HYDRODYNAMIC FORCES ON OSCILLATING SUBMERGED BODIES AT FORWARD SPEED

by

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1986
To My Parents

and

To My Wife
Abstract

This work attempts to solve the problem of a ship advancing in waves using potential theory. The results of resistance in otherwise calm water and hydrodynamic coefficients of oscillating submerged bodies at forward speed are presented. For the resistance problem the free surface condition is linearized but the body surface condition is exactly satisfied; while for the oscillating problem both conditions are linearized.

The linearized mathematical model is discussed in detail and a review of previous work on ship motions especially the strip theory is given in chapter one. Following the introduction, the mathematical equations are derived and the details of numerical procedure to be adopted are discussed in chapter two. This numerical method combines localized finite elements in the near region with representation by a boundary integral equation in the far field. The method is first examined at two dimensional level in chapter three by considering infinitely long cylinders. After a review of previous work on ship wave resistance, the method is then extended to general three dimensional submerged bodies in chapter four. The further extension of the present method to more general problems in ship hydrodynamics such as nonlinear wave resistance, the motion through an arbitrary time history, the added resistance in a regular sea, the elastic deformation of the ship, etc, is discussed. An analytical formulation for the potential problem of a submerged spheroid advancing in waves is presented for the purpose of providing a basis to check the numerical results. It is confirmed that the present method is one of the most promising methods in ship hydro-
dynamics.
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1. INTRODUCTION

1.1. The linearized potential theory of ship motions

An ocean going ship will be excited by wave forces and as a result be set into motion. This motion will affect the performance of the ship in many aspects and may raise some serious problems, such as increased resistance, deck wetting, slamming, vertical acceleration and propeller emergence, etc. While every aspect above has become a specialized subject in ship hydrodynamics, the fundamental problem remains as to estimate the overall motion of the ship in the ocean or the response of the ship to ocean waves.

In order to predict ship motions in waves, the ship is usually regarded as a rigid floating body having six degrees of freedom. The fluid loading on the ship is estimated by the linearized velocity potential theory. This theory assumes that the fluid is inviscid and incompressible, the fluid flow is irrotational, and both incoming wave elevation and the body oscillation are small. It yields that the velocity potential $\phi$ satisfies Laplace equations $\nabla^2 \phi = 0$ and the corresponding condition $\frac{\partial \phi}{\partial n} = L(\phi)+a\phi+b$ on the mean position of the fluid boundary, where $L$ is a linear differential operator, $a$ and $b$ are constants and $n$ is the normal of the boundary surface. In mathematics this equation is a familiar form and many theories concerning its solution have been established. These theories provide some principles and disciplines in seeking the solution in a very broad sense. Unfortunately, it is concluded that except for very few special cases, it is impossible to obtain the analytical solution of this equation. The special difficulty in ship hydrodynamics is that for most practical ships their surfaces
are described by the coordinates of some discretised points rather than being able to be described by some simple mathematical functions. This makes the analytical solution out of the question. The existence of the free surface and the infinitely large fluid domain make the problem more complicated. Therefore it seems inevitable that a numerical method has to be used to obtain the solution.

The essential spirit of the numerical method is to find a function which satisfies governing equations at discretised points or in a uniform sense rather than in the whole continuous fluid domain. This function is always an approximate solution in terms of rigorous mathematics; but it is possible in principle that the numerical solution can provide any wanted accuracy by imposing governing equations at more and more discretised points, if one can afford the computer time and the error due to the machine is always far less than acceptable. In this sense, the numerical method will give the exact solution.

As the strategy of the numerical analysis, it usually first considers some problems of special geometries, such as a circular cylinder or sphere. These geometries are by no means similar to practical ships, but as the semi-analytical solutions in the series form containing some special functions can be obtained, the results may provide a basis to check the numerical method.

There have been many computer programmes using numerical techniques to calculate ship motions in waves. Some of them are for ships of special geometries, such as slender or rectangular; and others are for ships of arbitrary geometries. These programmes have been successfully used in many cases. It is difficult to conclude that a particular
numerical method employed by some programmes is always better than the others. But in terms of the basic requirements for the numerical method, ie, accuracy and efficiency, it seems that the boundary integral method and coupled finite element method are most promising methods for most practical ships.

The boundary integral method transfers the Laplace equation into a boundary integral equation with the help of Green function. This reduces the problem in the whole fluid domain into that on the body surface. The numerical detail will be different corresponding to whether the direct or indirect method is used, but the basic idea is to divide the body surface into many small panels. On each of these panels the parameter to be found is assumed as constant or as a simple function containing some unknown coefficients. By invoking the body surface condition, the unknown constants or coefficients on each panel can be found. The advantage of this method is that it minimizes the unknown coefficients, since only the potential on the body surface is usually needed; the disadvantage is that the numerical integration of the Green function over the body surface is not always easy and the solution is non-unique at certain discrete frequencies for floating bodies, as known as the irregular frequency.

The coupled finite element method divides a domain surrounding the body known as the near region into many small sub-domains. In each of these sub-domains, the potential is expressed by a shape function containing some unknown coefficients. The potential in the other region or the far region is expressed in a series form or in a boundary element integral form. By a variational statement, the solution of the potential
can be obtained, which guarantees the continuity of the potential and its first order normal derivative on the boundary separating two regions. The advantage of this method is that the numerical difficulty of the integration of the Green function over the body surface is avoided; the irregular frequencies can be predicted if the boundary separating two regions is chosen as some simple forms and the irregular frequencies can be easily changed by altering the geometry of the boundary. The disadvantage is that the number of unknown coefficients is increased and the preparation of the input data needs more effort.

In general, these two methods give satisfactory results where linearized potential theory is valid. But their applications to the problems of ship motions in waves are strongly affected by the assumptions of the linearized potential theory. A serious limitation of the linear potential theory is due to the absence of the drift force. It has been observed that if the incoming waves contain two components of slightly different frequencies $\omega \pm \frac{\Delta \omega}{2}$, the higher order force on the ship will have the components of frequencies $2\omega$ and $\Delta \omega$. When the restoring force of the ship motion in mode $i$ is small or zero, the second component known as slowly-varying drift force will influence the ship motion in this mode significantly. Since there is no static restoring force on surface ships in horizontal plane, the consequence of this force is to increase the resistance and influence the course-keeping ability of ships. For compliant offshore structures employed in the oil industry this force can cause severe strain in the mooring line attached to the structure.

The accurate estimation of the slowly-varying drift force needs the
solution of the second order potential which is not an easy task. A closely related and rather easier problem is that of mean drift force in regular waves, which is also neglected by the linearized potential. This force is due to the fact that the time average of the product term of the linear potential in Bernoulli's equation is not equal to zero so that it yields a steady force on the ships or offshore structures. As it is easy to model and sheds considerable light on the behaviour of compliant systems in irregular waves, this steady drift force has received much attention.

The second effect of the application of the linearized potential theory is that it neglects the viscosity of the fluid (rigorously speaking, the limitation due to this effect applies to all potential theories which are not necessarily linearized). The real fluid flow is a very complicated physical process which can not be simply described by a velocity potential. As water is regarded as Newtonian fluid, this process in ship hydrodynamics follows the law described by the Navier-Stokes equation. This is a set of coupled nonlinear differential equations. To obtain the solution of these equations associated with the complicated free surface condition is extremely difficult if not impossible.

In fluid dynamics, however, it has been observed that when the flow of real fluid hits a body, a boundary layer will be created attached to the body surface. The fluid flow outside the boundary layer can be regarded as inviscid so that the potential theory is used; while the Navier-Stokes equation should be used for the viscous flow inside the boundary layer. Rigorously, the whole problem should be solved by a
potential theory outside the boundary layer matching the solution of the Navier-Stokes equation inside the boundary layer. But since the boundary layer is confined to a thin region, its existence will not affect the potential solution significantly. The solution outside the boundary may be obtained by ignoring the existence of the boundary layer; the solution inside the boundary may be obtained from the Navier-Stokes equation and by invoking the boundary condition on the body surface and matching the potential solution on the interface with the outside inviscid flow. The details and criteria of matching the inner solution and the outer solution have been discussed by Van Dyke.\textsuperscript{10}

The most important effect of viscosity on ship motions is that it can cause the separation of the fluid flow from the body, which will generate vortex shedding. It yields that the potential theory is not valid in a large region. Fortunately the separation usually happens or is significant when the body is small compared with the wavelength and wave amplitude. In this case the existence of the body will not affect the incoming wave significantly. The force on the body may be obtained by the Morison equation.\textsuperscript{70,96} When the body is large compared with the wavelength and wave amplitude, the effect of viscosity can be usually neglected and the potential theory may be used. As has been observed, the potential theory gives very satisfactory results for ship motions except that associated with roll.\textsuperscript{15}

The third limitation is that associated with the rigid body assumption, since the real ship is always a flexible structure. But its flexural deformation is usually far smaller than its rigid motion so that it can be neglected in the estimation of the ship motions. The important
effect of the flexural deformation is to induce the dynamic stress in the ship structure and this deformation can be also significant in offshore structures, such as risers. The subject of hydroelasticity concerns this problem in detail. A simplified mathematical model is to regard the ship as an elastic beam and the hydrodynamic loading may be obtained by the simplified two dimensional strip theory. A comprehensive discussion about this problem can be found in the book "Hydroelasticity of Ships" by Bishop and Price.

There are many other factors which will affect the applicability of the linearized potential theory such as the incompressible assumption of the fluid and neglect of the free surface tension. These effects on the ship motion however are far less important than those discussed above except for very specialized subjects. Therefore we are not going to discuss these effects in detail.

1.2. A review of previous work

A closely related problem to the motion of a ship advancing in waves is the resistance on the ship. It is part of our concern in this thesis, but the detailed discussion of this subject will be in later chapters. Specifically, a review of the previous work on ship wave resistance will be given in chapter four; while here we will concentrate on the review of the theory of ship motions in waves and mention some work on the resistance problem occasionally.

The theoretical research in this area was started by the work Froude and Krylov in the last century. They assumed that the existence of the ship would not affect the incoming wave structure. The
resultant force on the ship is known as the Froude-Krylov force later on. The first work taking into account the effect of the ship on the fluid flow is the work by Michell\textsuperscript{72} in analysing the wave resistance on the ship. He regarded the ship as a thin body so that the potential flow can be determined by a source distribution on the centerplane of the ship and the source strength can be obtained from the geometry of the ship directly without solving any complicated integral equations. These works provided some qualitative guidance in ship design but have never given any satisfactory quantitative results. It can be understood that the theoretical model used by them is for the need of mathematical simplification. It does not reflect the practical situation and therefore fails to give satisfactory quantitative results.

It is due to the appearance of the computer in the 1940s that the solution of the problem associated with the real ship can be obtained numerically. As a typical ship can be regarded as a slender body which means that its length is much larger than its width and draught, the mathematical problem can be solved based on the strip theory. The strip theory assumes that the fluid flow corresponding to each section of the ship is two dimensional and the effect from other sections can be neglected. For the two dimensional flow, the analysis had been well treated by Ursell\textsuperscript{105,106} for a floating circular cylinder and his method can be extended to an arbitrary cylinder by combining with the mapping technique\textsuperscript{67}. Therefore Korvin-Kroukovsky and Jacobs\textsuperscript{64} were able to give the first version of strip theory in 1957. Although they established this theory mainly based on physical intuition rather than a rigorous mathematical derivation, surprisingly the theory gives very good results comparing with experiment in many cases. However, since it lacks a firm
theoretical basis, this theory has a vital weakness that it does not satisfy Timman and Newman's symmetry relation which is regarded as a touchstone of strip theories. Therefore this theory inevitably has some limitations in applications specially when a ship has forward speed.

Although many extensions and modifications were made after this original strip theory, such as the work by Gerritsma and by Grim, they were still not consistent from a rigorous mathematical point of view. To avoid some difficulties and to make the theory have easy practical application, some inconsistent assumptions were usually made, which lead to the final equation containing terms of different orders. Ogilvie and Tuck observed this problem and started their strip theory from the order $\varepsilon^{1/2}$, where $\varepsilon$ is the ratio of body width to its length. In this, known as "rational theory", the influence of forward speed on the body surface and on the free surface is taken into account in a manner consistent with their assumptions. As a result, all equations have terms of order $O(\varepsilon^{1/2})$. From the mathematical point of view it is a rigorous theory. But as a strip theory, it is not widely used because of its complexity. In fact, the purpose of the strip theory is to provide a simple mathematical method to solve the complicated problem of ship hydrodynamics. On the other hand, although many simple strip theories are not consistent from the beginning, they do provide some satisfactory results compared with experiment. Therefore the practical significance of developing a very complicated strip theory is questionable otherwise a three dimensional theory may be used.

The most commonly used strip theory was developed by Salvesen, Tuck and Faltinsen. In this, known as the STF strip theory, the basic
assumption of original strip theory remained. The two dimensional problem analysed by them did not include any effect of forward speed. This effect was taken into account only in the relation between the pressure and potential. Although their theory is less rigorous than rational strip theory from a purely mathematical point view, it does satisfy the Timman and Newman relation and is much easier to use.

A novel modification of the conventional strip theory was made by Newman\textsuperscript{80}. According to his derivation the exciting force on each section of a ship should be determined from the hydrodynamic coefficients of this section at wave frequency rather than at encounter frequency. But apart from that this theory is quite similar to the conventional strip theory.

The assumptions of these conventional strip theories require that the frequency is high, so that the interaction of the fluid flow between the different sections of the ship can be neglected. When the frequency becomes low or even moderate it has been shown by Ursell\textsuperscript{108} considering horizontal slender bodies of revolution that the interaction is not negligible. His conclusion was extended by Newman\textsuperscript{76} to horizontal slender bodies of arbitrary shape. Rigorously speaking therefore, the conventional strip theory is not valid at low frequency. Fortunately, at low frequency the hydrodynamic force from the potential theory is of small order while the static force is of leading order which is exactly estimated by the strip theory. Thus as shown by the results of heave and pitch response of barges to waves of different frequencies at zero forward speed given in Fig.1 to Fig.3, the strip theory gives very good results.\textsuperscript{†} The surprising thing is that even if the ratio of beam to

\textsuperscript{†} These figures are reproduced from the results obtained in the
length reaches \( \frac{1}{3} \), which by no means corresponds to slender body, the strip theory still gives very good results in general.

As one of the serious limitations of the strip theory, the two dimensional theory gives infinitely large added mass when the frequency tends to zero, as the consequence of the neglect of the interaction between different sections of the ship. To remove such limitations of the conventional strip theory, Newman developed a unified strip theory in 1978. He divided the fluid domain into two domains: The near region surrounding the body and the far region tending to infinity. The flow corresponding to the slender body in the near region is regarded as two dimensional. The solution is obtained by the localized two dimensional method and is expressed as a sum of a particular solution and a homogeneous solution associated with a "constant" which is virtually a function of the body's longitudinal coordinate and can be interpreted as the interactions between sections. The flow in the far field is three dimensional in general. The solution for slender bodies is obtained by the source distribution along the axis of the body. By matching two solutions in their common region, the constant and source strength can be obtained. Using this theory, Sclavounos calculated the case without forward speed. His results show that added mass has been improved dramatically at low frequency. Later Sclavounos calculated the case with forward speed. It is understood that this theory is not easy to be used in this case unless a further assumption about the relative orders of magnitude of encounter frequency \( \omega \) and forward speed \( U \) is made such that \( U \ll \omega L \) or \( \frac{U}{\omega g} \ll \frac{L}{g} \), where \( L \) is the length of the body and

author's Msc report for Ocean Engineering, Department of Mechanical Engineering, University College London, 1984
\( g \) is gravitational acceleration. Although this assumption is not explicitly declared by Sclavonous in his work, it is essential, at least from a mathematical point of view. From his results of heave and pitch hydrodynamic coefficients, Sclavonous showed that unified theory provided an improvement over STF theory with the exception of cross-coupling added mass and gave better agreement with experimental data. We may notice however that comparison in this work is in the region \( F_n \ll \omega \sqrt{\frac{L}{g}} \) where the exclusion of the forward speed effect on the near field solution is acceptable. Therefore it is interesting to see what the results would be at low frequency and large Froude number. In the author's view, the present unified strip theory is superior to the conventional strip theory in many aspects, provides good results for slender ships advancing at low forward speed in waves, and its inexpensive computation effort is particularly attractive, but the assumption of the relative order of magnitude of encounter frequency and forward speed will limit the application of the unified strip theory to an appropriate region. Moreover it has been observed recently that it is difficult to conclude that the unified strip theory will give better results than conventional strip theory in finite water depth\(^{12}\).

Since the strip theory has its inherent limitations, it is therefore necessary to establish a more accurate mathematical model. To overcome the limitation of the slenderness assumption of the body, which is invalid for many unconventional ships such as barges and offshore platforms, the three dimensional theory has been developed. This theory has no such requirement for the body geometry and wavelength as the strip theory. It is valid for a general three dimensional body provided that the assumption of the linearized potential theory is acceptable. At zero
forward speed, there have been many works\cite{26,29,32,38} using either the source distribution method or coupled finite element method. A review was given by Mei\cite{169} and more recent developments were discussed by Yeung\cite{114} and Eatock Taylor\cite{23}. These three dimensional theories give very good results compared with experimental data except the rolling damping where the effect of the fluid viscosity is important (e.g. reference\cite{15}).

If the body has forward speed, the three dimensional theory becomes more difficult to implement. There are mainly two difficulties. The first is that the potential due to the oscillatory motions of the body contains the second order derivatives of the potential generated by forward speed in its body surface condition which needs to be accurately evaluated. The second difficulty is the waterline integration, since the Green function on the waterline of the surface ship in the boundary integral equation may not be convergent. Chang\cite{16} was the first person who challenged the problem of the surface ship advancing in waves using three dimensional theory. The conventional source distribution method was used. After paying the price of large computer time, better agreement than strip theory with experimental data was obtained. But the improvement was not as significant as one expected. This may raise the question whether this expensive price is worthwhile. However the difference could be due to the numerical process rather than the mathematical model itself, since the similar work by Guevel and Bougis\cite{144} using the source distribution and a mixture of source and dipole distribution improved Chang's results and the numerical results are in a better agreement with experimental data in general. This is further supported by Bougis and Vallier's work considering the problem of the barge
connecting its tug advancing in waves. The source distribution was also used by Inglis and Price and by Kobayashi. To avoid the difficulty of the waterline integral, Inglis and Price calculated this integration along a line close to the free surface while Kobayashi neglected this contribution.

As the purpose of this work, we are determined to establish a more efficient three dimensional method to calculate the ship response to waves at forward speed. For this particular problem, the advantage of the coupled finite element method becomes more apparent. The finite element representation in the near field enables us to calculate the second order derivatives of the potential generated by forward speed on the body surface using the shape function; the appropriate choice of the boundary of the localized element region enables us to calculate the waterline integral explicitly, so that these two difficulties can be overcome. Therefore the coupled finite element method is used in this work.

1.3. Discussion of the present work

The coupled finite element method was first introduced into ship hydodynamics about a decade ago. This was applied to the two dimensional harmonical motion problem then extended to the three dimensional problem. In terms of the representation in the far field, this method is usually classified as the boundary integral element method and the boundary series element method. The first one uses the Green's identity to obtain an integral equation on the localized finite element boundary. The second one expands the potential in the far field into a series. The detail discussion and comparison of these two methods has
been given by Eatock Taylor and Zietsman\textsuperscript{28}.

For the present problem, it seems very difficult to obtain an appropriate series in the far field. We will use the boundary integral element method as the Green function employed in the integral equation is available. Following this chapter, we will derive the mathematical equations for the linearized potential problem of ship motions in waves and discretize these equations in terms of the numerical technique of the boundary integral element method. In the third chapter, the problem is solved at the two dimensional level by considering the motions of infinitely long cylinders. We first calculate the resistance and lift on the cylinders then investigate the influence of forward speed on the hydrodynamic coefficients of cylinders. This two dimensional problem is of practical interest since many problems of elongated bodies can be approximated by a two dimensional solution. But the main purpose of this part is to assess the theoretical techniques and investigate the numerical methods at a simpler level than a fully three dimensional case. Having done that, we discuss some possible extensions of the present method to more general cases. The discussions of the non-linear wave resistance problem and the motion through an arbitrary time history are given in detail. The fourth chapter is the main purpose of this thesis. The detail discussion about the ship resistance is given. The wave resistance is calculated based on the linearized potential theory and using the present numerical method. Following that an efficient form of Green function for the problem of a ship advancing in waves is derived using the exponential integral. The numerical results of damping coefficients for a submerged sphere are provided and compared with those obtained from the first approximation solution. Further extensions are
discussed. The analytic formulations for a spheroid advancing in waves are derived. The principle is demonstrated by considering the diffraction problem in head seas or following seas. The second extension discussed is the added resistance, as the component of the drift force in the forward speed direction. The equations for calculation of the drift force and moment using the integral over the body are derived. The third extension is the discussion about the flexural deformation of the ship in waves. The boundary value problem is established and the equations for the hydrodynamic force are derived.

As the first step towards the purpose of developing a more efficient method to calculate the problem of the surface ship advancing in waves, the results given in this thesis are for the radiation problem of submerged bodies at forward speed. The principle and numerical procedure employed here may however be used to solve the general problem.
2. THE MATHEMATICAL MODEL

The potential flow problem has been discussed in detail in many text books. But most of them concentrate on the problem without free surface; give some general physical mechanism of the potential flow and introduce some concepts such as added mass. It was the book "Hydrodynamics" by Lamb which first discussed the free surface potential problem in detail and remains as a classic reference in ship hydrodynamics. The more recent discussion was given in the book "Marine Hydrodynamics" by Newman. Here we will not repeat the discussion of this well established basic material. Instead we will start our discussion immediately from the mathematical model of ship advancing in waves and many topics in this chapter have referred to the paper by Newman.

2.1. General

For convenience, we define the following three coordinate systems

(a) \( \vec{x}_0 = (x_0, y_0, z_0) \) is fixed in space with \( z_0 = 0 \) as the undisturbed free surface, and \( z_0 \) pointing upwards.

(b) \( \vec{x} = (x, y, z) \) is moving with the ship at the same forward speed \( U \), and \( x \) is pointing in the direction of forward speed.

(c) \( \vec{x}' = (x', y', z') \) is fixed on the ship and its mean position is \( \vec{x}' = \vec{x} \).

These coordinates then satisfy the following relations

\[
\vec{x} = (x_0 - Ut, y_0, z_0) \quad (2.1)
\]

\[
\vec{x}' = \vec{x} - \vec{a} \quad (2.2)
\]
where

\[ \overline{a} = (a_1, a_2, a_3) \]

\[ = \overline{\xi} + \overline{u} \times \overline{x}' \tag{2.3} \]

is the displacement of the point on the body relative to \( \overline{x} \); \( \overline{\xi} \) and \( \overline{u} \) are translation and rotation respectively. The validity of equation (2.3) is subject to that the rotational displacement is small. Based on the assumption that the fluid is inviscid and the flow is irrotational and incompressible, the potential \( \phi(\overline{x}_0, t) \) whose gradient is the velocity will satisfy the Laplace equation

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_0^2} + \frac{\partial^2 \phi}{\partial y_0^2} + \frac{\partial^2 \phi}{\partial z_0^2} = 0 \tag{2.4} \]

in whole fluid domain. Since

\[ [v^2]_{\overline{x}_0} = [v^2]_{\overline{x}} = [v^2]_{\overline{x}'} \tag{2.5} \]

which can be easily obtained from equations (2.1) and (2.2), the potential \( \phi \) will also satisfy the Laplace equation in coordinate systems (b), (c).

Mathematically there is an infinite number of solutions satisfying equation (2.4), unless some appropriate conditions are imposed on the boundaries of the fluid domain. In the potential flow problem, the kinematic condition is usually assumed to be that a fluid particle will not flow through or apart from the boundary surface of the fluid domain. Its mathematical interpretation can be written as

\[ \frac{\partial \phi}{\partial n} = \overline{v} \cdot \overline{n} \tag{2.6} \]

on the boundary \( S \) of the fluid, where \( \overline{v} \) is the velocity of the point on
S corresponding to the fluid particle; \( \mathbf{n} \) is the normal of \( S \) at the same point. It should be emphasized however that equation (2.6) is far more complicated than it looks. In hydrodynamics, the boundary of the fluid domain is mainly composed of free surface, body surface, etc. In the general case, the position of these boundaries will change with the time, and the shape of the free surface is not prescribed. Therefore the boundary surface \( S \) in equation (2.6) is unknown before the potential problem is solved, which in turn depends on how equation (2.6) is predefined. Furthermore it is not necessary that the Laplace equation will always have a solution, or the solution may be always unique for any given boundary conditions. In fact, in hydrodynamics it does happen for the mathematical model associated with some kinds of boundary conditions that there is no solution at all or maybe there is an infinite number of solutions. One typical example is the linearized potential problem of a two dimensional surface cylinder having constant horizontal forward speed in otherwise calm water. This argument however does not imply that the physical problem may not have a solution or may have an infinite number of solutions. There is no doubt that the physical problem always has a solution and the solution is always unique. But the mathematical model of it is an idealization which more or less may diverge from its origin. This may yield nonexistence or nonuniqueness of the mathematical solution.

Sometimes even though the mathematical model has been proved to have a unique solution, but the infinite number of solutions may come out due to the solution technique. The well known example is the irregular frequency phenomenon in the linearized potential problem of floating bodies, when the boundary integral method is used. But in this case the
problem is not the mathematical model itself, and it can be easily solved by either modifying the solution technique or using other techniques.

In this work, only the linearized problem is considered. It means that condition (2.6) will be satisfied on the mean position of the boundary and the higher order terms will be neglected, so that the potential problem can be solved at a simpler level. This process starts from the need for mathematical simplification, but the result has its own physical significance as discussed in the previous chapter. It is well known, when the sea condition is not too rough, or the incoming wave is an infinitesimal wave in more rigorously mathematical terms, that the corresponding linearized model is quite acceptable. Although the existence and uniqueness of the solution of the linearized potential have not been proved in the general case, it has been shown in many special cases\(^{61,100}\) that for the periodical motion without forward speed the solution exists and is unique. This is strongly supported by a lot of numerical evidence. In particular, it seems that there is little argument about there being a unique solution for a submerged body in periodical oscillation or having forward speed\(^ {19}\), although the conclusion still needs to be rigorously proved. But this is not the main purpose of this work, we will concentrate on finding the numerical solution for a given problem, rather than attempting to establish this firm mathematical theory.

2.2. Free surface condition

The free surface profile can be defined by a function \(z_0 = \eta(x_0, y_0, t)\). Based on the kinematic condition imposed on velocity
potential, which states that the velocity of the fluid particle should be equal to the velocity of the same point on the free surface, the relation between the potential and free surface profile may be written as

$$\frac{d}{dt}(z_0 - \eta) = \frac{dz_0}{dt} - \frac{\partial n}{\partial x_0} \frac{dx_0}{dt} - \frac{\partial n}{\partial y_0} \frac{dy_0}{dt} - \frac{\partial n}{\partial t}$$

$$= \frac{\partial \Phi}{\partial z_0} - \frac{\partial n}{\partial x_0} \frac{\partial \Phi}{\partial x_0} - \frac{\partial n}{\partial y_0} \frac{\partial \Phi}{\partial y_0} - \frac{\partial n}{\partial t}$$

$$= 0 \quad \text{(2.7)}$$

on \(z_0 - \eta\). As a boundary condition for the potential on the free surface, however, this equation is not complete since \(n(x_0, y_0, t)\) in it is unknown. The further equation may be obtained from the fact that the pressure on the free surface should be equal to atmospheric which can be assumed as constant. Then from the well known Bernoulli's equation, the other relation between the potential and free surface profile can be written as

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} v \Phi + g z_0 = 0 \quad \text{(2.8)}$$

on \(z_0 - \eta\). Combining equation (2.7) and (2.8) it is possible to eliminate \(n\) from them and obtain an appropriate boundary condition for the potential on the free surface. Since it is not the interest of this work to find an exact solution satisfying this complicated condition, we do not attempt to give this equation. Instead, a simplified condition will be derived by linearizing the problem based on some sort of assumptions. As this work concerns the problem of the body advancing in waves at constant forward speed \(U\), the potential in equations (2.7) and (2.8) is decomposed as
\[ \phi(\mathbf{x}_0, t) = U\phi(\mathbf{x}_0) + \phi(\mathbf{x}_0, t) \]  

where \( \phi(\mathbf{x}_0) \) is the steady potential due to unit forward speed and is taken as the solution without the oscillatory motion; \( \phi(\mathbf{x}_0, t) \) is the unsteady potential composed of the radiation and diffraction potential which will be affected by forward speed. Using

\[ \frac{\partial}{\partial t}\phi(\mathbf{x}_0, t) = (\frac{\partial}{\partial t} - U\frac{\partial}{\partial x})\phi(\mathbf{x}, t) \]  

and substituting equation (2.9) into (2.8), we obtain

\[ -U\frac{\partial \phi}{\partial x} + (\frac{\partial}{\partial t} - U\frac{\partial}{\partial x})\psi + \frac{1}{2}U(\nabla + \frac{\partial}{\partial x})U = 0 \]  

(2.11)
on \( z = n \), in which \( \frac{\partial}{\partial t}[\phi(\mathbf{x})] = 0 \) has been used. If the incoming wave is of infinitesimal height, the unsteady potential can be regarded as a small quantity. When the terms of order higher than \( O(\Delta) \) are neglected, equation (2.11) becomes

\[ -U\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial t} - U\frac{\partial}{\partial x} - \frac{U^2}{2}V\nabla \psi + U\nabla \nabla \psi + gz = 0 \]  

(2.12)

Correspondingly equation (2.7) becomes

\[ \frac{\partial \phi}{\partial z} + \frac{\partial}{\partial t} - \frac{U^2}{2}V\nabla \psi = 0 \]  

(2.13)
on \( z = n \) which is now an approximation of that in equation (2.11).

Since \( \phi \) is the potential without oscillatory motion, it satisfies the conditions by taking \( \phi = 0 \) in equations (2.12) and (2.13). We obtain

\[ -U\frac{\partial \phi}{\partial x} + \frac{U^2}{2}V\nabla \psi + gz = 0 \]  

(2.14)

and

\[ \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \]  

(2.15)
on \( z = n \) which is the wave profile generated by forward speed.
To find $\phi$, equations (2.12) and (2.13) should be used in general. However as the incoming wave is an infinitesimal wave, the difference of the potential $\phi$ on $\eta$ and $\bar{\eta}$ will be of higher order. Therefore conditions (2.12) and (2.13) can be satisfied on the free surface profile $\bar{\eta}$ without losing further accuracy, on which equation (2.14) and (2.15) can be used. Equation (2.12) then becomes

$$\frac{\partial \phi}{\partial t} - u \frac{\partial \phi}{\partial x} + U \frac{\partial \phi}{\partial \eta} + g \eta = 0$$

(2.16)

and equation (2.13) becomes

$$\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} = 0$$

(2.17)

on $z = \bar{\eta}$. It should be noticed that the $\eta$ in these two equations is due to unsteady potential $\phi$ only.

Equation (2.16) and (2.17) are consequence of the linearization of the potential $\phi$. However no assumptions have been made about the steady potential $\bar{\phi}$, therefore the product terms containing $\bar{\phi}$ are still retained in equation (2.14) and (2.15). If this potential generated by forward speed is also a small quantity, equations (2.14) and (2.15) can be further simplified as

$$-U^2 \frac{\partial^2 \phi}{\partial \eta^2} + g \eta = 0$$

(2.18)

$$\frac{\partial \bar{\phi}}{\partial z} + \frac{\partial \bar{\eta}}{\partial x} = 0$$

(2.19)

on $z = 0$. Substituting equation (2.18) into (2.19), we obtain

$$\frac{\partial \bar{\phi}}{\partial z} + U^2 \frac{\partial^2 \bar{\phi}}{\partial x^2} = 0$$

(2.20)

on $z = 0$. Equation (2.16) and (2.17) can be also be simplified as

$$\frac{\partial \bar{\phi}}{\partial t} - U \frac{\partial \bar{\phi}}{\partial x} + g \eta = 0$$

(2.21)
on \( z = 0 \). Substituting equation (2.21) into (2.22) we obtain

\[
\frac{\partial^2 \phi}{\partial z^2} + \left( \frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right)^2 \phi = 0
\]  

(2.23)

When these two linearized potentials are found, the corresponding wave profile can be obtained by

\[
\eta = \frac{U^2}{g} \frac{\partial \phi}{\partial x}
\]  

(2.18a)
due to the steady potential and

\[
\eta = - \frac{1}{g} \left( \frac{\partial \phi}{\partial t} - U \frac{\partial \phi}{\partial x} \right)
\]  

(2.21a)
due to the unsteady potential.

2.3. Body boundary condition

The kinematic condition for the potential on the body surface \( S_0(x,y,z,t) = 0 \) can be written as

\[
\frac{\partial \phi}{\partial n} \vec{V} \cdot \vec{n}
\]  

(2.24)
in coordinate system Oxyz, where \( \vec{V} \) is the velocity of the corresponding of the fluid particle on the body surface. For the present problem, it can be written as

\[
\vec{V} = \vec{U} i + \frac{d \vec{\alpha}}{dt}
\]  

(2.25)

where \( i \) is the unit vector in the \( x \) direction, \( U \) and \( \vec{\alpha} \) are defined as before. Equation (2.24) is the exact boundary condition on the instantaneous position of body surface \( S_0(x,y,z,t) \). For the linearized problem this condition may be transferred to the mean position \( \bar{S}_0(x,y,z) \). Substituting equations (2.9) and (2.25) into equation (2.24), we obtain

\[
\phi_n = \frac{d \vec{\alpha}}{dt} \cdot \hat{n} - \hat{W} \cdot \hat{n}
\]  

(2.25)
on $S_0(x,y,z,t)=0$, where
\[ \vec{W} = UV(\vec{\phi} - x) \]  
(2.26a)

From the kinematic condition of $\vec{\phi}$ on the body surface it is obvious that
\[ \vec{W}.\vec{n}=0 \]  
(2.26b)
on the mean position of the body surface $S_0$. In general case however, $\vec{W}.\vec{n}=0$ on the instantaneous position $S_0$, therefore it should be retained in equation (2.26). But as the unsteady motion is a small quantity we may write
\[ [(\vec{W}.\vec{n})].S_0 = \{[\vec{W} - \vec{V} \times \vec{W} + (\vec{a} \cdot \vec{V})\vec{W}].\vec{n}\} \]  
(2.27)
after the higher order terms are neglected. Similarly
\[ [(\vec{a}.\vec{n})].S_0 = [(\vec{a} - \vec{W}].\vec{n} \]  
(2.28)
then the linearized body boundary condition on potential $\phi$ is
\[ \phi_n = \frac{\partial \vec{a}}{\partial t} \cdot \vec{n} + [\vec{a} \times \vec{W} - (\vec{a} \times \vec{V})\vec{W}] \cdot \vec{n} \]
\[ = \frac{\partial \vec{a}}{\partial t} + \vec{V} \times (\vec{a} \times \vec{W}) \cdot \vec{n} \]  
(2.29)
on $S_0(x,y,z)=0$, after some straightforward vector calculation.

In this equation, the product term of the steady potential $\vec{\phi}$ with the displacement $\vec{a}$ is retained since we have not made any assumptions about its order on the body surface. To be consistent with equation (2.20) on the free surface, the steady potential should be deleted from equation (2.29), based on the assumption that it is a small quantity. Then this equation becomes
\[ \phi_n = \frac{\partial \vec{a}}{\partial t} - U(\vec{a} \times \vec{I}) \cdot \vec{n} \]  
(2.30)
For a deeply submerged body which is the main concern of this work, however, the disturbance due to forward speed on the free surface is a small quantity but it may be significant on the body surface. Therefore the contribution from the product terms containing $\phi$ on the free surface may be neglected but has to be retained on the body surface in general, so that equation (2.29) should be used.

Since we have completed the linearization of the body surface condition, we will use $S_0$ to denote $\overline{S}_0$ in latter part of this thesis unless it is specially declared otherwise.

2.4. Radiation condition

The radiation condition is that imposed on the potential at $|x^2+y^2+\omega|$. In reality, no fluid domain is unbounded, but in ship hydrodynamics the ocean is so large compared with ships that the effect of its boundary on the ship motion can be neglected. So the ocean can be regarded as a half (since it always has a free surface) unbounded fluid domain. When the ship is in a restricted water such as canal, the problem is treated differently.

Since the radiation condition is a condition artificially defined at $|x^2+y^2+\omega|$, its physical meaning is also quite artificial. Its justification is impossible to confirm by experiment directly. Experiment usually checks the mathematical model as a whole rather than every detail. Thus if the prediction of the ship motion by the potential theory agrees well with experiment, it only can be concluded that the mathematical model as a whole is acceptable, but it does not mean every detail in it is always correct, since one cannot deny mistakes in a
mathematical model can cancel each other or they may not be important in some cases. This seems meaningless in engineering, but it is important to remember that the justification of some assumptions has never been proved. When something goes wrong, it is then possible to establish the cause. Since the justification of the radiation condition has never been proved, it could be always this cause.

There is a lot of argument about what kind of radiation condition should be used in ship hydrodynamics, but it seems that for the linearized problem the general agreement has been reached. Considering the fundamental solution in ship hydrodynamics, which physically is the potential due to a source in a similar motion to the ship, it is found that this solution is not unique unless the radiation condition is imposed. For the steady potential, it is assumed that the gradient of this fundamental solution far ahead of the source is zero, then it yields a unique solution which has a wave term far behind the source. Since the rigid moving body can be mathematically represented by moving sources distributed over its surface, it is usually assumed that there is no wave far ahead of the body but there is a wave far behind the body.

For the periodically oscillatory motion, the procedure is similar. The radiation condition states that the wave generated by the body will travel in all directions and its amplitude at $\frac{1}{\sqrt{x^2+y^2}}$ decreases in proportion to the inverse square root of the distance from the origin of the body to the corresponding field point. For a body moving in regular waves, the problem is more complicated. Due to the effect of forward speed, it is found that there will be four waves instead of one in the
fundamental solution traveling with different speeds, when the motion is subcritical. The waves are assumed ahead of the body or behind the body in terms of whether their group velocities are larger or smaller than forward speed. If the forward speed is large enough so that the motion is supercritical, it is assumed that the body will overtake the wave far ahead of it and the corresponding wave terms in the mathematical equation should be deleted. The detail of this process will be discussed when the mathematical equations are derived in later chapters.

2.5. Decomposition of the unsteady potential

If the discussion is restricted to periodically oscillatory motion, the translation and rotation of the body relative to coordinate Oxyz may be written as

\[
\bar{\xi} = \text{Re}[(\xi_1, \xi_2, \xi_3)e^{i\omega t}] \\
\bar{\Omega} = \text{Re}[(\Omega_1, \Omega_2, \Omega_3)e^{i\omega t}] \\
= \text{Re}[(\xi_H, \xi_5, \xi_6)e^{i\omega t}] \\
\]

where \( \omega \) is the oscillatory frequency of the body. When the unsteady potential is correspondingly written as

\[
\phi(x, y, z, t) = \text{Re}[\phi(x, y, z)e^{i\omega t}] \\
\]

the body surface condition (2.29) becomes

\[
\frac{\partial}{\partial n}[\phi(x, y, z)] = i\omega \sum_j 6 \xi_j n_j + U \sum_j 6 \xi_j m_j \\
\]

where

\[
(n_1, n_2, n_3) = \bar{n} \\
\]

(2.35a)
\[(n_1, n_2, n_3) = (\vec{x} \cdot \vec{n})\] \hspace{1cm} (2.35b)

and

\[U(m_1, m_2, m_3) = U_\perp = -(\vec{n} \cdot \vec{v}) \vec{W}\] \hspace{1cm} (2.36a)

\[U(m_4, m_5, m_6) = -(\vec{n} \cdot \vec{v})(\vec{x} \cdot \vec{W})\] \hspace{1cm} (2.36b)

Therefore if \(\phi(x, y, z)\) is decomposed as

\[\phi(x, y, z) = [n_0 (\phi_0 + \phi_7) + \sum_{j=1}^{6} \xi_j \phi_j]\] \hspace{1cm} (2.37)

where \(\phi_0\) is the potential due to an incoming wave of amplitude \(n_0\); \(\phi_7\) is the scattered potential due to the reflection of \(\phi_0\) by the body and \(\phi_j\) \((j=1, 2, \ldots, 6)\) is the radiation potential corresponding to the motion of mode \(j\) respectively; then we obtain the boundary condition for each component of the unsteady potential as

\[\frac{\partial \phi_j}{\partial n} = i\omega n_j + Um_j\] \hspace{1cm} (2.38)

and

\[\frac{\partial \phi_7}{\partial n} = - \frac{\partial \phi_0}{\partial n}\] \hspace{1cm} (2.39)

on \(S_0(x, y, z) = 0\).

The corresponding free surface condition (2.23) becomes

\[\phi_{jz} + \frac{\tau}{\nu} \phi_{jxx} - 2i\tau \phi_jx - \nu \phi_j = 0\] \hspace{1cm} (2.40)

where

\[\tau = \frac{\omega U}{g}\] \hspace{1cm} (2.41a)

\[\nu = \frac{\omega}{g}\] \hspace{1cm} (2.41b)

These final two equations (2.38) and (2.40) suggest that the unsteady
radiation potential problem can be reduced to independent components corresponding to six degree motions of the body. This obviously is the consequence of the linearization. In fact, from equation (2.8), it can be easily seen that these components are coupled with each other due to product (or nonlinear) terms in the resulting expression.

In the process of decomposition no further assumptions are made, therefore all discussions made for \( \phi \) are also suitable for \( \phi_j \). Since the components \( \phi_j \) satisfy similar equations as \( \phi \), the existence and uniqueness theory for \( \phi \) will also hold for \( \phi_j \).

2.6. Hydrodynamic forces

From Bernoulli's equation the pressure in the fluid may be written as

\[
P = -\rho(\phi_t + \frac{1}{2} \nabla^2 \phi + g z_0)
\]  

(2.42)

where \( \rho \) is the density of the fluid. Substituting equations (2.9) and (2.10) into it we obtain

\[
P = -\rho(-U^2 \frac{2\phi}{\partial x} + \frac{U^2}{2} \nabla \nabla \phi + \frac{\partial \phi}{\partial t} - U \frac{2\phi}{\partial x} + U \nabla \nabla \phi + g z)
\]

\[
= -\rho(-U^2 \frac{2\phi}{\partial x} + \frac{U^2}{2} \nabla \nabla \phi + \frac{\partial \phi}{\partial t} + \nabla \phi) + g z)
\]  

(2.43)

after neglecting the higher order terms. We retain the terms \( \nabla \phi \) in this equation as \( \phi \) may not be a small quantity. To find the hydrodynamic force on the body, it is desirable to use the expression for the pressure on its mean position. It is obvious that the difference of the unsteady pressure

\[
p_{u} = -\rho(\phi_t + \nabla \phi)
\]  

(2.44)
on the instantaneous body surface and its mean position is of higher order; but the difference of pressure due to the steady potential

\[ p_s' = -\rho[-U^2\phi_x + \frac{1}{2}V^2\phi\phi] \quad (2.45) \]

may be significant. Using

\[ f(x+a_1, y+a_2, z+a_3) = f(x,y,z) + \bar{a}\cdot Vf + O(\bar{a}\bar{a}) \quad (2.46) \]
equation (2.45) may be written

\[ p_s' = -\rho[-U^2\phi_x - \bar{u}^2\bar{V}\phi_x + \frac{1}{2}[\bar{\phi}_x + \bar{\bar{a}}\cdot V\bar{\phi}_x]^2 + (\bar{\phi}_y + \bar{a}\cdot V\bar{\phi}_y)^2 \]
\[ + (\bar{\phi}_z + \bar{a}\cdot V\bar{\phi}_z)^2] \quad S_0 \]
\[ = -\rho[-U^2\phi_x + \frac{1}{2}V^2\phi\phi - \bar{u}^2\bar{V}\phi_x - (\bar{\phi}_x + \bar{a}\cdot V\bar{\phi}_x + \bar{\phi}_y + \bar{a}\cdot V\bar{\phi}_y + \bar{\phi}_z + \bar{a}\cdot V\bar{\phi}_z)] \quad S_0 \]
\[ = [p_s + \frac{1}{2}\bar{u}^2\bar{V}\phi\phi] \quad (2.47) \]

where \( S_0 \) is the mean position of the body as defined before; and

\[ p_s = -\rho[-U^2\phi_x + \frac{1}{2}V^2\phi\phi] \quad (2.48) \]
is the pressure due to the forward speed only which is identical to that in the case without oscillatory motion. Equation (2.43) then can be written as

\[ P = p_u + p_s \quad (2.49) \]

where

\[ p_u = p_u' + \frac{1}{2}\bar{u}^2\bar{V}\phi\phi \quad (2.50) \]
after the hydrostatic contribution is neglected. The hydrodynamic force \( \bar{F} \) and moment \( \bar{M} \) on the body can be found by the integration of the pressure over its surface. Thus
\[
\begin{align*}
\vec{F} &= \iint p \vec{n} \, dS \\
\vec{M} &= \iint p (\vec{x} \times \vec{n}) \, dS
\end{align*}
\]

Substituting equation (2.45) into (2.51), we obtain

\[
\begin{align*}
\vec{F} &= \vec{F}_s + \vec{F}_u \\
\vec{M} &= \vec{M}_s + \vec{M}_u
\end{align*}
\]

where

\[
\begin{align*}
\vec{F}_s &= \iint p_s \vec{n} \, dS \\
\vec{M}_s &= \iint p_s (\vec{x} \times \vec{n}) \, dS \\
\vec{F}_u &= \iint p_u \vec{n} \, dS \\
\vec{M}_u &= \iint p_u (\vec{x} \times \vec{n}) \, dS
\end{align*}
\]

are the force and moment due to the steady pressure, which are related to the problems of resistance, sinkage and trim of the ship; and

\[
\begin{align*}
\vec{F}_u &= \iint p_u \vec{n} \, dS \\
\vec{M}_u &= \iint p_u (\vec{x} \times \vec{n}) \, dS
\end{align*}
\]

are the force and moment due to the unsteady pressure, which are related to added masses and damping coefficients and wave exciting forces.

When the unsteady motion is periodic, substituting equations (2.33), (2.37), (2.44) and (2.50) into equation (2.54), we have the complex force

\[
\vec{F}_u = \rho \iint \left[ i\omega \eta_0 (\phi_0 + \phi_\gamma) + \sum_{j=1}^{6} \zeta_j \phi_j \right] \mathbf{V} \eta_0 (\phi_0 + \phi_\gamma)
\]
\[ + \sum_{j=1}^{6} \xi_j \phi_j \] \( \bar{n} \) \( ds \) \( e^{i\omega t} + \frac{1}{2} \rho \int \int (\bar{a} \cdot \nabla \bar{w}) \bar{n} ds \]

\[ = \left[ (\sum_{j=1}^{6} \xi_j \phi_j + \xi_j \phi_1 \right] \bar{I} + (\sum_{j=1}^{6} \xi_j \phi_2 \right] \bar{J} + \left[ (\sum_{j=1}^{6} \xi_j \phi_3 \right] \bar{K} \] \( e^{i\omega t} \) 

(2.55a)

Similarly, the complex moment

\[ \bar{M}_u = \left[ (\sum_{j=1}^{6} \xi_j \phi_j + \xi_j \phi_4 \right] \bar{I} + (\sum_{j=1}^{6} \xi_j \phi_5 \right] \bar{J} + \left[ (\sum_{j=1}^{6} \xi_j \phi_6 \right] \bar{K} \] \( e^{i\omega t} \) 

(2.55b)

where \( \tau_{ij} \) are composed of added mass \( m_{ij} \) and damping coefficient \( \lambda_{ij} \) and are defined as

\[ \tau_{ij} = \omega^2 m_{ij} - i\omega \lambda_{ij} \]

\[ = - \rho \int \int (i\omega \phi_j + \bar{w} \cdot \nabla \phi_j) \bar{n} ds \] 

(2.56)

c\( ^{'}_{ij} \) are the coefficients associated with the force proportional to the displacement; this can be regarded as stiffness and is defined by

\[ (\xi_1 c_{11}, \xi_2 c_{12}, \xi_3 c_{13}) = - \frac{1}{2} \rho \int \int (\xi_1 \bar{I} + \xi_2 \bar{J} + \xi_3 \bar{K}) \cdot \nabla \bar{w}^2 \bar{n} ds \] 

(2.57a)

\[ (\xi_4 c_{14}, \xi_5 c_{15}, \xi_6 c_{16}) = - \frac{1}{2} \rho \int \int [(\xi_4 \bar{I} + \xi_5 \bar{J} + \xi_6 \bar{K}) \cdot \nabla \bar{w}^2 \bar{n} ds \] 

(2.57b)

and \( \chi_1 \) is the exciting force defined as

\[ \chi_1 = - \rho n_0 \int [(i\omega \phi_0 + \phi_7) + \bar{w} \cdot \nabla (\phi_0 + \phi_7)] \bar{n} ds \] 

(2.58)

Equation (2.55) to (2.58) provide the means to find the hydro-
dynamic forces on the body from the velocity potential, which are valid provided the amplitude of the oscillatory motion is small. Generally, these results are not independent of each other, many relations between these hydrodynamic forces having been well developed in the case without forward speed. Some of them have been extended to the case with forward speed. The most well known is the Timman and Newman symmetry relation\textsuperscript{102} which states

$$\lambda_{ij}(-U) = \lambda_{j i}(+U) \quad (2.59a)$$

where $\lambda_{ij}(-U)$ on the left hand side represents the damping coefficients of the oscillatory motion with reversed forward speed. The validity of this equation is subject to the condition that

$$[\mathbf{V}(U) - U \mathbf{I}] = -[\mathbf{V}(+U) + U \mathbf{I}] \quad (2.60)$$

is approximately satisfied on the body surface. As a particular application of this relation, Timman and Newman considered the problem of the body of symmetry. They obtained that $\lambda_{ij}^T \lambda_{ji} = 0$ for all crossing-coupling coefficients except $\lambda_{15}^T \lambda_{51} = 0$ and $\lambda_{24}^T \lambda_{42} = 0$.

It can be understood that based on the present definition of $\tau_{ij}$ (without $c_{ij}$), the Timman and Newman relation can be extended to

$$\tau_{ij}(+U) = \tau_{ji}(-U) \quad (2.59b)$$

This was shown under condition (2.60) by Newman\textsuperscript{77} using a different procedure. Thus for the body of symmetry, we have $\tau_{ij}^T \tau_{ji} = 0$ except $\tau_{15}^T \tau_{51} = 0$ and $\tau_{24}^T \tau_{42} = 0$.

2.7. Numerical approach through the coupled element technique

The problem to be solved can be summarized as follows. The total
potential can be decomposed as a steady potential \( \phi \) which satisfies equations (2.4), (2.20) and (2.26b); and unsteady potential whose components satisfy equations (2.4), (2.38), (2.39) and (2.40). In the formulations of the coupled finite element method without forward speed, the differential equation is equivalently satisfied by a variational statement\(^2\). Unfortunately when there is forward speed, it seems difficult to obtain such equivalent statement. Therefore we use the Galerkin method to solve the present problem with forward speed\(^2\). Instead of satisfying equation (2.4) exactly, we may impose the condition

\[
\iiint_{R_1} \nabla^2 \phi_1 \psi \, d\sigma = 0
\]

where \( \psi \) is an arbitrary function (under some appropriate mathematical conditions); \( R_1 \) is the fluid domain surrounding the body; \( \phi_1 \) is the potential in \( R_1 \). Using Green's identity, equation (2.61) becomes

\[
\iiint_{R_1} \nabla \phi \psi \, d\sigma - \iint_{S_j} \frac{\partial \phi}{\partial n} \psi \, dS = \iint_{S_0} \frac{\partial \phi}{\partial n} \psi \, dS
\]

where \( S_j \) is a fully submerged boundary of \( R_1 \), and encloses but does not intersect the body. In this equation \( \frac{\partial \phi}{\partial n} \) on \( S_0 \) has been defined as the body boundary condition and \( \frac{\partial \phi}{\partial n} \) on \( S_j \) is to be determined by the boundary integral in the far field \( R_2 \). Since equation (2.62) is only a weak form it is only a necessary condition, not sufficient; the immediate question is whether the potential satisfying this equation for any given \( \psi \) is the solution of the original problem; how many functions will satisfy this equation. It is a complicated mathematical problem and is difficult to give an immediate answer. Some of theoretical background may refer to the book by Oden and Reddy\(^8\). In numerical analysis, it is quite often necessary to find the solution for any given problem directly rather
than waiting for these questions being answered. If one and only one (in the numerical sense) stable and converged numerical solution is obtained, this solution itself may give some heuristic evidence that the solution of this problem exists and is unique, although one may not generalize this conclusion to other problems.

The boundary integral representation in the far field to match equation (2.62) can be obtained with the help of a Green function which is defined as the potential due to a source in the corresponding motion to the ship's. It may be written as

$$G(x,y,z,a,b,c) = \frac{1}{r} + H(x,y,z,a,b,c)$$

for three dimensional problem, where $P(x,y,z)$ is the field point and $Q(a,b,c)$ is the location of the source and

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2};$$

$H$ is a harmonic function in the fluid domain and may be obtained by imposing appropriate free surface condition $(S_F)$, bottom condition $(S_B)$ and radiation condition $(S_\infty)$. Using the Green's identity, we obtain

$$\phi_2(P) = -\frac{1}{4\pi} \int_S \left[ \phi_2(Q) \frac{\partial G(Q)}{\partial n} - \frac{\partial \phi_2(Q)}{\partial n} G(Q) \right] dS$$

in $R_2$, where $S = S_J + S_F + S_\infty$. Using the condition for $\phi_2$ and $G$ on $S_F$ and $S_\infty$, this equation usually can be written as

$$\phi_2(P) = -\frac{1}{4\pi} \int_{S_J} \left[ \phi_2(Q) \frac{\partial G(Q)}{\partial n} - \frac{\partial \phi_2(Q)}{\partial n} G(Q) \right] dS$$

in $R_2$. When the field point $P$ is on $S_J$, $4\pi$ in this equation should be replaced by the subtended angle $\alpha$ at this point. Using the continuity of the potential and velocity on $S_J$, 

\[
\phi_1 = \phi_2 \quad (2.67a)
\]
\[
\frac{\partial \phi_1}{\partial n} = - \frac{\partial \phi_2}{\partial n} \quad (2.67b)
\]

where \( n \) points out of appropriate fluid domain, equation (2.66) becomes

\[
\phi_1(P) = \frac{1}{\alpha} \int \int \int [\phi_1(Q) \frac{\partial G(Q)}{\partial n} - \frac{\partial \phi_1(Q)}{\partial n} G(Q)] dS. \quad (2.68)
\]

From this equation \( \frac{\partial \phi_1}{\partial n} \) is determined by \( \phi_1 \) on \( S_j \). Substituting it into equation (2.62), we can obtain an equation involving \( \phi_1 \) only by discretization of these equations.

In the present numerical procedure of the discretization, the potential in equation (2.62) is expressed by means of isoparametric shape functions \( N_i \)

\[
\phi_1 = \sum_{i=1}^{n_e} \phi_{1i} N_i \quad (2.69)
\]

where \( \phi_{1i} \) is the value of the potential on the element nodes and \( n_e \) is the number of nodes. Similarly \( \frac{\partial \phi}{\partial n} \) on \( S_j \) may be also expressed by means of shape functions \( M_j \)

\[
\frac{\partial \phi_1}{\partial n} = \sum_{i=1}^{n_j} \frac{n_j}{\Delta n} \frac{\partial \phi_{1i}}{\partial n} M_i \quad (2.70)
\]

where \( n_j \) is the number of element nodes on \( S_j \). Since the choice of \( \psi \) is quite arbitrary, we take it as \( N_i \), so that equation (2.62) becomes

\[
\int \int \int \nabla \phi_1 \cdot \nabla N_i d\sigma - \int \int \int \frac{\partial \phi_1}{\partial n} N_i dS = \int \int \frac{\partial \phi_1}{\partial n} N_i dS; \quad (2.71)
\]

and equation (2.68) becomes

\[
\sum_{i=1}^{n_e} \left[ \delta_{ij} - \frac{1}{\alpha} \int \int \frac{\partial G}{\partial n} N_i dS \right] \phi_{1i} = \sum_{i=1}^{n_e} \left[ \frac{1}{\alpha} \int \int \frac{\partial G}{\partial n} M_i dS \right] \frac{\partial \phi_{1i}}{\partial n} \quad (2.72)
\]
on $S_j$, where $\delta_{ij}$ is the Kronecker function. In matrix form we have

$$[A][\phi] = [B][-\frac{\partial \phi}{\partial n}]$$  \hspace{1cm} (2.73)$$

where matrix $A$ has the coefficients

$$a_{ij} = \delta_{ij} - \frac{1}{\alpha} \int \int G_{ij} dS;$$ \hspace{1cm} (2.74)$$

and matrix $B$ has the coefficients

$$b_{ij} = -\frac{1}{\alpha} \int \int G_{ij} M_1 dS.$$ \hspace{1cm} (2.75)$$

Thus

$$[-\frac{\partial \phi}{\partial n}] = [C][\phi]$$ \hspace{1cm} (2.76)$$

where

$$[C] = [B]^{-1}[A].$$ \hspace{1cm} (2.77)$$

Substituting equations (2.60), (2.70) and (2.77) into equation (2.71), we have

$$\int \int \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \sum_{k=1}^{n_j} \phi_{ik} \int V_{N_{ij}} V_{N_{ik}} d\sigma + \int \int \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \sum_{k=1}^{n_j} \phi_{ik} \int M_{ik} M_{N_{ij}} dS$$

$$= \sum\sum_{j=1}^{n_j} \sum_{l=1}^{n_j} \sum_{k=1}^{n_j} (\int \int \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \sum_{k=1}^{n_j} \phi_{ij} (L) C_{ik} dS) + \int \int \sum\sum_{j=1}^{n_j} \sum_{l=1}^{n_j} \sum_{k=1}^{n_j} \phi_{ij} (L) C_{ik} M_{N_{ij}} dS$$

$$= \int \int \frac{\partial \phi_{ij}}{\partial n} N_{ij} dS$$ \hspace{1cm} (2.78)$$

where $j(L)$ is the relation between the numbering of the nodes on $S_j$ and
in $\mathbb{R}$. In matrix form, we have

$$[A_1 + A_1^*][\phi_1] = [P_1]$$

(2.79)

where $A_1$ is a symmetric matrix with coefficients

$$a_{ij} = \iint_{R_1} \mathbf{W}_{i} \mathbf{W}_{j} \, d\sigma;$$

(2.80)

$A_1^*$ is the square matrix with

$$a_{ij}^* = \iint_{R_1} \mathbf{W}_{i} \mathbf{W}_{j} \, d\sigma;$$

(2.81)

and $P_1$ is the matrix containing the appropriate body surface boundary condition.

For the two dimensional problem, the Green function has the form

$$G(x,z,a,c) = \ln r + H(x,z,a,c)$$

(2.82)

with

$$r = \sqrt{(x-a)^2 + (z-c)^2}.$$  

(2.83)

Following the same derivation, we may obtain a similar result except the minus sign should be deleted and $4\pi$ should be replaced by $2\pi$ in equation (2.65) and these changes should be retained throughout the derivation.\textsuperscript{27,28}
3. ANALYSIS OF THE TWO DIMENSIONAL PROBLEM

3.1. General

The two dimensional problem is defined so that the velocity of fluid particle at any point is parallel to a fixed plane, such as Oxz, and is independent of the distance of the point from this plane. In theory, such kind of motion can be generated only by infinitely long cylinders. In practice, the problem involving an elongated body usually can be approximated by assuming that the flow corresponding to each section of the body is two dimensional, such as the problem of wave resistance on long hydrofoils moving in transverse direction and the problem of slender ship response to waves. Thus, this theory is of great practical importance. Although the purpose of this work is to develop a fully three dimensional theory for the general body, it is valuable to consider first this simplified case. This enables theoretical techniques to be assessed and numerical methods investigated at a simpler level than for the fully three dimensional case.

As a direct result of the two dimensional assumption, the equations in the previous chapter can be simplified by taking \( \frac{\partial \phi}{\partial y} = 0 \). Thus the governing equations for the steady potential defined in equation (2.9) become

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{3.1}
\]

in whole fluid domain;

\[
\frac{\partial \phi}{\partial n} = n_1 \tag{3.2}
\]
on the body surface \( S_0 \);
on the free surface $S_F$, where $\frac{u}{U^*}$;

$$\frac{\partial \phi}{\partial z} = 0$$  (3.4)

on the bottom $S_b$ of the fluid; and

$$\frac{\partial \phi}{\partial x} = 0$$  (3.5a)

as $x \to \pm \infty$;

$$\left| \frac{\partial \phi}{\partial x} \right| < \infty$$  (3.5b)

as $x \to \pm \infty$.

The governing equations for unsteady potential $\phi_j$ ($j=1,3,5$) in equation (2.37) become

$$\nabla^2 \phi_j = \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_j}{\partial z^2} = 0$$  (3.6)

in whole fluid domain;

$$\phi_{jz} + \frac{1}{\nu} \phi_{jxx} - 2i\phi_{jx} - v\phi_j = 0$$  (3.7)

on the free surface;

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j + U m_j$$  (3.8)

on the body surface, with

$$\{m_1, m_3, m_5\}$$

$$= \left( n_1 \frac{\partial^2 \phi}{\partial x^2} + n_3 \frac{\partial^2 \phi}{\partial x \partial z}, n_1 \frac{\partial^2 \phi}{\partial x^2} + n_3 \frac{\partial^2 \phi}{\partial z^2},
\right.$$  (3.9)

$$\left. (n_1 \frac{\partial}{\partial x} + n_3 \frac{\partial}{\partial z})[z(\phi_x - 1) - x\phi_z] \right)$$;
and

\[ \frac{\partial \Phi}{\partial z} = 0 \quad (3.10) \]

on the bottom of the fluid.

As already seen, the starting point of solving these equations here is to find the solution of \( \Phi \) which is known as the two dimensional Neumann-Kelvin (N-K) problem. While it will contribute to the body surface condition on \( \Phi \), \( \Phi \) also has its own practical importance since it is related to the important problem of wave resistance on the ship. Thus we will begin the two dimensional analysis by solving the steady potential problem \( \Phi \) first.

3.2. Steady potential

3.2.1. A brief review of previous work

The first work on the two dimensional N-K problem was completed by Havelock in 1936\(^5\). He considered a particular problem of the submerged circular cylinder moving in infinite water depth. The potential was expressed by a series satisfying the Laplace equation, free surface condition and radiation condition. The unknown coefficients in the series were found by imposing the body surface condition on the potential, so that the analytic solution in a series form was obtained. The solution of this particular problem was also obtained by Wehausen and Laitone\(^{113}\) using an extension of another method suggested by Havelock as well\(^{50,51}\). Starting from a harmonic function \( f_0 \) satisfying the body surface condition, a harmonic function \( f_1 \) is found by imposing the free surface condition on \( f_0 + f_1 \); the \( f_2 \) is found by imposing the body surface condition on \( f_1 + f_2 \). This procedure is repeated by satisfying the body surface and
free surface condition alternately, until satisfactory accuracy is
reached. The solution then can be obtained as \( f = \sum_{n=1}^\infty f_n \). Wehausen and Laitone proved that this series will converge for \( \mu(h-a)>0.4 \), where \( h \) is the distance of the center of the cylinder to the free surface and \( a \) is the radius of the cylinder (it seems this condition is not necessary as far as the existence of the solution is concerned, although is suffi-
cient).

The problem of an arbitrary cylinder moving below the free surface remained unsolved until about thirty years later after Havelock's work. With the help of the computer, Giesing and Smith were able to obtain numerical solutions for arbitrary cylinders based on traditional representation by a source distribution over the surface of the cylinder. Quite similarly, Chang and Pien used a dipole instead a source distribution over the body. The advantage claimed for this method is that the potential can be found from the dipole strength without the calculation of any further complicated integrals.

About a decade ago, this problem was solved again by the coupled element method. Bai used the finite element method in the near field and a series in the far field; Mei and Chen transformed the problem into a fictitious radiation and diffraction problem; Yeung and Bouger used the Rankine source distribution over the complete boundary of the near field to represent the potential in this region, and combined with series expansion of the potential in the far field. An alternative representation of the far field using boundary element integral and a similar localized finite element representation to that of Bai in the near field was used by Eatock Taylor and Wu. However these works have
mainly concentrated on the submerged body problem since the N-K problem for a floating cylinder seems unattackable. There have been many attempts to challenge this problem but they usually end up with an uncertain conclusion\cite{18,31,101,109}. The main difficulty is due to the uncertainty of uniqueness of the N-K problem for a floating cylinder. This may raise the question whether the mathematical model of the N-K problem is valid for the floating cylinder. But as it is not our main interest to discuss the floating cylinder problem, we are not going to answer this question here.

3.2.2. Havelock's method and its extension

For a circular cylinder of radius $a$ and submergence $h$ (distance from the center of the cylinder to the free surface), Havelock\cite{54} obtained the complex potential $w$ as a series form satisfying all boundary conditions except on the body surface. It may be written as

$$w = \text{const} + UZ + \sum_{n=1}^{\infty} Ua^{2}(ia)^{n-1} b_{n-1} Z^{-n}$$

$$= \text{const} + \sum_{n=1}^{\infty} C_{n} Z^{n} + \sum_{n=1}^{\infty} D_{n} Z^{-n}$$

where

$$Z = x + i(z+h)$$

with the origin at the center of the cylinder;

$$C_{n} = \delta_{n}U + Ua^{2}(-i)^{n+1} n! \lim_{m \to 0} \frac{f^{(m)}(m)}{m!} e^{-i2mh}$$

$$D_{n} = \frac{1}{n!} \lim_{m \to 0} \frac{f^{(m)}(m)}{m!} e^{-i2mh}$$

\cite{54}
\[ D_n = Ua^2(na)^{n-1}b_{n-1} \] (3.14)

\[ f(m) = \sum_{n=0}^{\infty} \frac{b_n (ma)^n}{n!} \] (3.15)

with \( \mu = \frac{g}{u^2} \) as defined before and star denoting the conjugate of the complex. Imposing the body surface condition, the unknown coefficients may be obtained from

\[ D_n = a^{2n}c_n^* \] (3.16)

Having found \( b_n \), the horizontal force \( R_x \) and vertical force \( R_z \) may be obtained by the Blasius formula

\[
R_x - iR_z = \int_0^\infty \left( \frac{dw}{dz} \right)^2 \, dz
\]

\[
= 2\pi \rho \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+2}D_nD_{n+1}^*
\]

\[
= -2\pi \rho u^2 \sum_{n=1}^{\infty} \frac{n(n+1)b_n b_{n+1}^*}{a_n}
\] (3.17)

This method is limited to the case of a circular cylinder, but we may extend it to the case of multiple circular cylinders. Without losing generality, we consider the case of two cylinders of radii \( a_1, a_2 \) and submergences \( h_1, h_2 \) respectively. The complex potential may be written as

\[
w = \text{const} + UZ + \sum_{n=1}^{\infty} Ua_1^2(na_1)^{n-1}b_{1n-1}z_1^{-n}
\]

\[- iUa_1^2 \lim_{\kappa \to 0} \int_{m_0}^{m_1} \frac{e^{-i\kappa \epsilon^*_1(m)}}{m - \mu - i\kappa^*_1(m)} \, dm
\]

\[ + \sum_{n=1}^{\infty} Ua_2^2(na_2)^{n-1}b_{2n-1}z_2^{-n} \]
where $Z_1$ and $Z_2$ are the local coordinates corresponding to cylinder one and cylinder two respectively, satisfying the relation

$$Z_1 = Z_2 - \beta; \quad \text{(3.19)}$$

$\beta$ is the coordinate of the center of the cylinder one in $Z_2$. Similar to equations (3.13) to (3.15), $A_j$, $B_j$ ($j=1,2$) are defined by

$$A_j = Ua_j \lim_{0} \frac{m+\mu-ikZ_1}{0^m-\mu-1k} \frac{e^{-\gamma(m)}}{\gamma(m)} f_j(m) \quad \text{and} \quad B_j = D_a \lim_{0} \frac{m+\mu-ik}{m} e^{-\gamma(m)} f_j(m) \quad \text{for} \quad j=1,2,$$

where $Z_1$ and $Z_2$ are the local coordinates corresponding to cylinder one and cylinder two respectively, satisfying the relation

$$Z_1 = Z_2 - \beta; \quad \text{(3.19)}$$

$\beta$ is the coordinate of the center of the cylinder one in $Z_2$. Similar to equations (3.13) to (3.15), $A_j$, $B_j$ ($j=1,2$) are defined by

$$A_j = Ua_j \lim_{0} \frac{m+\mu-ikZ_1}{0^m-\mu-1k} \frac{e^{-\gamma(m)}}{\gamma(m)} f_j(m) \quad \text{and} \quad B_j = D_a \lim_{0} \frac{m+\mu-ik}{m} e^{-\gamma(m)} f_j(m) \quad \text{for} \quad j=1,2,$$

To impose the condition of cylinder one, we use Maclaurin series

$$Z_2^{-n} = (Z_1 + \beta)^{-n} = \sum_{k=0}^{\infty} C_{n+k-1}^{k} (-1)^{k} \beta^{-n-k} Z_1^{k}; \quad |Z_1| < |\beta| \quad \text{(3.23)}$$

Substituting them into equation (3.18), we have

$$w = \text{const} + UZ_1 + \sum_{n=1}^{\infty} A_n Z_1^{-n} + \sum_{n=1}^{\infty} B_n Z_1^n$$

$$- iUa_2 \lim_{0} \frac{m+\mu-\mu-1kZ_1}{0^m-\mu-1k} \frac{e^{-\gamma(m)}}{\gamma(m)} f_2(m) e^{-i\beta - 2mZ_2} \quad \text{for} \quad j=1,2,$$
\[ + \sum_{n=1}^{\infty} A_2 \sum_{k=0}^{n+k-1} (-1)^k \beta^{-n-k} Z_1^n \]
\[ = \text{const} + U Z_1 + \sum_{n=1}^{\infty} A_1 Z_1^{-n} + \sum_{n=1}^{\infty} B_1 Z_1^n + \sum_{n=1}^{\infty} B_2 Z_1^n \]
\[ + \sum_{n=0}^{\infty} A_2 (-1)^n C_2^{k+n-1} \beta^{-n-k} Z_1^n \]
\[ = \text{const} + \sum_{n=1}^{\infty} C_1 Z_1^{-n} + \sum_{n=1}^{\infty} D_1 Z_1^n \]
(3.24)

where

\[ D_1 = A_1 \]
(3.25)

\[ C_1 = \delta_1 n + B_1 + B_2 + \sum_{k=1}^{\infty} A_2 (-1)^n C_2^{k+n-1} \beta^{-n-k} \]
(3.26)

\[ B_2 = \frac{2 (-1)^{n-1}}{n!} \lim_{\kappa \to 0} \int_{m = \mu - i \kappa}^{m = \mu - i \kappa} \frac{e^{-2 \kappa^{-1} m} e^{-i \kappa \beta}}{f_2(m) \kappa} \]
(3.27)

The body surface condition on cylinder one suggests

\[ D_1 = a_n^2 C_1 \]
(3.28)

Similarly we may have

\[ w = \text{const} + \sum_{n=1}^{\infty} C_2 Z_2^{-n} + \sum_{n=1}^{\infty} D_2 Z_2^n \]
(3.29)

in coordinate system \( Z_2 \), where

\[ D_2 = A_2 \]
(3.30)

\[ C_2 = \delta_2 n + B_2 + B_1 + \sum_{k=1}^{\infty} A_1 (-1)^n C_2^{k+n-1} (\beta)^{-n-k} \]
(3.31)

\[ B_1 = \frac{2 (-1)^{n-1}}{n!} \lim_{\kappa \to 0} \int_{m = \mu - i \kappa}^{m = \mu - i \kappa} \frac{e^{-2 \kappa^{-1} m} e^{-i \kappa \beta}}{f_1(m) \kappa} \]
(3.32)

and with the relation
Equations (3.28) and (3.33) provide the means to find the unknown coefficients $b_n$. The forces on the cylinders may obtained by:

$$R_1x - iR_1z = 2\pi \rho \sum_{n=1}^{n(n+1)} \frac{D_n}{2n+1} \frac{D_n^*}{n+1}$$

and

$$R_2x - iR_2z = 2\pi \rho \sum_{n=1}^{2n+2} \frac{D_n}{2n+2} \frac{D_n^*}{2n+1}$$

These two equations are similar to that obtained by Havelock for a single cylinder (equation (3.17)). The difference is that $D_n$ in equations $(3.34)$ and $(3.35)$ contain the effect from the other cylinder since the solution of $b_n$ involves solving two infinite set of coupled equations. To obtain this solution, the procedure used by Havelock can be adopted here after some modifications in evaluating the coefficients of the linear equations. The results to be obtained in this analysis can provide some general phenomena of multi-body motion, which are important for some ships moving together and for twin hull vessels.

3.2.3. The Green function

The geometry of the fluid domain used in the present numerical analysis is shown in Fig. 4. The Green function employed to obtain $\Phi_2$ in $R_2$ discussed in section 2.7 can be obtained by imposing equation (3.1) on $H$, and equations (3.3), (3.4) and (3.5) on $G$ in equation (2.82). One may find the solution (except for an arbitrary additive constant)$^{113}$ as

$$G = \ln \frac{r_1}{d} + \ln \frac{r_2}{d} + \int_{0}^{m+\nu}\frac{me^{-md}}{\mu \sinh \mu d - \mu \cosh \mu d}$$
\[ [1 - \cosh m(z+d) \cos m(x-a)] dm \]
\[ - 2\pi \frac{\mu \cosh m_0(c+d)}{m_0(\mu d - \cosh^2 m_0 d)} \cosh m_0(z+d) \sin m_0(x-a) \] 

(3.36)

where

\[ r = \left[(x-a)^2 + (z-c)^2\right]^{1/2} \] 

(3.37a)

\[ r_2 = \left[(x-a)^2 + (z+2d+c)^2\right]^{1/2}. \] 

(3.37b)

\[ d \] is the water depth, \[ m_0 \] is the non-zero solution of

\[ \mu \sinh m d - m_0 \cosh m d = 0 \] 

(3.38)

and exists only if \( \frac{\mu}{\sqrt{g d}} < 1 \). The last term in equation (3.36) is associated with the singularity in the integrand at \( m = m_0 \), which should be deleted when \( \frac{\mu}{\sqrt{g d}} \geq 1 \). Numerically it is not difficult to evaluate this Green function for any given points \( P(x,z) \) and \( Q(a,c) \), apart from some tedious effort in dealing with the singularity. Here we adopt the procedure similar to that used in references [27,28], we write

\[ G = \ln \frac{r}{d} + \ln \frac{r_2}{d} + 2G_{pv} \]
\[ - 2\pi \frac{\mu \cosh m_0(c+d)}{m_0(\mu d - \cosh^2 m_0 d)} \cosh m_0(z+d) \sin m_0(x-a) \] 

(3.39)

where

\[ G_{pv} = \int_0^{2u_0} \frac{g(u) - g(u_0)}{u - u_0} du + \int \frac{f(u)}{\tanh u - u} du \] 

(3.40a)

\[ f(u) = \frac{u + u_0}{u} \frac{\cosh u(c+d)[1 - \cosh^2 u(z+d) \cos^2 u(x-a)]}{\cosh u} \] 

(3.40b)

\[ g(u) = \frac{f(u)(u - u_0)}{\tanh u - u} \] 

(3.40c)
\[ g(u_0) = \lim_{u \to u_0} g(u) = f(u_0) \frac{\sigma}{\sigma^2 - \sigma - u_0^2} \quad (3.40d) \]

\[ u = \mu d, u_0 = \mu_0 d, \sigma = \mu d \quad (3.40e) \]

In the case of infinite water depth, we have

\[ G = 1/r + 1/r_1 + \frac{c}{2} \int_0^\infty \frac{e^{m(z+c)}}{k - \mu} dm \]

\[ + 2\pi e^{u(z+c)} \sin(u(x-a)) \quad (3.41) \]

with

\[ r_1 = [(x - a)^2 + (z + c)^2]^{1/2} \quad (3.42) \]

And in a manner similar to equation (3.39), this one may be written as

\[ G = 1/r - 2G_{pv} + 2\pi e^{u(z+c)} \sin(u(x-a)) \quad (3.43) \]

where

\[ G_{pv} = \int_0^{2\mu} \frac{g(m) - g(\mu)}{m - \mu} dm + \int_{2\mu}^\infty \left[ \frac{f(m)}{\mu - m} - \frac{e^{-m}}{m} \right] dm \quad (3.44a) \]

\[ f(m) = (\mu + m) \frac{e^{m(z+c)}}{m} \cos(x-a) \quad (3.44b) \]

\[ g(m) = -f(m) - (m - \mu) \frac{e^{-m}}{m} \quad (3.44c) \]

\[ g(\mu) = -f(\mu) \quad (3.44d) \]

The evaluation of the integration in equations (3.40a) and (3.44a) may be fulfilled by the Gaussian method. The step is appropriately chosen by the careful consideration of the oscillatory behaviour of the function \( \cos(x) \) in the integrand. The integration is truncated at \( e^{m(z+c)} = 10^{-6} \). The detail of the analysis is discussed in references [118,119] in the case of harmonic motion and the principles are adopted...
As a particular advantage of this method, it is possible to choose the boundary of the localized finite element region $S_j$ so that the surface integral of the Green function can be evaluated explicitly. It is convenient to specify a rectangular boundary in the two dimensional case for $S_j$, and then the main task in evaluating the surface integral is the calculation of

$$\int x^pG\,dx, \quad \int x^pG\frac{\partial G}{\partial z}\,dx$$

(3.45a)
on a horizontal part of boundary $S_j$; and

$$\int z^p\,dz, \quad \int z^pG\frac{\partial G}{\partial x}\,dz$$

(3.46a)
on a vertical part of boundary $S_j$. Here $p=0,1,2$ for the quadratic shape functions used in this work. For the Green function given in equations (3.39) and (3.40), the calculation of equations (3.45a) and (3.46a) is to evaluate

$$\int x^p\cos(m(x-a))\,dx, \quad \int x^p\sin(m(x-a))\,dx$$

(3.45b)
on the horizontal part of boundary $S_j$, and

$$\int z^p\cosh(m(z+d))\,dz, \quad \int z^p\sinh(m(z+d))\,dz$$

(3.46b)
for finite water depth,

$$\int z^p e^{m(z+c)}\,dz$$

(3.46c)
for infinite water depth, on the vertical part of the boundary $S_j$. All these explicit integrals for the harmonic motion problem were worked out by Zietsman and Eatock Taylor\textsuperscript{119} and the results are extended to the present case in this work.
3.2.4. Discussion of results

To verify the present numerical approach, we first consider the case of a submerged circular cylinder at h=2a solved by Havelock analytically. Fig.5 gives a typical mesh of 12 elements employed in the present numerical analysis; while Table 1 gives the results of nondimensional coefficients of resistance and lift for several meshes from coarse (8 elements) to fine (24 elements). The analytical solution is obtained from equations (29) and (30) in reference[54] and the present numerical results are obtained by integrating the pressure in equation (2.48) over the cylinder surface. The comparison shows that the present method gives very accurate results.

<table>
<thead>
<tr>
<th>Fn</th>
<th>8</th>
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<th>16</th>
<th>20</th>
<th>24</th>
<th>analytic</th>
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<td>0.32113</td>
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<td>0.42045</td>
<td>0.42077</td>
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</tr>
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<td>0.43835</td>
<td>0.43849</td>
<td>0.43737</td>
<td>0.43647</td>
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<td>0.41130</td>
<td>0.41129</td>
<td>0.41011</td>
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<td>0.36721</td>
<td>0.36715</td>
<td>0.36722</td>
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<td>-0.27448</td>
</tr>
</tbody>
</table>

\[ C_R = \frac{\text{resistance}}{\rho g a^2} \quad C_L = \frac{\text{lift}}{\rho g a^2} \]

Having confirmed the justification of the present numerical method, we calculate several cases associated with a variety of problems. The
influence of water depth \( d \) on the resistance and lift of a circular cylinder is shown in Fig.6. It can be seen from Fig.6a and Fig.6b that this influence becomes important for a circular cylinder at \( h=2a \) when \( d<10a \). When the water is very shallow, the resistance becomes very sensitive to the depth and will be significantly increased by the reduction of water depth. Physically, the vertical flow of the fluid will be progressively blocked when the gap between the cylinder and even bottom becomes smaller. The only way for fluid to pass by the body is for its horizontal speed to increase. As the consequence, the wave amplitude will be increased, as given by the equation (2.18a). It is obvious that the body will suffer larger resistance when the wave amplitude is larger.

Fig.6c and Fig.6d give the curves of resistance and lift on the circular cylinder submerged in different depths against Froude number defined as \( F_n = \frac{U}{\sqrt{gd}} \). It can be seen that both resistance and lift are approximately linear functions of the Froude number. It seems that there might be a water depth between \( 4.5a \) and \( 7a \) in which the resistance on the cylinder is independent to the Froude number. It has been observed that there is a discontinuity of the lift at the critical Froude number \( F_n = \frac{U}{\sqrt{gd}} = 1 \), and the resistance on the cylinder is zero in the supercritical region.

As the linear theory is based on the small perturbation of the free surface as discussed in the previous chapters, the effect of nonlinearity of the free surface may become important in shallow water as the wave amplitude increase with the reduction of the water depth. However, from equation (3.38), it can be easily known that the length of the wave
far behind the moving body will also increase with the reduction of water depth. Thus it cannot be directly concluded that the nonlinearity will become severe in shallow water, before the careful analysis is made. Intuitively, if the body is sufficiently deeply submerged in infinite water depth so that its disturbance on the free surface is not severe, it is hard to believe that this disturbance due to the existence of an even fluid bottom below the body will be severe. But for the uneven bottom the situation is different; in this case the water depth cannot be taken as simply as influence factor in the N-K problem. Instead the obstruction on the bottom should be regarded as another body which may generate nonlinear waves.  

Fig.7 gives the results of nondimensional resistance and lift on elliptical cylinders with their major axis in the direction of forward speed. It can be seen the force will be significantly decreased by the reduction of the ratio of the half minor axis b to half major axis a and it can be anticipated the force will tend to zero when \( \frac{b}{a} \to 0 \). The interesting point is that the zero lift speed is far less sensitive to the ratio of \( \frac{b}{a} \).

To illustrate the flexibility of the method, Fig.9 gives the results of two submerged circular cylinders, while the geometry for this problem is shown in Fig.8. The results are from a mesh using two groups of elements similar to the group shown in Fig.5. It can be seen that the effect of the downstream body on the upstream body is to reduce both resistance and lift on the upstream body. Physically there is a region of high pressure before a moving body below the free surface and a region of low pressure behind it. The resistance is due to their differ-
ence. When two bodies are moving together, the low pressure behind the upstream body will be increased by the high pressure before the downstream body. This causes a reduction of the resistance on the upstream body. It can be understood that this reduction will disappear when the horizontal distance between the two cylinders tends to infinity, as can be seen in equations (3.26) and (3.27) for a circular cylinder. The situation for the downstream cylinder is rather different. It is found that the resistance and lift on the downstream body oscillates with the Froude number and the resistance can even be negative. This can be understood by realizing that there is always a wave behind a moving body, so that the downstream body is moving in the wave generated by the upstream body instead of in a uniform stream. In the two dimensional problem, since the wave generated by the moving cylinder will go to infinity without decreasing its amplitude gradually, the influence of the upstream body on the downstream body will always exist even when the horizontal distance between two cylinders tends to infinity. The most interesting point in Fig.9 is that the total resistance on two cylinders can be zero. This is a very important feature and needs further investigation. Some theoretical support for this feature might be obtained using the extension of Havelock’s method to multiple cylinder problem as discussed in section 3.1.2.

3.3. Unsteady potential

3.3.1. The hydrodynamic force on a circular cylinder in an unbounded fluid

From a rigorous mathematical point of view, the numerical results from the present method will converge to the exact solution in a uniform
sense. That is, if the exact solution of the Laplace equation is denoted by \( \phi^e \) and the solution from Galerkin method of \( n \) finite elements is denoted by \( \phi^n \), we have \( \lim_{n \to \infty} \left( \int_{\partial \Omega} \left\| \phi^n - \phi^e \right\|^2 \, ds \right)^{1/2} = 0 \), as the dimensions of all elements tend to zero. There is no guarantee for the potential that \( \phi^n + \phi^e \) at any points. Since the body surface condition for the unsteady potential \( \phi \) contains derivatives of the steady potential \( \phi \), one needs some assurance of the accuracy of \( \phi \) before the coupled finite element method can be used to solve this problem. Thus, we consider the problem of a circular cylinder in an unbounded fluid domain, as its analytical solution can be easily derived to check our numerical results.

To obtain the analytical solution for this particular problem we define the polar coordinate \((r, \theta)\) with its origin located at the center of the cylinder and \( \theta = 0 \) being the direction of forward speed. The solution of the steady potential for a circular cylinder at unit forward in the unbounded fluid may be obtained as\(^8\)

\[
\bar{\phi} = -\frac{a^2}{r} \cos \theta \tag{3.47}
\]

Using \( n_1 = -\cos \theta \), \( n_2 = -\sin \theta \) and \( \frac{\partial \phi}{\partial n} = -\frac{\partial \phi}{\partial r} \) the linearized body surface conditions (3.8), (3.9) then become

\[
\frac{\partial \phi_1}{\partial r} = -iwn_1 + U \frac{\partial^2 \phi}{\partial r \partial x} = i \omega \cos \theta + \frac{2U \cos 2\theta}{a} \tag{3.48}
\]

\[
\frac{\partial \phi_2}{\partial r} = -iwn_2 + U \frac{\partial^2 \phi}{\partial r \partial x} = i \omega \sin \theta + \frac{2U \sin 2\theta}{a} \tag{3.49}
\]
\[ \frac{\partial \phi}{\partial r} = \frac{\partial}{\partial r} [z(\frac{\phi}{x} - 1) - x\phi_z] \]

= 0 \quad (3.50)

The solutions can be obtained as

\[ \phi_1 = -i\omega \frac{a^2}{r} \cos \theta - \frac{U_{a}^2}{r^2} \cos 2\theta \quad (3.51) \]

\[ \phi_3 = -i\omega \frac{a^2}{r} \sin \theta - \frac{U_{a}^2}{r^2} \sin 2\theta \quad (3.52) \]

\[ \phi_5 = 0 \quad (3.53) \]

The hydrodynamic force may be obtained by substituting the solutions of \( \phi \) and \( \phi_j \) into equation (2.56). Using

\[ \bar{\omega} \cdot \nabla \phi_j = \frac{U}{a^2} \frac{\partial}{\partial \theta} (\phi - r \cos \theta) \frac{\partial \phi_j}{\partial \theta} \quad (3.54) \]

on the surface of the circular cylinder, we have

\[ \tau_{11} = -\rho \int [i\omega (-i\omega \cos \theta - U_{a} \cos 2\theta) + \frac{2U}{a} \sin \theta (i\omega \sin \theta \cos \theta - 2U \sin 2\theta)] \cos \theta \, d\theta \]

\[ + 2U \sin 2\theta] a \cos \theta \, d\theta \]

\[ = \rho a \int [\omega^2 \cos \theta - i\omega U \cos 2\theta + 2i\omega U \sin^2 \theta + \frac{4U^2}{a} \sin \theta \sin 2\theta] \cos \theta \, d\theta \]

\[ = \rho a \int [\omega^2 \cos \theta + i\omega U - 2i\omega U \cos 2\theta + \frac{2U^2}{a} (\cos \theta - \cos 3\theta)] \cos \theta \, d\theta \]

\[ = \rho a \int (\omega^2 a + \frac{2U^2}{a}) \cos^2 \theta \, d\theta \]

\[ = \rho \pi a (\omega^2 a + \frac{2U^2}{a}) \quad (3.55) \]

Thus
\[ \nu_{11} = \rho \pi \left( a^2 + \frac{2u^2}{\omega^2} \right) \]  
(3.56a)

\[ \lambda_{11} = 0 \]  
(3.56b)

Similarly we obtain other results as

\[ \tau_{33} = \tau_{11} \]  
(3.57)

\[ \tau_{13} = \tau_{31} = 0 \]  
(3.58)

\[ \tau_{15} = \tau_{51} = \tau_{35} = \tau_{53} = \tau_{55} = 0 \]  
(3.59)

Before starting the numerical analysis of this problem, it is interesting to briefly compare equations (3.56) to (3.59) with the results from the harmonic motion without forward speed. It can be seen that the difference is that the added masses \( \nu_{11} \) and \( \nu_{22} \) are no longer constants, but both frequency and speed dependent. The extra term in them shows that the added mass will tend to infinity when the frequency tends zero, while the added mass force \( \omega^2 \nu_{11} \) will tend to \( 2\pi \rho U^2 \) for unit oscillating amplitude. It is particularly interesting to notice that the influence of forward speed on added masses of a circular cylinder is independent of the radius of the cylinder, so that for a large cylinder this influence may be neglected.

From equations (2.71), (3.2) and (3.8), the equation used in the present numerical analysis can be written as

\[
\int \int V_{\phi} V_{N_1} d\sigma - \int \frac{\partial \Phi}{\partial n} N_1 dS = \int n_1 N_1 dS \]  
(3.60)

for the steady potential and

\[
\int \int V_{\phi j} V_{N_1} d\sigma - \int \frac{\partial \Phi_j}{\partial n} N_1 dS = \int (i\omega n_j + U m_j) N_1 dS \]  
for \( j \)
for the unsteady potential. The Green function employed to determine \( \frac{\partial \Phi}{\partial n} \) on \( S_j \) is \( \ln r \) in these two cases. Since the problem of the steady potential in equation (3.60) is a special case of that in the previous section, its solution can be straightforwardly obtained. However, great care is needed to deal with the last term in equation (3.61), which involves second order derivatives in the definition of \( m_j \) (see equation (3.9)). Generally, if the shape function used to represent the potential is of order \( n \) and typical length of the element is \( \delta \), the error of the \( m \)th order derivative of the potential will be of order \( O(\delta^{n+1-m}) \).

In the present work, the quadratic shape function is used so that on this basis the error of the second order derivatives of \( \Phi \) in equation (3.61) would be \( O(\delta) \), which requires very fine mesh. Indeed, we have observed that a large number of elements would be needed to obtain accurate results if equation (3.61) is used directly.

To avoid this difficulty, we may use the relation obtained by Ogilvie and Tuck\(^8\) for the general case of a floating body. In the present case, this relation can be derived in a slightly different manner. From equation (3.9), we obtain

\[
U \int m_j N_j dS = -U \int \left( \frac{\partial^2 \Phi}{\partial x^2} n_1 + \frac{\partial^2 \Phi}{\partial 3} n_3 \right) dS
\]

Using \( \nabla^2 \Phi = 0 \) (equation (3.1)), \( \bar{W} \cdot \bar{n} = 0 \) on \( S_0 \) (equation (2.26b)), \( dx = n_3 dS \) and \( dz = -n_1 dS \), and noticing that the second integration is zero for the submerged cylinder, we obtain
\[
U \int m_j n_1 \, dS = U \int \left[ -N_j \frac{\partial^2 \phi}{\partial z^2} \right] \, dz - U \int \frac{\partial N_j}{\partial x} \frac{\partial (\phi - x)}{\partial x} \, n_1 \, dS
\]
\[
= U \int \frac{\partial N_j}{\partial z} \frac{\partial \phi}{\partial z} \, dz - U \int \frac{\partial N_j}{\partial x} \frac{\partial (\phi - x)}{\partial x} \, n_1 \, dS
\]
\[
= -U \int \left[ \frac{\partial N_j}{\partial z} \frac{\partial \phi}{\partial z} + \frac{\partial N_j}{\partial x} \frac{\partial (\phi - x)}{\partial x} \right] n_1 \, dS
\]
\[
= -\int \overline{W} V_N n_1 \, dS \quad (3.62a)
\]

In general, we have

\[
U \int m_j n_1 \, dS = -\int \overline{W} V_N n_1 \, dS \quad (3.62b)
\]

after a similar derivation. The right hand side of this equation contains only the first derivatives of the steady potential, so that the accuracy of the numerical analysis can be improved.

Fortunately, in the present method, the body surface condition on \( \phi_j \) containing the derivatives of \( \phi \) is satisfied in an integral form (3.61), so we do not need to worry about the accuracy of \( \phi \) itself as discussed at the beginning of this section. Table 2 gives results of \( u_{11} \) and \( u_{33} \) from a mesh of 12 elements at \( F_n = \frac{U}{\sqrt{ga}} = 0.4 \). It can be seen that the error in the analytical solution obtained from equation (3.56a) in the whole frequency region we have calculated is about 1% or less. This gives strong evidence to guarantee that the problem of a body moving in waves can be solved by the coupled finite element method.
Table 2
The comparison of added mass of circular cylinder in the unbounded fluid domain

<table>
<thead>
<tr>
<th>( v a )</th>
<th>( \frac{u_{11}}{\rho \pi a^2} )</th>
<th>( \frac{u_{33}}{\rho \pi a^2} )</th>
<th>( \text{analytic} )</th>
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<td>0.2</td>
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<td>2.0375</td>
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</tr>
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<tr>
<td>1.0</td>
<td>1.3174</td>
<td>1.3095</td>
<td>1.3200</td>
</tr>
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</table>

\[ v = \frac{\omega^2}{g}, \quad F_n = \frac{U}{|ga|} = 0.4 \]

3.3.2. Green function

The Green function in the present problem of an oscillating cylinder at forward speed may be obtained by traditional Fourier techniques. We write equation (2.82) in a slightly different form

\[ G = \ln r - \ln r_1 + H(x, z, a, c) \quad (3.63) \]

for infinite water depth. The second term in this equation is physically the potential due to a sink located at the mirror image position \( Q_1(a, -c) \) of the source point \( Q(a, c) \) about the free surface. Since\(^{61}\)

\[ \ln r = \int_{0}^{\frac{e^{-m} - e^{-m|z-c|\cos(m(x-a))}}{m}} \, \text{dm} \quad (3.64) \]

\[ \ln r_1 = \int_{0}^{\frac{e^{-m} - e^{-m(z+c)\cos(m(x-a))}}{m}} \, \text{dm} \quad (3.65) \]

we may write \( H \) in the form
\[ H = \int_{-\infty}^{\infty} e^{n(z+c)} [A(m)\cos m(x-a) + B(m)\sin m(x-a)] \, dm \quad (3.66) \]

Imposing the free surface condition on \( G \), we obtain

\[
\int_{-\infty}^{\infty} \left[ e^{n} \left( -\frac{n^2}{\nu} m^2 A(m) - 2\imath m B(m) + m^2 A(m) - \nu A(m) + 2 \right) \cos m(x-a) \right] \, dm = 0 \quad (3.67)
\]

This equation can be satisfied by taking the coefficients of sine and cosine function as zero. Thus

\[
\left( -\frac{n^2}{\nu} m^2 + m-\nu \right) A(m) - 2\imath m B(m) = -2 \quad (3.68a)
\]

\[
2\imath m A(m) + \left( -\frac{n^2}{\nu} m^2 + m-\nu \right) B(m) = 0 \quad (3.68b)
\]

from which we obtain

\[
A(m) = \frac{-2\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)}{\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)^2 - 4\tau^2 m^2 \nu^2} \quad (3.69a)
\]

and

\[
B(m) = \frac{4\imath \nu^2 m}{\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)^2 - 4\tau^2 m^2 \nu^2} \quad (3.69b)
\]

For numerical evaluation, we decompose

\[
A(m) = \frac{-2\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)}{\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)^2 + 4\tau^2 m \nu^2} \quad \frac{-\nu}{\left( -\frac{n^2}{\nu} m^2 + \nu m - \nu^2 \right)^2 + 4\tau^2 m \nu^2}
\]

\[
= \left[ \frac{1}{(1-4\tau)^{1/2}} \left( \frac{1}{m-k_1} - \frac{1}{m-k_2} \right) \right] + \left[ \frac{1}{(1+4\tau)^{1/2}} \left( \frac{1}{m-k_3} - \frac{1}{m-k_4} \right) \right] \quad (3.70)
\]

where

\[
k_1, k_2 = \frac{\nu}{2\tau^2} \left( 1 - 2\tau \pm (1-4\tau)^{1/2} \right) \quad (3.71a)
\]
Similarly, we obtain

\[ B(m) = \frac{1}{(1-4\pi)^{1/2}} \left[ \frac{-1}{m-k_1} + \frac{1}{m-k_2} \right] + \frac{1}{(1+4\pi)^{1/2}} \left[ \frac{1}{m-k_3} - \frac{1}{m-k_4} \right] \]  

(3.72)

We notice that both \( A(m) \) and \( B(m) \) are singular at \( m = k_i \) \((i=1,2,3,4)\). The way to deal with these singularities depends on the radiation condition. It is hard to give a specific reason to define the radiation condition for this problem, as discussed in section 2.4. But it has to ensure that this Green function will recover the form used in the problem of harmonic motion only or forward speed only when \( U \rightarrow 0 \) or \( \omega \rightarrow 0 \) respectively. Thus it is assumed that there is a wave at \( x=\pm \infty \) with wave number \( k_2 \) and there are three waves at \( x=-\infty \) with wave numbers \( k_1, k_3, \) and \( k_4 \) respectively. Using the well known relations

\[
\lim_{x \rightarrow \infty} f(x) = \pm \pi f(x_0)
\]  

(3.73a)

\[
\lim_{x \rightarrow \pm \infty} f(x) \cos(x-x_0) = 0
\]  

(3.73b)

when \( x_0 > 0 \), the Green function \( G \) may be written as

\[
G = \ln r - \ln r_1 + \rho v f^m(z+c) [A(m)\cos m(x-a) + B(m)\sin m(x-a)] dm \\
+ \frac{\pi}{(1-4\pi)^{1/2}} \left[ e^{-k_1(z+c)} \sin k_1(x-a) + e^{-k_2(z+c)} \sin k_2(x-a) \right] \\
+ \frac{\pi}{(1+4\pi)^{1/2}} \left[ e^{-k_3(z+c)} \sin k_3(x-a) - e^{-k_4(z+c)} \sin k_4(x-a) \right] \\
+ \frac{i\pi}{(1-4\pi)^{1/2}} \left[ e^{k_1(z+c)} \cos k_1(x-a) + e^{k_2(z+c)} \cos k_2(x-a) \right]
\]
Haskind used a different procedure of derivation and obtained the Green function in a double complex form about space and time. It can be found that the present form is equivalent to the real part about space of Haskind's form. When \( \omega = 0 \), we have \( k_1 = k_3 = \nu \) and \( k_2 = k_4 = 0 \), and the Green function (3.74) becomes

\[
G = \ln r - \ln r_1 + 2 \pi \nu e^{\mu(z+c)} \cos \omega(x-a) \left( \frac{1}{m-\mu} - \frac{1}{m} \right) \, dm \\
+ 2 \pi \nu e^{\mu(z+c)} \sin \nu(x-a) + \text{const} \tag{3.75}
\]

Substituting equation (3.65) into equation (3.75), equation (3.41) will be recovered apart from a constant. Similarly, when \( \nu = 0 \) we find the result of harmonic motion only will be recovered.

From equation (3.71a), it can be seen that \( k_1 \) and \( k_2 \) are real only when \( \tau > 1/4 \). In other words, \( A(m) \) and \( B(m) \) in equation (3.69) have only two singularities at \( m=k_3, k_4 \) when \( \tau > 1/4 \) and have the form

\[
A(m) = \frac{\nu}{\tau - v(1-2\tau) m + v^2} + \frac{1}{(1+4\tau)^{1/2}} \left[ \frac{1}{m-k_3} - \frac{1}{m-k_4} \right] \tag{3.76a}
\]
\[
B(m) = \frac{-1/v}{\tau - v(1-2\tau) m + v^2} + \frac{1}{(1+4\tau)^{1/2}} \left[ \frac{1}{m-k_3} - \frac{1}{m-k_4} \right] \tag{3.76b}
\]

Correspondingly, the terms containing \( k_1 \) and \( k_2 \) in equation (3.74) should be deleted and the Green function can be written as

\[
G = \ln r - \ln r_1 + \pi \nu e^{\mu(z+c)} (A(m) \cos \omega(x-a) + B(m) \sin \omega(x-a)) \, dm \\
+ \frac{i\pi}{(1+4\tau)^{1/2}} e^{\mu(z+c)} \cos k_3(x-a) + e^{\mu(z+c)} \cos k_4(x-a) \tag{3.74}
\]
associated with the definition of \( A(m) \) and \( B(m) \) in equation (3.76). The physical interpretation of these mathematical results when \( \tau > 1/4 \) is that there is no wave far in front of the body and there are two waves with wave number \( k_3 \) and \( k_4 \) far behind the body.

Comparing equation (3.74) and (3.41), it can be found there is more similarity than difference between them. Thus we may borrow the numerical method from there to evaluate this Green function. To deal with the singularities at \( m=k_1 \), we may write the principal integration in (3.74) as

\[
G_{pv} = \sum_{i=1}^{4} G_{pv_i}
\]

where

\[
G_{pv_i} = \int_{0}^{2k_1} \frac{g_1(m) - g_1(k_1)}{m-k_1} dm + \int_{2k_1}^{\infty} \frac{g_1(m) - g_1(k_1)}{m-k_1} dm
\]

\[
+ \int_{0}^{2k_1} \frac{h_1(m) - h_1(k_1)}{m-k_1} dm + \int_{2k_1}^{\infty} \frac{h_1(m) - h_1(k_1)}{m-k_1} dm
\]

\[
g_1(m) = (-1)^{i+1} \frac{e^{m(z+c)}}{[1 + (-1)^{\alpha} 4\tau]^{1/2}} \sin m(x-a)
\]

\[
h_1(m) = (-1)^{\beta} \frac{e^{m(z+c)}}{[1 + (-1)^{\alpha} 4\tau]^{1/2}} \sin m(x-a)
\]

\[
\alpha = \frac{i(i+1)}{2}
\]

\[
\beta = \frac{i(i-1)}{2}
\]

Following the procedure discussed in section 3.2.3, the evaluation of
these equations and the integration of the Green function over the boundary \( S_j \) are straightforward.

However, attention should be paid to the critical point \( \tau = 1/4 \). For this particular case, the principal integration of the Green function becomes

\[
G_{pv} = \frac{\omega m[(z+c) - i(x-a)]}{0} \frac{1}{(\frac{1}{4}m - \nu)^2}dm + G_{pv_3} + G_{pv_4}
\]  
(3.80)

Because of the second order singularity at \( m = 4\nu \) in the first term in this equation, \( G_{pv} \) will tend to infinity. This could yield unstable numerical results when \( \tau \) is close to 0.25.

3.3.3. Discussion of the results

We first consider the problem of a circular cylinder submerged below the free surface, which was solved by Grue and Palm\(^{43}\) using the source distribution method. For this particular case they were able to write the source distribution \( \gamma(\theta) \) over the cylinder in Fourier series form

\[
\gamma(\theta) = \sum_{n=1}^{\infty} \left( C_n e^{in\theta} + C_n^* e^{-in\theta} \right)
\]  
(3.81)

with * denoting the complex conjugate. As a result the integral equation for \( \gamma(\theta) \) can be transformed into two infinite sets of linear uncoupled equations for \( C_n \). Table 3 to Table 5 give the comparison of the present damping results obtained from a similar mesh to that discussed in 3.2 with Grue and Palm's results\(^{42}\) from the truncated series after ten terms in equation (3.81). The nondimensional damping force is defined as \( \frac{\omega \lambda_{11}}{\rho g} \).
It can be seen in Table 3 that the present results converge very rapidly. Twelve elements can provide sufficient accuracy and so this number is used for obtaining Table 4 and Table 5. The comparison with analytic solution by Grue and Palm shows that the present numerical method gives very good results. It is found that the present numerical results become unstable when \( \tau \) is too close to the critical value of 0.25. However this instability is limited to a very small range of \( \nu \) as we can see from these tables.

Table 4 shows an important feature of the influence of forward speed. The damping force in some frequency region can be negative. This can be explained by noting that the \( k_3 \) wave has a negative wave energy flux, as discussed by Grue and Palm. Thus when the \( k_3 \) wave becomes dominant the damping force can be negative.

Fig. 10 and Fig. 11 show the results of the forward speed influence on the hydrodynamical coefficients of a circular cylinder. \( v_0 \) is defined so that \( F = \frac{1}{2} \). It can be found that for each given Froude number the results jump sharply in the region near the critical point; but the general feature is that the most significant influence of forward speed is in the region of low frequency and this influence seems to decrease with increase of the frequency (except in the region near the critical point). Theoretically, when the frequency becomes very low for a given
forward speed, the first term in equation (3.8) can be neglected so that we have the body surface condition on \( \phi_j \) as

\[
\frac{\partial \phi_j}{\partial n} = U m_j
\]  
(3.82)

Similarly the terms containing \( \omega \) in equation (3.7) may be also neglected, so the free surface condition becomes

\[
\phi_{jz} + u \phi_{jxx} = 0
\]  
(3.83)

From these two equations and the radiation condition on \( \phi_j \) as \( \omega \to 0 \), we may obtain the solutions

\[
\phi_1 = -u \frac{\partial \phi}{\partial x}
\]  
(3.84a)

\[
\phi_3 = -u \frac{\partial \phi}{\partial z}
\]  
(3.84b)

Therefore the solutions \( \phi_1, \phi_3 \) will be dominated by the steady motion of the body at low frequency.

Correspondingly, the terms containing forward speed in equations (3.7) and (3.8) can be neglected at high frequency, so that

\[
\phi_{jz} + v \phi_j = 0
\]  
(3.85)

on the free surface; and

\[
\frac{\partial \phi_j}{\partial n} = i \omega n_j
\]  
(3.86)

on the body surface. It can be seen that these conditions are identical to those in the harmonic motion problem. But the solutions will not be the same since the radiation conditions are different. For the present problem, when the frequency becomes so high that the fluid flow becomes supercritical \( (\gamma \frac{1}{a}) \), there will be two waves far behind the body but there is no wave far before the body; for the harmonic motion without
forward speed, there will be always a wave far before the body and a wave far behind the body. This yields different solutions for the two problems. However, when \( \omega \to \infty \), these waves in both cases will progressively disappear, so that the asymptotical behaviour of the two potentials as \( \omega \to \infty \) is similar. In other words, the radiation potential will not be affected by forward speed as \( \omega \to \infty \).

It can also be observed from the results shown in Fig.10 and Fig.11 that the Timman and Newman relation \( \tau_{13} = -\tau_{31} \) is satisfied when \( F_n = \frac{U}{\sqrt{2gA}} \leq 0.4 \), but the difference between \( \mu_{13} \) and \( -\mu_{31} \) becomes significant after that, which is caused by the violation of condition (2.60). This condition requires either \( \nabla_\sigma (-U) = \nabla_\sigma (+U) \) or \( ||\nabla_\sigma (\pm U)|| < \epsilon U \), but it is hard to conclude that one of these two equations will be always satisfied for arbitrary bodies at any Froude numbers. Therefore the Timman and Newman relation is not a general relation in the linearized potential problem. Its application will be strongly affected by condition (2.60). But it is not intended here to deny the significance of this relation in strip theory. Since the pre-condition of the validity of a strip theory is that the body must be slender and the longitudinal disturbance is of small order, the condition (2.60) will be satisfied automatically. This shows why the Timman and Newman relation is a touchstone of strip theories.
Table 3a. Convergence and comparison of damping force of surge on a submerged circular cylinder

\[ h = 2a \quad F_n = \frac{u}{\sqrt{g a}} = 0.4 \quad v_e = 0.3906 \]

<table>
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<th>( v_a )</th>
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<th>20</th>
<th>24</th>
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<td>0.0085</td>
<td>0.0085</td>
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</tr>
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<td>1.0041</td>
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<td>1.0005</td>
</tr>
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Table 3b. Convergence and comparison of damping force of heave on a submerged circular cylinder

\[ h = 2a, \quad F_n = \frac{U}{\sqrt{g}a} = 0.4, \quad v_c = 0.3906 \]

<table>
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<tr>
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Table 4. Comparison of damping force on a submerged circular cylinder

\[ h=2a \quad F_n = \frac{U}{\sqrt{g}a} = 1.0 \quad \nu_c=0.0625 \]

<table>
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<th>( \nu_a ) Grue &amp; Palm</th>
<th>surge present</th>
<th>Grue &amp; Palm</th>
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Table 5. Comparison of damping force on a submerged circular cylinder

\[ h = 3a \quad Fn = \frac{U}{\sqrt{g}a} = 0.4 \quad \nu_c a = 0.3906 \]

<table>
<thead>
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<th>heave (present)</th>
<th>heave (Grue &amp; Palm)</th>
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3.4. Discussion of some possible extensions

In the previous section the finite coupled element method has been used to solve the hydrodynamic problem of an oscillating cylinder advancing beneath the free surface. The results from the test cases are generally in good agreement with those obtained by others using different means. The influence of forward speed on hydrodynamic coefficients has been investigated at the two dimensional level. This may be regarded as a step towards to fully understanding the mechanism of the forward speed influence on the ship response to waves. The success in the previous sections further shows that the present method is one of
the most promising methods in ship hydrodynamics, and it also provides a basis for the solution of the more general three dimensional problem.

However since the results obtained above are from the linearized potential theory, their practical application and the conclusions drawn from them are subject to the validity of the linearized potential theory. While this theoretical model is a very successful representation for many practical problems, it is still necessary to develop a more general and accurate model to overcome its limitations. Therefore before we move to three dimensional analysis, we will discuss some more general problems and the potential applications of the coupled finite element method to these problems.

3.4.1. The effect of second order non-linearity at the free surface on the steady potential

It has been well known that the linearized potential does not provide satisfactory results of resistance for practical surface ships. There are many factors causing the discrepancy. One of them is the assumption of linearized potential theory on the free surface which unrealistically requires that the ship is either thin or deeply submerged. To remove this assumption, there have been many studies using a non-linear free surface condition. Two different procedures are usually used to find the corresponding solution. The first one uses iteration method to satisfy the exact free surface conditions (2.14) and (2.15)\(^3\). But here we will concentrate our discussion on the second procedure using the perturbation method. The essential spirit of this method is to write the potential as a polynomial series of a perturbation parameter \(\epsilon\).
\[ \phi = \phi_0 + \phi_1 + \phi_2 + \ldots + \phi_n + \ldots \]
\[ = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots + \varepsilon^n \psi_n + \ldots \quad (3.87) \]

Usually it is found that it is convenient to define \( \varepsilon \) so that it is small \( (\varepsilon \ll 1) \) and nondimensional. If a finite positive number \( M \) can be found so that \( |\psi_n| \leq M \), the series will converge at the speed \( \varepsilon \). The governing equation for \( \psi_n \) usually has terms containing \( \psi_i \) \( (i=1,2,\ldots,n-1) \). Thus the solution of \( \phi \) will be found by solving \( \psi_n \) one by one, until the terms afterwards are no more significant.

In the present problem, \( \varepsilon \) can be defined as the ratio of the amplitude of wave generated by forward speed to its length. In a similar manner to equation (3.87), the wave elevation \( \eta \) may be written as

\[ \eta = \eta_1 + \eta_2 + \ldots + \eta_n + \ldots \]
\[ = \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \ldots + \varepsilon^n \xi_n + \ldots \quad (3.88) \]

Substituting this and equation (3.87) into equations (2.14) and (2.15), rearranging the results systematically in the order of \( \varepsilon^0 \), we may obtain the free surface condition for \( \phi_1 \) and \( \phi_2 \) as

\[ U^2 \frac{\partial^2 \phi_1}{\partial x^2} + g \frac{\partial \phi_1}{\partial z} = 0 \quad (3.89a) \]

on \( z=0 \), with

\[ \eta_1 = \frac{U}{g} \frac{\partial \phi_1}{\partial x} \quad (3.89b) \]

and

\[ U^2 \frac{\partial^2 \phi_2}{\partial x^2} + g \frac{\partial \phi_2}{\partial z} = 2U \nabla \phi_1 \nabla \phi_1 x - \frac{U}{g} \nabla \phi_1 x \frac{\partial}{\partial z} (U^2 \phi_1 xx + g \phi_1 z) \quad (3.90a) \]

on \( z=0 \), with
while \( \phi_0 \) is identical to zero. In a similar way there is no difficulty in obtaining the free surface condition on \( \phi_n \) \((n>2)\), but the derivation is rather lengthy and the results are not so widely used as the first two for the time being. Therefore they will not be given here. Corresponding to equations (3.89) and (3.90), the body surface condition can be obtained as

\[
\frac{\partial \phi_1}{\partial n} = U_n \quad (3.91)
\]

and

\[
\frac{\partial \phi_2}{\partial n} = 0 \quad (3.92)
\]

Therefore \( \phi_1 \) here is identical to the linearized steady potential which has been solved in section (3.1); but \( \phi_2 \) is something new which is a correction towards a more accurate mathematical model. To find its solution by the coupled finite element method we substitute equation (3.92) into equation (2.62). The Galerkin form of the differential equation of \( \phi_2 \) in the near field \( R_1 \) then can be simplified to

\[
\int_{R_1} \int \nabla \phi_2 \cdot \nabla \psi d\sigma - \oint_{S_j} \frac{\partial \phi_1}{\partial n} \psi dS = 0 \quad (3.93)
\]

in the two dimensional case. But the integral equation in the far field \( R_2 \) will become more complicated due to the nonhomogeneous free surface condition on \( \phi_2 \) and the uncertainty of its radiation condition at infinity. In fact because of that one cannot jump from equation (2.65) to (2.66), if the same Green function as that in section 3.2 is used. Instead when the boundary conditions on \( \phi_2 \) and \( G \) are invoked, equation (2.65) becomes
\[
\phi_2(p) = \frac{1}{2\pi} \int_{\mathcal{S}_j} [\phi_2(Q) \frac{\partial G(Q)}{\partial n} - \frac{\partial^2 \phi_2(Q)}{\partial n^2} G(Q)] dS
\]

\[
- \frac{1}{2\pi \mu} \int_{\mathcal{S}_f} [\phi_2(Q) \frac{\partial^2 G(Q)}{\partial x^2} - (\frac{\partial \phi_2(Q)}{\partial x})^2 + \frac{\mu}{g \rho_1} G(Q)] dS
\]

\[
+ \frac{1}{2\pi} \int_{\mathcal{S}_\infty} [\phi_2(Q) \frac{\partial G(Q)}{\partial n} - G(Q) \frac{\partial^2 \phi_2(Q)}{\partial n^2}] dS
\]

\[
= \frac{1}{2\pi} \int_{\mathcal{S}_j} [\phi_2(Q) \frac{\partial G(Q)}{\partial n} - G(Q) \frac{\partial^2 \phi_2(Q)}{\partial n^2}]
\]

\[
+ \frac{1}{2\pi g} \int_{\mathcal{S}_f} G(Q) f_1 dS - \frac{1}{2\pi \mu} \int_{\mathcal{S}_f} [\frac{\partial G(Q)}{\partial x} \phi_2(Q) - G(Q) \frac{\partial \phi_2(Q)}{\partial x}]_{-\infty}^{+\infty}
\]

\[
+ \frac{1}{2\pi} \int_{\mathcal{S}_\infty} [\frac{\partial G(Q)}{\partial n} \phi(Q) \phi_2(Q) - \frac{\partial^2 \phi_2(Q)}{\partial n^2} G(Q)] dS
\] (3.94)

where \(\mu = \frac{g}{y^2}\) and \(f_1\) denotes the right hand side of equation (3.90a). The first term in this equation is the familiar form used in the linearized potential problem and the second term can be regarded as a constant after \(\phi_1\) has been found. But the difficult thing to deal with in this equation is the third and fourth terms which are contributions from infinity. Since there has not been a generally accepted explicit radiation condition for the nonlinear potential, the role of these two terms in the whole problem remains unknown. However careful analysis may suggest that \(f_1\) will tend to zero as \(|x|^\infty\), which yields that \(\phi_2\) satisfies the same free surface condition at infinity as \(\phi_1\). This may provide heuristic evidence that \(\phi_2\) has a similar asymptotical behaviour to \(\phi_1\) at
infinity, so that we may be able to assume that \( \phi_2 \) satisfies the same radiation condition as \( \phi_1 \). Then following the discussion by Ursell\(^{109}\) the third and fourth terms may be disregarded as in the formulation of \( \phi_1 \) by Eatock Taylor and Wu\(^{27}\). Thus we have

\[
\phi_2 (P) = \frac{1}{2\pi} \int_{S_j} \left[ \phi_2 \frac{\partial G}{\partial n} - \frac{\partial \phi_2}{\partial n} G \right] dS + \frac{1}{2\pi g_{-\infty}} \int_1^{+\infty} Gf_1 dx
\]  

(3.95)

The physical interpretation of this equation is that the nonlinearity of the free surface condition is localized and it will die out at infinity. This sounds reasonable but may not be necessarily true in the two dimensional case. Since the amplitude and length of the wave generated by forward speed of the body are constants for the given Froude number at infinity based on the present potential theory, there is no reason why the former is always far smaller than the latter so that the wave can be regarded as infinitesimal. Thus it is not our intention to conclude here that this radiation condition is correct, and this problem appears to need further investigation. But if we temporarily accept this condition, we may substitute equation (2.67) into (3.95) so that we have

\[
\phi_1 (P) = \frac{1}{\alpha} \int_{S_j} \frac{\partial \phi_2}{\partial n} dS + \frac{1}{\alpha g_{-\infty}} \int_1^{+\infty} Gf_1 dx
\]  

(3.96)

where \( P \) is located on \( S_j \) and \( \alpha \) is the subtended angle at \( P \). Combining this equation with equation (3.92), the solution of \( \phi_2 \) can be found by the coupled finite element method. The essential task in this process will be finding \( \phi_1 \) and its derivatives on the free surface, so that the second term in equation (3.95) can be evaluated. In principle this is not difficult. We may use equation (2.65) to find \( \phi_1 \) on the free surface if its value on the boundary \( S_j \) has been obtained. By appropriately
differentiating both sides of equation (2.65), the derivatives of $\phi_1$ on the free surface can be also found. However numerically it is not so straightforward, one has to ensure that the accuracy of $\phi_1$ and its derivatives on the free surface by this process is guaranteed, and the infinite integration should be appropriately truncated and oscillatory behaviour must be carefully dealt with.

Some theoretical background to $\phi_2$ has been investigated by Tuck\textsuperscript{104} considering a submerged circular cylinder. Extending Wehausen and Laitone's scheme\textsuperscript{113} for the linear potential, he found that the effect of non-linearity at the free surface on the force on the circular cylinder is important even when the submergence is twice as much as the radius. From this result, he strongly warned that one should not be too keen on the linearized N-K problem. While one may argue that the example given by him is a non-streamlined body so that the potential is invalid anyway, Salvesen\textsuperscript{114} considered a practical example of a submerged two dimensional hydrofoil. He found that the effect of $\phi_2$ on the force is also important in this case and the results after second order correction are in good agreement with experimental data at low and moderate Froude number. Therefore the solution of $\phi_2$ is of practical importance.

3.4.2. The motion through an arbitrary time history

While the application of the linearized potential is limited by its assumptions in general, the problem we have solved is a further specialized case since we assume that forward speed is constant and the oscillating motion is periodic. This idealized motion is based on the need of mathematical simplification and provides a basis for the Fourier analysis of arbitrary motion. In practice, ship motions in waves are a
random process, and can by no means be simplified as periodic motion. As one approach, the frequency domain method using the Fourier transform technique can be employed to analyse the non-periodic motions, but this method does not give the history of the motion. This history may be very important in many cases. One example is the wave resistance test in a wave tank. After the ship model is towed into motion, one can not start the measurement until the fluid flow becomes stable in the coordinate system fixed on the ship model. Therefore the essential problem is to know how long this process will take, which needs knowledge of the time history of the motion.

Some theoretical aspects of the arbitrary motion have been investigated by Havelock. Using a first approximation method, he obtained the solution of a circular cylinder starting suddenly from rest. From his results, it was discovered that the resistance on the cylinder oscillates with the distance travelled with a period of roughly $4\lambda_0$, where $\lambda_0$ is the wave length far behind the cylinder after the motion becomes stable. It was also discovered that the amplitude of this oscillation is approximately proportional inversely to the square root of the distance travelled; as $t \to \infty$, the resistance will tend to the first approximation solution in steady motion. At the same time Havelock considered the problem of a submerged circular cylinder starting from rest with a uniform acceleration; his results shows that at the same speed the resistance will significantly change with the acceleration, depending in other words on the history of the motion.

After Havelock's work, there have been many theoretical and numerical researches in this field usually called the transient problem. The
development was briefly reviewed by Euvrard\textsuperscript{30}. In these works, it is most interesting to notice the work by Jami\textsuperscript{60} coupling finite elements and an integral representation. Unlike the classic coupled element method, he used the boundary integration over the body surface to represent the the potential on the localized finite element boundary as denoted as $S_j$ in the present work, so that the boundary condition on $S_j$ becomes of Dirichlet type. The advantage of this method as discussed by Euvrard\textsuperscript{30} is that the integral representation is not singular (since there are not any common points on $S_0$ and $S_j$) and can be computed via Gauss quadratures, thus allowing finite element discretization of high order of accuracy. But in the author's view, this representation will still suffer the difficulties of the conventional source distribution method over the body surface. Because of the oscillatory behaviour of the integral part of the Green function, it is not easy to evaluate its numerical integration over a body surface of high irregularity. Using the present coupled element method, however, this difficulty can be avoided since the appropriate boundary $S_j$ can be chosen as discussed before. The singularity in evaluating the integration of $1/\rho$ (or $1/r$ in the three dimensional case) and its derivatives over the boundary $S_j$ will not appear in numerical analysis since the appropriate choice of this boundary enables this integration to be calculated explicitly.

Therefore, if the initial value of the potential and wave elevation is zero and there is no pressure on the free surface or the pressure is constant, the integral representation in the far field of the linearized potential through an arbitrary time history may be written as\textsuperscript{34,113}

\[
\phi_2 = \frac{1}{2\pi} \int_0^t d\tau \int_{S_j} [G\phi_{2\tau} - G_n\phi_{2\tau}] dS
\]  

(3.97)
for the two dimensional problem, while a similar equation to (2.71) can be used in the near field. The Green function used in this equation can be defined as the solution of the following equations

\[
\nabla^2_{x_0, z_0} G(x_0, z_0, a, b, t, \tau) = \delta(x_0-a)(z_0-c) \tag{3.98a}
\]

in whole fluid domain;

\[
G_{tt} + G_z = 0 \tag{3.98b}
\]

on the free surface;

\[
G = 0 \tag{3.98c}
\]

on the bottom of the fluid; and the initial condition

\[
G = G_t = 0 \tag{3.98d}
\]

for \( t=\tau \) and \( z_0=0 \). The solution has been obtained as

\[
G = \ln r_0 - \ln r_0 + \frac{4\pi f e^{-md}}{mcoshmd} \sinh m_z \sinh m_c \cosm(x_0-a)dm
\]

\[
+ 2f \frac{\cosm(z_0+c) \cosh(c+d)}{cosh^2 md} \frac{1 - \cos[m(tanhmd)^{(1/2)}(t-\tau)]}{mtanhmd} \cosm(x_0-a)dm \tag{3.99}
\]

in finite water depth of \( d \); and

\[
G = \ln r_0 - \ln r_0 - \frac{m(z_0+c)}{m} \frac{1 - \cos[m(t-\tau)]}{m} \cosm(x_0-a)dm \tag{3.100}
\]

in infinite water depth, where

\[
r_0 = \sqrt{[x_0-a]^2 + [z_0-c]^2} \tag{3.101a}
\]

\[
r_{01} = \sqrt{[x_0-a]^2 + [z_0+c]^2} \tag{3.101b}
\]
It can be seen from equation (3.97) that the solution for the potential at time $t$ depends on the results in $0 \leq t < \tau$. The contribution is in a form of continuous integration which is usually discretized using a finite difference scheme in numerical analysis. Thus, in principle we can find the potential from equation (3.97) and a corresponding finite element representation in the near field at each time step. However, numerically this process is much more complicated than for periodic motion with constant forward speed. One of the difficulties is the stability of solution when $t$ becomes large. Since the numerical error at each step will be summed to the next step, one has to ensure that the final error is within reasonable limit.
4. ANALYSIS OF THE THREE DIMENSIONAL PROBLEM

4.1. General

Having confirmed the justification of the present numerical method at the two dimensional level, we can be confident in trying to analyze the three dimensional problem. While the two dimensional theory provides the means to calculate the problem associated with elongated bodies, the three dimensional theory has no particular requirement for the relative order of body dimensions in different directions. It can be used for an arbitrary body provided that the linearized potential theory is valid.

The three dimensional theory is of particular importance in the calculation of the ship wave resistance. For practical ships, forward speed is usually in the longitudinal direction. The wave resistance on the ship due to forward speed is dominated by the pressure near the bow and stern. The flow in these areas is always three dimensional. There is no justification that the longitudinal flow can be neglected so that a two dimensional method like the strip theory can be used.

The extension of the numerical procedure in the previous chapter to the three dimensional problem is straightforward in principle. The potential flow in the near field is expressed by the twenty points three dimensional quadratic shape function. The localized finite element boundary is chosen as single planes. Each plane is perpendicular to one of the coordinate axes. On these planes, plane quadratic rectangular elements are used. This kind of mesh was previously used by Zietsman for harmonic motion problem, and the details have been discussed. The most difficult aspect of the present analysis is that the Green func-
tions in both steady and unsteady potential problems are of very compli-
cated form. Much effort is needed to obtain stable and converged
results.

The material in this chapter is organized in a similar way to the
previous chapter. In addition a detailed discussion of resistance on the
ship and a review of the work on the wave resistance is given at the
beginning of the next section; and discussion of the possible extension
of the present method to various problems, such as added resistance and
elastic deformation, is given at the end of this chapter.

4.2. Steady potential

4.2.1. A review of previous work on wave resistance on the ship

The resistance on the ship can be defined as the force against the
ship advancing at forward speed, which usually refers to the force in
otherwise calm sea. This force changes its value in waves and the
difference from the resistance without incoming waves is called added
resistance. Added resistance is an important subject in ship hydrodynam­
ics, but in this section we mainly discuss the ship resistance in other­
wise calm sea.

The resistance on the ship has a very complicated physical mechan­
ism. In the classic treatment, in the middle of the last century, Froude
divided the total resistance $R_T$ into frictional force $R_F$ and residual
force $R_R$. This division was motivated by the need for similarity in
experiments. The similarity between the real ship and its model
requires that the Froude number $\frac{U}{\sqrt{gL}}$ and Reynolds number $\frac{UL}{\nu}$ of the real
ship are equal to the corresponding numbers $\frac{U_m}{\sqrt{gL_m}}$ and $\frac{U_{mL_m}}{\nu_m}$ of the ship.
model, where $U$ and $L$ are forward speed and length of the ship respectively, $\nu$ is the viscosity coefficient of the sea water, and the subscript $m$ indicates those corresponding to the ship model. Since $\nu$ is approximately equal to $\nu_m$ and the gravitational acceleration $g$ is regarded as constant in ship hydrodynamics, the requirement that Froude number and Reynolds number of the ship model are equal to the corresponding numbers of the real ship is virtually impossible to meet. Therefore Froude assumed that the nondimensional residual resistance coefficients $C_R = \frac{1}{2} \rho U^2 S$ depend on Froude number only, where $\rho$ is the density of the fluid and $S$ is the wetted area of the ship, and the frictional resistance on the ship is equal to that on a flat plate of the same wetted area, length and forward speed. In a model test, the Froude number of the ship model equals that of the real ship. Having measured the total force and calculated the frictional force using an appropriate equation, the residual resistance coefficient can be obtained which equals that of the real ship. Using the same equation to calculate the frictional force on the real ship, the total resistance can be obtained (because of the different flow state inside the boundary layer between a real ship and its model, some artificial stimulation is needed in the experiment to disturb the laminar boundary layer so that a turbulent boundary layer is generated).

Froude's assumption laid solid foundations for future model testing. Although history has changed his original separation of the total resistance in many ways, his assumption remains as an essential principle in the wave tank. However Froude's assumption has an important disadvantage that it does not allow the mathematical calculation of the residual resistance which contains the contribution of the viscosity of
the fluid. But in some cases specially for a streamlined body it has been found that the residual resistance is dominated by the wave resistance, which is the force on the ship assuming that the fluid is inviscid and can be obtained by the potential theory.

The calculation of the wave resistance is one of the main subjects in ship hydrodynamics. It has been observed that the resistance due to viscosity is not very sensitive to the shape of ship hull for a given ship length and wetted area, except some considerations are needed in design to avoid or to postpone the separation of the fluid from the ship surface. However the wave resistance is extremely sensitive to the shape of ship hull. Thus the reduction of wave resistance is one of the main concerns in ship hull design.

The first mathematical calculation of wave resistance was by Michell\textsuperscript{72}. For mathematical simplification, he assumed that the ship is thin so that its width is far smaller than its length and draught. The potential is obtained from a source distribution over the centerplane of the ship, which satisfies the linearized free surface condition and radiation condition but does not satisfy the body surface condition. Although Michell theory never gives satisfactory quantitative results of the wave resistance, the curves of the results from his theory against Froude number exhibit humps and hollows resembling experimental curves of residual resistance. This usually gives some qualitative guidance in ship design. Because of that, specially because of the simplicity of Michell's theory, there has been always a temptation to develop and modify this original thin ship theory. The most significant work was by Guilloton\textsuperscript{47} suggesting a transformation of coordinates along isobars.
This transformation yields an "equivalent linearized ship hull", whose lateral $y_e$ offsets at $x=Ut$ are equal to the real ship hull $y$ offsets at $x=ut$, where $U$ is ship forward speed and $u$ is the local $x$ component velocity on the appropriate isobar along the hull. The details may be found in references\textsuperscript{37,46}.

Another approach to calculate the wave resistance is to attempt to satisfy the ship surface boundary condition more accurately, while retaining the linearized free surface condition. The attempt was initiated by Havelock. Some of his two dimensional work satisfying the exact body surface condition has been mentioned in the previous chapter. Although he did not obtain a solution satisfying the exact body surface condition in the three dimensional problem, his results of approximate solutions for submerged slender spheroids\textsuperscript{52} and ellipsoids\textsuperscript{53} are very accurate indeed. The solution satisfying the exact body surface boundary condition of the spheroid was obtained much later by Farell\textsuperscript{33} by expressing the source distribution on the body surface as a series of Legendre functions, or spherical harmonics.

To satisfy the body surface condition of the surface ship exactly, and retain the linearized free surface condition at the same time (i.e. the three dimensional Neumann-Kelvin (N-K) problem), Brard's work\textsuperscript{14} provided a milestone. He pointed out that the potential flow of a surface ship can not be equivalently represented by the source distribution over the body surface alone; an extra distribution along the waterline of the ship is needed. He demonstrated the importance of this line integral by considering the asymptotical case as forward speed tends to zero. His results for the elliptical cylinder showed that there is significant
difference between the solutions from source distribution and dipole
distribution over the body surface only, but the inclusion of the line
distribution will eliminate this difference. Like its importance how­
ever, the difficulty of evaluation of the waterline integration has been
widely recognized. Thus many traditional ways of exclusion of the
waterline integral survive and still receive much attention\textsuperscript{17,62,73,92}. However, the inclusion of the waterline integration in N-K problem is
quite likely to give better behaved numerical results, as discussed by
Gadd\textsuperscript{37}. The inclusion of the waterline integration analysis was
achieved by Tsutsumi\textsuperscript{103}. Bai\textsuperscript{4} used the localized finite element method
to solve the problem of a ship advancing in a canal, so that the line
integration was calculated along the waterline of the localized finite
element boundary.

For the N-K problem of a floating body, inclusion of the waterline
integral will give the exact solution. However, the N-K problem is not a
consistent mathematical model in general, since the body surface condi­
tion and free surface condition are imposed at different orders. The
importance of the nonlinearity of the free surface has been discussed at
the two dimensional level in the previous chapter. In the three dimen­
sional case, experimental data shows that the linearized potential does
not provide satisfactorily quantitative results for most practical
ships. This suggests that a more accurate mathematical model is needed,
but it does not deny the significance of the solution of the linearized
potential. The solution of the N-K problem is the first step towards a
more accurate mathematical model by perturbation analysis, and as argued
by Guevel, Delhommeau and Cordonnier\textsuperscript{45}, if a numerical method does not
succeed in the linearized potential problem it is hard to believe that
it will succeed in higher order potential problems.

As in the two dimensional problem, the methods of iteration and perturbation are essential methods to calculate the nonlinear wave resistance based on the three dimensional mathematical model. These methods are widely used in slender-ship theory and slow-ship theory. As proposed by Noblesse, the zeroth order potential \( \phi^{(0)} \) can be taken as zero for the slender ship. The boundary integral equation (similar to equation (2.65) after \( S_j \) is replaced by \( S_0 \)) then contains only \( \frac{\partial \phi}{\partial n} \) on the right hand side which is known from the boundary conditions. Thus the first order potential \( \phi^{(1)} \) on the left hand side may be obtained. Substituting \( \phi^{(1)} \) into the right hand side in the same equation, one may get the second order potential \( \phi^{(2)} \). This procedure can be repeated until satisfactory results are obtained. Noblesse proved later for the particular problem of an elliptical cylinder at low Froude number that the result from this iteration process converges to the solution of the slow-ship theory.

The slow-ship theory takes the double body potential as the zeroth order solution. The next order solution is a free surface layer potential which is obtained by imposing the free surface condition on the sum of these two potentials. The product terms of the double body potential are retained in the free surface condition but the product terms of the free surface layer potential are usually neglected.

There are many other methods to calculate the wave resistance such as ray theory and some special methods dealing with the flow around the bow and stern of ships. It is inappropriate to discuss all these theories in detail in this thesis. Some of earlier works have been
reviewed by Wehausen\textsuperscript{112}. More recent development may be found in ITTC (International Towing Tank Conference) proceedings and the proceedings of Workshop on Ship Wave-Resistance Computations. However despite almost one century's effort since Michell's pioneering work, the theoretical calculation still can not provide quantitatively satisfactory results for the wave resistance compared with experimental data for many practical ships. While further research is needed, some fundamental problems have to be kept in mind. The wave resistance is an artificial separation from the total resistance, required for mathematical calculation. It is virtually impossible to measure this resistance. The traditional methods of measurement are based on either momentum consideration or wave-pattern analysis\textsuperscript{112}. The first method measures the velocity distribution on a control plane behind a ship. The results will be affected by the viscosity and breaking wave, and the size of the plane is limited by the wave tank. The second method measures the elevation along an appropriate line. The results will be affected by the wall of the wave tank. Most ironically, the calculation of the wave resistance from the measured wave elevation is usually based on the relation obtained from the linearized potential theory and the use of this result to check the resistance from the potential theory.

Although the experimental data corresponding to this idealized wave resistance is not obtainable which makes it difficult to confirm the mathematical theory, it does not deny the significance of theoretical calculation of wave resistance. Experience has shown that if the wave resistance of a ship under certain conditions is reduced by the theoretical method, the total force on the ship will also be reduced. Thus the important thing for theoretical research is to improve the
mathematical model such as to consider the ship resistance as a whole so that the physical problem can be better represented.

This work, however, is one step towards that goal. The purpose of this part of the thesis is to provide a more efficient method to solve the existing N-K mathematical model and to try to establish a basis to solve the more sophisticated and accurate mathematical model in the future. Following the discussion above, we will start to solve the wave resistance problem by first introducing the three dimensional Green function for the three dimensional N-K problem.

4.2.2. The Green function

Like the two dimensional problem, the Green function may be obtained by imposing the Laplace equation on \( H(x,y,z,a,b,c) \) and the free surface condition and radiation condition on \( G(x,y,z,a,b,c) \) in equation (2.63). The classical method is to separate the variables using the Fourier transform. The solution may be found as

\[
G = \frac{1}{r} - \frac{1}{r_1} - \frac{4\mu}{\pi} \left( \frac{\pi/2}{\cos[m(x-a)\cos\theta]\cos[m(y-b)\sin\theta]} \right)_{\text{d}m} \frac{e^{m(z+c)}}{m\cos^2\theta - \mu} \]

\[
-4\mu \int e^{\mu(z+c)\sec^2\theta} \sin[\mu(x-a)\sec\theta]\cos[\mu(y-b)\sin\theta\sec^2\theta]\sec^2\theta d\theta
\]

in infinite water depth, where

\[
r = [(x-a)^2 + (y-b)^2 + (z-c)^2]^{1/2}
\]

\[
r_1 = [(x-a)^2 + (y-b)^2 + (z+c)^2]^{1/2}
\]

\[
\mu = \frac{g}{U^2}
\]
Numerically the evaluation of this Green function is much more difficult than that in the two dimensional problem. The position of the singularity and the oscillatory behaviour of the cosine function in the first integral in equation (4.1) depend on the second integral variable $\theta$. This yields that the result of the first integral is very sensitive to the value of $\theta$ so that a very small step of $\theta$ is needed in order to obtain converged results. Thus, this form of Green function is not widely used in practical calculation. However the advantage of this form of the Green function in the present method is that the integrand is a simple function of $x,y,z$. This enables the integration of the Green function over the rectangular boundary of localized finite elements to be calculated explicitly.

To deal with the singularity at $m=\mu \sec^2 \theta$, we write the Green function in a similar manner to the two dimensional case

\[ G = G_1 + G_2 + G_3 \]  

**where**

\[ G_1 = \frac{1}{r} - \frac{1}{r_1} \]  

\[ G_2 = -\frac{4\mu}{\pi} \int_0^\infty \left\{ PV \left[ \int_0^{2m_0} e^{m(z+c)} \frac{\cos[m(x-a)] 1 + t^2 \cos[m(y-b)t] 1 + t^2}{m - m_0} \cos[m(x-a)] 1 + t^2 \cos[m(y-b)t] 1 + t^2} \right] \right\} dm \]  

\[ + \int_0^{2m_0} e^{m(z+c)} \frac{\cos[m(z+c)] 1 + t^2 \cos[m(y-b)t] 1 + t^2}{m - m_0} \]  

\[ G_3 = -\frac{4\mu}{\pi} \int_0^\infty e^{z+c}(1 + t^2) \sin[\mu(x-a)] 1 + t^2 \cos[\mu(y-b)t] 1 + t^2] dt \]
\[ m_0 = \mu(1+\tau^2) \]  \hspace{1cm} (4.4d)

It has been found during the course of this work that the change of variable \( t = \tan \theta \) gives better performance of the numerical evaluation. As the integrands in \( G_2 \) and \( G_3 \) are function of \( x, y, z \) in the form \( e^{az}, \cos bx, \cos cy \), the integration of the Green function over the boundary \( S_j \) can be calculated using similar equations to (3.45) and (3.46).

To avoid the numerical difficulty of evaluation of the Green function in equation (4.1), there are many works using modified forms\(^{18,45,99}\). It seems that the form proposed by Noblesse is the most desirable. One may find the solution as\(^{83}\) (we write it in a slightly different form here)

\[ G = \frac{1}{r} - \mu N(x,y,z) - \mu W(x,y,z) \]  \hspace{1cm} (4.5)

where

\[ N = \frac{1}{r^1} - 2(1 + \frac{-z'}{r'|x'|}) + \frac{2}{\pi} \int_{-1}^{+1} \text{Im}[e^{Z(E_1(Z) + \ln Z + \gamma)]} dt \]  \hspace{1cm} (4.6)

\[ W = -H(x') \int_{-\infty}^{+\infty} [z'\tau + i(x'+y't)] \tau \, dt \]  \hspace{1cm} (4.7)

\[ Z = (z' - 1 - t^2 + y't + 1|x'|)/\sqrt{1-t^2} \]  \hspace{1cm} (4.8)

\[ x' = \mu(x-a) \]  \hspace{1cm} (4.9a)

\[ y' = \mu(y-b) \]  \hspace{1cm} (4.9b)

\[ z' = \mu(z+c) \]  \hspace{1cm} (4.9c)

\[ r' = [x'^2 + y'^2 + z'^2]^{1/2} \]  \hspace{1cm} (4.9d)

\[ \tau = \sqrt{1+t^2} \]  \hspace{1cm} (4.9e)
and \( E_1(Z) \) is the well known complex exponential integral; \( H(x') \) is the Heaviside step function defined as \( H(x') = 0 \) for \( x' < 0 \) and \( H(x') = 1 \) for \( x' > 0 \), \( \gamma = 0.5772156649... \) is Euler's constant.

This Green function is virtually equivalent to that defined in equation (4.1), but as discussed by Noblesse the integrand in \( N \) is well behaved for \(-1 \leq t \leq 1\). In the present numerical method, this advantage is gained at the expense of losing the explicit calculation of the surface integral over the boundary \( S_j \). Since the integrand in \( N \) is no more a simple function of \( x, y, z \), the integration of the Green function over the boundary \( S_j \) needs to be calculated numerically.

Corresponding to equation (4.6), the first order derivatives of \( N \) about \( x, y \) and \( z \) may be obtained as

\[
\text{sgn}(x')N_{x'} = \frac{-1}{r'} \frac{|x'|}{r'} - \frac{2}{r'^2 + |x'|^2} \frac{z'}{r'} \left[ \ln \frac{r'^2 + |x'|^2}{r'^2 + |x'|^2} + \gamma + \frac{|x'|}{r'^2 + |x'|^2} + \left( \frac{-z'}{r'^2 + |x'|^2} \right)^2 \right] \\
+ \frac{2}{\pi} \int_{-1}^{+1} \text{Re}[e^{z E_1(Z)} + \ln Z + \gamma] \sqrt{1-t^2} dt \quad (4.10a)
\]

\[
N_{y'} = \frac{-1}{r'} \frac{y'}{r'} + \frac{2}{r'^2 + |x'|^2} \frac{-z'}{r'} \left[ \ln \frac{r'^2 + |x'|^2}{r'^2 + |x'|^2} + \gamma \right] \\
+ \frac{2}{\pi} \int_{-1}^{+1} \text{Im}[e^{z E_1(Z)} + \ln Z + \gamma] \sqrt{1-t^2} dt \quad (4.10b)
\]

\[
N_{z'} = \frac{-1}{r'} \frac{z'}{r'} + \frac{2}{r'^2 + |x'|^2} \left( \ln \frac{r'^2 + |x'|^2}{r'^2 + |x'|^2} + \frac{z'}{r'} \right) \\
- \frac{2}{3} \left( z + \frac{-z'}{r'^2 + |x'|^2} \right) \left[ 3 - \frac{3}{2} \frac{|x'|}{r'^2 + |x'|^2} + \left( \frac{-z'}{r'^2 + |x'|^2} \right)^2 \right] 
\]
Thus, the essential task to calculate $N$ and its first order derivatives is the evaluation of $e^{Z E_1(Z) + \ln Z}$. This may be obtained by the series:

$$e^{Z E_1(Z) + \ln Z} = -\gamma - \sum_{n=1}^{\infty} \frac{\gamma}{n+1} \left( \frac{Z^n}{n!} \right)$$  \hspace{1cm} (4.11a)

The series is valid for $0 < |Z| < \infty$, but it does not offer a practical computation for large values of $|Z|$. Instead the asymptotical expansion of $E_1(Z)$ may be used. One may find the series as:

$$e^{Z E_1(Z) + \ln Z} = \ln Z + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma n!}{Z^{n+1}}$$  \hspace{1cm} (4.11b)

Alternatively, the exponential integral can be computed by the Hess and Smith formula:

$$e^{Z E_1(Z)} = \frac{M+N}{D} + \epsilon(Z); \hspace{1cm} 0 < |Z| < \infty; \hspace{1cm} |\epsilon(Z)| \leq 7.16^{-6}$$  \hspace{1cm} (4.12)

where

$$M = -(1+m_1 Z + m_2 Z^2 + m_3 Z^3 + m_4 Z^4) \ln Z$$

$$N = -v(n_0 + n_1 Z + n_2 Z^2 + n_3 + Z^3 + n_4 Z^4 + n_5 Z^5)$$

$$D = 1+d_1 Z + d_2 Z^2 + d_3 Z^3 + d_4 Z^4 + d_5 Z^5 + d_6 Z^6$$

$$v = 0.5772156649 \hspace{1cm} n_0 = 0.99999207 \hspace{1cm} d_1 = -0.76273617$$

$$m_1 = 0.23721365 \hspace{1cm} n_1 = -1.49545886 \hspace{1cm} d_2 = 0.28388363$$

$$m_2 = 0.020654300 \hspace{1cm} n_2 = 0.041806426 \hspace{1cm} d_3 = -0.066768033$$

$$m_3 = 0.00076329700 \hspace{1cm} n_3 = -0.03000591 \hspace{1cm} d_4 = 0.012982719$$

$$m_4 = 0.0000097087007 \hspace{1cm} n_4 = 0.0019387339 \hspace{1cm} d_5 = -0.00087008610$$
Similarly to equation (4.11), the first order derivatives of $W$ may be obtained as

$$W_x' = -H(x') \int_{-\infty}^{+\infty} \left[ z' \tau + i(x' + y' \tau) \right] t \, dt$$

(4.13a)

$$W_y' = -H(x') \int_{-\infty}^{+\infty} \left[ z' \tau + i(x' + y' \tau) \right] t \, dt$$

(4.13b)

$$W_z' = -H(x') \int_{-\infty}^{+\infty} \text{Im} \left[ z' \tau + i(x' + y' \tau) \right] t^2 \, dt$$

(4.13c)

Having obtained $G_x$, $G_y$, and $G_z$, the $\frac{\partial G}{\partial n}$ in equation (2.68) may be found from

$$\frac{\partial G}{\partial n} = \mu (n_1 G_x + n_2 G_y + n_3 G_z)$$

(4.14)

In the present work, the boundary $S_j$ consists of only the planes perpendicular to the axes, so that $\frac{\partial G}{\partial n} = \mu G_x$, on the planes $x=\text{const}$, $\frac{\partial G}{\partial n} = \mu G_y$, on the planes $y=\text{const}$ and $\frac{\partial G}{\partial n} = \mu G_z$, on the planes $z=\text{const}$.

4.2.3. Discussion of the results

We consider the problem of a submerged sphere of radius $a$ and submergence $h=2a$. A typical mesh of 24 elements with 4 on each face is shown in Fig.12. The problem is solved using both Green functions defined in equations (4.1) and (4.5) respectively. The numerical integration of the Green function (4.5) over the boundary $S_j$ is computed via the double Gaussian integration method. Nine points are used on each element with three on each side. The exponential integral is computed using equation (4.12). The comparison of the resistance from these two computations and that obtained by Guttmann using a coupled element
method similar to that used by Jami as discussed in section 3.4.2 is shown in Table 6. The results have been nondimensionalized by \( \pi \rho g a^3 \).

In the present calculation, 24 elements correspond to 174 nodes (74 nodes are on \( S_j \)) and 54 elements correspond to 384 nodes (164 on \( S_j \)). The results by Guttmann using quadratic isoparametric tetrahedron elements are from 178 nodes on a half mesh (symmetry is used), which is similar to the 54 elements in the present computation.

<table>
<thead>
<tr>
<th>Fn</th>
<th>G.f. (4.1) 24 elements</th>
<th>G.f. (4.1) 54 elements</th>
<th>G.f. (4.5) 24 elements</th>
<th>G.f. (4.5) 54 elements</th>
<th>Guttmann</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5000</td>
<td>0.00357</td>
<td>0.00393</td>
<td>0.00357</td>
<td>0.00391</td>
<td>0.0035</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.01764</td>
<td>0.01927</td>
<td>0.01767</td>
<td>0.01885</td>
<td>0.0174</td>
</tr>
<tr>
<td>0.7000</td>
<td>0.03731</td>
<td>0.03981</td>
<td>0.03683</td>
<td>0.03916</td>
<td>0.0350</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.05147</td>
<td>0.05420</td>
<td>0.05068</td>
<td>0.05376</td>
<td>0.0530</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.05712</td>
<td>0.06020</td>
<td>0.05623</td>
<td>0.05968</td>
<td>0.0590</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.05679</td>
<td>0.05997</td>
<td>0.05565</td>
<td>0.05916</td>
<td>0.0595</td>
</tr>
<tr>
<td>1.1000</td>
<td>0.05300</td>
<td>0.05632</td>
<td>0.05100</td>
<td>0.05499</td>
<td>0.0550</td>
</tr>
</tbody>
</table>

\[ Fn = \frac{U}{\sqrt{gh}} \quad \text{G.f. = Green function} \]

The table shows that for a given mesh the results from the two present computations are very close to each other. There is some difference between the results from 24 elements and 54 elements, which suggests that 24 elements are not enough for the sphere. It can be seen that when \( Fn \geq 0.8 \), the present results from 54 elements are in very good agreement with Guttmann's. One of the possibilities why the agreement is poorer at low forward speed is that the wavelength is smaller, so that the results are sensitive to the size of elements and a fine mesh is needed.

As a rough estimation, we may consider the wave pattern due to a
moving source. As is well known, the wave in this case is in the limited
region $a \leq \alpha \leq \pi$, where $a = \arcsin \frac{1}{3}$ and $x-a=R\cos\alpha$, $y-b=R\sin\alpha$. One may find
the solution as

$$n(R,\alpha) = 4\sqrt{\frac{\pi}{2\sin^2\alpha}} \left(\frac{\mu R}{1-9\sin^2\alpha}\right)^{1/2}$$

$$\left\{ \sec^{3/2} \theta_1 e^{\mu\sec^2\theta_1 \cos(\mu R\theta_1 - \frac{1}{4}\pi)} + \sec^{3/2} \theta_2 e^{\mu\sec^2\theta_2 \cos(\mu R\theta_2 + \frac{1}{4}\pi)} \right\}$$

(4.15a)

as $R \to \infty$, where $\theta_i (i=1,2)$ are the two roots of

$$\tan\theta = -\frac{1}{4} \cot\alpha \sqrt{1 - \left(1 - 8\tan^2\alpha\right)}$$

(4.15b)

with $\theta_2 > \theta_1$; $\mu_1$, $\mu_2$ are the corresponding values of $\sec^2\theta_1 \cos(\theta-\alpha)$. From
these results, one can see that the wave length changes with the value
of $\alpha$ and is inversely proportional to the square of the Froude number
(since $\mu = \frac{1}{\eta F n^2}$). Since $\mu_2 \to \infty$ as $\alpha \to \pi$ which yields zero wavelength, the
numerical solution of the three dimensional Neumann-Kelvin problem is
not easy, especially at low forward speed. The detailed discussion about
this wave pattern can be found in reference [107].

The present computation experience shows that computer time very
much depends on the form of Green function. Roughly, the computer time
when Green function (4.5) is used only a third of that when (4.1) is
used. This may question the advantage of the coupled finite element
method. Throughout this thesis, we have argued that one of the main
advantages of the coupled finite element method is that the integration
of Green function over the boundary $S_j$ may be calculated explicitly. But
the present result suggests that the computer time depends more on the integration in the Green function itself than on the integration of Green function over the surface. Since an efficient form of Green function needs less computer time to obtain the potential even though the surface integral has to be calculated numerically, it seems that the source distribution method over the body surface is more efficient. However, it is important to realize that the numerical integration over a surface is much easier to calculate on a simple finite element boundary than on an irregular body surface. Thus, the advantage of the coupled finite element method remains in the sense of a more efficient numerical surface integration.

We next consider a more practical problem, a submerged spheroid defined by

\[
\begin{align*}
x &= c \cosh \theta \cos \phi \\
y &= c \sinh \theta \sin \phi \\
z &= c \sinh \theta \cos \phi
\end{align*}
\]

with half the length of the major axis \(a = c \cosh \theta\), and half the length of the minor axis \(b = c \sinh \theta\). The results for the nondimensional resistance (\(R/\eta \rho g b^3\)) given in Table 7 for the spheroid of \(a=2.5b\) and submerged at \(h=2b\) are from a mesh of 54 elements and 384 nodes with 164 nodes on the localised finite element boundary. The detail of the mesh is shown in Fig. 13. Fig. 13a shows the elements on the body surface while Fig. 13b shows the whole mesh. The Green function in equation (4.5) is used to obtain the results in Table 7 while the exponential integral is computed using equation (4.12). The results from Guttmann in Table 7 were obtained from a similar mesh (with 178 nodes) to that he used for the
sphere (Table 6). The comparison with the results by Guttman gives very good agreement. The discrepancies in these results are believed to be associated with the meshes used. Greater accuracy could be achieved by increasing the number of elements.

The success of the present method suggests that N-K problem of a submerged body is not ill-posed mathematically. The failure of some analysis may be due to the numerical procedure. The Green function for the three dimensional N-K problem needs careful evaluation. The problem is further complicated by the practical ships. Most ships are slender, which means the longitudinal component of the surface normal is small, apart from the area near to the bow and stern. Therefore tedious numerical modelling is needed for the potential to satisfy the body surface condition accurately. As a result, a fine mesh of small elements is needed in the present analysis.
Table 7
Comparison of the resistance on a submerged spheroid (h=2b)

<table>
<thead>
<tr>
<th>Fn</th>
<th>Present</th>
<th>Guttmann</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.04254</td>
<td>0.0422</td>
</tr>
<tr>
<td>0.8</td>
<td>0.09548</td>
<td>0.0960</td>
</tr>
<tr>
<td>0.9</td>
<td>0.13396</td>
<td>0.136</td>
</tr>
<tr>
<td>1.0</td>
<td>0.15046</td>
<td>0.1542</td>
</tr>
<tr>
<td>1.1</td>
<td>0.15067</td>
<td>0.1560</td>
</tr>
<tr>
<td>1.2</td>
<td>0.14174</td>
<td>0.1484</td>
</tr>
</tbody>
</table>

\[
\frac{U}{\sqrt{2h}} \quad a=2.5b
\]

4.3. Unsteady potential

4.3.1. The hydrodynamic force on a sphere in an unbounded fluid

As in the analysis of the two dimensional problem, it is necessary and valuable to consider the case without free surface. To obtain the analytical solution for a sphere, the spherical coordinate system \((r, \phi, \theta)\) is defined as

\[
x = r \cos \phi \\
y = r \sin \phi \cos \theta \\
z = r \sin \phi \sin \theta
\]

We assume that \(\psi_j\) is the potential of the harmonic motion without forward speed satisfying the body surface condition \(\frac{\partial \psi_j}{\partial n_j} = n_j (j=1,2,3)\). From the detail discussion in section (3.3), it is straightforward to see that the potential \(\phi_j\) \((j=1,2,3)\) with forward speed may be written as

\[
\phi_j = \omega_j \psi_j - U \frac{\partial \phi_j}{\partial x_j}
\]

where
\( (x_1, x_2, x_3) = (x, y, z) \) \hspace{1cm} (4.18)

Since for the sphere of radius \( a \), we have

\[
\psi_1 = -\frac{1}{2} \frac{a^3}{r^2} \cos \theta \hspace{1cm} (4.19a)
\]

\[
\psi_2 = -\frac{1}{2} \frac{a^3}{r^2} \sin \theta \cos \alpha \hspace{1cm} (4.19b)
\]

\[
\psi_3 = -\frac{1}{2} \frac{a^3}{r^2} \sin \theta \sin \alpha \hspace{1cm} (4.19c)
\]

the solution of \( \phi_j \) can be obtained as

\[
\phi_1 = -\frac{i \omega a^3}{2} \frac{1}{r^2} \cos \theta - \frac{U a^3}{2} \frac{3}{r^3} (3 \cos^2 \theta - 1) \hspace{1cm} (4.20a)
\]

\[
\phi_2 = -\frac{i \omega a^3}{2} \frac{1}{r^2} \sin \theta \cos \alpha - \frac{3U a^3}{4} \frac{1}{r^3} \sin 2 \theta \cos \alpha \hspace{1cm} (4.20b)
\]

\[
\phi_3 = -\frac{i \omega a^3}{2} \frac{1}{r^2} \sin \theta \sin \alpha - \frac{3U a^3}{4} \frac{1}{r^3} \sin 2 \theta \sin \alpha \hspace{1cm} (4.20c)
\]

Substituting this equation into (2.56), we have

\[
\tau_{11} = -\rho \int S_0 \left[ -\frac{i \omega a^3}{2} \frac{1}{r^2} \cos \theta - \frac{U a^3}{2} \frac{3}{r^3} (3 \cos^2 \theta - 1) \right. \\
+ \left. \frac{3U}{2} a \sin \theta (\frac{1}{2} \omega \sin \theta + 3U \cos \theta \sin \theta) \right] \cos \theta dS \\
= \rho a^2 \frac{2 \pi}{2} \int_{0}^{2 \pi} \int_{0}^{\frac{\pi}{2}} \left[ -\frac{i \omega a^3}{2} \frac{1}{r^2} \cos \theta - \frac{i \omega}{2} (3 \cos^2 \theta - 1) \right. \\
+ \left. \frac{3U}{2} a \sin \theta (\frac{1}{2} \omega \sin \theta + 3U \cos \theta \sin \theta) \right] \sin \theta \cos \theta d\theta d\phi \\
= \rho a^2 \int_{0}^{2 \pi} \int_{0}^{\frac{\pi}{2}} \left[ -\frac{9U^2}{2} \cos \phi \sin^2 \theta - \frac{9U^2}{2} \cos \theta \sin \theta \right] d\phi d\theta \\
+ \rho a^2 \int_{0}^{2 \pi} \int_{0}^{\frac{\pi}{2}} \left[ -\frac{9U^2}{2} \cos \phi \sin^2 \theta - \frac{9U^2}{2} \cos \theta \sin \theta \right] d\phi d\theta
\[ p_n a^2 \left( \frac{2\omega^2}{3} - a + \frac{12}{5} \frac{U^2}{a} \right) \]  

Thus

\[ \mu_{11} = \rho \pi \left( \frac{2a^3}{3} + \frac{12}{5} \frac{U^2}{\omega^2} \right) \]  

\[ \lambda_{11} = 0 \]  

After a similar derivation, we have

\[ \mu_{22} = \mu_{33} = \rho \pi \left( \frac{2a^3}{3} + \frac{9}{5} \frac{U^2}{\omega^2} \right) \]  

\[ \lambda_{22} = \lambda_{33} = 0 \]

In the present numerical analysis, the finite element formulation in the near field (equation (2.62)) can be written as

\[ \iint_{R_1} V \phi N_1 d\sigma - \iint_{S_j} \frac{\partial \phi}{\partial n} N_1 dS = \iint_{S_0} n_1 N_1 dS \]  

for the steady potential and

\[ \iint_{R_1} V \phi_j N_1 d\sigma - \iint_{S_j} \frac{\partial \phi_j}{\partial n} N_1 dS = \iint_{S_0} \left( i\omega n_j + U_{mj} \right) N_1 dS \]

\[ = i\omega \iint_{S_1} n_j N_1 dS + \iint_{S_0} m_j N_1 dS \]  

for the unsteady potential. The Green function employed to determine \( \frac{\partial \phi}{\partial n} \) on \( S_j \) is \( \frac{1}{r} \) in both cases. Following a similar derivation to that used in obtaining equation (3.62), the second term on the right hand side of equation (4.25) becomes

\[ \iint_{S_0} m_j N_1 dS = -\iint_{S_0} \overline{V} \overline{\phi} N_1 n_j dS \]  

(4.26)
which will give better performance in the numerical analysis as discussed in detail in section (3.3.1).

Table 8 gives the comparison for the added masses of a sphere of radius \( a \). The analytic results are obtained from equations \((4.22a)\) and \((4.23a)\), while the numerical results are obtained from a similar mesh to that used for Table 6 of 54 elements. Since \( \mu_{33} = \mu_{22} \) in the present numerical calculation, it is not given in the table. It can be seen from Table 8 that the error of the numerical results from 54 elements compared with the analytic solution is much bigger than that in the results from 12 elements for the two dimensional circular cylinder given in Table 2. The error can even reach 10% at low frequency. It becomes smaller when the frequency is higher, where the contribution from \( \bar{\phi} \) is smaller, but it still remains a few per cent. This suggests that a finer mesh of more elements is needed. However as a means to investigate the numerical procedure, the results in this table are very encouraging. It gives sufficient evidence that the coupled finite element method is suitable for the problem of a body moving in surface waves.
Table 8
The comparison of added masses of the sphere in the unbounded fluid domain

<table>
<thead>
<tr>
<th>( \nu a )</th>
<th>( \frac{\mu_{11}}{\rho V_0} )</th>
<th>( \frac{\mu_{22}}{\rho V_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>numerical</td>
<td>analytic</td>
</tr>
<tr>
<td>0.1</td>
<td>3.3008</td>
<td>3.3800</td>
</tr>
<tr>
<td>0.2</td>
<td>1.8978</td>
<td>1.9400</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4300</td>
<td>1.4600</td>
</tr>
<tr>
<td>0.4</td>
<td>1.1962</td>
<td>1.2200</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0559</td>
<td>1.0760</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9623</td>
<td>0.9800</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8955</td>
<td>0.9114</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8454</td>
<td>0.8600</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8064</td>
<td>0.8200</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7752</td>
<td>0.7880</td>
</tr>
</tbody>
</table>

\( F_n = \frac{U}{\sqrt{ga}} = 0.4, \quad V_0 = \frac{4}{3} \pi a^3 \)

4.3.2. Green function

The Green function used to determine \( \frac{\partial \phi_j}{\partial n} \) on \( S_j \) in the case of general three-dimensional problem with a free surface may be obtained as

\[
G = \frac{1}{r} - \frac{1}{r_1} + \frac{2g \gamma}{\pi} \int_0^\infty \int_0^\pi \frac{d\theta dmF(\theta, m)}{} + \frac{2g}{\pi} \int_0^{\pi/2} \int_{L_1}^{L_2} \frac{d\theta dmF(\theta, m)}{}
\]

(4.27)

where \( \tau = \omega V/g \) defined as for the two dimensional problem;

\[
\gamma = 0 \quad (4.28a)
\]

if \( \tau < 1/4 \);

\[
\gamma = \cos^{-1} \left( \frac{1}{4 \tau} \right) \quad (4.28b)
\]

if \( \tau \geq 4 \); and \( \dagger \)

\( \dagger \) The difference in the present formulation of the Green function from that in Wehausen and Laitone's paper is due to the different definition of the periodic motion. In this thesis \( \epsilon^{*} \) is used,
\[ F(\theta, m) = \frac{me^{m[z+c-1(x-a)\cos \theta]}}{\sin^2 \theta} \cos [m(y-b)\sin \theta] \] (4.29)

There are two singular points at \( m = k_1 \) and \( m = k_3 \) in the second integration of equation (4.27) obtained by

\[ \sqrt{g_{k_1}^{-1} - \sqrt{g_{k_3}^{-1}}} = \frac{1 - \sqrt{1 - 4 \cos \theta}}{2 \cos \theta} \] (4.30a)

and also two singular points in the last integration obtained by

\[ \sqrt{g_{k_2}^{-1} - \sqrt{g_{k_4}^{-1}}} = \frac{1 + \sqrt{1 - 4 \cos \theta}}{2 \cos \theta} \] (4.30b)

The integration path is from 0 to \( \pi \) and at singular points it is defined as \( k_1 + i \epsilon, k_2 - i \epsilon, k_3 + i \epsilon \) and \( k_4 + i \epsilon \), as \( \epsilon \to 0 \).

For the present numerical evaluation, it is desirable to decompose the integrand \( F(\theta, m) \) in terms of its singularities. Thus, we write

\[ F(\theta, m) = \frac{1}{g_{k_1}^{-1} - \sqrt{g_{k_1}^{-1}}} [\frac{k_1}{m - k_1} - \frac{k_2}{m - k_2}] e^{m[z+c-1(x-a)\cos \theta]} \cos [m(y-b)\sin \theta] \] (4.31a)

for \( 0 \leq \theta \leq \pi/2 \); and

\[ F(\theta, m) = \frac{1}{g_{k_3}^{-1} - \sqrt{g_{k_3}^{-1}}} [\frac{k_3}{m - k_3} - \frac{k_4}{m - k_4}] e^{m[z+c-1(x-a)\cos \theta]} \cos [m(y-b)\sin \theta] \] (4.31b)

for \( \pi/2 < \theta \leq \pi \). The Green function then becomes

\[ G = \frac{1}{r} + \frac{2g}{r'} \int_0^{\pi} \int_0^\infty \frac{\gamma}{\pi} e^{\gamma} d\gamma d\theta \int_0^\infty \int_0^\infty F(\theta, m) \]

\[ + \frac{2g}{\pi} \int_0^{\pi/2} d\theta \int_0^\infty \int_0^\infty F(\theta, m) + \frac{2g}{\pi} \int_{\pi/2}^\pi d\theta \int_0^\infty \int_0^\infty F(\theta, m) \]

while \( e^{\gamma} \) is used there.
As discussed in section (4.1.2), the evaluation of this kind of Green function is not an easy task. In fact this is even more difficult here, since the four singularities \( k^q \) \((i = 1, 2, 3, 4)\) are much more complicated function of \( \theta \) than \( m_0 = \mu \sec^2 \theta \) in equation (4.1) for the steady potential. Thus the computer time for evaluation of Green function (4.32) will be at least four times that of evaluation of Green function (4.1). This may be an expensive price.

To reduce the computer time, an alternative of equation (4.32) is to use the exponential integral. We may write equation (4.27) as

\[
G = \frac{1}{r} - \frac{1}{r_1} + (I_{01} + I_{02}) + (I_{11} + I_{12}) - (I_{21} + I_{22}) + (I_{33} + I_{34}) - (I_{43} + I_{44})
\]

where

\[
I_{01} = \frac{1}{\pi} \int_{0}^{\pi/2} k_1 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_1} dm
\]

\[
I_{02} = \frac{1}{\pi} \int_{0}^{\pi/2} k_1 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_1} dm
\]

\[
I_{11} = \frac{1}{\pi} \int_{0}^{\pi/2} k_2 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_1} dm
\]

\[
I_{12} = \frac{1}{\pi} \int_{0}^{\pi/2} k_2 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_2} dm
\]

\[
I_{21} = \frac{1}{\pi} \int_{0}^{\pi/2} k_3 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_3} dm
\]

\[
I_{22} = \frac{1}{\pi} \int_{0}^{\pi/2} k_3 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_2} dm
\]

\[
I_{33} = \frac{1}{\pi} \int_{0}^{\pi/2} k_4 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_4} dm
\]

\[
I_{34} = \frac{1}{\pi} \int_{0}^{\pi/2} k_4 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_3} dm
\]

\[
I_{43} = \frac{1}{\pi} \int_{0}^{\pi/2} k_4 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_2} dm
\]

\[
I_{44} = \frac{1}{\pi} \int_{0}^{\pi/2} k_4 \frac{e^\gamma}{1 - 4 \mu \cos^2 \theta} d\theta \int_{0}^{\gamma} \frac{e^\gamma}{m - k_4} dm
\]
\[ I_{ij} = \frac{1}{\pi} \int_{\gamma} \frac{k_j}{\sqrt{1 - \alpha^2 \cos \theta}} \, d\theta \int_{0}^{m-k_j} \, dm \] (4.34c)

\[ \xi_1 = (z+c) + i[(x-a)\cos \theta + (y-b)\sin \theta] \] (4.35a)

\[ \xi_2 = (z+c) + i[(x-a)\cos \theta - (y-b)\sin \theta] \] (4.35b)

\[ \xi_3 = \xi_2^* \] (4.35c)

\[ \xi_4 = \xi_1^* \] (4.35d)

and * denotes the conjugate of the complex. The integration path at singularities in equation (4.34c) is defined as before. It can be understood that \( i > 0 \) in equation (4.34c) and \( \gamma' = \gamma, \) \( \alpha = 1 \) for \( i = 1,2, \) \( \gamma' = 0, \) \( \alpha = -1 \) for \( i = 3,4. \) The definitions of \( k_1 \) and \( k_2 \) in equations (4.34a) and (4.34b) are the extension of equation (4.30) to complex. They may be written as

\[ \sqrt{g_k^2} = \frac{1 \pm i \sqrt{4 \cos \theta - 1}}{2 \cos \theta} \] (4.36)

Performing the change of the integration variable

\[ u = -\xi_j(m-k_j) \] (4.37)

for the corresponding integral \( I_{ij} \) (\( i = 0,\ldots,4,j = 1,\ldots,4 \)), equations (4.34), equations (4.34a) and (4.34b) become

\[ I_{01} = \frac{1}{\pi} \int_{0}^{\gamma} \frac{k_1 \exp(k_1 \xi_1)}{1 \sqrt{4 \cos \theta - 1}} \, d\theta \int_{0}^{m-k_1} \, dm \] (4.38a)

\[ I_{02} = \frac{1}{\pi} \int_{0}^{\gamma} \frac{k_2 \exp(k_2 \xi_1)}{1 \sqrt{4 \cos \theta - 1}} \, d\theta \int_{0}^{m-k_2} \, dm \] (4.38b)
\[ E_p(Z) = E_q(Z) = E_1(Z) \]  
\[ \text{when Re}(Z) \geq 0; \]  
and

\[ E_p(Z) = E_1(Z) \quad \text{Im}(Z) \geq 0 \]  
\[ E_q(Z) = E_1(Z) - 2\pi i \quad \text{Im}(Z) < 0 \]  
\[ E_q(Z) = E_1(Z) + 2\pi i \quad \text{Im}(Z) \geq 0 \]  
\[ E_q(Z) = E_1(Z) \quad \text{Im}(Z) < 0 \]  
\[ \text{when Re}(Z) < 0. \]  
in equations (4.38a) and (4.38b) are given by equation (4.36a).

Similarly, equation (4.34c) becomes

\[ I_{1j} = \frac{1}{\pi} \int_0^{\pi/2} \frac{k_i \exp(k_i \xi_j)}{1 - 4\pi \cos \theta} E_{r_i}^{(1,2,3,4)}(k_i \xi_j) d\theta \]  
(4.40)

where

\[ E_{r_1}(Z) = E_1(Z) \quad \text{Im}(Z) \geq 0 \]  
\[ E_{r_1}(Z) = E_1(Z) - 2\pi i \quad \text{Im}(Z) < 0 \]  
for \( i = 1, 3, 4; \) and

\[ E_{r_1}(Z) = E_1(Z) + 2\pi i \quad \text{Im}(Z) \geq 0 \]  
\[ E_{r_1}(Z) = E_1(Z) \quad \text{Im}(Z) < 0 \]  
for \( i = 2. \) \( k_1 \) in equation (4.40) are given by

\[ k_2, k_1 = \frac{1 - 2\pi \cos \theta \pm \sqrt{1 - 4\pi \cos \theta}}{2\pi^2 \cos^2 \theta} \]  
(4.42a)
for $\gamma < 0 \leq \pi/2$, and

$$k_1, k_3 = \frac{1 + 2 \tau \cos \theta \pm \sqrt{1 + 4 \tau^2 \cos^2 \theta}}{2 \tau^2 \cos^2 \theta} \quad (4.42b)$$

for $0 \leq \theta \leq \pi/2$. It can be easily shown that the definition of $k_i$ ($i=1,2,3,4$) by equation (4.42) is equivalent to that by equation (4.30).

The computation of the exponential function $E_1(Z)$ is straightforward using equation (4.12) or equation (4.13) depending on the value of $Z$. It takes the value of $E_1(Z+0)$ when $\text{Im}(Z)=0$ in equations (4.39) and (4.41). As the computer time for the steady potential problem when Green function (4.5) is used is only about a third of that when (4.1) used, a significant reduction of computer time can be anticipated when the exponential integral is used for the present problem.

Fig.14 to Fig.17 give the comparison of the results from Green function (4.27) and (4.33) for different Froude numbers from 0.2 to 0.6. The source point is at $(0,0,-2)$ and the field point is at $(r \cos \phi, r \sin \phi, 0)$. Because the Green function is symmetric about $y=0$, the results given are in the range $0 \leq \phi \leq \pi$ with the step being $\pi/4$. These figures show that the results from the two different forms of Green function are in very good agreement. However, much effort has been spent to get the satisfactory results of Green function (4.27), especially at high frequency.

A common feature of these figures is the discontinuity of the Green function at $\tau=1/4$. Although it is not straightforward to obtain the value of Green function at $\tau=(1/4)\pm0$ or $\tau=(-1/4)\pm0$, it is easy to find the value of the jump. From equation (4.33), we have
Using equation (4.38) as \( \tau^{(1/4)_0} \) and \( \theta=0 \), equation (4.43) becomes

\[
G[(\frac{1}{4})_{+0}] - G[(\frac{1}{4})_{-0}] 
\]

\[
= \frac{1}{i\pi} [k_1 \exp(k_1 \xi_1) E_p(k_1 \xi_1) - k_2 \exp(k_2 \xi_1) E_q(k_2 \xi_1)] 
+ k_1 \exp(k_1 \xi_2) E_p(k_1 \xi_2) - k_2 \exp(k_2 \xi_2) E_q(k_2 \xi_2)] 
\]

\[
\lim_{\tau^{(1/4)}_+0} \frac{\cos^{-1}(\frac{1}{4\tau})}{\sqrt{4\tau-1}} = \lim_{\tau^{(1/4)}_+0} \frac{\cos^{-1}(\frac{1}{4\tau})}{\sqrt{4\tau-1}} 
\]

The limit in this equation can be obtained as

\[
\lim_{\tau^{(1/4)}_+0} \frac{\cos^{-1}(\frac{1}{4\tau})}{\sqrt{4\tau-1}} = \lim_{x \to 1^-} \frac{\cos^{-1}(x)}{\sqrt{1-x}} 
= \lim_{x \to 1^-} \frac{x}{\sqrt{2} \sin(x/2)} 
= \sqrt{2} 
\]

(4.45)

Noticing that \( k_1 = k_2 = 4\nu_c \) and \( \xi_1 = \xi_2 = \xi = (z+c)-i(x-a) \), when \( \tau=1/4 \) and \( \theta=0 \), equation (4.44) becomes

\[
G[(\frac{1}{4})_{+0}] - G[(\frac{1}{4})_{-0}] 
\]

\[
= \frac{2}{i\pi} \exp(4\nu_c \xi) E_p(4\nu_c \xi) - \exp(4\nu_c \xi) E_q(4\nu_c \xi) 
+ \exp(4\nu_c \xi) E_p(4\nu_c \xi) - \exp(4\nu_c \xi) E_q(4\nu_c \xi) 
\]

(4.46)
where \( \nu_c \) is defined such that \( \tau = \frac{1}{4} \). As \( \text{Re}(\xi) < 0 \) for submerged source, equation (4.46) becomes

\[
G[(\frac{1}{4})^+_{0}] - G[(\frac{1}{4})^-_{0}] = -16\sqrt{2}\nu_c\exp(4\nu_c\xi)
\]

based on the definition of \( E_p \) and \( E_q \) in equation (4.39).

It can be seen the jump of the Green function at the critical point is nonzero unless \( \nu_c = 0 \) or \( \nu_c = \infty \). Since \( \nu_c = \frac{U}{c_1} \), it can be concluded that there is a jump of the Green function at the critical point for any finite forward speed, but the Green function is continuous everywhere when \( U = 0 \) or \( U = \infty \). This agrees with the fact that the Green function without forward speed is continuous at any frequency. A special case of equation (4.47) is when \( \text{Im}(\xi) = 0 \) or \( x - a = 0 \) which yields that the right hand side of the equation is a purely real number. Thus the imaginary part of the Green function is continuous in this particular case of \( \text{Im}(\xi) = 0 \), while the jump of the real part reaches its maximum for given \( \nu_c \), \( z + c \) and \( r \).

### 4.3.3. Discussion of the results

#### 4.3.3.1. The solution by the first approximation method

Due to the complexity of the problem of oscillating bodies at forward speed, there has not been any exact analytic solution yet, even for a very simple geometry such as a sphere. There are a few numerical works mentioned in the first chapter, but they do not provide any comparable results. Therefore it is purpose of this section to derive an approximate solution for a deeply submerged body. The approximate analytical solution for the submerged sphere is given. This is used to check the numerical results.
As has been well known, the main difficulty in ship hydrodynamics is the existence of the free surface. However for the deeply submerged body, it is reasonable to assume that the effect of the free surface on the potential near the body is negligible. Thus the potential on the body can be taken as the solution without free surface, so that the difficulty due to the free surface can be avoided. This method, usually called the first approximation method, was initiated by Lamb\(^6\) and was widely used when the exact solution was not easy to obtain. Some applications by Havelock have been mentioned in chapter three. For the problem of an oscillating submerged body at forward speed, Newman\(^7\) used the extension of Havelock's method for the three dimensional steady potential\(^5\) and obtained the first approximation solution for the damping coefficients for ellipsoids. Here, this method is restated in a more general sense so that it can be applied to an arbitrary body. The principle is demonstrated by the solution of a sphere.

From the Green's identity, we have

\[
\phi = -\frac{1}{4\pi} \int \left[ \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} \right] dS \tag{4.48}
\]

where the Green function is defined by equation (4.27) or (4.33) for the present problem. Based on the assumption of the first approximation method, the potential on the right hand side of this equation can be taken as the solution \(\phi_0\) without the free surface but satisfying the same body surface condition as \(\phi\). Equation (4.48) then becomes

\[
\phi = -\frac{1}{4\pi} \int \left[ \frac{\partial G}{\partial n} - \frac{\partial \phi_0}{\partial n} \right] dS \tag{4.49}
\]

The numerical solution of \(\phi_0\) for general three dimensional bodies
can be obtained by solving a similar integral equation to (4.48) with \( G \) being simply taken as 1/r, or by many other techniques (such as the coupled finite element method discussed in detail in section (5.2.1)). For some special geometries, the availability of the analytical solution for \( \phi_0 \) enables the first approximation solution to be easily obtained. We may take the sphere as an example. From section (5.3.1), it is easy to have

\[
\phi_0 = \frac{i\omega^3}{2} \frac{\partial}{\partial x} \left( \frac{1}{r^2} \right) - \frac{a_3^2}{2r^3} \frac{\partial^2}{\partial x^2} \left( \frac{1}{r^2} \right)
\]

\[
\phi_0 = \frac{i\omega^3}{2} \frac{\partial}{\partial y} \left( \frac{1}{r^2} \right) - \frac{a_3^2}{4r^3} \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{r^2} \right)
\]

\[
\phi_0 = \frac{i\omega^3}{2} \frac{\partial}{\partial z} \left( \frac{1}{r^2} \right) - \frac{3a_3^3}{4r^3} \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{r^2} \right)
\]

where

\[
r = \left[ (x-\xi_1)^2 + (y-\xi_2)^2 + (z-\xi_3)^2 \right]^{1/2}
\]

with \((\xi_1, \xi_2, \xi_3)\) being the center of the sphere. Substituting equation (4.50) into (4.49) and using Green's identity inside the sphere, we obtain

\[
\phi_2 = -\frac{1}{4\pi^2} \frac{i\omega^3}{2} \frac{\partial}{\partial \xi_1} - \frac{a_3^3}{2r^3} \frac{\partial^2}{\partial \xi_1^2} G(x, y, z, \xi_1, \xi_2, \xi_3)
\]

\[
\phi_2 = -\frac{1}{4\pi^2} \frac{i\omega^3}{2} \frac{\partial}{\partial \xi_2} - \frac{3a_3^3}{4r^3} \frac{\partial^2}{\partial \xi_1 \partial \xi_2} G(x, y, z, \xi_1, \xi_2, \xi_3)
\]

\[
\phi_3 = -\frac{1}{4\pi^2} \frac{i\omega^3}{2} \frac{\partial}{\partial \xi_3} - \frac{3a_3^3}{4r^3} \frac{\partial^2}{\partial \xi_1 \partial \xi_3} G(x, y, z, \xi_1, \xi_2, \xi_3)
\]

where \((\xi_1, \xi_2, \xi_3)\) is the coordinates of the source (corresponding to \((a, b, c)\) in equation (4.29)) which is to be taken as the center of the sphere.
Having obtained the potential, the hydrodynamic forces can be found by equation (2.56). If only the damping coefficients are of interest, we may use the asymptotical expansion of $\phi_j$ at infinity. As $R=\sqrt{x^2+y^2+\omega}$, the Green function can be written as\textsuperscript{74,75}

$$G = \frac{1}{\pi R} \exp \left\{ \frac{\lambda_m(u_n) \sin^2 \theta}{\sin^2 u_n \frac{d}{du_n} \cos (u_n - \theta)} \right\} \frac{1}{2} s_m(u_n)$$

$$\exp \left\{ \lambda_m(u_n) [x + \xi_1 + i(x - \xi_1) \cos u_n + i(y - \xi_2) \sin u_n] \right\} (4.53)$$

where

$$x = R \cos \theta$$

$$y = R \sin \theta$$

$$\lambda_m(u) = \frac{g}{2U^2 \cos^2 u} \left[ 1 + 2 \cos u \pm (1 + 4 \cos u)^{1/2} \right]$$

$$s_1(u) = \frac{\cos u}{\cos u}$$

$$s_2(u) = -1$$

and the ($\pm$) sign in equation (4.53) is determined by the sign of

$$\frac{d}{du_n} \cos (u_n - \theta)$$

The second summation is over the $N$-roots of the equation

$$\cot \theta = -\frac{\sin^2 u_n \pm (1 + 4 \cos u_n)^{1/2}}{\sin u_n \cos u_n}$$

which is obtained from

$$\frac{d}{du_n} [\lambda_m \cos (u_n - \theta)] = 0$$

(4.59)
and \( u_n \) satisfies the inequality

\[
-\pi u_n \leq |\theta| \leq \frac{\pi}{2}, \quad |\theta| \leq \pi
\]  

(4.60)

Substituting equation (4.53) into (4.52), we obtain

\[
\phi_j = -\left( \frac{\lambda_j}{\pi R} \right)^{1/2} \sum_{m,n} \left[ \frac{\sin^2 \theta}{\sin \theta_n |\sin \theta_n - \cos \theta_n|} \right]^{1/2} 
\]

\[
\sum_{m,n} s_{m,n}(\lambda_j u_n(z+iR\cos(\theta_0 - \theta))) \frac{\pi^1}{4} \lambda_j \phi_j
\]

(4.61)

where

\[
P_1(u) = \frac{1}{4\pi} \left[ i\omega u + \frac{a^3 u}{2r^3} \cos^2 \theta \lambda_n(u) \right]
\]

\[
\lambda_n(u) \exp[-\lambda_n(u)n] 
\]  

(4.62a)

\[
P_2(u) = \frac{1}{4\pi} \left[ i\omega u - i\sin \theta \lambda_n(u) + \frac{3a^3 u}{4r^3} \cos \theta \lambda_n(u) \right]
\]

\[
\lambda_n(u) \exp[-\lambda_n(u)n] 
\]  

(4.62b)

\[
P_3(u) = \frac{i\omega u^3}{2} + \frac{3a^3 u}{4r^3} \cos \theta \lambda_n(u) \lambda_m(u) \exp[-\lambda_m(u)n] 
\]

(4.62c)

for the submerged sphere, \( h \) being the submergence of its center.

Having obtained the potential at infinity, the damping coefficients can be obtained from the energy conservation law. The results can be found as

\[
\lambda_{nj} = -\frac{\lambda_{m}^2(u)}{\sum_{m=1}^{m} 0} \left( 1 + 4r^2 \cos \theta \right) \frac{1}{2} \left| \lambda_j \phi_j \right|^2 s_m(u) du 
\]

(4.63)

where \( Y \) is defined by equation (4.28). Performing the change of
integration variable

\[ K = \frac{1 + 2t \cos u \pm \sqrt{1 + 4t \cos u}}{2t \cos u} \quad (4.64) \]

equation (4.63) becomes

\[ \lambda_{jj} = \frac{32\pi \rho a}{\omega} \int_{-\infty}^{+\infty} \frac{(Kt-1)^5 |Kt-1|}{\exp[-2\omega(Kt-1)^2]} \left[ (Kt-1)^4 - K^2 \right]^{1/2} \exp \left[ -2\omega(Kt-1)^2 \right] Q_2^2 dK \quad (4.65) \]

where

\[ Q_1 = -\frac{1}{4} \frac{\omega K}{Kt-1} \quad (4.66a) \]
\[ Q_2(K) = -\frac{1}{4} \frac{\omega(Kt-1)^4 - K^2}{Kt-1} \quad (4.66b) \]
\[ Q_3(K) = -\frac{1}{4} \omega(Kt-1) \quad (4.66c) \]

The prime in equation (4.65) indicates that the integration is calculated in the region where

\[ (Kt-1)^4 - K^2 \geq 0 \quad (4.67) \]

Thus the integration of equation (4.65) may be written as

\[ f' = f + f + f \]
\[ \int_{-\infty}^{+\infty} K_1 K_3 \quad (4.68a) \]

when \( \tau < 1/4 \); and

\[ f' = f + f \]
\[ \int_{-\infty}^{+\infty} K_2 K_4 \quad (4.68b) \]

when \( \tau \geq 1/4 \), where

\[ K_{2,1} = \frac{2\tau - 1 \pm \sqrt{1 - 4\tau}}{2\tau^2} \quad (4.69a) \]
\[ K_{4,3} = \frac{2\tau + 1 \pm \sqrt{1 + 4\tau}}{2\tau^2} \quad (4.69b) \]
Equation (4.65) shows that the damping coefficients of surge and heave tend to infinity as $t \to 0$, but the sway damping coefficient remains finite. This agrees with the result for ellipsoids by Newman. In fact, the present result for a sphere can be directly obtained by taking $a_1 = a_2 = a_3$ in Newman's solution for ellipsoids. We have followed a different procedure of derivation to demonstrate the principle of using equation (4.49) for the arbitrary body.

The reason for the continuous damping coefficient of sway at the critical point can be obtained from equation (4.47). It shows that the jump of the Green function at the critical point is independent of $y$. Thus from equation (4.52b) it can be seen that the potential by the first approximation method for sway is continuous. It can be also seen that there is a jump for the potentials for surge and heave. This yields the corresponding behaviour of the damping coefficients.

To demonstrate the applicability of the first approximation method, Fig. 17 gives a comparison of the damping coefficients with those from the coupled finite element solution for a submerged sphere of radius $a$ and submergence $h=2a$ at zero forward speed. As $\lambda_{11} = \lambda_{22} = \frac{1}{2} \lambda_{33}$ from equation (4.65) when $U=0$, which is confirmed by the numerical results, Fig. 17 gives only results for $\lambda_{11}$. It can been seen that the first approximation provides very good results in this case.

4.3.3.2. Numerical results at forward speed by the coupled finite element method

Fig. 18 compares the numerical results of damping coefficients $\lambda_{jj}$ ($j=1,3$) of the sphere obtained by the coupled finite element method with
the first approximate solution at Froude number $F_n = \frac{U}{\sqrt{g}a} = 0.4$ and the submergence $h = 2a$. The results have been nondimensionalized by $\frac{4}{3} \pi g a^3$. As the first step, the Green function (4.27) is used for the purpose of simplicity of the computer programme. As a result, the computation is rather expensive. Thus, as a test of numerical procedure and computer programme, we use a mesh of only 24 elements to obtain the results in Fig.18.

The figure shows that the results from the coupled finite element method and from the first approximation are in a reasonable agreement except in a region close to the critical point $\nu_c = 0.3906$. This is similar to the two dimensional case\textsuperscript{43}. The interesting thing is that although the Green function has a jump at $t = 1/4$ and the first approximation method gives infinite $\lambda_{11}$ and $\lambda_{33}$, it seems that the damping coefficients from the present numerical method are continuous at the critical point. This is a welcome result, since there is no experimental evidence that there is a jump of hydrodynamic force at $t = 1/4$, as discussed by Ogilvie and Tuck\textsuperscript{88}.

Because of the rather expensive computation, it is hard to give more results at this stage. But as we have derived the alternative form of Green function, the computer time could be significantly reduced by using Green function (4.33). Further reduction of computer time can be achieved by employing the symmetry of the body, which is a common feature for most practical ships. This would provide a means of more extensive calculation for a variety of practical problems.
4.4. Further extension

The extensions of the present numerical method to some more general problems have been discussed in detail at two dimensional level in the last chapter. All these discussions can be applied to the three dimensional problem. Here we will discuss some other extensions of the present numerical methods and some suggestions for future research.

4.4.1. The analytical formulations for submerged oscillating spheroids at forward speed

As has been discussed in the previous sections, it is necessary to derive some exactly analytical solutions to provide a basis to check the numerical results. Generally, the potential problem for the submerged body can be represented by a source distribution \( \sigma(\xi,\eta,\zeta) \) over the body surface, or

\[
\phi = \iiint \sigma(\xi,\eta,\zeta)G(x,y,z,\xi,\eta,\zeta)dS \quad (4.70)
\]

where the coordinates \((\xi,\eta,\zeta)\) of the source point are similar to \((a,b,c)\) in equation (4.27). The source distribution \( \sigma \) in this equation is to be found by the body surface condition on \( \phi \), or

\[
\frac{\partial \phi}{\partial n} = 2\pi \sigma + \iiint \sigma \frac{\partial G}{\partial n}dS \quad (4.71)
\]

In most cases, the solution of equation (4.71) is obtained by a numerical method of discretisation of the body surface, as discussed in detail in the first chapter. However for the spheroids considered presently, the source distribution can be obtained using spheroidal harmonics. To demonstrate the principle of the method, we consider the diffraction potential of the spheroid in head or following seas. This allows that
the symmetry can be used, but it does not imply that this method only applies to this case.

It can be found that

\[
\sigma = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{m+1} \frac{P_n^m(\cos \theta) \cos m \phi}{P_n^m(\cosh n \eta_0) \sinh n \eta_0 (\cosh^2 \eta_0 - \cos^2 \theta)^{1/2}}
\]  

(4.72)

for the spheroid defined by

\[
x = c \cosh n \cos \theta
\]

(4.73a)

\[
y = c \sinh n \sin \theta \sin \phi
\]

(4.73b)

\[
z = c \sinh n \sin \theta \cos \phi - h
\]

(4.73c)

when \(n = n_0\), where \(h\) is the submergence of the center of the spheroid. \(P_n^m\) in equation (4.72) is the well known associated Legendre function of the first kind\(^1\) (similarly \(Q_n^m\) is the associated Legendre function of the second kind; \(P_n, Q_n\) are the Legendre function of the first and second kind respectively, which will be used later). Substituting equations (4.27) and (4.72) into equation (4.70), we obtain

\[
\phi = \int \int \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{m+1} \frac{P_n^m(\cos \theta) \cos m \phi}{P_n^m(\cosh n \eta_0) \sinh n \eta_0 (\cosh^2 \eta_0 - \cos^2 \theta)^{1/2}} \right\} \right|_{y_0}^{y} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}
\]

\[
\int \frac{1}{r} \frac{1}{r_1} + \frac{2 \pi}{\eta} \int_{0}^{\infty} dt \int dk F(t,k)
\]

\[
+ \frac{2 \pi}{2} \int_{0}^{\infty} dt \int_{L_{2}}^{L_1} dk F(t,k) \right] dS
\]

(4.74)

where \(y, L_1, L_2\) and \(F(t,k)\) are defined as in section 4.3.2. Use is made of the following relations\(^3\)
\[ \int_0^1 \frac{1}{r} \text{d}S = - \sum_{n=0}^{\infty} \sum_{m=0}^{n-m} c(n-m)! \frac{P_m(\cos \theta) \cos m \phi Q_n^m(\cosh n)}{n! (n+m)!} \]  
(4.75a)

\[ \int \exp[k \xi - i k(\xi \cos t + \eta \sin t)] \text{d}S = \frac{c^2}{2^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n-m} (-1)^n \]  
(4.75b)

\[ I \rightarrow \int_0^1 \int_0^{2\pi} \frac{1}{r} \text{d}S = \int_0^\pi \int_0^{2\pi} \frac{1}{r} \text{d}S \]  
(4.75c)

\[ \Delta = k \cos \theta \]  
(4.75d)

\[ \exp[kz + ik(x \cos t + y \sin t)] = \exp(-kh) \sum_{n=0}^{\infty} \sum_{m=0}^{n-m} \frac{C_{n,m}(\cos \theta) P_n^m(\cosh n)}{2\pi} \]  
(4.75e)

where

\[ C_{n,0} = i^n (2n+1) j_n(\Delta) \]  
(4.76a)

\[ C_{n,m} = i^{n+m} \frac{(n-m)!}{(n+m)!} (2n+1)! \frac{((\sec+t\tan)^m}{(\sec+t\tan)^m} \]  
(4.76b)

\[ B_{n,m} = i^{n+m} \frac{(n-m)!}{(n+m)!} (2n+1)! \frac{((\sec+t\tan)^m}{(\sec+t\tan)^m} \]  
(4.76c)

and \( j_n \) is the spherical Bessel function of the first kind. We also use

\[ \int_0^1 \int_0^{2\pi} \frac{1}{r} \text{d}S \]  
(4.76d)

\[ \int_0^\pi \int_0^{2\pi} \text{d}S \]  
(4.76e)

\[ = \int_0^\pi \int_0^{2\pi} \exp[k(\xi + \zeta) + ik[(\xi \cos t + (y-\eta) \sin t)] \text{d}S \]  
(4.76f)

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \exp(-2kh) \sum_{n=0}^{\infty} \sum_{m=0}^{n-m} \frac{C_{n,m}(\cos \theta) \cos m \phi}{2\pi} \]  
(4.76g)
\[ P_n^m(\cos \theta) P_n^m(\cosh n) \{ \frac{c^2}{2} \sum_{n'=0}^{\infty} (-1)^{n'+1} n'+m' \} \]

\[ A_{n'}^m \left[ (\sec + \tan t)^{m'} + \frac{1}{(\sec + \tan t)^{m'}} \right] j_{n'}(\Delta) \]  

(4.78)

to obtain

\[ \Phi = -c \sum_{n=0}^{\infty} A_{n}^m (n+m)! P_n^m(\cos \theta) \cos \phi P_n^m(\cosh n) \]

\[ - \frac{1}{2\pi} \int_{-\pi}^{\pi} dt \int_{-\pi}^{\pi} \exp(-2\kappa h) \{ \sum_{n=0}^{\infty} C_{n}^m \cos \phi \}

\[ P_n^m(\cos \theta) P_n^m(\cosh n) \{ \frac{c^2}{2} \sum_{n'=0}^{\infty} (-1)^{n'+1} n'+m' \} A_{n'}^m A_{n'}^m(t) j_{n'}(\Delta) \}

\[ + \frac{2\kappa}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \gamma \left[ \int \frac{dt}{dk} + \int \frac{dt}{dk} + \int \frac{dt}{dk} \right] F'(t,k) \]

\[ \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_{n,m} \cos \phi P_n^m(\cos \theta) P_n^m(\cosh n) \right] \]

\[ + \frac{c^2}{2} \sum_{n'=0}^{\infty} \sum_{m'=0}^{\infty} (-1)^{n'+1} m'+m' A_{n',m'} A_{n',m'}(t) j_{n'}(-\Delta) \]

(4.79)

where

\[ F'(t,k) = \frac{k \exp(-2\kappa h)}{2[\kappa g - (\omega + k \cos \theta)]^2} \]  

(4.80a)

\[ D_{n,m}(k,t) = C_{n,m}(k,-t) = (-1)^{n+m} C_{n,m}(k,t) \]  

(4.80b)
\[ D'_{n,m}(k,t) = C_{n,m}(k,\pi+t) = (-1)^{n+m} C_{n,m}(k,t) \]  
(4.80c)

\[ T_{n}^{m}(t) = (\sec \theta + \tan \theta)^{m} + \frac{1}{(\sec \theta + \tan \theta)^{m}} \]  
(4.80d)

\[ S_{n}^{m}(t) = T_{n}^{m}(\pi-t) = (-1)^{m} T_{n}^{m}(t) \]  
(4.80e)

\[ R_{n}^{m}(t) = T_{n}^{m}(\pi+t) = (-1)^{m} T_{n}^{m}(t) \]  
(4.80f)

and \( m, n \) are from zero to infinity and \( P_{n}^{0} = P_{n}, Q_{n}^{0} = Q_{n} \). The terms of \( \sin \theta \) have been neglected in equation (4.79) because of symmetry.

Rearranging equation (4.79) in terms of \( P_{n}^{m}(\cos \theta) \cos m \), we obtain

\[ \phi = -c \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m}^{m}(n-m)! \cos \theta \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n}^{m}(\cos \theta) \cos m \]

\[ + \frac{c^{2}}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n}^{m}(\cos \theta) \]

\[ \left[ A_{n,m}^{m}(-1)^{n+1} \int_{0}^{\pi/2} dt \int_{0}^{\pi/2} dk \exp(-2kh) T_{n,m}^{m}(t) j_{n,m}(\Delta) C_{n,m} j_{n,m}(\Delta) \right] \]

\[ + \frac{c^{2}}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n}^{m}(\cos \theta) \]

\[ \left[ A_{n,m}^{m}(-1)^{n+1} \int_{0}^{\pi/2} dt \int_{0}^{\pi/2} dk \right] \]

\[ \left( D_{n,m}^{m}(t) + D_{n,m}^{m}(t) \right) F'(t,k) j_{n,m}(\Delta) \]

\[ \]  
(4.81)

The harmonic incident potential can be obtained as

\[ \phi = \frac{1}{\omega} \exp[k_{0}z - ik_{0}(x \cos \theta + y \sin \theta)] \]  
(4.82)

for unit amplitude, where \( \omega, k_{0} \) are incoming wave frequency and number respectively satisfying the relation
\[ \omega_0^2 = k_0 g \quad (4.83a) \]

\[ \omega = \omega_0 - k_0 U \cos \beta \quad (4.83b) \]

and \( \omega \) is the encounter frequency. The incoming wave incident angle \( \beta \) is defined so that \( \beta = 0 \) corresponds to a following sea and \( \beta = \pi \) corresponds to the head sea. Using equation (4.75d), we obtain

\[ \phi_0 = \frac{i g}{\omega_0} \exp(-k_0 h) \sum_{n,m} (n,m) (k_0,\beta) \frac{m}{n} \sinh n \cos \theta \quad (4.84) \]

with \( \beta = 0 \) or \( \beta = \pi \). Imposing the body surface condition (see section (2.5))

\[ \frac{\partial \phi}{\partial n} = -\frac{\partial \phi_0}{\partial n} \quad (4.85) \]

on \( n = n_0 \), we obtain an infinite set of equations

\[ -\frac{i g}{\omega_0} \exp(-k_0 h) D^*_{n,m}(k_0,\beta) \frac{d}{dn}[\frac{m}{n} \sinh n] = -c A^n_m (n+m)! \frac{d}{dn} [\frac{m}{n} \sinh n_0] \]

\[ -\frac{c^2}{4\pi} \frac{d}{dn}[\frac{m}{n} \sinh n_0] \sum_{n',m'} A^{n'}_{n,m} (-1)^{n'+1} n'^{1+m'} \int_{-\pi}^{\pi} \int_0^\infty [f \exp(-2k \xi) + f \exp(-2k \xi)] F'(k,t) j_n,(-\Delta) \]

\[ T_{n,m}^{'n'}(t) C_{n,m}^{n'}(\Delta) \]

\[ + \frac{c^2}{\pi} \frac{d}{dn}[\frac{m}{n} \sinh n_0] \sum_{n',m'} A^{n'}_{n,m} (-1)^{n'+1} n'^{1+m'} \int_{0}^{\pi/2} \int_0^\infty [f \exp(-2k \xi) + f \exp(-2k \xi)] F'(k,t) j_n,(-\Delta) \]

\[ D_{n,m}^{'n'}(t) S_{n,m}^{n'}(t) + D^*_{n,m} T_{n,m}^{n'}(t) \quad (4.86) \]

for \( n \neq m \neq 0 \).

Apparently, the essential task to solve equation (4.86) is the evaluation of the following two integrals

\[ I(n,m,n',m') = \int_{-\pi}^{\pi} \int_0^\infty [f \exp(-2k \xi) + f \exp(-2k \xi)] F'(k,t) j_n,(-\Delta) \]

\[ j_{n'}(-\Delta) dk \quad (4.87a) \]
\[ H(n,m,n',m') = g[\int dt/dk + \int^\pi dt/dk + \int^\pi dt/dk] \]

\[ 0 \quad 0 \quad 0 \quad 0 \quad Y \quad L_1 \quad L_2 \quad \pi/2 \]

\[ [D_{n,m}(t)S^m_{n'}(t) + D'_{n,m}(t)R^m_{n'}(t)]F'(k,t) \]

\[ J_n(-\Delta)j_n'(-\Delta) \]

\[ 0 = 2g[\int dt/dk + \int^\pi dt/dk + \int^\pi dt/dk] \]

\[ 0 \quad 0 \quad 0 \quad 0 \quad Y \quad L_1 \quad L_2 \quad \pi/2 \]

\[ [(-1)^{n+m+n'+m'}C_{n,m}(t)T^{m'}_{n'}(t)J_n(\Delta)j_n'(\Delta) \]

\[ F'(k,t) \]

\[ (4.87b) \]

In these two equations, we have defined

\[ C_{n,m} = C_{n,m}(k,t) = C_{n,m}(t)j_n(\Delta) \]  \hspace{1cm} (4.88a)

\[ D_{n,m} = D_{n,m}(k,t) = D_{n,m}(t)j_n(-\Delta) \]  \hspace{1cm} (4.88b)

\[ D'_{n,m} = D'_{n,m}(k,t) = D'_{n,m}(t)j_n(-\Delta) \]  \hspace{1cm} (4.88c)

and \[ C_{n,m}(k,t), D_{n,m}(k,t), D'_{n,m}(k,t) \] are given by equations (4.76) and (4.80). Since

\[ j_n(\Delta) = \sqrt{\frac{\pi}{2\Delta^{n+1/2}(\Delta)}} \]  \hspace{1cm} (4.89)

where \[ j_{n+1/2} \] is the Bessel function of the first kind, and

\[ j_{n+1/2}(\Delta)j_{n'+1/2}(\Delta) = \sum_{s=0} (-1)^{s} \frac{\Gamma(n+n'+1+2s)\Gamma(n+n'+2s+2)}{8^{s}\Gamma(n+n'+s+2)\Gamma(n+n'+s+3/2)\Gamma(n+n'+s+3/2)} \]  \hspace{1cm} (4.90)

where \( \Gamma \) is the Gamma function, we obtain
\[ I'(n, n', t) = \int_0^\infty \exp(-2kh) j_n(t) j_{n'}(t) dk \]

\[
= \int_0^\infty \exp(-2kh) \frac{\pi}{2k \cos t} \left( -1 \right)^{s} \cos t \sum_{s=0}^\infty \frac{1}{s! \Gamma(n+n'+s+2) \Gamma(n+s+3/2) \Gamma(n'+s+3/2)} \Gamma(n+n'+2s+1) \Gamma(n+n'+2s+2) \Gamma(n+s+3/2) \Gamma(n'+s+3/2) \frac{\Gamma(n+n'+2s+1)}{(2h)^{n+n'+2s+1}} \frac{1}{(2h)^{2s}} \\
\]

\[
= \pi \frac{\cos \theta}{n+n'+2s+2} \sum_{s=0}^\infty \frac{(-1)^s \cos \theta}{s! \Gamma(n+n'+s+2) \Gamma(n+s+3/2) \Gamma(n'+s+3/2)} \Gamma(n+n'+2s+1) \Gamma(n+n'+2s+2) \Gamma(n+s+3/2) \Gamma(n'+s+3/2) \frac{\Gamma(n+n'+2s+1)}{(2h)^{n+n'+2s+1}} \frac{1}{(2h)^{2s}} \\
\]

Using

\[
\Gamma(n+s+3/2) = \frac{\Gamma(2n+2s+2)(2\pi)^{1/2}}{2^{2n+2s+2-1/2} \Gamma(n+s+1)} \\
\]

equation (4.91) becomes

\[ I(n, n', t) = \frac{1}{(2h)^{n+n'+2s+1}} \sum_{s=0}^\infty \frac{(-1)^s \cos \theta}{s! \Gamma(n+n'+s+2) \Gamma(n+s+3/2) \Gamma(n'+s+3/2)} \frac{\Gamma(n+n'+2s+1)}{(2h)^{n+n'+2s+1}} \frac{1}{(2h)^{2s}} \\
\]

Substituting \( I'(n, n', t) \) into (4.87a), \( I(n, m, n', m') \) can be obtained by means of

\[
I(n, m, n', m') = 4 \int_0^{\pi/2} C_{n, m}^m(t) n_{m'}(t) I'(n, n', t) dt \\
\]

for \( n+m+n'+m' \) being even;

\[ I(n, m, n', m') = 0 \]
for \(n+m+n'+m'\) being odd.

The series in equation (4.93) converges absolutely and can be accurately computed numerically when \(c \cos \theta < h\). If \(c \cos \theta \geq h\), the direct integration in equation (4.91) may be more efficient.

To calculate the integration over \(k\) in equation (4.87b), we write

\[
H(n,m,n',m') = (-1)^{n+m+n'+m'} \int \frac{c}{n,m}(t)T_{n',t}H_0'(n,n',t)dt
\]

\[+ \int C_{n,m}(t)T_{n',t}[H_1'(n,n',t)-H_2'(n,n',t)]dt\]

\[+ \int \frac{\pi}{2} C_{n,m}(t)T_{n',t}[H_3'(n,n',t)-H_4'(n,n',t)]dt (4.95)\]

where

\[
H_0'(n,n',t) = \frac{1}{\pi} k_e \exp(-2kh) \int \frac{1}{k - k_1} j_n(\Delta)j_{n'}(\Delta)dk
\]

\[
+ k_2 \exp(-2kh) \int j_n(\Delta)j_{n'}(\Delta)dk
\]

(4.96a)

and

\[
H_1'(n,n',t) = \frac{1}{\pi} k_1 \exp(-2kh) \int \frac{1}{k - k_1} j_n(\Delta)j_{n'}(\Delta)dk
\]

(4.96b)

for \(i \geq 1\). The definition of \(k_1\) and the integration path at the singular point in equation (4.96) are defined as in section (4.3.2).

Substituting equations (4.89) and (4.90) into equation (4.96), and also using equation (4.92), we obtain
\[ H_{0,2s+n+n'}(k_1) \]
\[ = \frac{k_2}{1 - 4a \cos t} \sum_{s=0}^{\infty} (-1)^s (\cos t)^{n+n'+s+2} \Gamma(n+n'+2s+2) \]
\[ \times \Gamma(n+s+3/2) \Gamma(n'+s+3/2) \]

\[ H_{0,2s+n+n'}(k_2) \]
\[ = \frac{k_1(2\cos t)^{n+n'}}{1 - 4a \cos t} \sum_{s=0}^{\infty} (-1)^s (\cos t)^{2s} \]
\[ \times \Gamma(n+n'+2s+2) \Gamma(n+s+1) \Gamma(n'+s+1) \]
\[ \times \Gamma(n+n'+s+2) \Gamma(2n+2s+2) \Gamma(2n'+2s+2) \]

\[ H_{0,2s+n+n'}(k_1) \]
\[ = \frac{k_2(2\cos t)^{n+n'}}{1 - 4a \cos t} \sum_{s=0}^{\infty} (-1)^s (\cos t)^{2s} \]
\[ \times \Gamma(n+n'+2s+2) \Gamma(n+s+1) \Gamma(n'+s+1) \]
\[ \times \Gamma(n+n'+s+2) \Gamma(2n+2s+2) \Gamma(2n'+2s+2) \]

\[ H_{0,2s+n+n'}(k_2) \]

and

\[ H_1'(n,n',t) = \frac{k_1}{1 - 4a \cos t} \sum_{s=0}^{\infty} (-1)^s (\cos t)^{n+n'+s+2} \Gamma(n+n'+2s+2) \]
\[ \times \Gamma(n+s+3/2) \Gamma(n'+s+3/2) \]

\[ H_{1,2s+n+n'}(k_1) \]
\[ = \frac{k_1(2\cos t)^{n+n'}}{1 - 4a \cos t} \sum_{s=0}^{\infty} (-1)^s (\cos t)^{2s} \]
\[
\frac{\Gamma(n+n'+2s+2)\Gamma(n+s+1)\Gamma(n'+s+1)}{s!\Gamma(n+n'+s+2)\Gamma(2n+2s+2)\Gamma(2n'+2s+2)}
\]

\[H'_{s+n+n'}(k_1)\]  \hspace{3cm} (4.97b)

for \(i\geq1\), where (see equations from (4.33) to (4.42))

\[H_0,^{s+n+n'}(k_1) = \int_{0}^{\infty} \frac{k^{s+n+n'}}{k-k_1} \exp(-2kh)dk \]

\[= \int_{0}^{\infty} \frac{k^{s+n+n'-1}}{k-k_1} \exp(-2kh)dk + k_1H_0,^{s+n+n'-1}(k_1) \]

\[= \frac{\Gamma(s+n+n')}{(2h)^{s+n+n'}} + k_1H_0,^{s+n+n'-1}(k_1) \]  \hspace{3cm} (4.98a)

\[H_0,^{0}(k_1) = \exp(-2k_1h)E_p(-2k_1h) \]  \hspace{3cm} (4.98b)

Similarly

\[H_0,^{s+n+n'}(k_2) = \frac{\Gamma(s+n+n')}{(2h)^{s+n+n'}} + k_2H_0,^{s+n+n'-1}(k_2) \]  \hspace{3cm} (4.98c)

\[H_0,^{0}(k_2) = \exp(-2k_2h)E_q(-2k_2h) \]  \hspace{3cm} (4.98d)

\[H_1,^{s+n+n'}(k_1) = \frac{\Gamma(s+n+n')}{(2h)^{s+n+n'}} + k_1H_1,^{s+n+n'-1}(k_1) \]  \hspace{3cm} (4.98e)

\[H_1,^{0}(k_1) = \exp(-2k_1h)E_{r_1}(-2k_1h) \]  \hspace{3cm} (4.98f)

It can be understood that \(\alpha=1\) for \(i=1,2\) and \(\alpha=-1\) for \(i=3,4\) in equation (4.97b). The definition of \(E_p(Z)\), \(E_q(Z)\) and \(E_{r_1}(Z)\) are given in section (4.3.2).

Apparently, for large \(c\) or \(k_1\), the series expression for
H(n,m,n',m') does not offer practical computation. The direct numerical integration of equation (4.96) is more efficient.

The formulations above provide a basis to obtain the analytic solution of the scattering potential of a submerged spheroid in head or following sea. The remaining problem is the existence and uniqueness of a finite solution of the infinite sets of equations (4.86). A sufficient condition is that the sum of the modulus of the coefficients in the equation is finite (e.g. Hulme^{58}). Further research is needed to prove that equation (4.86) satisfies this condition. However, it is not unreasonable to assume that the solution exists and is unique, as stable and converged results of the wave resistance on the spheroid have been obtained by Farrell^{33} using this method. Thus we may be able to find the solution directly without the solid proof of its existence and uniqueness first.

4.4.2. The added resistance - the x component of the drift force

One of the influences of ocean waves on a ship is to increase the resistance. The mechanism of this increase has not yet been fully understood. Based on the potential theory, this is usually explained by the nonzero time average of the pressure due to the unsteady potential on the body surface, so that a steady force and moment on the body is generated. The component of this force in the forward speed direction is usually regarded as added resistance. In the linearized potential theory, this steady force is neglected in estimation of a ship's oscillatory response to waves as being of higher order. But even in the case of small wave height and small body motion amplitude, the steady force can be important where the restoring force is small. As discussed in the
introduction, since the ship has no restoring force in the horizontal plane, this steady force cannot be disregarded.

In the general case, calculation of the steady force and moment on a ship needs exact solution of the nonlinear potential problem. By use of the perturbation theory, the steady force and moment are calculated up to the second order at the present (in fact this is the first order that the steady force begins to contribute). Maruo\textsuperscript{68} derived an equation for the added resistance from the momentum consideration

\[ \Delta R = 2\pi \left( \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{3\pi/2} |H(k_2, \alpha)|^2 \frac{k_2^2 (k_2 \cos \alpha - v \cos \beta)}{\sqrt{1 - 4 \tau \cos \alpha}} d\alpha d\gamma + \int_{-\pi/2}^{\pi/2} |H(k_1, \alpha)|^2 \frac{k_1^2 (k_1 \cos \alpha - v \cos \beta)}{\sqrt{1 - 4 \tau \cos \alpha}} d\alpha \right) \]

(4.99)

where \( \beta \) is the direction of the incoming wave, \( k_1, k_2 \) are given by (4.42a), and \( H(k, \theta) \) is the Kochin function and can obtained by

\[ H(k, \theta) = \int \int \left( \phi \frac{\partial^2}{\alpha^2} - \phi \frac{\partial^2}{\alpha^2} \right) \exp(kz + ikx \cos \theta + ikysin \theta) dS \]

(4.100)

for a submerged body. The potential \( \phi \) in equation (4.100) includes the linearized radiation potential and diffraction potential (see equation (2.37)). The computation of \( H(k, \theta) \) needs the integration over the body surface. Based on the coupled finite element method, however, this integration can be calculated over the localized finite element boundary \( S_j \), or

\[ H(k, \theta) = \int \int \left( \phi \frac{\partial^2}{\alpha^2} - \phi \frac{\partial^2}{\alpha^2} \right) \exp(kz + ikx \cos \theta + ikysin \theta) dS \]

(4.101)

In an experiment, the Kochin function \( H(k, \theta) \) can be obtained by the measurement of wave elevation \( \zeta \) along a line \( y=\text{const.} \). It is found that\textsuperscript{89,90}
\[ H(k_2, \theta) = \frac{g}{2\omega} \int_{-\infty}^{\infty} \exp(ik_2 \cos \theta) dx \]

\[ \frac{\exp(ik_2 y \sin \theta \text{sgn}(\cos \theta))}{2\pi \sin \theta \text{sgn}(\cos \theta) (1 + k_2/k_1 \cos \theta)} \]  \hspace{1cm} (4.102a)

\[ H(k_1, \theta) = \frac{g}{2\omega} \int_{-\infty}^{\infty} \exp(ik_1 x \cos \theta) dx \frac{\exp(ik_1 y \sin \theta)}{2\pi \sin \theta (1 + k_1/k_2 \cos \theta)} \]  \hspace{1cm} (4.102b)

As an alternative to equation (4.98), we will derive the equation to calculate the added resistance by the time average of the integration of the pressure over the wetted body surface. Generally, the hydrodynamic force \( \vec{F} \) and moment \( \vec{M} \) on a body can be written as

\[ \vec{F}(t) = \rho \int_{S(t)} \bar{p} n dS \]  \hspace{1cm} (4.103a)

\[ \vec{M}(t) = \rho \int_{S(t)} \bar{p} \bar{x} \times \bar{n} dS \]  \hspace{1cm} (4.103b)

where \( S(t) \) is the instantaneous position of the body surface. Substituting equation (2.42) into (4.103), we obtain

\[ \vec{F}(t) = -\rho \int_{S(t)} \left[ \text{Re}(\Phi_t) + \frac{1}{2} \text{Re}(V \Phi) \text{Re}(\Phi) + gz \right] n dS \]  \hspace{1cm} (4.104a)

\[ \vec{M}(t) = -\rho \int_{S(t)} \left[ \text{Re}(\Phi_t) + \frac{1}{2} \text{Re}(V \Phi) \text{Re}(\Phi) + gz \right] (\bar{x} \times \bar{n}) dS \]  \hspace{1cm} (4.104b)

where \( \Phi \) is the total potential including the steady potential \( \Phi(x,y,z) \) and unsteady potential \( \phi(x,y,z,t) \); the partial derivative with respect to \( t \) is calculated in the coordinate system fixed in space (see section 2.1). Using equations (2.9), (2.10) and (2.33), equation (4.104) becomes

\[ \vec{F}(t) = -\rho \int_{S(t)} \left\{ -U \frac{\partial \Phi}{\partial x} + \text{Re} \left( (i\omega \phi - \frac{\partial \phi}{\partial x}) e^{i\omega t} \right) \right. \\
\left. + \frac{U^2}{2} \frac{\partial^2 \phi}{\partial x^2} + U \phi \text{Re}(\phi e^{i\omega t}) + \frac{1}{2} \text{Re}(\phi e^{i\omega t}) \text{Re}(\phi e^{i\omega t}) \right\} n dS \]
\[ \overline{M}(t) = -p \iint_{S(t)} \left(-U^2 \frac{\partial \overline{\phi}}{\partial x} + \text{Re}[(i\omega - \frac{\partial \overline{\phi}}{\partial x})e^{i\omega t}] \right) dS \]

\[ + \frac{U^2}{2} \frac{\partial \overline{\psi}}{\partial x} + U \overline{\psi} \text{Re}(\overline{\psi}e^{i\omega t}) + \frac{1}{2} \text{Re}(\overline{\psi}e^{i\omega t}) \text{Re}(\overline{\psi}e^{i\omega t}) \]

\[ + gz \overline{(\mathbf{x} \times \mathbf{n})} dS \]

(4.105a)

From Taylor series, we have

\[ f(x + \alpha_1, y + \alpha_2, z + \alpha_3) = f(x, y, z) + \left( \frac{\partial f}{\partial x} \alpha_1 + \frac{\partial f}{\partial y} \alpha_2 + \frac{\partial f}{\partial z} \alpha_3 \right) + O(|\alpha|^3) \]

(4.106a)

\[ \overline{n}_S(t) = [\overline{n} + \overline{n} \times \mathbf{n} + \varepsilon^2 H\overline{n}]_{S_0} \]

(4.106b)

\[ [\overline{\mathbf{x}} \times \overline{n}]_{S(t)} = [\overline{\mathbf{x}} \times \overline{n} + \overline{\xi} \times \overline{n} + \overline{\Omega} \times (\overline{\mathbf{x}} \times \overline{n}) + \varepsilon^2 H(\overline{\mathbf{x}} \times \overline{n})]_{S_0} \]

(4.106c)

where \( \overline{\mathbf{x}}, \overline{\zeta}, \overline{\Omega} \) are defined by equations (2.3), (2.31) and (2.32); \( \varepsilon^2 H \) contains the terms \( u_i u_j \) of order \( O(|\alpha|^2) \) as defined by Ogilvie. Substituting equations (4.106) into equations (4.105), the integration over the instantaneous body surface can be transformed to that over the mean position, or

\[ \overline{F}(t) = -p \iint_{S_0} \left[ 1 + \left( \frac{\partial f}{\partial x} \alpha_1 + \frac{\partial f}{\partial y} \alpha_2 + \frac{\partial f}{\partial z} \alpha_3 \right) + \frac{1}{2} \left( \frac{\partial f}{\partial x} \alpha_1 + \frac{\partial f}{\partial y} \alpha_2 + \frac{\partial f}{\partial z} \alpha_3 \right)^2 \right] \]

\[ + \left(-U^2 \frac{\partial \overline{\phi}}{\partial x} + \text{Re}[(i\omega - \frac{\partial \overline{\phi}}{\partial x})e^{i\omega t}] \right) \]

\[ + \frac{U^2}{2} \frac{\partial \overline{\psi}}{\partial x} + U \overline{\psi} \text{Re}(\overline{\psi}e^{i\omega t}) + \frac{1}{2} \text{Re}(\overline{\psi}e^{i\omega t}) \text{Re}(\overline{\psi}e^{i\omega t}) \]

\[ + \frac{1}{2} \text{Re}(\overline{\psi}e^{i\omega t}) \text{Re}(\overline{\psi}e^{i\omega t}) + gz \overline{(\mathbf{x} \times \mathbf{n})} + \varepsilon^2 H dS \]
Taking the time average of equation (4.107), noticing that only the terms associated with \( \cos^2 \omega t \) or \( \sin^2 \omega t \) and of order \( |\bar{a}|^2 \) will contribute, we obtain the steady force \( \bar{F} \) and moment \( \bar{M} \) on the submerged body

\[
\bar{F} = \bar{F}_1 + \bar{F}_2 \quad (4.108a)
\]

\[
\bar{M} = \bar{M}_1 + \bar{M}_2 \quad (4.108b)
\]

where

\[
\bar{F}_1 = -\rho \iint_{S_0} \left[ -U^2 \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \bar{V} \bar{V} \phi \phi + gz \right] \, n \, dS
\]

\[
\bar{F}_2 = -\rho \iint_{S_0} \left[ \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right] \left( -U^2 \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \bar{V} \bar{V} \phi \phi + gz \right) \, n \, dS
\]

\[
\bar{M}_1 = -\rho \iint_{S_0} \left[ \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right] \left( -U^2 \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \bar{V} \bar{V} \phi \phi + gz \right) \, n \times n \, dS
\]

\[
\bar{M}_2 = \rho \iint_{S_0} \left[ \frac{1}{2} \left( \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right) \left( -U^2 \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \bar{V} \bar{V} \phi \phi + gz \right) \right] \, n \times n \, dS
\]
\[ -\rho \iint_{S_0} \left[ -U \frac{\partial \Phi}{\partial x} + \frac{U^2}{2} \nabla \Phi \nabla \Phi + gz \right] \text{Re}(\varepsilon^2 H_0) \text{ndS} \quad (4.109a) \]

\[ \overline{F}_2 = -\rho \iint_{S_0} \frac{1}{4} \nabla \Phi \nabla \Phi + \frac{1}{2} \text{Re}\left( (\alpha \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial z}) (i\omega \Phi - U \frac{\partial \Phi}{\partial x} + UV \Phi) \right) \text{ndS} \]

\[ -\rho \iint_{S_0} \text{Re}\left( (i\omega \Phi - U \frac{\partial \Phi}{\partial x} + UV \Phi)(\vec{n}_0 \times \vec{n}) \right) \text{dS} \quad (4.109b) \]

and

\[ \overline{M}_1 = -\rho \iint_{S_0} \left[ -U \frac{\partial \Phi}{\partial x} + \frac{U^2}{2} \nabla \Phi \nabla \Phi + gz \right] \vec{x} \cdot \vec{n} \text{dS} \]

\[ -\rho \iint_{S_0} \frac{1}{4} \left( \alpha \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial z} \right)^2 \left( -U \frac{\partial \Phi}{\partial x} + \frac{U^2}{2} \nabla \Phi \nabla \Phi + gz \right) \vec{x} \cdot \vec{n} \text{dS} \]

\[ -\rho \iint_{S_0} \text{Re}\left( (\alpha \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial z}) ( -U \frac{\partial \Phi}{\partial x} + \frac{U^2}{2} \nabla \Phi \nabla \Phi + gz ) \right) \vec{x} \cdot \vec{n} \text{dS} \]

\[ \left[ (\vec{\xi}_0 \times \vec{n} + \vec{n} \times (\vec{x} \times \vec{n})) \right] \text{dS} \]

\[ -\rho \iint_{S_0} \left[ -U \frac{\partial \Phi}{\partial x} + \frac{U^2}{2} \nabla \Phi \nabla \Phi + gz \right] \text{Re}\left[ \varepsilon_0^* (\vec{n}_0 \times \vec{n}) \right] \text{dS} \]

\[ -(\vec{x} \times \vec{n}) \text{dS} \quad (4.110a) \]

\[ \overline{M}_2 = -\rho \iint_{S_0} \frac{1}{4} \nabla \Phi \nabla \Phi + \frac{1}{2} \text{Re}\left( (\alpha \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial z}) (i\omega \Phi - U \frac{\partial \Phi}{\partial x} + UV \Phi) \right) \text{dS} \]

\[ (\vec{x} \times \vec{n}) \text{dS} \]

\[ -\rho \iint_{S_0} \text{Re}\left( (i\omega \Phi - U \frac{\partial \Phi}{\partial x} + UV \Phi) [\vec{n}_0^* \times \vec{n} + \vec{n}_0^* \times (\vec{x} \times \vec{n})] \right) \text{dS} \quad (4.110b) \]

where \( \vec{n} = \text{Re}(\vec{\alpha} e^{i\omega t}) \); \( \vec{n}_0, \vec{\xi}_0, H_0 \) are similarly defined; and the star * denotes the conjugate of the complex.
The first integrations in equations (4.109a) and (4.110a) are the steady force and moment on a submerged body moving in calm water. The other three integrations in equations (4.109a) and (4.110a) are also contributions from the steady potential, but they exist only when there are oscillatory motions. These terms arise from the transform of the integration of the steady potential over the instantaneous body surface to the mean position. We retain them in the equations since the steady potential may not be a small quantity (see the comments after equation (2.29) and (2.30)).

If the steady potential is also a small quantity, equations (4.109) and (4.110) can be simplified as

\[ \bar{F}_1 = -\rho \int_S \left( -\frac{U^2}{2} \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \nabla \phi \cdot \nabla \phi + gz \right) \overline{n} \, dS \]  

(4.111a)

\[ \bar{F}_2 = -\rho \int_S \left[ \frac{1}{4} \nabla \phi \cdot \nabla \phi \overline{\alpha}^* + \frac{1}{2} \text{Re} \left( \alpha_0 \cdot \nabla (i\omega \phi - U \frac{\partial \phi}{\partial x}) \right) \right] \overline{n} \, dS \]  

(4.111b)

\[ \bar{M}_1 = -\rho \int_S \left( -\frac{U^2}{2} \frac{\partial \phi}{\partial x} + \frac{U^2}{2} \nabla \phi \cdot \nabla \phi + gz \right) \overline{\alpha \times n} \, dS \]  

(4.112a)

\[ \bar{M}_2 = -\rho \int_S \left[ \frac{1}{4} \nabla \phi \cdot \nabla \phi \overline{\alpha}^* + \frac{1}{2} \text{Re} \left( \alpha_0 \cdot \nabla (i\omega \phi - U \frac{\partial \phi}{\partial x}) \right) \right] \overline{n} \, dS \]  

(4.112b)

These equations apply only to submerged bodies. For the floating body, the extra terms due to the change of the wetted body surface, \( S_\eta \), by the free surface wave elevation and the body motion in the vertical...
direction should be included. Thus, while retaining equation (4.107), for the floating body, we have an extra term

\[
\bar{F}_{\eta}(t) = -\int p \, ndS \tag{4.113}
\]

Using the equation (4.106), equation (4.113) becomes

\[
\bar{F}_{\eta}(t) = -\int_{\text{WL}}^{\text{WL}} dC \int \left[ p + \bar{\alpha} \bar{V} p \right] ndz + O(\alpha^3) \tag{4.114}
\]

where WL denotes the waterline of the floating body in the calm water and \( \eta \) is the wave elevation along the waterline. Using Bernoulli's equation, we obtain

\[
\bar{F}_{\eta}(t) = \frac{\rho g}{2} \int_{\text{WL}} \left[ (n - \alpha_3)^2 n_{WL} \, dC + O(n^3) \right] \tag{4.115}
\]

where \( n_{WL} \) is the normal of the body surface along the waterline and the steady potential has been regarded as a small quantity. Since

\[
n = -\frac{1}{g} \left\{ \left( \frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) [U \bar{\phi} + \text{Re}(\phi e^{i\omega t})] + \frac{1}{2} \bar{V} \left[ U \bar{\phi} + \text{Re}(\phi e^{i\omega t}) \right] \bar{V} \bar{\phi} + \text{Re}(\phi e^{i\omega t}) \right\}
\]

\[
= -\frac{1}{g} \left\{ -U^2 \bar{\phi} \frac{\partial}{\partial x} + \frac{U^2}{2} \bar{V} \bar{\phi} + \text{Re}[(i\omega - U^2 \bar{\phi}) e^{i\omega t}] + U \bar{V} \bar{\phi} \text{Re}(\phi e^{i\omega t}) \right\}
\]

\[
+ \frac{1}{2} \text{Re}(V e^{i\omega t}) \text{Re}(V e^{i\omega t}) \right\}
\]

\[
= -\frac{1}{g} \left\{ (i\omega - U \bar{\phi}) e^{i\omega t} \right\} + U \bar{V} \bar{\phi} \text{Re}(\phi e^{i\omega t})
\]

\[
+ \frac{1}{2} \text{Re}(V e^{i\omega t}) \text{Re}(V e^{i\omega t}) \right\} \tag{4.116}
\]

where \( \bar{\eta} \) is the wave elevation due to forward speed of the ship in calm water, equation (4.115) becomes

\[
\bar{F}_{\eta}(t) = \frac{\rho g}{2} \int_{\text{WL}} \bar{n}_{WL} \left\{ n^2 - \frac{2}{g} \text{Re}[(i\omega - U \bar{\phi}) e^{i\omega t}] \right\}
\]
Taking the time average, we obtain

\[ \bar{F} = \bar{F}_1 + \bar{F}_2 \] (4.118)

where

\[ \bar{F}_1 = \frac{\rho}{2} \int_{WL} \bar{\eta}^2 \, dC \] (4.119a)

\[ \bar{F}_2 = \frac{\rho}{4} \int_{WL} |\eta_0 - \alpha_{03}|^2 \, dC \] (4.119b)

Here \( \eta_0 \) in equation (4.119b) is the wave elevation due to the unsteady potential only and is given by

\[ \eta_0 = -\frac{1}{g} (i\omega \phi - U \frac{\partial \phi}{\partial x}) \] (4.120)

After a similar derivation, we obtain

\[ \bar{M}_1 = \frac{\rho}{2} \int_{WL} (\bar{\eta} \times \bar{\eta}_n) \, \eta^2 \, dC \] (4.121a)

\[ \bar{M}_2 = \frac{\rho}{4} \int_{WL} (\bar{\eta} \times \bar{\eta}_n) |\eta_0 - \alpha_{03}|^2 \, dC \] (4.121b)

Combining equations (4.111) and (4.120), equations (4.112) and (4.121), the steady force and moment on the floating body can be written as

\[ \bar{F} = \bar{F}_0 + \Delta \bar{F} \] (4.122a)

\[ \bar{M} = \bar{M}_0 + \Delta \bar{M} \] (4.122b)

where
\[ F_0 = F_1 + F_{n1} \]  
\[ M_0 = M_1 + M_{n1} \]

are the contributions from the steady potential and the \( x \) component of \( F_q \) is the wave resistance; and

\[ \Delta F = F_2 + F_{n2} \]  
\[ \Delta M = M_2 + M_{n2} \]

are the contributions from the unsteady potential and the \( x \) component of \( \Delta F \) is the added resistance.

A special case of equations (4.124) is when \( U=0 \). We have

\[ \Delta F = \rho \int_{S_0} \left\{ \frac{1}{4} \nabla \phi \cdot \nabla \tilde{\phi} + \frac{1}{2} \text{Re} [\alpha_{0} \cdot \nabla (i \phi)] \right\} \mathbf{n} dS \]

\[ - \rho \int_{S_0} \frac{1}{2} \text{Re} [(i \phi) (\tilde{n}_0 \times \mathbf{n})] dS + \frac{\rho}{q_{WL}} n_{WL} \left| \eta_0 - \alpha_{30} \right|^2 dC \]  

\[ \Delta M = \rho \int_0 \left\{ \frac{1}{4} \nabla \phi \cdot \nabla \tilde{\phi} + \frac{1}{2} \text{Re} [\alpha_{3} \cdot \nabla (i \phi)] \right\} (\mathbf{x} \times \mathbf{n}) dC \]

\[ - \rho \int_{S_0} \frac{1}{2} \text{Re} [(i \phi) (\mathbf{x} \times \mathbf{n} + \tilde{n}_0 \times (\mathbf{x} \times \mathbf{n}))] dS \]

\[ + \frac{\rho}{q_{WL}} (\mathbf{x} \times \mathbf{n}) \left| \eta_0 - \alpha_{03} \right|^2 dC \]

It can be shown that these equations are identical to those for the drift force and moment without forward speed obtained by Pinkster and Oortmerssen\(^1\).

A more general case of equation (4.124) is for a surface piercing and articulated structure in current and waves. An extra term due to the bottom of the fluid should be included. This term was first observed
by Drake, Eatock Taylor and Matsui in the case without current\(^{20}\). Their results can be straightforwardly extended to the case with current following a similar derivation above. But since the disturbance in the ocean decays exponentially in the vertical direction, the contribution from the sea bed is much smaller than that from the free surface, as has been noticed by Drake et al. The contribution from the sea bed may become important in shallow water, but it has little practical significance for the present offshore industry, since articulated structures are designed to operate in deep water. Thus the extension of equation (4.124) to include the sea bed contribution is omitted.

4.4.3. The elastic deformation of the ship

While the six degrees of freedom motion of a ship dominates its overall response to waves, the flexural deformation dominates the dynamic stress in the ship hull. The dynamic stress is superposed on the stresses generated by hydrostatic pressure and various loads on the ship. A ship can be split as a consequence of the total stress being excessive. Thus it is important to provide some reliable method to predict the dynamic stress.

However the theoretical prediction of the dynamic stress is not an easy task. Even the vibration problem for a practical ship hull in vacuo needs tedious numerical modelling. The problem of a ship advancing in waves is further complicated by the nonlinear interaction between the structure deformation and fluid loading. Therefore, like the rigid body analysis, the problem of ship structure deformation is usually linearised. This permits the structure and fluid problems to be solved independently (e.g. Bishop and Price\(^{8}\), Eatock Taylor\(^{22}\)).
Generally, the governing equation for the elastic deformation \( \bar{X} \) of the structure can be written

\[
M \ddot{\bar{X}} + B \dot{\bar{X}} + K \bar{X} = \bar{F} \tag{4.126}
\]

where \( M, B, K \) is the mass, damping and stiffness of the structure respectively; \( \bar{F} \) is the external exciting force including fluid loads. Equation (4.126) assumes that the continuum properties of the structure have been discretized in some approximate manner (finite element method for example)\(^2^1\). The solution of equation (4.126) can be usually found by the so called modal analysis\(^7\). Corresponding to equation (4.126), the undamped free vibration of the structure in vacuo can be obtained from

\[
M \ddot{\bar{X}} + K \bar{X} = 0 \tag{4.127}
\]

If the structure is approximated by \( N \) degrees of freedom, \( N \) natural frequencies \( \omega_r \) (\( r=1,\ldots,N \)) and \( N \) corresponding mode shapes \( \bar{\psi}_r \) can be obtained from equation (4.127). The general solution of equation (4.121) then can be expressed by the sum of principal coordinates \( P_r \) associated with the mode shapes \( \bar{\psi}_r \)

\[
\bar{X} = \sum_{r=1}^{N} P_r(t) \bar{\psi}_r \tag{4.128}
\]

Substituting this into equation (4.121), we can get the governing equation for \( r \)th principal coordinate

\[
M \ddot{P}_r(t) + 2\omega_r \dot{P}_r(t) + \omega_r^2 P_r(t) = -\bar{\psi}_r \bar{F} \tag{4.129}
\]

The equation has employed the orthogonality relation

\[
\delta_{ij} K_{ij} = \delta_{ij} \omega_i^2 = \bar{\psi}_i \bar{\psi}_j \quad \text{(e.g. Bishop et al\(^{10}\))}; \quad \text{and the assumption is made}
\]
that the structural damping has a similar property to the structure stiffness so that the orthogonality relation can be also applied to it.

The equation (4.129) is based on the dry modes. The remaining problem is to find the exciting forces. The exciting forces may have many different components. One of them commonly concerned is the wave loading which can be obtained from the potential theory.

The main difference between the potential theories of the rigid body and a flexible structure is that vibrations will contribute extra terms to the body surface condition. Generally, from equation (4.128), equation (2.3) should be written as

$$a = \bar{\xi} + \sum_{r=1}^{N} p_r(t)\bar{\psi}_r + \bar{\eta}\times\bar{X}$$

(4.130)

if both amplitudes of rigid oscillation and flexural vibration are small. After a similar derivation to that in chapter two, the body surface condition (2.34) becomes

$$\frac{\partial}{\partial n}[\phi(x,y,z)] = i\omega \sum_{j=1}^{6} \xi_j \bar{n}_j + U \sum_{j=1}^{6} \xi_j \bar{m}_j$$

$$+ i\omega \sum_{r=1}^{N} p_r\bar{\psi}_r \cdot \bar{n} + U \sum_{r=1}^{N} p_r\bar{\psi}_r \cdot \bar{m}$$

(4.131)

where $\bar{\xi}_j$, $\bar{n}$, $\bar{m}$ are defined by equations (2.31), (2.35a), (2.36a) respectively; and

$$p_r(t) = \text{Re}[p_r e^{i\omega t}]$$

(4.132)

Thus, if the total potential $\phi(x,y,z)$ is decomposed as

$$\phi = \eta_0(\phi_0 + \phi_7) + \sum_{j=1}^{N+7} \xi_j \phi_j + \sum_{j=8}^{N+7} p_j \phi_j$$

(4.133)

the body surface condition for each component can be written as
\[ \frac{\partial \Phi_j}{\partial n} = i \omega \psi_j + U \psi_j \quad j=1, \ldots, 6 \]  
\[ \frac{\partial \Phi_j}{\partial n} = - \frac{\partial \Phi_0}{\partial n} \quad j=7 \]  
\[ \frac{\partial \Phi_j}{\partial n} = i \omega \bar{\psi}_{j-7} \cdot \bar{n} + U \bar{\psi}_{j-7} \cdot \bar{m} \quad j=8, \ldots, N+7 \]  

Here, the first eight components \((j=0, \ldots, 7)\) of the potential are associated with the modes of rigid motion, while the rest of the components are the contribution from the flexural vibration. It can be easily seen that the inclusion of the elastic defromation in the body surface condition does not offer any difficulty to the potential theory. If the principal modes \( \bar{\psi}_p \) have been found by structural dynamics, the method used obtaining the solution for the rigid body can be directly extended to find the solution for the flexural vibration.

Having obtained the potential and following the derivation in section (2.6), we find the hydrodynamic loading in equation (4.129)

\[ \bar{F} = \bar{F}_s + \bar{F}_u + \bar{F}_u' \]  
\[ \bar{M} = \bar{M}_s + \bar{M}_u + \bar{M}_u' \]  

where \( \bar{F}_s \), \( \bar{M}_s \) are given by equation (2.54) and contribute a steady force and moment; \( \bar{F}_u \) and \( \bar{M}_u \) are given by (2.55); and \( \bar{F}_u' \) and \( \bar{M}_u' \) are defined by

\[ \bar{F}_u' = \sum_{j=8}^{N+7} \tau_{1j} \bar{I} + \tau_{2j} \bar{J} + \tau_{3j} \bar{K} \]  
\[ \bar{M}_u' = \sum_{j=8}^{N+7} \tau_{4j} \bar{I} + \tau_{5j} \bar{J} + \tau_{6j} \bar{K} \]  

where

\[ \tau_{1j} = \omega^2 \nu_{1j} - i \omega \lambda_{1j} \]
\[ (4.137) \]
\[
\int_{S} \rho \Phi_{j} + \Phi_{j} \int_{S} n_{i} dS = 0
\]
for \( i = 1, \ldots, 6, j = 8, \ldots, N+7. \)

As \( \Phi_{j} \) \( (j > 8) \) satisfy the same free surface condition and radiation condition, it can be easily shown that \( \tau_{ij} \) satisfy the Timman and Newman relation provided the condition (2.60) is satisfied.

A complete solution (probably the only one so far) of this problem has been obtained by Bishop et al\(^{9}\) using the source distribution method. As an approximation, they neglected the contribution from \( \Phi \) and took \( \Phi = -u_{i} \) (equation (2.26)). As a consequence, forward speed will not affect the body surface condition associated with translation. This may be justified for a slender ship. In the general case however, we may see from equations (3.51), (3.52) and (3.56), (3.57) in the two dimensional case and equations (4.20) and (4.22), (4.23) in the three dimensional case that the contribution from \( \Phi \) is significant.
5. Conclusions

This thesis has analysed the hydrodynamic problems of a ship advancing in waves at constant forward speed. The problem is solved based on the linearized potential theory and using the method of coupling finite elements and the boundary integral equation. The theoretical basis and numerical process have been tested for the case of submerged bodies. The success of these tests shows that the principle of the present method can be applied to the general case of a surface ship advancing in waves. It also shows that present method can be employed in more sophisticated mathematical models, such as the nonlinear potential theory. From the present numerical analysis and results for submerged bodies, the following conclusions can be drawn:

1. The coupled finite element method is one of the most promising methods in hydrodynamics. The use of the shape function avoids the integration of the Green function over the body surface, which is difficult especially when the body shape is highly irregular; and it also avoids the calculation of the second order derivatives of the steady potential on the body surface which appear in the body surface condition on the unsteady potential.

2. Desirable accuracy is much easier to get for the two dimensional problems. The computer time in the three dimensional problem very much depends on the form of the Green function. The forms in Wehausen and Laitone's classic paper, both for uniform forward speed motion and for periodic oscillatory motion with uniform forward speed are not suitable for numerical analysis. The author strongly warns that these forms
should not be used directly. The Green function for uniform forward speed motion obtained by Noblesse, and the form of the Green function for the periodic motion with uniform forward speed obtained by the author have much better numerical performance.

3. The two dimensional results show that the wave resistance and lift on cylinders are significantly affected by the water depth. The resistance is increased with the reduction of the water depth. The interaction between two cylinders moving together is important. For two identical circular cylinders moving one behind the other, it is observed numerically that the total resistance can be zero.

4. The two dimensional results also show that the most important influence of forward speed on the hydrodynamic coefficients of the cylinder is mainly in the low frequency region and the effect will progressively disappear when the frequency tends to infinity. But the behaviour at the critical point is highly irregular.

5. The numerical solution of the three dimensional N-K problem is further complicated for practical ships. The slenderness of most conventional ships requires a fine mesh so that the potential can satisfy the body surface condition at bow and stern accurately, where the pressure dominates the resistance. In this sense the three dimensional N-K problem is much more expensive than the oscillatory motion problem.

6. As the problem of a ship advancing in waves has the steady potential due to forward speed in its body surface condition, it needs an even finer mesh than the three dimensional N-K problem. Careful con-
sideration is required in the numerical process to reduce the computer
time. It is particularly important to use an appropriate Green function
form.

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Fig. 1a The comparison of the pitch response; dimension of the barge LxH=2.4x0.2x0.185

Fig. 1b The comparison of heave response; dimension of the barge LxH=2.4x0.2x0.185

Fig. 2a The comparison of the pitch response; dimension of the barge LxH=2.4x0.4x0.185

Fig. 2b The comparison of heave response; dimension of the barge LxH=2.4x0.4x0.185
Figure 3a: The comparison of pitch response
dimensional of the barge LX6W2.4x8.8x8.185

Figure 3b: The comparison of the heave response;
dimensional of the barge LX6W2.4x8.8x8.185

Figure 4. Definition of geometry and fluid regions
Figure 5. 12 element mesh for submerged circular cylinder
I

\[ \text{Froude number } F_n = \frac{U}{(gh)^{1/2}} \]

Fig. 5c Resistance on a circular cylinder in water of different depth, \( h/2a \)

\[ \text{Froude number } F_n = \frac{U}{(gh)^{1/2}} \]

\[ d = 4.5a, d = 4a, d = 3.5a, d = 4.5a, d = 4a \]

\[ b/a = 0.75, b/o = 0.5, b/a = 0.75, b/o = 0.5 \]

Fig. 6a Lift on a circular cylinder in water of different depth, \( h/2a \)

Fig. 6b Lift on elliptical cylinders in infinite water depth, \( h/2a \)
Figure 8. Geometry for multi-body problem
Fig. 10a: The influence of forward speed on added mass of a submerged circular cylinder: sway

Fig. 10b: The influence of forward speed on added mass of a submerged circular cylinder: heave

Fig. 10c: The influence of forward speed on added mass of a submerged circular cylinder: sway
Fig. 11a. The influence of forward speed on damping coefficient of a submersed circular cylinder: sway

Fig. 11b. The influence of forward speed on damping coefficient of a submersed circular cylinder: heave

Fig. 11c. The influence of forward speed on damping coefficient of a submersed circular cylinder: sway

Fig. 11d. The influence of forward speed on damping coefficient of a submersed circular cylinder: heave
Figure 12. A typical mesh of 24 elements for the sphere
Fig. 13  A mesh for the spheroid
Fig. 1a: $r = 1, z = 2.8, F_0 = 0.2, z = 1.5625$
Fig. 1b: $r = 1, z = 2.8, F_0 = 0.2, z = 1.5625$
Fig. 1c: $r = 1, z = 2.8, F_0 = 0.2, z = 1.5625$
Fig. 1d: $r = 1, z = 2.8, F_0 = 0.2, z = 1.5625$
Green Function (1.33)
\* Green Function (1.27)

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Fig.1a: \( r = l \), \( \mu_c = -2.0 \), \( F_n = 0.2 \) \( \psi_c = 1.5625 \)

Fig.1b: \( r = l \), \( \mu_c = -2.0 \), \( F_n = 0.2 \) \( \psi_c = 1.5625 \)

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Fig.15a: \( r = l \), \( \mu_c = 2.0 \), \( F_n = 0.1 \) \( \psi_c = 0.3906 \)

Fig.15b: \( r = l \), \( \mu_c = 2.0 \), \( F_n = 0.1 \) \( \psi_c = 0.3906 \)
Fig. 16a $r=1, \theta=2.2$, $\mathcal{F} n=0.6, \nu=0.1736$

Fig. 16b $r=1, \theta=2.2$, $\mathcal{F} n=0.6, \nu=0.1736$

Fig. 16c $r=1, \theta=2.2$, $\mathcal{F} n=0.6, \nu=0.1736$

Fig. 16d $r=1, \theta=2.2$, $\mathcal{F} n=0.6, \nu=0.1736$
Fig. 16a r = 1 \phi \pi/2 \ zc = 2.8, \ Fr = 0.5 \ \nu = 0.1736

Fig. 16b r = 1 \phi \pi/2 \ zc = 2.8, \ Fr = 0.5 \ \nu = 0.1736

Fig. 16c r = 1 \phi \pi/2 \ zc = 2.8, \ Fr = 0.5 \ \nu = 0.1736

Fig. 16d r = 1 \phi \pi/2 \ zc = 2.8, \ Fr = 0.5 \ \nu = 0.1736
Fig. 16a: \( r = 1, \theta = \pi, z = 2, h, \quad F_n = 0,6, \quad v_x = 0,1736 \)

Fig. 16b: \( r = 1, \theta = \pi, z = 2, h, \quad F_n = 0,6, \quad v_x = 0,1736 \)

Fig. 17: The comparison of surge damping coefficients by the finite element method and by the first approximation. \( F_n(0) \times 10^{-3} = 0,8 \)

Fig. 18a: The comparison of surge damping coefficients by the finite element method and by the first approximation. \( F_n(0) \times 10^{-3} = 0,4 \)
Fig. 18a: The comparison of epoxy damping coefficients by the finite element method and by the first approximation, for $\alpha^2 = -0.4$.

Fig. 18b: The comparison of beng damping coefficients by the finite element method and by the first approximation, for $\alpha^2 = -0.4$. 

* Results by Finite element method
— Results by First approximation