Three-Dimensional Vortex Flows in Distorted Pipes:
Theory & Computation

PhD Thesis

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To my late father, Haji Nazir Ahmad
ABSTRACT

Three related pipeflow problems are studied analytically and computationally. Firstly the development of an arbitrary three-dimensional (starting) velocity profile is addressed for flow in a straight pipe. Results are presented for different values of initial disturbance. A Hagen-Poiseuille starting condition is also considered for this geometry with the addition of forcing terms which then set up a three-dimensional flow field downstream. Secondly, the influence of curvature on a pipeflow is discussed for a pipe that starts bending uniformly after an initial straight section. The motion depends on a parameter, the alternative Dean number $K$. The relative curvature $\delta$ is taken to be small. Thirdly, the fluid motion through a straight pipe which experiences an abrupt small angular bend or corner is considered. In all three cases the pipe is of circular cross-section, and the Reynolds number is taken to be large. The starting condition for the distorted pipes is that of Hagen-Poiseuille flow, which then becomes three-dimensional however due to the pipes' distortions. Two numerical techniques are developed to solve the three-dimensional vortex equations, making use of a forward marching scheme in the streamwise direction, $x$. In all three geometries, the flow starts in a boundary-layer fashion for small values of $x$. The results are presented for different values of $K$ for the curved pipe and different angle values of the normalized pipe bend $\alpha$ for the cornered pipe. Both short-scale and long-scale adjustments of the pipe flow, due to the presence of the abrupt curving or cornering, are examined, yielding upstream- and downstream-influence properties. Although this work covers flow characteristics arising from curvature and cornering, the computational scheme(s) derived can be used to study flow in pipes of a general cross-section. The equations of motion are developed and use of conformal mappings is suggested to obtain the flow field arising from pipeflows of any general cross-sectional area, as a starting point for future work.
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ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Professor F.T. Smith F.R.S for his support and encouragement throughout the course of this work. Without his continual help, guidance and above all patient & friendly manner, this thesis would not have been completed.

I am grateful to Dr. B.T. Dodia and Mr. A. Sanmuganathan for generously sharing their computing knowledge with me. Mr. M. Adiseshiah M.S., F.R.C.S (consultant vascular surgeon, UCL Hospitals) is thanked for his interest shown in my work and giving me an insight into the potential clinical applications of this study. I am also grateful to Mr. M. Ahmed for the diagrams in this thesis.

I would like to thank my family for both their financial and moral support throughout.

Financial support from E.P.S.R.C., U.K is gratefully acknowledged.
Chapter 1

INTRODUCTION

In 1808 Thomas Young introduced his Croonian lecture to the Royal Society on the function of the heart and arteries with the words:

_The mechanical motions, which take place in an animal body, are regulated by the same general laws as the motions of inanimate bodies ... and it is obvious that the inquiry, in what manner and in what degree, the circulation of the blood depends on the muscular and elastic powers of the heart and of the arteries, supposing the nature of those powers to be known, must become simply a question belonging to the most refined departments of the theory of hydraulics._

During his lifetime Young was both a practising physician and a professor of physics, so this was a natural approach to physiology; like many other scientists in the nineteenth century, he paid scant attention to the distinction between biological and physical science. Although Young is remembered today mainly for his work on the wave theory of light (in particular the double slit experiment) and because the elastic modulus of materials is named after him, he also wrote authoritatively about optic mechanisms, colour vision, and the blood circulation, including wave propagation in arteries.

This polymath tradition seems to have been particularly strong among the early students of the circulation, as names like Borelli, Hales, Bernoulli, Euler, Poiseulle, Helmholtz, Fick, and Frank testify; but as science developed, so did specialization and the study of the cardiovascular system became sep-
arated from physical science. This process was not, of course, complete because collaborative work between scientists from different disciplines has always gone on. However, its scale was quite limited, and many medical and physiological workers found it difficult to comprehend because of their inadequate background in mathematics and mechanics, just as physical scientists found the complexity and empiricism of physiological studies, as well as the terminology, forbidding.

The separation caused by specialization has now assumed new importance. Over the last forty years physical scientists and engineers have made considerable contributions to the understanding of the mechanics of the circulation. These have resulted in strongly stimulated collaborative research, but at the same time have made the field increasingly difficult for those without a background in mathematics and physics.

This thesis addresses certain fundamental problems in internal flows, with many real applications. There is clear relevance to plumbing and oil-pipe-line configurations as well as to physiological flows.

The motivation for this work was due to its relevance to physiological situations. It is becoming increasingly accepted that haemodynamic factors play a role in the initiation of atherosclerosis, the commonest form of arterial disease. Certain sites in the arterial system are particularly prone to atherogenesis. Examples include the inner bend of the aortic arch, the carotid bulb, the curved coronary arteries and bifurcations (both natural and the junctions with coronary by-pass grafts). Many of these sites occur where there is rapid expansion of vessel cross-sectional area or sharp streamwise curvature of the wall, with the consequence that steady flow through the system would be associated with a sharp adverse pressure gradient at the wall and hence flow separation. Moreover, when the motion is steady, regions of separated flow are invariably associated with low fluid velocities, in nearly stagnant eddies, and hence with small values of wall shear stress.

Early experimental work by Fry (1968) and Cornhill & Roach (1976) who
put animals (particularly rabbits and dogs) on a high cholesterol diet for a few weeks, suggested that fatty streaks developed on the outer wall of the aortic arch. Caro et al. (1971) found that a normal, or a slightly higher than normal cholesterol diet leads to fatty layers being deposited on the inner wall of the aortic arch.

Blood flow in arteries is pulsatile, with the consequence that both the pressure gradients and the wall positions vary with time. Modelling arterial blood flow therefore requires a good understanding of time-dependent flow through pipes of complicated geometries, thus providing the fluid dynamicist with a formidable task. However, in this study we wish to address some basic pipeflows.

In theoretical, computational and experimental terms problems associated with the fluid motion through an abruptly cornered tube or through a uniformly curved tube have attracted much attention, mainly because of their relevance to many engineering or physiological situations. In the steady fully-developed motion for the curved pipe case, for example, the centrifugal force tends to drive the fluid in the interior towards the outside bend and a return motion is then instigated near the tube wall (see, for example, Dean 1927, 1928; McConalogue & Srivastava 1968; Collins & Dennis 1975; Smith 1975; Van Dyke 1978; Dennis 1980; Kitoh 1987). The axial shear-stress therefore tends to be greatest at the outside bend. On the other hand, oscillatory or pulsatile flows can produce secondary streaming in the opposite direction (Lyne 1971; Smith 1975; Blennerhassett 1976), while Singh (1974) has shown that, for certain initial profiles, the shear in the entry region of a curved pipe may be greatest, initially, at the inside bend in steady flow, before crossing over to shear maximum at the outside. Smith (1976v) has shown that, in a related branching-flow problem, a more realistic initial profile, i.e. one with no slip at the wall, initiates flow features quite different from those of a uniform entry, including fast secondary vortex motions near the wall. Further related theoretical work is by Smith (1976iii).
Here we discuss primarily the steady laminar motion of an incompressible fluid through a circular pipe that is initially straight but, at some point, starts curving along the arc of a circle (curved pipe inlet) or abruptly changes direction (cornered pipe inlet). Far upstream the flow is supposed to be unidirectional straight pipe flow, while sufficiently far downstream the fully developed state for the curved or straight pipe must presumably emerge. Between these two states it is the presence of the bending alone that promotes a gradual development of the secondary flow normal to the pipe-axis. This formulation contrasts with previous work in at least two vital ways: first, it appears to give physically realizeable situations: second, the distortion of the oncoming flow is due solely to the introduction of the bending in the pipe. In addition, the formulation, or a simple modification of it provides a better representation of the shape of the human aorta or a household pipe, for instance, then does a uniformly bending tube.

The theory in the present study assumes for the curved-inlet case that the Reynolds number and curvature are large and small respectively, but for definiteness we take the Dean number (whose value characterizes the eventual full-developed state downstream) to be finite. The motion of the interior core of fluid is found to persist in the axial direction until the bending starts, after which it drifts virtually across the curved tube, so that the maximum of the axial velocity is shifted towards the outer bend. The three-dimensional boundary-layer thus provoked near the wall is of a jet-like nature in its return motion, but, most significantly perhaps, it first develops even before the bending is reached. This is due to a type of upstream response that takes place, between the axial and azimuthal pressure forces and the swirling boundary-layer, in the effective absence of any core-displacement in the straight pipe, and the shear maximum is sited at the inner bend. Beyond the onset of the curvature, however, the outward centrifuging in the core eventually overrides the upstream effect and establishes crossover, to shear maximum at the outside, at 1.51 pipe-radii downstream of the start.
of bending (Smith 1976iv). A similar analysis and flow properties hold for
the cornered-inlet case. The limitations and extensions of the theory are also
examined. The further long adjustment to fully developed flow downstream
is discussed, involving computational solutions obtained for the vortex-like
flow between the inlet and fully-developed state downstream for both the
curved and cornered cases; these require very careful computation as the
starting behaviour is irregular, especially for the cornered inlet. The non­
linear properties that arise if the bend is severe enough (but still small) are
also considered by Smith (1976iv); and we consider some of the effects of a
general initial straight-pipe profile. Specifically, it has been shown that, to a
certain extent, the crossover point (1.51 radii) is independent of the starting
profile provided this profile is of a realistic 'interior' type. A corresponding
result applies in the cornered-inlet case, again implying wide application. Fi­
nally, the effects of smooth junctions between the original straight and the
new curved or straight sections, and of pulsatility in the inlet flow, may also
be incorporated.

The numerical task near the start of the vortex flows (the long-scale flows)
tends to be difficult because of the singular starting behaviour of the solu­
have addressed this in various ways in different contexts of boundary layers,
trailing edges, wakes, and rotating flows, usually within aerodynamical ap­
lications. We seek a flexible and relatively simple computational approach
as distinct from the multi-structured approaches of Keller & Cebecci (1970),
Smith (1973) for example. The approach used here is described and tested
in chapters 2, 3, 4, 5 and 6. It’s results are found to agree well with the local
small-distance analytical results also given in chapters 5 and 6.

In the numerical work on the vortex flows, it is vital to ensure that the small­
starting responses are of the correct form, and so we focus on these at first.
As part of that investigation we also study the flow behaviour as a shorter

\[ x \]

where \( x \) is the streamwise distance
$O(1)$, scale of $x$ closer to the start of the pipe distortion and there the major upstream influence presence is found.

The computational (vortex) approach has the advantages of being parabolic in $x$, and relatively fast, provided the flow remains forward in $x$, and of being able to accommodate a wide variety of tube (pipe or channel) shapes in principle. A possible disadvantage is the lack of upstream influence.

The upstream influence ahead of any pipe bend, corner or other distortion is in anticipation of the need for the incident flow to change direction downstream. This can produce positive and/or negative alterations in the wall pressure or shear stresses for example.

In chapter 2 the equations of motion are derived for fluid flow in a straight pipe of circular cross-section. The three-dimensional Navier-Stokes equations are non-dimensionalized. By considering long axial and time scales of $O(Re)$, where $Re$ is the Reynolds number and performing suitable scalings on the velocity and pressure terms (neglecting time dependent terms) we obtain the three-dimensional steady vortex equations, which become the governing equations of motion upon which our work is based on. The numerical approach, which makes the vortex equations amenable to computation is described. The first order derivatives in the streamwise direction $x$ are replaced by an expression involving the velocities at the previous $x$ step. The velocity and pressure terms are expanded in Fourier series form thus eliminating the dependency on the variable in the azimuthal direction $\theta$. We are thus left with a set of linear o.d.e’s for the velocity and pressure unknowns as functions of the variable in the radial direction $r$.

We discuss in detail, in chapter 3 the numerical technique required to solve the equations obtained in the preceding chapter (in Fourier mode form). Finite differencing in $x$ and $r$, of first order and second order accuracy is used respectively. The method of solution is iterative and our governing equations are written as matrix equations. The solutions to these are used to update the unknown terms on the right hand side and the equations solved again.
This procedure is repeated until we obtain convergence, within a specified
tolerance of $\epsilon$, (say) and march forward in $x$. However, a shooting method
is used to solve the continuity equation for the zeroth mode of the radial ve-
locity. Inverting block tridiagonal matrices requires considerable work and is
fairly time consuming, hence the steps used in inverting the matrix containing
coefficients of the non-zero modes for the continuity, radial momentum and
azimuthal momentum equations are presented. The problem encountered
in this work was in developing the computer code for inverting the block
tridiagonal matrix. As time was a considerable factor here, it was suggested
that we seek a simpler method for solving the vortex system. Solutions of
the three-dimensional vortex system are also obtained for a Hagen-Poiseulle
starting condition which is forced by the addition of two forcing terms $F_1$
and $F_2$ in the $r$ and $\theta$ momentum equations, in turn. This brings about
three-dimensional flow effects as we travel downstream.

In chapter 4 we present an alternative method which is explicit in nature
for obtaining solutions to the governing equations. The technique is simpler
than the former and provides a method for comparing solutions obtained
by the main numerical method given in chapter 3. Eliminating the pressure
terms from the radial and azimuthal momentum equations results in a rather
complicated expression for the non-zero modes of the radial velocity, which
is then solved by matrix inversion.

Chapter 5 is a study of fluid flow into a curved pipe of circular cross-section,
which is initially straight but at some point starts bending uniformly to
form the arc of a circle. The Navier-Stokes equations for this geometry are
expressed. The flow depends on a parameter known as the Dean number $K$,
which is a quantity governing exactly steady flow through curved pipes. For
large $x$ of $O(\delta^{-\frac{1}{2}})$, where $\delta$ defines the relative curvature of the pipe and $O(1)$
values of $K$, the Navier-Stokes equations reduce to the Dean equations, which
become our governing equations. The starting condition is that of Hagen-
Poiseulle flow, however which then becomes three-dimensional due to the
pipes curvature. A large part of the chapter is then devoted to expressing the
governing equations in a form amenable to the numerical methods discussed
in chapters 2 and 3. Computations for different values of the Dean number
are performed.

The reason why the flow in a curved pipe is difficult to calculate lies in the
fact that the motion cannot be everywhere parallel to the curved axis of the
tube but transverse (or secondary) components of velocity must be present.

This follows because in order for a fluid particle to travel in a curved path of
radius $R$ with speed $u$ it must be acted on by a lateral force (provided by the
pressure gradients in the fluid) to give it a lateral acceleration $\frac{u^2}{R} \left(= \frac{\partial p}{\partial s}\right)$. Now
the pressure gradient acting on all particles will be approximately uniform,
but the velocity of these particles near the wall will be much lower than that
of the particles in the core as a result of the no-slip condition. Therefore
the radius of curvature of the path of particles in the core must be greater
than that of the particles near the wall - i.e. fluid in the core is swept to
the outside of the bend, and that near the wall returns to the inside; a
secondary circulation is set up, as shown in appendix 1. These secondary
motions themselves influence the distribution of axial velocity and result in
a complicated interaction of the two.

At the same time, because the faster moving fluid has been swept to the
outside of the bend, the axial velocity profile is distorted from its original
(Hagen-Poiseulle) symmetric shape, and the greatest velocities occur near
the outside wall. Near the entrance of the tube there are thin boundary-
layers, but because of the higher core velocity towards the inner wall, the
boundary layer is thinner there, and hence the velocity gradients (and the
wall shear rate) are greater. However the higher pressures which occur at the
outside of the tube to force the rapidly moving core fluid round the bend,
also act on the slower moving fluid in the boundary-layer. This is therefore
forced round the walls towards the inside (as in fully developed flow) so that
the boundary-layer thickness increases relative to that on the outside bend,
and the shear rate decreases on the inside.

Another related pipeflow problem is studied in chapter 6, that of flow through a cornered pipe. We consider Hagen-Poiseulle flow in a straight pipe described in chapter 2, which at some point experiences an abrupt corner, the pipes cross section remaining circular. We present suitable starting conditions, which are used to solve the straight pipeflow problem, discussed in chapter 2. Numerical results are presented for different values of normalized pipe bend $\alpha$. The flow becomes three-dimensional due to the pipes distortion. For small $x$, the flow is seen to develop in a boundary-layer fashion, as is the case (also) of the curved pipeflow problem. Far downstream, as $x \to \infty$ the flow becomes that of Hagen-Poiseulle.

In chapters 5 and 6, both short-scale and long-scale adjustments of the pipeflow, due to the presence of the abrupt curving or cornering, are examined, yielding upstream- and downstream-influence properties.

We extend the work in the earlier chapters to flow through pipes of a general cross-section, in chapter 7. The equations of motion are developed, which may be used to solve for the velocity and pressure field in a pipe of any cross-sectional area, including pipeflows distorted by constrictions, dilatations and bifurcations. Use is made of conformal mappings $w = f(z)$ to transform the interiors of the cross sectional area of interest from the $Z$-plane (real geometry) in $(r, \theta)$ onto the unit circle in the $W$-plane (computational plane) in $(\bar{r}, \bar{\theta})$, the basis of this being the Riemann mapping theorem. If such a mapping $w$ exists, then fluid flow through the particular pipe can be modelled. Due to the complicated structure of the equations we derive, coupled with the time constraint, solving the equations is not attempted here, but is left for future study. The chapter ends with examples of pipe cross-sections which appear to have physical applications, in particular for modelling blood flow through diseased arteries.
Chapter 2

Equations of Motion

2.1 Background

We start by considering incompressible fluid flowing in a straight pipe of circular cross-section and radius $a$. This case of the straight pipe with three-dimensional flow through it provides the basis for all the subsequent computational and analytical work on various kinds of pipe flows studied in the rest of the thesis. The distance along the axis of the pipe is measured by $x^*$. Here starred variables denote dimensional quantities. An arbitrary point $P$ inside the pipe is given by the cylindrical co-ordinates $(x^*, r^*, \theta)$; within the cross-section of the pipe, polar co-ordinates $r^*, \theta$ are used and the pipe centre is taken as the origin. The variables $x^*, r^*$ and $\theta$ represent the axial, radial and azimuthal co-ordinates respectively. The velocity vector $u^*$ has components $(u^*, v^*, w^*)$ corresponding to the spatial co-ordinates $(x^*, r^*, \theta)$, and we take $p^*$ to denote the pressure, $\rho^*$ the density, $t^*$ the time and $\nu^*$ the kinematic viscosity.

The unsteady three-dimensional Navier-Stokes equations can be written in vector form as

$$
\frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla) u^* = -\frac{1}{\rho^*} \nabla p^* + \nu^* \nabla^2 u^* \quad (2.1.1a)
$$

\begin{equation}
\nabla \cdot u^* = 0 \quad (2.1.1b)
\end{equation}
The dimensional starred variables, namely the lengths \( (x^*, r^*) \), velocity components \( (u^*, v^*, w^*) \), pressure/density ratio \( p^*/\rho^* \) and time \( t^* \) are now made non-dimensional with a representative length \( l^* \), representative speed \( U^* \), \( U^* \) and \( l^* \) respectively. Thus

\[
(x, r, \theta) = \frac{(x^*, r^*, l^* \theta)}{l^*}; \quad (u, v, w) = \frac{(u^*, v^*, w^*)}{U^*}; \quad (2.1.2a - d)
\]

\[
p = \frac{p^*}{\rho^* U^*}; \quad t = \frac{t^* U^*}{l^*}
\]

The Reynolds number \( Re \), which is taken subsequently to be large, is defined as

\[
Re = \frac{U^* l^*}{\nu^*} \quad (2.1.3)
\]

Using (2.1.2), (2.1.3) the Navier-Stokes equations (2.1.1a,b) can be written in the non-dimensional component form as follows

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial \theta} &= - \frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right), \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial v}{\partial \theta} - \frac{w^2}{r} &= - \frac{\partial p}{\partial r} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right) \\
&+ \frac{1}{Re} \left( - \frac{v}{r^2} - 2 \frac{\partial w}{\partial \theta} \right), \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial r} + \frac{w}{r} \frac{\partial w}{\partial \theta} + \frac{vw}{r} &= - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \\
&+ \frac{1}{Re} \left( - \frac{w}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right)
\end{align*}
\]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0, \]

consisting of the \(x, r, \theta\) momentum equations together with the equation of continuity, in turn.

The next step is to scale the Reynolds number \(Re\) out of the Navier-Stokes equations, by considering long axial and time scales of \(O(Re)\), which are appropriate for long vortex-type flows through the pipe. We put

\[ \frac{\partial}{\partial t} = Re^{-1} \frac{\partial}{\partial t} + \ldots \quad \frac{\partial}{\partial x} = Re^{-1} \frac{\partial}{\partial x} + \ldots \quad (2.1.5) \]

and then perform the following scalings on the velocities

\[ u = \bar{u} \left( \bar{x}, r, \theta, \bar{t} \right) + \ldots \quad (2.1.6a - c) \]

\[ v = Re^{-1} \bar{v} \left( \bar{x}, r, \theta, \bar{t} \right) + \ldots \]

\[ w = Re^{-1} \bar{w} \left( \bar{x}, r, \theta, \bar{t} \right) + \ldots \]

The pressure expansion becomes

\[ p = -Re^{-1} \bar{G} \left( \bar{t} \right) x + Re^{-2} \bar{q} \left( \bar{x}, r, \theta, \bar{t} \right) + \ldots \quad (2.1.7) \]

Upon substitution of these expansions \((2.1.5)-(2.1.7)\) into the Navier-Stokes equations \((2.1.4a-d)\) and neglecting any time dependent derivatives, we have the steady three-dimensional vortex equations (removing the \('-'s from the velocity and spatial co-ordinate terms for convenience) and neglecting terms of \(O(Re^{-2})\)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial \theta} = \bar{G} + \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \quad (2.1.8a - d) \]
\[
\begin{align*}
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial v}{\partial \theta} - \frac{w^2}{r} &= -\frac{\partial \bar{q}}{\partial r} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \\
&+ \left( -\frac{v}{r^2} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w}{\partial x} + v \frac{\partial w}{\partial r} + \frac{w}{r} \frac{\partial w}{\partial \theta} + \frac{vw}{r} &= -\frac{1}{r} \frac{\partial \bar{q}}{\partial \theta} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \\
&+ \left( -\frac{w}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r} \frac{\partial w}{\partial \theta} &= 0.
\end{align*}
\]

Here we are concerned with studying flow features arising from steady motion (the pressure \(\bar{q}\) is independent of time); hence the above (2.1.8a-d) now become the governing equations of motion upon which the remainder of this work is based.
2.2 Numerical Approach

We begin by representing the three-dimensional vortex equations (2.1.8a-d) consisting of the $x$, $r$, $\theta$ momentum equations and continuity equation respectively as

$$\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta} &= \tilde{G} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \\
\frac{\partial v}{\partial x} + \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta} - \frac{\partial w}{\partial r} &= -\frac{\partial q}{\partial r} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \left( -\frac{u}{r^2} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right), \\
\frac{\partial w}{\partial x} + \frac{\partial w}{\partial r} + \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial r} &= \frac{1}{r} \frac{\partial q}{\partial \theta} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \left( -\frac{w}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right),
\end{align*}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0.$$

They will be considered in this order henceforth.

The velocity terms $\tilde{u}, \tilde{v}, \tilde{w}$ above denote the velocities at the previous $x$-step, for the computational scheme to be described below. The method employed is, basically to solve (2.2.1) at each $x$-step for the velocity and pressure terms (unbarred) and then to march forward, using the previous solutions at $x$ as an initial guess for the unknowns at $(x + \Delta x)$, where $\Delta x$ is the small step size in the $x$ direction. Since the barred terms are taken to be known the system (2.2.1a-d) gives a linear implicit system for the unbarred quantities. For computational purposes, the vortex equations in...
the form (2.2.1) should be relatively economical and efficient to solve due to the missing viscous terms in the $x$-direction rendering the system parabolic in $x$. Suitable starting conditions are assumed at a finite $x$ station, say $x = 0$.

The first order derivatives in $x$ are replaced by

$$\frac{\partial u}{\partial x} = \frac{1}{\Delta x} (u - \bar{u}), \quad (2.2.2a-c)$$

$$\frac{\partial v}{\partial x} = \frac{1}{\Delta x} (v - \bar{v}),$$

$$\frac{\partial w}{\partial x} = \frac{1}{\Delta x} (w - \bar{w}).$$

Substituting (2.2.2a-c) into (2.2.1a-d) gives

$$\frac{\bar{u}(u - \bar{u})}{\Delta x} + \bar{v} \frac{\partial u}{\partial r} + \bar{w} \frac{\partial u}{\partial \theta} = \bar{G} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \quad (2.2.3a-d)$$

$$\frac{\bar{u}(v - \bar{v})}{\Delta x} + \bar{v} \frac{\partial v}{\partial r} + \bar{w} \frac{\partial v}{\partial \theta} - \frac{\bar{w} w}{r} = -\frac{\partial \bar{q}}{\partial r} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2},$$

$$-\frac{\bar{v}}{r^2} - \frac{2}{r^2} \frac{\partial \bar{w}}{\partial \theta},$$

$$\frac{\bar{u}(w - \bar{w})}{\Delta x} + \bar{v} \frac{\partial w}{\partial r} + \bar{w} \frac{\partial w}{\partial \theta} + \frac{\bar{w} w}{r} = -\frac{1}{r} \frac{\partial \bar{q}}{\partial \theta} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2},$$

$$-\frac{\bar{w}}{r^2} + \frac{2}{r^2} \frac{\partial \bar{v}}{\partial \theta},$$

$$\frac{1}{\Delta x}(u - \bar{u}) + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0.$$

This yields an implicit system for the unknowns $u, v, w, \bar{q}$ at $x + \Delta x$ as functions of $r, \theta$. To solve we now expand the velocity and pressure terms in the following Fourier-series form
\[ u(r, \theta) = u_0(r) + \sum_{m=1}^{N} \left( u_m(r)E^m + u_m^*(r)E^{-m} \right), \quad (2.2.4a - d) \]

\[ v(r, \theta) = v_0(r) + \sum_{m=1}^{N} \left( v_m(r)E^m + v_m^*(r)E^{-m} \right), \]

\[ w(r, \theta) = w_0(r) + \sum_{m=1}^{N} \left( w_m(r)E^m + w_m^*(r)E^{-m} \right), \]

\[ \tilde{q}(r, \theta) = \tilde{q}_0(r) + \sum_{m=1}^{N} \left( \tilde{q}_m(r)E^m + \tilde{q}_m^*(r)E^{-m} \right) \]

where \( E = e^{i\theta} \), \( N \) is the total number of \( \theta \) modes taken in the computation and * denotes the complex conjugate.

Upon substitution of (2.2.4a-d) into (2.2.1a-d), the \( \theta \) dependence is eliminated and we have at \( O(E^m) \) for \( m = 0 \) and \( m = 1, \ldots, N \) the following system of equations

\[
\begin{align*}
\frac{d^2 u_0}{dr^2} + \left( \frac{1}{r} - \bar{u}_0 \right) \frac{du_0}{dr} - \frac{\bar{u}_0}{\Delta x} u_0 &= \bar{G} - \frac{1}{\Delta x} \bar{u}_0^2 + \\
\sum_{l=1}^{N} \left[ \left( \frac{i l}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l^* \right) u_l - \frac{1}{\Delta x} \bar{u}_l \bar{u}_l^* + \bar{u}_l \frac{du_l}{dr} \right] + \\
\sum_{l=1}^{N} \left[ \left( \frac{-i l}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_l^* - \frac{1}{\Delta x} \bar{u}_l \bar{u}_l^* + \bar{u}_l \frac{du_l}{dr} \right], \quad (2.2.5a) \\
\frac{d^2 u_m}{dr^2} + \left( \frac{1}{r} - \bar{u}_0 \right) \frac{du_m}{dr} - \left( \frac{m}{r} \right)^2 + \frac{\bar{u}_0}{\Delta x} + \frac{im}{r} \bar{u}_0 \right) u_m = \\
\frac{1}{\Delta x} \bar{u}_m u_0 + \bar{u}_m \frac{du_0}{dr} - \frac{2}{\Delta x} \bar{u}_0 \bar{u}_m + \\
\sum_{l=1, l \neq m}^{m-1} \left[ \left( \frac{i(m-l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m-l} + \bar{u}_l \frac{du_{m-l}}{dr} \right] + \\
\sum_{l=m+1, l \neq m}^{N} \left[ \left( \frac{-i(l-m)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{l-m} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{l-m} + \bar{u}_l \frac{du_{l-m}}{dr} \right] + 
\end{align*}
\]
\[
\sum_{l=1, l\neq m}^{N-m} \left[ \left( \frac{i(m + l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{m+l} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m+l} + \bar{u}_l \frac{d u_{m+l}}{d r} \right]
\]

\[(m = 1, \ldots, N), \quad (2.2.5b)\]

\[
\frac{d v_0}{d r} + \frac{1}{r} v_0 = -\frac{1}{\Delta x} (u_0 - \bar{u}_0), \quad (2.2.5c)
\]

\[
\frac{d^2 v_m}{d r^2} + \left( \frac{1}{r} - \bar{v}_0 \right) \frac{d v_m}{d r} - \left( \frac{m^2 + 1}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{i m}{r} \bar{w}_0 \right) v_m + \left( \frac{-2 i m + 1}{r^2} \bar{w}_0 \right) w_m
\]

\[-\frac{d q_m}{d r} = \frac{1}{\Delta x} \bar{u}_m v_0 - \frac{1}{\Delta x} \bar{u}_0 \bar{u}_m - \frac{1}{r} \bar{u}_m \bar{u}_0 + \bar{v}_m \frac{d v_0}{d r} - \frac{1}{r} \bar{w}_m w_0 +
\]

\[
\sum_{l=1, l\neq m}^{m-1} \left[ \left( \frac{i(m - l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{m-l} - \frac{1}{r} \bar{w}_l w_{m-l} + \bar{v}_l \frac{d v_{m-l}}{d r} \right] +
\]

\[
\sum_{l=m+1, l\neq m}^{N} \left[ \left( -\frac{i(l - m)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{i-l} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{i-l} - \frac{1}{r} \bar{w}_l w_{i-l} + \bar{v}_l \frac{d v_{i-l}}{d r} \right] +
\]

\[
\sum_{l=1, l\neq m}^{N-m} \left[ \left( \frac{i(m + l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{m+l} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{m+l} - \frac{1}{r} \bar{w}_l w_{m+l} + \bar{v}_l \frac{d v_{m+l}}{d r} \right]
\]

\[(m = 1, \ldots, N), \quad (2.2.5d)\]

\[
\frac{d^2 w_0}{d r^2} + \left( \frac{1}{r} - \bar{v}_0 \right) \frac{d w_0}{d r} - \left( \frac{1}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{1}{r} \bar{v}_0 \right) w_0 = -\frac{1}{\Delta x} \bar{u}_0 \bar{w}_0 +
\]
The initial or starting conditions used to solve (2.2.5a-g) are assumed to take the particular form
\[ u_0(r) = 1 - r^2 \quad \bar{v}_0(r) = 0 \quad \bar{w}_0(r) = r^2(1-r)^2RF \quad (2.2.6a - c) \]

\[ u_1(r) = r(1-r^2)RF \quad \bar{v}_1(r) = (1-r^2)^2RF \quad (2.2.6d - f) \]

\[ \bar{w}_1(r) = -i \left[ r^2(1-r^2) - (1-r^2)^2 \right] RF \]

\[ \bar{u}_m(r) = \frac{r}{m^2}(1-r^2)RF \quad \bar{v}_m(r) = 0 \quad \bar{w}_m(r) = 0 \quad (m = 2, \ldots, N) \quad (2.2.6g - i) \]

where \( RF \) is an initial disturbance factor.

The appropriate boundary conditions to be used consist of the regularity conditions at the pipe centre \( r = 0 \) (to avoid singular behaviour there) and the condition of no-slip at the pipe wall \( r = 1 \).

The regularity conditions take the form

\[ \frac{du_0}{dr}(0) = 0 \quad v_0(0) = 0 \quad w_0(0) = 0 \quad u_m(0) = 0 \quad (m = 1, \ldots, N), \quad (2.2.7a - i) \]

\[ v_1(0) = -iw_1(0) ; \quad \frac{dv_1}{dr}(0) = 0 \quad v_m(0) = 0 \quad w_m(0) = 0 \quad (m = 2, \ldots, N) ; \]

\[ \hat{g}_m(0) = 0 \quad (m = 1, \ldots, N) \]

and the no-slip conditions are

\[ u_0(1) = 0 \quad v_0(1) = 0 \quad w_0(1) = 0 \quad (2.2.8a - f) \]
Thus we are left with the task of solving equations (2.2.5a-g). This task is addressed in the following chapter.
3.1 Computational Scheme

In this section we present the main computational method for the solution of (2.2.5a-g) subject to the initial conditions (2.2.6a-i) and boundary conditions (2.2.7a-i), (2.2.8a-f).

We use finite differencing in \( x \) and \( r \), of first order and second order accuracy, respectively and march forward in \( x \). At each \( x \)-step, we solve by sweeping the \((r, \theta)\) plane \( M \) times to obtain convergence, within a specified tolerance of \( \varepsilon \).

Thus (2.2.5a,b) become of the form

\[
\Phi_j u_{oj-1} + \Omega_j u_{oj} + \varphi_j u_{oj+1} = R_{oj}
\]  \hspace{1cm} (3.1.1a)

\[
\sigma_j u_{mj-1} + \Theta_{mj} u_{mj} + \delta_j u_{mj+1} = \Gamma_{mj}
\]  \hspace{1cm} (3.1.1b)

respectively, with \( u_{oj} \equiv u_0(r_j) \) and \( u_{mj} \equiv u_\nu(r_j) \) and where the coefficients are

\[
\Phi_j = \frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r} \left( \frac{1}{r_j} - \bar{u}_{oj} \right),
\]  \hspace{1cm} (3.1.2a - d)

\[
\Omega_j = -\frac{2}{(\Delta r)^2} - \frac{1}{\Delta x} \bar{u}_{oj},
\]
\[ \varphi_j = \frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r} \left( \frac{1}{r_j} - \bar{v}_{0j} \right), \]

\[ R_{\varphi j} = \bar{G} - \frac{1}{\Delta x} \bar{u}_{0j}^2 + \]

\[ \sum_{l=1}^{N} \left[ \left( \frac{i l}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right) u_{lj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{u}_{lj} + \frac{1}{2\Delta r} \bar{v}_{lj} \left( u_{lj+1} - u_{lj-1} \right) \right] + \]

\[ \sum_{l=1}^{N} \left[ \left( \frac{i l}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right) u_{lj}^* - \frac{1}{\Delta x} \bar{u}_{lj} \bar{u}_{lj} + \frac{1}{2\Delta r} \bar{u}_{lj} \left( u_{lj+1}^* - u_{lj-1}^* \right) \right] \]

and

\[ \sigma_j = \frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r} \left( \frac{1}{r_j} - \bar{v}_{0j} \right) \quad (3.1.3a - d) \]

\[ \Theta_{mj} = -\frac{2}{(\Delta r)^2} - \left( \frac{m^2}{r_j^2} + \frac{1}{\Delta x} \bar{u}_{0j} + \frac{im}{r_j} \right) \]

\[ \delta_j = \frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r} \left( \frac{1}{r_j} - \bar{v}_{0j} \right) \]

\[ \Gamma_{mj} = -\frac{2}{\Delta x} \bar{u}_{0j} \bar{u}_{mj} + \frac{1}{\Delta x} \bar{u}_{mj} u_{0j} + \frac{1}{2\Delta r} \bar{w}_{mj} \left( u_{0j+1} - u_{0j-1} \right) + \]

\[ \sum_{l=1, l \neq m}^{m-1} \left[ \left( \frac{i(m-l)}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right) u_{(m-l)j} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{u}_{(m-l)j} + \right. \]

\[ + \frac{1}{2\Delta r} \bar{u}_{lj} \left( u_{(m-l)j+1} - u_{(m-l)j-1} \right) \]

and

\[ \sum_{l=m+1, l \neq m}^{N} \left[ \left( \frac{-i(l-m)}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right) u_{(l-m)j}^* - \frac{1}{\Delta x} \bar{u}_{lj} \bar{u}_{(l-m)j}^* + \right. \]

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\[
\frac{1}{2\Delta r} \bar{v}_{ij} \left( u_{(l-m)j+1}^* - u_{(l-m)j-1}^* \right) + \\
\sum_{l=1, l \neq m}^{N-m} \left[ \left( \frac{i(m+l)}{\Delta x} \bar{u}_{ij}^* + \frac{1}{\Delta x} \bar{u}_{ij}^* \right) u_{(m+l)j} - \frac{1}{\Delta x} \bar{u}_{ij}^* u_{(m+l)j} + \\
\frac{1}{2\Delta r} \bar{v}_{ij} \left( u_{(m+l)j+1} - u_{(m+l)j-1} \right) \right]
\]

\( m = 1, \ldots, N \)

for \( j = 1, \ldots, J - 1 \). Here \( J \) is the total number of steps taken, and \( \Delta r = r_j - r_{j-1} \) represents the constant mesh size. So \( r_j = j\Delta r \).

We now have a set of matrix equations in (3.1.1a,b). For instance, (3.1.1b) can be represented in the form

\[
A_m^{(\infty)} \Gamma_m = \Gamma_m
\]

where \( u_m = (u_{m1}, \ldots, u_{mj-1})^T \) and \( \Gamma_m = (\Gamma_{m1}, \ldots, \Gamma_{mj-1})^T \)

\( m = 1, \ldots, N, \quad j = 1, \ldots, J - 1 \) (3.1.4)

as shown in appendix 2, \( j = 0 \) and \( J \) is given by the boundary conditions (2.2.7d) and (2.2.8d) representing the regularity condition at the pipe centre and no-slip at the pipe wall, respectively. The matrix \( A_m^{(\infty)} \Gamma_m \) if the matrix \( B_{i\omega<3,\lambda} \) then (3.1.1a) is expressed in similar form as (3.1.4) where \( u = (u_0, \ldots, u_{j-1})^T \) and \( R = (R_0, \ldots, R_{j-1})^T \)

\[
B u_0 = R_0
\]

(3.1.5)

The non-zero term \( B_{01} \) comes from the regularity condition (2.2.7a), which gives \( u_{01} = u_{00} \). We use (3.1.4) to illustrate the method of solution, which involves the inversion of the matrix \( A \), by a form of Gaussian elimination known as Thomas's algorithm. The initial conditions (2.2.6a-i) provide suitable guesses for the unknown terms in \( \Gamma_m \). Gaussian elimination is performed and the \( u_m \) terms are then used to update the unknown terms in
Inversion is then performed to obtain new $\hat{u}_m$'s and subsequent updating of $\Gamma_{mj}$ after each iteration. This is repeated until we obtain a converged solution for the complete set of $\hat{u}_m$'s. Consistency between successive iterations of $O(10^{-8})$ is taken as a sufficient condition for convergence.

The following operations are performed

$$\Theta_{mj} := \Theta_{mj} - \left( \frac{\sigma_j}{\Theta_{mj-1}} \right) \delta_{j-1} \quad (3.1.6a)$$

$$\Gamma_{mj} := \Gamma_{mj} - \left( \frac{\sigma_j}{\Theta_{mj-1}} \right) \Gamma_{mj-1} \quad (3.1.6b)$$

$m = 1, \ldots, N, \quad j = 2, \ldots, J - 1$

We sweep down to $J - 1$ and then obtain $u_{mj}$ from the relations for $m = 1, \ldots, N$

$$u_{mJ-1} = \frac{\Gamma_{mj-1}}{\Theta_{mj-1}} \quad (3.1.7)$$

because of the no slip condition $u_{mj} = 0$. We now sweep back up using for $j = J - 2, \ldots, 1$

$$u_{mj} = \frac{\Gamma_{mj} - \delta_{j}u_{mj+1}}{\Theta_{mj}} \quad (3.1.8)$$

and finally $u_{m0} = 0$ $(m = 1, \ldots, N)$ is the imposed regularity condition. This gives us $u_{0j}$ and $u_{mj}$ for $j = 0, \ldots, J$ and $m = 1, \ldots, N$.

We now proceed onto the radial velocity by first considering (2.2.5c) to obtain $v_0$, by rewriting it as for $j = 0, \ldots, J$

$$\left. \frac{dv_0}{dr} \right|_j = -\frac{1}{\Delta x} (u_{0j} - \tilde{u}_{0j}) - \frac{1}{r_j} v_{0j} \quad (3.1.9)$$

A shooting method is the most appropriate mode of solution. A fourth order Runge Kutta scheme is used to solve the above, together with the regularity condition $v_{0\theta} = 0$. The second boundary condition (2.2.8b), that of no-slip, is
not satisfied by shooting once. (3.1.1a) is solved twice for $u_0$ using different values of $\bar{G}$, typically $\bar{G}_{01} = -4$ and $\bar{G}_{02} = -5$ and the values of $u_0$ ($u_{01}$ and $u_{02}$, in turn) are substituted into (3.1.9) to obtain the corresponding $v_0$'s, i.e. $v_{01}$ and $v_{02}$ respectively. We find at the wall, $v_{01} = v_0^+$ and $v_{02} = v_0^-$. So we perform a linear interpolation on $\bar{G}$, such that $v_0$ satisfies the no-slip condition at the wall. The appropriate value of $\bar{G}$, $\bar{G}_c$ is thus obtained from

$$\bar{G}_c = -\frac{v_{02}}{(v_{01} - v_{02})} \bar{G}_{01} - \frac{v_{01}}{(v_{02} - v_{01})} \bar{G}_{02} \quad (3.1.10)$$

after which $u_0$, $u_m$ ($m = 1, \ldots, N$) are again obtained and used to solve (3.1.9) for the correct $v_0$, which satisfies both boundary conditions. The procedure is performed at each $x$ step. We take $J = 600$ to obtain accurate results. Decreasing this value produces a spike effect at the pipe centre. Performing matrix inversion yields good results for much coarser grids, say $J = 100$. For a smaller sized grid, the velocity profile obtained for $v_0$ is greatly distorted. To obtain the same accuracy from the shooting scheme entails performing calculations on a much finer grid, therefore using a larger number of points ($J = 600$) becomes necessary.

In a similar fashion to (3.1.1a,b), we write (2.2.5e) as

$$\psi_{0j}w_{0j-1} + \vartheta_{0j}w_{0j} + \Xi_{0j}w_{0j-1} = (RHS)_{0j} \quad (3.1.11)$$

where

$$\psi_{0j} = \frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r} \left( \frac{1}{r_j - \overline{v}_{0j}} \right), \quad (3.1.12a - d)$$

$$\vartheta_{0j} = -\frac{2}{(\Delta r)^2} - \left( \frac{1}{r_j^2} + \frac{1}{\Delta x} \overline{u}_{0j} + \frac{1}{r_j} \overline{v}_{0j} \right),$$

$$\Xi_{0j} = \frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r} \left( \frac{1}{r_j} - \overline{v}_{0j} \right),$$

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(RHS)\textsubscript{0j} = -\frac{1}{\Delta x} \bar{u}_{0j} \bar{w}_{0j} + \\
\sum_{l=1}^{N} \left[ \left( \frac{il}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} + \frac{1}{r_j} \bar{v}_{lj} \right) w_{lj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{w}_{lj} + \frac{1}{2\Delta r} \bar{v}_{lj} (w_{lj+1} - w_{lj-1}) \right] + \\
\sum_{l=1}^{N} \left[ \left( -\frac{il}{r_j} \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} + \frac{1}{r_j} \bar{v}_{lj} \right) w_{lj}^* - \frac{1}{\Delta x} \bar{u}_{lj} \bar{w}_{lj}^* + \frac{1}{2\Delta r} \bar{v}_{lj} (w_{lj+1}^* - w_{lj-1}^*) \right]

and use the method described for the inversion of (3.1.4), to obtain \(w_{0j}\) (\(j = 0, \ldots, J\)).

We now proceed onto solving for the non zero modes of \(u_m, w_m\) and \(\bar{q}_m\). (2.2.5d,f,g) are the \(r, \theta\) momentum and continuity equations, in turn. (2.2.5d,f) both contain 3 unknown terms. We begin by writing (2.2.5d,f,g) in finite difference form

\[ \bar{\ddot{A}}_j v_{mj-1} + B'_m v_{mj} + C_j v_{mj+1} + D_m w_{mj} + E_j \bar{q}_{mj-1} + F_j \bar{q}_{mj+1} = \bar{\xi}_{mj} \]

\[ \bar{A}_m v_{mj} + \bar{B}_j w_{mj-1} + \bar{C}_m w_{mj} + \bar{D}_j w_{mj+1} + \bar{E}_m \bar{q}_{mj} = \Psi_{mj} \]

\[ \alpha_j v_{mj-1} + \beta_j v_{mj} + \gamma w_{mj-1} + \lambda w_{mj} = -\frac{1}{2\Delta x} (u_{mj} + u_{mj-1} - \bar{u}_{mj} - \bar{u}_{mj-1}) \]

\[ (m = 1, \ldots, N) \]

respectively with \(v_{mj} \equiv v_m(r_j), w_{mj} \equiv w_m(r_j)\) and \(\bar{q}_{mj} \equiv \bar{q}_m(r_j)\), where

\[ \bar{\xi}_{mj} = -\frac{1}{\Delta x} \bar{u}_{0j} \bar{v}_{mj} + \frac{1}{\Delta x} \bar{u}_{mj} \bar{v}_{0j} - \frac{1}{\Delta x} \bar{u}_{mj} \bar{v}_{0j} - \frac{1}{r_j} \bar{w}_{mj} \bar{w}_{0j} + \bar{v}_{mj} \frac{1}{2\Delta r} (v_{0j+1} - v_{0j-1}) \]
\[
+ \sum_{l=1, l \neq m}^{m-1} \left[ \frac{i}{r_j} (m - l) \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right] v_{m-lj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{v}_{m-lj} - \frac{1}{r_j} \bar{w}_{lj} w_{m-lj} +
\]
\[
\bar{v}_{lj} \frac{1}{2\Delta r} (v_{m-lj+1} - v_{m-lj-1}) +
\]
\[
\sum_{l=m+1, l \neq m}^{N} \left[ -\frac{i}{r_j} (l - m) \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right] v_{l-mj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{v}_{l-mj} - \frac{1}{r_j} \bar{w}_{lj} \bar{v}_{l-mj} +
\]
\[
\bar{v}_{lj} \frac{1}{2\Delta r} (v_{l-mj+1} - v_{l-mj-1}) +
\]
\[
\sum_{l=1, l \neq m}^{N-m} \left[ \frac{i}{r_j} (m + l) \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} \right] v_{m+l-j} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{v}_{m+l-j} - \frac{1}{r_j} \bar{w}_{lj} \bar{v}_{m+l-j} +
\]
\[
\bar{v}_{lj} \frac{1}{2\Delta r} (v_{l+m-j+1} - v_{l+m-j-1}) \tag{3.1.14a}
\]

and
\[
\Psi_{mj} = -\frac{1}{\Delta x} \bar{u}_{0j} \bar{w}_{mj} - \frac{1}{\Delta x} \bar{u}_{mj} \bar{w}_{0j} + \frac{1}{\Delta x} \bar{u}_{mj} \bar{w}_{0j} + \frac{1}{r_j} \bar{v}_{mj} \bar{w}_{0j} +
\]
\[
\bar{v}_{mj} \frac{1}{2\Delta r} (w_{0j+1} - w_{0j-1}) +
\]
\[
\sum_{l=1, l \neq m}^{m-1} \left[ \frac{i}{r_j} (m - l) \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} + \frac{1}{r_j} \bar{v}_{lj} \right] w_{m-lj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{w}_{m-lj} +
\]
\[
\bar{v}_{lj} \frac{1}{2\Delta r} (w_{m-lj+1} - w_{m-lj-1}) +
\]
\[
\sum_{l=m+1, l \neq m}^{N} \left[ -\frac{i}{r_j} (l - m) \bar{w}_{lj} + \frac{1}{\Delta x} \bar{u}_{lj} + \frac{1}{r_j} \bar{v}_{lj} \right] w_{l-mj} - \frac{1}{\Delta x} \bar{u}_{lj} \bar{w}_{l-mj} +
\]

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\[
\bar{v}_{ij} \frac{1}{2 \Delta r} \left( w_{i-mj+1} - w_{i-mj-1} \right) + \\
\sum_{l=1, l \neq m}^{N-m} \left[ \frac{i}{r_j} (m + l) \bar{w}_{lj}^* + \frac{1}{\Delta x} \bar{w}_{lj}^* + \frac{1}{r_j} \bar{v}_{lj}^* \right] w_{m+lj} - \frac{1}{\Delta x} \bar{w}_{lj}^* \bar{w}_{m+lj} + \\
\bar{v}_{ij} \frac{1}{2 \Delta r} (w_{m+lj+1} - w_{m+lj-1})
\]

(3.1.14b)

(3.1.13a,b) are centred at \( j \), whereas (3.1.13c) is centred at \( j - \frac{1}{2} \). We now have a set of 3 equations in 3 unknowns, which can be solved simultaneously for \( m = 1, \ldots, N \). The total number of modes \( N = 6 \), which is sufficient for our computations. Increasing \( N \) beyond this has an insignificant effect upon the solutions for the velocity and pressure field.

The coefficients of (3.1.13c,a,b), in turn form the rows of a block tridiagonal matrix \( \tilde{A} \), each block being a \((3 \times 3)\) matrix. The system can be represented in the form

\[
\tilde{A}_{k,k,j,j} \tilde{\mu}_{kj} = \xi_{kj}
\]

\((m = 1, \ldots, N), \quad (j = 0, \ldots, J), \quad (k = 1, 2, 3)\)

(3.1.15)

The square matrix \( \tilde{A}_{k,k,j,j} \) consists of \((J - 2)\) rows, each containing three non-zero matrices, i.e. \( j = 1, \ldots, J - 1 \). The rows \( j = 0 \) and \( J \) consist of the boundary conditions (2.2.7e-i) and (2.2.8e,f). The non-zero structure of each row \( j = 1, \ldots, J - 1 \) is:

- **continuity**
  \[
  \alpha_j \quad \beta_j \quad \gamma_{mj} \quad \lambda_{mj} \quad 0 \quad 0 \quad 0 \quad 0
  \]

- **\( r \) momentum**
  \[
  A_j \quad 0 \quad E \quad B_{mj} \quad D_{mj} \quad 0 \quad C_j \quad 0 \quad F
  \]

- **\( \theta \) momentum**
  \[
  0 \quad 0 \quad \bar{B}_j \quad \bar{A}_{mj} \quad \bar{C}_{mj} \quad \bar{E}_{mj} \quad 0 \quad \bar{D}_j \quad 0
  \]

Fig. 3.1.1  

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So \( \tilde{A}_{k,k,j,j} \) has a total of \( [(J + 1)x(J + 1)] \) blocks of (3x3) matrices, i.e. a sum of \( 9(J + 1)^2 \) elements.

The system (3.1.15) is given in Appendix 3, together with the definitions of the elements in Fig. 3.1.1. (3.1.5) is not so easily solved, but as in the preceding matrix equations, the method of solution is iterative. We now present the numerical procedure for solving (3.1.15). Gaussian elimination is used to invert \( \tilde{A}_{k,k,j,j} \), due to its structure this provides us with quite a formidable task.

We begin by defining the row \( R_{k,j} \) to be the \( k^{th} \) within the (3x3) matrix, which itself lies in the \( j^{th} \) row, e.g. \( R_{2,1} \) and \( R_{1,4} \) would be, in turn

\[
\begin{array}{cccc}
0 & - & - & 0 \\
0 & - & - & 0
\end{array}
\begin{array}{cccc}
\tilde{A}_1 & 0 & E' & B_{m1}' \\
\alpha_4 & \gamma_{m4} & 0 & \beta_4
\end{array}
\begin{array}{cccc}
D_{m1} & 0 & C_1 & 0 \\
\lambda_{m4} & 0 & 0 & 0
\end{array}
\begin{array}{cccc}
F & 0 & - & - \\
0 & - & - & 0
\end{array}
\]

Similarly \( \tilde{A}_{1,1,2,2} \), \( \tilde{A}_{2,3,3,2} \) and \( \tilde{A}_{3,2,4,5} \) would be \( \beta_2 \), \( E_3 \) and \( D_4 \) respectively, using Fig. 3.1.1.

The operations performed on (3.1.15) are for \( j = 1, \ldots, J - 1; \ m = 1, \ldots, N \)

1. \[
R_{1,j} = R_{1,j} - [A_{1,1,j,j-1}] \ R_{1,j-1}
\]

2. If \( m = 1 \) and \( j = 1 \) then

\[
R_{3,j} = R_{3,j} - \left[ \frac{A_{3,2,j,j-1}}{A_{1,2,j,j-1}} \right] R_{1,j}
\]

\[
\xi_{3,j} = \xi_{3,j} - \left[ \frac{A_{3,2,j,j-1}}{A_{1,2,j,j-1}} \right] \xi_{1,j}
\]

3.
4. 
\[ R_{1,j} = R_{1,j} - [A_{1,2,j,j-1}] R_{2,j-1} \]
\[ \xi_{1,j} = \xi_{1,j} - [A_{1,2,j,j-1}] \xi_{2,j-1} \]

5. 
\[ R_{2,j} = R_{2,j} - [A_{2,1,j,j-1}] R_{1,j-1} \]
\[ \xi_{2,j} = \xi_{2,j} - [A_{2,1,j,j-1}] \xi_{1,j-1} \]

6. If \( m = 1 \) and \( j = 1 \)
\[ R_{1,j} = R_{1,j} - \left[ \frac{A_{1,2,j,j-1}}{A_{2,2,j,j-1}} \right] R_{2,j-1} \]
\[ \xi_{1,j} = \xi_{1,j} - \left[ \frac{A_{1,2,j,j-1}}{A_{2,2,j,j-1}} \right] \xi_{2,j-1} \]
Else \( (j \neq 1) \)
\[ R_{3,j} = R_{3,j} - [A_{3,2,j,j-1}] R_{2,j-1} \]
\[ \xi_{3,j} = \xi_{3,j} - [A_{3,2,j,j-1}] \xi_{2,j-1} \]
Else \( (m \neq 1) \)
\[ R_{3,j} = R_{3,j} - [A_{3,2,j,j-1}] R_{2,j-1} \]
6. \( J = J_{j-1} \)

7. If \( m = 1 \) and \( j = 1 \)

\[
R_{1,j} = R_{1,j} - [A_{1,1,j,j-1}] R_{1,j-1}
\]

\[
\xi_{1,j} = \xi_{1,j} - [A_{1,1,j,j-1}] \xi_{1,j-1}
\]

8. \( R_{3,j} = R_{3,j} - [A_{3,3,j,j-1}] R_{3,j-1} \)

\[
\xi_{3,j} = \xi_{3,j} - [A_{3,3,j,j-1}] \xi_{3,j-1}
\]

9. \( R_{2,j} = R_{2,j} - \left[ \frac{A_{1,1,j,j}}{A_{1,1,j,j}} \right] R_{1,j} \)

\[
\xi_{2,j} = \xi_{2,j} - \left[ \frac{A_{1,1,j,j}}{A_{1,1,j,j}} \right] \xi_{1,j}
\]

10. \( R_{3,j} = R_{3,j} - \left[ \frac{A_{3,1,j,j}}{A_{1,1,j,j}} \right] R_{1,j} \)

\[
\xi_{3,j} = \xi_{3,j} - \left[ \frac{A_{3,1,j,j}}{A_{1,1,j,j}} \right] \xi_{1,j}
\]

11. \( R_{3,j} = R_{3,j} - \left[ \frac{A_{2,1,j,j}}{A_{2,1,j,j}} \right] R_{2,j} \)

\[
\xi_{3,j} = \xi_{3,j} - \left[ \frac{A_{2,1,j,j}}{A_{2,1,j,j}} \right] \xi_{2,j}
\]
12. \[ R_{1,j} = R_{1,j} - \left[ \frac{A_{1,2,i,j}}{A_{2,2,j,j}} \right] R_{2,j} \]
\[ \xi_{1,j} = \xi_{1,j} - \left[ \frac{A_{1,2,i,j}}{A_{2,2,j,j}} \right] \xi_{2,j} \]

13. \[ R_{1,j} = R_{1,j} - \left[ \frac{A_{1,3,i,j}}{A_{3,3,j,j}} \right] R_{3,j} \]
\[ \xi_{1,j} = \xi_{1,j} - \left[ \frac{A_{1,3,i,j}}{A_{3,3,j,j}} \right] \xi_{3,j} \]

14. \[ R_{2,j} = R_{2,j} - \left[ \frac{A_{2,3,i,j}}{A_{3,3,j,j}} \right] R_{3,j} \]
\[ \xi_{2,j} = \xi_{2,j} - \left[ \frac{A_{2,3,i,j}}{A_{3,3,j,j}} \right] \xi_{3,j} \]

15. \[ R_{1,j} = \frac{R_{1,j}}{A_{1,1,j,j}} \quad \xi_{1,j} = \frac{\xi_{1,j}}{A_{1,1,j,j}} \]

16. \[ R_{2,j} = \frac{R_{2,j}}{A_{2,2,j,j}} \quad \xi_{2,j} = \frac{\xi_{2,j}}{A_{2,2,j,j}} \]

17. \[ R_{3,j} = \frac{R_{3,j}}{A_{3,3,j,j}} \quad \xi_{3,j} = \frac{\xi_{3,j}}{A_{3,3,j,j}} \]

This now brings us down to \( j = J \), where the following operations are performed for \((m = 1, \ldots, N)\)

a. 36
\[ R_{1,J} = R_{1,J} - [A_{1,1,J,J-1}] R_{1,J-1} \]

\[ \xi_{1,J} = \xi_{1,J} - [A_{1,1,J,J-1}] \xi_{1,J-1} \]

b.

\[ R_{1,J} = R_{1,J} - [A_{1,2,J,J-1}] R_{2,J-1} \]

\[ \xi_{1,J} = \xi_{1,J} - [A_{1,2,J,J-1}] \xi_{2,J-1} \]

We obtain \( \tilde{x}_{k,J} \) \( (k = 1, 2, 3) \) from the boundary conditions at the wall. \( \tilde{x}_{1,J} \), \( \tilde{x}_{2,J} \) and \( \tilde{x}_{3,J} \) denotes \( v_j \), \( w_j \) and \( \bar{q}_j \) in turn and these are given by the relations for \( m = 1, \ldots, N \)

\[ v_{m,J} = \frac{\xi_{2,J}}{A_{2,1,J,J}} \quad w_{m,J} = \frac{\xi_{3,J}}{A_{3,2,J,J}} \quad \text{and} \]

\[ \bar{q}_{m,J} = \frac{[\xi_{1,J} - A_{1,1,J,J} v_{m,J} - A_{1,2,J,J} w_{m,J}]}{A_{1,3,J,J}} \]

This now allows us to sweep back up (back substitution), using

\[ \tilde{x}_{k,J} = \xi_{k,J} - (A_{k,1,J,J+1}) \tilde{x}_{1,J+1} - (A_{k,2,J,J+1}) \tilde{x}_{2,J+1} - (A_{k,3,J,J+1}) \tilde{x}_{3,J+1} \]

for \( j = J - 1, \ldots, 1 \); \( k = 3, 2, 1 \); \( m = 1, \ldots, N \), which leaves the regularity conditions. If \( m = 1 \), then we have

\[ \tilde{x}_{1,0} = \tilde{x}_{1,1} + \xi_{1,0} \]

\[ \tilde{x}_{2,0} = \frac{[\xi_{2,0} - (A_{2,1,0,0}) \tilde{x}_{1,0}]}{A_{2,2,0,0}} \]

\[ \tilde{x}_{3,0} = \xi_{3,0} \]

and for \( m \geq 2 \), simply
\[ \bar{x}_{1,0} = \xi_{1,0} ; \quad \bar{x}_{2,0} = \xi_{2,0} ; \quad \bar{x}_{3,0} = \xi_{3,0} \]

so this gives us \( v_m, w_m, \bar{q}_m \) for \( j = 0, \ldots, J \) and \( m = 1, \ldots, N \).
3.2 Results

We begin by showing results for the flow field at \( x = \Delta x \), for an initial disturbance \( RF = 0.01 \). Profiles are given for the velocity modes \( u_1, v_1, w_1, v_\rho \) and pressure \( p_1 \) (Fig. 3.2.1-3.2.5), \( \Delta x = 0.001 \).

If we now keep the initial disturbance factor \( RF \) constant and vary the step length in \( x, \Delta x \) (Figs. 3.2.6-3.2.8 & 3.2.10-3.2.13), show that the values of \( v_1, w_1 \) change by a factor 10. So if the step length \( \Delta x \) is reduced by a factor \( 10^a \), then it’s effect on \( v_1, w_1 \) is that they increase by a multiple of \( 10^a \), where \( a \) is a positive integer. Decreasing \( \Delta x \) requires the use of a much finer grid in \( r \).

The effects of varying \( RF \), for a fixed \( \Delta x \) is clearly seen from Figs. 3.2.9 & 3.2.14-3.2.15. Increasing \( RF \) to \( \tilde{a}RF \), where \( \tilde{a} \) is any positive real variable, increases \( v_1, w_1 \) to \( \tilde{a}v_1, \tilde{a}w_1 \) respectively. For small \( \Delta x, w_1 \) develops in a boundary layer fashion (Fig. 3.2.4 & 3.2.12). We investigate the reason for the sudden start at \( x = 0 \) of the large values of the radial velocity \( v_1 \).

We consider the two-dimensional case, which is equivalent to the axi-symmetric case near the wall \( r = 1 \) or \( y = 0 \). We wish to address what the starting behaviour (\( x \ll 1 \)) if, say, the \( x \)-pressure-gradient is not the Plane Poiseuille Flow (PPF) value. This may be answered by looking at the wall layer where

\[
y = x^{\frac{1}{3}} \eta, \quad \text{with } \eta = O(1)
\]

and

\[
u = \lambda \left( x^{\frac{1}{3}} \eta \right) + x^{\frac{2}{3}} u_2(\eta) + \ldots \quad (3.2.1)
\]
Here \( u_2 \sim \lambda_2 \eta^2 \) as \( \eta \to \infty \) (giving \( u \approx \lambda y + \lambda_2 y^2 + \ldots \ldots \) where the first term in this expansion is the input profile \( U_0(y) \)), and \( \lambda = 1 \) for example, but \( \lambda_2 \neq \lambda_{2PF} \); the value of \( \lambda_2 \) is given by the input profile. Now (3.2.1) implies, where \( \psi \) is the stream function and \( \lambda_2 \) is a scale factor,

\[
\psi = \frac{1}{2} \lambda x^2 \eta^2 + x \psi_2(\eta) + \ldots \ldots \quad (3.2.2)
\]

\[
v = - \left\{ \psi_2 - \frac{1}{3} \frac{1}{\eta} \psi'_2 \right\}.1 + \ldots \ldots \quad (3.2.3)
\]

and suppose

\[
\frac{dp}{dx} = K_1 + \ldots \ldots \quad (3.2.4)
\]

Substituting (3.2.1)-(3.2.4) into the two-dimensional vortex equations we get from the continuity and momentum equations, in turn

\[
u_2 = \psi'_2,
\]

\[
\lambda \eta \left\{ \frac{2}{3} u_2 - \frac{1}{3} \eta u'_2 \right\} - \left\{ \psi_2 - \frac{1}{3} \frac{1}{\eta} \psi'_2 \right\} \lambda = -K_1 + u''_2.
\]

Which gives

\[
\psi''_2 + \frac{1}{3} \lambda \eta^2 \psi''_2 - \lambda \left( \eta \psi'_2 - \psi_2 \right) = K_1 \quad (3.2.5)
\]

together with the boundary conditions

\[
\psi_2(0) = \psi'_2(0) = 0; \psi_2 \sim \frac{1}{3} \lambda_2 \eta^3 \text{ as } \eta \to \infty \quad (3.2.6)
\]
The solution of (3.2.5) is

\[ \psi_2 = a_2 \eta + b_2 \left( \eta^3 - \frac{6}{\lambda} \right) + c_2 f_2 (\eta) + \frac{K_1}{\lambda} \]  

(3.2.7)

with three arbitrary constants \( a_2, b_2, c_2 \), and \( f_2 \) decays exponentially as \( \eta \to \infty \). Imposing (3.2.6) gives three equations for \( a_2, b_2, c_2 \)

\[
\begin{align*}
-6b_2 \lambda^{-1} + c_2 f_2 (0) + K_1 \lambda^{-1} &= 0, \\
a_2 + c_2 f'_2 (0) &= 0, \\
b_2 &= \frac{1}{3} \lambda_2 .
\end{align*}
\]

The three constants can be calculated (by obtaining \( f_2 \)), but the main point is that usually \( a_2 \) will be non-zero. Hence

\[ u_2 \sim \lambda_2 \eta^2 + a_2, \quad \frac{v}{\psi} \sim -a_2 \eta, \quad \text{as } \eta \to \infty. \]  

(3.2.8)

Hence in the core where \( y = O(1) \) (i.e. \( \eta \to x^{-\frac{1}{2}} y \)) we get

\[
\begin{align*}
u &= U_0 (y) + O \left( x^\frac{1}{2} \right), \\
v &= O \left( x^{-\frac{1}{2}} \right),
\end{align*}
\]

(3.2.9)

which is the cause of the large values of the radial velocity, therefore we cannot impose the initial \( v \) as a \( V_0 (y) \), say. We would expect a similar response at small \( x \) in the three-dimensional vortex system.
Solution for the zeroth mode of the radial velocity $v_0$ (real part), $RF = 0.01$
Fig. 3.2.2

Solution for the first mode of the streamwise velocity $u_1$ (real part), $RF = 0.01$
Solution for the first mode of the radial velocity $u_1$ (real part), $RF = 0.01$
Solution for the first mode of the azimuthal velocity $w_4$ (imaginary part), $RF = 0.01$
Fig 3.2.5

Solution for the first mode of the pressure $\hat{\mathbf{x}}_1$(real part), $RF = 0.01$
Fig 3.2.6

Solutions for the first mode of the streamwise velocities $u_4$ (real part) using

$\Delta x = 0.01$ and $0.1$, $RF = 0.01$
Solution for the first mode of the radial velocity $v_1$ (real part) using 

$\Delta x = 0.01$, $RF = 0.01$
Fig 3.2.8

Solution for the first mode of the azimuthal velocity $w_1$ (imaginary part) using

$\Delta x = 0.01, RF = 0.01$
Fig 3.2.9

Solution for the first mode of the pressure $\tilde{p}$ (real part) using

$\Delta x = 0.01$, $RF = 0.00001$
Fig 3.2.10

Solution for the first mode of the streamwise velocity $u_1$ (real part) using

$\Delta x = 0.0001, \ R F = 0.01$
Solution for the first mode of the radial velocity $v_1$ (real part) using

$\Delta x = 0.0001, RF = 0.01$
Fig 3.2.12

Solution for the first mode of the azimuthal velocity $w_1$ (imaginary part) using

$\Delta x = 0.0001, RF = 0.01$
Solution for the first mode of the pressure $\hat{q}_1$ (real part) using

$\Delta x = 0.0001, RF = 0.01$
Fig 3.2.14

Velocity profiles for $v_1$ (real part) and $w_1$ (imaginary part)

using $RF = 0.1, \Delta x = 0.001$
Fig 3.2.15

Velocity profiles for $v_1$ (real part) and $w_1$ (imaginary part)

using $RF = 0.05, \Delta x = 0.001$
3.3 $F_1$, $F_2$ Forcing

Here we show a different method of solving our equations of motion by adding extra (known) terms $F_1$ and $F_2$ which force the $r$, $\theta$ momentum equations respectively, in the case of the initial starting condition being that of Hagen-Poiseulle. The $x$, $r$, $\theta$ momentum equations (2.2.1a – c) become

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = \frac{\partial w}{\partial \theta} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r \frac{\partial}{\partial r}} + \frac{1}{r^2 \frac{\partial^2}{\partial \theta^2}}, \quad (3.3.1a - c)
\]

\[
\frac{\partial v}{\partial x} + \frac{\partial v}{\partial r} + \frac{\partial w}{\partial r} = -\frac{\partial q}{\partial r} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r \frac{\partial}{\partial r}} + \frac{1}{r^2 \frac{\partial^2}{\partial \theta^2}} + \left(- \frac{v}{r^2} \frac{\partial w}{\partial \theta} \right) + F_1,
\]

\[
\frac{\partial w}{\partial x} + \frac{\partial w}{\partial r} + \frac{\partial v}{\partial r} = -\frac{1}{r \frac{\partial}{\partial r}} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r \frac{\partial}{\partial r}} + \frac{1}{r^2 \frac{\partial^2}{\partial \theta^2}} + \left(- \frac{w}{r^2} \frac{\partial v}{\partial \theta} \right) + F_2,
\]

in turn. So in Fourier mode form the latter two are written as

\[
\frac{d^2 v_m}{dr^2} + \left(\frac{1}{r} - \frac{v_m}{r} \right) \frac{dv_m}{dr} - \left(\frac{m^2 + 1}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{1}{r \bar{w}_0} \right) v_m + \left(\frac{-2}{r^2} \frac{im}{r} + \frac{1}{r \bar{w}_0}\right) w_m
\]

\[
-\frac{d \bar{q}_m}{dr} = \frac{1}{\Delta x} \bar{u}_m v_0 - \frac{1}{\Delta x} \bar{u}_0 v_m - \frac{1}{\Delta x} \bar{u}_m \bar{v}_0 + \bar{v}_m \frac{dv_0}{dr} - \frac{1}{r} \bar{w}_m w_0 +
\]

\[
\sum_{l=1, l \neq m}^{m-1} \left[ \left(\frac{i(m-l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{m-l} - \frac{1}{r} \bar{w}_l w_{m-l} + \frac{\bar{v}_l}{\Delta x} \frac{dv_{m-l}}{dr} \right]
\]

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So the effect of these forcing terms is that as we march forward in \( x \), \( F_1 \) and \( F_2 \) bring in the three-dimensional effects (see Fig. 3.3.1 - 3.3.4). If we take for example

\[
F_1 = \begin{cases} 
0 \\
(\sin \theta) r^2 (1 - r^2)^2 x^3 
\end{cases} \tag{3.3.3}
\]
then in Fourier modes

\[ F_{2n} = \frac{1}{2i} r^2 \left(1 - r^2\right)^2 (ndx)^3 \quad n = 1, 2, \ldots \]  

(3.3.4)

where \( n = 0 \) denotes the starting position. So the extra term forces the \( \theta \) momentum equation for \( m = 1 \), similarly for \( F_1 \) in the case of a non-zero forcing term. We solve subject to the initial conditions

\[ \bar{u}_0 (r) = 1 - r^2; \quad \bar{v}_0 (r) = \bar{w}_0 (r) = 0 \]  

(3.3.5)

\[ \bar{u}_m (r) = \bar{v}_m (r) = \bar{w}_m (r) = 0 \quad (m = 1, \ldots, N) \]  

(3.3.6)

and boundary conditions (2.2.7) and (2.2.8).

In (3.3.2a,b) \( F_1 \) and \( F_2 \) are given in turn by

\[ F_1 = F_{10} + \sum_{m=1}^{N} \left( F_{1m} E^m + F_{1m}^* E^{-m} \right), \]

\[ F_2 = F_{20} + \sum_{m=1}^{N} \left( F_{2m} E^m + F_{2m}^* E^{-m} \right). \]
3.4 Summary

So the method employed is an iterative technique to solve system (2.2.5), with the exception of (2.2.5c). The matrices on the right hand sides are updated after each iteration with the latest values of the velocity components, and solved again. This process is repeated until successive iterates are consistent within a specified tolerance. An $O(10^{-9})$ consistency is used as a test for convergence. The matrix inversion method allows us to use a fairly coarse grid, but solving for (2.2.5c) for $v_0$ using a shooting technique requires us to make the grid considerably finer.

Equations (2.2.5a,c) are solved 3 times at each $x$ step. By shooting once, the no-slip condition is not satisfied by $v_0$. Two different values of $\frac{dp}{dx}$ ($G_1$, $G_2$ say) are used to solve (2.2.5a,c). A linear interpolation is then performed using $G_1, G_2$ together with the corresponding radial velocities, to obtain $\frac{dp}{dx}$ for which the vanishing condition at the wall is satisfied. The third set of $u_0, v_0$ are used to solve the remaining equations in (2.2.5), before marching on to the next $x$ step.

Results are presented for varying step lengths $\Delta x$ and initial disturbance $RF$. We see that for small $x$, the flow develops in a boundary-layer fashion. We also see that the initial three-dimensional profile which is the cause for the large values of $v_1, u_1$ and our analysis confirms this.

The case of Hagen-Poiseulle flow in a straight pipe is also studied with the addition of forcing terms $F_1$ and $F_2$ which sets up a three-dimensional flow field downstream.
Chapter 4

Alternative Method Of Solution

4.1 Numerical Approach

In this chapter we present another method for the solution of the vortex system (2.2.1a-d). This alternative method is used later to compare with and/or check the results obtained from the original method of the previous chapter. We begin by rewriting (2.2.1a-d) as

\[ \frac{\partial u}{\partial x} = \tilde{G} + \left( \nabla^2 \tilde{u} - \tilde{v} \frac{\partial \tilde{u}}{\partial r} - \frac{\tilde{w}}{r \partial \theta} \right) \]

(4.1.1a)

\[ \frac{\partial v}{\partial x} = -\frac{\partial \tilde{q}}{\partial r} + \left( \nabla^2 \tilde{v} - \frac{\tilde{v}}{r^2} - \frac{2}{r^2 \partial \theta} \right) \frac{\partial \tilde{u}}{\partial r} - \frac{\tilde{w} \partial \tilde{v}}{r \partial \theta} + \frac{1}{r} \tilde{w} \]

(4.1.1b)

\[ r \frac{\partial w}{\partial x} = -\frac{\partial \tilde{q}}{\partial \theta} + \left( \nabla^2 \tilde{w} - \frac{\tilde{w}}{r^2} + \frac{2}{r^2 \partial \theta} \right) \frac{\partial \tilde{u}}{\partial r} - \frac{\tilde{v} \partial \tilde{w}}{r \partial \theta} - \frac{\tilde{w}}{r} \]

(4.1.1c)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r \partial \theta} \frac{\partial w}{\partial \theta} = 0 \]

(4.1.1d)

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \partial \theta^2} \]

The method is a more explicit one, basically of the Euler kind. We examine (4.1.1a) first to obtain \( u \). Using (2.2.2a) and the Fourier series expansions (2.2.4a-d), the expression for the \( x \) momentum (4.1.1a) becomes
\[
    u_0 = \frac{\Delta x}{u_0} \left\{ \frac{d^2 u_0}{dr^2} + \left( \frac{1}{r} - \bar{v}_0 \right) \frac{du_0}{dr} + \frac{1}{\Delta x} \bar{v}_0^2 + \bar{G} \right\} + \frac{\Delta x}{u_0} \sum_{l=1,l \neq m}^{N_l} \left( \bar{u}_l \left[ \frac{1}{\Delta x} \bar{u}_l^* - \frac{i l}{r} \bar{w}_l \right] - \bar{v}_l \frac{d\bar{u}_l}{dr} - \frac{1}{\Delta x} \bar{u}_l^* \bar{u}_l + c.c. \right) \quad (4.1.2a)
\]

where \(c.c\) denotes the complex conjugate.

\[
    u_m = \frac{\Delta x}{u_0} \left\{ \frac{d^2 u_m}{dr^2} + \left( \frac{1}{r} - \bar{v}_0 \right) \frac{d\bar{u}_m}{dr} - \left( \frac{m^2}{r^2} + \frac{i m}{r} \bar{w}_0 - \frac{2}{\Delta x} \bar{u}_0 \right) \bar{u}_m \right\} - \frac{\Delta x}{u_0} \bar{u}_m \frac{d\bar{u}_0}{dr} - \frac{\bar{u}_m u_0}{\bar{u}_0} +
\]

\[
    \frac{\Delta x}{u_0} \sum_{l=1,l \neq m}^{m-1} \left( \bar{u}_{m-l} \left[ \frac{1}{\Delta x} \bar{u}_{m-l}^* - \frac{i l}{r} (m - l) \bar{w}_l \right] - \bar{v}_{m-l} \frac{d\bar{u}_{m-l}}{dr} - \frac{1}{\Delta x} \bar{u}_{m-l}^* \bar{u}_{m-l} \right) +
\]

\[
    \frac{\Delta x}{u_0} \sum_{l=m+1,l \neq m}^{N} \left( \bar{u}_{l-m} \left[ \frac{1}{\Delta x} \bar{u}_{l-m}^* - \frac{i (l - m)}{r} \bar{w}_l \right] - \bar{v}_{l-m} \frac{d\bar{u}_{l-m}}{dr} - \frac{1}{\Delta x} \bar{u}_{l-m}^* \bar{u}_{l-m} \right) +
\]

\[
    \frac{\Delta x}{u_0} \sum_{l=1,l \neq m}^{N-m} \left( \bar{u}_{m+l} \left[ \frac{1}{\Delta x} \bar{u}_{m+l}^* - \frac{i (m + l)}{r} \bar{w}_l \right] - \bar{v}_{m+l} \frac{d\bar{u}_{m+l}}{dr} - \frac{1}{\Delta x} \bar{u}_{m+l}^* \bar{u}_{m+l} \right) \quad (m = 1, \ldots, N) \quad (4.1.2b)
\]

This allows us to solve for \(u_0\) and \(u_m\) \((m = 1, \ldots, N)\) respectively. The zeroth mode for the radial velocity \(v_0\) is obtained from the first method described in chapter 3.

We now wish to obtain an expression to evaluate the non-zero modes of the radial velocity \(u_m\).

To do this, we begin by eliminating \(\bar{q}\) from (4.1.1b,c) by the following
\[
\frac{\partial}{\partial \theta}(4.1.1b) - \frac{\partial}{\partial r}(4.1.1c)
\]

which gives

\[
\frac{1}{\Delta x} \left[ \bar{u} \left( \frac{\partial \bar{v}}{\partial \theta} - \bar{w} - r \frac{\partial \bar{u}}{\partial r} \right) + \bar{v} \frac{\partial \bar{u}}{\partial \theta} - r \bar{w} \frac{\partial \bar{u}}{\partial r} \right] =
\]

\[
\left\{ \frac{\partial^3 \bar{u}}{\partial r^2 \partial \theta} + \frac{1}{r} \frac{\partial^2 \bar{u}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^3 \bar{v}}{\partial \theta^3} - \frac{1}{r^2} \frac{\partial \bar{v}}{\partial \theta} - 2 \frac{\partial \bar{w}}{\partial r} - 2 \frac{\partial^2 \bar{w}}{\partial r^2} \right\} +
\]

\[
\left\{ \frac{1}{r^2} \frac{\partial^2 \bar{w}}{\partial \theta^2} - \frac{1}{r} \frac{\partial^3 \bar{w}}{\partial \theta^2 \partial \theta} - \frac{\bar{w}}{r^2} + \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} + \frac{2}{r^2} \frac{\partial \bar{u}}{\partial \theta} - \frac{2}{r} \frac{\partial^2 \bar{u}}{\partial \theta \partial r} \right\} +
\]

\[
\left\{ \frac{\partial \bar{v}}{\partial \theta} \frac{\partial \bar{v}}{\partial r} - \bar{v} \frac{\partial^2 \bar{v}}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} \frac{\partial \bar{w}}{\partial \theta} - \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} + \frac{2}{r^2} \frac{\partial \bar{w}}{\partial \theta} + 2 \frac{\bar{w}}{\partial r} + r \frac{\partial \bar{w}}{\partial \theta \partial r} \right\} +
\]

\[
\left\{ r \bar{w} \frac{\partial^2 \bar{w}}{\partial r^2} + \frac{\partial \bar{w}}{\partial \theta} \frac{\partial \bar{w}}{\partial \theta} + \frac{\partial^2 \bar{w}}{\partial \theta \partial r} + \frac{\partial \bar{w}}{\partial r} \right\} -
\]

\[
\frac{1}{\Delta x} \left[ \bar{u} \left( - \frac{\partial \bar{v}}{\partial \theta} + \bar{w} + r \frac{\partial \bar{u}}{\partial r} \right) - \bar{v} \frac{\partial \bar{u}}{\partial \theta} + r \bar{w} \frac{\partial \bar{u}}{\partial r} \right] = (4.1.4)
\]

Using (2.2.2b,c) and the Fourier expansions (2.2.4a-d), we have for \( v_m \) \((m = 1, \ldots, N)\)

\[
\frac{1}{\Delta x} \sum_{l=1, l \neq m}^N \left\{ \bar{u}_l \left[ i (m - l) v_{m-l} - w_{m-l} - r \frac{dv_{m-l}}{dr} \right] + il \bar{u}_l v_{m-l} - rw_{m-l} \frac{d\bar{u}_l}{dr} \right\} +
\]

\[
\frac{1}{\Delta x} \left\{ \bar{u}_0 \left[ im \bar{u}_m - w_m - r \frac{dw_m}{dr} \right] - rw_m \frac{d\bar{u}_0}{dr} \right\} +
\]

\[
\frac{1}{\Delta x} \left\{ \bar{u}_m \left[ -w_0 - r \frac{d\bar{u}_0}{dr} \right] + im \bar{u}_0 v_0 - rw_0 \frac{d\bar{u}_m}{dr} \right\} =
\]

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\[
\left\{ \begin{align*}
& im \frac{d^2 \overline{u}_m}{dr^2} + \left( \frac{im}{r} - \frac{2im}{r} \right) \frac{d \overline{v}_m}{dr} + \left( \frac{im}{r^2} - \frac{im^3}{r^2} \right) \overline{v}_m - \frac{r^3 \overline{w}_m}{dr^3} - 2 \frac{d^2 \overline{w}_m}{dr^2} + \\
& \frac{(m^2 + 1)}{r} \frac{d \overline{w}_m}{dr} + \frac{(m^2 - 1)}{r^2} \overline{w}_m \right\} + \\
& \sum_{l=0}^{N} \left\{ \overline{w}_{m-l} \left[ 2 \frac{d \overline{u}_l}{dr} + \frac{r^2 \overline{w}_l}{dr^2} + \frac{(m-l)^2}{r} \overline{u}_l - i(m-l) \frac{d \overline{u}_l}{dr} \right] + \frac{r}{dr} \frac{d \overline{u}_l d \overline{w}_{m-l}}{dr} \right\} + \\
& \sum_{l=0}^{N} \left\{ \overline{w}_{m-l} \left[ \frac{l(m-l)}{r} \overline{u}_l + \frac{2i(m-l)}{r} \overline{v}_l + im \frac{d \overline{w}_l}{dr} \right] + \frac{r}{dr} \frac{d \overline{w}_{m-l} \overline{v}_l}{dr} - \frac{r}{dr} \frac{d \overline{w}_{m-l} \overline{u}_l}{dr} \right\} \right.
\] 

where \( (4.1.5) \) contains the term for the azimuthal velocity, which we can eliminate using the equation of continuity \((4.1.1d)\) in the form

\[
\overline{w}_m = \frac{i}{m} \left( r \overline{S}_{1m} + \overline{u}_m + r \frac{d \overline{v}_m}{dr} \right) \tag{4.1.6}
\]

so \((4.1.5)\) becomes

\[
(-ir^2 \overline{u}_0) \frac{d^2 \overline{u}_m}{dr^2} - \left( 3ir \overline{u}_0 + ir^2 \frac{d \overline{u}_0}{dr} \right) \frac{d \overline{v}_m}{dr} + \left( i \overline{u}_0 [m^2 - 1] - ir \frac{d \overline{u}_0}{dr} \right) \overline{v}_m = \\

m \Delta x \left\{ -r \frac{d^3 \overline{w}_m}{dr^3} + im \frac{d^2 \overline{v}_m}{dr^2} + \left( -\frac{im}{r} + \overline{w}_0 - im \overline{v}_0 \right) \frac{d \overline{v}_m}{dr} + im \frac{d \overline{u}_m}{dr} + \frac{im}{\Delta x} \overline{u}_m \overline{v}_0 + \frac{im}{\Delta x} \overline{v}_m \overline{v}_0 \right\}
\]
\[
\left(-\frac{im^3}{r^2} + \frac{im}{r^2} - \frac{im}{r} \frac{d\bar{v}_0}{dr} + \frac{im}{\Delta x} \bar{u}_0 + \frac{m^2}{r} \bar{w}_0 + 2 \frac{d\bar{w}_0}{dr} + r \frac{d^2 \bar{w}_0}{dr^2}\right) \bar{v}_m + \\
\left(\frac{(m^2 - 1)}{r^2} + \frac{2im}{r} \bar{w}_0 + \frac{im}{\Delta x} \frac{d\bar{w}_0}{dr} - \frac{1}{\Delta x} \bar{u}_0 - \frac{r}{\Delta x} \frac{d\bar{u}_0}{dr}\right) \bar{w}_m - 2 \frac{d^2 \bar{w}_m}{dr^2} + \\
\left(\frac{(m^2 + 1)}{r} + \frac{r}{\Delta x} \frac{d\bar{w}_m}{dr} - \frac{r}{\Delta x} \frac{d\bar{u}_m}{dr}\right) \frac{d\bar{w}_m}{dr} \right) - m \left\{ \bar{u}_m \left( \frac{\bar{w}_0 + r \frac{d\bar{w}_0}{dr}}{dr} + r \frac{d\bar{w}_0}{dr} \right) + \\
m \Delta x \left\{ \frac{\bar{u}_0}{\Delta x} \left( \frac{-2ir^2 \bar{S}_{1m}}{m} - \frac{ir^2 d\bar{S}_{1m}}{m} \right) - \frac{1}{\Delta x} \frac{d\bar{u}_0}{dr} \frac{d\bar{w}_m}{dr} \right\} - \\
m \left\{ \bar{u}_m \left( \bar{w}_0 - r \frac{d\bar{w}_0}{dr} \right) + im \bar{u}_0 - r \frac{d\bar{u}_0}{dr} \right\} + \\
\sum_{l=1, l \neq m}^{m-1} m \Delta x \left\{ \bar{v}_{m-l} \left[ \frac{2}{r} \frac{d\bar{w}_l}{dr} + \frac{r}{l} \frac{d^2 \bar{w}_l}{dr^2} + \frac{(m-l)^2}{r} \bar{w}_l - \frac{1}{m} \frac{d\bar{w}_l}{dr} \right] + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m-l}}{dr} \right\} + \\
\left[ \left( \frac{d\bar{v}_{m-l}}{dr} \left( \bar{w}_l - i(m-l)\bar{v}_l \right) + \bar{w}_{m-l} \left[ \frac{1}{r} \frac{\left( m-l \right) \bar{v}_l + 2i (m-l) \bar{w}_l + im \bar{w}_l}{dr} \right] \right) \right] + \\
\sum_{l=m+1, l \neq m}^{N} m \Delta x \left\{ \bar{v}_{l-m} \left[ \frac{2}{r} \frac{d\bar{w}_l}{dr} + \frac{r}{l} \frac{d^2 \bar{w}_l}{dr^2} + \frac{(l-m)^2}{r} \bar{w}_l + i(l-m) \frac{d\bar{v}_l}{dr} \right] + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{l-m}}{dr} \right\} + \\
\left\{ \frac{d\bar{v}_{l-m}}{dr} \left( \bar{w}_l + i(l-m)\bar{v}_l \right) + \bar{w}_{l-m} \left[ -\frac{i}{r} \frac{(l-m) \bar{v}_l - 2i (l-m) \bar{w}_l + im \bar{w}_l}{dr} \right] \right\} + \\
\sum_{l=1, l \neq m}^{N-m} m \Delta x \left\{ \bar{v}_{m+l} \left[ \frac{2}{r} \frac{d\bar{w}_l}{dr} + \frac{r}{l} \frac{d^2 \bar{w}_l}{dr^2} + \frac{(m+l)^2}{r} \bar{w}_l - i(m+l) \frac{d\bar{v}_l}{dr} \right] + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m+l}}{dr} \right\} + \\
\sum_{l=1, l \neq m}^{N-m} m \Delta x \left\{ \bar{v}_{m+l} \left[ \frac{2}{r} \frac{d\bar{w}_l}{dr} + \frac{r}{l} \frac{d^2 \bar{w}_l}{dr^2} + \frac{(m+l)^2}{r} \bar{w}_l - i(m+l) \frac{d\bar{v}_l}{dr} \right] + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m+l}}{dr} \right\} + \\
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\[
\left\{ \frac{d \bar{v}_{m+l}}{dr} \left( \bar{w}_l^r - i(m + l)\bar{v}_l^r \right) + \bar{w}_{m+l} \left[ -\frac{l}{r} (m + l) \bar{v}_l^r + \frac{2i}{r} (m + l) \bar{w}_l^r + im \frac{d\bar{w}_l^r}{dr} \right] \right\} - \\
\sum_{l=1, l \neq m}^{m-1} m \left\{ \bar{u}_l \left[ -\frac{2ir}{(m - l)} \mathcal{S}_1(m-l) - \frac{i\nu^2}{(m - l)} \frac{d\mathcal{S}_1(m-l)}{dr} \right] - \frac{i\nu^2}{(m - l)} \frac{d\bar{u}_l}{dr} \mathcal{S}_1(m-l) \right\} + \\
\sum_{l=m+1, l \neq m}^{N} m \left\{ -\bar{u}_l \left[ \frac{2ir}{(l - m)} \mathcal{S}_1(l-m) + \frac{i\nu^2}{(l - m)} \frac{d\mathcal{S}_1(l-m)}{dr} \right] + \frac{i\nu^2}{(l - m)} \frac{d\bar{u}_l}{dr} \mathcal{S}_1(l-m) \right\} + \\
\sum_{l=1, l \neq m}^{N-m} m \left\{ -\bar{u}_l \left[ \frac{2ir}{(m + l)} \mathcal{S}_1(m+l) - \frac{i\nu^2}{(m + l)} \frac{d\mathcal{S}_1(m+l)}{dr} \right] - \frac{i\nu^2}{(m + l)} \frac{d\bar{u}_l}{dr} \mathcal{S}_1(m+l) \right\} - \\
\sum_{l=1, l \neq m}^{m-1} m \left\{ \bar{u}_l \left[ -i(m - l)\bar{v}_{l-m} + \bar{w}_{l-m} + r \frac{d\bar{w}_{l-m}}{dr} \right] - i(m - l)\bar{u}_{l-m}\bar{v}_l + r\bar{w}_{l-m} \frac{d\bar{u}_l}{dr} \right\} - \\
\sum_{l=m+1, l \neq m}^{N} m \left\{ \bar{u}_l \left[ i(l - m)\bar{v}_{l-m} + \bar{w}_{l-m} + r \frac{d\bar{w}_{l-m}}{dr} \right] + i(l - m)\bar{u}_{l-m}\bar{v}_l + r\bar{w}_{l-m} \frac{d\bar{u}_l}{dr} \right\} - \\
\sum_{l=1, l \neq m}^{N-m} m \left\{ \bar{u}_l \left[ -i(m + l)\bar{v}_{l+m} + \bar{w}_{l+m} + r \frac{d\bar{w}_{l+m}}{dr} \right] - i(m + l)\bar{u}_{l+m}\bar{v}_l + r\bar{w}_{l+m} \frac{d\bar{u}_l}{dr} \right\} + \\
\sum_{l=1, l \neq m}^{m-1} m \left\{ \left\{ \frac{i\nu}{(m - l)} \frac{d\bar{u}_l}{dr} \left( \bar{v}_{l-m} + r \frac{d\bar{v}_{l-m}}{dr} \right) \right\} - im\bar{u}_l\bar{v}_{l-m} \right\} - \\
\bar{u}_l \left\{ -\frac{i}{(m - l)} \left( \bar{v}_{l-m} + r \frac{d\bar{v}_{l-m}}{dr} \right) - \frac{i\nu}{(m - l)} \left( \frac{d\bar{v}_{l-m}}{dr} + r \frac{d^2\bar{v}_{l-m}}{dr^2} \right) \right\} + \\
\sum_{l=m+1, l \neq m}^{N} m \left\{ \left\{ -\frac{i\nu}{(l - m)} \frac{d\bar{u}_l}{dr} \left( \bar{v}_{l-m} + r \frac{d\bar{v}_{l-m}}{dr} \right) \right\} - im\bar{u}_l\bar{v}_{l-m} \right\} - \\
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\]
So this rather complicated expression gives us a method for obtaining $v_m$. In the derivation of (4.1.7), $w_m$ is eliminated using (4.1.5). To obtain the swirl velocity $w_m$, $(m = 1, \ldots, N)$ we simply use (4.1.6). So we have $u_m, v_m, w_m$ for all $m$, leaving us to evaluate the pressure terms $\hat{q}_m$, for the non-zero modes.

If we recall (2.2.5f), the $\theta$-momentum equation in Fourier mode form and re-arrange it, then we have

$$\frac{im}{r} \hat{q}_m = \frac{d^2w_m}{dr^2} + \left(1 - \frac{1}{r}\right) \frac{dw_m}{dr} - \left(\frac{m^2 + 1}{r^2} + \frac{1}{\Delta x} \frac{dw_m}{dr} - \frac{1}{r} \hat{w}_m w_0 + \frac{1}{\Delta x} \frac{w_0}{w_m} + \frac{1}{\Delta x} \frac{w_m w_0}{\Delta x}\right) w_m +$$

$$\frac{2}{r^2} im \Delta x u_m w_0 - \frac{1}{\Delta x} \frac{dw_0}{dr} - \frac{1}{r} \frac{w_m w_0}{\Delta x} - \frac{1}{\Delta x} \frac{w_0}{w_m} + \frac{1}{\Delta x} \frac{w_m w_0}{\Delta x} -$$

$$\sum_{l=1, l \neq m}^{m-1} \left\{ \frac{1}{\Delta x} (\hat{w}_l w_{m-l} - \hat{u}_l \hat{w}_{m-l}) + \left[ \frac{i}{r} (m - l) \hat{w}_l w_{m-l} + \frac{1}{r} \frac{dw_{m-l}}{dr} + \frac{1}{r} \frac{w_{m-l}}{\hat{w}_l}\right] \right\} -$$

$$\sum_{l=m+1, l \neq m}^{N} \left\{ \frac{1}{\Delta x} (\hat{w}_l w_{l-m} - \hat{u}_l \hat{w}_{l-m}) + \left[ \frac{i}{r} (l - m) \hat{w}_l w_{l-m} + \frac{1}{r} \frac{dw_{l-m}}{dr} + \frac{1}{r} \frac{w_{l-m}}{\hat{w}_l}\right] \right\} -$$
\[
\sum_{l=1, l \neq m}^{N-m} \left\{ \frac{1}{\Delta x} ( \bar{u}_l^* w_{m+l} - \bar{u}_l^* \bar{w}_{m+l}) + \left[ \frac{i}{r} (m + l) \bar{w}_l^* w_{m+l} + \bar{v}_l^* \frac{d w_{m+l}}{dr} + \frac{1}{r} \bar{v}_l^* w_{m+l} \right] \right\}
\]

\[(m = 1, \ldots, N) \quad (4.1.8)\]

which gives \( \bar{q}_m \).
4.2 Results

We present results for \( u_1, v_1, w_1 \) and \( q_1 \), (see Figs. (4.2.1-4.2.4)), and compare them with those obtained from the main method, given in chapter 3. This method is not as economical as the former as it gives reasonable solutions for very fine grids. These have to be used to overcome the singular type behaviour exhibited at the pipewall for coarser grids. Velocity profiles obtained from the second method (described in this chapter) are shown, using grids containing 1000 points. Profiles marked "1" and "2" denote the method used to obtain the solution.

The mesh was refined in \( x \) and \( r \) simultaneously but no better agreement could be achieved.
Fig. 4.2.1
Solutions obtained from both methods for the zeroth mode of the streamwise velocity $u_0$ (real part), $RF = 0.01$
$\Delta x = 0.001$
Fig. 4.2.2

Solutions obtained from both methods for the first mode of the streamwise velocity $u_1$ (real part), $RF = 0.01, \Delta x = 0.000001$
Solutions obtained from both methods for the first mode of the radial velocity $v_1$ (real part), $RF = 0.05, \Delta x = 0.00001$
Fig. 4.2.4

Solutions obtained from both methods for the first mode of the azimuthal velocity $w_1$ (real part), $RF = 0.05, \Delta z = 0.00001$
4.3 Summary

The work in this thesis is based on the first (numerical) method, described in chapter 3, which forms the main part of our study. The main reason for developing the second (explicit) method described in this chapter, is that the first method caused problems in developing due to the block tridiagonal matrix. As time was a major factor, another (simpler) method was sought. Whilst the second technique was being developed, work continued on the former and finally, both methods produced results which agreed.

The second method is less economical as it can only be used for very fine grids in the streamwise and radial directions. For larger values of $\Delta r$ and $\Delta x$ the solutions exhibit singularity type behaviour at the pipe wall. However this scheme can be used for validating the results obtained from the main method.

The technique used here is fairly simple for obtaining $u_0, u_m$ ($m = 1, ..., N$). The main task here is obtaining (4.1.7) for non-zero modes of the radial velocity $v_m$ which is clear from the work required in deriving it. Solving (4.1.7) is the only part of the method which requires an iterative technique for solution. This then allows us to simply obtain non-zero modes of the swirl velocity $w_m$ from the continuity equation for ($m = 1, ..., N$). Thus leaving us to solve for non-zero modes of the pressure terms $\tilde{q}_m$, by making use of the $\theta$ momentum equation, as expressed in chapter 2.
Chapter 5

Fluid Flow Into A Curved Pipe

5.1 Equations of Motion

We begin here by considering the flow through a pipe of circular cross-section, radius \( a \) that is straight at \( x < 0 \) but at \( x = 0 \) suddenly bends, its own axis being coiled to form the arc of a circle of radius \( R \)(see appendix 4). Here the Dean (1927, 1928) co-ordinates \( ax (= \Phi \rho ) \), \( ar \) and \( \theta \) will be adopted to denote distances down the pipe (where \( \Phi \rho \) measures the angular displacement around the arc), distances from the curved tubes axis of symmetry and angles around that axis, respectively. The corresponding velocity components are \( \bar{U}(u, v, w) \) in turn, \( \bar{U} \) being the characteristic (downpipe) velocity of the contained fluid.

Using the vector identities

\[
(u \cdot \nabla) u = (\nabla \wedge u) \wedge u + \nabla \left( \frac{1}{2} u^2 \right)
\]  

(5.1.1a)

\[
\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \wedge (\nabla \wedge \vec{u})
\]  

(5.1.1b)

the momentum equation (2.1.1a) becomes

\[
\nabla \left( \frac{1}{2} u^2 \right) - u \wedge (\nabla \wedge u) = -\frac{1}{\rho} \nabla p - \nabla \cdot (\nabla \wedge u)
\]  

(5.1.2)

If the pressure is taken to be \( \rho \bar{U}^2 \), the Navier-Stokes equations using (5.1.2) have the form (Dean 1927, 1928; Lyne 1971; Smith 1976iv)
\[ J \left( v \frac{\partial u}{\partial r} + \frac{w \partial u}{r} \right) + \delta u (v \cos \theta - w \sin \theta) + u \frac{\partial u}{\partial x} = -\frac{\partial p(x)}{\partial x} + \]

\[ Re^{-1} \left[ J \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left( \frac{\partial u}{\partial r} + \frac{\delta u}{J} \cos \theta \right) + \frac{J \partial^2 u}{r^2 \partial \theta^2} - \frac{J \partial}{r \partial \theta} \left( \frac{\partial u}{J} \sin \theta \right) \right] \]

\[ = Re^{-1} \left[ J \frac{\partial}{\partial r} \left( \frac{1}{J} \frac{\partial w}{\partial x} \right) + J \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left( \frac{1}{J} \frac{\partial v}{\partial x} \right) \right] \quad (5.1.3a) \]

\[ J \left( \frac{\partial v}{\partial r} + \frac{w \partial v}{r \partial \theta} - \frac{w^2}{r} \right) + u \frac{\partial v}{\partial x} - \delta u^2 \cos \theta = -J \frac{\partial p}{\partial r} + \]

\[ Re^{-1} \left[ \frac{1}{J} \frac{\partial^2 w}{\partial x^2} + \left( \frac{\delta \sin \theta - \frac{J \partial}{r \partial \theta}}{J} \right) \left( \frac{\partial w}{\partial r} + \frac{w}{r} - \frac{1}{r} \frac{\partial v}{\partial \theta} \right) - \frac{\partial^2 u}{\partial r \partial x} - \frac{\delta \partial u}{J x \cos \theta} \right] \]

\[ J \left( \frac{\partial w}{\partial r} + \frac{w \partial w}{r \partial \theta} + \frac{vw}{r} \right) + u \frac{\partial w}{\partial x} + \delta u^2 \sin \theta = -\frac{J \partial p}{r \partial \theta} + \]

\[ Re^{-1} \left[ \frac{1}{J} \frac{\partial^2 w}{\partial x^2} + \left( \frac{\delta \cos \theta + \frac{J \partial}{r \partial r}}{J} \right) \left( \frac{\partial w}{\partial r} + \frac{w}{r} - \frac{1}{r} \frac{\partial v}{\partial \theta} \right) - \frac{1}{r} \frac{\partial^2 u}{\partial x \partial \theta} + \frac{\delta \partial u}{J x \sin \theta} \right] \]

\[ = \frac{J \partial v}{r \partial r} + (2J - 1) \frac{v}{r} + \frac{J \partial w}{r \partial \theta} - \delta w \sin \theta + \frac{\partial u}{\partial x} = 0 \quad (5.1.3d) \]

The parameters appearing in (5.1.3a-d) are defined as

\[ Re = \frac{aU}{v}, \quad \delta = \frac{a}{R} \quad (5.1.4) \]

being the Reynolds number, assumed to be large and the relative curvature taken to be small, respectively. Also \( J = 1 + r \delta \cos \theta \).

The steady problem of this kind was first analyzed by Dean(1927,1928), who found that the motion depended on the parameter \( D \), where
\[ D = 2Re^2 \delta \] (5.1.5)

The analysis employed by Dean was restricted to small values of \( D \), but this has been extended numerically to moderately large values of \( D \) by McConologue & Srivastava (1968), Van Dyke (1978) and Dennis (1980). The study of steady flow in a curved pipe is thus quite extensive, although full understanding remains elusive.

The first main study of time-dependent viscous flows in a curved pipe was a theoretical treatment by Lyne (1970). Lyne assessed the motion induced in a circular tube by a pressure gradient that varies sinusoidally with time about a zero mean. Smith (1975) examined a number of the unsteady laminar flow features arising when a pulsatile pressure variation is imposed along a pipe of arbitrary but symmetric cross-section.

An extensive computational study of unsteady flows is carried out by Shirayama and Kuwahara (1986). They use a finite differencing approach to look at the unsteady flow features arising from curved pipeflow of circular cross-section with a straight inlet and outlet, covering a large range of Dean number.

We introduce a third parameter (as part of this study), the alternative Dean number \( K \), given by

\[ K = Re \delta^{\frac{1}{2}}. \] (5.1.6)

The other notable parameters adopted previously are given in Lyne (1970).

We consider the case of the flow far from the pipe-inlet first and then solve for the entry-flow.

When \( x \) is large and \( O \left( \delta^{-\frac{1}{2}} \right) \), for order one values of \( K \), the entire flow in the curved pipe is inertial and viscous, and the expressions (Smith 1976)

\[ u = u_c (\bar{x}, r, \theta) + O(\delta), \] (5.1.7a–d)
\[ v = \delta^\frac{1}{3} v_c(x, r, \theta) + O(\delta^\frac{1}{3}) \]

\[ w = \delta^\frac{1}{3} w_c(x, r, \theta) + O(\delta^\frac{1}{3}) \]

\[ p = \Delta p_c(x, r, \theta) + O(\delta^2) \]

describe the motion, where \( \Delta x = \delta^\frac{1}{3} x \). Upon substitution of (5.1.7a-d) into the Navier-Stokes equations (5.1.3a-d), and dropping the subscript \( c \) for convenience, we obtain the modified Dean equations

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{v}{r} \frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial \theta} &= -\frac{d \Delta p_c(x)}{dx} + K^{-1} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right), \\
\frac{\partial v}{\partial x} + \frac{v}{r} \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial v}{\partial \theta} &= -\frac{\partial q}{\partial r} + K^{-1} \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right) + K^{-1} \left( -\frac{v}{r^2} + \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right), \\
\frac{\partial w}{\partial x} + \frac{v}{r} \frac{\partial w}{\partial r} + \frac{w}{r} \frac{\partial w}{\partial \theta} &= \frac{\partial \Delta q}{\partial r} + K^{-1} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) + K^{-1} \left( \frac{w}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + v + \frac{1}{r} \frac{\partial w}{\partial \theta} &= 0,
\end{align*}
\]

with boundary conditions

\[ u = v = w = 0 \text{ at } r = 1 \quad (5.1.9) \]

In this chapter we use the analytical work by Smith (1976iv) as the basis for our computational study. The notation of the barred variables denoting
velocities at the previous \( \hat{x} \) step is again adopted here. It is worth noting that (5.1.8a-d) is the three dimensional vortex equations with the addition of the curvature terms \(-\vec{u}^2 \cos \theta \) and \( \vec{u}^2 \sin \theta \) in the \( r, \theta \) momentum equations (5.1.8b,c) respectively, and the factor of the Dean number in the viscous terms. Using the Fourier series expansions (2.2.4a-d) for the velocity and pressure terms, the modified Dean equations give

\[
K^{-1} \frac{d^2 u_0}{d r^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{d u_0}{d r} - \frac{\bar{u}_0}{\Delta x} u_0 = \bar{G}_0 - \frac{1}{\Delta x} \bar{w}_0^2 + \sum_{l=1}^{N} \left[ \left( \frac{i l}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_l - \frac{1}{\Delta x} \bar{u}_l \bar{u}_l + \bar{v}_l \frac{d u_l}{d r} \right] + \sum_{l=1}^{N} \left[ \left( \frac{-i l}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_l^* - \frac{1}{\Delta x} \bar{u}_l \bar{u}_l^* + \bar{v}_l \frac{d u_l^*}{d r} \right] \quad (5.1.10a)
\]

\[
K^{-1} \frac{d^2 u_m}{d r^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{d u_m}{d r} - \left( \frac{K^{-1} \left[ \frac{m}{r} \right]}{r} + \bar{u}_0 + \frac{im}{r} \bar{w}_0 \right) u_m = -\frac{2}{\Delta x} \bar{u}_0 u_m + \frac{1}{\Delta x} \bar{u}_m u_0 + \bar{v}_m \frac{d u_0}{d r} +
\]

\[
\sum_{l=1 \neq m}^{m-1} \left[ \left( \frac{i(m-l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m-l} + \bar{v}_l \frac{d u_{m-l}}{d r} \right] + \sum_{l=m+1 \neq m}^{N} \left[ \left( \frac{-i(l-m)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{l-m}^* - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{l-m}^* + \bar{v}_l \frac{d u_{l-m}^*}{d r} \right] + \sum_{l=1 \neq m}^{N-m} \left[ \left( \frac{i(m+l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) u_{m+l} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m+l} + \bar{v}_l \frac{d u_{m+l}}{d r} \right]
\]

\((m = 1, \ldots, N)\) \quad (5.1.10b)
for the streamwise velocity $u_0$ and $u_m$. The term $\tilde{G}_0 = -\frac{d}{d\bar{x}} \tilde{\rho}(\bar{x})/d\bar{x}$.

For the radial velocity components $v_0$ and $v_m$ ($m = 1, \ldots, N$) we have in turn

\[
\frac{dv_0}{dr} + \frac{1}{r} v_0 = -\frac{1}{\Delta x} (u_0 - \bar{u}_0) \tag{5.1.11a}
\]

\[
K^{-1} \frac{d^2 v_m}{dr^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{dv_m}{dr} - \left( K^{-1} \frac{(m^2 + 1)}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{im}{r} \bar{w}_0 \right) v_m +
\]

\[
\left( -\frac{2K^{-1}}{r^2} im + \frac{1}{r} \bar{w}_0 \right) w_m - \frac{d\tilde{g}_m}{dr} =
\]

\[
\frac{1}{\Delta x} \bar{u}_m v_0 - \frac{1}{\Delta x} \bar{u}_0 \bar{v}_m - \frac{1}{\Delta x} \bar{u}_m v_0 + \bar{v}_m \frac{dv_0}{dr} - \frac{1}{r} \bar{w}_m w_0 +
\]

\[
\sum_{l=1, l \neq m}^{m-1} \left[ \left( \frac{i(m-l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{m-l} - \frac{1}{r} \bar{w}_l w_{m-l} + \bar{v}_l \frac{dv_{m-l}}{dr} \right] -
\]

\[
\frac{1}{2} \left[ \sum_{l=0}^{m-1} \bar{u}_l \bar{u}_{m-l-1} + \sum_{l=0}^{m+1} \bar{u}_l \bar{u}_{m-l+1} \right] +
\]

\[
\sum_{l=m+1, l \neq m}^{N} \left[ \left( -\frac{i(l-m)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l \right) v_{l-m} - \frac{1}{\Delta x} \bar{u}_l \bar{v}_{l-m} - \frac{1}{r} \bar{w}_l w_{l-m} + \bar{v}_l \frac{dv_{l-m}}{dr} \right] -
\]

\[
\frac{1}{2} \left[ \sum_{l=m}^{N} \bar{u}_l \bar{u}_{l-m+1} + \sum_{l=m+1}^{N} \bar{u}_l \bar{u}_{l-m-1} \right] +
\]

\[
\sum_{l=1, l \neq m}^{N-m} \left[ \left( \frac{i(m+l)}{r} \bar{w}_l^* + \frac{1}{\Delta x} \bar{u}_l^* \right) v_{m+l} - \frac{1}{\Delta x} \bar{u}_l^* \bar{v}_{m+l} - \frac{1}{r} \bar{w}_l^* w_{m+l} + \bar{v}_l^* \frac{dv_{m+l}}{dr} \right] -
\]

\[
\frac{1}{2} \left[ \sum_{l=1}^{N-m+1} \bar{u}_l^* \bar{u}_{m+l-1} + \sum_{l=0}^{N-m-1} \bar{u}_l^* \bar{u}_{m+l+1} \right]
\]
The expressions for the swirl velocity components \( w_0 \) and \( w_m \) \((m = 1, \ldots, N)\) are

\[
K^{-1} \frac{d^2 w_0}{dr^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{dw_0}{dr} - \left( \frac{K^{-1}}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{1}{r} \bar{v}_0 \right) w_0 = -\frac{1}{\Delta x} \bar{u}_0 \bar{w}_0 + \sum_{l=1}^{N} \left[ \left( \frac{i}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l + \frac{1}{r} \bar{v}_l \right) w_l - \frac{1}{\Delta x} \bar{u}_l \bar{w}_l + \bar{v}_l \frac{dw_l}{dr} \right] + \frac{1}{2i} \left[ \sum_{l=m}^{N} \bar{u}_l \bar{u}_{l+1} - \sum_{l=1}^{N} \bar{u}_l \bar{u}_{l-1} \right] + \sum_{l=1}^{N} \left[ \left( \frac{-i}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l + \frac{1}{r} \bar{v}_l \right) \bar{w}_l^* - \frac{1}{\Delta x} \bar{u}_l \bar{w}_l^* + \bar{v}_l \frac{d(w_l^*)}{dr} \right] + \frac{1}{2i} \left[ \sum_{l=1}^{N+1} \bar{u}_l \bar{u}_{l-1} - \sum_{l=0}^{N-1} \bar{u}_l \bar{u}_{l+1} \right] \tag{5.1.12a}
\]

\[
K^{-1} \frac{d^2 w_m}{dr^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{dw_m}{dr} - \left( \frac{K^{-1}(m^2 + 1)}{r^2} + \frac{\bar{u}_0}{\Delta x} + \frac{im}{r} \bar{w}_0 + \frac{1}{r} \bar{v}_0 \right) w_m + \frac{2K^{-1}}{r^2} im v_m - \frac{im}{r} \bar{g}_m = \frac{1}{\Delta x} \bar{u}_m w_0 - \frac{1}{\Delta x} \bar{u}_0 \bar{w}_m - \frac{1}{\Delta x} \bar{u}_m \bar{w}_0 + \bar{v}_m \frac{dw_0}{dr} + \frac{1}{r} \bar{v}_m w_0 + \sum_{l=1, l \neq m}^{m-1} \left[ \left( \frac{i(m-l)}{r} \bar{w}_l + \frac{1}{\Delta x} \bar{u}_l + \frac{1}{r} \bar{v}_l \right) w_{m-l} - \frac{1}{\Delta x} \bar{u}_l \bar{w}_{m-l} + \bar{v}_l \frac{d(w_{m-l})}{dr} \right] + \frac{1}{2i} \left[ \sum_{l=0}^{m-1} \bar{u}_l \bar{u}_{m-l-1} - \sum_{l=0}^{m+1} \bar{u}_l \bar{u}_{m-l+1} \right] + \frac{1}{2i} \left[ \sum_{l=0}^{m-1} \bar{u}_l \bar{u}_{m-l-1} - \sum_{l=0}^{m+1} \bar{u}_l \bar{u}_{m-l+1} \right] + \frac{1}{2i} \left[ \sum_{l=0}^{m-1} \bar{u}_l \bar{u}_{m-l-1} - \sum_{l=0}^{m+1} \bar{u}_l \bar{u}_{m-l+1} \right] + \frac{1}{2i} \left[ \sum_{l=0}^{m-1} \bar{u}_l \bar{u}_{m-l-1} - \sum_{l=0}^{m+1} \bar{u}_l \bar{u}_{m-l+1} \right].
\]
and the continuity equation becomes

\[
\frac{1}{\Delta x} (u_m - \bar{u}_m) + \frac{dv_m}{dr} + \frac{1}{r} v_m + \frac{im}{r} w_m = 0
\]

\[ (m = 1, \ldots, N) \quad (5.1.12c) \]

The initial conditions used to solve (5.1.10), (5.1.11) and (5.1.12) take the form, at \( x = 0 \)

\[
\bar{u}_0(r) = 1 - r^2; \quad \bar{v}_0(r) = 0; \quad \bar{w}_0(r) = 0 \quad (5.1.13)
\]

\[
\bar{u}_m(r) = \bar{v}_m(r) = \bar{w}_m(r) = 0 \quad (m = 1, \ldots, N) \quad (5.1.14)
\]

which describe the Hagen-Poiseulle motion far upstream in the straight part of the pipe in cylindrical polars \((a\vec{x}, a\vec{r}, \vec{\theta})\) and corresponding velocities \( U(\bar{u}, \bar{v}, \bar{w}) \).

The boundary conditions to be used are those of regularity (2.2.7a-i) and no-slip (2.2.8a-f).

The numerical procedure used to solve the system (5.1.10) to (5.1.12) inclusive is that given in chapter 3.
5.2 Alternative Method Of Solution

As in chapter 4, we rewrite our equations of motion (5.1.8a-d) as

\[\frac{\partial u}{\partial x} = -\frac{dp}{dx} + K^{-1}\left(\nabla^2u - \frac{v}{r^2} - \frac{\partial v}{\partial \theta}\right) - \frac{\partial \bar{v}}{\partial r} - \frac{\bar{w}}{r} \frac{\partial \bar{v}}{\partial \theta} \quad (5.2.1a)\]

\[\frac{\partial v}{\partial x} = -\frac{\partial \bar{q}}{\partial r} + K^{-1}\left(\nabla^2\bar{v} - \frac{\bar{v}}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \theta}\right) - \frac{\partial \bar{v}}{\partial r} - \frac{\bar{w}}{r} \frac{\partial \bar{v}}{\partial \theta} + \frac{1}{r} \bar{w}^2 + \bar{u}^2 \cos \theta \quad (5.2.1b)\]

\[\frac{\partial w}{\partial x} = -\frac{\partial \bar{q}}{\partial \theta} + K^{-1}\left(\nabla^2\bar{w} - \frac{\bar{w}}{r^2} + \frac{2}{r^2} \frac{\partial w}{\partial \theta}\right) - \frac{\partial \bar{v}}{\partial r} - \frac{\bar{w}}{r} \frac{\partial \bar{v}}{\partial \theta} - \bar{\omega} \quad (5.2.1c)\]

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0 \quad (5.2.1d)\]

where

\[\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\]

Using (2.2.2a) and the Fourier series expansions (2.2.4a-d), we begin by writing the \(x\) momentum (5.2.1a) as

\[u_0 = \frac{\Delta x}{u_0} \left\{ K^{-1} \frac{d^2 u_0}{dr^2} + \left(\frac{K^{-1}}{r} - \bar{u}_0\right) \frac{du_0}{dr} + \frac{1}{\Delta x} \bar{u}_0^2 + \bar{G}_0\right\} + \]

\[\frac{\Delta x}{u_0} \left\{ \sum_{i=1,i\neq m}^N \left( u_i \left[ \frac{1}{\Delta x} \bar{u}_i - \frac{i \bar{u}_i}{r} \bar{w}_i \right] - \bar{u}_i \frac{du_i}{dr} - \frac{1}{\Delta x} \bar{u}_i^* u_i + c.c.\right) \right\} \quad (5.2.2a)\]

where \(c.c\) denotes the complex conjugate, and for \(m = 1, \ldots, N\)
\[ u_m = \frac{\Delta x}{u_0} \left\{ K^{-1} \frac{d^2 \bar{u}_m}{dr^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{d\bar{v}_m}{dr} - \left( \frac{K^{-1} m^2}{r^2} + \frac{im}{r} \bar{w}_0 - \frac{2}{\Delta x} \bar{u}_0 \right) \bar{u}_m \right\} \]

\[ - \frac{\Delta x}{u_0} \frac{d\bar{u}_0}{dr} - \frac{\bar{u}_m u_0}{\bar{u}_0} + \]

\[ \frac{\Delta x}{u_0} \left\{ \sum_{l=1}^{m-1} \left( \bar{u}_{m-l} \left[ \frac{1}{\Delta x} \bar{u}_l - \frac{i}{r} (m-l) \bar{w}_l \right] - \bar{v}_l \frac{d\bar{u}_{m-l}}{dr} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m-l} \right) \right\} \]

\[ + \frac{\Delta x}{u_0} \left\{ \sum_{l=m+1}^{N} \left( \bar{u}_{l-m} \left[ \frac{1}{\Delta x} \bar{u}_l + \frac{i}{r} (l-m) \bar{w}_l \right] - \bar{v}_l \frac{d\bar{u}_{l-m}}{dr} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{l-m} \right) \right\} \]

\[ + \frac{\Delta x}{u_0} \left\{ \sum_{l=1}^{N-m} \left( \bar{u}_{m+l} \left[ \frac{1}{\Delta x} \bar{u}_l - \frac{i}{r} (m+l) \bar{w}_l \right] - \bar{v}_l \frac{d\bar{u}_{m+l}}{dr} - \frac{1}{\Delta x} \bar{u}_l \bar{u}_{m+l} \right) \right\} \]

which gives us \( u_m \) for all \( m \). The zeroth mode of the radial velocity, \( v_0 \) is obtained by the method prescribed in the preceding section 5.1.

We obtain our expression to evaluate the non-zero modes of the radial velocity \( u_m \) by eliminating the pressure term \( \bar{q} \) from (5.2.1b) and (5.2.1c) by performing

\[ \frac{\partial}{\partial \theta} (5.2.1b) - \frac{\partial}{\partial r} (5.2.1c) \quad (5.2.3) \]

the operation gives

\[ \frac{1}{\Delta x} \left[ \bar{u} \left( \frac{\partial v}{\partial \theta} - w r \frac{\partial w}{\partial r} \right) + v \frac{\partial \bar{u}}{\partial \theta} - w \frac{\partial \bar{u}}{\partial r} \right] = \]

\[ K^{-1} \left\{ \frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \bar{v}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^3 \bar{v}}{\partial \theta^3} - \frac{1}{r^2} \frac{\partial \bar{v}}{\partial \theta} - \frac{2}{r^2} \frac{\partial^2 \bar{w}}{\partial \theta^2} - r \frac{\partial^3 \bar{w}}{\partial \theta^2} - 2 \frac{\partial^2 \bar{w}}{\partial r^2} \right\} + \]

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Using (2.2.2b,c) and the Fourier expansions (2.2.4a-d), we have for \( \nu_m \) (\( m = 1, \ldots, N \))

\[
\frac{1}{\Delta x} \sum_{l=1, l \neq m}^{N} \left\{ \bar{v}_l \left[ i(m-l)\nu_{m-l} - \nu_{m-l} - r \frac{d\nu_{m-l}}{dr} \right] + il\nu_{m-l} - ru_{m-l} \frac{d\nu_{0}}{dr} \right\} +
\]

\[
+ \frac{1}{\Delta x} \left\{ \bar{u}_0 \left[ im\nu_m - \nu_m - r \frac{d\nu_m}{dr} \right] - ru_m \frac{d\nu_0}{dr} \right\} +
\]

\[
- \frac{1}{\Delta x} \left\{ \bar{u}_m \left[ -w_0 - r \frac{d\nu_0}{dr} \right] + im\nu_m - ru_0 \frac{d\bar{v}_m}{dr} \right\} =
\]

\[
\left\{ \frac{im}{r} \frac{d^2\bar{v}_m}{dr^2} + \left( \frac{im}{r} - 2im \right) \frac{d\bar{v}_m}{dr} + \left( \frac{im}{r^2} - \frac{im^3}{r^2} \right) \bar{v}_m - r \frac{d^3\bar{v}_m}{dr^3} -
\right.
\]

\[
\left. 2 \frac{d^2\bar{w}_m}{dr^2} + \left( \frac{m^2}{r} + 1 \right) \frac{d\bar{w}_m}{dr} + \left( \frac{m^2}{r^2} - \frac{1}{r^2} \right) \bar{w}_m \right\} K^{-1} +
\]

\[
\sum_{l=0}^{N} \left\{ \bar{v}_{m-l} \left[ \frac{d\bar{w}_l}{dr} + r \frac{d^2\bar{w}_l}{dr^2} + \frac{(m-l)^2}{r} \bar{w}_l - i(m-l) \frac{d\bar{v}_l}{dr} \right] + r \frac{d\nu_l}{dr} \frac{d\bar{w}_{m-l}}{dr} \right\} +
\]

\[
- \sum_{l=0}^{N} \left\{ \bar{w}_{m-l} \left[ \frac{l(m-l)}{r} \bar{v}_l + \frac{2i(m-l)}{r} \bar{w}_l + i \frac{d\bar{v}_l}{dr} \right] + \frac{d\bar{v}_{m-l}}{dr} \left[ \bar{w}_l - i(m-l)\bar{v}_l \right] \right\} -
\]

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\[
\frac{1}{\Delta x} \sum_{i=0}^{N} \left\{ \bar{u}_i \left[ -i (m-l) \bar{w}_{m-l} + \bar{w}_{m-l} + r \frac{d\bar{w}_{m-l}}{dr} \right] - i (m-l) \bar{u}_{m-l} \bar{v}_l + r \bar{w}_{m-l} \frac{d\bar{v}_l}{dr} \right\} \\
+ \frac{i}{2} \left\{ \sum_{i=0}^{N} \left[ \bar{u}_i \bar{w}_{m-l} - \bar{u}_i \bar{w}_{m+l+1} \right] - \left\{ \sum_{i=0}^{N} \left[ \frac{d\bar{u}_{m-l}}{dr} - \bar{u}_i \frac{d\bar{w}_{m+l+1}}{dr} \right] \right\} \right\} \tag{5.2.5}
\]

(5.2.5) contains the term for the azimuthal velocity, which we can eliminate using the equation of continuity (5.2.1d) in the form

\[
u_m = \frac{i}{m} \left( r \bar{S}_{1m} + v_m + r \frac{dv_m}{dr} \right) \tag{5.2.6}
\]

where

\[
\bar{S}_{1m} = \frac{1}{\Delta x} (u_m - \bar{u}_m)
\]

so (5.2.5) becomes

\[
(-i r^2 \bar{u}_0) \frac{d^2 \nu_m}{dr^2} - \left( 3i r \bar{u}_0 + i r^2 \frac{d\bar{u}_0}{dr} \right) \frac{d\nu_m}{dr} + \left( i \bar{u}_0 \left[ m^2 - 1 \right] - i r \frac{d\bar{u}_0}{dr} \right) \nu_m =
\]

\[
m \Delta x \left\{ -r K^{-1} \frac{d^3 \bar{w}_m}{dr^3} + i m K^{-1} \frac{d^2 \bar{w}_m}{dr^2} + \left( \frac{im K^{-1}}{r} + \bar{w}_0 - im \bar{v}_0 \right) \frac{d\bar{w}_m}{dr} + \frac{im}{\Delta x} \bar{u}_m \bar{v}_0 \right\}
\]

\[
+ \left( \frac{im K^{-1}}{r^2} + \frac{im K^{-1}}{r^2} - im \frac{d\bar{u}_0}{dr} + \frac{im}{\Delta x} \bar{u}_0 + \frac{m^2}{r} \bar{w}_0 + 2 \frac{d\bar{w}_0}{dr} + r \frac{d^2 \bar{w}_0}{dr^2} \right) \bar{v}_m +
\]

\[
\left( \frac{(m^2 - 1) K^{-1}}{r^2} + \frac{2im}{r} \bar{u}_0 + im \frac{d\bar{w}_0}{dr} - \frac{1}{\Delta x} \bar{u}_0 - \frac{r}{\Delta x} \frac{d\bar{u}_0}{dr} \right) \bar{w}_m - 2K^{-1} \frac{d^2 \bar{w}_m}{dr^2} +
\]

\[
\left( \frac{(m^2 + 1) K^{-1}}{r} + r \frac{d\bar{u}_0}{dr} - \frac{r}{\Delta x} \bar{u}_0 \right) \frac{d\bar{w}_m}{dr} \right\} - m \left\{ \bar{u}_m \left( \bar{w}_0 + r \frac{d\bar{w}_0}{dr} \right) + r \bar{w}_0 \frac{d\bar{u}_m}{dr} \right\} +
\]

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\[ m \Delta x \left\{ \bar{u}_0 \left( 2 \frac{d\bar{w}_m}{dr} + r \frac{d^2 \bar{w}_m}{dr^2} \right) + r \frac{d\bar{v}_m}{dr} \frac{d\bar{w}_0}{dr} + i m \bar{w}_0 \frac{d\bar{w}_m}{dr} + \frac{d\bar{v}_0}{dr} \frac{d\bar{w}_m}{dr} \right\} - \\
\]
\[ m \Delta x \left\{ \frac{\bar{u}_0}{\Delta x} \left( \frac{-2i r S_{1m}}{m} \right) - \frac{\bar{v}_0}{\Delta x} \frac{dS_{1m}}{dr} \right\} - \\
\]
\[ m \left\{ \bar{u}_m \left( -\bar{w}_0 - r \frac{d\bar{w}_0}{dr} \right) + i m \bar{u}_m \bar{v}_0 - r \bar{w}_0 \frac{d\bar{u}_m}{dr} \right\} + \\
\]
\[ \sum_{l=1, l \neq m}^{m-1} m \Delta x \left\{ \bar{v}_{m-l} \left( 2 \frac{d\bar{w}_l}{dr} + r \frac{d^2 \bar{w}_l}{dr^2} + \frac{(m-l)^2}{r} \bar{w}_l - i(m-l) \frac{d\bar{v}_l}{dr} \right) + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m-l}}{dr} \right\} + \\
\]
\[ \left\{ \frac{d\bar{u}_{m-l}}{dr} \left( \bar{w}_l - i(m-l)\bar{v}_l \right) + \bar{w}_{m-l} \left[ \frac{l}{r} (m-l) \bar{v}_l + \frac{2i}{r} (m-l) \bar{w}_l + i m \frac{d\bar{w}_l}{dr} \right] \right\} \right\} + \\
\]
\[ \sum_{l=m+1, l \neq m}^N \left( \bar{v}_{m-l} \left( 2 \frac{d\bar{w}_l}{dr} + r \frac{d^2 \bar{w}_l}{dr^2} + \frac{(l-m)^2}{r} \bar{w}_l - i(l-m) \frac{d\bar{v}_l}{dr} \right) + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m-l}}{dr} \right\} + \\
\]
\[ \left\{ \frac{d\bar{u}_{m-l}}{dr} \left( \bar{w}_l + i(l-m)\bar{v}_l \right) + \bar{w}_{m-l} \left[ -\frac{l}{r} (l-m) \bar{v}_l - \frac{2i}{r} (l-m) \bar{w}_l + i m \frac{d\bar{w}_l}{dr} \right] \right\} \right\} + \\
\]
\[ \sum_{l=1, l \neq m}^{N-m} \left( \bar{v}_{m+l} \left( 2 \frac{d\bar{w}_l}{dr} + r \frac{d^2 \bar{w}_l}{dr^2} + \frac{(m+l)^2}{r} \bar{w}_l - i(m+l) \frac{d\bar{v}_l}{dr} \right) + r \frac{d\bar{v}_l}{dr} \frac{d\bar{w}_{m+l}}{dr} \right\} + \\
\]
\[ \left\{ \frac{d\bar{u}_{m+l}}{dr} \left( \bar{w}_l - i(m+l)\bar{v}_l \right) + \bar{w}_{m+l} \left[ -\frac{l}{r} (m+l) \bar{v}_l + \frac{2i}{r} (m+l) \bar{w}_l + i m \frac{d\bar{w}_l}{dr} \right] \right\} - \\
\]
\[ \sum_{l=1, l \neq m}^{m-1} \left( \bar{u}_l \left[ -\frac{2ir}{(m-l)} S_{1(m-l)} - \frac{ir^2}{(m-l)} \frac{dS_{1(m-l)}}{dr} \right] - \frac{ir^2}{(m-l)} \frac{d\bar{u}_l}{dr} \frac{S_{1(m-l)}}{dr} \right) + \\
\]
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\[ \sum_{l=m+1, l \neq m}^{N} m \left\{ -\bar{u}_l \left[ \frac{2ir}{(l-m)} \overline{S}^{*}_{1(l-m)} + \frac{ir^2}{(l-m)} \frac{d\overline{S}^{*}_{1(l-m)}}{dr} \right] + \frac{ir^2}{(l-m)} \frac{d\bar{u}_l}{dr} \overline{S}^{*}_{1(l-m)} \right\} + \]

\[ \sum_{l=1, l \neq m}^{N-m} m \left\{ -\bar{u}_l \left[ \frac{2ir}{(m+l)} \overline{S}^{*}_{1(m+l)} - \frac{ir^2}{(m+l)} \frac{d\overline{S}^{*}_{1(m+l)}}{dr} \right] + \frac{ir^2}{(m+l)} \frac{d\bar{u}_l}{dr} \overline{S}^{*}_{1(m+l)} \right\} - \]

\[ \sum_{l=1, l \neq m}^{m-1} m \left\{ \bar{u}_l \left[ -i(m-l)\overline{V}_{m-l} + \overline{V}_{m-l} + r \frac{d\overline{V}_{m-l}}{dr} \right] - i(m-l)\bar{u}_{m-l} \bar{u}_l + r\bar{V}_{m-l} \frac{d\bar{u}_l}{dr} \right\} - \]

\[ \sum_{l=m+1, l \neq m}^{N} m \left\{ \bar{u}_l \left[ i(l-m)\overline{V}^{*}_{m-l} + \overline{V}^{*}_{m-l} + r \frac{d\overline{V}^{*}_{m-l}}{dr} \right] + i(l-m)\bar{u}_{m-l} \bar{u}_l + r\bar{V}^{*}_{m-l} \frac{d\bar{u}_l}{dr} \right\} - \]

\[ \sum_{l=1, l \neq m}^{N-m} m \left\{ \bar{u}_l \left[ -i(m+l)\overline{V}_{m+l} + \overline{V}_{m+l} + r \frac{d\overline{V}_{m+l}}{dr} \right] - i(m+l)\bar{u}_{m+l} \bar{u}_l + r\bar{V}_{m+l} \frac{d\bar{u}_l}{dr} \right\} + \]

\[ m \sum_{l=1, l \neq m}^{m-1} \left\{ \frac{ir}{(m-l)} \frac{d\bar{u}_l}{dr} \left( \overline{V}_{m-l} + r \frac{d\overline{V}_{m-l}}{dr} \right) - i\bar{u}_l \bar{u}_{m-l} \right\} - \]

\[ m \sum_{l=1, l \neq m}^{m-1} \bar{u}_l \left\{ i(m-l)\overline{V}_{m-l} - \frac{i}{(m-l)} \left( \overline{V}_{m-l} + r \frac{d\overline{V}_{m-l}}{dr} \right) - \right\] \[ \frac{ir}{(m-l)} \left( 2 \frac{d\overline{V}_{m-l}}{dr} + r \frac{d^2\overline{V}_{m-l}}{dr^2} \right) \right\} + \]

\[ m \sum_{l=m+1, l \neq m}^{N} \left\{ -\frac{ir}{(l-m)} \frac{d\bar{u}_l}{dr} \left( \overline{V}^{*}_{m-l} + r \frac{d\overline{V}^{*}_{m-l}}{dr} \right) - i\bar{u}_l \bar{u}_{m-l} \right\} - \]

\[ m \sum_{l=m+1, l \neq m}^{N} \bar{u}_l \left\{ -i(l-m)\overline{V}^{*}_{m-l} + \frac{i}{(l-m)} \left( \overline{V}^{*}_{m-l} + r \frac{d\overline{V}^{*}_{m-l}}{dr} \right) + \right\] \[ \frac{ir}{(l-m)} \left( 2 \frac{d\overline{V}^{*}_{m-l}}{dr} + r \frac{d^2\overline{V}^{*}_{m-l}}{dr^2} \right) \right\} + \]

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Again this rather complicated expression gives us a method for obtaining $v_m$.
In the derivation of (5.2.7), $w_m$ is eliminated using (5.2.6). To obtain the swirl velocity $w_m$, ($m = 1, ..., N$) we simply use (5.2.6). So we have $u_m, v_m, w_m$ for all $m$, leaving us to evaluate the pressure terms $\hat{q}_m$, for the non-zero modes.
If we recall (2.2.5f), the $\theta$-momentum equation in Fourier mode form and re-arrange it, then we have
\[
\frac{\im}{r} \tilde{q}_m =
\]

\[
K^{-1} \frac{d^2 w_m}{dr^2} + \left( \frac{K^{-1}}{r} - \bar{v}_0 \right) \frac{dw_m}{dr} - \left( \frac{K^{-1} (m^2 + 1)}{r^2} + \frac{1}{\Delta x} \bar{u}_0 + \frac{\im}{r} \bar{w}_0 + \frac{1}{r} \bar{v}_0 \right) w_m
\]

\[
+ \frac{2K^{-1}}{r^2} \im v_m - \frac{1}{\Delta x} \bar{u}_m w_0 - \bar{v}_m \frac{dw_0}{dr} - \frac{1}{r} \bar{v}_m w_0 - \frac{1}{\Delta x} \bar{u}_0 \bar{w}_m + \frac{1}{\Delta x} \bar{v}_m w_0 -
\]

\[
\sum_{l=1, l \neq m}^{m-1} \left\{ \frac{1}{\Delta x} (\bar{u}_l w_{m-l} - \bar{u}_l \bar{w}_{m-l}) + \left[ \frac{i}{r} (m - l) \bar{w}_l w_{m-l} + \bar{u}_l - \frac{1}{r} \bar{v}_l w_{m-l} \right] \right\} -
\]

\[
\sum_{l=m+1, l \neq m}^{N} \left\{ \frac{1}{\Delta x} (\bar{u}_l^* w_{l-m} - \bar{u}_l^* \bar{w}_l) + \left[ -\frac{i}{r} (l - m) \bar{w}_l^* w_{l-m} + \bar{u}_l^* - \frac{1}{r} \bar{v}_l^* w_{l-m} \right] \right\}
\]

\[
\sum_{l=1, l \neq m}^{N-m} \left\{ \frac{1}{\Delta x} (\bar{u}_l^* w_{m+l} - \bar{u}_l^* \bar{w}_{m+l}) + \left[ \frac{i}{r} (m + l) \bar{w}_l^* w_{m+l} + \bar{u}_l^* - \frac{1}{r} \bar{v}_l^* w_{m+l} \right] \right\}
\]

\[
m = 1, \ldots, N
\] (5.2.8)

which gives \( \tilde{q}_m \). Again the numerical scheme used is that given in chapter 4.

The vortex system here is solved subject to the initial conditions (5.1.13), (5.1.14) and the boundary conditions (2.2.7), (2.2.8).
5.3 Solution Of The Entry-Flow Problem

The entry-flow problem on the shorter, $O(1)$, scale in $x$ (rather than the longer $x$-scale of section 5.4 below) is governed by the following viscous equations (Smith 1976iv)

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial \zeta} + \frac{\partial W}{\partial \theta} + \mu K^2 \frac{dN(x)}{dx} \cos \theta = 0, \quad (5.3.1a)
\]

\[
\mu V + \mu \zeta \left\{ \frac{\partial U}{\partial x} + \mu K^2 \frac{dN(x)}{dx} \cos \theta \right\} = -\frac{\partial P(x, \theta)}{\partial x} + \frac{\partial^2 U}{\partial \zeta^2}, \quad (5.3.1b)
\]

\[
\mu \zeta \frac{\partial W}{\partial x} = -\frac{\partial P(x, \theta)}{\partial \theta} + \frac{\partial^2 W}{\partial \zeta^2}, \quad (5.3.1c)
\]

where $\zeta$ is the boundary-layer coordinate and is related to $r$ by

\[
r = 1 + \delta N(x) \cos \theta - Re^{-\frac{1}{2}} \zeta, \quad (5.3.2)
\]

and

\[
N(x) = \begin{cases} 
\frac{1}{2}x^2 & : x < 0, \\
0 & : x > 0. 
\end{cases} \quad (5.3.3)
\]

so that the wall is given by $\zeta = 0$ for all values of $x$.

The governing equations (5.3.1a-c) are solved subject to the boundary conditions

\[
U = V = W = 0 \text{ at } \zeta = 0, \quad (5.3.4a)
\]
\[ U \to \mu K^2 \left[ \frac{1}{2} x^2 - N(x) \right] \cos \theta, \quad V \sim -\mu \zeta x K^2 \cos \theta, \quad W \to 0 \text{ as } \zeta \to \infty, \]  
(5.3.4b - d) 

where \( \mu = \frac{1}{2} \) (Smith 1976iv) and the unknown pressure term \( P \) has the property \( \partial P / \partial \zeta = 0 \) which follows from the radial momentum balance.

The solution of (5.3.1a-c) may be obtained by first eliminating the \( \theta \)-dependence. If we put

\[ (U, V, P) = (\bar{U}, \bar{V}, \bar{P}(x)) \cos \theta, \quad W = \bar{W} \sin \theta \]  
(5.3.5) 

where \( \bar{U}, \bar{V}, \bar{W}, \bar{P} \) are independent of \( \theta \), substituting in (5.3.1a-c) gives, in turn

\[ \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial \zeta} + \bar{W} + \mu K^2 \frac{dN(x)}{dx} = 0, \]  
(5.3.6a) 

\[ \mu \bar{V} + \mu \zeta \left\{ \frac{\partial \bar{U}}{\partial x} + \mu K^2 \frac{dN(x)}{dx} \right\} = -\frac{d\bar{P}(x)}{dx} + \frac{\partial^2 \bar{U}}{\partial \zeta^2}, \]  
(5.3.6b) 

\[ \mu \zeta \frac{\partial \bar{W}}{\partial x} = \bar{P}(x) + \frac{\partial^2 \bar{W}}{\partial \zeta^2}. \]  
(5.3.6c) 

Then eliminating \( \bar{V} \) between (5.3.6a) and (5.3.6b), our equations of motion reduce to

\[ \frac{\partial^2 \bar{U}}{\partial \zeta^3} + \mu \bar{W} = \mu \zeta \frac{\partial^2 \bar{U}}{\partial \zeta \partial x}, \]  
(5.3.7a)
\[
\mu \zeta \frac{\partial \tilde{W}}{\partial x} = \tilde{\bar{P}} + \frac{\partial^2 \tilde{W}}{\partial \zeta^2}, \tag{5.3.7b}
\]

which can be solved by use of the Fourier Transform approach. Letting * signify the Fourier Transform (F.T) with respect to \(x\), we define

\[
\tilde{U}^*(\omega, \zeta) = \int_{-\infty}^{\infty} \tilde{U}(x, \zeta) e^{-i\omega x} \, dx, \tag{5.3.8}
\]

eq etc. We now obtain the transformed version of the reduced momentum equations in (5.3.7)

\[
\frac{d^3 \tilde{U}^*}{d\zeta^3} + \mu \tilde{W}^* = i\omega \mu \zeta \frac{d\tilde{U}^*}{d\zeta}, \tag{5.3.9a}
\]

\[
i\mu \zeta \omega \tilde{W}^* = \tilde{\bar{P}}^* + \frac{d^2 \tilde{W}^*}{d\zeta^2}. \tag{5.3.9b}
\]

The solution for \(\tilde{W}^*\), satisfying the conditions of no-slip \((\zeta = 0)\) and vanishing at the edge of the boundary-layer \((\zeta = \infty)\), obtained from (5.3.9b) is

\[
\tilde{W}^*(\omega, \zeta) = -(O + i\mu \omega)^{-\frac{3}{2}} \left(\tilde{\bar{P}}^*(\omega)\right) \ell(t), \tag{5.3.10}
\]

where

\[
\ell(t) = Ai(t) \int_{0}^{t} \frac{\, dq}{Ai^2(q)} \int_{q}^{\infty} Ai(\xi) \, d\xi, \quad t = (O + i\mu \omega)^{\frac{1}{2}} \zeta, \tag{5.3.11}
\]

\(Ai\) is the Airy function and \(\ell(t)\) satisfies

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\[ \frac{d^2 \ell(t)}{dt^2} - t \ell(t) = 1, \quad \ell(0) = \ell(\infty) = 0. \]  

(5.3.12)

Here \( t \) is obtained from the balance on (5.3.9b), i.e.

\[ i \mu \omega \zeta \bar{W}^* \sim \frac{\bar{W}^*}{\zeta^2}, \]

giving \( \zeta \sim (i \mu \omega)^{-\frac{1}{2}} \), so we put \( t \equiv (i \mu \omega)^{\frac{1}{2}} \zeta \) and seek \( Q(t) \), such that \( \ell(t) = Q(t) \sqrt{\zeta} \) in (5.3.12), using

\[ \frac{d^2 Ai(t)}{dt^2} - t Ai(t) = 0. \]

The notation here implies that \((0 + i \mu \omega)^{\frac{1}{2}}\) has a branch cut extending from \( t0^+ \) along the positive imaginary \( \omega - \) axis. This now allows us to obtain \( U^* \), again vanishing at \( \zeta = 0 \) and bounded by the outer condition (5.3.4b). Substitution for \( \bar{W}^* \) in (5.3.9a), gives the solution in the form

\[ \frac{\partial U^*}{\partial t} = B(\omega)Ai(t) + \frac{\mu \bar{P}^*(\omega)}{(0 + i \mu \omega)^{\frac{3}{2}}} \left\{ \frac{d \ell}{dt} + \frac{Ai(t)}{3 Ai^2(0)} \right\}. \]

(5.3.13)

The unknown functions \( \tilde{B}(\omega) \) and \( \bar{P}^*(\omega) \) are determined from the following conditions:

(a) Setting \( \zeta = 0 \) in the axial momentum equation (5.3.1b), after obtaining its fourier transform yields

\[ \frac{\partial^2 U^*}{\partial \zeta^2} \bigg|_{\zeta=0} = i \omega \bar{P}^*, \]
giving

\[ \bar{B}(\omega) \frac{d}{dt} \exp(i \omega) (0 + \mu i \omega)^{\frac{3}{2}} = i \omega \tilde{P}^*(\omega) - \frac{\tilde{P}^*(\omega)}{i \omega} \left\{ 1 + \frac{d}{dt} \exp(0i \omega) \right\} \]  

(5.3.14)  

(b) The outer boundary condition on \( \bar{U}^* \) (from (5.3.4b)) gives

\[ \frac{1}{3} \bar{B}(\omega) + \frac{\mu \tilde{P}^*(\omega)}{9 \exp(2(0 + \mu i \omega)^{\frac{3}{2}})} = \frac{\mu K^2}{(i \omega)^{\frac{3}{2}}} \]  

(5.3.15)

Eliminating \( \bar{B} \) between (5.3.14) and (5.3.15) gives us an expression for the pressure transform, which is

\[ \tilde{P}^*(\omega) = \frac{\mu^\frac{3}{2} K^2 (0 + i \omega)^{-\frac{3}{2}}}{\omega^2 + 1} \]  

(5.3.16)

and \( \gamma = -3 \frac{d}{dt} \exp(0i \omega) \approx 0.7765 \). We note that (5.3.16) has two simple poles at \( \omega = \pm i \), and it is the presence of the pole in the lower half of the \( \omega \)-plane which is responsible for the upstream response in the solution. Upon inverting (5.3.16), we have

\[ \tilde{P}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mu^\frac{3}{2} K^2 \gamma (i \omega)^{-\frac{3}{2}}}{\omega^2 + 1} e^{ix} d\omega \]  

(5.3.17)

For \( x < 0 \), (5.3.17) is easily inverted (by calculating the residue at the simple pole \( \omega = -i \)) to give

\[ \tilde{P}(x) = \frac{1}{2} \mu^\frac{3}{2} K^2 \gamma e^x \]  

(5.3.18)

For \( x > 0 \), obtaining a solution of (5.3.17) requires more work because of the branch cut along the positive imaginary axis together with the simple pole at \( \omega = +i \). We begin by noting that (5.3.17) can be written as
\frac{1}{\omega^2 + 1} = 1 + \frac{(i\omega)^2}{\omega^2 + 1},

which allows us to write (5.3.17) as

\begin{equation}
\mu^{-\frac{5}{8}} K^{-2} \gamma^{-1} \tilde{P}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix\omega}}{(i\omega)^{\frac{1}{2}}} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix\omega}(i\omega)^{2}}{(\omega^2 + 1)(i\omega)^{\frac{1}{2}}} d\omega . \tag{5.3.19}
\end{equation}

The second term in (5.3.19) is evaluated by integrating around the contour \( C \) (see appendix 5) and letting the radii (see appendix 5) \( \delta \rightarrow 0 \) and \( R \rightarrow \infty \) which gives, for \( x > 0 \)

\begin{equation}
\tilde{P}(x) = -\frac{1}{4\pi} \mu^{\frac{5}{8}} K^{2} \gamma \left\{ \pi \psi - 6\sqrt{3} \Gamma \left( \frac{2}{3} \right) x^{\frac{1}{3}} + 2\sqrt{3} \int_{0}^{\infty} \frac{e^{-\psi x} \psi^{\frac{2}{3}}}{(1 - \psi^{2})} d\psi \right\} . \tag{5.3.20}
\end{equation}

The integral term in (5.3.20) assumes its principal value and is complicated to evaluate so we can obtain the behaviour of \( \tilde{P}(x) \) by reducing it to

\begin{equation}
-6x^{\frac{1}{3}} \int_{0}^{\infty} \frac{\cos(t^{-3} - 2\pi \beta)}{(m^2 + t^{-3}x^{-1})} dt . \tag{5.3.21}
\end{equation}

Here (5.3.21) suggests that \( \tilde{P}(x) \propto x^{\frac{3}{2}} \), which agrees with the analytical study of Smith (1976iv).
5.4 Results

We present results in this section for the velocity modes on the longer (vortex) scale downstream. For different values of the Dean number $\mathcal{K}$ we show profiles of the velocity modes $v_1$, $w_1$ near the starting position $\hat{x} = 0$ where the pipe begins to bend (Figs. 5.4.a-5.4.c). Varying the step-length $\Delta x$ (and hence $\hat{x}$) gives us the behaviour near the starting position, i.e. $0 < \hat{x} \ll 1$. Away from the wall Smith (1976iv) for $0 < x \sim C$ gives

\[ v_c = U_0(r) - \frac{1}{2} \hat{x}^2 U_v(r) \cos \theta + O\left(\hat{x}^\frac{3}{2}\right) \]  
\[ (5.4.1a - d) \]

\[ v_c = U_0(r) \hat{x} \cos \theta + O\left(\hat{x}^\frac{3}{2}\right) \]

\[ w_c = -U_0(r) \hat{x} \sin \theta + O\left(\hat{x}^\frac{3}{2}\right) \]

\[ p_c = O\left(\hat{x}^\frac{3}{2}\right) \]

where $U_0(r)$ is the Hagen-Poiseuille velocity profile and $'$ denotes differentiation with respect to $r$. (5.4.1a - d) allows us to validate the numerical results which our computations yield, near $x = 0$. In the table below

| $\Delta x$ | $|v_1|_{r=0}$ | subtract from 0.5 | $\log_{10}(1)$ | $\log_{10}(2)$ |
|-----------|--------------|-------------------|----------------|----------------|
| $10^{-3}$ | 0.2327       | 0.2673            | -0.6332        | -0.5730        |
| $10^{-4}$ | 0.2662       | 0.2338            | -0.5748        | -0.6312        |
| $10^{-5}$ | 0.2883       | 0.2117            | -0.5402        | -0.6742        |
| $10^{-6}$ | 0.303        | 0.197             | -0.5186        | -0.7055        |
| $10^{-7}$ | 0.309        | 0.191             | -0.5100        | -0.7190        |

the value of $|v_1|_{r=0}$ is noted from profiles for different values of $\Delta x$ and denoted by (1), it is subtracted from 0.5 and denoted by (2). $\log_{10}(1)$ and $\log_{10}(2)$ are plotted against $\log_{10} \Delta x$. We find that $\log_{10}(1)$ tends to the theoretical limit $\log_{10} 0.5 \ (= -0.3010)$ and we compare $\log_{10}(2)$ with the
theoretical relative error $O\left(\bar{x}^{\frac{1}{2}}\right)$. These together with Fig. 5.4.9 show that our numerical results agree with the analysis of Smith (1976iv). For small $\bar{x}$ the flow develops in a boundary-layer fashion (Fig. 5.4.10, 11). Marching forward in $\bar{x}$, i.e. $\bar{x} \to \infty$ we find that $w_c, v_c, w_c, p_c$ tend to the limits $U_c, V_c, W_c, P_c$ respectively (see Figs 5.4.1-5.4.8). The limit $W$ that we obtain agrees with the analytical result given by Van Dyke (1978).
Fig 5.4.a

Profiles for the radial & azimuthal velocities $v_1$ (real part) & $w_1$ (imaginary part), $K = 8$
Fig 5.4.b

Profiles for the radial & azimuthal velocities $v_1$ (real part) & $w_1$ (imaginary part), $K = 2$
Profiles for the radial & azimuthal velocities $v_1$ (real part) & $w_1$ (imaginary part), $K = 20$
Solution for the streamline velocity $u(x, y)$

Figure 5.4.1
Solution for the radial velocity $v_r$ (real part) $I = r \phi$

**Fig. 5.4.2**
Fig 5.4.3

Solution for the azimuthal velocity $u_1$ (imaginary part), $K = 1$
Solution for the pressure $\frac{\pi}{2}$ (real part).
Solution for the streamwise velocity u (real part).

Fig. 5.4.5

\( \nu = 4 \)
Solution for the radial velocity with real part

Figure 5.4.6
Fig 5.4.7

Solution for the azimuthal velocity $w_1$ (imaginary part), $K = 4$
Fig 5.4.9

log\(w_1(\text{max/min})\) against \(\log \Delta x\)
The flow begins to exhibit boundary-layer behaviour for small $\tilde{e}$. 

**Fig 5.4.10**
The flow develops in a boundary-layer fashion for small $\varepsilon$. 

Fig 5.4.11
Chapter 6

Flow In A Cornered Pipe

6.1 Equations of Motion

In this chapter we study a new pipeflow related problem, by considering flow through a straight pipe of circular cross-section, radius $a$, that is straight for $x < 0$, but at $x = 0$ experiences an abrupt angular bend $\alpha$.

(see appendix 6).

The equations of motion to be solved are (2.2.5a-g) together with the regularity and no-slip boundary conditions (2.2.7) and (2.2.8) respectively, using the methods described in chapters 3 and 4. The appropriate initial conditions used to solve (2.2.5a-g) are

$$\bar{u}_0 = 1 - r^2; \quad \bar{v}_0 = 0; \quad \bar{w}_0 = 0$$ (6.1.1a - c)

$$\bar{u}_1 = 0; \quad \bar{v}_1 = -\frac{\alpha}{2}(1 - r^2); \quad \bar{w}_1 = \frac{\alpha}{2i}(1 - r^2)$$ (6.1.1d - f)

$$\bar{u}_m = \bar{v}_m = \bar{w}_m = 0 \quad \text{for } m = 1, \ldots, N$$ (6.1.1g - i)

where $\alpha$ is the normalized pipe bend, given by

$$\alpha = \frac{\hat{\alpha}}{Re}$$ (6.1.2)
The pipe bend $\alpha$ in the initial conditions (above) behaves like the initial disturbance factor $RF$ given in chapter 3 for the case of flow in a straight pipe.
6.2 Solution Of The Entry-Flow Problem

In the viscous wall layer, \( \zeta = (1 - r) Re^{\frac{1}{2}} \) is \( O(1) \) on the shorter, \( O(1) \), scale present in \( x \) and the motion there is governed by the steady three-dimensional boundary-layer equations (Smith 1976v)

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial \zeta} + \frac{\partial W}{\partial \theta} = 0 , \tag{6.2.1a}
\]

\[
U \frac{\partial U}{\partial x} + V \frac{\partial V}{\partial \zeta} + W \frac{\partial U}{\partial \theta} = -\frac{\partial P(x, \theta)}{\partial x} + \frac{\partial^2 U}{\partial \zeta^2} , \tag{6.2.1b}
\]

\[
U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial \zeta} + W \frac{\partial W}{\partial \theta} = -\frac{\partial P(x, \theta)}{\partial \theta} + \frac{\partial^2 W}{\partial \zeta^2} , \tag{6.2.1c}
\]

subject to the boundary conditions

\[
U = V = W = 0 \quad \text{at} \quad \zeta = hF(x, \theta) , \quad \tag{6.2.2a - c}
\]

where \( h = \alpha Re^{\frac{1}{2}} \)

\[
(U - \mu \zeta), \quad \frac{\partial V}{\partial \zeta}, \quad W \longrightarrow 0 \quad \text{as} \quad \zeta \longrightarrow \infty , \]

\[
(U - \mu \zeta), \quad V, \quad W, \quad P \longrightarrow 0 \quad \text{as} \quad x \longrightarrow -\infty ,
\]

where \( \mu = \frac{1}{2} \) (Smith 1976v), \( F(x, \theta) \) is the indentation in the shape of a corner and \( h \) is an \( 0(1) \) constant. The unknown pressure term \( P \) has the property \( \partial P/\partial \zeta = 0 \) which follows from the radial momentum balance.

For \( h \ll 1 \), we can seek an analytical solution of (6.2.1a-c) in the form

\[
U = \mu \zeta + hU , \quad (V, W, P) = h \left( \overline{V}, \overline{W}, \overline{P} \right) \quad \tag{6.2.3}
\]

If we work to \( O(h) \) in the governing equations (6.2.1a-c) we have then
\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial \zeta} + \frac{\partial W}{\partial \theta} = 0 \ , \quad (6.2.4a - c) \]

\[ \mu \zeta \frac{\partial U}{\partial x} + \mu V = - \frac{\partial P}{\partial x} + \frac{\partial^2 U}{\partial x^2} \ , \]

\[ \mu \zeta \frac{\partial W}{\partial x} = - \frac{\partial P}{\partial \theta} + \frac{\partial^2 W}{\partial \theta^2} \ , \]

with

\[ U = -\mu \pi(x, \theta), \ V = W = 0 \ at \ \zeta = 0 \ , \quad (6.2.5a - c) \]

\[ U, \ \frac{\partial V}{\partial \zeta}, \ W \to 0 as \ \zeta \to \infty or \ x \to -\infty, \ F(-\infty, \theta) = 0 \ . \quad (6.2.5d - g) \]

Eliminating \( V \) between (6.2.4a) and (6.2.4b), our equations of motion reduce to

\[ \frac{\partial^3 U}{\partial \zeta^3} + \mu \frac{\partial W}{\partial \theta} = \mu \zeta \frac{\partial^2 U}{\partial \zeta \partial x} \ , \quad (6.2.6a) \]

\[ \zeta \mu \frac{\partial W}{\partial x} = - \frac{\partial P}{\partial \theta} + \frac{\partial^2 W}{\partial \theta^2} \ , \quad (6.2.6b) \]

which can be solved by use of the Fourier Transform approach described in section 5.3. Letting \( \ast \) signify the Fourier Transform (F.T) with respect to \( x \), we define as in section 5.3

\[ U^\ast (\omega, \zeta, \theta) = \int_{-\infty}^{\infty} U(x, \zeta, \theta) e^{-i\omega x} dx \ , \quad (6.2.7) \]
etc. We now obtain the transformed version of the reduced momentum equations in (6.2.6)

\[
\frac{\partial^3 U^*}{\partial \zeta^3} + \mu \frac{\partial W^*}{\partial \theta} = i\omega \mu \zeta \frac{\partial U^*}{\partial \zeta}, \\
i\mu \zeta \omega W^* = -\frac{\partial F^*}{\partial \theta} + \frac{\partial^2 W^*}{\partial \zeta^2}.
\] (6.2.8a, 6.2.8b)

These are now similar to the equations in (5.3.9) obtained in section (5.3) and the method employed from this point on, follows from that. Repeating the working yields from (6.2.8b)

\[
W^* = (0 + \mu i \omega)^{\frac{5}{2}} \frac{\partial F^*}{\partial \theta} \ell(t),
\] (6.2.9a)

where the notation of \((0 + \mu i \omega)^{\frac{5}{2}}\) and definition of \(\ell\) is given in section (5.3) and \(\ell(t)\) is defined by (5.3.11)

Substitution for \(W^*\) in (6.2.8a), gives the solution

\[
\frac{\partial U^*}{\partial t} = \bar{B}(\omega, \theta) Ai(t) + \frac{\mu}{(0 + \mu i \omega)^{\frac{5}{2}}} \frac{\partial F^*}{\partial \theta} \left[ -\frac{d\ell}{dt} \right],
\] (6.2.9b)

with the unknown function \(\bar{B}(\omega, \theta)\) satisfying

\[
\bar{B} \frac{d}{dt} Ai(0) - (0 + \mu i \omega)^{-\frac{5}{2}} \mu \frac{\partial^2 F^*}{\partial \theta^2} = (0 + \mu i \omega)^{\frac{5}{2}} \mu^{-1} F^*.
\] (6.2.10)

Integrating (6.2.9b) with respect to \(t\) and applying the constraints on \(\bar{U}^*\) at \(\zeta = 0\) and as \(\zeta \to \infty\), then yields the expression

\[
\frac{1}{3} \bar{B} = \mu F^*(\omega, \theta).
\] (6.2.11)
which now gives us two expressions for the unknown terms $\tilde{B}$ and $F^*(\omega, \theta)$.

We now suppose the pipe shape $F(x, \theta)$ is of Fourier analyzable form so

$$ F(x, \theta) = \sum_{m=0}^{\infty} G_m(x) \cos m\theta $$

(6.2.12)

Then the linearized solution is expressible as Fourier series

$$ U(x, \zeta, \theta) = \sum_{m=0}^{\infty} U_m(x, \zeta) \cos m\theta , \quad P(x, \theta) = \sum_{m=0}^{\infty} P_m(x) \cos m\theta $$

(6.2.13a, b)

e etc. Expanding (6.2.10) using (6.2.12) and (6.2.13) gives the expression

$$ -\gamma \mu G_m^*(\omega) \cos m\theta + (0 + \mu i\omega)^{-\frac{3}{2}} \mu m^2 \bar{P}_m^*(\omega) \cos m\theta \approx $$

$$ (0 + \mu i\omega)^{\frac{1}{2}} \mu^{-1} \bar{P}_m^*(\omega) \cos m\theta. $$

(6.2.14)

and $\gamma$ is defined in section 5.3.

Equating coefficients in $\cos m\theta$ and following further simplification, we have for each pressure component

$$ \bar{P}_m^*(\omega) = -\frac{\mu \gamma G_m^*(\omega)(0 + \mu i\omega)^{\frac{3}{2}} i\omega}{\omega^2 + m^2}. $$

(6.2.15)

The case of a cornered pipe is given by

$$ F(x, \theta) = \begin{cases} x \cos \theta : x > 0, \\ 0 : x < 0 \end{cases} $$

(6.2.16)

and hence $G_1(x) = x$, $G_m(x) = 0$ for $m=\theta$ and $m=2,3,\ldots$

$$ G_1^*(\omega) = \frac{1}{(i\omega)^2}. $$

so (6.2.15) becomes
\[
\overline{P}^*(\omega) = -\frac{\mu \gamma (0 + \mu \omega) \frac{3}{2} (i\omega)^{-\frac{1}{2}}}{\omega^2 + m^2} .
\] (6.2.17)

Upon inverting (6.2.17) we find that

\[
\overline{P}_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mu \gamma (i\omega)^{-\frac{1}{2}}}{\omega^2 + m^2} e^{i\omega x} d\omega ,
\] (6.2.18)

where \(m = 1\) is the only non-zero component. We can write \(\overline{P}_1(x) = \overline{K} \overline{P}_1(x)\) where

\[
\overline{K} = \frac{\mu \gamma}{2\pi} .
\]

Upon evaluating (6.2.18) for \(x < 0\) the only contribution comes from the pole \(\omega = -i\) and gives

\[
\overline{P}(x, \theta) = \frac{1}{2} \mu \gamma e^x \cos \theta \quad (x < 0) ;
\] (6.2.19)

see Fig. 6.2.1 .

For \(x > 0\), we write (6.2.18) as

\[
\overline{K}^{-1} \overline{P}_1(x) = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(i\omega)^{\frac{3}{2}}} d\omega + \int_{-\infty}^{\infty} \frac{e^{i\omega (i\omega)^2}}{(\omega^2 + 1)(i\omega)^{\frac{1}{2}}} d\omega .
\] (6.2.20)

The second integral is evaluated by integrating around \(C\) given in appendix 5 , again letting \(\delta \to 0\) and \(\overline{K} \to \infty\). However, we can obtain a numerical evaluation of (6.2.18) by reducing it to
\[ \tilde{P}_0(x) = \frac{3}{x^3} \int_0^\infty \cos \left( \frac{t^3}{x^2} - \frac{\pi}{6} \right) dt , \]  
\text{(6.2.21)}

and evaluating this by the trapezoidal rule. The result for (6.2.21) is given in Fig. (6.2.2). By using the expressions

\[ \int_0^\infty \cos t^p dt = \frac{1}{p} \cos \frac{\pi}{2p} \Gamma \left( \frac{1}{p} \right) , \quad \int_0^\infty \sin t^p dt = \frac{1}{p} \sin \frac{\pi}{2p} \Gamma \left( \frac{1}{p} \right) \quad x > 0 , \]  
\text{(6.2.22)}

we obtain an asymptotic solution to (6.2.21)

\[ \frac{\sqrt{3}}{x^3} \Gamma \left( \frac{2}{3} \right) , \quad x > 0 , \]  
\text{(6.2.23)}

which is also given in Fig. (6.2.2) and used to validate the accuracy of the numerical solution to (6.2.21). The present solutions incidentally are in a sense derivatives with respect to \( x \) of those in section 5.3. From that property, or otherwise, the behaviour of the skin-friction perturbation for example can also be deduced, showing enhancement and depletion at different \( x \) stations, but here we are concerned mainly with the pressure.
Fig. 6.2.1
Solution to the integral for the pressure term $\frac{\bar{F}(z, \theta)}{\bar{K} \cos \theta}$ for $z < 0$
Comparison between numerical & asymptotic solutions

Solution to the integral for the pressure term $F(z)$ given by (6.2.21) for $z > 0$, using numerical and asymptotic approach.

- $n$: numerical solution
- $a$: asymptotic solution

Graph shows the comparison between numerical and asymptotic solutions for pressure as a function of $x$, with $x$ ranging from 0 to 50 and pressure ranging from 0 to 30.
6.3 Results

We begin by presenting results (concerning the longer scale, downstream) for the velocity field, for flow near the start $x = 0$. Various plots of the velocity profiles for $u_1$, $v_1$, $w_1$ (see Figs. 6.3.a-6.3.d) are given for different values of the normalized pipe bend angle $\tilde{\alpha}$. We see from the velocity profiles that the starting condition of Hagen-Poiseulle flow becomes three-dimensional due to the pipe's bend.

The three-dimensional flow development is investigated and is given for different $\tilde{\alpha}$ values (see Figs. 6.3.1-6.3.12), for which we march forward in the streamwise direction $x$. Far from the inlet, as $x \to \infty$ the three-dimensional effects which are brought about at $x = 0$ due to the pipe's distortion die out and the flow returns to that of Hagen-Poiseulle. For larger angular bends, the cornering initiates more considerable three-dimensional effects and thus the flow has to travel further downstream to return to its original starting condition, a case for which we use a greater number of steps in $x$.

Fig. 6.3.d shows, as in the curved pipe case, that for small $x$ the flow develops in a boundary-layer fashion. Near the pipe inlet for $0 < x \ll 1$ we obtain for different values of the step-length $\Delta x$ the quantity

$$\left. \frac{\partial w_1}{\partial r} \right|_{\text{wall}}.$$

If we plot values of $\log \left. \frac{\partial w_1}{\partial r} \right|_{\text{wall}}$ against $\log \Delta x$, we obtain a curve (Fig. 6.3.13) which tends to a limiting value. The constant term is used to validate the starting behaviour at $x = 0$ obtained from the computational scheme.
Profile for the streamwise velocity $u_1$ (real part) for $\alpha = 0.8$
Fig 6.3.b

Profiles for the radial and azimuthal velocities $v_1$(real part) & $w_1$(imaginary part) for $\alpha = 0.8$
Fig 6.3.c

Profile for the streamwise velocity $u_4$ (real part) for $\tilde{\alpha} = 10.0$
Fig 6.3.d
Profiles for the radial and azimuthal velocities $v_1$ (real part)
& $w_1$ (imaginary part) for $\alpha = 10.0$
Fig 6.3.1

Solution for the streamwise velocity $u_1$ (real part) for $\alpha = 0.01$

three-dimensional effects die away downstream
Three-dimensional effects due away downstream

Solution for the radial velocity v (real part) for \( g = 0.1 \)

**Figure 6.3.2**
Fig 6.3.3

Solution for the azimuthal velocity $u_1$ (imaginary part) for $\alpha = 0.01$

three-dimensional effects die away downstream
Figure 6.3a

Solution for the pressure mode \( \xi \) (real part) for \( \alpha = 0.05 \)
Solution for the streamwise velocity $u$ (real part) for $a = 0.5$

**Fig. 6.3.5**
Fig 6.3 6
Solution for the radial velocity $v_r$ (real part) for $\sigma = 0.5$

- ABOVE = 0.003
- 0.000 = 0.003
- -0.003 = 0.000
- -0.007 = -0.003
- -0.010 = -0.007
- -0.014 = -0.010
- -0.017 = -0.014
- -0.021 = -0.017
- -0.024 = -0.021
- -0.028 = -0.024
- -0.031 = -0.028
- BELOW = -0.031
Fig 6.3.7

Solution for the azimuthal velocity $w_1$(imaginary part) for $\bar{\alpha} = 0.5$
Solution for the pressure mode $\xi$ (real part) for $\xi = 0.5$

Fig. 6.3.8
Solution for the streamline velocity v (real part) for $\omega = 5.0$
Solution for the radial velocity $v^1$ (real part) for $\eta = 0, \infty$.
Solution for the azimuthal velocity $w_1$ (imaginary part) for $\alpha = 5.0$
Solution for the pressure mode $g(\chi)$ (real part) for $\theta = 5.0$.

Figure 6.3.12
Fig 6.3.13

Plot of $\log \frac{\partial w_1}{\partial r}_{wall}$ against $\log \Delta x$
Chapter 7

Flow In A Pipe Of General Cross-Section.

7.1 Equations Of Motion

In this chapter we present the working in principle for obtaining flow solutions through pipes of any general cross-sectional area. In chapters 5 and 6 we obtained the flow fields in straight pipes of circular cross section but distorted by curvature and cornering, in turn. We now develop the equations of motion to solve for the velocity field in any shaped/distorted pipe, again concerning the longer-scale motion of vortex type on the $O(Re)$ scale in the streamwise direction.

We begin by re-writing the three-dimensional vortex equations (2.2.1a-d) as

\[
\left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial}{\partial \theta} \right\} u = \bar{g} + \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right\} u ; \quad (7.1.1a - d)
\]

\[
\left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial}{\partial \theta} \right\} v - \frac{w^2}{r} = -\frac{\partial \bar{q}}{\partial r} + \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2} \right\} v - \frac{2}{r^2} \frac{\partial w}{\partial \theta} ;
\]

\[
\left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial}{\partial \theta} \right\} w + \frac{vw}{r} = -\frac{1}{r} \frac{\partial \bar{q}}{\partial r} + \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2} \right\} w + \frac{2}{r^2} \frac{\partial w}{\partial \theta} ;
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0
\]

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where

\[-\tilde{G} = -\frac{dp(x)}{dx}\]

We now introduce our co-ordinate system to handle a general cross-sectional area of pipe. Within the pipe, currently we are using the polar co-ordinates \((r, \theta)\) and denote this as the complex \(Z\)-plane and refer to it as the real geometry. Our new system, namely the complex \(W\)-plane has the polar co-ordinates \((\tilde{r}, \tilde{\theta})\). This is represented diagrammatically in Fig. 7.1.1

The geometry in the \(Z\)-plane is the shape of pipe of interest and we perform a conformal map from the \(Z\)-plane onto the unit disc in the \(W\)-plane, which becomes our computational plane.

The basis of this work is the Riemann Mapping Theorem which states that there exists a one to one conformal mapping which maps any geometry from the \(Z\)-plane onto the unit disc in the \(W\)-plane, the inverse mapping also being conformal. For our study, if a conformal map exists for the interior cross-sectional shape of pipe, which we are interested in, then we can model the flow of fluid through it. Having defined our computing plane and new co-ordinates we can proceed further with the development of the equations of motion.

We introduce transformations for the derivatives

\[
\frac{\partial}{\partial r} \equiv \frac{\partial \tilde{r}}{\partial r} \frac{\partial}{\partial \tilde{r}} + \frac{\partial \tilde{\theta}}{\partial r} \frac{\partial}{\partial \tilde{\theta}}
\]

(7.1.2)

\[
\frac{\partial}{\partial \theta} \equiv \frac{\partial \tilde{r}}{\partial \theta} \frac{\partial}{\partial \tilde{r}} + \frac{\partial \tilde{\theta}}{\partial \theta} \frac{\partial}{\partial \tilde{\theta}}
\]

(7.1.3)

to express (7.1.1a-d) in terms of our new variables \((\tilde{r}, \tilde{\theta})\).

We now define a differential operator

\[\mathfrak{G} \equiv u \frac{\partial}{\partial x} + v \frac{\partial}{\partial \tilde{r}} + w \frac{\partial}{\partial \tilde{\theta}}\]

(7.1.4)

where
\[ \nu = v \frac{\partial \bar{r}}{\partial r} + \frac{w}{r} \frac{\partial \bar{r}}{\partial \theta} \]  \hspace{1cm} (7.1.5a)

\[ \bar{w} = v \frac{\partial \bar{\theta}}{\partial r} + \frac{w}{r} \frac{\partial \bar{\theta}}{\partial \theta} \]  \hspace{1cm} (7.1.5b)

Now \( W = \bar{r} e^{\bar{\theta}} \) is a complex function of \( Z = r e^{\theta} \). So the Cauchy-Riemann (C-R) equations can be written as

\[ \frac{\partial \bar{r}}{\partial r} = \frac{1}{r \partial \theta}, \quad \frac{\partial \bar{\theta}}{\partial r} = -\frac{1}{r \partial \theta} \]  \hspace{1cm} (7.1.6a, b)

Clearly (7.1.6a,b) satisfy Laplace's equation

\[ \frac{\partial^2 \bar{r}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{r}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{r}}{\partial \theta^2} = 0, \quad \frac{\partial^2 \bar{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\theta}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{\theta}}{\partial \theta^2} = 0 \]  \hspace{1cm} (7.1.7)

for \( \bar{r} \) and \( \bar{\theta} \) in turn. Now we define

\[ J = \left( \frac{\partial \bar{r}}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \bar{r}}{\partial \theta} \right)^2 \]  \hspace{1cm} (7.1.8a)

and similarly from (7.1.6a, b)

\[ J = \left( \frac{\partial \bar{\theta}}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \bar{\theta}}{\partial \theta} \right)^2 \]  \hspace{1cm} (7.1.8b)

and so it is clear that \( J \) is positive in general. Also from (7.1.6a,b) we find that

\[ \frac{\partial \bar{r}}{\partial r} \frac{\partial \bar{\theta}}{\partial r} + \frac{1}{r^2} \frac{\partial \bar{r}}{\partial \theta} \frac{\partial \bar{\theta}}{\partial \theta} = 0 \]  \hspace{1cm} (7.1.9)

This now allows us to write equations (7.1.1a-d) as

\[ \Omega [u] = -\bar{G} + \frac{\partial u}{\partial \bar{r}} \left\{ \frac{\partial^2 \bar{r}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \bar{r}}{\partial \theta^2} \right\} + \frac{\partial u}{\partial \bar{\theta}} \left\{ \frac{\partial^2 \bar{\theta}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \bar{\theta}}{\partial \theta^2} \right\} + \\
J \left\{ \left[ \frac{\partial^2 u}{\partial r^2} \right] + \left[ \frac{\partial^2 u}{\partial \theta^2} \right] \right\} \]  \hspace{1cm} (7.1.10a)
To replace \( \mathfrak{v}(v) \), \( \mathfrak{z}(w) \) by \( \mathfrak{v}(\bar{v}) \) and \( \mathfrak{z}(\bar{w}) \) respectively, we use (7.1.5a,b).

By performing

\[
\frac{\partial \bar{v}}{\partial \theta} \quad (7.1.5a) - \frac{\partial \bar{r}}{\partial \theta} \quad (7.1.5b) \quad \text{and} \quad \frac{\partial \bar{\theta}}{\partial r} \quad (7.1.5a) - \frac{\partial \bar{r}}{\partial r} \quad (7.1.5b) \quad (7.1.11a - b)
\]

we can express \( v \) and \( w \) in terms of \( \bar{v} \) and \( \bar{w} \), thus giving us the expressions

\[
v = \frac{1}{rJ} \left\{ \bar{v} \frac{\partial \bar{\theta}}{\partial \theta} - \bar{w} \frac{\partial \bar{\theta}}{\partial \bar{r}} \right\} \quad (7.1.12a)
\]

\[
w = \frac{1}{J} \left\{ \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} - \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{r}} \right\} \quad (7.1.12b)
\]

So now we can express the equations of motion (7.1.1a-d) in terms of the new velocity variables \( \bar{v}(r, \theta) \) and \( \bar{w}(r, \bar{\theta}) \). After considerable simplification, the three-dimensional vortex equations become
\[
\begin{align*}
    u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial \rho} + w \frac{\partial u}{\partial \theta} &= -\tilde{G} + \frac{\partial u}{\partial \rho} \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \rho \partial \theta} \right) + \frac{\partial u}{\partial \theta} \left( \frac{\partial^2 \tilde{G}}{\partial \rho^2} + \frac{1}{r^2} \frac{\partial^2 \tilde{G}}{\partial \rho \partial \theta} \right) + \\
    J \left\{ \left[ \frac{\partial^2 u}{\partial \rho^2} \right] + \left[ \frac{\partial^2 u}{\partial \theta^2} \right] \right\}  
\end{align*}
\]  

(7.1.13a)

\[
\begin{align*}
    \frac{1}{rJ} \left\{ u \left[ \frac{\partial \tilde{G}}{\partial x} - \frac{\partial w}{\partial x} \right] + v \left[ \frac{\partial \tilde{G}}{\partial \rho} + \frac{\partial^2 \tilde{G}}{\partial \rho \partial \theta} - \frac{w}{\partial \rho^2} \right] + w \left[ \frac{\partial \tilde{G}}{\partial \theta} - \frac{w}{\partial \rho \partial \theta} \right] \right\} + \\
    \frac{1}{rJ} \left\{ w \left[ \frac{\partial^2 \tilde{G}}{\partial \theta^2} + \frac{\partial^2 \tilde{G}}{\partial \theta^2} - \frac{w}{\partial \theta^2} \right] \right\} - \\
    \frac{1}{(rJ)^2} \left\{ v \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial w}{\partial \rho} \right] \left[ \left( \frac{\partial \tilde{G}}{\partial \rho} + r \frac{\partial \tilde{G}}{\partial \theta} \right) v + r \frac{\partial \tilde{G}}{\partial \theta} w \right] - \frac{1}{J^2} \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial \tilde{G}}{\partial \theta} \right]^2 = \\
    - \left\{ \frac{\partial q}{\partial r} \frac{\partial \tilde{G}}{\partial \rho} + \frac{\partial q}{\partial \rho} \frac{\partial \tilde{G}}{\partial \theta} \right\} + J \left\{ \left[ \frac{\partial \tilde{G}}{\partial r} \right]^2 + \left[ \frac{\partial \tilde{G}}{\partial \theta} \right]^2 \right\} - \frac{1}{r^2} \left\{ \frac{1}{rJ} \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial \tilde{G}}{\partial \theta} \right] \right\} - \\
    \left\{ \left[ \frac{\partial^2 \tilde{G}}{\partial \rho^2} + \frac{1}{r^2} \frac{\partial^2 \tilde{G}}{\partial \theta^2} \right] \frac{\partial \tilde{G}}{\partial r} + \left[ \frac{\partial^2 \tilde{G}}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \tilde{G}}{\partial \theta^2} \right] \frac{\partial \tilde{G}}{\partial \theta} \right\} \left[ \frac{1}{rJ} \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial \tilde{G}}{\partial \theta} \right] \right] - \\
    \frac{2}{r^2} \left\{ \frac{\partial \tilde{G}}{\partial \rho} \frac{\partial \tilde{G}}{\partial \rho} \right\} \left[ \frac{1}{J} \left( \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial \tilde{G}}{\partial \theta} \right) \right]  
\end{align*}
\]  

(7.1.13b)

\[
\begin{align*}
    \frac{1}{J} \left\{ u \left[ \frac{\partial \tilde{G}}{\partial x} - \frac{\partial w}{\partial x} \right] + v \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial w}{\partial \rho} \right] + w \left[ \frac{\partial \tilde{G}}{\partial \theta} - \frac{w}{\partial \rho} \right] \right\} + \\
    \frac{1}{J} \left\{ w \left[ \frac{\partial \tilde{G}}{\partial \rho} - \frac{w}{\partial \theta} \right] \right\} - \\
    \frac{1}{J^2} \left( \frac{\partial \tilde{G}}{\partial r} - \frac{\partial \tilde{G}}{\partial \rho} \right) \left[ \frac{\partial J}{\partial \theta} - \frac{\partial J}{\partial \rho} + \frac{1}{r^2} \left( \frac{\partial \tilde{G}}{\partial \rho} - \frac{\partial \tilde{G}}{\partial \theta} \right) \right] = 
\end{align*}
\]
\[-\frac{1}{r} \left\{ \frac{\partial^2 \theta}{\partial \theta \partial \theta} + \frac{\partial \bar{\phi}}{\partial \bar{r} \partial \theta} \right\} + \int \left\{ \left[ \frac{\partial \bar{r}}{\partial \bar{r}} \right]^2 + \left[ \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \right]^2 \right\} - \frac{1}{r^2 J} \left[ \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} - \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{r}} \right] + \right. \\
\left\{ \left[ \frac{\partial^2 \bar{r}}{\partial \bar{r}^2} + \frac{1}{r^2} \frac{\partial^2 \bar{r}}{\partial \bar{\theta}^2} \right] \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \right. \left[ \frac{\partial^2 \bar{\theta}}{\partial \bar{\theta}^2} + \frac{1}{r^2} \frac{\partial^2 \bar{\theta}}{\partial \bar{\theta}^2} \right] \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \left\{ \frac{1}{J} \left( \frac{\partial \bar{r}}{\partial \bar{r}} - \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{r}} \right) \right\} + \right. \\
\left. \frac{2}{r^2} \left[ \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} + \frac{\partial \bar{r}}{\partial \bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \right] \left\{ \frac{1}{r J} \left[ \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} - \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} \right] \right\} \right) \right) \tag{7.1.13c} \\
\frac{\partial \bar{u}}{\partial \bar{x}} + \left[ \frac{\partial \bar{r}}{\partial \bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} + \frac{\partial \bar{\theta}}{\partial \bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \right] \left\{ \frac{1}{r J} \left[ \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} - \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} \right] \right\} + \frac{1}{r^2 J} \left[ \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} - \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} \right] + \right. \\
\left. \frac{1}{r} \left[ \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \frac{\partial \bar{r}}{\partial \bar{\theta}} + \frac{\partial \bar{r}}{\partial \bar{\theta}} \frac{\partial \bar{\theta}}{\partial \bar{\theta}} \right] \left[ \frac{1}{J} \left( \bar{w} \frac{\partial \bar{r}}{\partial \bar{r}} - \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{r}} \right) \right] = 0 \right) \tag{7.1.13d} \]
7.2 Examples

As we discussed in the previous section, conformal mappings are used to map the interiors of the geometry of interest from the complex $Z$-plane by the function $W = f(Z)$ onto the unit circle in the complex $W$-plane which becomes our working plane where $Z = z + iy$ and $W = \xi + i\eta$. So the main problem encountered is to find the function $f(Z)$ with the above mentioned properties. In this section we look at mappings of interest, which have physical applications. We begin by considering the flow through a pipe of elliptic cross-section. This entails obtaining a conformal mapping, which maps the interior of an ellipse in $Z$ onto the interior of the unit circle in $W$.

If we take the ellipse in the $Z$-plane given by

$$\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1$$

see Fig. 7.2.1

Then the mapping $W = f(Z)$ maps the ellipse and the points $Z = 0; \sqrt{a^2 - b^2}; -\sqrt{a^2 - b^2}; a; ib$, (Fig. 7.2.1) conformally onto the interior of the unit circle and the points $w = 0; \sqrt{k}; -\sqrt{k}; 1; i$ (Fig. 7.2.2) respectively, and the required transformation is

$$W = \sqrt{k} \sin \left\{ \frac{2\widetilde{K}}{\pi} \sin^{-1} \frac{Z}{\sqrt{a^2 - b^2}} \right\}$$

where

$$k = \left[ \frac{\theta_2(\tau)}{\theta_3(\tau)} \right]^2 \quad ; \quad Im(\tau) > 0 \quad ; \quad \tau = \frac{2i}{\pi} \ln \left| \frac{a + b}{a - b} \right|$$

and $\widetilde{K}$ obtained from the elliptic integral.

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\[
\bar{K} = \int_0^\frac{\pi}{2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad \left( = F(k, \frac{\pi}{2}) \right)
\]

\[\theta_i(\tau) \quad (i = 1, 2, 3)\] is a theta function and defined by \(\theta_2(\tau) = \vartheta_2(0/\tau)\) and \(\theta_3(\tau) = \vartheta_3(0/\tau)\) where

\[
\vartheta_2(\tau Z) = 2 \sum_{n=1}^{\infty} Q^{\frac{1}{2}(2n-1)^2} \cos(2n-1) Z
\]

\[
\vartheta_3(\tau Z) = 1 + 2 \sum_{n=1}^{\infty} Q^n \cos(2n) Z
\]

with \(Q = e^{i \pi \tau}\).

As discussed previously, the flow development in pipes has attracted much attention due to its potential help in trying to obtain a better understanding of the dynamics of blood flow and the initiation of arterial disease. A survey carried out using ultrasound to study the effect on the reduction of arterial cross section due to atherosclerotic plaque formation (Hennerici & Steinke 1988) shows the most common shapes. A common cross-section for soft plaques results in blood flowing through a D-shaped area approximately, see Fig.7.2.3a. So the problem for us can be interpreted or modelled as that of obtaining a conformal mapping which transforms the interior of a semi-circle onto the unit disk, see Fig.7.2.3b. The required transformation is

\[
W = \frac{i(1-Z)^2-(1+Z)^2}{i(1-Z)^2+(1+Z)^2}
\]

where \(w = f_2 \circ f_1(Z)\), with
\[ f_1(Z) = \left( \frac{1+Z}{1-Z} \right)^2 \quad \text{and} \quad f_2(Z) = \frac{i-Z}{i+Z} \]

\( f_1(Z) \) maps the interior of a semi-circle onto the upper half plane, which is then mapped by \( f_2(Z) \) onto the interior of the unit circle.
Fig. 7.1.1 a, b

Z-plane

\[ z = 0 \]

W-plane

\[ \eta \text{-axis} \]

\[ \xi \text{-axis} \]
Fig. 7.2.1

Fig. 7.2.2
Fig. 7.2.3a, b
Chapter 8
Summary & Conclusion

This work is a theoretical and computational study of steady flow features at high Reynolds number, arising when fluid passes through various shaped pipes of circular cross-section. The computational work is based upon obtaining numerical solutions of the three-dimensional vortex equations. These are derived from the Navier Stokes equations in chapter 2 and the geometry of a straight pipe (in cylindrical polars) with three-dimensional steady fluid flowing through it is described. The numerical approach is given which allows us to write the governing equations in a form amenable to computation. By eliminating the dependency in $x$ and $\theta$ we are left with a system of second order linear ordinary differential equations. The initial conditions for the three-dimensional starting flow are given.

The computational scheme is described in chapter 3, which makes use of a finite differencing in $r$. A large part of the work is devoted to inverting the block tridiagonal matrix for $v_m$, $w_m$ and $\bar{\eta}_m$ for $m = 1, \ldots, N$. Results are shown for the velocity and pressure modes. Profiles are given for various values of the initial disturbance $RF$, showing the effects of altering this factor. The large values of $v_1$ and subsequently $w_1$ and $\bar{\eta}_m$ is a cause for concern. Analysis shows that imposing the radial velocity as an initial condition produces the large values of $v_1$. Another related case is studied which involves a Hagen-Poiseulle starting flow which is then forced by the addition of forcing terms $F_1$ and $F_2$, which initiate a three-dimensional flow as we travel down
Developing this numerical procedure was time consuming which prompted us to seek a simpler (explicit) method, which is given in chapter 4. An alternative technique for obtaining solutions of our governing equations (given in chapter 2) is described. The method is explicit in nature and relatively simple to compute. The terms $v_0 (r)$ and $w_0 (r)$ are obtained from the first method. Eliminating the pressure terms from the $r$, $\theta$ momentum equations results in a complicated expression for the radial velocity term $v_m (m = 1, \ldots, N)$. This is the only term which is evaluated iteratively from this method, solutions of which then allow us to obtain $u_m$ and $q_m$ in turn. Velocity profiles are plotted on the same axes as those obtained from the first method, which allows us to validate the accuracy of the former and allows us to make a good comparison between the two methods. However as a forward marching technique, the method is not very economical as we must use small $\Delta x$. For moderate sized values of $\Delta x$, the solutions exhibit singular type behaviour at the pipewall (which is then eliminated by reducing $\Delta x$ considerably).

We consider fluid flow through a curved pipe, in chapter 5. The work is based upon the analytical study by Smith (1976iv). The starting flow is that of Hagen-Poiseuille, through a straight pipe which at some finite distance starts bending uniformly to form the arc of a circle. The flow depends on the Dean number $K$ and the relative curvature of the pipe $\delta$ is taken to be small. We begin by solving for the long-scale flow. The Navier-Stokes equations reduce to the Dean equations for flow far from the curved pipe inlet, which govern the flow. Solutions of the Dean equations are obtained numerically using the methods described in chapters 3 & 4. Results are given for the flow field for various $K$. Numerical values taken from the velocity profiles for the velocity and pressure modes at $\tilde{x} = 0$ and $0 < \tilde{x} \ll 1$ agree with the theoretical results obtained by Smith (1976iv) with $p_1 \propto x^{\frac{1}{4}}$. As $\tilde{x} \to \infty$, we find that the velocity and pressure modes $u_m$, $v_m$, $w_m$ and $p_m$ tend to the limits $U_m$, $V_m$, $W_m$ and $P_m$. The limit $u_m \to W_m$ agrees with the solution obtained by
Van Dyke (1978), in the form of series for the stream function $\Psi$. Solution of the entry-flow problem yields the behaviour of the pressure.

The third pipeflow problem studied in chapter 6 is that of flow through a straight pipe which experiences an abrupt angular bend. The governing equations for flow far from the inlet is the vortex system for the straight pipe given in chapter 2 with a Hagen-Poiseulle starting profile, which becomes three-dimensional due to the cornering of the pipe. For small distances away from the pipe inlet we observe the flow developing in a boundary-layer fashion (as in the curved pipe case). Results are shown for different values of the normalized pipe bend $\alpha$. As we travel downstream ($x \to \infty$), the three-dimensional effects die out. Increasing $\alpha$ requires the flow to travel further downstream before the three-dimensional flow begins to return to the original Hagen-Poiseulle starting condition. Solving for the flow field at the entry to the cornered pipe inlet shows that the pressure $p \propto x^{3/2}$, which agrees with the asymptotic solution obtained.

We conclude the work by considering flow through pipes of any general cross-sectional area. The equations of motion are derived from the vortex equations and use is made of conformal mappings to transform the interior of the cross-section of the pipe from the $Z$-plane on to the unit disk in the $W-$plane. This chapter is of interest for modelling physiological flows, e.g. blood flow through healthy/diseased arteries which entails studying fluid flow through pipes of a general non-uniformly cross-sectional area distorted by constriction/dilatation. No attempt is made at solving the equations of motion, which are left as a starting point for future work.

Although the main numerical method (in chapter 3) required a lot of effort to develop, it seems to have been worthwhile for a number of reasons. It is now ready to be applied to other internal flows of interest (e.g. chapter 7) as well as those of chapters 5 & 6; it is flexible in the sense of being able to accommodate many different wall shapes and incident flow conditions; and it
could be made second-order accurate in the streamwise direction $x$ relatively easily as in the external flow work of Smith & Timoshin (1996).
Appendix 1

axial velocity profile

outer wall

secondary motions
Appendix 2

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & - & - & - & - & 0 \\
\sigma_{m1} & \Theta_{m1} & \delta_{m1} & 0 & - & - & - & - & - & 0 \\
0 & \sigma_{m2} & \Theta_{m2} & \delta_{m2} & 0 & 0 & - & - & - & 0 \\
0 & 0 & \sigma_{m3} & \Theta_{m3} & \delta_{m3} & 0 & 0 & - & - & 0 \\
0 & 0 & 0 & \sigma_{m4} & \Theta_{m4} & \delta_{m4} & 0 & 0 & - & 0 \\
0 & - & 0 & - & - & 0 & - & - & - & 0 \\
0 & - & - & 0 & - & - & 0 & - & - & 0 \\
0 & - & - & - & - & 0 & - & - & - & - \\
0 & - & - & - & - & 0 & - & - & - & - \\
0 & - & - & - & - & 0 & - & - & - & - \\
\end{pmatrix}
\times
\begin{pmatrix}
\begin{pmatrix}
\gamma_m \ 0 \\
\gamma_m \ 1 \\
\gamma_m \ 2 \\
\gamma_m \ J-2 \\
\gamma_m \ J-1 \\
\gamma_m J \\
\end{pmatrix}
\end{pmatrix}

= \begin{pmatrix}
\Gamma_{m0} \\
\Gamma_{m1} \\
\Gamma_{m2} \\
\Gamma_{mJ-2} \\
\Gamma_{mJ-1} \\
\Gamma_{mJ} \\
\end{pmatrix}
\]
Appendix 3

\[ \begin{bmatrix}
0 & 0 & 0 & 0 \\
\Delta_m & \Pi_m & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_m^0 \\
\epsilon_m^1 \\
\epsilon_m^2 \\
\epsilon_m^3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Where

\[ \bar{z}_m^0 = \begin{pmatrix} u_m^0 \\ \bar{y}_m^0 \\ \bar{g}_m^0 \end{pmatrix}, \quad \bar{z}_m^j = \begin{pmatrix} v_m^j \\ \bar{y}_m^j \\ \bar{g}_m^j \end{pmatrix}, \]

and for \( j = 1, \ldots, J - 1 \)
\[
\tilde{z}_{m,j} = \begin{pmatrix} 
u_{m,j} \\ w_{m,j} \\ q_{m,j} \end{pmatrix}, \quad \xi_{m,j} = \begin{pmatrix} \frac{-1}{2\Delta x} (u_{m,j} - u_{m,j} + u_{m,j-1} - u_{m,j-1}) \\ \xi_{m,j} \\ \psi_{m,j} \end{pmatrix},
\]
\[
\omega_{m,j} = \begin{pmatrix} \frac{-1}{2\Delta x} (u_{m,j} - u_{m,j} + u_{m,j-1} - u_{m,j-1}) \\ 0 \\ 0 \end{pmatrix}
\]

\[
I_{m0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (m \neq 1), \quad 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Lambda_{m,j} = \begin{pmatrix} \alpha_j & \gamma_{m,j} & 0 \\ \hat{A}_j & 0 & E \\ 0 & \hat{B}_j & 0 \end{pmatrix},
\]

\[
\Pi_{m,j} = \begin{pmatrix} \beta_j & \lambda_{m,j} & 0 \\ \bar{B}_{m,j} & \bar{D}_{m,j} & 0 \\ \bar{A}_{m,j} & \bar{C}_{m,j} & \bar{E}_{m,j} \end{pmatrix}, \quad \Upsilon_{m,j} = \begin{pmatrix} 0 & 0 & 0 \\ C_j & 0 & F \\ 0 & D_j & 0 \end{pmatrix}
\]

\[
\chi_{m,j} = \begin{pmatrix} \alpha_j & \gamma_{m,j} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa_{m,j} = \begin{pmatrix} \beta_j & \lambda_{m,j} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

If \( m \neq 1 \) then \( \tilde{0} = 0 \), else for \( m = 1 \)

\[
I_{m0} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

The coefficients are defined as follows

\[
\alpha_j = \frac{1}{r_j + r_{j-1}} - \frac{1}{\Delta r}, \quad \gamma_{m,j} = \frac{im}{r_j + r_{j-1}}, \quad \tilde{A}_j = \frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r} \left[ \frac{1}{r_j} - \bar{v}_{ij} \right],
\]

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\[
E = \frac{1}{2\Delta r}, \quad \tilde{B}_j = \frac{1}{(\Delta r)^2} - \frac{1}{2\Delta r} \left[ \frac{1}{r_j} - \bar{v}_{0j} \right], \quad \beta_j = \frac{1}{r_j + r_{j-1}} + \frac{1}{\Delta r},
\]

\[
\lambda_{mj} = \frac{im}{r_j + r_{j-1}}, \quad B'_{mj} = -\frac{2}{(\Delta r)^2} - \left\{ \frac{m^2 + 1}{r_j^2} + \frac{1}{\Delta x} \bar{v}_{0j} + \frac{im}{r_j} \bar{w}_{0j} \right\},
\]

\[
D_{mj} = -\frac{2}{r_j^2} im + \frac{1}{r_j} \bar{w}_{0j}, \quad \tilde{C}_{mj} = -\frac{2}{(\Delta r)^2} - \left\{ \frac{m^2 + 1}{r_j^2} + \frac{1}{\Delta x} \bar{v}_{0j} + \frac{im}{r_j} \bar{w}_{0j} + \frac{1}{r_j} \bar{v}_{0j} \right\},
\]

\[
\tilde{E}_{mj} = -\frac{im}{r_j}, \quad C_j = \frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r} \left[ \frac{1}{r_j} - \bar{v}_{0j} \right], \quad F = -\frac{1}{2\Delta r},
\]

\[
\tilde{D}_j = \frac{1}{(\Delta r)^2} + \frac{1}{2\Delta r} \left[ \frac{1}{r_j} - \bar{v}_{0j} \right], \quad \tilde{A}_{mj} = \frac{2im}{r_j^2}
\]
Appendix 5
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