THEORY OF PLASTICITY
APPLIED TO THE
SLANT SHEAR TEST

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Thesis submitted to the
University of London for the degree of
Doctor of Philosophy

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING
UNIVERSITY COLLEGE LONDON
SEPTEMBER, 1994
ABSTRACT

This thesis provides a critical analysis of the use of the theory of plasticity in the interpretation of Slant Shear Test (SST) results.

The SST is a testing method of evaluating efficacy of bonding systems used both in new structures and in repair services. Among other similar tests, the SST is considered as the most appropriate when structural performance is important. Nevertheless, a theoretically supported analysis of SST results is still necessary.

The theory of plasticity, originally derived for ductile materials such as metal, has also been applied to concrete which behaves almost like a brittle material. Hence, this work concisely describes features of ductile and brittle materials and applicable failure criteria. Principal theorems of limit analysis and basic concepts such as plastic potential flow are likewise presented.

The upper bound method applied to the SST is examined fully in this thesis. This study covers missing points found in previous work, such as, restrictions regarding the flow vector direction, by providing alternative solutions. However, theoretical approaches were unable to make predictions matching the reported data. This outcome led to further investigations into material performance.

An experimental investigation was carried out using specially designed specimens and apparatus to encourage and to monitor plastic response. Experimental results though, demonstrated virtually no plasticity. In order to identify whether ductility is required for redistribution of stress within SST specimens, lower bound analyses were considered by using finite element methods. Resulting stress profiles showed large differences of stress over failure zones which would require great ductility to allow a fully developed plastic mechanism.

In conclusion, plastic upper bound methods are shown inadequate for the analysis of the Slant Shear Test. Recommendations are therefore made for further work in two areas: empirical bases for assessing SST results and appropriate failure criterion for brittle materials.
To

Daniel and our lovely Bryan
ACKNOWLEDGEMENTS

First, I would like to thank Dr. J.R. Eyre, Head of Structures of the Department of Civil and Environmental Engineering at University College London, for his invaluable support, encouragement and his careful guidance during my research work.

I gratefully acknowledge the financial support conceded by the CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico, entity of the Brazilian government, for the programme of research described in this thesis.

Thanks are also due to Mr. D. Vale, for his professional and technical advice on the design of the experimental apparatus and Mr. O. Bourne, Mr. M. Saytch, Mr. I. P. Sturtevant and Mr. L. Wade, for their ready assistance in the laboratory work. I also wish to thank Dr. M. Nokhasteh for many helpful discussions during this study and Dr. E. Yarimer for his assistance in computer usage.

I wish to express my gratitude to Ms. A. Xydias for helping in reviewing this thesis.

Special thanks to Ms. L. Bordonalli and Ms. V. Pinto for looking after my son, Bryan.

I would also like to thank my parents for their general support, my son for his existence and, finally, a very special thank you to Daniel, my husband, for his support and encouragement during these four years away from our country.
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<td>$\alpha$</td>
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<td>$\alpha_s$</td>
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<td>$\beta$</td>
<td>inclination of the yield line</td>
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<td>$\beta_R$</td>
<td>shear retention factor (FE)</td>
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<td>$\dot{\gamma}$</td>
<td>shear strain rate</td>
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<td>$\gamma_{xy}$</td>
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<tr>
<td>$\lambda$</td>
<td>factor of proportionality (Eq. 2-10)</td>
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<tr>
<td>$\nu$</td>
<td>velocity of sliding</td>
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\( \nu_{\sigma}, \nu_{\tau} \) - components of the velocity of sliding in \( \sigma - \tau \) coordinates

\( \phi \) - angle of friction

\( \mu \) - coefficient of friction

\( \sigma \) - direct stress

\( \sigma_n \) - normal stress

\( \sigma_{\text{min}} \) - minimum direct stress

\( \sigma_y \) - yield stress

\( \sigma_1, \sigma_2, \sigma_3 \) - principal stresses

\( \tau \) - shear stress

\( \tau_{\text{max}} \) - maximum shearing stress

\( \tau_{xy} \) - in-plane shear stress (FE)

\( A \) - cross-sectional area

\( \overline{A} \) - area along a yield plane

\( B \) - parameter \( B=(1+r-rk)/(1-r-rk) \)

\( b \) - width (Fig. 3-7)

\( C \) - parameter \( C=(1+r)/(1-r) \)
c - cohesion

c_{con} - cohesion of concrete

d - length (Fig. 3-3a)

E - modulus of elasticity or Young’s modulus

\( f_c \) - concrete material strength in compression (factor multiplied by laboratory cylinder or cube strength)

\( f_c^* \) - effective concrete compressive strength

\( f_t \) - concrete material strength in tension

G - shear modulus

k - strength parameter of concrete (Eq. 3-17)

l, m, n - strength parameters of concrete (Eqs. 4-28, 4-29 and 4-30)

M_1, M_2 - mortar mixes by weight

\( N_{M_1}, N_{M_2} \) - maximum compressive load for control specimens made of mixes \( M_1 \) and \( M_2 \), respectively

\( \overline{N}_{M_1}, \overline{N}_{M_2} \) - average maximum compressive load for control specimens made of mixes \( M_1 \) and \( M_2 \), respectively

\( \overline{N} \) - maximum compressive load for slant shear test specimens

P - normal compressive load
Q - generalized stress
q - generalized strain
R - radius of curvature
r - ratio between the tensile and the compressive strength of concrete ($r = \frac{f_t}{f_c}$)
t - depth (Eq. 3-25)
$\dot{u}$ - displacement rate
W - work
$\dot{W}_I$ - rate of dissipation of internal energy
$\dot{W}_E$ - rate of external work
w/c - water/cement ratio
1.1 - GENERAL POSITION OF THE SLANT SHEAR TEST

1.1.1 - Repair & new materials

Repair services have been in significant demand even before the end of concrete structure design life (Long, et al, 1986). This fact is due to a wide variety of factors such as: inappropriate structure design, misapplications of construction techniques and different types of environmental aggression. Faced with the problem of ensuring security of the structure and economic factors, remedial action to stop further deterioration and maintain structural safety must be taken.

The high cost of maintenance and repair of subsequent failures determine that repair systems and materials be selected with care to ensure that durable and structurally safe solutions are provided. Before designing a repair system though, it is necessary to define among many options which is the most convenient material for a specific application. The main requirement of materials used to repair concrete structures is that they should have properties and dimensions which will make them compatible with the substrate for the application in hand (Plum, 1990).

Material properties such as: elastic modulus; creep and skrinkage; effects of temperature and humidity and bond strength must be defined. However, in the attempts to produce new materials which combine these properties, an assessment of bond quality is perhaps of prime importance to ensure durability.
1.1.2 - Bond needs & applications

The durability of concrete repair services relies on the adequacy of the bond at the surface between the old concrete and the applied volume repair. As an attempt to improve the bonding of concrete and of other materials to concrete, both in new structures and during the process of repairing existing constructions, bond materials, such as epoxy bonding agent, have been used.

Depending upon the type of application, the bond surface will be subjected to different loading conditions and stress states. For example, flat paved surfaces are likely to require bond properties under high compressive direct stress conditions. While vertical surfaces, such as surfaces originating from repair services in columns, require bond properties under tension stress conditions and soffits of beams and slabs demand bond properties under shear stress conditions. Hence, there is a need to investigate the bond strength of repair and new materials by selecting an appropriate testing method which anticipates loading conditions and stress states compatible with the in-service repair material conditions. Many attempts have been made to find a test procedure for evaluating the bonding strength. A brief description and discussion of common testing methods available is presented below.

1.1.3 - Common testing methods available

In order to assess the bonding mechanism many testing procedures have been proposed. Destructive test methods with the bond surface subjected to pure normal or shear stress present the following limitations:

(i) *direct compression* - This is an unproductive type of test method as zero bond is required to transfer stress (Eyre & Domone, 1985).

(ii) *direct tension* - This type of test method is of little value as the concrete tensile strength is not high and in most cases the tensile strength of repair materials surpasses the tensile strength of the original concrete. Hence, this type of test method
often measures the concrete tensile strength, but not that of the bond or repair material (Eyre & Domone, 1985). In addition, axial alignment in the direct tension testing method is difficult and very small deviations will result in a non-negligible bending moment (Saucier, et al, 1991). Indirect tension tests such as many variations of the flexure test are easier to carry out correctly. However, they are not effective as the bond area is submitted to a gradient of stress which determines the rupture of just a small part of the area subjected to maximum stress.

(iii) pure shear - In this type of test method where shear loading is applied exactly at the bond line, high stress concentration is induced at the bond area ends (Fig. 1-1(a)). In order to generate a pure shear stress, an elaborate set-up is required to avoid bending moments and tensile stresses. Base’s apparatus to obtain a pure shear condition is an example (Fig. 1-1(b)), which is perhaps too sophisticated to be considered as a standard procedure (Base, 1963).

![Diagram of Pure Shear Test](image)

**FIGURE 1-1 - Pure shear test**

Shear - compression testing method, where shear stress is combined with compressive stress, is an attempt to reduce the influence of local stress concentration verified in pure shear tests. Moreover, this testing method submits the bond line to a type of regime likely to be encountered in concrete structures. Reinforced concrete members are made of concrete - basically designed to sustain compressive stresses - and reinforcement, introduced to accommodate tensile stresses. Shear stresses are then generated when redistributing the tensile stresses within concrete. Hence,
concrete is often subjected to a combination of compressive and shear stresses, a situation to which the bond surface is subjected when a shear-compression test is performed.

Saucier, et al. (1991) developed, for research purposes, a type of shear-compression testing method. Some procedures were adopted in order to reduce stress concentrations (a disadvantage of the pure shear testing method) and to submit the bond line to a more representative loading system. However, complex apparatus needed to be built up, described as follows.

The device used in this method is schematically shown in Fig. 1-2. The following numbers in brackets refer to pieces pointed out in that figure. Specimens are subjected to a constant normal compressive stress generated by a gravitational load, applied by a mobile head [1]. The shearing load is applied by a hydraulic jack [2] on the upper half of the specimens. The bottom half of the specimens is tightened by a complex mechanism [3] comprising a mobile weight connected to a wheel pulling a steel cable. The cable pulls a pair of mobile rods that push a tightening head compressing the base of the specimens. Finally, magnesium pads are introduced to the metallic surface contact areas [4].

A much more practical version of a shear-compression test which is called Slant Shear Test was previously devised and is described as follows:
1.1.4 - The Slant Shear Test (SST)

The slant shear test consists of testing, in compression, a composite specimen which presents a diagonal bonded plane at an angle of 30° from the vertical axis. This method is standardized by the British Standards Institution (BS 6319 - Part 4, 1984) and by the American Society for Testing and Materials (ASTM - C-882, 1983). This type of testing method presents advantages such as:

- it is rather simple;
- it utilizes the standard compression test machine and other equipment usually available in a concrete testing laboratory;
- the diagonal bonded plane angle can vary and a large range of shear to normal stress ratios may be applied to the bond surface.

The SST has been well accepted in the technical community as a meaningful testing method which can be used to access the efficacy of bonding systems particularly where structural performance is important. It can also be used in assessment of repair proposals for deciding suitable applications for new formulations (Eyre, 1988).

1.2 - SST - Previous uses and methods of analysis

Some previous uses of the SST are briefly described below as well as the respective empirical attempts to analyse SST data.

1.2.1 - Empirical approaches

In order to investigate glued joints for structural concrete Johnson (1963, 1966 and 1970) carried out shear-compressive tests on prismatic specimens in which two concrete halves were joined with different types of epoxy adhesive at various joint angles.
His first experiments (1963, 1966) were developed principally to measure creep and little information was obtained on the effect of creep upon strength. In Johnson’s further tests (1970), the strength of specimens comprising joints (composite specimens) was compared to the strength of the parent control strength and there was no evidence that composite specimens were weakened by the presence of a joint.

Kriegh, 1976 suggested the SST be performed on standard cylinder specimens with the purpose of evaluating the mechanical capacity of epoxy bonding agents used to improve the bond between new or old concrete to existing concrete. His proposal was incorporated by the ASTM - C882/83. In order to analyse the SST results, he considered two empirical types of acceptability criterion.

The first one was by observing the failure mode. In this case, the epoxy system would be accepted either if testing two halves of different strength concrete, the specimen failed through the concrete on a plane adjacent to the bond surface or when testing two halves of concrete of or of nearly the same strength, the specimen failed like a monolithic one.

The other criterion suggested, was to consider the epoxy compound adequate when the composite cylinder strength was not less than 90% of the control parent cylinder strength. However, this figure (90%) was not properly justified by Kriegh.

Tabor, 1978 proposed the SST with a joint angle of 30° to the longitudinal axis, be performed in rectilinear prisms. He added to the SST the possibility of verifying the behaviour of repair systems using resin injection methods, by creating a crack as the joint line. His proposal was incorporated by the BS 6319 Part 4, 1984. Tabor suggested an analysis of the bond surface based upon the specimen failure mode, similar to Kriegh’s first proposition.

Eyre & Domone, 1985 pointed out some missing considerations of the BS 6319 Part 4, 1984, such as the use of a smooth joint surface to allow the comparison of SST results carried out in different laboratories and the use of a range of joint
inclinations. Based upon the latter suggestion, they proposed to apply Coulomb's criterion to analyse the results of the SST performed at different joint angles. The parameters c and \( \mu \) of Coulomb's equation would be obtained by adopting the linear extrapolation of the SST data corresponding to each particular repair system under analysis. In this way, different types of repair systems could be compared giving a more realistic meaning to the SST.

Similarly to Eyre & Domone, Franke, 1986 proposed the investigation of repair material performance by using rectilinear prisms and a varied range of joint inclination. However, he insisted on using roughened surfaces. He also suggested the use of Coulomb's criterion to the SST results but the parameters c and \( \mu \) were obtained, perhaps inappropriately, by using the average data provided by distinct authors under different testing conditions. In the end, having considered the effects on the repair material performance of sustained load; damp conditions and increased temperature, he proposed that the repair material would be accepted if the composite stress strength was not less than the concrete compressive strength.

Clímaco, 1989 applied the SST to rectilinear prisms. He used Coulomb's criterion just to determine the "critical angle" and the minimum strength by using empirical values of c and \( \mu \) proposed by Regan, 1986 for different joint roughness. In the experimental programme described by Clímaco, joint inclinations of 30° to the vertical axis (according to BS 6319, Part 4) and 20° to the vertical axis (according to his proposal) were used. He proposed an acceptability criterion to repair systems subjected to the SST based upon the specimen failure mode.

To summarize, some empirical attempts to interpret SST results have been made. Trying to evaluate the effectiveness of epoxy bonding agent, Kriegh proposed without a proper explanation, a figure of 90% to compare the strength of the composite specimen with the strength of the control parent specimen. Other attempts either analysing the failure mode, as proposed by Kriegh, Tabor and Clímaco or using the parameters c and \( \mu \) of Coulomb's criterion by extrapolation of the SST data, as suggested by Eyre & Domone and Franke, were also made. However, there
is a lack of statistical proof as these methods were defined either based upon just a little test data or by using mismatched data provided by distinct authors under different test conditions.

Although these empirical methods of analysis provide some guidance to SST users, a more critical and theoretically supported analysis is required. For this purpose, a method of analysis of plain concrete construction joints was suggested by a research worker of the Technical University of Denmark (Jensen, 1975) based upon the theory of plasticity.

1.2.2 - Analytical approach

Jensen, 1975 applied the concept of line of discontinuity for displacements in the theory of plasticity to plain concrete construction joints. Rectilinear prismatic specimens comprising inclined joints, tested in compression, were considered. By equating the energy dissipated along the joint line with the external work of the moving load, he determined an upper bound solution which was used to analyse the test results. A modification of the Mohr-Coulomb criterion was utilised as the yielding criterion. Agreement between theory and practice was found for a small number of tests. However, more work on the formulation of the theory is required. Although this application appears to be an incomplete approach, it inspired the present research to use the theory of plasticity as explained below.

1.3 - DECISION TO CHOOSE PLASTICITY

The theory of plasticity was chosen to be applied to the analysis of the slant shear test as an attempt to find a theoretically supported method to evaluate SST results. This decision was motivated by Jensen’s paper (Jensen, 1975). However, this application presents some points of consideration missing which directed the line of the present research. For example, choice of flow vector direction and adequacy of experimental data to support the theory.
In order to cover the complete range of plastic flow vector directions allowed by the Mohr-Coulomb modified yielding criterion, only partially used by Jensen, alternative upper bound solutions were derived. Also, the square yield locus was assumed as an alternative yielding criterion for the derivation of another upper bound solution. The adequacy of those solutions to analyse SST data was then discussed.

As the applicability of the theory of plasticity to concrete-like materials is quite limited, experimental proof of its validity for the purpose at hand, was also necessary. Therefore, an experimental investigation was carried out to determine whether internal stresses can be redistributed within a specimen subjected to the SST which is the essential condition for the validity of the application of limit analysis. In addition, lower bound solutions were considered, by using finite element (FE) methods, in order to identify the distribution of stress within SST specimens to show the level of ductility required by such specimens for redistribution of stress at first material yield.

1.4 - OUTLINE OF THIS THESIS

The theory of plasticity was originally derived for ductile materials such as metal. However, it has also been applied to concrete which presents behaviour closer to that of brittle materials. Therefore, a brief discussion on behaviour of ductile and brittle materials and respective applicable failure criteria is presented in Chapter 2. In order to support the investigation into the theory of plasticity, its principal theorems and basic concepts such as rigid, perfectly plastic material and plastic potential flow are also included in this chapter together with the proposal of a plastic analysis of SST data.

Previous applications of the upper bound theorem to the SST are described in Chapter 3 where limitations, such as the use of a flow vector direction arbitrarily fixed when an infinite range is available, were pointed out. An extension and completion of the use of the upper bound theorem is provided by means of the derivation of alternative upper bound solutions presented in Chapter 4.

It is also explained in Chapter 4 that the mechanism assumed by plastic
approaches to yield in the SST considers a yield line and a flow vector direction. It is not assumed that a pure shear failure, precipitated by loss of bond, takes place along the yield line. It is rather supposed that failure occurs along the yield line in the direction defined by the position of the flow vector on the yield criterion. A comparison between the solutions obtained here with previous ones is made and the suitability of these solutions to analyse the SST is discussed. However, there is an element of doubt because SST seems not to have a ductile response and plastic methods may not be applicable.

A programme of laboratory work was then carried out as an attempt to reproduce the failure pattern assumed in the theory. It should be stated here that it is not the purpose of the experimental programme to try to scrutinise the bond failure observed in real shearing joints but rather to create a zone of plastic failure to encourage the theoretically assumed mechanism and monitor its performance. As described in Chapter 5, special specimens were designed to model failure in a weak joint zone as it is conceived in theory. Attempts were made to avoid failure at the material interface in order to allow any yield process to take place inside the weak layer introduced in the designed SST specimen. Appropriate recording and loading operating systems were specially set up to monitor the experiments. Thus, the laboratory work was designed to examine the theory of plasticity not the usual failure pattern. Despite providing the conditions which most encourage a plastic response, experimental results demonstrated that the behaviour of SST specimens is closer to that of brittle materials with virtually no ductility.

In view of the experimental results, it was necessary to identify to what extent ductility is required by SST specimens to allow redistribution of stress from a lower bound satisfying equilibrium everywhere to an upper bound with a fully developed failure mechanism. For this purpose, lower bound solutions were considered by using FE methods, as described in Chapter 6. This is an attempt to provide a qualitative indication of stress distribution rather than a quantitative prediction of lower bound at failure.

As a result of this investigation, upper bound plastic methods are demonstrated inadequate for the analysis of SST data, as summarized in Chapter 7. Recommendations are therefore made for further research in two areas: (i) empirical bases for assessing the SST and (ii) development of an appropriate failure criterion for brittle materials.
In the previous chapter the SST was introduced, among other available testing methods of evaluating the efficacy of bonding systems used in repair services, as the most appropriate where structural performance is important. However, a theoretically supported analysis of SST results is still required. For this purpose, the theory of plasticity was suggested. This theory analyses the behaviour of materials at failure.

The behaviour of materials when failing depends upon the nature of their composition and the stress or strain conditions that a material is subjected to. Failure in materials is always out of the elastic range where Hooke's stress vs strain law governs the response of different materials under compressive or tensile loading conditions. It is during plastic or irreversible deformations that failure takes place and that the theory of plasticity is applied.

The theory of plasticity was originally derived for ductile materials such as metal, but it has also been applied to concrete-like materials which present behaviour closer to that of brittle materials. Therefore, a brief description of features of ductile and brittle materials and respective applicable failure criteria is presented in this chapter.

The definition of ideal plastic materials and the corresponding stress vs strain curves, together with the description of the principal concepts and theorems of the theory of plasticity are also presented in this chapter. Conditions and limitations of the applicability of the theory of plasticity to other materials are then considered and finally, the proposal of a plastic analysis of SST data is made.
2.1 - DUCTILE & BRITTLE MATERIALS - Features and failure criteria

2.1.1 - Ductile materials - Physical characteristics

Ductile materials are identified by the plastic deformations or the yielding process which takes place in this type of material just after the load exceeds the elastic limit. At this point, called yield point, the molecules of ductile materials such as metal, begin to "slide" across each other characterizing the flow or yielding process. The ability of metals to flow, depends upon defects in the metal crystal lattice. When a metal is stressed so that plastic deformation occurs, the crystal planes slip over one another, at material imperfections. Molecule bonds are broken and then remade. If the metal is repeatedly deformed, its resistance to plastic deformation increases. This is called work hardening. It happens when dislocations encounter other types of imperfections, such as an impurity atom, and tend to become tangled. The more the solid is deformed, the greater the amount of entanglement. Thus, dislocations become more difficult and so the strength of the metal increases (work hardening) until it breaks.

Figure 2-1 illustrates the behaviour of a ductile material (mild steel) under uniaxial tensile stress (Case & Chilver, 1972).

---

**Notation**

- a - elastic limit (upper yield point)
- b - lower yield point
- bc - constant yield
- cd - work hardening
- de - "necking"
- e - fracture point

**Figure 2-1 - Uniaxial tensile stress vs strain curve for a mild steel - ductile material - (Case & Chilver, 1972)**
Figure 2-1 shows that a specimen of a mild steel when axially loaded in tension presents a linear range up to Point a which is called the elastic limit. As the yielding process starts from this point, it is also called the upper yield point. However, the stress is almost immediately reduced until Point b (the lower yield point) in order to maintain equilibrium. From Point b to c, the yielding process is developed at a roughly constant stress. Beyond Point c, work hardening takes place and attains a maximum value at Point d when a reduction of the cross-section area of the specimen, necking, begins. From d to e, the nominal stress is reduced until fracture occurs.

2.1.2 - Failure criteria for ductile materials - Tresca and von Mises

The consideration that metal yields along planes inclined at 45° to the direction of the principal stresses, i.e. planes of maximum shearing stress, was first suggested by Tresca, in 1878. In a two-dimensional stress space, this yielding criterion is represented by a hexagon as illustrated in Fig. 2-2 (b).

The hexagon ABCDEF of Fig. 2-2(b) represents the yield locus of any combination of $\sigma_1$ and $\sigma_2$ (Fig. 2-2(a)) giving yielding according to the maximum shearing stress criterion. For example, the yielding condition corresponding to a
simple tensile test when \( \sigma_2 = 0 \) and \( \sigma_1 = \sigma_y \) (Fig. 2-2(a)) is represented by the Point A in Fig. 2-2(b). Line AB (Fig. 2-2(b)) represents the yielding condition when both principal stresses are tensile but \( \sigma_1 > \sigma_2 \). In this case, the maximum shearing stress at yielding, from the simple tensile test, is given by \( \tau_{\text{max}} = \frac{1}{2} \sigma_y = \frac{1}{2} (\sigma_1 - 0) = \frac{1}{2} \sigma_1 \) then \( \sigma_1 = \sigma_y \). The yielding for these stress conditions is unaffected by \( \sigma_2 \) and occurs in the 3-1 plane. The combination of stresses represented by the Line AF (Fig. 2-2(b)) corresponds to \( \sigma_1 \) in tension and \( \sigma_2 \) in compression (Fig. 2-2(a)). In this case, yielding occurs in the 1-2 plane, the maximum shearing stress is given by \( \tau_{\text{max}} = \frac{1}{2} \sigma_y = \frac{1}{2} (\sigma_1 - \sigma_2) \) and then \( (\sigma_1 - \sigma_2) = \sigma_y \).

Another failure or yielding criterion for ductile materials was suggested by von Mises, in 1913. This criterion considers that the yielding is governed by a critical value of the strain energy of distortion (Case & Chilver, 1972), i.e. by a critical value of the energy dissipated by the distorting or shearing strains. For a two-dimensional stress system, yielding occurs when

\[
\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_y^2.
\]

The yielding locus given by Eq. 2-1 is an ellipse with major and minor axes inclined at 45° to the direction of the principal stresses \( \sigma_1 \) and \( \sigma_2 \), see Fig. 2-3(a).

![Figure 2-3](image-url)
For a comparison between the two yielding criteria for ductile materials previously described, von Mises’s yielding criterion is superimposed on Tresca’s yielding criterion in Fig. 2-3(b).

2.1.3 - Failure criteria for brittle materials

Unlike ductile materials, failure in brittle materials happens after the elastic limit is reached with little, or no yielding, at relatively low strains. A brittle material, such as concrete, contains a great number of microscopic cracks in its structure giving rise to high stress concentrations and causing local failure of the material. Molecular bonds are then lost by separation and are not remade as in ductile materials. Failure in brittle materials is mainly governed by the maximum principal tensile strain and is followed by a complete falling off in stress.

2.1.4 - Coulomb’s failure description

Coulomb’s essay published in 1776 reports his rupture criterion which was developed by considering the frictional hypothesis (Heyman, 1972). In order to derive this criterion, a masonry pier under uniaxial compressive stress was considered. The strength of such a column was supposed to be evaluated by determining its load-carrying capacity as well as the angle at which it would fracture. Then, Coulomb defined that there is a plane of failure along which, in the limit state, one portion of an axially loaded column will slide over the other. The resistance against the sliding is determined by cohesion and friction which Coulomb called "passive forces". According to Coulomb’s essay, cohesion "is measured by the resistance that solid bodies offer to the simple separation of their parts". If a solid body is considered as homogeneous, "each element has the same strength against failure and the total cohesion is proportional to the number of parts to be broken, and then to the area of rupture". The strength due to friction was assumed to be proportional to the normal compressive forces acting on the failure surface. In terms of stresses, this criterion is known by the following equation:
where \( \tau \) and \( \sigma_n \) are respectively the shear and the normal stresses acting on the failure surface with \( \sigma_n \) negative in compression and \( \mu = \tan \phi \) is the coefficient of friction. From Eq. 2-2 it is seen that the shear stress is determined by the parameters \( c \) and \( \mu \) which are known as the shear strength parameters. In this case, cohesion \( c \) can be interpreted as the part of the shear strength independent of the normal stress and the coefficient of friction \( \mu \), defined as the tangent of the internal angle of friction (\( \phi \)), as the slope of the straight-line obtained from Coulomb's equation. Coulomb derived his solution by the equilibrium of the forces presented as follows.

\[
\tau = c - \mu \sigma_n
\]

\[2-2\]

The failure mechanism illustrated in Fig. 2-4 was considered. Failure conditions \( \sigma_n \) and \( \tau \) (normal and shear stresses, respectively) assumed to be fully developed over total area \( \overline{A} \). The horizontal and vertical equilibria were respectively found by:

\[
\sigma_n \sin \beta = \tau \cos \beta
\]

\[2-3\]

\[
P = (\tau \sin \beta + \sigma_n \cos \beta) \frac{\overline{A}}{\cos \beta}
\]

\[2-4\]
where $A$ is the cross-sectional area.

If the values of the normal and the shear stresses obtained from Eqs. 2-2 and 2-3 are replaced in Eq. 2-4, the load is given by:

$$P = \frac{c A}{\cos \beta (\sin \beta - \tan \phi \cos \beta)}$$

This was only Coulomb's first approach. In order to find the minimum load to cause collapse in the considered body, Eq. 2-5 had to be minimised with respect to the angle of inclination of the plane of failure. In this case, it was found that $\beta = \pi/4 + \phi/2$ and the minimum load was given by the expression:

$$P_{\text{min}} = 2 c A \tan (\pi/4 + \phi/2)$$

Although Coulomb's criterion was derived by considering a masonry pier which is a brittle material, this criterion considers that the failure mechanism happens through a sliding plane. However, this fact is not supported by experiments as Coulomb does not report any compression tests of his own and just quotes a single result for brick tested by Musschenbroek (Heyman, 1972). It is in the microscopical behaviour of ductile materials that sliding is verified. This fact perhaps explains the major use of Coulomb's criterion for materials presenting ductility such as particular applications in soil mechanics (Scott, 1980). Nevertheless, Coulomb's criterion has also been used as a yield condition for concrete as described in Section 2.3.

Some concepts such as the yield condition and plastic flow as well as the description of the principal theorems of the theory of plasticity are presented below, as they are fundamental points of discussion in this research.

### 2.2 - THEORY OF PLASTICITY - BASIC CONCEPTS

In order to simplify the analysis of the behaviour of plastic solids, the definition of ideal materials and the corresponding stress vs strain curves under uniaxial tensile
and compressive stresses are required (Prager, 1959). For this purpose, not only a rigid, perfectly plastic material but also an elastic, perfectly plastic material were defined. The corresponding stress vs strain curves are illustrated in Figs. 2-5(a) and 2-5(b), respectively.

As can be seen, when the absolute value of $\sigma$ equals $\sigma_Y$, the material flows plastically under constant stress and $\varepsilon = \varepsilon_Y$. Approximations of stress vs strain curves of real plastic materials to the stress vs strain curves presented in Figs. 2-5(a) and 2-5(b) are made when the theory of plasticity is considered.

The term generalized stress ($Q_i$) will be used to indicate the variables which characterize a state of stress in a continuum. The corresponding generalized strains ($q_i$) are defined by the condition that

$$W_i = \Sigma Q_i q_i$$  \hspace{1cm} 2-7$$

is the work done by stresses on infinitesimal increments of strain.

The yield condition of the form:

$$F(Q_i) = F(Q_1, \ldots, Q_n) = 0$$  \hspace{1cm} 2-8$$
defines a combination of generalized stresses \((Q_i)\) which determine continuing flow without any change in the stresses. For a rigid, perfectly plastic material, the condition \(F(Q_i) < 0\), corresponds to a stress combination where no strains are developed and the condition \(F(Q_i) > 0\) cannot occur.

If Eq. 2-8 is represented in an \(n\)-dimensional stress space, the yield condition describes a surface which is called yield surface or yield locus. The set of points representing states of stress at or below the yield condition is proved to be convex (Prager, 1959). Therefore, the yield locus as a boundary of this set is a convex surface.

If \((Q_1, \ldots, Q_n)\) represents a state of stress that satisfies the yield condition, and this state of stress is subjected to stress increments \((dQ_1, \ldots, dQ_n)\), it will lead from the considered state to a neighbouring state of stress at the yield limit. In this case, both the original and the modified states of stress must satisfy Eq. 2-8 and the expression presented below can be derived:

\[
\frac{dF}{dQ_1} dQ_1 + \ldots + \frac{dF}{dQ_n} dQ_n = 0
\]

If Eq. 2-9 is geometrically interpreted, the orthogonality of the vectors \((dF/dQ_1, \ldots)\) and \((dQ_1, \ldots)\) is expressed. The latter of these vectors is tangential to the yield surface, as it joins neighbouring points of this surface. Thus, the vector with components \((dF/dQ_1, \ldots)\) is normal to the yield surface at the considered stress point. In addition, if a variation in \(F(Q_i)\) is considered, changes from negative to positive values are observed coming from the interior to the exterior of the yield surface. Then, \(dF/dQ_1, \ldots\) have the direction of the exterior normal.

According to Prager's principle, "the stress increment does no work on the increment of plastic strain" as the vector that represents increments of stress and plastic strain are defined as orthogonal. Since the direction of the vector of stress increment is tangent to the yield locus, the corresponding plastic strain increment must be normal to the yield locus at the considered stress point. Then, the flow rule has the form:
\[ dq_i = \lambda \frac{\partial F}{\partial Q_i} \]  

where \( \lambda \) is an arbitrary factor of proportionality. In order to satisfy the condition that plastic flow always involves dissipation of mechanical energy (see Eqs. 2-7 and 2-10), factor \( \lambda \) is restricted to being positive. Hence, the plastic flow vector has the same direction as \( \frac{\partial F}{\partial Q_i} \), i.e. it is also an outward-directed vector normal to the yield surface. Then, the flow rule (Eq. 2-10) is also called the normality condition.

As defined before, the strain rate direction depends upon the curvature of the yield surface (1/R) and the following conditions can be observed:

(i) \( 0 < 1/R < \infty \)

In this case, the strain rate vector or flow vector direction can be uniquely defined by the position at the yield surface and vice-versa. This condition is verified for any point considered at the curve defined by von Mises's yielding criterion (Fig. 2-3(a)).

(ii) \( 1/R = \infty \)

This situation corresponds to singular points like edges or corners of the yield loci where the direction is undefined by the position of the strain rate vector at the yield surface. For example, at Points A, B, C, D, E or F of the curve defined by Tresca's yielding criterion (Fig. 2-2(b)).

(iii) \( 1/R = 0 \)

For certain extensions of the yield surface where this condition is verified, the position of the strain rate is undefined by the direction of the flow vector. It is the case of any point considered at the curve defined by Tresca's yielding criterion along AB, BC, CD, DE, EF or FA (Fig. 2-2(b)).

In the last two cases, if the least upper-bound is to be found, minimisation is required based upon the chosen failure mechanism.

Limit analysis, in the theory of plasticity, furnishes the load-carrying capacity of plastic materials by using the theorems stated overleaf.
- The **Lower Bound Theorem** determines a load for which a statically admissible stress distribution is found. Equilibrium is found everywhere and stresses do, at no point, exceed the yield limit. Loads for which these conditions are satisfied are less than or equal to the yield load.

- The **Upper Bound Theorem** defines a load for which kinematically permissible mechanisms of rigid elements are connected by fully plastic discontinuities, i.e. a field of compatible generalized strain rates. An upper bound is derived by equating the rate of external work with the rate of internal work dissipated in the material, according to the flow rule. Loads for which these conditions are satisfied are greater than or equal to the yield load.

- The **Uniqueness Theorem** defines the complete solution for the yield load, i.e. when the least upper bound and the highest lower bound coincide. In this case, not only a statically admissible stress distribution, but also a kinematically permissible mechanism is satisfied by the generalized strain rate and stress fields.

In the derivation of upper bound solutions, it is convenient to use failure mechanisms where displacement discontinuities are observed in a yield surface. For plane elements, the intersection of the yield surface with the plane containing the displacement vector is a kinematic discontinuity called the displacement discontinuity line or simply the yield line. Fig. 2-6 illustrates the corresponding strain rate vectors,

![Figure 2-6 - Strain rates in a discontinuity line in plane stress state](image-url)
where $\dot{e}_n$ is the normal strain rate, $\dot{\gamma}_s$ is the shear strain rate and $\dot{e}_p$, the strain perpendicular to the plane in reference, has been considered to be zero (Braestrup & Nielsen, 1983).

As previously commented, the behaviour of ductile materials is appropriate for a plastic approach. However, the theory of plasticity has also been applied to other types of materials under certain conditions discussed in the next section.

### 2.3 - VALID CONDITIONS FOR THE USE OF PLASTICITY

The applicability of the theory of plasticity to a particular type of material depends upon the knowledge of the behaviour of such a material at yielding. For example, in ductile materials such as mild steel and other metals, yielding is sufficiently well known for a sound use of plasticity. These materials have an almost perfect plastic behaviour, i.e. they present a continued support of stress at high strains (Fig. 2-7(a)). However, yielding can occur as a result of the type of stress a specific material is subjected to. Certain types of soil for example, when subjected to directly applied shear stress (Fig. 2-7(b)) present a stress vs strain curve compatible with plastic approaches. As far as concrete is concerned, it can present a practically brittle response under normal tensile stress with no yielding (Fig. 2-7(c)). However, when subjected to a simple compressive test within rigid platens, the behaviour of concrete can neither be characterized strictly as brittle nor as clearly plastic (Fig. 2-7(d)).

![Characteristic stress vs strain curves](image-url)

FIGURE 2-7 - Characteristic stress vs strain curves
Nevertheless, some attempts to describe the failure of concrete as a yielding process have been made. For instance, a yield criterion has been defined by upper and lower limits on direct stress. In this case, it is called the Square yield criterion as can be seen in Fig. 2-8(a) where the corresponding strain rate vector directions are superimposed. It is considered in a plane stress state or plane strain state, i.e. $\sigma_3 = 0$ or $\varepsilon_3 = 0$, respectively, and defined by the limit stresses.

This yield criterion can be simplified by neglecting the tensile strength, i.e. $f_t = 0$, as illustrated in Fig. 2-8(b).

![FIGURE 2-8 - Square yield locus](image)

Both Coulomb's rupture criterion, described in Section 2.1.4, and the Mohr-Coulomb modified criterion, presented below, have been used as a yield condition for concrete (Jensen, 1975; Nielsen, 1984). The Mohr-Coulomb modified criterion is defined when the condition of failure by tension is considered together with Coulomb's criterion and when an association with Mohr's circle is made. The envelope thus formed is known as the Mohr-Coulomb modified criterion (Fig. 2-9). In this case, the envelope is limited by the shear stress linked to normal compressive stress by the coefficient of friction and also by the part of Mohr's circle which is tangent to Coulomb's criterion and the limit tensile strength is one of the principal stresses. An undefined range can be observed on the left-hand side of the straight line defined by this criterion.
An essential condition for the validity of the upper bound theorem applied to concrete or any other material is that internal stresses can be redistributed within the structure during failure (Braestrup & Nielsen, 1983).

The ideal plastic model, predicts arbitrary high deformations at constant stress level. However, ductility in concrete under compression, for example, is quite limited. Furthermore, the corresponding stress vs strain curve for a monolithic concrete body (Fig. 2-7(d)) presents a falling branch, i.e. the redistribution of stresses is characterized by strain softening and loss in strength as witnessed in laboratory tests on small samples between rigid platens.

If the theory of plasticity is applied to concrete, an effectiveness factor has been proposed to be used in order to reduce the concrete strength (Nielsen, 1984). Thus, effective concrete strength or plastic strength of concrete is determined.

The effectiveness factor which is primarily a measure of concrete ductility must be evaluated by tests for each particular application. This factor takes into account the falling branch of the concrete stress vs strain curve. In addition, it must also incorporate all the deficiencies of theoretical modelling such as the initial state of stress, stiffness of the materials, type of loading, geometrical effects and it assumes that the restraining effect of the platens which exists in laboratory tests is also present in the concrete structure.
In the particular case of joints, a weakness is frequently introduced to the structure. As in any case, special parameters such as cohesion (c) and the coefficient of friction (μ) are required when Coulomb's rupture criterion is considered. According to Nielsen (1984), cohesion in construction joints is often reduced but the angle of friction can be less, equal to or greater than in monolithic concrete.

2.4 - VALID CONDITIONS FOR THE USE OF PLASTICITY IN THE SST

The objective of this work is to verify whether the theory of plasticity can be applied to analyse slant shear test data. This test is a type of compressive test in which the specimen presents an inclined joint section subjected to shear and compressive stresses (Fig. 2-10), where the bond is to be assessed.

If limit analysis can be applied to this test, some points must be observed, for instance:

(i) that there is stress redistribution during loading up to failure in the considered test;

(ii) a yield criterion must be defined and the corresponding flow vector.
position and direction are also required for the joint;

(iii) upper and lower bound solutions must be derived and,

(iv) as previously discussed, if the theory of plasticity can be applied to concrete, an appropriate effectiveness factor must be obtained in order to reduce the concrete strength.

In the next chapter, the application of plastic flow to Coulomb’s criterion as considered by Heyman, 1972, is described. Also, the upper bound method applied to concrete bodies under simple compressive stress, found in articles published by research workers at the Technical University of Denmark (Jensen, 1975; Nielsen, 1984), is presented. However, some apparent misconceptions can be observed in this previous work as a consequence of assumptions related to the flow vector direction. Hence, a summary of their principal assumptions is also given.
3.1 - HEYMAN'S PLASTIC APPROACH USING COULOMB'S CRITERION

Coulomb's rupture criterion, presented in Section 2.1.4, had a plastic insight into its derivation given by Heyman, 1972. Originally, Coulomb's solution was developed by the balance of forces considering a masonry pier under uniaxial compressive load. However, Heyman assumed that Coulomb's treatment of the problem of the fracture of columns was, in fact, an upper bound approach. In this case, two situations were analysed.

Firstly, a frictionless material ($\phi = 0$) was considered, for which Eq. 2-2 gives $\tau = c$. In order to derive an upper bound solution, Heyman defined the relative "velocity of sliding" as $v$ and the rate of dissipation of energy was obtained by multiplying the shear force ($c A/\cos \beta$) (see Fig. 2-4) by $v$:
The rate at which the external load works was given by

$$W_e = P v \sin \beta$$  \hspace{1cm} 3 - 2

Equating expressions 3-1 and 3-2 led to

$$P = \frac{c A}{\sin \beta \cos \beta}$$  \hspace{1cm} 3 - 3

The minimum load found by the minimisation of Eq. 3-3 and for $\beta = 45^\circ$ was

$$P = 2 c A$$  \hspace{1cm} 3 - 4

The case of frictionless materials is illustrated in Fig. 3-1. As can be seen in Fig. 3-1(c), the planes of sliding for the collapse load occurs along planes at $45^\circ$ to the direction of the principal stresses. Hence, it can be said that Coulomb's criterion for frictionless materials (presented in 1776) anticipated Tresca's maximum shear stress criterion derived for plastic materials (1878, see Section 2.1.2).
For frictional materials, when an inclined plane was considered as the failure surface, there was a general possibility of displacement with components in two directions, corresponding to the shear and normal stresses acting on the surface. The corresponding relative velocities or flow vectors were represented as $v_\tau$ and $v_\sigma$, respectively. Coulomb's criterion was used as a yield criterion of the form:

$$f(\tau, \sigma) = 0$$

3 - 5

i.e., $f(\tau, \sigma) = \tau + \mu \sigma - c = \tau + (\tan \phi) \sigma - c = 0$.

Applying the normality rule to Coulomb's criterion, it was found that:

$$\frac{\nu_\tau}{\nu_\sigma} = \frac{\delta f/\delta \tau}{\delta f/\delta \sigma} = \frac{1}{\mu \tan \phi} \Rightarrow \nu_\sigma = \nu_\tau \tan \phi$$

3 - 6

i.e., the components of the flow vector in both directions were independent of the values of normal and shear stress (see Fig. 3-2(a)).

In this case, the dissipation of energy per unit area was presented as:

$$\dot{W}_f = \tau v_\tau + \sigma v_\sigma = (\tau) v_\tau + (\mu \sigma) v_\tau = c v_\tau$$

3 - 7

which also becomes independent of the actual magnitudes of the stresses acting on the failure surface. Considering the failure mechanism presented in Fig. 3-2(b), the total rate of dissipation of energy was obtained by multiplying Eq. 3-7 by the area of failure $A/\cos \beta$ which gives Eq. 3-1. The corresponding rate of external work was determined by

$$\dot{W}_E = P (v \sin \beta - v \tan \phi \cos \beta)$$

3 - 8

Equating expressions 3-1 and 3-8 led to Eq. 2-5 found by Coulomb for the value of $P$ (see Section 2.1.4). Minimizing Eq. 2-5 with respect to the inclination of the failure surface ($\beta$), the least upper bound was found for $\beta = \frac{\pi}{4} + \frac{\phi}{2}$ and the minimum load was given by Eq. 2-6 (Section 2.1.4).
Heyman commented that Coulomb's linear criterion used as a yield condition is restrictive and a "more general yield criterion" such as von Mises's yield criterion should be used. The author supported this comment comparing how least upper bound solutions would be encountered using each of these two yield criteria. For example, the least upper bound solution found when Coulomb's criterion was used, was directly obtained from the minimisation of the corresponding upper bound solution with respect to p, due to Coulomb's linear form. On the other hand, when a non-linear criterion such as von Mises's yield criterion was used, the corresponding least upper bound solution could only be found by trial and error, assigning arbitrary values to the components of the flow vector.

However, it must be pointed out that only if plastic flow takes place during failure of a considered body, Heyman's choice of using von Mises's criterion as the yield condition is appropriate. As seen in Section 2.1.2, von Mises's criterion was postulated for ductile materials which are compatible with plastic approaches.

Coulomb's criterion is apparently just a rupture criterion which was derived considering a brittle material (a masonry pier) which does not present a plastic flow when failing (see Section 2.1.4). Hence, the main restriction is perhaps the application of the plastic flow rule to Coulomb's rupture criterion.
Criticism of the use of Coulomb’s criterion as a yield criterion was formerly pointed out by Schofield & Wroth (1968) in the application of the Critical State Theory in soil mechanics. In this situation, the objection to using the flow rule to Coulomb’s criterion was justified by the pore-water pressure, present in soils.

In addition, observations of tests performed on concrete specimens, such as a simple compressive test within rigid platens, show that failure takes place by cracking across lines of principal tensile stress rather than by formation of slide planes.

3.2 - JENSEN’S APPLICATION OF THE UPPER BOUND THEOREM

In order to analyse plain concrete construction joints, Jensen, 1975 utilized the upper bound theorem. It was used as an example of the application of the concept of lines of discontinuity for displacements in the theory of plasticity. Before presenting Jensen’s approach, some concepts assumed by the author must be detailed. For example:

(i) the concept of discontinuity line:

Jensen idealized a plane, homogeneous displacement field occurring in a narrow zone of high straining (d) between two rigid parts of a body (Fig. 3-3(a)). Converting Jensen’s displacement to a displacement rate \( \dot{u} \), the strains rates which take place at a line of discontinuity, were defined by:

\[
\varepsilon_n = \frac{\dot{u} \sin \alpha}{d}; \quad \gamma_{nt} = \frac{\dot{u} \cos \alpha}{d}; \quad \varepsilon_t = 0 \quad 3 - 9
\]

The principal strain rates, in the plane of the paper, whose directions were considered to coincide with the principal directions of stress, were defined by:
The principal strain perpendicular to the plane of the paper was assumed to be zero.

A general equation for the dissipation of energy per unit area of discontinuity for plane stress fields or plane strain fields, i.e. for $\sigma_3 = 0$ or $\varepsilon_3 = 0$, respectively, was given by:

$$ W_I = (\sigma_1 \varepsilon_{1} + \sigma_2 \varepsilon_{2}) d $$

Equation 3-12 becomes independent of dimension $d$ by inserting Eqs. 3-10 and 3-11.

(ii) yield criterion for concrete:
As the yield criterion for concrete, Jensen considered what he called Coulomb's modified yield criterion, presented in Fig. 3-4(a).

This yield criterion consisted of Coulomb's failure criterion which was classified as the sliding condition given by:

\[
\tau = c - \mu \sigma = c - \sigma \tan \phi
\]  

limited by tensile strength \( f_t \), which was assumed as a separation condition of the form:

\[
\sigma_i = f_t
\]  

Coulomb's criterion (Eq. 3-13) expressed by means of the maximum and minimum principal stresses was given by (see Fig. 3-5):

\[
\frac{\sigma_1 - \sigma_2}{2} = c \cos \phi - \left( \frac{\sigma_1 + \sigma_2}{2} \right) \sin \phi
\]  

which was rewritten as
\[ k \sigma_1 - \sigma_2 = 2c \sqrt{k} \quad 3-16 \]

where \( k \) was given by the expressions:

\[ k = \left( \frac{\cos \phi}{1 - \sin \phi} \right)^2 = \frac{1 + \sin \phi}{1 - \sin \phi} \quad 3-17 \]

When the maximum and minimum principal stresses were considered as \( \sigma_1 = 0 \) and \( \sigma_2 = -f_c \), i.e. For the uniaxial compressive test as presented in Fig. 3-6, Eq. 3-16 was rewritten as:

\[ f_c = 2c \sqrt{k} \quad 3-18 \]

or

\[ f_c = \frac{2c \cos \phi}{1 - \sin \phi} \quad 3-19 \]

where \( f_c \) was defined as the compression strength of concrete and referred to the uniaxial compressive strength.
Thus, by inserting $f_c$ given by Eq. 3-18 in Eq. 3-16, the yielding condition for materials which obey Coulomb’s criterion was given by:

$$k\sigma_1 - \sigma_2 = f_c$$

representing a constitutive equation for Coulomb materials also satisfying the condition $\sigma_1 > \sigma_3 > \sigma_2$.

According to the normality or flow rule (see Section 2.2), the flow vector given by Eq. 3-9 must be perpendicular to the yield criterion when represented in a $\sigma - \tau$ co-ordinate system. It is seen from Fig. 3-4(a) that $\alpha = \phi$ along the straight line defined as the sliding condition and $\alpha > \phi$ along the curve defined as the separation condition.

(iii) the derivation of the dissipation of energy:

The dissipation of energy per unit area valid for $\alpha = \phi$, i.e. anywhere along the straight line (Fig. 3-4(a)), was obtained by the vector product $\overline{OP} \cdot \overline{u}$

$$\dot{W}_f = \dot{u} c \cos \phi$$
According to Jensen, from the geometry of Fig. 3-4(a) the dissipation of energy per unit area valid anywhere along curve A'B', i.e. when $\alpha > \phi$, was obtained by the vector product $\overrightarrow{OP'} \cdot \overrightarrow{u}$

$$\dot{W}_f = \dot{u} f_c \left( \frac{1 - \sin \alpha}{2} + r \frac{\sin \alpha - \sin \phi}{1 - \sin \phi} \right) 3 - 22$$

where $r = f_t / f_c$.

Coulomb's modified yield criterion was also represented as a function of the principal stresses for plane stress fields as described in Fig. 3-4(b). In this case, if the normality or flow rule is obeyed, the flow vector given by Eqs. 3-10 and 3-11 must be perpendicular to the yield criterion when represented in a $\sigma_1 - \sigma_2$ co-ordinate system.

Considering the general equation for the dissipation of energy given by Eq. 3-12, special cases were derived. For example:

(a) for $\alpha = 90^\circ$, i.e. along 1-2 in Fig. 3-4(b) where $\sigma_1 = f_t$, Eq. 3-11 gave $\dot{e}_2 = 0$ and Eq. 12 became:

$$\dot{W}_f = \dot{u} f_t 3 - 23$$

(b) for $\phi \leq \alpha \leq 90^\circ$, i.e. at Point 2 of Fig. 3-4(b), the principal stresses were defined as $(\sigma_1, \sigma_2) = (f_t, k_f - f_t)$ and the principal strains were given by Eqs. 3-10 and 3-11. In this case, Eq. 3-12 became:

$$\dot{W}_f = \dot{u} f_c \left( \frac{1 - \sin \alpha}{2} + r \frac{\sin \alpha - \sin \phi}{1 - \sin \phi} \right) 3 - 22$$

where $r = f_t / f_c$.

(c) for $\alpha = \phi$, i.e. along 2-3 in Fig. 3-4(b) and for $0 \leq \alpha \leq \phi$, i.e. at Point 3 in Fig. 3-4(b) where $(\sigma_1, \sigma_2) = (0, -f_t)$, the dissipation of energy per unit area was given by:
\[ W_i = \dot{u} f_c \frac{1 - \sin \alpha}{2} \]  

3 - 24

It can be observed that the flow vector direction can assume positions \( \alpha < \phi \) when the yield criterion is represented for plane stress fields as a function of the principal stresses (Fig. 3-4(b)). However, this condition is not verified when the yield criterion is represented in terms of shear and normal stresses (Fig. 3-4(a)). Figs. 3-4(a) and 3-4(b) replicates Figs. 2 and 3 of Jensen’s paper (1975). It can be noticed that the construction of those yield surfaces is not appropriate. For example, the coordinates of Point 2 (Fig. 3-4(b)) are not associated with the position on Mohr’s circle (Point E’ - Fig. 3-4(a)). A discussion which will clarify these matters is included in Chapter 4 - Section 4-1.

From these assumptions, Jensen presented an analysis of plain construction joints considering prismatic specimens of concrete as a modified Coulomb material. He assumed the failure mechanism presented in Fig. 3-7 taking the form of a plane displacement field along the joint line (AB).

As can be seen in Fig. 3-4(b), for any combination of positive \( \sigma_1 \) and negative \( \sigma_2 \) except between Points 2 and 3, the inclination of the displacement vector could be \( \alpha \geq \phi \). However, the flow vector direction was assumed to be equal to the angle of

\[ \dot{u} \]

\[ f_c \]

\[ \frac{1 - \sin \alpha}{2} \]

\[ A \]

\[ B \]

\[ C \]

\[ D \]

\[ E \]

\[ F \]

\[ G \]

\[ H \]

\[ I \]

\[ J \]

\[ K \]

\[ L \]

\[ M \]

\[ N \]

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]
friction, i.e. $\alpha = \phi$, considering "pure sliding failure". The dissipation of energy per unit area was given by Eq. 3-21. By inserting inclined length $AB = b/cos \beta$ and depth $t$ of the body into Eq. 3-21, it was found that:

$$W_i = \dot{u} c \cos \phi \frac{b t}{cos \beta}$$  \hspace{1cm} 3 - 25

Considering the adopted displacement field (see Fig. 3-7(c)), the external stress $\sigma$ does the work:

$$W_e = \sigma \dot{u} \sin (\beta - \phi) b t$$  \hspace{1cm} 3 - 26

By equating the expressions 3-25 and 3-26, Jensen achieved the following solution:

$$\sigma = \frac{c \cos \phi}{cos \beta \sin (\beta - \phi)}$$  \hspace{1cm} 3 - 27

In order to apply Jensen's solution (Eq. 3-27) special parameters were utilized. He considered that in construction joints made "reasonably rough", the angle of friction is not modified but the cohesion is reduced. Thus, he proposed a reduction of 60% of the value of the cohesion adopted for the surrounding concrete ($c_{con}$) (in the example given by Jensen, $c_{con} = 7.5$) and the angle of friction for concrete was considered as $\phi = 37^\circ$.

In addition to these special parameters, Jensen introduced the condition that $\sigma \leq f_c$, i.e. Eq. 3-27 should give results less than or equal to the corresponding concrete compressive strength. From these assumptions, he concluded that: "if Eq. 3-27 gives $\sigma > f_c$, the concrete will fail in the same way as monolithic concrete, outside the construction joint for $\sigma = f_c$".

Figure 3-8 presents Jensen's solution (Eq. 3-27) compared with the data from a very limited number of tests described in his paper where $f_c = 30N/mm^2$, $c = 0.4 c_{con} = 3N/mm^2$ and $\phi = 37^\circ$, limited by $f_c$ (concrete compressive strength).
In this case Eq. 3-27 is only valid for values of the inclination of the joint section in the interval $43.5^\circ < \beta < 83.5^\circ$ (approximately). According to Jensen’s assumptions, only for joints inclined in the range defined above, specimens would fail along the joint line. Otherwise, specimens, even those comprising joints, would fail as monolithic ones without any influence due to the presence of the joint.

Jensen supposed that if there were no construction joint, $\beta$ could be varied. In this case, the minimum would occur for $\beta = \pi/4 + \phi/2$ and the corresponding least upper bound solution would be:

$$\sigma_{\text{min}} = \frac{c \cos \phi}{1 - \sin \phi} = f_c$$

i.e., the concrete compressive strength.

Some limitations were found in Jensen’s solution which are presented as follows.
Firstly, the restriction of the flow vector direction to be equal to the angle of friction $\alpha = \phi$ is not properly justified. Regarding the yield criterion adopted by Jensen, the inclination of the flow vector could be $\alpha \geq \phi$ (see Figs. 3-4(a) and 3-4(b)). Besides, Jensen was dealing with concrete specimens whose failure pattern cannot be characterized as a "pure sliding failure", as considered by Jensen. Failure in brittle materials, such as concrete, is mainly governed by separation (see Section 2.1.3).

As a result of the assumption that the displacement vector should be equal to the angle of friction, an arbitrary upper bound solution was achieved by Jensen (Eq. 3-27).

Furthermore, Jensen's solution is not supported by enough data to satisfy a rigorous statistical analysis. As shown in Fig. 3-8, just a little test data was considered. Finally, the reduction of 60% of the value considered as the cohesion of the surrounding concrete and used for the construction of the graph presented in Fig. 3-8 (Jensen's solution), was arbitrarily chosen.

As an extension of Jensen's analysis an acceptability criterion for adhesive materials subjected to the slant shear test was suggested by Campos, 1989. It was proposed to compare:

a) the value calculated by Eq. 3-28 by using $c$ and $\phi$ obtained by means of linear regression of slant shear test data performed in at least two different inclinations of the joint line with

b) a reduced value of the average concrete compressive strength ($f'_c$) provided by compressive tests on parent control specimens.

The tested adhesive material could be considered as adequate if $\sigma_{\min} > f'_c$.

As far as the basic assumptions are concerned, this proposal has the same limitations as Jensen's solution. However, the parameters $c$ and $\phi$ were provided by test data and not by arbitrary values as suggested by Jensen. According to Nielsen's assumptions (1984) the reduced concrete strength $f'_c$ could be considered as a plastic concrete compressive strength or effective concrete compressive strength. Finally, a lack of statistical proof is also observed in this proposal.
3.3 - NIELSEN's APPLICATION OF THE UPPER BOUND THEOREM

Nielsen, 1984 described three different approaches for monolithic bodies subjected to a uniformly distributed compressive stress $\sigma$ as examples of the application of the upper bound theorem. These approaches are presented as follows.

3.3.1 - monolithic body of a Coulomb material in a plane strain field:

The failure mechanism and the sign convention as presented in Fig. 3-7 (see Section 3.2) was assumed. As can be observed, the angle of the displacement vector with the yield line was assumed to be equal to the angle of friction ($\phi$).

The least upper bound solution achieved in this example was given by:

$$(\sigma)_{\text{min}} = f_c$$  \hspace{1cm} 3 - 29$$

where $f_c$ is the concrete compressive strength.

This expression was obtained by minimisation of the corresponding upper bound solution with regard to $\beta$ which assumed the value:

$$\beta = \frac{\pi}{4} + \frac{\phi}{2}$$  \hspace{1cm} 3 - 30$$

However, the corresponding upper bound solution was not presented in Nielsen's work. Instead, the dissipation was suggested to be determined from the expressions:
\[ W_I = \dot{u} \cdot b \cdot f_c \cdot \frac{1 - \sin \phi}{2} \] 3 - 31

or

\[ \dot{W}_I = \dot{u} \cdot b \cdot c \cdot \cos \phi \] 3 - 32

where \( c = f_c / 2 \sqrt{k} \) and \( k \) is obtained from Eq. 3-17.

According to the selected failure mechanism (see Fig. 3-7), failure takes place along yield line AB positioned at angle \( \beta \). Hence, Eq. 3-31 is rewritten as

\[ W_I = \dot{u} \cdot b \cdot \frac{f_c}{\cos \beta} \cdot \frac{1 - \sin \phi}{2} \] 3 - 33

Also, regarding Fig. 3-7(c), the external work is given by

\[ W_E = \sigma \cdot \dot{u} \cdot \sin (\beta - \phi) \cdot b \] 3 - 34

Thus, although not presented by Nielsen, the corresponding upper bound solution for this example can be obtained by equating expressions 3-33 and 3-34 which leads to

\[ \frac{\sigma}{f_c} = \frac{1 - \sin \phi}{2 \cos \beta \sin (\beta - \phi)} \] 3 - 35

3.3.2 - monolithic body of a modified Coulomb material for \( f_i \neq 0 \) in a plane strain field:

In this case, the axisymmetrical failure mechanism presented in Fig. 3-9 was considered.
Nielsen presented as the achieved upper bound solution the expression:

$$\sigma = f_c + kf_f$$

3 – 36

without setting up the work equations or at least suggesting how such an "upper bound" was derived. Nielsen commented that the "exact solution" would be:

$$\sigma = f_c$$

3 – 37

which could only be obtained for $f_i = 0$.

In order to find Nielsen’s solution, the following assumptions are here adopted:

1) as both tensile and compressive strength were considered, the principal stresses are assumed to be $(\sigma_1, \sigma_2) = (f_t, -f_c)$;

2) the principal strains as given by Eqs. 3-10 and 3-11 (see Section 3.2);

3) the dissipation of energy per unit length as given by the expression:

$$\tilde{W}_1 = (\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2) \frac{b}{\cos \beta} = \left( f_t \left( \frac{1 + \sin \phi}{2} \right) + f_c \left( \frac{1 - \sin \phi}{2} \right) \right) u \frac{b}{\cos \beta}$$

3 – 38
4) the external work as given by Eq. 3-34.

From these assumptions, the upper bound solution is obtained by equating expressions 3-34 and 3-38 which lead to:

\[
\frac{\sigma}{f_c} = \frac{r (1 + \sin \phi) + (1 - \sin \phi)}{r \cos \beta \sin (\beta - \phi)}
\]

where \( r = \frac{f_t}{f_c} \).

And then, the solution given by Eq. 3-36 can only be found by minimizing Eq. 3-39 with respect to \( \beta \) which assumes the value given by Eq. 3-30. Hence, Nielsen's solution given by Eq. 3-36 is in fact a least upper bound. However, what is called the "exact solution" by Nielsen (Eq. 3-37) cannot be found directly from the minimisation of the corresponding upper bound (Eq. 3-39). The condition \( f_t = 0 \) also has to be introduced.

This problem arises as a consequence of an arbitrarily fixed flow vector direction \( \alpha = \phi \). According to Nielsen, the yield condition and flow rule for a modified Coulomb material in plane strain, hypothesis considered in the application described in this section, assume the configuration shown in Fig. 3-10. In this case, the assumption of a fixed flow vector direction is not properly justified as the flow vector could assume any position in the range \( \alpha \geq \phi \). This matter is also discussed in the next chapter.

**FIGURE 3-10 - Yield condition and flow rule for a modified Coulomb material in plane strain (Nielsen, 1984)**
3.3.3 - **monolithic body of a modified Coulomb material for** $f_v = 0$ **in a plane stress field:**

The failure mechanism adopted in this case is shown in Fig. 3-11. It was represented by a straight yield line at angle $\beta$ with sides $b$ and it was considered in a plane stress state where the flow vector direction $\alpha$ was assumed as a variable.

![Sign Convention](image)

- $\sigma$: negative in compression
- $\alpha$: positive in the clockwise

**FIGURE 3-11** - (a) Failure mechanism of a monolithic body under compression in a plane stress field; (b) $n$-$l$ coordinates; (c) $v$-$h$ coordinates (Nielsen, 1984)

The work equations set up for the considered failure mechanism were given by:

$$\sigma b \ u \sin(\beta - \alpha) = f_c \ u \left(\frac{1 - \sin \alpha}{2}\right) \frac{b}{\cos \beta}$$  \hspace{1cm} 3 - 40

Then,

$$\frac{\sigma}{f_c} = \frac{1 - \sin \alpha}{2 \cos \beta \sin(\beta - \alpha)}$$  \hspace{1cm} 3 - 41

The least upper bound solution was found by minimizing Eq. 3-41 with respect to $\alpha$ and $\beta$ which was given by:

$$\langle \sigma \rangle_{\text{min}} = f_c$$  \hspace{1cm} 3 - 42
where \( f_c \) is the concrete compressive strength and angle \( \beta \) was determined by:

\[
\beta = \frac{\pi}{4} + \frac{\alpha}{2}
\]

3.4 - FINAL COMMENTS

A missing point was found when Coulomb's modified yield criterion (Jensen, 1975) was considered for plane stress states as a function of the principal stresses (see Section 3.2). Condition \( \alpha < \phi \), allowed by the yield surface presented in Fig. 3-4(b) is not found in Coulomb's modified yield criterion in terms of the shear and normal stresses (Fig. 3-4(a)). An attempt to establish a complete correspondence between the yield surface drawing in the two coordinate systems regarding the flow vector direction is presented and discussed in the next chapter.

The consideration that the displacement vector direction is equal to the angle of friction \( (\alpha = \phi) \), caused by the assumption of Coulomb's (modified) criterion as the yield condition, is perhaps a limitation in previous work. As mentioned before, the displacement vector direction \( \alpha \) can be assumed to be a variable either if the complete range allowed by Coulomb's modified criterion is considered or assuming other yield conditions which would suit the problem better. In order to investigate this matter, an alternative upper bound solution is derived by considering the square yield locus (Fig. 2-8(a)) as the yield criterion. This alternative upper bound solution is presented in the next chapter where a comparison with previous work is also made.
CHAPTER 4

Alternative upper bound solutions

Previous applications of the upper bound theorem to the SST were described in the preceding chapter. Some restrictions were found such as the use of an arbitrarily fixed flow vector direction when an infinite range is available. In order to examine the upper bound theorem fully, alternative upper bound solutions are derived and discussed in this chapter.

First of all, a modified Mohr-Coulomb envelope defined as a function of the normal and shear stresses is considered together with the corresponding yield surface for plane stress states derived by Jensen (1975) in terms of the principal stresses. The main purpose of this approach is to establish the correspondence between those two yield surfaces and that of the flow vector direction in each case (Section 4.1).

In the absence of evidence of a sliding condition, concrete failure is sometimes simply defined by a maximum direct stress condition represented by the square yield locus. This hypothesis is considered to derive an upper bound solution for a variable flow vector direction (Section 4.2).

A comparison of previous theoretical predictions with the solutions presented in this work is made. Also, the suitability of all those theoretical approaches with available test results is discussed in Section 4.3.

Finally, Section 4.4 presents the conclusions obtained from the study of theoretical modelling which leaves some doubt about the applicability of plastic methods and the need for an experimental investigation is shown.
4.1 - A MODIFIED MOHR-COULOMB ENVELOPE AS THE YIELD CONDITION

When Coulomb’s modified yield criterion (Fig. 3-4(a)) was presented for plane stress fields as a function of the principal stresses (Fig. 3-4(b)) (Jensen, 1975), a lack of correspondence between yield surfaces drawing in the two coordinate systems regarding flow vector directions in the range $\alpha < \phi$ (see Section 3.2.ii) was observed.

Also, the coordinates of Point 2 (Fig. 3-4(b)) are not associated with the position on Mohr’s circle (Point E’ - Fig. 3-4(a)) (Figs. 2 and 3 of Jensen’s paper). A clearer construction is shown in Figs. 4-1(a) and 4-1(b) in which prescribed values of $c$, $\phi$ and $f_i$ show the general case with Mohr’s circle through $f_i$ is not centred on the origin and correctly positioning Points 2 and E’ at equal distances from axes $\sigma_1$ and $\tau$, respectively.

In order to allow the flow vector to assume directions in the range $\alpha < \phi$, it is not possible to consider Coulomb’s criterion from Point C’ (Fig. 4-1(a)) upwards. A yield surface such as the Mohr-Coulomb envelope shown in Fig. 4-1(a) has to be considered. It must be pointed out though, that a Mohr diagram is simply a transformation diagram and its validity as a yield surface is somewhat questionable.

![Diagram](image-url)
In order to establish the correspondence between the yield surfaces presented in Figs. 4-1(a) and 4-1(b) and that of the flow vector direction in each case, upper bound solutions are derived. The failure mechanism presented in Fig. 4-2 is considered together with the following conditions: a specimen of concrete as a rigid, perfectly plastic material subjected to uniformly distributed load and which yields in accordance with Coulomb’s criterion. The angle of friction is taken as equal to $\phi = 37^\circ$, as reported by Nielsen (1984) and others.

According to the adopted failure mechanism (Fig. 4-2), the plastic strain rates are defined by:

\[
\dot{\epsilon}_n = \dot{u} \sin \alpha \quad 4-1
\]
\[
\dot{\gamma}_{nt} = \dot{u} \cos \alpha \quad 4-2
\]

And then, the principal strain rates are:

\[
\dot{\epsilon}_1 = \frac{\dot{u}}{2} (1 + \sin \alpha) \quad 4-3
\]
\[
\dot{\epsilon}_2 = -\frac{\dot{u}}{2} (1 - \sin \alpha) \quad 4-4
\]
Based upon these assumptions, Point 3 at diagram 4-1(b) is initially considered. At that point, the strain rate can assume any direction in the range: $-90^\circ \leq \alpha \leq \phi (= 37^\circ)$ and the principal stresses are $(\sigma_1, \sigma_2) = (0, -f_c)$.

It is important to define $f_c$ at this stage. This compressive strength, $f_c$, is not that of the materials of either of the two halves (parts I and II, Fig. 4-2) but of the joint. This may be considered to be modelled by a monolithic specimen of material $J$ controlled to yield along a line at the angle of the joint in the test, where $J$ has the properties of the joint at plastic yield. This is explained in great detail in Section 4.3 (pp 85) in the discussion of the upper bound solutions.

The dissipation of energy per unit length is calculated by the general expression:

$$W_f = \overline{A} (\sigma_1 \dot{\varepsilon}_1 + \sigma_2 \dot{\varepsilon}_2)$$  \hspace{1cm} 4 - 5

where $\overline{A}$ is the area along the yield plane ($\overline{A} = A / \cos \beta$). Thus, it is found that:

$$W_f = \frac{A \dot{u}}{2 \cos \beta} f_c (1 - \sin \alpha)$$  \hspace{1cm} 4 - 6

The rate of external work done on the mechanism by the applied load is determined by taking into account the vertical component of the plastic flow vector $\dot{u}_v = \dot{u} \sin (\beta - \alpha)$ (see Fig. 4-2(c)). In this case, the following equation is derived:

$$W_E = \sigma A \dot{u} \sin (\beta - \alpha)$$  \hspace{1cm} 4 - 7

By equating the rate of external work given by Eq. 4-7 to the dissipation of energy (Eq. 4-6), the general upper bound solution is given by:

$$\frac{\sigma}{f_c} = \frac{1 - \sin \alpha}{2 \cos \beta \sin (\beta - \alpha)}$$  \hspace{1cm} 4 - 8
The least upper bound solution is then found to be:

\[
\left( \frac{\sigma}{f_c} \right)_{\min} = 1.0 \quad \Rightarrow \quad \sigma_{\min} = f_c
\]  

which is obtained by minimisation of Eq. 4-8 with respect to the flow vector direction \( \alpha \) defined by:

\[
\alpha = 2 \beta - 90^\circ
\]

Point 3 at the yield surface (Fig. 4-1(b)) corresponds to inclinations of the yield line in the range, \( 0^\circ \leq \beta \leq 45^\circ + \phi/2 \) (= 63.5°) found by Eq. 4-10 for the considered range \(-90^\circ \leq \alpha \leq \phi \) (= 37°). This situation corresponds to points along D'C' on the Mohr-Coulomb envelope (Fig. 4-1(a)). For whatever inclination of the yield line in the considered range, the same least upper bound solution is found, given by Eq. 4-9.

Special cases can be pointed out, such as:

- a horizontal yield line \( \beta = 0^\circ \) and the corresponding displacement vector direction which is perpendicular to the yield line, i.e. \( \alpha = -90^\circ \) are presented in Fig. 4-3(a). This particular solution corresponds to Point D' in Fig. 4-1(a).

- Fig. 4-3(b) presents a yield line at 45°. In this case, the corresponding displacement vector is parallel to this yield line, i.e. \( \alpha = 0^\circ \).

- a yield line at \( 45^\circ + \phi/2 \) (= 63.5°) corresponds to the flow vector direction equal to the angle of friction, i.e. \( \alpha = \phi \) (= 37°) (see Fig. 4-3(c)). This solution represents Point C' in Fig. 4-1(a).
Between Points 3 and 2 at the yield surface (Fig. 4-1(b)) $\alpha = \phi (= 37^\circ)$ as it is between Points C' and B' on the Mohr-Coulomb envelope (Fig. 4-1(a)). The relation between the principal stresses is given by:

$$\sigma_2 = -f_c + k \sigma_1 = -f_c + \left( \frac{1 + \sin \phi}{1 - \sin \phi} \right) \sigma_1 \quad 4-11$$

The dissipation is obtained from Eq. 4-5, by replacing the principal strain rates given by Eqs. 4-3 and 4-4 considering $\alpha = \phi$ and the principal stresses satisfying Eq. 4-11. In this case, the following expression is found:

$$\dot{W}_f = \frac{A \dot{u}}{2 \cos \beta} f_c (1 - \sin \phi) \quad 4-12$$

By equating the dissipation of energy (Eq. 4-12) to the external work, obtained from Eq. 4-7 considering $\alpha = \phi$, the following solution is obtained:

$$\sigma = \frac{1 - \sin \phi}{f_c} \frac{1 - \sin \phi}{2 \cos \beta \sin (\beta - \phi)} \quad 4-13$$

which is simply a special case of Eq. 4-8.
In order to provide a better visualization of the obtained results Figs. 4-4 and 4-5 were drawn.

Fig. 4-4 presents the relation between the inclination of the yield line (\(\beta\)) and the flow vector direction (\(\alpha\)) which minimises the corresponding upper bound solutions.
Fig. 4-5 presents the least upper bound solutions $(\sigma/f_c)_{\min}$ as a function of inclinations of the yield line in the range $0^\circ \leq \beta \leq 90^\circ$.

As previously observed, for inclinations of the yield line in the range $0^\circ \leq \beta \leq 45^\circ + \phi/2 (= 63.5^\circ)$, the minimum compressive stress of SST specimens is always the same and equal to the uniaxial compressive strength of the $J$ material control specimen $f_c$ (Eq. 4-9).

It can be noticed that results provided by Eq. 4-13 are greater than a unity. This is a result of forcing a material in a two dimensional stress field to fail under imposed principal strain rates and therefore, in this case, there are contributions from both compressive and tensile stresses.

The comparison of these solutions not only with previous work but also with available test data is presented in Section 4.3.
4.2 - SQUARE YIELD LOCUS AS THE YIELD CONDITION

In the previous section, Coulomb's sliding failure mechanism was assumed as the failure mode for concrete. However, in the absence of evidence of a sliding condition, concrete failure is sometimes simply defined by a maximum direct stress condition represented by the square yield locus. This hypothesis is considered to derive an upper bound solution for bodies of concrete, assumed as a rigid, perfectly plastic material under uniaxial compressive stress. Point A of the square yield locus presented in Fig. 2-8(a) is considered. In this case, the principal tensile and compressive stresses are \((\sigma_1, \sigma_2) = (f_t, -f_c)\). It should be restated here that \(f_t\) and \(f_c\) are assumed as the strength in tension and in compression of the material representing the joint. The failure mechanism, as presented in Fig. 4-2, is assumed to be in a plane stress state where the inclination \((\alpha)\) of the plastic strain rate or plastic flow \((\dot{\varepsilon})\) is a variable.

From these assumptions, the plastic strain rates are defined by Eqs. 4-1 & 4-2 and the principal strains by Eqs. 4-3 & 4-4. The dissipation of energy is calculated by replacing the principal strains (Eqs. 4-3 & 4-4) and the principal stresses (as defined above) into Eq. 4-5. Thus,

\[
W = \frac{A \dot{\varepsilon}}{\cos \beta} \left( f_c \frac{(1 - \sin \alpha)}{2} + f_t \frac{(1 + \sin \alpha)}{2} \right)
\]  \hspace{1cm} (4-23)

By equating the rate of external work done on the mechanism by the applied load (Eq. 4-7) to the dissipation of energy (Eq. 4-23), the upper bound solution is found by:

\[
\frac{\sigma}{f_c} = \frac{(1 + r) - (1 - r) \sin \alpha}{2 \cos \beta \sin (\beta - \alpha)}
\]  \hspace{1cm} (4-24)

where \(r = f_t / f_c\).

As a consequence of the considered position at the yield surface and the chosen failure mechanism, least upper bound solutions require minimisation of Eq. 4-24. If the failure mechanism presented in Fig. 4-2 is considered for monolithic bodies, both \(\alpha\) and \(\beta\) are variables. In this case, Eq. 4-24 can be minimized with respect to \(\alpha\) or \(\beta\).
In order to make a comparison with previous work possible, Eq. 4-24 is initially minimized with respect to the inclination of the yield line $\beta$, only considering monolithic bodies. In this case, it is found that:

$$\beta = \frac{\pi}{4} + \frac{\alpha}{2} \Rightarrow \alpha = 2\beta - \frac{\pi}{2} \quad 4 - 25$$

Then, the corresponding least upper bound solution in terms of the flow vector direction is found by:

$$f_c = \frac{1 + \sin \alpha}{1 - \sin \alpha} \quad 4 - 26$$

and as a function of the inclination of the yield line is given by:

$$f_c = 1 + r \left( \frac{1 + \cos 2\beta}{1 - \cos 2\beta} \right) \quad 4 - 27$$

However, if bodies comprising joints are considered, which is the case of SST specimens, $\beta$ is no longer a variable. In this case, Eq. 4-27 can be simplified by using the parameters:

$$l = \frac{1 + r}{2 \cos^2 \beta} \quad 4 - 28$$

$$m = \frac{1 - r}{2 \cos^2 \beta} \quad 4 - 29$$

$$n = \tan \beta \quad 4 - 30$$

which becomes:

$$\frac{\sigma}{f_c} = \frac{l - m \sin \alpha}{n \cos \alpha - \sin \alpha} \quad 4 - 31$$

As a result, the corresponding least upper bound solution can only be obtained by minimisation of this solution (Eq. 4-31) with regard to the flow vector direction ($\alpha$), which assumes the value given by:

$$\alpha = \beta - \arccos \left( \frac{\sin \beta}{C} \right) \quad 4 - 32$$
where \( C = \frac{1 + r}{1 - r} \).

The least upper bound is then found by replacing \( \alpha \) in Eq. 4-24 by its corresponding expression obtained from Eq. 4-32, giving:

\[
\left( \frac{\sigma}{f_c} \right)_{\text{min}} = \frac{\sqrt{(1 + r)^2 - (1 - r)^2 \sin^2 \beta} + (1 - r)}{2 \cos \beta} \quad \text{4-33}
\]

In order to illustrate the solutions derived above, Figs. 4-6 and 4-7 were drawn. Fig. 4-6 presents the relation between the inclination of the yield line (\( \beta \)) and the flow vector direction (\( \alpha \)) which minimises the corresponding upper bound solution.

**FIGURE 4-6**
Flow vector direction vs Yield line inclination
Square yield locus

Fig. 4-7 presents the least upper bound solutions \((\sigma/f_c)_{\text{min}}\) given by Eq. 4-33, as a function of inclinations of the yield line in the range \(0^\circ < \beta < 90^\circ\). Non-dimensionalised strength is presented against inclination of the joint line where the failure mechanism is supposed to occur.
Special cases are presented in Fig. 4-8.

a) when $0^\circ < \beta < 90^\circ$, the least upper bound solution for each inclination of the joint line considered is provided by Eq. 4-33. The minimum stress ratio is found for $\beta = 0^\circ$, when there is no contribution from the tensile strength, i.e. $(\sigma / f_c)_{\text{min}} = 1.0 \Rightarrow \sigma_{\text{min}} = f_c$ (Fig. 4-8(a)).
b) Fig. 4-8(b) shows a flow vector direction parallel to the yield line, i.e. $\alpha = 0^\circ$, which corresponds to an inclination of the yield line equal to $\beta = \arctan C \ (= 50.7^\circ)$, for $r = 0.10$ and $C = 1.22$ considered in this example.

c) in a theoretical limit condition where $\beta = 90^\circ$ (vertical yield line), the corresponding uniaxial stress becomes infinite. Although the same result ($\beta = 90^\circ, \sigma = \infty$) has been found both by previous researchers and in the solutions obtained here, this type of failure would only be theoretically possible for concrete if, for example, an infinitely long concrete specimen was considered (Fig. 4-8(c)).

4.3 - COMPARISON WITH PREVIOUS WORK

In order to compare the obtained solutions with results achieved in previous work Figs. 4-9 and 4-10 were drawn.

Figure 4-9 shows upper bound solutions of previous researchers in which the flow vector direction was arbitrarily fixed. The flow vector direction was taken, in these examples, as $37^\circ$.
For the purpose of the comparison shown in this figure the control strength \( f_{cc} \) is that of the parent concrete material as assumed by Jensen and Clark.

Values of stress ratios between the bodies under analysis and the corresponding material strength are plotted against inclinations of the yield line in the range \( \phi < \beta < 90^\circ \). For Curves B and C no effectiveness factor was included. For Curve A, the effectiveness factor adopted by Jensen equal to 0.4 was considered. Finally, \( r \) was taken as 0.10 for Nielsen’s solution (Eq. 3-39) represented by Curve C.

Curve A in Fig. 4-9 represents Jensen’s solution (Eq. 3-27) derived for specimens comprising joints which were assumed to coincide with the yield lines. Jensen considered an arbitrary effectiveness factor equal to 0.4 applied to the cohesion of the surrounding concrete and imposed a restriction that if the specimen fails along the joint line, test data should be less than or equal to the compressive strength \( f_c \) of the concrete. As a consequence, compressive tests are limited to a small range \( \sim 43.5^\circ < \beta < \sim 83.5^\circ \).

Curve B in Fig. 4-9 represents Nielsen’s solution derived for monolithic bodies of Coulomb materials in plane strain state (Eq. 3-35 - Section 3.3.1) in which the tensile strength of concrete was neglected.

Curve C in Fig. 4-9 shows Nielsen’s solution derived for monolithic bodies of modified Coulomb materials also in plane strain state (Eq. 3-39 - Section 3.3.2). In this case, the concrete tensile strength was taken into account. The corresponding least upper bound solution (Eq. 3-36) provides exactly the same result as the alternative least upper bound solution, given by Eq. 4-26, if the flow vector direction is assumed to be equal to the angle of friction, as considered in these examples.

As a consequence of the use of a fixed flow vector direction, the minimum stress ratio for all graphs presented in Fig. 4-9 always occurs for a particular inclination of the yield line equal to \( \beta = 45^\circ + \phi/2 \) (\( = 63.5^\circ \)). However, the value of the minimum stress ratio varies according to the assumptions made by respective authors to the corresponding solutions. For example:

- for Curve A (Fig. 4-9), the minimum compressive stress is 40% of \( f_c \) due to the use
of Jensen’s effectiveness factor;

- for Curve B (Fig. 4-9), the minimum compressive stress is equal to the compressive strength \( f_c \) of the concrete which corresponds to a solution in which the tensile strength of concrete was neglected and

- for Curve C (Fig. 4-9), the minimum compressive stress given by Eq. 3-36 includes a contribution from the tensile strength of the concrete and is equal to 1.4 in this example. In this case, if a comparison with the alternative solution (Eq. 4-26) is made, the following comment is applicable. Although the tensile strength of the concrete was taken into account in both solutions, lower results are provided by the alternative solution as the flow vector direction is not arbitrarily fixed. For example, if the flow vector direction was considered in the range \( 0° < \alpha < \phi \), Eq. 4-26 would provide lower results, i.e. \( \sigma / f_c \) min < 1.4.

Fig. 4-10 shows the least upper bound solutions presented and those derived in this thesis together with available SST data.
Chapter 4 - Alternative upper bound solutions

Curve A in Fig. 4-10 represents the alternative least upper bound solutions (Eqs. 4-9 and 4-13) in which the Mohr-Coulomb envelope (Fig. 4-1(a)) was considered as the yield condition. The minimum compressive stress ratio is equal to 1.0 and is the same for any inclination of the yield line in the range $0^\circ \leq \beta \leq 45^\circ + \phi/2$. For inclinations of the yield line in the range $45^\circ + \phi/2 \leq \beta \leq 90^\circ$ the solution is given by Eq. 4-13. Special cases are presented in Fig. 4-3.

Curve B in Fig. 4-10 shows the alternative least upper bound solutions (Eq. 4-33) in which the square yield locus was considered as the yield condition. Critical stress ratios are provided for each inclination of the yield line considered. Special cases are presented in Fig. 4-8.

Curve C in Fig. 4-10 represents Nielsen’s least upper bound solutions derived for monolithic bodies of a modified Coulomb material in plane stress state (Eq. 3-42, Section 3.3.3). A stationary result of $(\sigma / f_c)_{\min} = 1.0$ was obtained. As the concrete tensile strength was neglected in this solution, lower results than the alternative least upper bound solutions (Curves A and B) were found. However, the same stationary result would be achieved by those solutions if the condition $f_t = 0$ had been considered in the derivation of the corresponding upper bound solutions.

Some available SST results are also presented in Fig. 4-10. Points referred to by a circular solid mark were reported by Jensen, 1975; those referred to by a square mark were performed by Clark & Gill, 1985 and those referred to by a triangular solid mark were carried out by Campos, 1989. In order to use data provided in Clark’s paper, it was considered that $f_t = 0.8 f_{cu}$ and that the normal and shear stresses used to define the stress ratios $\sigma / f_{cu}$ and $\tau / f_{cu}$, plotted in the graphs presented on the mentioned paper, had been calculated to satisfy equilibrium in the failure mechanism assumed by the authors. It can be observed that test results are lower than the solutions presented (Curves A, B and C in Fig. 4-10). However, lower results than the presented least upper bound results would have been found if a proper effectiveness factor had been applied to those solutions.

If a regression curve of SST data, such as the Curve A in Figs. 4-11 or 4-11A, is considered, it can be said that the trend of such curves do not fit any of the solutions presented in Fig. 4-10 (Curves A, B and C). Apparently, it is not a matter
of using an effectiveness factor. Jensen’s proposal (Eq. 3-27 - Curve B in Fig. 4-11) includes the use of an arbitrary effectiveness factor. However, just by adding some more SST results to the data reported by Jensen, it can be observed that the apparent suitability of Eq. 3-27 shown in Fig. 3-8 is no longer the case. Hence, Jensen’s solution also appears as inappropriate for the analysis of SST data.
The test results presented were non dimensionalised with respect to the parent concrete compressive strength, $f_{cc}$ and not that of the unknown joint zone assumed in the plastic analysis.

It is worth extending the discussion of the joint material and its strength here. Consider a specimen made of two materials: concrete C and a repair material R of strength $f_{cc}$ and $f_{cr}$, respectively. Control specimens made of each of these materials alone and forced to fail along a line at angle $\beta$ may produce curves C and R in Fig. 4-12 where C is stronger than R. Neither of these may represent the properties of the joint. The joint will have some other strength $f_c$ and it would be the purpose of a plastic analysis of test data to furnish this value.

4.4 - CONCLUSIONS

Apparent inadequacies regarding flow vector direction observed in previous applications of the upper bound theorem were reported. The need for a rational re-draft of upper bound solutions was demonstrated and a comparison with previous work and available SST results was made.
A missing point regarding the flow vector direction was observed in Coulomb's modified yield criterion defined by Jensen, 1975 for concrete in plane stress states (see Section 3.2). Condition $\alpha < \phi$, allowed by the adopted criterion, when defined in terms of the principal stresses (Fig. 3-4(b)) was not verified when the same criterion was represented as a function of the shear and normal stresses (Fig. 3-4(a)).

In order to satisfy this condition, the use of a curved yield surface such as the Mohr-Coulomb envelope presented in Fig. 4-1(a) was necessary. Alternative upper bound solutions were derived for some special points at the related yield surface in terms of the principal stresses (Fig. 4-1(b)). The correspondence between those two yield surfaces (Figs. 4-1(a) & (b)) and that of the flow vector direction, in each case, was obtained. Those alternative solutions were then minimised with respect to the flow vector direction and the achieved least upper bound solutions (see Fig. 4-5) were compared not only with previous solutions but also with available test data.

The consideration that the displacement vector direction is equal to the angle of friction $\alpha = \phi$ is perhaps a restriction in previous work. The assumption that the flow vector is fixed in the direction equal to the angle of friction was made, considering both Coulomb's (modified) criterion as the yield condition and the corresponding failure mechanism in plane strain states.

However, an infinite number of flow vector directions is possible. Also, in the absence of evidence of sliding conditions, concrete failure is sometimes simply defined by a maximum direct stress condition represented by the square yield locus.

Facing all these considerations, the need for further investigation into the upper bound theorem was shown. For this purpose, an alternative upper bound solution was derived (Eq. 4-24), considering the square yield locus as the yield criterion for a variable flow vector direction (Point A, Fig. 2-8(a)). The corresponding failure mechanism was assumed to be in a plane stress state and the tensile strength of the concrete was taken into account. The obtained upper bound solution (Eq. 4-24) included two variables, i.e. the inclination of the yield line and the direction of the flow vector.
If least upper bound solutions were required just regarding monolithic specimens, the inclination of the yield line would be assumed as the variable for minimisation. In this case, Eq. 4-26 would provide the minimum stress ratios for the alternative solution. Comparison with Nielsen’s least upper bound solution (Eqs. 3-36, Section 3.3.2), where the concrete tensile strength was also considered, was then possible. For example, if $\alpha$ could be considered to be equal to the angle of friction in Eq. 4-26, the same result as Nielsen’s solution (Eq. 3-36) would be found, i.e. $(\sigma/f_c)_{\text{min}} = 1.4$, for $r = 0.10$ and $k = 4$. However, lower results could be obtained by the alternative least upper bound solution (Eq. 4-26) as a consequence of the use a variable flow vector direction. For example, for $0 < \alpha < \phi$ Eq. 4-26 would provide $(\sigma/f_c)_{\text{min}} < 1.4$.

The objective of this research is to analyse SST specimens which comprise joints. Assuming that the joint line coincided with the yield line, the alternative upper bound (Eq. 4-24) was minimised with respect to the flow vector direction as, unlike previous work, there was no imposed restriction. Thus, the obtained least upper bound solution (Eq. 4-33) provides a specific and critical stress ratio for each inclination of the yield line considered (see Fig. 4-7). Focusing on the SST, for each inclination of the joint line considered, there is a unique solution provided by Eq. 4-33. The minimum achieved stress ratio $(\sigma/f_c)_{\text{min}} = 1$ was found for a horizontal yield line (see Fig. 4-8(a)).

In order to verify the applicability of previous least upper bound solutions and those obtained in this thesis to the analysis of SST results, a comparison with available test data was made (see Fig. 4-10).

It was observed that experimental results were lower than the presented solutions. Lower results to the least upper bound would have been obtained if a proper effectiveness factor had been applied to those solutions. However, when a regression curve of SST data, such as the Curve A in Fig. 4-11, was made, it was verified that the trend of such a curve did not fit any of the solutions presented in Fig. 4-10 (Curves A, B and C). Apparently, it was not a matter of using an effectiveness factor.
Jensen's proposal included the use of an arbitrary effectiveness factor. However, just by adding some more SST results to the data reported by Jensen (Fig. 4-11 (A)), it was observed that the apparent suitability previously encountered (Jensen's proposal - Fig. 3-8) was no longer the case. Hence, Jensen's solution also appeared as inapplicable to SST data. Summarizing, it can be said that it was not possible to establish any correspondence between the test data and the solutions presented here.

In conclusion, the theoretical modelling left some doubt on the applicability of plastic methods to analyse the slant shear test and some points are not very clear, for example: limited knowledge about the failure mode and adequacy of test data.

In order to make a possible decision on the applicability of plasticity, it must be verified whether internal stresses can be redistributed within a specimen subjected to the SST which is the essential condition for the validity of the application of limit analysis. Considering the standardized form of the test (ASTM-C-882, 1978 and BS 6319 - Part 4, 1984), the failure process happens suddenly and in a very small area. As a consequence, it is impossible to characterize the failure mode, as well as the type of deformation which takes place during the failure process.

As explained in the Introduction, a programme of laboratory work was carried out as an attempt to reproduce the failure pattern assumed in theory in order to investigate the failure mechanism of SST specimens, as described in the next chapter. For this purpose, a specially designed specimen was used, partly to encourage a plastic failure and partly, with the aid of specially tailored apparatus, to allow a better visualization of the deformations developed during the failure process.
CHAPTER 5

Experimental investigation of the failure process in Slant Shear Test specimens

Material performance has been put under scrutiny in this experimental investigation through the analysis of the failure process in specially designed specimens when subjected to the SST. As previously commented, when tests are carried out in accordance with existing standards (ASTM-C-882, 1978 and BS 6319 - Part 4, 1984) failure occurs suddenly and in a very small area. This is unsatisfactory for a careful observation of the failure mode and for the definition of deformations which take place during the failure mechanism. Therefore, the speed of tests was reduced and specially designed specimens were created. The objective of these alterations was to deliberately encourage a plastic failure as well as to allow a better visualization of deformations by monitoring load during observed material damage to detect evidence of stress redistribution.

It should be restated here that the objective of this laboratory work is to test a theoretical hypothesis and not to model completely a real specimen or a joint. For this purpose, special composite specimens were created by the insertion of a 13 mm wide joint layer made of a low strength material at an inclination of 30° to the uniaxial compressive stress direction. The reason for this modification was to provoke failure mechanism inside these joint layers as an attempt to reproduce the model conceived in theory.

Some preliminary tests were carried out in order to obtain appropriate procedures, equipment and mixes to pursue the principal objectives of this investigation. In this initial stage, the following machines were used: the Hounsfield Tensometer, type W, No 7638; the Instron, model 1114, serial No. H 1395 and the Denison model T42.B4. A brief description and discussion of the use of these machines is included in Sections 5.1.1.1 and 5.1.2.1. In order to slow down the failure process, tests were carried out at the minimum loading or straining rate
allowed by the equipment specifications. Also, some preliminary tests carried out on
the three different machines were recorded on high-speed video, which was edited in
slow-motion to provide a clearer identification of the failure mechanism. Photographs were taken of frames of the video produced showing significant
changes happened at the start of the failure process of the recorded tests (see
Appendix A).

In order to obtain accurate load vs displacement curves both for control and
SST specimens and to allow consistent results in this experimental investigation,
specially tailored apparatus had to be set up for the main tests of this experimental
investigation, as described in the Section 5.2.2.

A description of this experimental investigation is presented in this chapter. The
preliminary tests which consisted of two phases are reported in Sections 5.1.1 and
5.1.2. The preparation and the testing procedures of the specimens as well as the
presentation and discussion of the main test results are described in Section 5.2. Finally, the obtained conclusions are presented in Section 5.3.

5.1 - PRELIMINARY TESTING PROCEDURES AND RESULTS

Preliminary tests were carried out in order to obtain suitable procedures,
equipment and mixes to pursue the main objectives of this experimental
investigation. The preliminary tests consisted of two phases as described in Sections
5.1.1 and 5.1.2.

5.1.1 - First phase of the preliminary tests

In order to create the special specimens comprising the previously mentioned
joint layers, moulds and casting procedures were specially designed.
Moulds used in the first phase of the preliminary tests comprised a removable,
smooth, internal partition as shown in Fig. 5-1.
Due to factors such as equipment specifications and handling, the external dimensions of the test specimens used in both phases of the preliminary tests were: 50 mm wide; 25 mm deep and 132 mm high. All dimensions and characteristics of the composite specimens are presented in Fig. 5-2.

**FIGURE 5-1** - Photo of the moulds used in the first phase of the preliminary tests (removable smooth partition)

**FIGURE 5-2** - Characteristics and dimensions of the composite specimens used in the first phase of the preliminary tests

1. Dimension in mm.
2. $M_1$ and $M_2$ - mixes of each casting stage
The casting procedure consisted of two stages. In the first stage, the triangular parts of the specimen were made of mortar whose composition is referred to here as $M_1$. Moulds with the removable, smooth partition inside were filled with mix $M_1$. However, the surfaces of the triangular parts in contact with this section were made rough by mechanical abrasion.

The second casting stage consisted of filling the joint layers with a lower strength mortar which is referred to here as $M_2$. In this case, the triangular parts of the specimens were placed inside the moulds without the removable partition and the joint layers were filled.

In this first phase of the preliminary tests, five composite specimens were made and tests were only carried out on the Hounsfield Tensometer (see Section 5.1.1.1). The composition of the first two specimens was as follows: $M_1$ - 1:3 mortar (one part of ordinary Portland cement to three parts of Thames Valley sand), water/cement ratio ($w/c$) = 0.5 and $M_2$ - 1:6 mortar, $w/c = 0.8$. In order to verify if the mixes used to make the first two specimens had influenced the failure process found, the mixes were changed. For the remaining three specimens of these preliminary tests, the adopted composition was as follows: $M_1$ - 1:3 mortar, $w/c = 0.4$ and $M_2$ - 1:3 mortar, $w/c = 0.5$.

All mixing procedures were carried out in accordance with the BS 4551:1980 (Methods of Testing Mortars Screeds and Plasters). For each batch of mortar, two (25 x 50 x 132) mm monolithic specimens were made for quality control. The compressive tests and the SST were performed seven days after the gaps had been filled.

5.1.1.1 - Equipment - Hounsfield Tensometer

This machine was initially chosen as it is a hand-operated machine and it allows tests to be performed at a slow loading rate under operator control. In addition, load vs displacement graphs can be drawn by the operator during tests.

The principles on which this machine operates are illustrated in Fig. 5-3 where
a device available for compressive tests used in this experimental investigation, is included. The compressive test modification is held at one end by a force measuring system and at the other by a system which applies extension, both positively linked. As this device consists of two connected steel structures which slide over each other, the extension applied at the opposite ends is transformed into compression on the specimen held by the internal parts of the device (see Fig. 5.4).

![Hounsfield Tensometer - Type W - Operating Scheme](image)

The measuring system is based upon a spring beam which deflects under the resistive force of the specimen transmitted by other parts of this system. The deflections which are within the elastic limit of the corresponding spring beam are linearly related to the applied force which can be directly read from a scale positioned near the mercury column.

The recording drum rotates at the same rate as specimens are compressed. The corresponding displacement which can be recorded on the autographic recording drum are amplified by one of the three available magnification levels (4:1, 8:1 and 16:1) depending on the position chosen for the gearing on the end of the drum. Movements within the jaws and linkages of the machine are included in the displacements recorded.
The compressive testing modification has the disadvantage of not providing a proper seat, such as a ball seat, to avoid loading eccentricity. In order to overcome this difficulty, pieces of hard rubber were initially used between the platens of the machine and at both ends of the specimen. However, it was verified that the control specimens failed through vertical splitting as movements perpendicular to the loading direction were allowed by this flexible end condition. This end condition is referred to here as a frictionless end condition. As minor irregularities were found on the top and bottom surfaces of the specimens and to keep the frictional end condition as the SST is standardized (ASTM-C-882, 1978 and BS 6319 - Part 4, 1984), the "seats" were removed and specimens were then tested directly in contact with the platens of the machine.

Dealing with this machine, the operator has both hands busy as one has to move the operating handle which puts the specimen under compression and the other one presses the cursor to record the loads on the recording drum which rotates proportionally to the corresponding displacements. As the operator has to look at the mercury column to record the loads, it is not possible for her to observe the specimen (see Fig. 5-4). This fact was considered as another disadvantage in using this machine.

**FIGURE 5-4** - Photo of an operator dealing with the Hounsfield Tensometer.
In order to obtain an approximation of the actual displacement of the specimens, graphs were drawn without a specimen inside the machine, just pressing the platens against each other (with and without the above-mentioned pieces of rubber - see Fig. 5-5).

Load vs displacement curves of this equipment, as a whole, were then obtained and can be seen in Fig. 5-6 (Curve A - frictional end condition and Curve B - frictionless end condition). The stiffness corresponding to each end condition, calculated without taking the initial parts of the curves into account (as they are considered to comprise the initial adjustment of the internal parts of the equipment), are presented as follows: $S_A = 7.2$ KN/mm (frictional end condition) and $S_B = 6.2$ KN/mm (frictionless end condition).

Thus, a better approximation of the real displacements was obtained and the following procedure was adopted. At each load in the range considered and in accordance with the end condition adopted, an approximation of the real displacements of the specimens was found by reducing the displacements of the machine from the ones obtained from graphs drawn while composite specimens were tested. The results found were then divided by the corresponding magnification level.

FIGURE 5-5 - Close-up of platens of the Hounsfield Tensometer being compressed.
Finally, the greatest disadvantage in using this machine is its elastic loading system. This disadvantage is due to the fact that this system stores potential energy in the spring beam during loading which cannot be controlled when specimens fail. Although tests can be performed at a very low loading rate, when specimens start to fail, the release of energy cannot be controlled. As a result, the objective to slow down the failure process is not attained.

5.1.1.2 - Presentation and discussion of the first phase of the preliminary tests

Data and testing results of the control specimens corresponding to the mixes used to make the composite specimens are presented in Tables 1 and 2.
Table 1 - Data and testing results of control specimens made of mix M₁
First phase of the preliminary tests

<table>
<thead>
<tr>
<th>Specimen Index</th>
<th>Control Specimen Index</th>
<th>Maximum Load ((N_{\text{w1}})) (KN)</th>
<th>Average Load ((N_{\text{w2}})) (KN)</th>
<th>Mix by Weight</th>
<th>End Condition</th>
<th>Failure Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>HTP1</td>
<td>HCP1A</td>
<td>15.3</td>
<td>16.8</td>
<td>1 : 3</td>
<td>frictionless end condition</td>
<td>vertical splitting (Fig. 5-7)</td>
</tr>
<tr>
<td></td>
<td>HCP1B</td>
<td>18.3</td>
<td></td>
<td></td>
<td>w/c = 0.5</td>
<td></td>
</tr>
<tr>
<td>HTP2</td>
<td>HCP1C</td>
<td>13.0</td>
<td>13.4</td>
<td>1 : 3</td>
<td>frictional end condition</td>
<td>diagonal</td>
</tr>
<tr>
<td></td>
<td>HCP1D</td>
<td>13.8</td>
<td></td>
<td></td>
<td>w/c = 0.4</td>
<td></td>
</tr>
<tr>
<td>HTP3</td>
<td>HCP1E</td>
<td>25.9</td>
<td></td>
<td></td>
<td>frictionless end condition</td>
<td>vertical splitting (Fig. 5-7)</td>
</tr>
<tr>
<td></td>
<td>HCP1F</td>
<td>26.4</td>
<td></td>
<td></td>
<td>w/c = 0.8</td>
<td></td>
</tr>
<tr>
<td>HTP4</td>
<td>HCP1G</td>
<td>26.3</td>
<td>26.3</td>
<td>1 : 3</td>
<td>end condition</td>
<td>cracks</td>
</tr>
<tr>
<td></td>
<td>HCP1H</td>
<td>26.1</td>
<td></td>
<td></td>
<td>w/c = 0.4</td>
<td></td>
</tr>
<tr>
<td>HTP5</td>
<td>HCP1I</td>
<td>26.6</td>
<td></td>
<td></td>
<td>condition</td>
<td></td>
</tr>
<tr>
<td></td>
<td>HCP1J</td>
<td>26.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2 - Data and testing results of control specimens made of mix M₂
First phase of the preliminary tests

<table>
<thead>
<tr>
<th>Specimen Index</th>
<th>Control Specimen Index</th>
<th>Maximum Load ((N_{\text{w2}})) (KN)</th>
<th>Average Load ((N_{\text{w2}})) (KN)</th>
<th>Mix by Weight</th>
<th>End Condition</th>
<th>Failure Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>HTP1</td>
<td>HCP2A</td>
<td>5.1</td>
<td>4.7</td>
<td>1 : 6</td>
<td>frictionless end condition</td>
<td>vertical splitting (Fig. 5-7)</td>
</tr>
<tr>
<td></td>
<td>HCP2B</td>
<td>4.3</td>
<td></td>
<td></td>
<td>w/c = 0.8</td>
<td></td>
</tr>
<tr>
<td>HTP2</td>
<td>HCP2C</td>
<td>3.8</td>
<td>3.9</td>
<td>1 : 3</td>
<td>frictional end condition</td>
<td>diagonal</td>
</tr>
<tr>
<td></td>
<td>HCP2D</td>
<td>4.0</td>
<td></td>
<td></td>
<td>w/c = 0.5</td>
<td></td>
</tr>
<tr>
<td>HTP3</td>
<td>HCP2E</td>
<td>25.8</td>
<td></td>
<td></td>
<td>frictionless end condition</td>
<td>vertical splitting (Fig. 5-7)</td>
</tr>
<tr>
<td></td>
<td>HCP2F</td>
<td>26.3</td>
<td></td>
<td></td>
<td>w/c = 0.8</td>
<td></td>
</tr>
<tr>
<td>HTP4</td>
<td>HCP2G</td>
<td>26.5</td>
<td>25.8</td>
<td>1 : 3</td>
<td>end condition</td>
<td>cracks</td>
</tr>
<tr>
<td></td>
<td>HCP2H</td>
<td>26.0</td>
<td></td>
<td></td>
<td>w/c = 0.5</td>
<td></td>
</tr>
<tr>
<td>HTP5</td>
<td>HCP2I</td>
<td>25.3</td>
<td></td>
<td></td>
<td>condition</td>
<td></td>
</tr>
<tr>
<td></td>
<td>HCP2J</td>
<td>24.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conventions:

<table>
<thead>
<tr>
<th>HTPX</th>
<th>SST specimen tested on the Hounsfield Tensometer sequential number</th>
</tr>
</thead>
<tbody>
<tr>
<td>HCYZ</td>
<td>control specimen tested on the Hounsfield number corresponding to mixes M₁ or M₂ sequential letter</td>
</tr>
</tbody>
</table>

Note: (25 x 50 x 132) mm specimens.
Depending upon the end conditions, control specimens presented two different failure modes as shown in Figs. 5-7 and 5-8, resulting from frictionless and frictional end conditions, respectively.

**FIGURE 5-7** - Photo of the failure mode of control specimens under frictionless end condition.
(First phase of the preliminary tests on the Hounsfield Tensometer)

**FIGURE 5-8** - Photo of the failure mode of control specimens under frictional end condition.
(First phase of the preliminary tests on the Hounsfield Tensometer)
Although tests were performed at a low loading rate, specimens failed suddenly as a consequence of the elastic nature of the machine used (the Hounsfield Tensometer). Hence, the failure mechanism could not be observed, i.e. the formation and propagation of fissures could not be identified.

As an attempt to observe the failure process, tests were recorded on video in the second phase of the preliminary tests. Also, other machines were used as described in Section 5.1.2.1.

5.1.2 - Second phase of the preliminary tests

Some of the objectives of this experimental programme such as, provoking failure inside the joint layers and slowing down the failure process were not achieved in the first phase of the preliminary tests. In order to pursue these objectives and to
make a comparison with the first phase results possible, some changes were introduced in the second phase. But some procedures were kept the same as in the first phase.

As far as specimens were concerned, the casting procedures, shape and external dimensions were kept the same as described in Section 5.1.1. However, the roughness of the joints was substantially increased by means of a removable, "rough" partition specially designed for this phase (see Fig. 5-10).

**FIGURE 5-10** - Photo of the moulds used in the second phase of the preliminary tests (removable rough partition)

The dimensions and characteristics of the composite specimens used in this phase are shown in Fig. 5-11.

Five groups of tests were carried out and all the composite specimens were made with the same mixes, i.e. $M_1$ - 1:3 mortar, $w/c = 0.4$ and $M_2$ - 1:6 mortar, $w/c = 1.1$. 

5.1.2.1 - Equipment used in the second phase of the preliminary tests

The Instron model 1114, serial number H 1395 (see Fig. 5-12) was used in this phase in order to check the first phase results and to overcome some difficulties, which are described overleaf, encountered when using the Hounsfield Tensometer.
This machine has a plotter which automatically draws load vs displacement graphs while tests are being performed. However, the recorded displacements include movements within the jaws and linkages of the machine which in this case, were not possible to identify. This fact was not considered as a disadvantage as, at least, the shape of the load vs displacement curves could be obtained. Furthermore, by using this automatic machine, the operator was free to observe the failure process.

The lowest straining rate provided by this machine was chosen, i.e. 0.5 cm/min and two different speeds for chart drawing were used, i.e. 0.5 and 1.0 cm/min. The ratio between the speed at which the graphs were drawn and the corresponding straining rate is referred to here as chart magnification which was considered in order to obtain the curves presented in Section 5.1.2.2.

A compressive testing modification was also used and the same problem of not having a proper seat for the specimens to be tested under compression, was encountered. As there was still doubt about the alignment of loading and that any eccentricity may have affected the specimen failure pattern, further tests were performed on Denison model T42.B4 which applies compression through a ball seat.

5.1.2.2 - Presentation and discussion of the second phase of the preliminary tests

Data and testing results of control specimens corresponding to each mix used to make the composite specimens are presented in Tables 4 and 5. The average maximum load of specimens made of mix $M_1$ was 23.2 KN and 6.7 KN for those made of mix $M_2$.

Regardless of the machine used to carry out the compressive tests on the control specimens, the same failure patterns were found, i.e. specimens failed through diagonal cracks (see Fig. 5-13). However, due to the elastic nature of the Hounsfield Tensometer, whenever tests were performed on this machine, the failure process progressed faster than the ones developed by specimens tested on the other
two machines. In order to make it possible to observe the failure process in
specimens tested on the Hounsfield machine, some of the tests were recorded on
high-speed video (see Figs. A-1 and A-2 - Appendix A).

### Table 4 - Data and testing results of control specimens made of mix M\(_1\)
Second phase of the preliminary tests

<table>
<thead>
<tr>
<th>Group of tests Index</th>
<th>Equipment Used</th>
<th>Control Specimen mix M(_1) Index</th>
<th>Maximum Load (N_{M_1}) (KN)</th>
<th>Average Load (N_{M_1}) (KN)</th>
<th>Mean Load (KN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Instron</td>
<td>C1A</td>
<td>26.8</td>
<td>21.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1B</td>
<td>17.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1C</td>
<td>24.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Hounsfield</td>
<td>C1D</td>
<td>21.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Denison</td>
<td>C1E</td>
<td>21.4</td>
<td>21.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1F</td>
<td>21.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Denison</td>
<td>C1G</td>
<td>22.4</td>
<td>23.0</td>
<td>23.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1H</td>
<td>23.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Instron</td>
<td>C1I</td>
<td>22.5</td>
<td>22.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1J</td>
<td>23.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Instron</td>
<td>C1K</td>
<td>23.1</td>
<td>26.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C1L</td>
<td>30.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Mortar mix M\(_1\) - 1:3 (mix by weight) - w/c = 0.4.

### Table 5 - Data and testing results of control specimens made of mix M\(_2\)
Second phase of the preliminary tests

<table>
<thead>
<tr>
<th>Group of tests Index</th>
<th>Equipment Used</th>
<th>Control Specimen mix M(_2) Index</th>
<th>Maximum Load (N_{M_2}) (KN)</th>
<th>Average Load (N_{M_2}) (KN)</th>
<th>Mean Load (KN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Instron</td>
<td>C2A</td>
<td>5.0</td>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2B</td>
<td>4.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2C</td>
<td>7.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Hounsfield</td>
<td>C2D</td>
<td>6.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Denison</td>
<td>C2E</td>
<td>6.1</td>
<td>8.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2F</td>
<td>8.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Denison</td>
<td>C2G</td>
<td>5.4</td>
<td>5.9</td>
<td>6.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2H</td>
<td>6.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Instron</td>
<td>C2I</td>
<td>5.1</td>
<td>5.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2J</td>
<td>6.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Instron</td>
<td>C2K</td>
<td>7.6</td>
<td>8.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>C2L</td>
<td>9.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: 1. Mortar mix M\(_2\) - 1:6 (mix by weight) - w/c = 1.1.
Table 6 shows data and testing results of composite specimens (SST). The average maximum load obtained for composite specimens was $\bar{N} = 8.1$ KN which is above the average load of mix $M_2$ used to make the joint layers $\left(\frac{\bar{N}}{\bar{N}_{M_2}}\right) = 1.17$ and much lower than the average load of mix $M_1$ $\left(\frac{\bar{N}}{\bar{N}_{M_1}}\right) = 0.35$.

The procedure used to calculate displacement for tests performed on the Hounsfield Tensometer was the one adopted in the first phase of the preliminary tests (see Section 5.1.1.1). However, for tests performed on the Instron machine, displacements were obtained by dividing the total displacement by the corresponding chart magnification. In this case, the total displacement obtained from the charts drawn during the performance of tests includes displacements within the Instron machine.

The graphs presented in Figs. 5-14 and 5-15 were drawn based upon characteristic results of load and displacement obtained from tests performed on the Hounsfield and Instron machines, respectively.
Table 6 - Composite Specimen Data and Testing Results

Second phase of the preliminary tests

<table>
<thead>
<tr>
<th>Group of tests Index</th>
<th>Composite Specimen Index</th>
<th>Maximum SST Load N (KN)</th>
<th>Average Load (N ( N_{m1} )) (KN)</th>
<th>Maximum ( N / N_{m1} ) ratio</th>
<th>Average Load (N ( N_{m2} )) (KN)</th>
<th>Maximum ( N / N_{m2} ) ratio</th>
<th>Maximum Displacement (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>ITP1</td>
<td>7.4</td>
<td>21.9</td>
<td>0.34</td>
<td>5.0</td>
<td>1.48</td>
<td>50.16</td>
</tr>
<tr>
<td></td>
<td>HTP6</td>
<td>7.8</td>
<td>23.2</td>
<td>0.34</td>
<td>6.7</td>
<td>1.16</td>
<td>0.50</td>
</tr>
<tr>
<td>1</td>
<td>HTP7</td>
<td>6.9</td>
<td>21.5</td>
<td>0.32</td>
<td>8.2</td>
<td>0.84</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>HTP8</td>
<td>7.1</td>
<td>0.33</td>
<td>1.03</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>HTP9</td>
<td>7.6</td>
<td>0.35</td>
<td>1.15</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>HTP10</td>
<td>6.9</td>
<td>0.32</td>
<td>1.03</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP1</td>
<td>8.6</td>
<td>0.40</td>
<td>1.05</td>
<td>-</td>
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<tr>
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<td>HTP11</td>
<td>7.5</td>
<td>23.0</td>
<td>0.33</td>
<td>8.2</td>
<td>0.84</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>ITP4</td>
<td>6.1</td>
<td>0.27</td>
<td>1.03</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>ITP5</td>
<td>6.8</td>
<td>0.30</td>
<td>1.15</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP2</td>
<td>6.1</td>
<td>0.27</td>
<td>1.03</td>
<td>-</td>
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<td></td>
<td>DTP3</td>
<td>8.6</td>
<td>0.37</td>
<td>1.46</td>
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<td>-</td>
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<td>3</td>
<td>HTP12</td>
<td>8.1</td>
<td>22.8</td>
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<td>1.37</td>
<td>0.54</td>
</tr>
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<td>ITP2</td>
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<td>59.40</td>
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<td>-</td>
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<td></td>
<td>ITP3</td>
<td>6.9</td>
<td>0.30</td>
<td>1.17</td>
<td>36.96</td>
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<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP4</td>
<td>7.8</td>
<td>0.34</td>
<td>1.32</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP5</td>
<td>7.9</td>
<td>0.35</td>
<td>1.34</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>DTP6</td>
<td>11.8</td>
<td>26.6</td>
<td>0.44</td>
<td>8.6</td>
<td>1.37</td>
<td>-</td>
</tr>
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<td>DTP7</td>
<td>11.6</td>
<td>0.44</td>
<td>1.35</td>
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<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP8</td>
<td>9.8</td>
<td>0.37</td>
<td>1.14</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP9</td>
<td>7.9</td>
<td>0.30</td>
<td>0.92</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>DTP10</td>
<td>11.8</td>
<td>0.44</td>
<td>1.37</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>8.1</td>
<td>23.2</td>
<td>0.35</td>
<td>6.7</td>
<td>1.17</td>
<td>-</td>
</tr>
</tbody>
</table>

Conventions: YTPX | YTP | X
- YTPX - specimen tested on the Hounsfield Tensometer
- YTP - specimen tested on the Instron machine
- X - sequential number

Notes: 1. (25x50x132) mm SST specimens made of mixes M₁ & M₂ (Tables 4 & 5).
2. Frictional end condition.
3. Rough removable partition.
Figure 5-14 presents the load vs displacement curves corresponding to test results obtained on the Hounsfield Tensometer of composite specimens: HTP8, HTP10 and HTP11 and control specimen C2C.

In spite of the attempt to identify displacements within the Hounsfield Tensometer, their influence can still be observed due to the apparent increasing stiffness along the initial parts of the resulting curves obtained from both composite and control specimens.

The displacements developed in control specimens made of mix M₂ were approximately twice those in the composite specimens, when tests were performed on the Hounsfield. This might be justified by the different compositions of the two types of specimens together with the presence of joints in composite specimens. The fact that the SST specimens were also constituted of mix M₁ might have provided stiffer testing pieces than control specimens exclusively made of mix
M2. On the other hand, the presence of joints introduced flexibility to the composite specimens. As a consequence of the combination of these features, both types of specimens failed at approximately the same load but more deformation was found in the control specimens.

Load vs displacement curves both of composite and control specimens presented in Fig. 5-14, show a brittle mode owing to the full and sudden release of the load at maximum capacity. Monolithic specimens should unload with a falling branch to the curve after maximum load. The Hounsfield Tensometer disguises this by uncontrolled release of its stored energy and therefore, it cannot be said that the obtained curves are representative of the nature of the testing material nor of this particular test (SST). Hence, if there is any difference in the failure mode of SST specimens, this will not be shown on this apparatus.

The graphs presented in Fig. 5-15 are the load vs displacement curves corresponding to test results obtained on the Instron machine of composite specimens: IPT1, IPT2 and IPT3 and control specimen C2A.

**FIGURE 5-15 - LOAD VS DISPLACEMENT CURVES**

Instron machine

Second phase of the preliminary tests

![Graph showing load vs displacement curves]
As in Fig. 5-14, the shape of the curves of both the composite and control specimens are similar. But in this case, the resulting load vs displacement curves obtained from the Instron machine present some data after the maximum load was reached, represented by the falling branches (Fig. 5-15).

For the purpose of determining the complete stress vs strain relation of concrete specimens in uniaxial compression, Barnard (1964) used a constant strain rate machine especially designed by Turner & Barnard in 1962, to be stiff enough to allow load to fall off in a failing specimen. Barnard’s stress vs strain curves of concrete prisms and necked specimens are presented in Figs. 5-16(a) and 5-16(b), respectively. These graphs show a much more gradual reduction in load resistance than the curves corresponding to tests performed on both the Hounsfield Tensometer (Fig. 5-14) and the Instron machine (Fig. 5-15). Although Barnard provided stress vs strain relations for concrete, similar results might have been expected, at least, for the monolithic specimens made of mortar, utilised in this laboratory work, if a stiff constant strain rate testing machine had been used.
Like the composite specimens tested on the Hounsfield machine, failure took place along one of the joint lines which will be referred to here as the failing joint or yield line. Although the failure process of SST specimens did not happen suddenly when tests were performed on the Instron machine, the beginning of the process could not be easily identified. In order to facilitate the analysis of the failure process, some of tests performed on the Instron machine were recorded on high-speed video (see Figs. A-3 and A-4, Appendix A). The characteristic failure mechanism is described, as follows. Failure commenced suddenly, revealing a line along the failing joint. The failure process progressed along that joint by breaking small pieces of the specimen, releasing the bond and allowing the upper part of the specimen to slide along the yield line until it fell off.

A similar failure mode was obtained when tests were performed on the Denison machine, which has a proper ball seat for specimens to be tested under compression (see Figs. A-5 and A-6, Appendix A). Additionally, in certain SST specimens, cracks spread over other parts of the specimens (see Fig. 5-17(a)), as described below. Cracks found in such specimens are schematically shown in Fig. 5-17(b). The following numbers in brackets refer to pieces pointed out in that figure. Cracks propagated diagonally from the yield line to the upper part of specimens [1] and were also seen in parts of the other joint line (some vertical cracks in the middle of the joint [2], some progression along one of the joint ends [3] and also some diagonal cracks to the bottom part [4]).

Although tests performed on the Denison machine reiterated the failure mechanism previously encountered when tests were carried out on the Instron machine, load vs displacement curves cannot be automatically plotted from the Denison machine. As explained in the introduction of this chapter, specially tailored apparatus had to be set up for the main tests in order to obtain accurate load vs displacement curves both for control and SST specimens and to allow a proper completion of this experimental investigation.
FIGURE 5-17 (a) - Photo of a composite specimen during failure process (Denison T42.B4).

FIGURE 5-17(b) - Schematic failure mode shown in Fig. 5-17(a)
5.2 - MAIN TESTS

The principal objective of the experiments described in this section was to obtain accurate load vs displacement curves of both SST (Slant Shear Test) and control specimens, in order to allow a clear identification of deformations developed during the failure process.

Specially designed SST specimens as described in Section 5.1.1 were used. The specimens comprised a 13 mm wide joint layer made of a low strength material at an inclination of 30° to the uniaxial compressive load direction. Exaggerated rough surfaces between the weak material and the other parts of the specimens, as utilized in the second phase of the preliminary results (Section 5.1.2), were introduced in order to allow a yielding or plastic behaviour to take place.

Specially tailored apparatus as described in Section 5.2.2 was assembled to allow rigorous monitoring and recording of load and displacement data developed in the specimens while tests were being performed.

A description of the preparation and testing procedures of the specimens; details of the equipment used and the presentation and discussion of tests results are included in this section.

5.2.1 - Preparation and testing procedures of specimens used in the main tests

In order to produce specimens with accurate dimensions, a steel, multiple mould was used including a specially fabricated device made of acetal to create the joint layers of the SST specimens (see Fig. 5-18).

The external dimensions of both the control specimens and the SST specimens were (50 x 50 x 150) mm. The characteristics and all dimensions of the composite specimens are shown in Fig. 5-19.
The casting procedure consisted of two stages. Firstly, the triangular parts of the composite specimens were made of mortar whose composition is referred to here as $M_1$ (mortar 1:3 by weight, w/c=0.5). Then, the end bays of the steel, multiple mould with the specially fabricated device, previously mentioned, inside were filled with mix $M_1$. In addition, two control specimens were produced by filling the central bays of the mould with the same mix.
The second casting stage consisted of filling the joint layers with a lower strength mortar which is referred to here as $M_2$ (mortar 1:6 by weight, w/c = 1.08). In this case, the triangular parts of the specimen were placed inside the mould without the specially fabricated device and the joint layers were filled. As in the first casting stage, two control specimens were also produced by filling the two central bays of the multiple mould but, this time, with mix $M_2$.

Four groups of tests were carried out. Each group of tests comprised two SST specimens and four control specimens (two of each mix used to make the SST specimens). All mixing procedures were carried out in accordance with BS 4551:1980 (Methods of Testing Mortars Screeds and Plasters). The compressive tests and the SST were performed seven days after the gaps had been filled.

5.2.2 - Apparatus used in the main tests

**Denison model T42B4** was chosen to be used as part of the apparatus including a 50KN load cell, four LVDTs and a Macintosh computer - MacLab - (see Fig. 5-20).

![Figure 5-20 - Photo showing the general view of the apparatus used in the main tests.](image-url)
The Denison machine was chosen, as it is a hand-operated machine where tests can be performed at a constant straining rate. It has a proper ball-seat which avoids load eccentricity and a load range compatible with the experiments to be carried out. However, it does not provide any facilities to record or plot both load and displacement data while tests are being performed. The following procedures were carefully adopted in order to set up an experiment to record load and specimen displacement, exclusive of machine movement, as accurately as possible.

First of all, a test rig was designed which consisted of an auxiliary steel structure comprising two platens (top and bottom) and four columns as shown in Figs. 5-21 and 5-22. The objective of this device was to support the load cell; to sustain and locate the LVDTs; to hold the 50x50x150mm specimens and to avoid creating any eccentricity. The latter was assured by designing a top platen which could act as a ball seat allowing movement of the specimens when loaded (see Detail 1 in Fig. 5-22). The 50KN load cell connected to a conditioning PSU (Power Supply Unit), provided input to one of the three channels used in the MacLab.

![FIGURE 5-21 - Auxiliary Steel Structure](image-url)
In order to record the specimen displacements, four LVDTs (A, B, C and D) were used as shown in Figs. 5-21 and 5-22. Each LVDT was connected to an amplifier (D7) and also to a conditioning unit. Average displacements between each two of the four LVDTs used ((A+B)/2 and (C+D)/2) were obtained through an average channel unit which provided input to the other two channels utilised in the MacLab. The scheme presented in Fig. 5-23 shows how the experimental data were obtained.
"Chart" was the software used on the MacLab which recorded the readings of the three channels used simultaneously (one for load data and two for average displacement data). Also, the corresponding time at which each sample of data obtained from the three channels used, was registered. Although the user of this software could not see the actual curves while tests were being performed, it allowed data to be recorded, which made it possible for the load vs displacement curves to be obtained.

5.2.3 - Presentation and discussion of the main test results

Data and testing results of control specimens corresponding to each mix used to make the composite specimens are presented in Tables 7 and 8. The average maximum load for specimens made of mix M₁ was \( \overline{N}_{M_1} = 72.4 \) KN and \( \overline{N}_{M_2} = 19.4 \) KN for those made of mix M₂. Tables 7 and 8 present the results obtained from direct readings of the Denison loading scale.

<table>
<thead>
<tr>
<th>Table 7 - Data and testing results of control specimens made of mix M₁</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Main Tests</strong></td>
</tr>
<tr>
<td>Group of tests Index</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

* Direct reading of the Denison loading scale.

Conventions: CIX | CI X | control specimen made of mix M₁, sequential letter

Notes: 1. (50x50x150) mm control specimens made of mix M₁.
2. Mix M₁ - mortar 1:3 by weight, w/c = 0.5.
Table 8 - Data and testing results of control specimens made of mix M₂

<table>
<thead>
<tr>
<th>Group of tests Index</th>
<th>Control Specimen Index</th>
<th>Maximum Load *(KN)</th>
<th>Average Load *(KN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C2A (C1)</td>
<td>17.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C2B (C2)</td>
<td>19.3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C2C</td>
<td>22.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C2D</td>
<td>21.8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>C2E (C3)</td>
<td>19.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C2F</td>
<td>19.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>C2G (C4)</td>
<td>17.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C2H (C5)</td>
<td>18.4</td>
<td></td>
</tr>
</tbody>
</table>

* Direct reading of the Denison loading scale.

Conventions: C2X C2 - control specimen made of mix M₂
X sequential letter
CY control specimen selected for the drawing of graphs
C sequential number
Y sequential number

Notes: 1. (50x50x150) mm control specimens made of mix M₂.
2. Mix M₂ - mortar 1:6 by weight, w/c = 1.08.

As expected, failure patterns of control specimens under compressive tests were characterized by diagonal cracks (see Fig. 5-24).

FIGURE 5-24 - Photo of the characteristic failure mode of 50x50x150 mm control specimens
Table 9 - Composite Specimen Data and Testing Results

<table>
<thead>
<tr>
<th>Group of tests Index</th>
<th>Composite Specimen Index</th>
<th>Maximum Load * (KN)</th>
<th>Average Load (KN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CP1 (SST1)</td>
<td>18.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CP2</td>
<td>15.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>CP3 (SST2)</td>
<td>23.2</td>
<td>19.8</td>
</tr>
<tr>
<td></td>
<td>CP4 (SST3)</td>
<td>22.6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>CP5 (SST4)</td>
<td>20.6</td>
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</tr>
<tr>
<td></td>
<td>CP6</td>
<td>18.6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>CP7</td>
<td>20.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CP8 (SST5)</td>
<td>19.4</td>
<td></td>
</tr>
</tbody>
</table>

* Direct reading of the Denison loading scale

Conventions: CPX SSTX CP SST X composite specimen composite specimen selected for the drawing of graphs sequential letter

Note: 1. (50x50x150) mm composite specimens made of mixes M1 & M2 (Tables 7 & 8).

Load vs displacement curves of control and SST specimens were obtained by using the data recorded on the Maclab (in volts) transformed into KN and mm regarding the corresponding calibration of load cell and LVDTs as described below.

The calibration of the load cell was obtained by applying load within the 0-50 KN range on the Denison machine and recording the corresponding reading in volts. A linear regression of these readings was obtained and used to convert the data recorded on the Maclab, from volts to KN.

The range of the LVDTs used was ± 2.5 mm = ± 5 V. In order to ensure operation on the linear part of its output curves, the mechanical zero (i.e., the reading of zero volts) had to be set before the start of each test. For the LVDTs used, the mechanical zero corresponded to a distance of 19 mm (see Fig. 5-25).

Based upon characteristic results of load and displacement obtained from tests carried out as described above, the graphs presented in Figs. 5-26 and 5-28 were drawn.
Figure 5-26 presents the load vs displacement curves corresponding to test results of the Slant Shear Test performed on composite specimens: SST1, SST2, SST3, SST4 and SST5.

Note: Data obtained from Table 9.
The graphs presented in Fig. 5-26 show irregular shapes of the initial branches of the load vs displacement curves of composite specimens. These results might be a consequence of non-homogeneous specimens - specimens not only composed of two different mixes but also comprising an exaggerated rough bond between the two different mixes. Some extra points on the falling branches (Fig. 5-26) were obtained due to the accuracy of the data recording system used. However, after the maximum load was reached, the strength of the specimen was still drastically reduced.

It can be noticed that composite specimens failed when tests were carried out on the Denison machine at a slightly higher load than control specimens made of the weak, joint layer filling material. The average result was confirmed when tests were performed both on the Hounsfield and Instron machines. However, as observed in the previous experiments (see Sections 5.1 and 5.2), the failure process did not take place inside the joint layers but along one of the joint lines (see Fig. 5-27).

FIGURE 5-27 - Photo of the characteristic failure mode of 50x50x150 mm SST specimens
Figure 5-28 presents the load vs displacement curves corresponding to results of compressive test carried out on control specimens C1, C2, C3, C4 and C5.

Figure 5-28 shows the complete load vs displacement curves including the appropriate falling branch, presenting the known shape (Barnard, 1964 and others) of monolithic concrete (mortar) specimens. This fact assures that curves of composite specimens were also correctly obtained. Hence, as a result of an accurate system of recording load and displacement data together with suitable apparatus (see Section 5.2.2), the appropriate shapes of the falling branches of both control and composite specimen were obtained.

In order to investigate the obtained load vs displacement curves and to make the comparison between control and composite specimen results clearer, particular points of each curve were considered and presented in Tables 10 and 11. The target of the experiments here described, was the analysis of the falling branches. Hence, points at the top and base of the falling branch of each curve obtained were
considered (marked points on curves presented in Figs. 5-26 and 5-28) together with the corresponding time data recorded. Figure 5-29 presents one example curve of an SST specimen (SST2) and a control specimen (C4) showing the complete set of points of each mentioned curve. It includes the marked points also represented on the graphs in Figs. 5-26 and 5-28. But it excludes lines connecting points in order to make the comparison of the above mentioned curves clearer.

### Table 10 - SST Specimen Data

<table>
<thead>
<tr>
<th>SST specimen</th>
<th>Falling branch - top</th>
<th>Falling branch - base</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Load (KN)</td>
<td>Displ. (x 10^-3 mm)</td>
<td>Time (s)</td>
</tr>
<tr>
<td>SST1 (CP1)</td>
<td>18.15</td>
<td>243.75</td>
<td>20.85</td>
</tr>
<tr>
<td>SST2 (CP3)</td>
<td>23.00</td>
<td>217.50</td>
<td>83.27</td>
</tr>
<tr>
<td>SST3 (CP4)</td>
<td>22.35</td>
<td>483.75</td>
<td>75.40</td>
</tr>
<tr>
<td>SST4 (CP5)</td>
<td>19.90</td>
<td>470.00</td>
<td>69.65</td>
</tr>
<tr>
<td>SST5 (CP8)</td>
<td>19.30</td>
<td>494.50</td>
<td>28.93</td>
</tr>
<tr>
<td><strong>AVERAGE</strong></td>
<td>20.54</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Obs.: Data recorded on the MacLab computer (Main Tests)

### Table 11 - Control Specimen Data

<table>
<thead>
<tr>
<th>Control Specimen</th>
<th>Falling branch - top</th>
<th>Falling branch - base</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Load (KN)</td>
<td>Displ. (x 10^-3 mm)</td>
<td>Time (s)</td>
</tr>
<tr>
<td>C1 (C2A)</td>
<td>17.41</td>
<td>349.50</td>
<td>109.00</td>
</tr>
<tr>
<td>C2 (C2B)</td>
<td>19.19</td>
<td>354.25</td>
<td>48.15</td>
</tr>
<tr>
<td>C3 (C2E)</td>
<td>19.04</td>
<td>444.50</td>
<td>54.43</td>
</tr>
<tr>
<td>C4 (C2G)</td>
<td>17.20</td>
<td>453.25</td>
<td>19.83</td>
</tr>
<tr>
<td>C5 (C2H)</td>
<td>18.38</td>
<td>506.75</td>
<td>45.18</td>
</tr>
<tr>
<td><strong>AVERAGE</strong></td>
<td>18.24</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Obs.: Data recorded on the MacLab computer (Main Tests)
Considering the SST data, for a fall in load of about 20 KN, a displacement of 0.45 mm took place in the space of approximately 1 second. In the control specimens, for a fall in load of 17 KN, a displacement of nearly 0.9 mm was recorded in the space of approximately 6 seconds.

The results obtained show that, on average, for a similar fall in load, composite specimens subjected to the slant shear test developed half the displacement of control specimens subjected to the compressive test and this happened in one sixth of the time.

The graphs presented in Fig. 5-29 emphasise this result, as, not only was much less data obtained for the SST specimen than for the control specimen in the interval considered, but also the difference regarding displacement is clear.

**FIGURE 5-29 - LOAD VS DISPLACEMENT CURVES**

50 X 50 X 150 mm SST and control specimens
Example Curves
As a summary, the following comments can be made:

(i) the apparatus used in the main tests allowed the load vs displacement curves of both SST and control specimens to be obtained fully.

(ii) the different response of specimens comprising joints and of monolithic specimens under the same test conditions and the same type of loading was pointed out, not only through the shape of the corresponding load vs displacement curves (Figs. 5-26 and 5-27) but also through the recorded time, load and displacement data (Tables 10 & 11).

(iii) the obtained shape of the control specimen load vs displacement curves presented the known shape of monolithic specimens under the uniaxial compressive test with gradual loss in strength after the maximum load was reached (Fig. 5-28). On the other hand, the plotted results of SST specimens showed a very rapid loss in strength after the maximum load was reached (Figure 5-26).

(iv) comparison of time, load and displacement data corresponding to the top and bottom of the falling branches both of SST and control specimens (Tables 10 & 11), emphasised the different behaviour presented by these two types of specimen. In the considered interval, for a similar fall in load SST specimens developed half the displacement of control specimens and it occurred in one sixth of the time.

(v) Finally, results of main tests also demonstrated a brittle response.

5.3 - CONCLUSIONS

Material performance was assessed in this experimental investigation through the analysis of the failure mechanism and load vs displacement curves of specially designed SST specimens. The summary of the principal procedures and results are
reported and discussed below.

Special composite specimens comprising joint layers of low strength material at an inclination of 30° to the uniaxial compressive stress direction were created as described in Sections 5.1.1 and 5.2.1. The purpose of introducing these joint layers was to provoke failure inside them, where the failure mechanism could be observed more easily. This objective, though, was not attained and the resulting failure process developed along one of the joint lines, in spite of making the above mentioned joint layers much weaker than other parts of the specimens and despite the adoption of exaggerated rough joint surfaces (see Figs. 5-11 and 5-19).

In order to obtain suitable procedures, equipment and mixes to pursue the principal objectives of this experimental investigation some preliminary tests were carried out. Different machines were used and the following comments can be made.

The Hounsfield Tensometer was initially chosen as tests could be performed at a very low loading rate under operator control and load vs displacement graphs could be obtained simultaneously. However, this machine has a great disadvantage owing to its elastic nature. The loading system is based upon a spring beam which stores energy during loading and when maximum load is reached and specimens start to fail, energy cannot be controlled and failure occurs very quickly. As a consequence, the failure process could not be analysed during the performance of tests as it happened too suddenly. Observation of the failure mechanism was possible because some of the tests were recorded on high-speed video, edited at a speed of 1/200 of real time (see photos of important parts of this video in Appendix A). In addition, the right-hand side of the curves obtained of both control and SST specimens revealed instantaneous collapse (see Fig. 5-14) rather than gradual unloading with increasing deformation. Hence, other machines also had to be used.
In order to check the results obtained from tests performed on the Hounsfield Tensometer and to overcome the great disadvantage of its elastic loading system, Instron model 1114 (see Section 5.2.1) was used. This machine comprises a mechanical loading system and has a plotter which automatically draws load vs displacement graphs while tests are being performed.

Resulting failure mechanisms of SST specimens performed on the Instron machine also took place along one of the joint lines. But, in this case, failure could be seen while tests were being carried out and it could be observed more clearly on the video produced at a speed of 1/30 of real time (photos - Appendix A). Some data was provided after the maximum load was reached represented by the falling branches shown on load vs displacement curves obtained for both composite and control specimens (Fig. 5-15). However, at least for the monolithic specimens, the shapes of the curves were not the ones expected (Fig. 5-16). In order to obtain complete falling branches, tests would have to be carried out at a constant straining rate and the testing machine would have to be stiff enough to allow load to fall off in a failing specimen (Barnard, 1964).

As there was still doubt about the alignment of loading and that any eccentricity might have affected the specimen failure pattern, further tests were carried out on Denison, model T42.B4, which applies compression through a ball seat. Failure mechanisms found in specimens tested on the Denison machine also occurred along one of the joint lines, repeating previous results. Some of the tests were also recorded on video, edited at a speed 1/30 of real time (photos - Appendix A).

As previously indicated, the speed at which the video was produced for this experimental investigation depended upon the equipment used. However, features of the failure mechanism obtained from tests performed on the different machines were
similar, regardless of the speed. The failure process always began suddenly, revealing a line along the failing joint. The failure process progressed by breaking small pieces of the specimen, releasing the joint bond and allowing the upper part of the specimen to slide along the yield line until it fell off.

In order to conceive an appropriate experiment able to monitor and record load and displacement data accurately, special apparatus had to be designed for the main tests. As part of the apparatus, Denison machine (model T42B4) was used and specially arranged as described in Section 5.2.2. In summary, arrangements included the use of four LVDTs and a 50 KN load cell giving input to a Macintosh computer which allowed load vs displacement curves to be obtained fully.

The same type of composite specimen as the one used in the previous experiments was produced. The preparation and testing procedures of the specimens are presented in Section 5.2.1. A further improvement was made through the use of steel precision moulds for a more accurate production of specimens.

As a result of an effective system of recording load and displacement data together with suitable apparatus, the appropriate shapes of the falling branches of both control and composite specimens were obtained (see Figs. 5-26 and 5-28). It was now revealed that load vs displacement curves of monolithic specimens were different from the ones obtained for composite specimens.

The shape of the load vs displacement curves of composite specimens obtained from the main tests (Fig. 5-26) was not as sharp as the curves previously obtained when the Instron machine was used (Fig. 5-15). Nevertheless, the curves obtained in the main tests also demonstrated that the behaviour of specimens comprising joints at 30° to the uniaxial compressive load direction, characteristic specimen of the SST, was close to an entirely brittle behaviour with a very rapid loss in strength at failure.
The shape of the curves achieved for the control specimens (Fig. 5-28) were the appropriate and known shape of curves obtained for monolithic specimens under compressive loading (Fig. 5-16). This ensures that the curves of the composite specimens were also adequately obtained.

The analysis of load vs displacement curves of control and SST specimens was also made by comparing time, load and displacement data corresponding to the top and bottom of the falling branches (see Tables 10 and 11). Results showed that, on average, for a similar fall in load, composite specimens subjected to the slant shear test developed half the displacement of control specimens subjected to the compressive test and in one sixth of the time.

To summarize, it can be said that in spite of providing conditions which most encourage a plastic response, experimental results demonstrated that behaviour of Slant Shear Test specimens is close to an entirely brittle behaviour. As a consequence, the use of upper bound plastic methods to analyse the SST seems inappropriate.

In view of the results of this experimental investigation, it was necessary to identify to what extent ductility is required by SST specimens to allow redistribution of stress from a lower bound, at first material yield satisfying equilibrium everywhere, to a fully developed failure mechanism. For this purpose, lower bound solutions were considered by using finite element (FE) methods, as presented in the next chapter. It must be emphasised that it was not the objective of the lower bound consideration to provide a quantitative prediction of the results encountered in the experimental work. It was rather an attempt to provide a qualitative indication of the stress distribution at lower bound loads.
CHAPTER 6

Lower bound analysis of SST specimens

6.1 - INTRODUCTION

The upper bound method applied to SST specimens was examined fully in Chapter 4. This study covered missing points encountered in previous applications, such as restrictions regarding the flow vector direction (see Chapter 3) by providing alternative upper bound solutions. However, the theoretical approaches, both presented and derived in this work, were unable to make predictions matching the reported test data of previous researchers. The suitability of such approaches was not confirmed. This outcome led to further investigations into material performance. It was essential to identify whether stresses could be redistributed within a specimen subjected to the SST for the validity of the application of limit analysis.

For this purpose, a programme of laboratory work was carried out using specially designed specimens and apparatus to encourage and to monitor plastic response (Chapter 5). Experimental results though, demonstrated virtually no ductility. It was shown that the behaviour of SST specimens is close to an entirely brittle behaviour with a very rapid loss in strength at failure.

Not only is material ductility required to enable an upper bound fully developed plastic mechanism but also sufficient ductility to allow full redistribution called for in the structure from a lower bound approach (at first material yield satisfying equilibrium everywhere) to a fully developed plastic mechanism. In order to identify to what extent ductility is required by specimens subjected to the SST, lower bound solutions were considered. For this purpose, finite element (FE) methods were used, as described in this chapter.
The modelling of SST specimens was designed for two distinct conditions. In order to provide some information about practical situations of repair services, such as the use of mismatched repair materials, a SST specimen comprising concrete and a repair material with Young's modulus which was substantially different from the original concrete, was considered. In addition, a similar specimen to the ones used in the main tests of the experimental investigation was modelled. The distribution of stress from a finite element analysis was carried out to show the approximate level of ductility required by such specimens at first material yield.

LUSAS was the software used for the finite element modelling and analysis combined with the MYSTRO program which provides the graphical output. A brief description of the modelling of the material behaviour and structural discretisation is given together with the presentation and criticism of the "nonlinear concrete material properties" used. Results of the finite element analysis are then presented and discussed.

6.2 - MODELLING OF SST SPECIMENS

The process of discretisation of a structure into finite element models comprises the definition of size, shape and composition of model, supporting and loading conditions and type of behaviour to be modelled, based upon the real structure. In the following sections, the procedures to define the finite element models used in this work are described.

6.2.1 - Shape and composition of the SST models

As explained in the introduction, two different SST specimens were modelled. The model which reproduces the situation of incompatible repair materials will be referred to here as Mesh A and the one which emulates the specimens used in the main tests will be referred to here as Mesh C.
6.2.1.1 - Model for incompatible repair materials - Mesh A

The arrangements comprising elements, supporting and loading conditions of Mesh A are shown in Fig. 6.1.
Features of Mesh A, such as external size, shape of elements (except along joint line) and loading conditions, are attributed to Wall & Shrive (1988) who provided a FE linear analysis of SST specimens with the purpose of verifying the factors affecting the bond between new and old concrete. The definition of elements used in Mesh A is presented as follows. Four noded two-dimensional plane stress, isoparametric quadrilateral membrane elements (QPM4 - total 154 elements) were used to represent both the concrete and the repair material. Also, some three noded two-dimensional plane stress, isoparametric triangular membrane elements (TPM3 - total 24 elements) were used to cover the intersection between the 30° inclined lines, parallel to the joint line and the quadrilateral elements.

An almost uniform, concentrated elemental mesh was used for the construction of the model except at both ends of the joint line, where a concentration of stress could be expected and therefore, a more refined mesh was utilised.

As it could be supposed that failure would occur along the joint line, the choice of quadrilateral elements (QPM4) parallel to the joint (along both sides of the joint line) was made because this type of element provides output of stress and strains at each node. Whereas triangular elements only give average results at the centre of gravity of the element. In this case, more data and, as a consequence, more precision would be provided for the definition of the contour of stresses along the joint line.

However, the link between the concrete and the repair material was considered as presenting a perfect or infinitely strong bond and no special elements or parameters were used to define the characteristics (friction and cohesion) of the joint. This choice was made because a perfect bond would be the ideal performance of any joint pursued in practice. In addition, friction and cohesion are not only difficult parameters to determine, but also, they would only apply for quite a specific problem as they vary from case to case.

6.2.1.2 - Model for the specimens used in the main tests - Mesh C

The arrangements comprising elements, supporting and loading conditions of Mesh C are shown in Fig. 6.2.
The SST specimens used in the main tests of the experimental investigation carried out in this research (see Fig. 5-9 - Chapter 5) were modelled using Mesh C. The test pieces consisted of composite specimens comprising a 13 mm wide joint layer at an angle of 30° to the uniaxial compressive stress direction, made of mortar whose composition is referred to here as $M_2$ (low strength material) and the other parts of the specimen were made of mortar whose composition is referred to here as
M₁ (stiff material). In order to verify whether the choice of mesh discretisation affected finite element results, Mesh C, as described below, was also considered for the modelling of the monolithic specimens, composed solely of a single mix.

A similar element topology to Mesh A was adopted for Mesh C. In Mesh C, quadrilateral elements (QPM4 - total 166 elements) were used to represent both low strength and stiff materials. Triangular elements (TPM3 - total 24 elements) were used to cover the intersection between the 30° joint lines and the quadrilateral elements made of mix M₁.

As in Mesh A, an almost uniform, concentrated elemental mesh was used for the construction of the SST experimental model except at both ends of the joint layer where a more refined mesh was utilised.

As explained in Section 6.2.1.1, quadrilateral elements provide more accuracy for the definition of the contour of stresses than triangular elements, as more output data is furnished. Considering that, in this case, failure would occur inside the joint layer, just quadrilateral elements (QPM4) parallel to the joint lines were chosen for the composition of the joint layer of Mesh C.

Similarly to Mesh A, the link between weak and strong materials was considered as presenting a perfect or infinitely strong bond and no special elements or parameters were used to define the characteristics (friction and cohesion) of the joints. In this case, the choice could be justified simply due to the fact that exaggerated rough joint surfaces were adopted in the experimental investigation as an attempt to achieve a performance close to that of a perfect bond.

6.2.2 - Supports and loading conditions

The choice of supports was made to reproduce the frictional end conditions that occur in SST specimens caused by contact with the top and bottom platens. Hence, support conditions consisted of suppressing horizontal movements (x direction in the global axis of the model) both at the top and at the bottom of the models (top nodes:
183 to 193 - Mesh A; 196 to 206 - Mesh C and bottom nodes: 1 to 11 - Meshes A and C) (see Figs. 6-1 and 6-2). Vertical movements (y direction in the global axis of the model) were also suppressed at the bottom nodes. Restriction on vertical movements at the top nodes was just made to control uniform displacements as part of the incremental loading system.

It is not possible to reproduce the specimen end conditions exactly as platten rotation during the test were not measured. This becomes a more important consideration for Mesh A. Fig. 6-2(a) shows an exaggeration of the anticipated platten displacements for a specimen comprised of materials of different stiffness.

\[ E_1 > E_2 \]

**FIGURE 6-2(a) - Exaggeration of platten displacement for specimens comprised of materials of different stiffness.**

The support displacement assumptions used in the model provide end reactions consisting of direct forces and moments and affect the distributions of stress along the joint. Mesh A would represent a specimen of two different materials in an experiment performed on the Instron testing machine which has no ball-seat.

An analysis assuming nonlinear material properties was used (see Section 6.2.3). For nonlinear analysis, since it is no longer possible to directly obtain a stress distribution which equilibrates a given set of external load, i.e. if the stresses integrated through the volume of the structure do not satisfy equilibrium with the external forces, a solution procedure is usually adopted in which the total required load is applied in a number of increments. Within each increment, a linear prediction of the nonlinear response is made, and subsequent iterative corrections are made to
restore equilibrium by the elimination of the residual or *out of balance* forces. The iterative corrections are referred to some form of *convergence* criteria which indicates to what extent an equilibrate state has been achieved. Such a solution procedure is therefore commonly referred to as an *incremental-iterative* method.

The nonlinear solution utilised for the analysis of SST models, presented in this work, was based upon the modified Newton-Raphson procedure. The way in which loads were increased was to apply incremental displacements at the top nodes (183 to 193 - Mesh A; 196 to 206 - Mesh C) of the finite element structure which are adequate conditions under which the SST can be simulated.

### 6.2.3 - Modelling of material behaviour

Considering the difficulties of a proper mathematical modelling of concrete due to the complex and variable chemical composition of concrete mixes, the choice of a concrete model must be made cautiously. Also, close observation of the limitations of the adopted model (described in the following sections) must be made when analysing the results. In view of these mentioned restrictions, "nonlinear concrete material properties" developed for plane stress applications was adopted. Features and limitations of using this concrete model are presented as follows.

#### 6.2.3.1 - General description of LUSAS nonlinear concrete material properties

The nonlinear concrete constitutive model represents the nonlinear material effects associated with the cracking of concrete. This concrete model includes the effects of strain softening ("tension-stiffening") and shear retention. However, the aspect of concrete behaviour related to concrete crushing is not taken into account. Hence, in the region of compression the material is assumed to behave elastically. Concrete cracking is defined by the tensile cracking model as described in Section 6.2.3.2. The characteristic stress vs strain curve of this model is shown in Fig. 6-3.
6.2.3.2 - Tensile cracking model

The single cracking model was adopted in this analysis, cracking is assumed to occur when the major principal stress exceeds the specified tensile strength of concrete \( f_t \). Crack plane(s) form in the direction(s) normal to the tensile principal stresses and are represented by the smeared crack approach. In this method a single discrete crack is represented by a number of finely spaced or smeared cracks. Once cracks form they are assumed to remain fixed throughout the remainder of the analysis, however, they may open or close in response to load reversals.

The post cracking response represented by a gradual release of stress from cracked concrete modelled by a strain-softening curve (see Fig. 6-3) is controlled by the specification of a softening parameter, \( \alpha_s \). This relates the initial cracking strain \( \varepsilon_{cr} \) to the ultimate tensile strain \( \varepsilon_{ult} \) by

\[
\varepsilon_{ult} = \alpha_s \times \varepsilon_{cr}, \quad \text{where} \quad \varepsilon_{cr} = \frac{f_t}{E}
\]

Nodes at which the strain is in excess of this ultimate tensile strain transmit no normal stress.

Typical values of the softening parameter \( \alpha_s \) suggested in the LUSAS "Theory Manual" (1990) range from 5 to 50 with a recommendation for lower values of \( \alpha_s \) where brittle responses are expected to occur. After a sensitivity analyses for \( \alpha_s \) it was observed that results were not influenced by the choice of this parameter. Hence, the softening parameter was simply specified to be \( \alpha_s = 10 \), in this FE work.
6.2.3.3 - Shear retention factor

After cracking occurs, the shear transfer between cracked surfaces is reduced but it is not completely lost owing to aggregate interlock. This reduction is modelled in the LUSAS program by a constant shear retention parameter, $\beta_R$, which modifies the in-plane shear modulus, $G$, by the following condition:

$$\tau_{xy} = \beta_R \cdot G \cdot \gamma_{xy} \quad 6-2$$

where $\tau_{xy}$ and $\gamma_{xy}$ are the in-plane shear stress and the in-plane shear strain, respectively, and $\beta_R$ is defined by

$$\beta_R = 1 \quad \text{if} \quad \varepsilon < \varepsilon_{cr} \quad 6-3a$$

$$\beta_R = \beta_{R(\text{input})} \quad \text{if} \quad \varepsilon \geq \varepsilon_{cr} \quad 6-3b$$

As the material utilized to make the specimens used in the experimental programme was mortar which has a substantially lower aggregate interlock than concrete due to its composition, a low value for the shear retention parameter was specified, $\beta_R = 0.3$. Different values for the shear retention parameter were tried before the definitive finite element analysis was carried out, as presented herein. Results were not affected by the choice of this parameter, in the case at hand.

6.3 - EXAMPLE INPUT DATA FOR ANALYSIS

Typical input data files (LUSAS User Manual, Vol. 1, Version 11, 1993) are provided in Appendices B and C. The input data used for the analysis of Mesh A (Model for incompatible repair materials) is presented in full in Appendix B and of Mesh C (Model for SST specimens used in the main tests) is shown in Appendix C.

6.4 - PRESENTATION AND DISCUSSION OF THE FE ANALYSIS

The significant results of the finite element (FE) analysis found for Mesh A are shown in Section 6.4.1 and are presented in Section 6.4.2 for Mesh C.

6.4.1 - Mesh A - FE results and criticism
Figures 6-4, 6-5(a) and 6-6(a) show the resulting maximum principal (SMax) stress distribution of Mesh A at prescribed displacements at top nodes of 0.07 mm, 0.09 mm and 0.10 mm, respectively, and Figs. 6-5(b) and 6-6(b) present the corresponding cracking pattern.

**FIGURE 6-4 - Maximum principal stress distribution**

Mesh A - incompatible repair materials
prescribed displacement at the top nodes = 0.07 mm
FIGURE 6-5 - (a) Maximum principal stress distribution; (b) cracking pattern

Mesh A - incompatible repair materials
prescribed displacement at the top nodes = 0.09 mm
CONTOURS OF SMax
A -1.351
B -1.017
C -0.6833
D -0.3494
E -0.1539E-01
F 0.3186
G 0.6525

FIGURE 6-6 - (a) Maximum principal stress distribution; (b) cracking pattern
Mesh A - incompatible repair materials
prescribed displacement at the top nodes = 0.10 mm
Although all the presented contours of maximum principal stress are quite similar, some small changes in shape can be observed after the start of the cracking process, as prescribed displacement increases, for example, on contours D and E above the joint line. It can be said that if it were a purely elastic analysis, exactly the same contours re-labelled would be found. Changes in the shape of stress contours mean that some convergence procedure was necessary to find the solutions after the cracks began.

Maximum principal stress contours show concentrations of stress in two zones of the model: (1) at the bottom end of the line parallel to and above the joint line and (2) in a region below the joint line, within the low strength material. These stress contours represent average nodal results. The cracking pattern, though, just exposes areas of the model where unaveraged nodal maximum tensile stresses exceed the minimum specified tensile strength of the model ($f_{t2}$). This fact shows why it is not possible to identify the beginning and the propagation of cracks with zones of stress concentration of maximum principal stress contours.

In order to observe the stress distribution along the joint line (where cracks were represented) and along the line parallel to and above the joint line (where the highest average principal stress was found), example profiles were drawn for a prescribed displacement at the top nodes of 0.09 mm, as presented in Fig. 6-7.

Along the joint line (Curve A-A - Fig. 6-7), tensile stresses are observed at both ends of the joint line presenting higher values at the bottom where the first cracking was predicted. However, failure in this model (incompatible repair materials, joined at 30° with a perfect bond) would start in the stiff material, at the bottom of a line parallel to and above the joint line as demonstrated by Curve B-B (Fig. 6-7). This outcome is expected owing to the variation in specimen stiffness across its width; the material represented by the elements in the upper part of the mesh shown in Fig. 6-1 is stiffer than the remainder and therefore the specimen stiffness is at its greatest on the left-hand side. It also corroborates an empirical interpretation of the failure mode of SST specimens comprising concrete and a repair material of lower elastic modulus presented by Tabor (1978). In this case, this type of failure (diagonal failure in the concrete parallel to the joint) was considered to indicate an excellent bond.

It can be said that both stress profiles (Curves A-A and B-B - Fig. 6-7) present quite an uneven profile of stresses which would require material ductility to give redistribution of stresses, chance to occur.
6.4.2 - Mesh C - FE results and criticism

Figures 6-8, 6-9(a) and 6-10(a) show the resulting maximum principal stresses (SMax) of Mesh C at prescribed displacements at top nodes of 0.0255 mm, 0.026 mm and 0.02625 mm, respectively, and Figs. 6-9(b) and 6-10(b) present the corresponding cracking pattern.

Similarly to Mesh A, resulting contours of the maximum principal stress of Mesh C show some changes in shape after the start of the cracking process, as the prescribed displacement increases, for example, on contours F (at joint lines) and G (inside joint lines). Hence, it is clear that some convergence procedures must have been necessary, to find the solutions after cracking had started.

Maximum principal stress contours - represented by average nodal results - show high concentrations of stress inside the joint layer made of the low strength material. As previously commented, cracks are identified with parts of the model where unaveraged maximum tensile stresses exceed the minimum specified tensile strength of the model (f_{ct}). As a consequence, the beginning and the propagation of
cracks, encountered for Mesh C, were predicted at the joint lines and not inside the joint layer where the average maximum principal stresses were found (see Figs. 6-9 and 6-10).

CONTOURS OF SMαx

- A: -0.7963
- B: -0.6111
- C: -0.4258
- D: -0.2406
- E: -0.5531E-01
- F: 0.1299
- G: 0.3152

FIGURE 6-8 - Maximum principal stress distribution
Mesh C - experimental SST specimen
prescribed displacement at the top nodes = 0.0255 mm
CONTOURS OF $S_{\text{Max}}$

- A: -0.8114
- B: -0.6213
- C: -0.4312
- D: -0.2411
- E: -0.5096E-01
- F: 0.1391
- G: 0.3292

**FIGURE 6-9** - (a) Maximum principal stress distribution; (b) cracking pattern

Mesh C - experimental SST specimen

prescribed displacement at the top nodes = 0.0260 mm
CONTOURS OF S_{\text{Max}}

A - 0.8217
B - 0.6348
C - 0.4480
D - 0.2611
E - 0.7423E-01
F  0.1126
G  0.2995

FIGURE 6-10 - (a) Maximum principal stress distribution; (b) cracking pattern

Mesh C - experimental SST specimen
prescribed displacement at the top nodes = 0.02625 mm
In order to observe stress distribution along the joint line (where cracks were represented) and along the middle joint layer (where the highest average principal stress was found), example profiles were drawn for a prescribed displacement at the top nodes of 0.02625 mm, as presented in Fig. 6-11.

Along the bottom joint line (Curve B-B - Fig. 6-11), concentrations of stress are observed where the cracks appeared (marked points on Curve B-B - Fig. 6-11) which correspond to the cracking pattern represented in Fig. 6-10(b). Failure in this model (comprising a perfect bond) started in the middle and half way along the joint layer made of the low strength material, as demonstrated by Curve A-A (Fig. 6-11). However, in the main tests of the experimental investigation (Chapter 5), SST specimens failed through one of the joint lines proving the weakness of the joint section.

As in the case of FE results encountered for Mesh A, the profiles of stresses found for Mesh C (Curves A-A and B-B Fig. 6-11) present a great range between the maximum and the minimum stress. Lower bound analyses showing large differences of stress over all the plastic zones required for a full mechanism, indicate that great ductility is required to permit an upper bound with fully developed plastic yielding.
However, as reported in Chapter 5, experimental results demonstrated that the behaviour of SST specimens is close to an entirely brittle behaviour, as virtually no ductility was found during the failure process of such specimens.

6.5 - CONCLUSIONS

In the present chapter, lower bound analyses of SST specimens were made by the use of finite element methods. LUSAS software (LUSAS FEA Ltd., 1993) together with MYSTRO, the graphical output program, were used for the modelling of SST specimens. As described in the introduction of this chapter, two different conditions were considered: the case of mismatched repair materials (Mesh A) and the specially designed specimen (Mesh C) used in the main tests of the experimental investigation (Chapter 5). The main purpose was to identify the distribution of stress within SST specimens to show the level of ductility required by such specimens at first material yield.

Contours of maximum principal stresses resulting from the nonlinear FE analysis and corresponding to the beginning of the yield process were presented for both meshes under analysis. Profiles of maximum principal stresses along lines where the highest principal tensile stresses were predicted were likewise presented.

Maximum principal stress distributions referring to Mesh A - which consider the case of a repair material with lower elastic modulus than the original concrete - show two zones of stress concentration: (1) at the bottom end of the line parallel to and above the joint line and (2) in a region below the joint line, within the repair material (Figs. 6-5 and 6-6). Failure in this model started in the stiff material in the stress concentration zone (1), referred to above, as demonstrated by Curve B-B in Fig. 6-7. This result corroborates an empirical interpretation of the failure mode of SST specimens, similar to the ones considered in Mesh A, as presented by Tabor (1978). In this case, this type of failure (diagonal failure in the concrete parallel to the joint) was considered to indicate an excellent bond.
A high concentration of stress was found for Mesh C inside the joint layer made of the low strength material (see Figs. 6-9 and 6-10). Failure of this model (comprising a perfect bond) started in the middle and half way along the joint layer, as shown by Curve A-A in Fig. 6-11. In the main tests though, SST specimens failed through one of the joint lines proving the weakness of the joint section.

Finally, the principal outcome of the FE analysis is summarized as follows. Maximum principal stress profiles, drawn along failure zones at first material yield, showed a great range between the minimum and the maximum stresses (see Fig. 6-7 (Mesh A) and Fig. 6-11 (Mesh C)). In this case, a great level of ductility would be required to allow an upper bound with a fully developed plastic yielding. However, experimental results have shown that SST specimens behave almost like a brittle material with virtually no ductility.
CHAPTER 7
Conclusion

7.1 - THESIS REVIEW

The use of the theory of plasticity in the interpretation of SST data has been investigated in this thesis.

The SST has been well accepted in the technical community as a meaningful testing method which has been applied to access the efficacy of bonding systems. This test has been used both in new structures and in repair services, particularly when structural performance is important. Nevertheless, a theoretically supported analysis of SST results is still required.

Although originally derived for ductile materials such as metal, the theory of plasticity has also been applied to other types of materials such as concrete. In this case, knowledge of such a material at yielding is required. As plain concrete presents a behaviour closer to that of a brittle material, a brief discussion about behaviour of ductile and brittle materials and applicable failure criteria was included in Chapter 2. In order to support the investigation into the theory of plasticity, the principal theorems of limit analysis and basic concepts such as the yield condition, the rigid, perfectly plastic material and the plastic potential flow were likewise presented in Chapter 2.

A literature review was made in order to identify previous work on limit analysis applied to bodies of concrete-like material under uniaxial compressive stress which is the stress condition that occurs when a specimen is subjected to the SST. Some applications of the upper bound theorem were found which were reported in Chapter 3.
It was observed that Coulomb’s (modified) criterion was considered, in previous work, as the *yielding* condition and that the plastic flow rule was applied to this criterion.

The consideration that the plastic vector direction is equal to the angle of friction, ($\alpha = \phi$), assumed in previous work for plane strain applications, was found restrictive as an infinite number of flow vector directions is possible.

Additionally, the use of Coulomb’s criterion as a yield criterion has been subjected to criticism. For example, Heyman (1972) commented that Coulomb’s criterion used as a yield condition is restrictive owing to its linear equation. Schofield & Wroth (1968) objected to using the flow rule with Coulomb’s criterion due to the pore-water pressure, present in soils. Indeed, the main restriction is perhaps the application of the plastic flow rule to Coulomb’s criterion which is, apparently, just a rupture criterion. In the derivation of such a criterion, a sliding failure mode was assumed for a brittle material (a masonry pier) which does not present a plastic flow when failing (see Section 2.1.4).

Facing all these considerations, the need for further research into the upper bound theorem was demonstrated. For this purpose, alternative upper bound solutions were considered, as presented in Chapter 4. First, upper bound solutions were derived observing the full range of flow vector directions allowed by Coulomb’s modified criterion, only partially considered in previous work. In the absence of evidence of a sliding condition, concrete failure is sometimes simply defined by a maximum, direct stress condition represented by the square yield locus. This hypothesis was considered in the derivation of another upper bound solution.

However, theoretical approaches, both those presented and derived in this work, were unable to make predictions matching the reported data of previous researchers. The suitability of these approaches was not confirmed. This outcome led to further investigations into material performance. It was imperative to identify whether stresses could be redistributed within a specimen subjected to the SST for the validity of the application of limit analysis.
As explained in the introduction of this thesis a programme of laboratory work was carried out as an attempt to reproduce the failure pattern assumed in the theory of plasticity. For this purpose, a specially designed specimen was used, partly to encourage a plastic failure and partly, with the aid of specially tailored apparatus, to allow a better visualization of the deformations developed during the failure process (see Chapter 5). Results from the experimental investigation are summarized as follows:

(i) the apparatus used in the main tests allowed load vs displacement curves of both control and SST specimens to be obtained fully;

(ii) different responses of specimens comprising joints and of monolithic specimens under uniaxial compressive stress was pointed out, not only through the shape of the corresponding load vs displacement curves (Fig. 5-29) but also through the recorded time, load and displacement data (Tables 10 & 11);

(iii) load vs displacement curves obtained for monolithic specimens presented the known shape of a concrete-like material under uniaxial compressive stress with gradual loss in strength after maximum load. On the other hand, plotted results of SST specimens showed a very rapid loss in strength after the maximum load was reached.

(iv) comparison of time, load and displacement data referring to the top and bottom of the falling branches both of control and SST specimens, emphasized the different behaviour of these two types of specimen. In the considered interval, for a similar fall in load, SST specimens developed half the displacement of control specimens and in one sixth of the time.

As a conclusion to the programme of laboratory work, it can be said that despite providing the conditions which most encourage a plastic response, experimental results demonstrated that the behaviour of SST specimens is close to an entirely brittle behaviour. As a consequence, using upper bound methods to analyse SST data seems inappropriate.
In view of the almost brittle response of SST specimens, it was necessary to identify to what extent ductility is required to allow redistribution of stress from a lower bound, satisfying equilibrium everywhere, to an upper bound with a fully developed failure mechanism. Lower bound solutions were then considered by using finite element (FE) methods for the purpose of providing a qualitative indication of the stress distribution, as described in Chapter 6. The principal outcome of this study is presented as follows. Maximum principal stress profiles, drawn along failure zones at first material yield predicted by the performed nonlinear FE analysis (Figs. 6-7 and 6-11), showed large differences of stress. Hence, a great level of ductility would be required to allow an upper bound with a fully developed plastic yielding. However, as previously mentioned, SST specimens demonstrated virtually no ductility.

7.2 - CONTRIBUTIONS OF THIS RESEARCH PROGRAMME

The main contributions of this programme of research are pointed out below.

(i) *This thesis extended and complemented the use of the upper bound theorem applied to concrete-like materials under uniaxial compressive stress.*

(ii) For the first time, complete load vs displacement curves were obtained experimentally for SST specimens. This achievement was possible owing to accuracy of the apparatus specially tailored for the programme of laboratory work carried out during this research.

(iii) *Lower bound solutions were considered by the use of a nonlinear FE analysis* for the purpose of identifying the distribution of stress within SST specimens in order to show the level of ductility required by such specimens at first material yield.

(iv) The main results of this work are presented as follows.

Upper bound solutions previously obtained and those developed in this research programme did not produce the results matching the SST data.
Experimental results demonstrated that SST specimens behave almost like a brittle material with a very rapid loss in strength at failure.

Resulting maximum principal stress profiles from the FE analysis which displayed large differences in stress, showed that great ductility would be called for in SST specimens for the redistribution of stress required by a plastic upper bound.

In conclusion, both the results of the theoretical approaches and those of the experimental work demonstrated upper bound plastic methods inadequate for the analysis of SST data. Hence, alternative lines of research must be followed in order to achieve a critical method of analysis and interpretation of SST results. Recommendations are given in the following section.

7.3 - FUTURE WORK

Not only are suggestions about empirical approaches to SST presented in this section, but also the need for future research into failure criteria for brittle materials, particularly concrete, is demonstrated.

7.3.1 - Empirical approaches to SST

Empirical methods should be pursued as a means of comparing SST results and such methods could also be used in the assessment of repair proposals for deciding on suitable applications for new formulations. Tests should then be carried out in sufficient number to satisfy a rigorous statistical analysis and performed in a way which could be reproduced in any other laboratory. For this purpose, conditions such as, the use of a smooth joint surface and a range of joint inclinations should be considered in order to produce a standardized test specimen and for the procedure to be reproducible.
7.3.2 - Alternative further research

Failure in ductile materials is developed by a well-known yielding process characterized by the formation of sliding planes within a metal crystal lattice (see Section 2.1.1). In this failure process, molecule bonds are broken and then remade providing a continued support of stress at high strains. This failure mechanism is properly represented by existing failure criteria, such as those of Tresca and von Mises.

Brittle materials, such as concrete, contain a great number of microscopic cracks in their structure giving rise to stress concentrations and causing local failure of the material. Molecule bonds are then lost by separation and are not remade as in ductile materials. Failure of brittle materials is mainly governed by maximum principal tensile strains and is followed by a complete falling off in stress.

Attempts have been made to model concrete behaviour, for example in FE programs, by the definition of special parameters which could account for the cracking of concrete, the reduction in capacity for shear transfer between cracked surfaces and concrete crushing. A complex range of parameters are generally introduced in such programs in order to replicate stress vs strain curves resulting from compressive tests carried out on concrete specimens under rigid platens. These conditions generally apply and are, therefore, usually satisfactory for most reinforced concrete structures.

Although concrete is known as a brittle material, it displays some ductility under various circumstances, i.e. when restrained by platens or stirrup reinforcement. However, failure of small elements is completely non-ductile. Hence, a more general and appropriate failure criterion for concrete which embodies the features of such a material at failure is still a subject for further research.

A single criterion should then be developed for the material being able to predict the performance of concrete in cubes and cylinders as well as under combined states of stress in structures.
7.4 - FINAL REMARKS

For at least the last two decades, engineering work has been supported by advanced techniques and supplied by an unlimited number of new materials. Nevertheless, factors such as, different types of environmental aggression, misapplications of construction techniques, etc, has caused premature life to concrete structures and, subsequently, the need for repair services. Considering the high cost of such services, safe, durable solutions must be provided.

The durability of concrete repair services relies on the adequacy of the bond at the surface between the old concrete and the applied volume repair. In order to ensure the durability of a bonding system, a testing procedure which could evaluate bond strength is of prime importance. Whenever structural performance is considered, the bond surface must be subjected to a stress condition likely to be encountered in the concrete structure. In this context, the slant shear test has already been accepted as a meaningful testing method by the technical community. Some empirical attempts to interpret SST results have been made without satisfying a rigorous statistical analysis and without considering reproducible specimens and testing procedures. Suggestions for a comprehensive empirical approach are provided in Section 7.3.1.

In order to find a theoretically supported analysis to the SST, the theory of plasticity was considered in this thesis. However, the use of plastic methods was shown to be inappropriate for the interpretation of the slant shear test as a consequence of the brittle failure mode observed in SST specimens.
APPENDIX A

Photos of important parts of the video produced for the experimental investigation

In the second phase of the preliminary tests of the experimental investigation described in Chapter 5, some of the tests performed on machines: Hounsfield Tensometer, Instron and Denison, were recorded on high-speed video. Photographs were taken of frames of this video showing that significant changes occurred at the start of the failure process of the recorded tests. Regardless of the machine used, photographs were taken of SST specimens under compression in two different conditions: (i) just before any visible cracks were identified and (ii) the frame in which the line along the failing joint suddenly appeared.

The digital numbers which appear at the top of the photographs presented in this appendix are explained as follows. The first three digits are the seen code which simply represents the order in which tests were recorded, in sequential numbers. The time display shows the elapsed time during recording in the three columns next to the seen code. It counts time in units of five milliseconds (1/200 secs.) which corresponds to each picture at the recording speed of 200 pictures/sec. The display in each picture therefore changes 005, 010, 015, and so on. In the middle column a unit corresponds to one second and in the left hand side column a unit corresponds to one hundred seconds.

The photographs taken from the above mentioned video are described as follows. Figures A-1 and A-2 show SST specimen HTP12 (Table 6) being tested on the Hounsfield Tensometer. In the space of 5 milliseconds the instantaneous collapse took place owing to the elastic nature of this machine. Figures A-3 and A-4 show SST specimen ITP2 (Table 6) being tested on the Instron machine. In the space of about one second (1.035 seconds) the line along the failing joint was observed. Figures A-5 and A-6 show SST specimen DTP3 (Table 6) being tested on the Denison machine. In the space of about one second (1.035 seconds) the line along the failing joint was observed. The difference in time encountered above is due to the varying stiffness of the machines used.
A.1 - HOUNSFIELD TENSOMETER

Figure A-1 - SST specimen on the Hounsfield Tensometer just before failure.

Figure A-2 - SST specimen showing instantaneous collapse (Hounsfield Tensometer)
A-2 - INSTRON MACHINE

Figure A-3 - SST specimen on the Instron machine (before starting the failure process).

Figure A-4 - SST specimen showing a line along the failing joint (Instron machine)
Figure A-5 - SST specimen on the Denison machine (before starting the failure process).

Figure A-6 - SST specimen showing a line along the failing joint (Denison machine)
APPENDIX B

Input data for Mesh A - LUSAS FEA
SST model for incompatible repair materials

PROBLEM TITLE "NONLINEAR ANALYSIS OF A 100 X 100 X 300 mm COMPOSITE PRISM"

UNITS N mm
OPTIONS 17 18 27 30 44 55 146 187

TPM3 ELEMENT TOPOLOGY

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QPM4 ELEMENT TOPOLOGY

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Appendix B - Input data for Mesh A - LUSAS FEA

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145  146  148  160  158
146  136  147  159  148
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INC  10  11  11  11  11  2
SOLUTION ORDER AUTOMATIC
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INC  11  0  23.04  3
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FIRST  35  0  63.4
INC  1  10  0  11
46  5  72.06
FIRST  47  0  80.72
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FIRST  59  0  98.04
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INC  11  0  17.32  7
136  95  210.62
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INC    1  10  0  11
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TPM3 GEOMETRIC PROPERTIES
1  100  100  100  (depth at each node)

QPM4 GEOMETRIC PROPERTIES
2  100  100  100  100  (depth at each node)

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Example of Material Properties Nonlinear 24

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MATERIAL PROPERTIES NONLINEAR 24

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131 134 1 2
146 146 0 2
33 33 0 1
45 48 1 1
57 58 1 1
68 71 1 1
79 83 1 1
90 95 1 1
101 107 1 1
112 120 1 1
123 130 1 1
135 145 1 1
147 178 1 1

SUPPORT NODES
1 11 1 R R
183 193 1 R R

LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.01 mm
PDSP 2

ELEMENT OUTPUT ASCENDING TITLE NODAL OUTPUT
11 11 0 1
21 24 1 1
33 37 1 1
45 50 1 1
57 62 1 1
69 74 1 1
81 86 1 1
93 98 1 1
105 110 1 1
117 122 1 1
129 134 1 1
142 146 1 1
155 158 1 1
168 168 0 1

NODE OUTPUT TITLE REACT. AT SUPP., LOAD POINT AND DISPL.
183 193 1 3
Appendix B - Input data for Mesh A - LUSAS FEA

NONLINEAR CONTROL
ITER 20 3 .5
NR
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  INC (20)
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  OUTP 20 1 1 0 0
LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.02 mm
PDSP 2
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LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.03 mm
PDSP 2
  183 193 1 0 -0.01
LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.04 mm
PDSP 2
  183 193 1 0 -0.01
LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.05 mm
PDSP 2
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LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.11 mm

PDSP 2

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END

MESH A

100x100x300 mm prism
APPENDIX C

Input data for Mesh C - LUSAS FEA
SST model for the specimens used in the main tests

PROBLEM TITLE "NONLINEAR ANALYSIS OF A 50 X 50 X 150 mm
COMPOSITE PRISM"

UNITS N mm
OPTIONS 17 18 27 30 44 55 146 187

TPM3 ELEMENT TOPOLOGY

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95 96 108 107
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108 111 112 123
116 120 132 131
123 123 124 135
131 132 145 144
133 135 136 137
134 136 148 137
142 145 158 157

See Mesh C
element & node numbers
in page 159.
155 158 171 170
168 171 173 183
169 173 184 183

QPM4 ELEMENT TOPOLOGY

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### QPM4 GEOMETRIC PROPERTIES

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SUPPORT NODES
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  196 206 1 R R

LOAD CASE TITLE PRESCRIBED DISPL. AT TOP TOTAL DISPL. = 0.01 mm
PDSP 2
  196 206 1 0 -0.01

ELEMENT OUTPUT ASCENDING TITLE NODAL OUTPUT
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NODE OUTPUT TITLE REACT. AT SUPP., LOAD POINT AND DISPL.
   196 206 1 3

NONLINEAR CONTROL
ITER 20 3 .5
NR
FIRST MNR
   INC (20)
CONV 0 0 5 0 0
OUTP 20 1 1 0 0
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PDSP 2
  196 206 1 0 -0.00125
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