cycle is unstable.

As the damping for the system is increased the steady state motion of the limit cycle is still present, but the physical size of the actual limit cycle decreases gradually. Figures 5.4 (b)-(c) also show that all transient motions outside or inside of the limit cycle are still attracted onto the limit cycle.

When the damping reaches a value of \( c = 0.4 \), as shown in figure 5.4 (d), the
Figure 5.4: Phase portraits of the damped system at $v_{dr} = 0.6$ with increasing damping. The stable periodic motion is shown as a thick line and transient motion as thin lines. The fixed point when unstable is shown as $\circ$, and when stable as $\bullet$. 
stable periodic motion of the limit cycle can still be observed. A trajectory from outside of the limit cycle will still spiral inwards onto the stable periodic solution and undergo a stick-phase.

However a trajectory originating from somewhere within the limit cycle now either spirals around the fixed point for a finite period of time, and then is attracted to the stick-slip attractor (limit cycle), or it might spiral around the fixed point and be attracted to it, indicating that the fixed point has now become stable. Figure 5.4(d) shows two trajectories which are started from slightly different starting positions within the limit cycle, one is attracted towards the stable periodic motion of the limit cycle whilst the other one is attracted to the fixed point.

Thus it seems that all trajectories originating from outside of the limit cycle form part of the basin of attraction of the stable periodic solution, while some areas within the limit cycle have another basin attraction, of which the now stable fixed point is the epicentre. Similar behaviour is observed in figure 5.4 (e) where $c = 0.5$.

Since a trajectory inside the stable periodic motion of the limit cycle can spiral outwards to the limit cycle, or inwards to the fixed point, which has now become stable, the nature of the system has changed and now there must exist a boundary of attraction between the two attracting sets. Thus it can be deduced that an unstable periodic attractor exists within the limit cycle, occupying the phase space between the fixed point and limit cycle.

The damping is now increased to $c = 0.6$ in figure 5.4 (f) and now all trajectories are now attracted to the fixed point. The stable periodic motion of the stick-slip attractor is no longer present.

The disappearance of the stable periodic motion of the stick-slip attractor and the fixed point eventually emerging as the only attractor, has been brought about by some form of global bifurcation of the system dynamics.

The change in the stability of the fixed point from unstable to stable and the
appearance of the unstable periodic attractor are prime components of this bifurcation and their role will be fully investigated.

If increasing damping is plotted against the maximum velocity of the motion observed, figure 5.5 is obtained.

It becomes apparent from observing figure 5.5 that at some damping values, the system has coexisting behaviour. For damping values $c < H$ just the stable periodic motion of the stick-slip attractor is observed with the unstable fixed point, as in figures 5.4 (a)-(c).

For damping values $H < c < F$ the stable periodic motion of the stick-slip attractor along with the unstable periodic attractor and the now stable fixed point is observed. At damping values $c > F$ just the stable fixed point of the system is observed.

To get a better understanding of what is actually happening to the system figures 5.6 (a)-(d) show the phase portraits of the system as the damping is slowly
increased towards point $F$.

When figures 5.6 (a)-(d) are observed in conjunction with the bifurcation figure 5.5, the systems behaviour becomes clearer.

Figure 5.6 (a) shows the phase portrait of the system when $c = 0.38$, and shows only the steady state motion of the limit cycle and the position of the unstable fixed point. The presence of the unstable periodic attractor has not been detected.

Figure 5.6 (b) now shows the phase portrait of the system when the damping has been slightly increased to $c = 0.45$. The stable periodic motion of the stick-slip attractor is still present, and the fixed point of the system, at this damping value has now become stable and the presence of the unstable periodic attractor has been detected.

When the damping reaches a value of $c = 0.55$ as in figure 5.6 (c) the orbit of the unstable periodic attractor gets very close to the limit cycle, and takes on its general shape.

At the damping value of $c = 0.6$ shown in figure 5.6 (d) the stable periodic motion of the stick-slip attractor and unstable periodic attractor is no longer present, leaving just the stable fixed point of the system as the sole attractor of the system.

5.2 Bifurcation paths of the damped system

If the non-dimensionalised version of equation 5.1 is considered, and expressed as a system of first order differential equations, namely:

\[
\begin{align*}
\dot{X} &= V \\
\dot{V} &= -CV - X + F_d(V - v_{dr})
\end{align*}
\]  

(5.11)
Figure 5.6: Phase portraits of the system at $V_{dr} = 0.6$ with increasing damping coefficient values. The stable periodic motion of the stick-slip attractor is shown as a thick line and the unstable periodic attractor as a dashed line. The fixed point when it is unstable is shown as a $\circ$, and when stable as $\bullet$. 

(a) $c = 0.38$  

(b) $c = 0.45$  

(c) $c = 0.55$  

(d) $c = 0.6$
The system described by equation 5.11 can be examined as a continuous system without a stick phase. The above system possesses a fixed point for \( X = F_d(\dot{X} - V_{dr}) \), \( \dot{X} = 0 \). The associated Jacobian matrix for the system described by equation 5.11 is:

\[
\begin{vmatrix}
0 & 1 \\
-1 & -C + \frac{\partial F_d}{\partial V}
\end{vmatrix}
\]  

(5.12)

The associated eigenvalues of the Jacobian 5.12 are:

\[
\lambda_{1,2} = \frac{1}{2} \left( \frac{\partial F_d}{\partial V} - C \pm \sqrt{\left( \frac{\partial F_d}{\partial V} - C \right)^2 - 4} \right)
\]  

(5.13)

If \( C = 0 \) i.e. the system is undamped then the two eigenvalues are complex-conjugate and their real parts are positive and as a result the fixed point is unstable.

If \( (C - \frac{\partial F_d}{\partial V}) \) is slightly greater than 0 the two eigenvalues again are complex-conjugate with real parts that are positive and again the fixed point is unstable.

As \( C \) increases in value the two eigenvalues continue to be complex-conjugate until their real parts vanish at a critical value of the damping coefficient where:

\[
C = C_1 = \frac{\partial F_d}{\partial V}
\]  

(5.14)

In this case \( \lambda_{1,2} = \pm i \) and the system undergoes a bifurcation as shown in figure 5.5 as the letter \( H \). It is from this point onwards that the nature of the fixed point changes.

As \( C \) gets larger, relation 5.13 clearly shows that the two eigenvalues are now real and negative, consequently the fixed point is now stable, and thus now attracting.
CHAPTER 5. THE DAMPED SINGLE DEGREE OF FREEDOM SYSTEM

Figure 5.7: Diagram showing the phase space of the damped system as damping is increased. The evolution of the subcritical Hopf bifurcation at $H$ can be seen, along with the birth of the unstable periodic attractor. $F$ indicates the bifurcation, where the unstable periodic motion coalesces with the periodic motion of the stick-slip attractor, leaving just the stable fixed point attractor.

The sequence of events that have been described suggest a bifurcation that is characterised by a change in the stability of a fixed point, and as shown by the phase portraits of the system in figures 5.6 (a)-(d), accompanied by the sudden emergence of an unstable periodic orbit and disappearance of a limit cycle.

The most likely candidate for the kind of bifurcation at $H$ is the Hopf bifurcation. More specifically it is the subcritical Hopf bifurcation that lends itself to this type of behaviour. Figure 5.7 shows the phase space of the type of subcritical Hopf bifurcation that is occurring as the damping is varied for this particular system showing bifurcation points $H$ and $F$.

At the bifurcation point $H$ in figure 5.7, as the fixed point becomes stable an unstable periodic orbit emerges which acts as a separatrix between the now attracting stable fixed point and the attracting periodic motion of the stick-slip attractor.

The unstable periodic orbit generated by the subcritical Hopf bifurcation exists
for a range of values of the damping coefficient that are larger than the bifurcation value $H$. As a consequence in this system as the damping coefficient increases, the amplitude of the unstable periodic orbit in turn also increases.

The unstable periodic orbit gets larger and takes on the general shape of the periodic stick-slip attractor. Eventually at point $F$ in figure 5.7, the stable periodic attractor and unstable periodic orbit of the system come together and coalesce, annihilating each other at point $F$, leaving just the stable fixed point.

The type of bifurcation that occurs at $F$ seems reminiscent of a fold bifurcation. The fold bifurcation is one of the simplest bifurcations that can occur. This type of bifurcation is responsible for how fixed points are created and destroyed.

### 5.3 AUTO: Analysis

Whilst we have shown that the stability of fixed points depends upon the eigenvalues at the equilibrium point, the stability of a periodic orbit is characterised by its Floquet multipliers. Floquet multipliers can be thought of as the eigenvalues of an associated discrete map (the Poincaré map) which quantifies the local stability properties of a point on the periodic orbit. The Floquet multipliers of a periodic orbit must have a moduli less than one for the periodic orbit to be a stable orbit. When the Floquet multipliers of a periodic orbit cross the unit circle, global bifurcations result.

Analysis of the system using the computer program AUTO gave a more in depth explanation of the bifurcational behaviour of the system. This program written by Eusebius Doedel (Doedel, Wang & Fairgrieve 1994), allows the analysis of stable and unstable periodic orbits, giving the specific start and end point of any branch's in any bifurcation diagram obtained. More importantly in this case, it allows the behaviour of the Floquet multipliers of the system to be analysed around the unit circle at specific points along the branch of the bifurcation diagram.
For AUTO to analyse the unstable periodic orbit, the initial starting conditions had to lie exactly on the unstable orbit. However in this case as the system was being analysed as a continuous system without a stick-phase, initial starting coordinates very close to point $F$ had to be chosen. This was required as the start point for the integration process, as beyond this point, the discontinuous nature of the system would commence.

Once the initial conditions had been obtained, the integration process was then started from this point upto its branch end point. This gave the nearest approximation to point $F$ that would be possible. This gave a small portion of the diagram that was required. The rest of the diagram was obtained by running the integration process backwards from the unstable periodic orbit branch end point, towards point $H$.

The diagram obtained is shown in figure 5.8

The \textit{birth} of the unstable periodic orbit at point $H$ can clearly be seen, upto its annihilation when it coalesces with the stable periodic orbit of the stick-slip attractor at point $F$. For reference, the circles that define the shape of the unstable
periodic orbit signify to the user that this particular branch is unstable. Using this diagram it is possible to move along this unstable branch of the bifurcation diagram and determine the Floquet multipliers at that point.

Figures 5.9(a) to 5.9(d) show the behaviour of the Floquet multipliers around the unit circle, as specific points along the path of the unstable branch of the bifurcation diagram are traversed. The Floquet multipliers can be observed in the box in the bottom left hand corner of each of the diagrams, as points in or around the unit circle. As mentioned before they indicate stability; those in the circle are stable and those that are out of the circle are unstable. Bifurcation’s occur on the circle, that is, Floquet multipliers crossing the unit circle indicate that a global bifurcation has occurred.

Figure 5.9(a) shows that the Floquet multipliers for the system at this point where \( c = 0.3827 \) lie exactly on the edge of the unit circle. The significance of this point will become apparent later in this section, but for time being it signifies that the system is poised on the transition point between stability and instability. AUTO labels this point as \( LP \). This label is reserved for special points that AUTO wants to highlight and in this case \( LP \) represents the Limit Point or turning point of a branch.

Figure 5.9(b) shows the Floquet multipliers for the system at \( c = 0.3967 \). We now see that one of the Floquet multipliers has crossed the unit circle through +1, whilst the other one lies on the edge of the unit circle. This indicates that the orbit at this point is unstable. Hence a bifurcation has occurred since the last point traversed in figure 5.9(a), which has lead to a state of instability from a state of stability. It should be noted that one of the Floquet multipliers associated with a periodic solution of an autonomous system is always unity (Nayfeh & Mook 1979).

Figure 5.9(c) shows the Floquet multipliers for the system at \( c = 0.4837 \). It is seen that the Floquet multiplier that crossed the unit circle has moved further away
CHAPTER 5. THE DAMPED SINGLE DEGREE OF FREEDOM SYSTEM

Figure 5.9: Diagrams showing the behaviour of the Floquet multipliers around the unit circle as specific points along the path of the unstable periodic orbit are traversed. NOTE: Floquet multipliers within the unit circle signify that the orbit is stable and those outside the unit circle signify an unstable orbit.
CHAPTER 5. THE DAMPED SINGLE DEGREE OF FREEDOM SYSTEM

Figure 5.10: Bifurcation diagram obtained with AUTO, showing the path of the fixed point with increasing damping.

from the unit circle, whilst the other one stays on the unit circle. Hence the orbit at this point is still unstable.

The last figure 5.9(d) shows the Floquet multipliers for the system at $c = 0.5563$. The Floquet multipliers stay in the same position as before, that is, one on the unit circle at $+1$ and one outside of the unit circle.

In order to establish the a more complete picture of the behaviour of the system with varying damping the stability of the fixed point would need to be addressed. This has already been done using the Jacobian of the continuous system, but the stability analysis will be done again using AUTO for confirmation and demonstration purposes.

To carry out a stability analysis of the fixed point, AUTO was provided with initial conditions, which corresponded with those of the system at a steady state. AUTO seems to work well when starting from a steady state and the bifurcation diagram shown in figure 5.10, was produced quite easily.

The bifurcation diagram, if we follow the path of the fixed point, has two different
Figure 5.11: Diagrams showing the behaviour of the eigenvalues around the unit circle as specific points along the path of the fixed point are traversed. NOTE: Eigenvalues within the unit circle signify that the fixed point is stable and those outside the unit circle signify an unstable fixed point.

Since we are now dealing with the stability of fixed points, eigenvalues will be used to determine the stability of the fixed point. Figures 5.11(a) to 5.11(d) show the behaviour of the eigenvalues around the unit circle, as specific points along the bifurcation diagram are traversed.
Figure 5.11(a) shows the eigenvalues for the system when $c = 0.04$. It can clearly be seen that both eigenvalues lie outside of the unit circle, indicating an unstable fixed point.

Figure 5.11(b) shows the eigenvalues for the system when $c = 0.16$. Again it can clearly be seen that both eigenvalues lie outside of the unit circle indicating an unstable fixed point, but both seem to be closer to the unit circle than they were in figure 5.11(a).

Figure 5.11(c) shows the eigenvalues for the system when $c = 0.3827$. At this point we can observe that both eigenvalues now lie on the unit circle, preparing to cross it, indicating that the stability of the fixed point is about to change. AUTO highlights this point as a special point and labels it as HB, which in this case stands for Hopf Bifurcation. This point is the exact point from where the unstable periodic orbit emerges, when the fixed point becomes stable. The unstable periodic orbit can also be see in figure 5.9(a), emerging from this point. Using AUTO we can take this co-ordinate as an initial condition and trace the unstable periodic orbit, producing figure 5.12 which shows both the stability of the fixed point and the emergence of the unstable periodic orbit.

The point $F$, as seen in figure 5.7, is the bifurcation point where the unstable periodic orbit coalesces with the stable periodic attractor, is reminiscent of a fold bifurcation. This assumption can be made since it has been established that as the damping is varied and the unstable branch of the bifurcation diagram is followed (see figure 5.11 (a)-(d)), the second Floquet multiplier leaves the unit circle through $+1$ and to infinity (Nayfeh & Mook 1979; Thompson & Stewart 1986). The fold bifurcation is in many ways the most fundamental bifurcation in non-linear dynamics, and is recognised as the primary example of discontinuous and catastrophic bifurcation's (Foale & Thompson 1991). Once the bifurcation has taken place, there is no way of following the path past point $F$ and thus the attractor bifurcation in this case is discontinuous.
In detailed analysis by Leine et al (Leine, Van Campen & Van de Vrande 2000) on the stick slip system that is presented here they have investigated the bifurcation at point $F$ by using a combination of Floquet theory and Filippov theory. The results also determine that the stable and unstable periodic branches of the bifurcation diagram shown in figure 5.5 are connected through a fold bifurcation point. The observed bifurcation resembles a continuous fold bifurcation of a smooth system. The second Floquet multiplier jumps through from $\lambda = 0$ to $\lambda = \infty$ at point $F$, and therefore through $+1$ on the unit circle. Since the path after the bifurcation is not continuous, it is therefore called a discontinuous fold bifurcation. This jumping of a Floquet multiplier to infinity (through $+1$) resulted in periodic solutions which Leine et al termed infinitely unstable periodic solutions.

### 5.4 Summary

The inclusion of damping is a natural extension of the single degree of freedom system towards a more physically realistic model. The inclusion of damping does
not produce a more accurate model of tectonic plate movement, or any other system which exhibits stick-slip oscillations, but rather a better understanding of the dynamic processes involved.

The investigation of the dynamic processes involved yielded some dynamically rich and interesting behaviour. A Subcritical Hopf bifurcation and discontinuous fold bifurcation were both identified, in the course of the investigations.

To determine the nature of these bifurcation's classical methods of analysis as well as the computer program AUTO were used. AUTO for all its faults and difficult user interface, allowed the analysis of the unstable periodic orbit to proceed without having to get deeply acquainted with Floquet Theory.

Since in reality we cannot be sure of the exact nature of the friction characteristic, a non-smooth approximation has been used. It should be pointed out that this non-smooth approximation did provide a very good approximation to the experimental data obtained in chapter 3. However due to the non-smooth nature of the friction characteristic, in areas around the discontinuity where the slope is very steep, led to non-smooth dynamics. Hence it was not surprising to find non-standard or discontinuous bifurcation's.
Chapter 6

The Externally Forced Single Degree of Freedom System

In all the various systems that have been investigated so far, only self-excitation induced by dry friction has been considered as the mechanism for oscillations that evolve within systems such as the single degree of freedom system.

Since in most mechanical systems there is some form of external forcing present, that causes the system to exhibit some form of periodic behaviour which is dependent upon the frequency, amplitude and period of the external forcing.

It therefore becomes imperative to understand how external forcing may affect the dynamical behaviour of a system which exhibits self sustained oscillations, and how these two types of oscillatory behaviour superimpose and interact with each other.

Fig 6.1 shows the single degree of freedom system with self and external excitation that will now be investigated.

Energy is transferred from the moving belt to the system via the friction force. The dynamic friction force $F_d$ depends upon the relative velocity between the moving
Figure 6.1: Schematic of the forced single degree of freedom model with damping.

belt and mass, and for the purpose of this investigation can be modelled by means of the friction characteristic:

\[ F_d = \frac{F_s}{1 + \gamma |(\dot{x} - v_{dr})|} \]  

(6.1)

The general equation for the motion of the forced system is as follows:

\[ m\ddot{x} + c\dot{x} + kx = F_d(\dot{x} - v_{dr}) + u_0 \cos \omega t \]  

(6.2)

where \( u_0 \) is the excitation amplitude and \( \omega \) is the excitation frequency.

The phase portrait for the system when unforced, produces a limit cycle which is shown in figure 6.2.

In previous chapters it was found that with the undamped system, all trajectories initiated outside or inside of the limit cycle would be attracted onto the limit cycle, independent of the underlying friction characteristic. The conclusions drawn from this behaviour was that there existed an unstable fixed point within the limit cycle,
which repelled trajectories onto the limit cycle, whilst the limit cycle itself, being a stable periodic solution of the system, attracted trajectories. The stability of this fixed point alters with increasing damping and is discussed in chapter 5.

In engineering applications stick-slip oscillations are highly undesirable and should be avoided at all costs since they diminish the precision of motion and safety of operation. They also cause noise and promote wear and tear.

In the case of simultaneous self and external excitation, due to the non-smooth non-linearity of dry friction, rich bifurcational and chaotic behaviour has been found (Popp & Stelter 1990b; Oestreich, Hinrichs & Popp 1996; Galvanetto, Bishop & Briseghella 1995).

The aims of the chapter are to investigate the single degree of freedom system show in figure 6.1 under self and external excitation and to show the transition from regular to chaotic motions and the corresponding parameter dependencies. A secondary aim, if possible, would be to determine whether the very robust limit cycle of a typical stick-slip oscillation can be broken up by an external harmonic disturbance, and if this behaviour can be controlled.
6.1 System behaviour

As before, numerical simulations will be used to determine the behaviour of the system. The system's equations of motion are integrated numerically with an adaptive step-size Runge-Kutta routine.

The phase portraits obtained for the system with different forcing frequencies $\omega$ are shown in figures 6.3 to 6.5.

In immediate contrast with the results obtained for the other single degree of freedom systems that have been investigated without external forcing in chapters 2, 4 and 5, the system with external forcing can exhibit single-periodic, double-periodic, multiple-periodic and chaotic behaviour.

This type of behaviour can be more clearly observed with the bifurcation diagram in figure 6.6, for the system with varying forcing frequency ($\omega$).

For this investigation, the point $X_A$, which represents the position of the stick-to-slip transition point is plotted with the bifurcation parameter $\omega$. For varying forcing frequency the bifurcation diagram in figure 6.6, shows $X_A$ against increasing forcing frequency. It shows single periodic solutions, period doubling and windows of periodic behaviour within areas of chaos.

When the various phase portraits in figures 6.3 to 6.5 are seen in conjunction with the bifurcation diagram, the complexity of the system becomes apparent.

6.2 The forced system: A closer look

The bifurcation diagram produced by varying forcing frequency in figure 6.6 shows that simply by varying the frequency of the external excitation it is possible to alter the period of oscillation of the system, and even induce chaotic motion.
CHAPTER 6. THE EXTERNALLY FORCED SINGLE DEGREE OF FREEDOM SYSTEM

Figure 6.3: A single periodic solution, $\omega = 0.30$

Figure 6.4: A period 2 solution, $\omega = 0.41$

Figure 6.5: Chaotic motion, $\omega = 0.20$
This in itself is a very crude form of control, and would, in certain applications be very useful if feedback was available giving details about the state of the mass. Given that a method of altering the period of the system is now available, in many engineering applications stick-slip behaviour is highly undesirable and a method of preventing stick-slip behaviour altogether would useful.

Feeny et al (Feeny & Moon 2000) refer to this process as quenching stick-slip behaviour with dither, which is their description for the application of a high frequency excitation to affect the behaviour of the system. Dither in their examples has been used to couple the behaviour of various oscillatory systems, such as in quenching limit cycle behaviour, stabilising equilibria, such as the inverted pendulum and improving friction behaviour.

Thomsen (Thomsen 1999) in his investigations, shows that a high frequency harmonic excitation with a small amplitude can prevent stick-slip oscillations from occurring. To test these hypotheses, the same numerical simulations were carried out with the current forced system model. The results are shown in figure 6.7.

Figure 6.7 shows the time vs. displacement time histories for both the quenched
The thin line represents the unquenched system, where the external excitation has a frequency within the range used in the bifurcation diagram, and a small amplitude. The resulting oscillation of the system has a constant amplitude throughout the simulation, consistent with stick-slip behaviour.

The thick line represents the quenched system, where a high frequency excitation force is applied with a slightly larger amplitude. The results are markedly different. Although the quenched system starts off having motion which seems to be stick-slip behaviour, it soon decays. Eventually the mass seems to perform tiny oscillations about a non zero equilibrium, and there is no longer any evidence of any stick-slip behaviour.

The time vs. velocity time histories are shown for the quenched and the unquenched system in figure 6.8.

The thin line represents the unquenched system which shows the classic signs of stick-slip behaviour, namely, the constant velocity for a short period followed by
oscillatory motion in an even shorter period and so on. The thick line represents the quenched system, which has a markedly smaller amplitude than the unquenched system. There are no apparent signs of stick-slip behaviour.

Although this system of control works well, it still is far from being an efficient system. The main problem is that although a harmonic excitation force is applied to the system, there are always going to be periods when the excitation force will be helping the mass in its oscillatory motion instead of impeding it. To compensate for this helping effect the frequency is increased.

In terms of energy considerations this is very wasteful as we have apply a large frequency, when we would not have to, if we knew the state of the system. Thus in order to successfully quench stick-slip behaviour applying just a harmonic excitation force to the mass is not enough.

In order for the quenching process to be efficient it would be highly desirable if the excitation force was a phase shifted version of the oscillation velocity. Thus the maximum effect would be attained as the force would always work against the motion of the mass. This would be hard to accomplish with just a harmonic
excitation force which also happened to be a phase-shifted version of the oscillation velocity. So another approach will be taken.

6.3 The controlled quenching of the single degree of freedom system

The previous section dealt with one method of quenching stick-slip behaviour. There are probably many methods to reduce or quench friction induced vibrations, but it is a common fact that oil or grease works well. Lubrication has the effect of changing the dominating mechanism of energy dissipation from dry-friction into viscous type damping. While oil may stop a rusty door hinge from squeaking, this cure is inadequate for cases where dry friction is necessary for proper operation of a device, an example being vehicle brakes. The problem here is that once the lubrication has been applied, it cannot be quickly removed.

The main emphasis of this section will be to highlight ways of quenching friction induced vibrations in a more controlled and energy efficient manner.

We assume the same friction characteristic as expressed in equation 6.1, as in all other investigations. For small oscillation amplitudes where the velocity of the mass does not reach the sticking section of the friction characteristic, the friction characteristic can be approximated by its' linearised form. This linear approximation will be sufficient to carry out a prediction of the stability behaviour of the system.

Firstly the unforced system will be analysed, using the method adopted Heckl (Heckl & Abrahams 1996).

The equation of the motion for the unforced system is
\[ m\ddot{x} + c\dot{x} + kx = F_y \]  

(6.3)

If we then assume the linear approximation for the friction characteristic

\[ F_y = F_0 - \gamma v \]  

(6.4)

where \( F_0 \) is the friction force at zero relative velocity. The parameter \( \gamma \) is as in the non-linear form of the curve the slope of the slip section of the friction characteristic.

Equation 6.3 becomes

\[ m\ddot{x} + (c - \gamma)\dot{x} + kx = F_0 \]  

(6.5)

The general solution of equation 6.5 is of the form

\[ x(t) = ae^{-\Omega t} + b \]  

(6.6)

where \( \Omega \) is the complex eigenfrequency of the single degree of freedom model, \( a \) and \( b \) are constants. \( \Omega \) and \( b \) can be determined by the insertion of equation 6.6 into equation 6.5. The result obtained is

\[
\begin{align*}
b &= \frac{F_0}{k} \\
\Omega &= \pm \sqrt{\left(\frac{k}{m}\right) - \left(\frac{1}{4m^2}\right)(\gamma - c)^2 + i(\gamma - c)/2m}
\end{align*}
\]  

(6.7)

The real and imaginary parts of \( \Omega \)

\[ \Omega = \omega + i\delta \]  

(6.8)
are physically meaningful. The real part $\omega$ is the frequency of the stick-slip oscillation. It is close to the resonance frequency $\omega_0 = \sqrt{k/m}$ of the free mass-spring oscillator if $(\gamma-c)^2 \ll mk$, that is if the damping of the system is small. The imaginary part $\delta$ represents the growth rate of the oscillation. The sign of $\delta$ provides a method of categorising the stability behaviour of the system,

$$
\begin{align*}
\delta > 0, & \quad \text{unstable} \\
\delta = 0, & \quad \text{critically stable} \\
\delta < 0, & \quad \text{stable}
\end{align*}
$$

(6.9)

The second solution in equation 6.7 for $\Omega$ with the negative real part is of no physical interest and is therefore ignored.

### 6.3.1 The control method

The single degree of freedom model can be extended to allow some form of control over the oscillatory behaviour of the system. If a small force was applied to the mass which was a phase-shifted signal of the oscillation frequency of the mass, then this could reduce or neutralise the motion of the mass depending upon the correct magnitude of the force being applied. A theoretical set up could be one that is shown below in figure 6.9.

The mass has a small transducer attached to it, which allows the calculation of the velocity of the mass. This velocity signal is then sent to a frequency filter to separate the oscillation frequency from any other contaminating signal. A phase shifter then applies an adjustable phase shift to the filtered signal. The signal is then amplified to the desired magnitude and then sent to a shaker on the mass. This shaker then oscillates out of phase with the mass and superimposes it’s out of phase oscillations upon those of the mass and hence diminish the movement of the mass.
The force exerted by the *shaker* is for this purpose called the control force and is denoted by $F_c$. As described above it is a phase shifted and amplified version of the oscillation velocity $v$. Following conventions adopted by Heckl (Heckl & Abrahams 1996), the control force can be described mathematically by

$$F_c = \alpha e^{-i\phi} v$$

where $\alpha$ is a measure for the amplification and $\Phi$ is the phase shift in radians.

When $F_c$ is added to friction force, the equation of motion becomes

$$m\ddot{x} + (c - \gamma)\dot{x} + kw = F_0 + \alpha e^{-i\phi}\dot{x}$$

The complex eigenfrequency $\Omega$ of the *controlled* single degree of freedom model is determined as before, by assuming a general solution of the form 6.6 and inserting it in the equation of motion above. The result is

$$m\Omega^2 - i\Omega(c - \gamma - \alpha e^{-i\phi}) + k = 0$$

which has the solution

Figure 6.9: A theoretical experiment to control the single degree of freedom system.
\[ \Omega = \sqrt{\left(\frac{k}{m}\right) - \left(\frac{1}{4m^2}\right)(\gamma - c + \alpha e^{-i\Phi})^2 + i(\gamma - c + \alpha e^{-i\Phi})/2m} \quad (6.13) \]

This result now gives the frequency and the growth rate of the controlled single degree of freedom model in terms of the control parameters \( \alpha \) and \( \Phi \). In the absence of control, i.e. if \( \alpha = 0 \), the eigenfrequency reduces to equation 6.7. Looking at equation 6.13 it is clear that both \( \alpha \) and \( \Phi \) have an influence on the real part and on the imaginary part of the eigenfrequency. They can even change the sign of the imaginary part of the eigenfrequency to give a negative growth rate. In this instance control has been achieved and a previously unstable oscillation is made to decay exponentially.

Furthermore it is of interest to find which values of \( \alpha \) and \( \Phi \) give a particular growth rate. Hence \( \alpha \) and \( \Phi \) are written as explicit functions of \( \omega \) and \( \delta \),

\[ \alpha = \sqrt{R^2 + I^2} \]
\[ \Phi = \arctan(-I/R) \quad (6.14) \]

where

\[ R = \frac{\delta (\omega^2 + \delta^2) m + k}{\omega^2 + \delta^2} + c - \gamma \]
\[ I = \omega \frac{-(\omega^2 + \delta^2) m + k}{\omega^2 + \delta^2} \quad (6.15) \]

These parameters can now be used to plot curves \( \delta = \text{constant} \) and \( \omega = \text{constant} \) in the \( \alpha - \Phi \) plane. The curve \( \delta \) is of particular interest as it the separating line between stability and instability for the control system.

However the way in which \( \alpha \) and \( \Phi \) influence the oscillation frequency \( \omega \) has not been investigated.
This is an important question when we consider that the frequency filter has to filter out contaminating signals from the oscillation frequency, and it too has a limited range of bandwidths in which it can effectively function. If the oscillation frequency does not lie within the filters bandwidth, then control will not be possible.

Increasing the bandwidth of the filter may be possible but it also increases the susceptibility of the signal to contamination by noise and is therefore not a viable option.

A suitable filter bandwidth setting, consisting of a narrow frequency band and which contains the oscillation frequency $\omega$ can be quite easily chosen. If $\omega_{\min}$ and $\omega_{\max}$ are minimum and maximum frequencies of the oscillation, then the area in the $\alpha - \Phi$ plane between the curves $\omega = \omega_{\min}$ and $\omega = \omega_{\max}$ indicates the stability range imposed by the filter.

Equations 6.15 can be evaluated numerically to give results in the form of curves $\delta = \text{constant}$ and $\omega = \text{constant}$ in the $\alpha - \Phi$ plane.

Figure 6.10 shows the curve $\delta = 0$ in the $\alpha - \Phi$ plane, and as can be observed, without any frequency restrictions, the stability region in the $\alpha - \Phi$ plane is very large. The optimum phase shift is at $\pi$ degrees, achieving stability if the amplification $\alpha > 0.3$. The oscillator is actually stable for any phase shift $\Phi$ between $\pi/2$ and $3\pi/2$, so long as the amplification is sufficiently large. Other curves of $\delta = -0.1\omega_0$ and $\delta = -0.2\omega_0$ are also shown and both these curves represent stable cases. The parameter $\omega_0$ in this case is the resonance frequency of the freely oscillating mass/spring system.

The need to filter the control signal from any potentially contaminating signal reduces the size of the stability region. The reduction, as mentioned before, depends upon the the filter bandwidth. Various curves of constant frequency in the $\alpha - \Phi$ plane are shown in figure 6.11. The middle curve is $\omega = \omega_0$ and the outer curves are for frequency values of $\omega = 0.9\omega_0$ and $\omega = 1.1\omega_0$ which represents 10% above
and below the resonance frequency and $\omega = 0.5\omega_0$ and $\omega = 2\omega_0$ which represent a factor of 2 above and below the resonance frequency.

Examining the curves for $\delta = constant$ and $\omega = constant$, it becomes apparent that from a practical point of view it is best to choose the control parameters $\alpha$ and $\omega$ from the lower end of the stability region. The main reason for this is that there are a greater range of phase values that are available that give stability. This can be more clearly seen with increasing amplification $\alpha$ the phase range becomes narrower. Even when a filter bandwidth as restrictive as $\omega = \pm 10\%\omega_0$ is chosen there still is almost $\pi/2$ phase range within which control can be achieved.

### 6.4 Controlling behaviour of the single degree of freedom system

Since it has been shown that quenching of stick-slip behaviour is possible, the original idea that it may be possible to control the system, and alter it's state...
from chaotic to periodic behaviour is intriguing.

Initially the presence of chaos in the behaviour of dynamical systems was considered to be something that induced extra difficulties. Chaotic motion was thought to be impossible to predict or to control. Chaos in certain conditions is desirable, such as in combustion processes as it enhances mixing of fuel and air and hence better performance. On the other hand, in most mechanical conditions such as in aerodynamic and hydrodynamic applications, chaos in the form of turbulence is undesirable because it dramatically increases the drag of vehicles and results in increased operating costs. This forced engineers and designers to take special care, so that the systems they built would not behave chaotically.

This view of chaos has changed and around 1950 John Von Neumann suggested that carefully planned local perturbations in the atmosphere, induced by chemicals sprayed by aeroplanes, could lead to desired, large scale changes in the weather pattern. He seemed to understand how the behaviour of a chaotic system, such as the weather could be influenced by small perturbations in the systems parameters. As always Von Neumann always seemed to be three steps ahead of everybody else!
Following these ideas, (Hubler & Luscher 1989) showed that it was possible to control chaos and made some first attempts. These methods succeeded only in some cases and they had severe limitations. The first robust method for controlling chaotic motion was presented by (Ott, Grebogi & Yorke 1990). Since then many other successful attempts to produce efficient algorithms for control of chaos have been made.

The previous section has shown that the single degree of freedom system when behaving chaotically, can have its oscillatory motion quenched by applying perturbations to the system. Control methods that exploit this property of the system may also be used to determine the type of behaviour of the system by using the same perturbations but in a more controlled fashion.

6.4.1 Theory of control methods

The control of chaotic systems is based on exploitation of certain properties of chaotic motion. The first such property is that any chaotic attractor has embedded within it an infinite number of unstable periodic orbits.

This can be seen easily in the transition of many dynamical systems to chaos through a series of period doubling bifurcation's. An example of such a transition to chaos being the logistic map.

Each bifurcation creates a new stable orbit having twice the previous period, while the previous orbit becomes unstable. These unstable orbits do not cease to exist even when the system is well into the chaotic regime.

Usually, these embedded unstable orbits are the ones that control techniques aim to stabilise. However, some methods allow stabilisation of orbits that are not embedded in the original attractor. This is mainly done by applying large perturbations on the system which may significantly alter the dynamics of the original system.
The other feature of chaotic motion exploited by control techniques is the sensitive dependence on initial conditions: very small perturbations will, in finite time produce motion that is very different from motion produced if the system is not perturbed. This feature of chaotic systems, previously seen as a disadvantage, can be used to drive the system towards any desired state. This is called targeting (Shinbrot, Ott, Grebogi & Yorke 1990; Baretto, Kostelich, Grebogi, Ott & Yorke 1995; Arecchi & Bocaletti 1997).

Ergodicity states that every chaotic trajectory will eventually come arbitrarily close to any point within the region of the phase space occupied by the attractor. Thus when the trajectory is given time to approach the area of interest, control can be activated. This enables the required stabilising perturbations to be very small.

Control can be achieved by applying perturbations to an accessible system parameter or a variable, in a controlled manner, so that the next iterate falls closer to the desired state. Thus, the motion of the chaotic system can be stabilised close to any unstable periodic orbit embedded within the chaotic attractor.

These developments have greatly affected the way engineers think of chaos. There are practical reasons for controlling chaos as in some systems chaos can lead to catastrophic situations, such as wing flutter in aircraft. Controlling chaos in some systems can bring the richness of chaos into use, and building chaotic dynamics purposely into an engineering system can now be seen as an advantage. With successful control, a system can follow any desirable orbit and thus any type of behaviour, within its state space. This can be very cost effective because applying small perturbations on an accessible system parameter is probably cheaper than changing the systems configuration in order to obtain the desired behaviour. On the other hand, controlling systems that are not chaotic is not as effective, because in order to achieve a large change in the behaviour of the system it is likely that very large perturbations will be required.
6.4.2 The OGY algorithm

In 1990 Ott, Grebogi & Yorke presented a computer algorithm now known as the *OGY algorithm*. This was a method which showed how chaos can always be suppressed by shadowing and stabilising a chaotic trajectory of the system onto a periodic orbit one of the infinitely many unstable periodic orbits (or perhaps steady states) embedded in the system's chaotic attractor.

The algorithm operates on a Poincaré section of the desired periodic orbit. Using a Poincaré section, the unstable periodic orbit is reduced to a fixed point. A local linearisation around the fixed point is made. By ergodicity, any trajectory of the system will eventually come close enough to the periodic orbit, so that the linearisation is valid.

The region on the section, for which the linearisation is valid, can be visualised as the area enclosed by a circle with the fixed point in the centre. When a chaotic trajectory passes close enough to the desired orbit, successive intersections with the section give points within the region for which linearisation is valid. When a point falls within this region, the control procedure is activated.

The trajectory must pass close to the unstable periodic orbit in order to activate the control. This is because, linearised approximations are only valid for a small region around the unstable orbit and also, the stabilising perturbations required for a trajectory that is far from the unstable periodic orbit would be very large.

The perturbations are applied on an accessible system parameter, have to be very small and are time dependent. They are chosen so that the next point on the section will fall closer to the stable manifold of the fixed point. Eventually, the chaotic trajectory will coincide with the stable manifold of the required periodic motion. From this point onwards, no further control is required, and the evolving trajectory will eventually approach the desired periodic orbit.
6.4.3 Other control techniques

Following the presentation of the OGY method, many other methods were published aiming to control chaotic motion.

The first techniques to evolve were non-feedback techniques. The control perturbations are applied either on the parameters of the dynamical system or directly on its variables. Non-feedback methods that apply perturbations on the system variables, instead of its parameters, essentially involve coupling of a chaotic system to a simple, autonomous one.

Other methods (Molgedey, Schuchhardt & Schuster 1992; Fahy & Hamann 1992; Rajasekar & Lakshmanan 1993) use external noise to suppress chaotic motion. The effectiveness of these methods lies in the observation that the addition of a suitable external noise term affects the sensitive dependence of chaotic systems to initial conditions.

The most effective methods, considering their simplicity, are those using proportional feedback (Hunt 1991; Pyragas 1992; Roy, Murphy, Maier, Gills & Hunt 1992; Johnson & Hunt 1993; Bielawski, Derozier & Glorieux 1994; Kittel, Pyragas & Richter 1994). In these methods, calculation of the perturbations is done using the difference of the current state of the system to a time-delayed state. This is essentially equivalent to synchronising the system with its time history.

The advantage of proportional feedback is that it does not require explicit knowledge of a system's dynamical equations. Therefore it is suitable for experiments and for analysis from a time series.

Another strategy is to use conventional linear control to transform chaotic motion to periodic. The technique attempts to determine a feedback weight, such that all the eigenvalues of the Jacobian matrix of the dynamical system will be less than unity (Chen & Dong 1992; Paskota & Mees 1994).
Finally, there is a group of methods using geometric approaches to control (Toroczkai 1994; Bishop, Xu & Clifford 1996; Sass & Toroczkai 1996; Xu & Bishop 1996). Algorithms using such techniques, usually involve procedures for automatic search and location of periodic orbits and fixed points.

### 6.4.4 The control method

The method of control that will be used to locate and lock onto specific unstable periodic orbits, will be an amalgamation of the various methods discussed. The reason for this is that one specific method would not allow control for this type of system. Thus the most appropriate techniques will be used where necessary.

The first step in the control technique is the identification of the unstable periodic orbits. To accomplish this, a OGY type of method will be used where the algorithm operates upon the Poincaré section of the desired periodic orbit. The Poincaré section in this case will be the section obtained at $v = 0$ on the phase portrait of the system, since the flow at this point is transverse to the section. The intersections of the chaotic trajectory with the chosen section are represented by vectors $x_1, x_2, \ldots, x_n$.

A periodic orbit corresponds to a finite number of points on the section. Therefore, to determine the locations of the unstable periodic orbits from the above points of the intersection of the trajectory with the section, a small positive number $\varepsilon$ is chosen. Then for each vector $x_i$, the smallest index $j > i$ is identified such that:

$$| x_i - x_j | < \varepsilon \quad (6.16)$$

Then, typically there is a period-$p$ orbit, where $p = j - i$, near $x_i$ and $x_j$. For example, there is a period-one orbit near $x_{30}$ if $| x_{30} - x_{31} | < \varepsilon$, and there is a period-two orbit near $x_{30}$ if $| x_{30} - x_{32} | < \varepsilon$ but $| x_{30} - x_{31} | > \varepsilon$. In this way the
approximate locations of the periodic orbits are identified.

Following this method, if the criteria for a specific periodic orbit is not satisfied, then the control parameter is perturbed. Eventually the control parameter is perturbed up to a critical point where the criteria for a specific periodic orbit is satisfied and the trajectory locks onto it. Once the desired periodic orbit has been locked onto, the algorithm removes the perturbation and allows the trajectory to continue unhindered and eventually makes its way to the stable manifold.

Figure 6.6 shows the bifurcation diagram obtained with increasing forcing frequency. There are many values of the forcing frequency where the system exhibits chaotic behaviour. The area where $\omega = 1.6$ is an area where dense chaotic motion and most probably an infinite number of unstable periodic orbits reside. This is probably an ideal area to test out the algorithm.

The algorithm was required to locate and lock onto a period-2 orbit in the region where $\omega = 1.6$. Figure 6.12 shows the phase portrait of the final limit cycle which the algorithm located. The final amplitude perturbation that the algorithm applied to bring the chaotic trajectory onto the period-2 orbit was 0.0006 or 0.3% of the actual forcing amplitude.

The algorithm was then required to locate and lock onto a period-5 periodic orbit when $\omega = 1.1$. This again was another area which was dense in chaotic motion. Figures 6.15 to 6.17 show the phase portrait, perturbation time series and velocity time series for the located orbit.

The final amplitude perturbation that the algorithm applied to bring the chaotic trajectory onto the period-5 orbit was 0.0047 or 2.3% of the actual forcing amplitude.
Figure 6.12: A period-2 periodic orbit located and locked onto when $\omega = 1.6$.

Figure 6.13: The perturbation applied to the system to enable the system to lock onto a period-2 orbit when $\omega = 1.6$. 
Figure 6.14: The velocity time series of the period-2 periodic orbit.

Figure 6.15: A period-5 periodic orbit located and locked onto when $\omega = 1.1$. 
Figure 6.16: The perturbation applied to the system to enable the system to lock onto a period-5 orbit when $\omega = 1.1$.

Figure 6.17: The velocity time series of the period-5 periodic orbit.
6.5 Summary

The dynamical behaviour of the single degree of freedom system under self and external excitation has been investigated. Numerical simulations of the system resulted in \( p \)-periodic solutions of any period up to \( p = 9 \) as well as chaotic motion. The routes to chaos are similar to intermittency and most motion displayed stick-slip behaviour.

These investigations have also provided a theoretical method for understanding how friction induced oscillations are affected by external excitation. According to the numerical simulations that have been run, external excitation can effectively cancel the discontinuity in the friction characteristic, and under certain parameter values actually prevent self-excited oscillations from occurring.

Fast oscillations have been shown to quench friction induced oscillations by changing the effective friction characteristic of the system. High-frequency excitation turns dry friction into a viscous like form of damping similar to the kind of dissipation that would occur when using a lubricating oil.

However it should be stressed that this technique does not simply turn off self-excited oscillations into absolute rest but rather transform them into small-amplitude vibrations at very high frequency. This is clearly visible from the results shown in figures 6.7 and 6.8.

If this frequency is outside the audible range then one problem has been solved, however these vibrations can be quite energetic and may introduce other problems such as heat generation.

The method of controlling the behaviour of the system has been demonstrated successfully in two different regions of dynamical motion. The technique shows that unstable periodic orbits within an area of intense chaotic motion can be stabilised by small perturbations to the system. The perturbations are of the
same form as the forcing applied to the system, but at a considerably smaller amplitude.

The final perturbations that were applied were very small when compared to the actual forcing amplitude. In some instances the applied perturbation amplitude was only 0.3% of the forcing amplitude. This type of control method is a viable alternative to changing the configuration of a mechanical system which was experiencing chaotic behaviour.
Chapter 7

Conclusions

Although non-linear dynamical systems exhibit extremely complex phenomena, we conclude that a non-linear analysis has important engineering relevance, especially when simplicity can be derived out of such complexity. Since many engineering systems are not linear, and as such are modelled by non-linear equations for which closed-form analytical solutions are unobtainable, a detailed numerical analysis may be a daunting prospect for the typical engineer.

Many mechanical systems display non-smooth characteristics and in recent years they have become increasingly important, as the non-smooth effects are genuinely taken into account and are not smoothed out. Many small systems of differential equations demonstrate phenomena related to discontinuities, and due to the lack of smoothness, classical methods can only be applied in limited amounts. Thus a reliance on linear equations may be simpler, but would not give a realistic representation of the system.

This was borne out by the analysis carried out in chapter 2 where a number of different models for dry friction were investigated. Since many of the equations of motion, for the particular numerical models would have been difficult to analyse using classical methods, they were integrated using an adaptive step-size Runge-
Kutta integration scheme. However in order to make the numerics as efficient as possible, a number of algorithms are used which make use of a number of system properties. To determine the end of a slip-phase of a mass, an algorithm is used which has its roots in Hénons methods for calculating Poincaré maps (Hénon 1982). This algorithm allows the location where the mass stops slipping to be determined without any intensive computational effort, and since it is very accurate, very few errors are carried forwards to the next integration step. Thus a considerable reduction in time and effort have been saved by applying these algorithms as well as being more accurate than any brute force method.

It was found in the course of these numerical simulations that in order for the system to exhibit distinct stick-slip behaviour, the gradient of the slip section of the friction characteristic was found to be a crucial parameter. Only a non-zero positive slope would lead to a distinct stick-slip oscillation of constant amplitude.

This immediately ruled out the models that possessed a constant linear friction force during a slip phase, such as those based on a Coulomb friction characteristic. Had one of these models been used in an analysis of the system then it is all too likely that the results would have been misleading and much of the behaviour that has been observed would not have emerged.

The influence of the stick-phase of the friction characteristic was also looked at during this investigation. It was found that if the gradient of the stick section was changed from $\infty$, as was in the case of the various discontinuous friction models, to a negative gradient, as in the continuous friction models, the following two effects were observed:

- The velocity during a stick phase, which for a discontinuous friction characteristic was constant now fluctuated within a range of sticking velocities.
- The maximum displacement attained by the mass was found to be considerably lower in the continuous models than the discontinuous models.
These observed effects are consistent with behaviour which would be expected from discontinuous and continuous friction models.

In the discontinuous case the transition point between sticking and slipping on the friction characteristic is the point where $\dot{x} = v_{dr}$. However for the continuous model the transition between sticking and slipping starts at the point $\dot{x} = v_{dr1}$. Since $v_{dr1}$ is less than $v_{dr}$ the mass will start its path towards a stick phase more quickly in the continuous model than in the discontinuous model. The discontinuous model would require the mass to attain the higher belt velocity before it would begin a stick phase, thus it would travel more, hence greater displacement.

The experimental investigations in chapter 3 revealed that the system is far more complicated than what had been presented in any of the numerical simulations. The phase portrait obtained from the experiment shows that not all trajectories follow the same path, even if they show the same behaviour. The numerical simulations on the other hand suggest that all trajectories are exactly the same and follow the same path.

Qualitatively the results produced from the experiment, did resemble some of the numerical simulations. There are some differences between numerical and experimental results but that is to be expected. The numerical model epitomises perfect laboratory conditions where everything remains constant, conditions which we all know are not achievable in the real world. It emerges that a friction characteristic which posses a non-linear slip section with a non-zero positive gradient, is the most realistic for this particular system. This particular friction characteristic allowed the system to exhibit the classic behaviour of a stick-slip system, namely long periods of sticking with relatively short periods of slipping.

Since the mass does not slip or stick at the same displacements, as was indicated by the numerical simulations, the experiment seems to gives results which are inherently noisy, where trajectories in the experimental model do not follow the same paths as those before them.
This shifting of the positions where sticking and slipping takes place is primarily due to the friction characteristics of the belt and the contact surface of the mass altering with time. The mass is becoming more smoother as time progresses as it is being rubbed and polished by the belt, which is also getting smoother as particles of saw dust and glue get embedded into its surface, so altering the distribution of contact points between the two surfaces. Also during the slipping process, the temperature of the contacting surfaces may vary, which may in due course induce a variation in the friction properties, and would alter the position of the transition points for the change in behaviour from slipping to sticking and vice versa.

Another factor is the joint in the conveyor belt. In the numerical simulations it is assumed that the belt is seamless and perfect. The slight raising of the belt surface at the joint causes the mass to momentarily lose contact with the belt when the experiment is in progress. This would obviously effect the overall dynamics of the system, as the jumping of the mass when it came into contact with the joint would be similar to a perturbation being applied at periodic intervals to the mass in the single degree of freedom system.

From these experimental investigations it was established that any realistic friction model, for a system such as this, should include variability in the static friction between the two surfaces. This variability is not a simple reduction in the static friction as the experiment progress, but is more complex than that. If the mass slips at displacement X, then the next time it may slip at X-2 and the next time it may slip at X+4. This variability could be dependent upon many factors, however further investigations into the overall characteristics of the friction model were pursued without this variability.

Thus the fact that certain experimental parameters change with time indicates that the experiment is not actually very good. However it should be made clear that the main reason why experimental results were desirable was to get a physical feel for the single degree of freedom system and its associated difficulties, so as to build up an experimental insight to match our numerical experiences.
The two degree of freedom model in chapter 4 was felt to be a logical extension to the single degree of freedom model. The system was now able to exhibit chaotic behaviour since the phase space dimension was greater than 3, and the extent of the rich complex dynamical behaviour of the system was evident.

The nonsmooth nature of the system and of its dynamical behaviour is further complicated by the fact that the phase space dimension of the system varies. Thus classical methods of analysis were not especially applicable. Typically Poincaré maps would be the principal way of recognising chaotic behaviour of such a system. However in light of the difficulties outlined above, new numerical techniques for determining Poincaré maps and Lyapunov exponents had to be developed to analyse the system.

Since the system itself is autonomous without periodic external forcing, stroboscopic periodic sampling of the phase space trajectory was not an option. A new definition for a Poincaré section for this system was developed. A Poincaré section is defined as an $n - 1$ dimensional hypersurface in the state space that is transverse to the flow in the $n$ dimensional space spanned by $x$ (Foale & Thompson 1991; Thompson & Stewart 1986; Nayfeh & Balachandran 1995). Examining the equations for the two degree of freedom system, it can be shown that the four dimensional flow is transverse to the three dimensional volume at when the velocity for the first mass is equal to zero i.e. $V_1 = 0$, which suggests that this may be chosen as a Poincaré section. The motions of this four dimensional system can be classified according to the Poincaré map defined in this way so that if the map has an attracting fixed point then this corresponds to an attracting limit cycle for the global system and the time between two intersections with the Poincaré section is the period of the cycle.

Using this definition for a Poincaré section, analysis of the dynamical behaviour of the system proceeded. Bifurcation diagrams showing the Poincaré point for the first mass, $X_{1P}$, verses increasing belt velocity showed the existence of two distinct attractors for the system. A number of bifurcation's could be seen including period
doubling, reverse period doubling, chaotic explosions and jumps to remote attractors as well as chaotic zones with periodic windows. Thus using this definition for the Poincaré section it has been possible to identify attractors and bifurcation’s of the system which would have been difficult to otherwise locate.

The exploration of the multi-dimensional phase space of such a system would be difficult and computationally expensive. However since it was possible to classify certain events of the system, the introduction of a one-dimensional map to classify any bifurcation’s of the system that may occur was explored. The reduction of the four dimensional phase space of the system to the one-dimensional map allowed the extensive exploration of the dynamical behaviour of the system without being computationally intensive (Moon 1987; Ott 1994). The different attractors of the system were located and bifurcation’s of the system identified by use of the lower dimensional map.

Using the one-dimensional map, bifurcation’s such as crises, reverse period doubling and fold bifurcation’s are identified as the belt velocity is increased. In keeping with the theme of the one-dimensional map, it can also be utilised to calculate the Lyapunov exponents of the system using methods applied in other investigations (Oestreich, Hinrichs & Popp 1996; Hagedorn 1988). The Lyapunov exponent is a convenient indicator of the sensitivity to small perturbations characteristic of chaotic systems. The Lyapunov exponent measures the exponential rates of divergence or convergence of nearby trajectories of an attractor in the state space. When the Lyapunov exponent calculated at particular belt velocities is then compared to the bifurcation diagram there is a good degree of agreement between the results. Areas of chaos in the bifurcation diagram have a calculated Lyapunov exponent which is positive and areas where periodic behaviour is prevalent Lyapunov exponents with a negative value are calculated.

Hence the method of analysis used in this chapter, has shown that there is some merit in considering methods where a reduction in the dynamics of higher dimensional systems to lower order maps, is made. This form of analysis is especially
applicable to experimental data, where the dimension of the system, location of attractors and their basins are not implicitly known. Since these lower dimensional maps are directly applicable to the more complex system, yet provide various advantages, such as easier analysis, reduced computational time and effort and more straightforward implementation for experimental investigations, they are viewed as an important and powerful method of analysing the behaviour of such systems which behave chaotically.

Chapter 5 dealt with the analysis of the damped single degree of freedom system subjected to dry friction forces. The inclusion of damping was seen as a natural extension of the single degree of freedom system towards a more physically realistic model. The inclusion of damping does not produce a more accurate model of systems which exhibits stick-slip oscillations, but rather a better understanding of the dynamic processes involved.

The impending slip condition for the damped system had to be altered due to the inclusion of damping as the energy dissipation caused by damping had to be taken into account. The behaviour of the system showed a new type of phenomena. The normally unstable fixed point at the centre of the limit cycle would at a certain levels of damping become stable and able to attract trajectories. Thus a bifurcation of the system dynamics had taken place. A bifurcation diagram showing increasing damping verses maximum velocity of the mass showed that coexisting behaviour at certain levels of damping was present. Classical methods of analysis were used to determine the type of bifurcation that occurred, and it was found that the most likely candidate was a Subcritical Hopf bifurcation. Eventually at a critical level of damping the limit cycle of the system would disappear leaving just the now stable fixed point attracting all trajectories. Since this type of bifurcation involves the stability of a periodic attractor, the Floquet multipliers of the periodic orbit concerned would need to be analysed.

However this analysis was made easier by using the computer program AUTO. This program written by Eusebius Doedel (Doedel, Wang & Fairgrieve 1994), allows the
system behaviour to be followed whilst increasing a bifurcation parameter. More importantly it shows the behaviour of the Floquet multipliers as the bifurcation parameter is varied. For AUTO to analyse the periodic orbit, the initial starting conditions had to lie exactly upon this orbit. Once the integration process for the program had been initiated it calculated the bifurcation diagram upto its branch end point.

The Floquet multipliers of the periodic orbit can be observed in a box in the bottom left hand corner of the bifurcation diagram, as points in or around the unit circle. As mentioned before they indicate stability; those inside the circle are stable and those that are outside of the circle are unstable. Bifurcation's occur on the circle, that is, Floquet multipliers crossing the unit circle indicate that a global bifurcation has occurred.

The birth of an unstable periodic orbit at a damping value where the unstable fixed point at the centre of the limit cycle becomes stable, is observed. The eventual annihilation of the limit cycle is also observed as the unstable periodic orbit coalesces with it. Various points along the branch of the bifurcation diagram were traversed and the corresponding Floquet multipliers of the system at those points were determined.

By examining the behaviour of the Floquet multipliers of points leading upto the bifurcation point, where the stable stick-slip attractor and the unstable periodic orbit collide, it was found that the first Floquet multiplier stays on the unit circle whilst second Floquet multiplier leaves the unit circle through +1. This behaviour of the second Floquet multiplier suggests that a fold bifurcation (Nayfeh & Mook 1979) has occurred.

The fold bifurcation is in many ways the most fundamental bifurcation in non-linear dynamics, and is recognised as the primary example of discontinuous and catastrophic bifurcation's (Foale & Thompson 1991). Once the bifurcation has taken place, there is no way of following the path past the bifurcation point and
hence the bifurcation in this case is discontinuous.

The investigation of the dynamic processes involved in the damped single degree of freedom system yielded some dynamically rich and interesting behaviour. A Sub-critical Hopf bifurcation and a discontinuous fold bifurcation were both identified, in the course of our investigations.

A number of techniques have been used to determine the nature of these bifurcation's, which include classical methods of analysis as well as the computer program AUTO. AUTO for all its faults and difficult user interface, allowed the analysis of the unstable periodic orbit to proceed without having to get deeply acquainted with Floquet theory.

Since in reality we cannot be sure of the exact nature of the friction characteristic, a non-smooth approximation has been used. It should be pointed out that this non-smooth approximation did provide a very good approximation to the experimental data obtained in chapter 3. However due to the non-smooth nature of the friction characteristic, in areas around the discontinuity where the slope is very steep, non-smooth dynamics were observed. Hence it was not surprising to find non-standard or discontinuous bifurcation's.

In most mechanical systems there is some form of external forcing present, that causes the system to exhibit some form of periodic behaviour which is dependent upon the frequency, amplitude and period of the external forcing. It therefore becomes imperative to understand how external forcing may affect the dynamical behaviour of a system which exhibits self sustained oscillations, and how these two types of oscillatory behaviour superimpose and interact with each other.

Therefore in chapter 6, dynamical behaviour of the single degree of freedom system under self and external excitation was seen as a valid investigation into the engineering complexities that could arise in systems where friction induced oscillations are affected by external excitation.
Numerical simulations of the system resulted in p-periodic solutions of any period up to $p = 9$ as well as chaotic motion. The routes to chaos are similar to intermittency and the majority of the motion displayed stick-slip behaviour. Results from the numerical simulations show that external excitation can effectively cancel the discontinuity in the friction characteristic, and may under certain parameter values actually prevent self-excited oscillations from occurring (Thomsen 1999; Feeny & Moon 2000).

Fast oscillations have been shown to quench friction-induced oscillations by changing the effective friction characteristic of the system. High-frequency excitation turns dry friction into a viscous like form of damping similar to the kind of dissipation that would occur when using a lubricating oil.

Lubricating oil as mentioned before may disrupt the proper functioning of a device that specifically relies upon friction. An example may be a brake disc or drum, the squeal that is given off during braking may decrease but so may the maximum braking force.

So unlike lubrication, the damping effect created by fast vibrations can be controlled quickly and easily be removed when unwanted. In using high-frequency excitation described in the manner here, to control self excited oscillations, there are costs to pay in the form of extra monitoring devices, energy, complexity and new problems.

However it should be stressed that this technique does not simply turn off self excited oscillations into absolute rest but rather transform them into small-amplitude vibrations at very high frequency. This is clearly visible from the results shown in figures 6.7 and 6.8.

If this frequency is outside the audible range then one problem has been solved, however these vibrations can be quite energetic and may introduce other problems such as heat generation.
Since it has been shown that quenching of stick-slip behaviour is possible, the idea that it may be possible to alter the state of the system from chaotic to periodic behaviour, or vice versa was intriguing. In engineering applications stick-slip oscillations are highly undesirable and should be avoided at all costs since they diminish the precision of motion and safety of operation. They also cause noise and promote wear and tear. So a method of controlling these oscillations would be highly desirable.

Initially the presence of chaos in the behaviour of dynamical systems was considered to be something that induced extra difficulties. Chaotic motion was thought to be impossible to predict or to control. However the control of chaotic systems is based on exploitation of certain properties of chaotic motion. The first such property is that any chaotic attractor has embedded within it an infinite number of unstable periodic orbits.

These embedded unstable orbits are the ones that control techniques aim to stabilise. However, some methods allow stabilisation of orbits that are not embedded in the original attractor. This is mainly done by applying large perturbations on the system which may significantly alter the dynamics of the original system.

The other feature of chaotic motion exploited by control techniques is the sensitive dependence on initial conditions: very small perturbations will, in finite time produce motion that is very different from motion produced if the system is not perturbed. This feature of chaotic systems, previously seen as a disadvantage, can be used to drive the system towards any desired state. This is called targeting (Shinbrot et al. 1990; Baretto et al. 1995; Arecchi & Bocaletti 1997).

Ergodicity states that every chaotic trajectory will eventually come arbitrarily close to any point within the region of the phase space occupied by the attractor. Thus when the trajectory is given time to approach the area of interest, control can be activated. This enables the required stabilising perturbations to be very small.
Control can be achieved by applying perturbations to an accessible system parameter or a variable, in a controlled manner, so that the next iterate falls closer to the desired state. Thus, the motion of the chaotic system can be stabilised close to any unstable periodic orbit embedded within the chaotic attractor.

The method of control that was used to locate and lock onto specific unstable periodic orbits, was an amalgamation of the various methods discussed. The reason for this is that one specific method would not allow control for this type of system. Thus the most appropriate techniques were used where necessary.

First of all an unstable periodic orbit had to be identified. For this, an OGY type of method was used where the algorithm operated upon the Poincaré section of the desired periodic orbit. The Poincaré section in this case was the section obtained at $V = 0$ on the phase portrait of the system, since the flow at this point is transverse to the section. This method worked very well and it was possible to identify a range of unstable periodic orbits within areas of dense chaotic motion.

If the criteria for locating a particular unstable periodic orbit is not satisfied, then the control parameter, which in this case was the increase in the amplitude of the external forcing, was slightly perturbed. Eventually the control parameter was perturbed up to a critical point where the criteria for locating the periodic orbit was satisfied and the trajectory locks onto it. Once the desired periodic orbit has been locked onto, the algorithm removes the perturbation and allows the trajectory to continue unhindered, which eventually makes its way to the stable manifold.

This method of targeting a particular unstable periodic orbit was slow, but was very robust and worked well. There was no real way in which the trajectory of the periodic orbit could be followed and monitored continuously as information regarding the trajectory was lost as it entered the stick phase and the velocity component reverted to a constant i.e. when $v = v_{dr}$. Thus perturbations could not be applied as and when required, but had to be applied until the desired periodic orbit was located and locked onto. This did not pose too much of a problem as the
perturbations are of the same form as the forcing applied to the system, but of a considerably smaller amplitude. The periodic orbits that were located and locked onto in areas of dense chaotic motion, showed that the system worked well and the final perturbations that were applied were very small when compared to the actual forcing amplitude. In some instances the applied perturbation amplitude was only 0.3% of the forcing amplitude.

This method of controlling the behaviour of the system has been demonstrated successfully in regions of chaotic dynamical motion. The technique shows that those unstable periodic orbits within an area of intense chaotic motion can be stabilised by small perturbations to the system. Thus this type of control method is a viable alternative to changing the configuration of a mechanical system that was experiencing chaotic behaviour.

The ideas developed here during the course of this thesis could be extended on both the theoretical and practical levels. On the practical side, a more in depth investigation into the variability of the friction force should be carried out. It is important to establish how this variability is instigated, how it develops over long periods and determine what other dependencies are involved, such as material properties and temperature of contact surfaces.

This would lead to an overall assessment of how this could be applied to the theoretical model leading to more accurate simulations, so that engineering systems can be designed accordingly. Such an analysis could then be refined by applying this accurate model to higher degrees of freedom models which would give a more realistic representation of a real engineering system operating in a real environment. The methods developed for the two-degree of freedom model could be extended and then be applied to these systems to aid analysis.

The control of such systems has shown that exploiting certain properties of chaotic behaviour, can enable systems to operate within designed safety limits. In terms of energy, the control method is quite efficient as it only uses a small percentage
of the total external excitation force, and it is only applied when it is required and is removed once control has been achieved.

We have shown that the solution of the problem and the interpretation of the mathematical results in the context of practical experience can then lead to an improved and revised model. This leads to a more realistic correlation between the observed and predicted results, as well as a greater understanding of the dynamic processes involved. As with many real world problems, in our modelling process we are trying to produce mechanisms from observations, one of the great tasks of modern science.
References


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