On the combinatorics of convex polytopes

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Abstract

A *convex polytope* may be defined as the convex hull of a finite set, or as a bounded intersection of closed halfspaces. Its *faces* are defined as its intersection with hyperplanes which do not intersect its relative interior, and the set of these form a *lattice*. While these definitions are purely geometric, the associated *face lattice* is rather more combinatorial. Study of the face lattice inspired the introduction of *abstract polytopes*, which share many of the properties of convex polytopes, but are defined purely combinatorially.

A number of properties of the face lattice of convex polytopes are known, but are found using convexity in some way. It is desirable to ignore the geometric properties such as convexity in order to find purely combinatorial proofs of known results. In three dimensions, for example, the family of polytopes and the family of 3-connected planar graphs are in one-to-one correspondence (due to Steinitz [38]). The face lattice is uniquely determined by the graph of a 3-polytope, and so questions of combinatorics of 3-polytopes may all be answered combinatorially. There is no known analogue of Steinitz's Theorem in higher dimensions however.

A new proof of Balinski's Theorem is given here, which is valid for all abstract poly-

topal lattices. It is conjectured that higher dimensional analogues of Balinski's Theorem hold for certain subsets of polytope lattices, by way of generalizing Menger's Theorem on the connectivity of graphs. A corollary of Menger's Theorem is proved using a construction rather than using Menger's Theorem itself; this suggests that similar constructions may be found to prove our conjecture.

A corollary of the g-theorem is that the h-vector of a simple d-polytope satisfies the equations

$$h_r(P) = h_{d-r}(P)$$

for $r = 0, \ldots, d$, and the inequalities

$$h_{d/2} \geqslant \cdots \geqslant h_d = 1.$$

Stanley conjectured that these inequalities also hold for non-simple polytopes. This was recently proved for polytopes which have at most one non-simple vertex in each facet by Timorin [40], and was settled for all polytopes recently by Karu [11]. Here, the result is proved for polytopes which have at most one non-simple vertex in each face of dimension at most $(d+1)/2$. Although this does not cover the generality of Karu's, the proof uses far more elementary ideas. The proof extends the known algebraic properties of weights to these non-simple polytopes: the Hodge-Riemann-Minkowski (HRM) inequalities and the Lefschetz decomposition extend to describe their polytope algebras.

A possible framework for a simplified proof of the g-theorem is described. The idea is to avoid proving the stronger HRM inequalties, and prove the injectivity of multiplicative maps directly. There seem to be technical difficulties, however, which prevent the proof from working. Nevertheless, the construction is of interest, and it is hoped that the difficulties may be overcome.

The relationship between the face-ring and weights on polytopes is detailed. Many results proved for weights using geometric arguments, may be translated into more algebraic proofs which hold for more general complexes. The purpose of this section is to give some indication of where convexity is used, and where it may be ignored.

If P is a simple d-polytope, with d odd, then the set of 1-weights for which multiplication induces a singular map between $\Omega_{(d-1)/2}(P)$ and $\Omega_{(d+1)/2}(P)$ is shown to be an algebraic surface $Z(P)$ of degree $h_{(d-1)/2}(P)$. The exact form of these surfaces is laborious to compute. Matrices are therefore constructed to allow the calculation of $Z(P)$ by a computer. Some low dimensional examples are given.

Finally, straight-line graphs are introduced. These relatively simple objects may be used to determine information about the type-cones of polytopes. In particular, a result of Smilansky is re-proved: no 3-polytope with more facets than it has vertices has a 1-dimensional type-cone.

Contents

Chapter 1

Introduction

The combinatorial properties of polygons are rather trivial. It is easy to see that a polygon has as many vertices as it has edges, and these edges form a cycle. There is little more to be said in this two-dimensional case. In three or more dimensions, there is a great deal more complexity. The 3-polytopes include the familiar Platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron. The number of vertices of a 3-polytope is not determined by its number of facets. For example, if we slice off two vertices of a cube, to give two triangular facets, then the resulting polytope has eight facets and twelve vertices. The octahedron however has eight facets, but only six vertices. Nevertheless, the combinatorics of 3-polytopes are relatively well understood. Steinitz's Theorem states that a graph is isomorphic to the graph of a 3-polytope if and only if it is planar and 3-connected. Euler's Theorem states that the sum of the number of vertices and the number of facets of a 3-polytope is equal to the number of edges plus two.

In four or more dimensions, visualization becomes more tricky. There are various techniques available. Schlegel diagrams are a way of representing a polytope in one fewer dimension. In three dimensions, for example, the Schlegel diagram of a polytope may be 'drawn' in the plane with no edge crossings. This is the picture you would see if you were positioned close to a facet. In higher dimensions, the same construction is used. We project the $(d-2)$ faces onto a hyperplane from a point close to one of the facets. In four dimensions, we obtain the image of the close facet (a 3-polytope) subdivided into cells, each of which is the image of another facet.

Even in four dimensions, we do not have a result comparable to Steinitz's Theorem. A subdivision of a 3-polytope into cells is not necessarily the Schlegel diagram of a 4-polytope. Indeed, it may be that there is no 4-polytope which shares the combinatorial structure of some particular subdivision.

There are many other questions which may be asked of d-polytopes, which have a positive answer in three dimensions. Is the graph d-connected? Is there an analogue of Euler's Theorem? May we determine how many faces a polytope can have of each dimension? The proofs in three dimensions may be rather trivial using intuitively obvious properties. In higher dimensions, our intuition may be quite misleading, and it is only by reducing our arguments to the bare essentials that we may have any success.

By simplifying proofs as much as we can, we frequently ask whether properties hold for a more general class of objects. For example, the boundary of a polytope is a topological sphere, and so it is natural to ask whether properties which hold for polytopes also hold for all topological spheres. In order even to ask these more general questions, the setting must itself be more general. In Section 4 we shall discuss *abstract polytopes*. These are objects which possess some of the combinatorial properties of polytopes. Any properties of polytopes which also hold for these purely combinatorial objects have proofs which can not rely on geometric properties such as convexity. In this section we re-prove Balinski's theorem, which states that the graph of a d-polytope is d-connected. This proof also holds for far more general *polytope lattices*, and as such is able to avoid any mention of convexity. Perles and Prabhu [29] extended Balinski's Theorem to show that the removal of a family $\mathcal{X}$ of faces of a polytope P, with

$$\sum_{X \in \mathcal{X}} (\dim X + 1) \leqslant d - 1$$

does not disconnect the graph of P. We prove this result for abstract polytope lattices, and extend it to show that, if

$$\sum_{X \in \mathcal{X}} (\dim X + 1) \leqslant d - k,$$

then the removal of $\mathcal{X}$ leaves a strongly connected k-skeleton. Balinski's Theorem states that the removal of d vertices does not disconnect the graph of P, but to show that there are d independent paths between any given pair of vertices, we must appeal to the far more general Menger's Theorem [17]. In his thesis, Lockeberg conjectures an extension of Balinski's Theorem: for any pair of vertices u and v of a polytope P, there are strong chain of faces in the boundary of P, the sum of whose dimension is d, which intersect only at u and v. Our previous result shows that an analogue of Menger's Theorem would supply a proof of this conjecture.

One of the most important breakthroughs in the combinatorics of convex polytopes is the g-theorem, conjectured by McMullen [19] in 1970. This states that for a sequence to be given by the numbers of faces of each dimension of a simplicial polytope, it is necessary and sufficient for it to satisfy a purely combinatorial condition. Sufficiency was proved by Billera and Lee [4, 5], by constructing a suitable polytope for each sequence, which satisfied this combinatorial condition. Necessity was proved by Stanley [36] in 1979 using rather deep algebraic geometry. Our discussion shall not venture into this realm. In 1993, McMullen [24] used the polytope algebra to give a new, more elementary proof. He simplified this further in [25] by introducing weights on polytopes.

The weight algebra is an elegant construction, and in the dual it is a quotient of the *face ring.* We illustrate this property in Chapter 6. There is a connexion between weights on polytopes and mixed volumes, and so some constructions follow quite naturally in the context of weights, but not in the context of the face ring. In Section 5, we present some of these constructions in a more general setting, and show that many of the properties of weights, in particular the separation properties, generalize.

The g-theorem refers only to simple polytopes. A corollary of the g-theorem is the *unimodality condition*

$$1 = h_0(P) \leqslant \cdots \leqslant h_{\lceil d/2 \rceil} \geqslant h_{\lceil d/2 \rceil + 1} \geqslant \cdots \geqslant h_{d-1} \geqslant h_d = 1.$$

In a recent paper [40], Timorin proved that, if P has simple edges, then

$$h_r(P) \leqslant h_{d-r}(P).$$

Further, if each facet of a polytope P has at most one non-simple vertex, then the latter half of these inequalities hold:

$$h_{\lceil d/2 \rceil} \geqslant h_{\lceil d/2 \rceil + 1} \geqslant \cdots h_{d-1} \geqslant h_d = 1.$$

We shall show that the first part is an immediate corollary of the Lefschetz decomposition. We shall also weaken the extra condition from the second part, and give a proof that these inequalities hold for polytopes which have no more than one non-simple vertex in each $\lceil (d+1)/2 \rceil$-face. Our proof generalizes the Lefschetz decomposition and the Hodge-Riemann-Minkowski inequalities of [24].

Karu [11] recently proved that these inequalities hold for general polytopes. His proof deduces properties of a intersection cohomologies on the normal fans of polytopes. If the polytope is simple, then these intersection cohomologies are isomorphic to the weight-spaces. This is not true in general, and so the results that we prove for weights are combinatorially weaker, but are algebraically quite different.

We shall describe a construction which uses weights as in McMullen's proof of the g-theorem, but uses a geometric construction to show how weights on a pyramid over a polytope P restrict to P in a way which mirrors weight multiplication. We propose a method for proving the g-theorem, by considering the intersections of weight-spaces on simple approximants of a pyramid. We note however that there are technical difficulties which are yet to be overcome, to yield a new proof.

All current proofs of the g-theorem find a Lefschetz decomposition of the weight algebra (or an isomorphic quotient of the face ring). This is more than a proof requires. The crucial part is that there is an element of the first grade which, by multiplication, induces an injective map between the middle two grades. We shall calculate exactly which elements have this property (actually it is more interesting to find which elements *do not* have this property) for some examples.

Finally, *straight-line graphs* are introduced. These provide us with useful ways of estimating the dimension of the type-cone of a polytope from combinatorial data. It gives perhaps the most general definition of a type-cone, which may apply to any complex embedded in real space.

Chapter 2

Notation and definitions

2.1 Algebra

We begin by giving some elementary definitions of convex polytope theory. Books written by Grünbaum [8] and Ziegler [42] both provide a good introduction to the field; they are the source of our notation and terminology. Recently, Kaibel, Klee, and Ziegler [9] produced a second edition of Grünbaum's book, which updates some of the material.

Let us first describe the space in which we shall work. Let $\mathbb{V}$ be d-dimensional vector space over an ordered field $\mathbb{F}$, and let $\mathbb{V}^*$ be the *dual* space of linear functionals on $\mathbb{V}$. We denote by $\langle x, y \rangle$ the image of x under a linear functional y^*. If $B = (b_1, \ldots, b_d)$ is a basis for $\mathbb{V}$, then there corresponds a dual basis $B^* = (b_1^* \ldots, b_d^*)$ for $\mathbb{V}^*$ such that

$$\langle b_j, b_k^* \rangle = \delta_{jk} \quad \text{for } j, k = 1, \ldots, d,$$

where

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

is the usual Kronecker delta function. We may now identify $\mathbb{V}$ and $\mathbb{V}^*$, identifying $x = \sum_{i=1}^{d} x_i b_i$ with $x^* = \sum_{i=1}^{d} x_i b_i^*$.

Identifying $\mathbb{V}$ and $\mathbb{V}^*$ is not actually necessary. Indeed, it is often a good idea to distinguish the two, and keep track of which space one is working in. However, the identification makes the constructions easier to visualize. It will not confuse the reader to think in a coordinatized d-dimensional real vector space, with the usual inner product

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i,$$

where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$.

With this identification, we say that two vectors u and v are *orthogonal* if $\langle u, v \rangle = 0$. If $\mathbb{X}$ is a subspace of $\mathbb{V}$, then the *orthogonal complement* $\mathbb{X}^\perp$ of $\mathbb{X}$ is the subspace

$$\mathbb{X}^\perp = \{y \in \mathbb{V} : \langle x, y \rangle = 0 \text{ for all } x \in \mathbb{X}\}$$

of $\mathbb{V}$. To a linear map

$$\Phi \colon \mathbb{U} \longrightarrow \mathbb{V},$$

between vector spaces $\mathbb{U}$ and $\mathbb{V}$, there corresponds a *dual* linear map

$$\Phi^* \colon \mathbb{V} \longrightarrow \mathbb{U},$$

given by

$$\langle u, v\Phi^* \rangle = \langle u\Phi, v \rangle,$$

for each $u \in \mathbb{U}$ and $v \in \mathbb{V}$. If we again identify spaces with their duals, then

$$(\mathbb{X}\Phi)^{\perp} = \mathbb{X}^{\perp}\Phi^{*}.$$

2.2 Basic definitions

To prevent the reader getting lost, we shall keep this section to a minimum and provide other definitions in their appropriate contexts. However, there are some definitions which we need to begin with.

We shall use familiar set theoretic notation: $\subseteq$, $\subset$, $\supseteq$ and $\supset$ for inclusion relations; $\cap$ and $\cup$ for intersections and unions repectively; $\{x_1, \ldots, x_n\}$ will denote a set of n elements; $(x_1, \ldots, x_n)$ will be used if the set is ordered. The algebraic notation for the *Minkowski sum* of sets A and B, and the dilatate of a set A by a scalar λ should also be familiar:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\}, \\ \lambda A &= \{\lambda a : a \in A\}. \end{aligned}$$

We shall work in a finite dimensional vector space $\mathbb{V}$. An *affine* (*linear*) *hyperplane* of $\mathbb{V}$ is an affine (linear) subspace of codimension 1. An affine (linear) hyperplane H may be defined as

$$H = \{x \in \mathbb{V} : \langle x, u \rangle = \eta\},$$

for some $u \in \mathbb{V}^*$ and $\eta \in \mathbb{R}$ ($\eta = 0$). The vector u is said to be *normal* to H. An affine subspace may be thought of as a translate of a linear subspace. A hyperplane divides the space into two closed *affine* (*linear*) *halfspaces* $H^+ = \{x \in \mathbb{V} : \langle x, u\rangle \geqslant \eta\}$ and $H^- = \{x \in \mathbb{V} : \langle x, u\rangle \leqslant \eta\}$. Usually we refer to affine halfspaces as simply 'halfspaces'. We say that H *supports* a set A if A touches, and lies to one side of H. In other words, the linear functional u is maximized or minimized over A at $H \cap A$.

A set K is *convex* if and only if, for each pair of distinct points $a, b \in K$, the closed line segment with endpoints a and b is contained in K. It is trivial to show that the intersection of a family of convex sets is convex.

The *affine*, *convex*, *linear*, and *positive* hulls of a finite set A are the intersections of all affine subspaces, affine halfspaces, linear subspaces, and linear halfspaces respectively, which contain A. We denote them by $\operatorname{conv} A$, $\operatorname{aff} A$, $\operatorname{lin} A$, and $\operatorname{pos} A$ respectively. These concepts may alternatively be defined in terms of affine or linear combinations:

$$
\begin{aligned}
\operatorname{aff} A &= \left\{\sum \lambda_i a_i : a_i \in A, \sum \lambda_i = 1\right\}, \\
\operatorname{conv} A &= \left\{\sum \lambda_i a_i : a_i \in A, \sum \lambda_i = 1, \lambda_i \geqslant 0\right\}, \\
\operatorname{lin} A &= \left\{\sum \lambda_i a_i : a_i \in A\right\}, \\
\operatorname{pos} A &= \left\{\sum \lambda_i a_i : a_i \in A,\ \lambda_i \geqslant 0\right\}.
\end{aligned}
$$

The set of points $x = \sum \lambda_i a_i \in \operatorname{conv} A$ such that $\lambda_i > 0$ for each i, is called the *relative interior* of A, which we denote by $\operatorname{relint} A$.

An *affine map* is a linear map followed by a translation. We may think of an affine map

$$x \to x\phi + t$$

as a linear map

$$(x, 1) \to (x\phi + t, 1)$$

in the space $\mathbb{V} \times \mathbb{F}$ given by a matrix of the form

$$\begin{pmatrix} \phi & o \\ t & 1 \end{pmatrix}.$$

It is then natural to consider the maps obtained from the whole class of linear maps in this way, particularly since the maps dual to affine maps are not themselves affine. Let

$$\Phi \colon \mathbb{V} \times \mathbb{F} \longrightarrow \mathbb{V} \times \mathbb{F}$$

be a linear map given by the matrix

$$\begin{pmatrix} \phi & u \\ t & \delta \end{pmatrix}.$$

Then

$$x\Phi = (x\phi + t, \langle u, x\rangle + \delta)$$

for some $t \in \mathbb{V}$, $u \in \mathbb{V}^*$ and $\delta \in \mathbb{F}$. The *projective* map or *transformation* on $\mathbb{V}$ associated with Φ is given by

$$x \to \frac{x\phi + t}{\langle u, x\rangle + \delta}.$$

Thus for each x the line (x, λ), $(\lambda \in \mathbb{F})$ is identified with the point $(x/\lambda, 1)$. For $\lambda = 0$, the map is not defined. The hyperplane

$$H_\infty = \{x \in \mathbb{V} : \langle u, x \rangle + \delta = 0\}$$

is said to be *sent to infinity*. A projective transformation is said to be *permissible* for a set S if

$$H_\infty \cap S = \emptyset.$$

If $\pi_v : \mathbb{V} \longrightarrow H$ is a linear projection in direction $v \in \mathbb{V}^*$ onto a hyperplane H of $\mathbb{V}$, and Φ is of the form

$$\Phi = \begin{pmatrix} \pi_v & o \\ & \\ u & \delta \end{pmatrix},$$

then the projective map associated with Φ is called a *projection* from w to H, where w is the intersection point of $\operatorname{lin} v$ and H_∞.

2.3 Convex polyhedra

There are essentially two ways of defining a *convex polyhedron* P. The first is as the closed convex hull of a finite set of points V and a finite set of halflines L.

The second is as the intersection of a finite set of (affine) halfspaces $H_i = \{x \in \mathbb{V} : \langle x, u_i \rangle \leqslant \eta_i\}$, $i = 1, \ldots, n$, say. Thus

$$P = \{x \in \mathbb{V} : \langle x, u_i \rangle \leqslant \eta_i, \text{ for } i = 1, \ldots, n\}.$$

The equivalence of these definitions was proved by Motzkin [28]. It is useful to have these two definitions because we may prove some basic results easily.

Theorem 2.3.1 *If P and Q are polyhedra, then the following are also polyhedra:*

i) $P \cap Q$,

ii) $\operatorname{cl}\operatorname{conv}\{P \cup Q\}$,

iii) the image of P under a permissible projective map.

The *dimension* of a polyhedron is the dimension of its affine hull. For brevity, we refer to a d-dimensional convex polyhedron as a d-polyhedron. A bounded polyhedron is called a polytope. A polytope is therefore the convex hull of a finite set of points; it is a d-polytope if and only if a maximal, affinely independent set of these points has $d+1$ elements. The simplest type of d-polytope is the convex hull of $d+1$ affinely independent points; such a polytope is called a d-simplex.

2.4 Faces and normal fans

Let P be a d-polyhedron in a space $\mathbb{V} = \operatorname{aff} P$. A set F is a *face* of P if it is either empty, equal to P, or the intersection of P with a supporting hyperplane H. In this latter case, F is said to be a *proper* face of P. We can identify a proper face of P with a vector $u \in \mathbb{V}^*$. We define $F(P, u)$ to be the face of P which maximizes u:

$$F(P,u) = \{x \in P : \langle y, u\rangle \leqslant \langle x, u\rangle \text{ for all } y \in P\}.$$

The intersection of any two faces of P is a face. Since a hyperplane is a convex polyhedron, then each face of P is also a polyhedron. A face of dimension $r \geqslant 0$ is called an r-face (by convention, the dimension of the empty set is -1). The family of all r-faces of P is denoted $\mathcal{F}_r(P)$, and we write

$$\mathcal{F}(P) = \bigcup_{i=-1}^{d} \mathcal{F}_r(P).$$

The faces of P are partially ordered by inclusion. Indeed, $\mathcal{F}(P)$ is a *lattice*: every pair of elements of $\mathcal{F}(P)$ has a unique upper and lower bound in $\mathcal{F}(P)$. We shall refer to $\mathcal{F}(P)$ as the *face lattice* of P. Faces of dimension 0 and 1 are called *vertices* and *edges* respectively; faces of codimension 2 and 1 are called *ridges* and *facets* respectively.

If each facet of a d-polytope P has d vertices (and hence is a $(d-1)$-simplex), then we say that P is *simplicial.* Thus each face of P is a simplex (except P itself). If each vertex is contained in d facets, then we say that P is simple. Thus each r-face of P is a simple r-polytope. The only polytope that is both simple and simplicial is a simplex.

2.5 Duality

We have presented two alternatives for some of the definitions in this chapter. In each case, the definitions are *dual* to each other. All geometric statements have duals. Non-singular projective maps have non-singular dual projective maps. On the other hand, a projective map with a k-dimensional kernel is dual to intersecting with a k-dimensional subspace.

If S is a full-dimensional convex set in $\mathbb{V}$, that contains the origin in its interior, then we define the *polar* S^Δ by

$$S^\Delta = \{x \in \mathbb{V} : \langle x, y\rangle \leqslant 1, \text{ for all } y \in S\}.$$

The polar of a polyhedron is a polyhedron. Explicitly, if

$$P = \{x \in \mathbb{V} : \langle x, u_i \rangle \leqslant \eta_i \text{ for } i = 1, \dots, n\},$$

with $\eta_i \geqslant 0$ for $i = 1, \dots, n$, then

$$P^{\Delta} = \operatorname{conv}\left(\{o\} \cup \left\{\eta_i^{-1} u_i : \eta_i \neq 0\right\} \cup \{\operatorname{pos} u_i : \eta_i = 0\}\right).$$

If P is bounded and contains the origin in its interior, then P^{Δ} has a face lattice dual to $\mathcal{F}(P)$. Note that if we translate P by a small displacement t such that the origin still lies within the interior of P, then the polar of $P+t$ is projectively equivalent to P^{Δ}. This does not affect the combinatorial type.

Since the polar of a polytope is again a polytope, the face lattice of a facet F of P is dual to some sublattice of $\mathcal{F}(P^{\Delta})$. The dual sublattice is the set of faces of P^{Δ} which contain the vertex v which corresponds to F. If we intersect P^{Δ} with a hyperplane H which separates v from the other vertices, then we obtain a polytope whose r-faces are in one-to-one correspondence with the $(r+1)$ faces of this sublattice. This polytope is called the *vertex-figure* of v and is denoted P/v. Different choices of hyperplane H will give different projectively equivalent polytopes. However, they are all combinatorially isomorphic. This process may be repeated to obtain a polytope dual to any face of P.

As we stated earlier, projections are dual to sections and this duality can be seen here. If we *project* P from a point *close* to a facet F to a hyperplane, then we will obtain a polytope projectively equivalent to F. The dual map is the *section* of P^{Δ} with a hyperplane *close* to v.

Chapter 3

Diagrams and representations

Gale diagrams provide a way to represent a polytope P in a way that is independent of affine (or even projective) equivalence. Although they were invented by Gale in 1956 [7], they were later developed by Perles, and documented in detail in Grünbaum's book [8]. As predicted by Grünbaum, Gale diagrams have yielded many results since then.

Representations were introduced by McMullen [20] in 1973. They share many of the properties of Gale diagrams, but they are more versatile: they can be used to represent not just polytopes, but also unbounded polyhedra. They also have the advantage that they can be used to investigate properties such as volume. For this reason, we shall use them far more than diagrams. We shall therefore ignore the chronology, and describe representations first, and in more detail. Before that, however, we must introduce some terminology.

3.1 The type-cone

If $P = \lambda Q + t$ for some $\lambda > 0$ and some vector t, then P and Q are said to be *homothetic*. If $P = Q + R$ is the Minkowski sum of polyhedra Q and R, then we say that Q is a *summand* of P. Note that

$$F(P,u) = F(Q,u) + F(R,u)$$

for each $u \in \mathbf{V}^*$. We write $Q \preccurlyeq P$ if Q is homothetic to a summand of P. In this case, it is clear that $\dim Q \leqslant \dim P$, and further that $\dim F(Q,u) \leqslant \dim F(P,u)$. Actually the reverse is also true.

Theorem 3.1.1 *Let P and Q be polyhedra in $\mathbf{V}$. Then $Q \preccurlyeq P$ if and only if*

$$\dim F(P,u) \leqslant \dim F(Q,u)$$

for each $u \in \mathbf{V}$.

If $P \preccurlyeq Q$ and $Q \preccurlyeq P$, then we say P and Q are *strongly isomorphic*, and we write $P \approx Q$. It is clear that then P and Q have isomorphic face lattices. The relation $\approx$ is an equivalence relation; the equivalence class of a P is called the *type-cone* of P and is denoted $\mathcal{K}(P)$ (see [20]). As the name suggests, $\mathcal{K}(P)$ is isomorphic to an open euclidean cone under Minkowski summation and dilatation. Since any polyhedron is strongly isomorphic to a translate, then $\mathcal{K}(P)$ contains a subspace of dimension d.

3.2 Representations

Let $\mathcal{P}(U)$ denote the family of polyhedra whose facet outer normals form the ordered set $U = (u_1, \ldots, u_n)$, with $\operatorname{lin}(U) = \mathbb{V}$. Recall that P may be written in the form

$$P = \{x \in \mathbb{V} : \langle x, u_i \rangle \leqslant \eta_i \text{ for } i = 1, \ldots, n\},$$

where $\eta_1, \ldots, \eta_n \in \mathbb{F}$. We call $\eta_1, \ldots, \eta_n$ the *support parameter* of facets of P We can identify $P \in \mathcal{P}(U)$ with the vector

$$y = (\eta_1, \ldots, \eta_n) = \sum_{i=1}^{n} \eta_i e_i,$$

where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{F}^n$.

If P is identified with y, then $P + t$ is identified with the vector

$$y + (\langle u_1, t \rangle, \ldots, \langle u_n, t \rangle).$$

Let σ be a linear map on $\mathbb{F}^d$ with kernel T, where T is the subspace

$$T = \left\langle (\langle u_1, t \rangle, \ldots, \langle u_n, t \rangle) : t \in \mathbb{F}^d \right\rangle.$$

Thus σ associates translates of P. This map is said to be a *representation* associated with $\mathcal{P}(U)$. The subspace T has dimension d, and so the image space of σ is $(n-d)$-dimensional.

We shall write $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_n)$, where $\overline{u}_i = e_i\sigma$. If P is identified with the vector y, then we say that $p = y\sigma$ is the *representative* of P. Notice that the definition of representation does not rely on convexity in any way, only on U, and is only defined up to linear equivalence. It is the representative p that gives the information that is specific to P. The representation (together with the representative p) presents the combinatorial properties of P in a useful way.

Theorem 3.2.1 *If P has facets $F_1, \ldots, F_n$, with outer normals $u_1, \ldots, u_n$ respectively, then there exists a face F which lies in (and only in) each element of the set $\{F_i : i \in I\}$ if and only if $p \in \operatorname{relint} \operatorname{pos}\{\overline{u}_i : i \notin I\}$.*

Proof. Let $F = \bigcap_{i \in I} F_i$ be a face such that $F \not\subset F_i$ for $i \notin I$. The facets F_i of a translate of P with $o \in \operatorname{relint} F$ will have strictly positive support parameters for $i \notin I$, and the remaining facets will contain the origin. Hence $p \in \operatorname{relint} \operatorname{pos} \{\overline{u}_i : i \notin I\}$.

Conversely, if $p \in \operatorname{relint} \operatorname{pos} \{\overline{u}_i : i \notin I\}$, then there is a translate of P such that $o \in F_i$ for $i \in I$, and the remaining facets have strictly positive support. Thus $F = \bigcap_{i \in I} F_i$ is a face. Since it contains the origin, and facets with strictly positive support cannot, no facet F_i with $i \notin I$ contains F. □

The sets U and $\overline{U}$ are related to each other in a purely algebraic way. Let $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$ be vector spaces of dimension $n - d$, n and d respectively, with maps Ψ and Φ such that

$$\mathbb{O} \longrightarrow \mathbb{U} \xrightarrow{\Psi} \mathbb{V} \xrightarrow{\Phi} \mathbb{W} \longrightarrow \mathbb{O}$$

is a *short exact sequence*: the image of Ψ is the kernel of Φ. There is a dual short exact sequence given by

$$\mathbb{O} \longleftarrow \mathbb{U}^* \xleftarrow{\Psi^*} \mathbb{V}^* \xleftarrow{\Phi^*} \mathbb{W}^* \longleftarrow \mathbb{O},$$

between the dual spaces $\mathbb{U}^*$, $\mathbb{V}^*$ and $\mathbb{W}^*$, where Ψ^* and Φ^* are the dual maps of Ψ and Φ respectively. If $U = (u_1, \ldots, u_n)$ is the image of a basis B of $\mathbb{V}$ under the map Φ, then the image $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_n)$ of the dual basis B^* under the map Ψ^* is a *linear transform* of U. If $u_1 \ldots, u_n$ are the facet normals of a polyhedron P, then Ψ^* is a representation of U.

In practice, we relabel if necessary so that $u_1, \ldots, u_d$ is linearly independent, and let $U = (u_1, \ldots, u_d)$. Then, expressed as a matrix,

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} I_d \\ A \end{pmatrix}$$

for some $(n-d) \times d$ matrix A. A linear transform is then given by

$$\overline{U} = \begin{pmatrix} A & -I_{n-d} \end{pmatrix} = (\overline{u}_1, \ldots, \overline{u}_n),$$

where $\overline{u}_i$ is the ith column of the matrix. We illustrate with an example. Let P be the quadrilateral

$$P = \left\{ x \in \mathbb{R}^2 : \langle x, u_i \rangle \leqslant \eta_i \text{ for } i = 1, \ldots, 4 \right\}$$

shown in Figure 3.1 with outer normals

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}$$

and support parameters 1, 1, 2, and 2 respectively.

We obtain the matrix

$$\overline{U} = (\overline{u_1}, \overline{u_2}, \overline{u_3}, \overline{u_4}) = \begin{pmatrix} -1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}$$

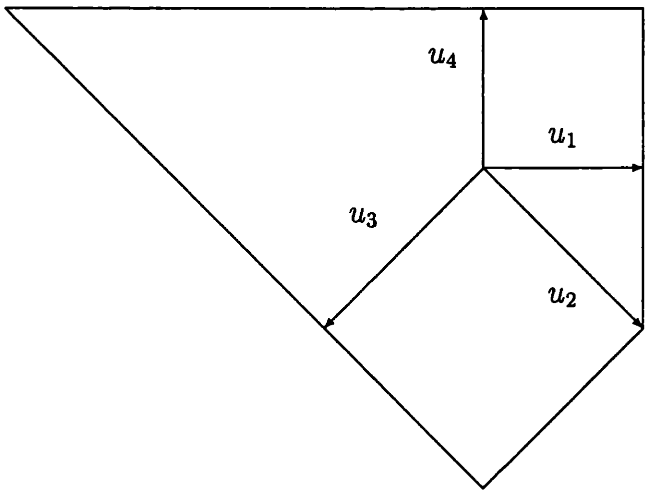


Figure 3.1: A quadrilateral

which we plot to give Figure 3.2. We also plot the point

$$p = \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4}$$

3.3 Diagrams

The Gale diagram of a d-polytope P may be constructed in a similar way to the representation of its polar. If $U = (u_1, \ldots, u_n)$, with $o \in \operatorname{int}\operatorname{conv} U$, then we can identify a polytope $P = \operatorname{conv}\{\lambda_1 u_1, \ldots, \lambda_n u_n\}$, $(\lambda_i > 0)$ with the vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{F}^n$. We identify projectively isomorphic polytopes P and $Q = P\Phi$ if

$$x\Phi = \frac{x}{\langle u, x \rangle + 1}$$

for some $u \in \mathbb{V}^*$. The association between sets of identified polytopes and a point in a representation $\overline{U}$ of U is then identical to that described before.

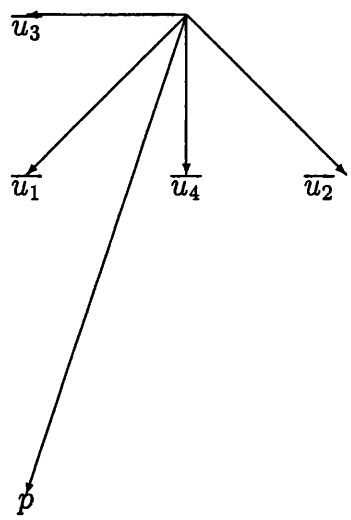


Figure 3.2: Representation of P

A *Gale diagram* of a polytope $P = \operatorname{conv}\{u_1, \ldots, u_n\}$ lies in the projective space obtained from $\operatorname{lin} \overline{U}$ by identifying lines through the origin. The *Gale vertices* are those lines that contain elements of $\overline{U}$ (with multiplicity). By identifying lines through the origin, we do not distinguish between dilatates of a polytope. Hence a Gale diagram has dimension $n - d - 1$.

For example, if Q is the polar of P from our previous example, then the lines in Figure 3.2 that contain $\overline{u_1}, \ldots, \overline{u_4}$ are the Gale vertices of a Gale diagram of Q.

Theorem 3.2.1 has an analogue for diagrams.

Theorem 3.3.1 *If P has vertices $u_1, \ldots, u_n$, then $\{u_i : i \in I\}$ is the vertex set of a face of P if and only if $o \in \operatorname{relint} \operatorname{conv} \{\overline{u}_i : i \notin I\}$.*

Gale diagrams are particularly useful for examining polytopes of large dimensions, when the number of vertices only exceeds the dimension by a small number. For example, if a d-polytope P has $d+3$ vertices, then the Gale diagram of P is 2-dimensional, for each d. Similarly, representations may be used to examine polytopes with small numbers of facets compared to d.

To construct d-polytopes with n facets, Gale diagrams are quite powerful.

Theorem 3.3.2 *A set $\overline{U} = \{\overline{u}_1, \ldots, \overline{u}_n\} \subset \mathbb{F}^{n-d-1}$ is the Gale diagram of a d-polytope with n vertices if and only if every linear halfspace contains at least two elements of $\overline{U}$.*

For example, we may construct polytopes (up to linear equivalence) with $d+3$ vertices by arranging points in the plane. If we are interested only in polytopes up to projective equivalence, then we may simplify things still further.

Theorem 3.3.3 *If $(v_1, \ldots, v_n)$ is the image of $(u_1, \ldots, u_n)$ under a projective transformation Φ, then $\overline{v}_i = \lambda_i \overline{u}_i$ for $\lambda_i \neq 0$, $i = 1, \ldots, n$. Moreover, Φ is permissible for* $\mathrm{conv}(u_1, \ldots, u_n)$ *if and only if $\lambda_i > 0$ for $i = 1, \ldots, n$ (or $\lambda_i < 0$ for $i = 1, \ldots, n$).*

For a proof, we refer the reader to [8]. The *affine* Gale diagram of a d-polytope $P = \mathrm{conv}\,\{u_1, \ldots, u_n\}$ may be defined as the set $\{\mathrm{pos}\,\overline{u}_1, \ldots, \mathrm{pos}\,\overline{u}_n\}$ of half-lines in $\mathbb{F}^{n-d-1}$. It is usually more useful to consider the intersection of the lines $\{\mathrm{lin}\,\overline{u}_1, \ldots, \mathrm{lin}\,\overline{u}_n\}$ with some general hyperplane (which does not contain the origin), place a positive sign on the intersection point if it lies in $\mathrm{pos}\,\overline{u}_i$, and a negative sign otherwise. This gives a set of signed points in $\mathbb{F}^{n-d-2}$ which determines P up to projective equivalence.

Chapter 4

Abstract polytopes

4.1 Introduction

Although our definitions are geometric, there are numerous combinatorial properties of polyhedra. In particular, the face lattice is a purely combinatorial object, even if it is associated with a geometric object. When examining combinatorial properties of face lattices, we may instead focus on certain classes of posets with some of the properties possessed by face lattices. The study of abstract polytopes is an example of such an approach. These are very general and, with no topological constraints, can behave in very different ways from convex polytope lattices. We shall impose extra constraints to establish some results for a class of abstract polytopes which includes all convex polytopes.

Balinski showed that the 1-skeleton of a convex n-polytope remains connected if at most $n-1$ vertices are removed. Menger showed that this property of a graph is equivalent to the existence of n paths between any given pair of vertices, that intersect only at their endpoints.

For a polytopal cell complex S, Lockeberg [15] defines a *suspension* of S as a prism over S, with each end face contracted into a single *suspension* vertex. For example, a suspension of a triangle has two vertices, three edges, and three 2-faces. Combinatorially, the dimension of each face is increased by one, and two new vertices are introduced: these *suspension vertices* are the only two vertices of the suspension, and they are contained in every face of dimension at least 1. Balinski's Theorem then shows that, with pre-assigned suspension vertices, the 1-skeleton of a convex n-polytope contains a *refinement* of a suspension of a set of n vertices: we combine the edges in each path into one *pseudo-edge*, and these become the edges of the suspension.

In three dimensions, three paths between vertices u and v divide the boundary of a polytope into three regions. Each region is a strong family of facets, which we combine into a *pseudo-facet.* The intersection of two pseudo-facets is a path, or *pseudo-edge.* Hence the *pseudo-faces* are the faces of a suspension of a triangle, with suspension vertices u and v.

It is natural to ask whether some higher dimensional analogue holds: given a pair of vertices, is every polytope a refinement of a suspension of a $(n-1)$-simplex? This would immediately answer a number of questions regarding disjoint strong chains of faces between pairs of vertices, which as yet remain a mystery. It is easy to show that, for the boundary of an n-simplex, the answer is yes. Thus we could give a positive answer for a general n-polytope P, if we could show that its boundary was a refinement of the boundary of a n-simplex with a pre-assigned pair of principal vertices. However, Lockeberg [15] gave a counterexample, showing that at most one principal vertex could be pre-assigned in general.

4.2 Abstract polytopes

McMullen and Schulte [26] define an *abstract polytope* $\mathcal{P}$ of rank n, or abstract n-polytope, as a poset which satisfies the properties $(P1), \ldots, (P4)$ below. The elements of $\mathcal{P}$ are the *faces* of $\mathcal{P}$. A *chain* of $\mathcal{P}$ is a totally ordered subset of $\mathcal{P}$. Two faces F and G of $\mathcal{P}$ are said to be *incident* if $F \leqslant G$ or $G \leqslant F$. A chain has length i if it contains exactly $i+1$ faces. The maximal chains are called the *flags* of $\mathcal{P}$.

$(P1)$ $\mathcal{P}$ contains a least face F_{-1} and a greatest face F_n.

$(P2)$ Each flag of $\mathcal{P}$ has length $n+1$.

These first two properties allow us to define the *rank* of $\mathcal{P}$ as one less than the length of a flag of $\mathcal{P}$. The faces F_{-1} and F_n are called the *improper* faces of $\mathcal{P}$. All other faces of $\mathcal{P}$ are *proper*. For any two faces F and G of $\mathcal{P}$, with $F \leqslant G$, we call

$$G/F = \{H : H \in P, F \leqslant H \leqslant G\}$$

a *section of* $\mathcal{P}$. Observe that $(P1)$ and $(P2)$ also hold for a section of $\mathcal{P}$. The *rank* of a face F is the rank of F/F_{-1}. Indeed, we shall identify F with F/F_{-1} when there is no ambiguity.

We use the language of convex polytopes, and refer to the faces of rank $0, 1, n-2$, and $n-1$ as vertices, edges, ridges, and facets respectively; an r-face is a face of rank r.

If $\mathcal{P}$ satisfies $(P1)$, and each pair of faces F, G has a unique infimum $F \wedge G$ and a unique supremum $F \vee G$, then we say that $\mathcal{P}$ is a *lattice.* This is not a defining property of an abstract polytope, but note that for a convex polytope, the faces form a lattice. Note also that a section of a lattice is itself a lattice.

If a lattice $\mathcal{P}$ satisfies $(P2)$, and each face (except F_{-1}) can be expressed as the supremum of a set of vertices, then $\mathcal{P}$ is said to be *atomic.* Similarly, if each face (except F_n) can be expressed as the infimum of a set of facets, then $\mathcal{P}$ is said to be *coatomic.*

The next property concerns the connectivity of $\mathcal{P}$. A poset of rank n with properties $(P1)$ and $(P2)$ is called *connected* if either $n \leqslant 1$, or $n \geqslant 2$ and, for any two proper faces F and G of $\mathcal{P}$, there exists a finite sequence of proper faces $F = H_0, \ldots, H_k = G$ of $\mathcal{P}$ such that H_{i-1} and H_i are incident for $i = 1, \ldots, k$. We say that $\mathcal{P}$ is *strongly connected* if each section of $\mathcal{P}$ is connected. The next defining property is

$(P3)$ $\mathcal{P}$ is strongly connected.

We present two alternatives $(P3')$ and $(P3'')$ which are both equivalent to $(P3)$. Given a poset with properties $(P1)$ and $(P2)$, we call two flags of $\mathcal{P}$ *adjacent* if one differs from the other by exactly one face; if this face has rank i then the two flags are said to be *i-adjacent.* Then $\mathcal{P}$ is *flag-connected* if any two distinct flags Φ and Ψ of $\mathcal{P}$ can be *joined* by a sequence

$$\Phi = \Phi_0, \ldots, \Phi_k = \Psi$$

of flags, such that Φ_{j-1} and Φ_j are adjacent for $j = 1, \ldots, k$. Further, $\mathcal{P}$ is strongly *flag-connected* if each section of $\mathcal{P}$ is flag-connected.

($P3'$) $\mathcal{P}$ is strongly flag-connected.

Given a poset $\mathcal{P}$ of rank n which satisfies ($P1$) and ($P2$), we call the subposet of $\mathcal{P}$ which contains all faces of rank k or $k-1$ the *k-stratum* of $\mathcal{P}$. We often refer to the 1-stratum of a polytope $\mathcal{P}$ as the *graph* of $\mathcal{P}$. We say that $\mathcal{P}$ is *stratum-connected* if the k-stratum of $\mathcal{P}$ is connected for $k = 1, \ldots, n-1$. Further, $\mathcal{P}$ is *strongly stratum-connected* if each section of $\mathcal{P}$ is stratum-connected.

($P3''$) $\mathcal{P}$ is strongly stratum-connected.

Conditions ($P3$) and ($P3'$) are shown to be equivalent in [26]. To see that ($P3'$) implies ($P3''$), notice that, if Φ and Ψ are adjacent flags, then the k-stratum of $\Phi \cup \Psi$ is connected. Every k-face is contained in a flag and so the k-stratum of $\mathcal{P}$ is connected. For the converse, it is easier to show that ($P3''$) implies ($P3$). Each face is contained in a flag and is therefore connected to the 1-stratum of $\mathcal{P}$. The 1-stratum is connected and so $\mathcal{P}$ is connected.

The last defining property is the so-called *diamond property.*

($P4$) Each section of length 2 has exactly four elements.

Lemma 4.2.1 *If a lattice $\mathcal{P}$ satisfies ($P4$), then $\mathcal{P}$ is atomic and coatomic.*

Proof. Since ($P4$) also holds for $\mathcal{P}^*$ (obtained by reversing the partial ordering on $\mathcal{P}$), it suffices to show that $\mathcal{P}$ is coatomic. Suppose that $F \neq F_n$ is a maximal face which is not the infimum of coatoms. If $k = r(F)$, then $k < n-1$. Since F is contained in some

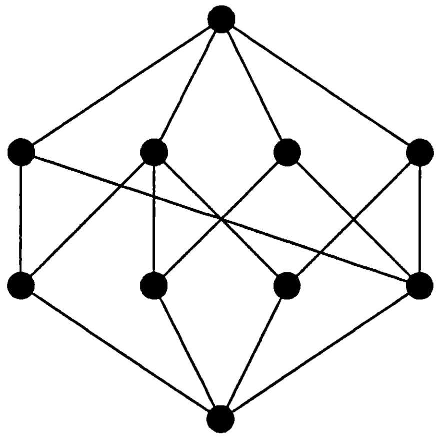


Figure 4.1: A counterexample

flag of S, then, by $(P4)$, there exist distinct $(k+1)$-faces G and H such that $F \leqslant G$ and $F \leqslant H$. If $G = C_1 \wedge \cdots \wedge C_r$ and $H = C_{r+1} \wedge \cdots \wedge C_s$, for coatoms $C_1, \ldots, C_s$, then $F \leqslant C_i$ for $i = 1, \ldots, s$. Hence

$$F \leqslant \bigwedge_{i=1}^{s} C_i < \bigwedge_{i=1}^{r} C_i = G.$$

But F and G differ in rank by 1, and so the inequality must in fact be an equality. Hence F is the infimum of coatoms $C_1, \ldots, C_s$. □

The converse is not true. Figure 4.1 is atomic and coatomic, but does not satisfy $(P4)$.

If $\mathcal{X}$ is a subset of a poset $\mathcal{P}$, then we write

$$\mathcal{P}\backslash\mathcal{X} = \{F \in \mathcal{P} : F \nleqslant X \text{ for any } X \in \mathcal{X}\};$$

for each face F in $\mathcal{X}$, we remove F and every face of F from $\mathcal{P}$ to obtain $\mathcal{P}\backslash\mathcal{X}$.

If $\mathcal{X}$ is a subset of a lattice $\mathcal{P}$, then let

$$\mathcal{P}!\mathcal{X} = \{F \in \mathcal{P} : F \wedge G = F_{-1}, \text{ for all } G \in \mathcal{X}\}.$$

For example, if $\mathcal{X}$ is a vertex v of a polytope $\mathcal{P}$, then $\mathcal{P}!\mathcal{X}$ is obtained by removing the faces which contain v.

Theorem 4.2.2 *If $\mathcal{P}$ is a coatomic lattice, and $\mathcal{X}$ is a subset of $\mathcal{P}$ that satisfies*

$$\sum_{X \in \mathcal{X}} (r(X) + 1) \leqslant n - k,$$

then there exists some $F \in \mathcal{P}!\mathcal{X}$ of rank k.

Proof. For $k = n$, there is nothing to prove. If $X \neq F_{-1}$ for some $X \in \mathcal{X}$, then a facet F exists such that $X \not\leqslant F$, since $\mathcal{P}$ is coatomic, and $r(X \wedge F) < r(X)$. Thus

$$\sum_{X \in \mathcal{X}} (r(X \wedge F) + 1) \leqslant n - k - 1,$$

and, by induction, there exists some $y \in F!X$ with $r(y) = k$. □

We shall now prove a generalized version of Balinski's Theorem. This turns out to be a purely combinatorial result, which is just as applicable to abstract polytopes as convex polytopes. A graph is *k-connected* if it is either a complete graph on $k + 1$ vertices (each pair of vertices is contained in an edge) or it has at least $k + 2$ vertices, and the removal of any set of $k - 1$ vertices leaves a connected graph.

Lemma 4.2.3 *The graph of an n-polytope $\mathcal{P}$ with at least three vertices is 2-connected for $n \geqslant 2$.*

Proof. For $n = 2$, $(P4)$ implies that each vertex is the infimum of a unique pair of facets, and each facet is the supremum of a unique pair of vertices. $(P3)$ says that the graph of $\mathcal{P}$ is connected, and hence the graph of $\mathcal{P}$ is a cycle with at least three vertices, which is 2-connected.

For larger n, if X and Y are 2-faces, and V is a vertex, then the 1-strata of $X!V$ and $Y!V$ are connected. If X and Y have an edge in common, then X and Y have a vertex $V' \neq V$ in common. Hence the 1-strata of $X!V$ and $Y!V$ are connected to each other at V'.

For $n > 2$, let W be a vertex of $\mathcal{P}$. For each pair of other vertices U, V, there exist 2-faces X and Y such that $U \leqslant X$ and $V \leqslant Y$, by property $(P2)$. By property $(P3)$, there exists a sequence of 2-faces $X = X_0, \ldots, X_m = Y$ of $\mathcal{P}$ such that X_i and X_{i+1} have an edge in common for $i = 0, \ldots, m$. In particular, by $(P4)$, X_i and X_{i+1} have a vertex $W' \neq W$ in common. The graph of $X_i!W$ is connected (the case $n = 2$) for $i = 0, \ldots, m$, and the 1-strata of $X_i!W$ and $X_{i+1}!W$ are connected to each other for $i = 0, \ldots, m-1$. Hence U and V are connected to each other by the 1-strata of $X_0!W, \ldots, X_m!W$. □

We make the observation that a 2-polytope lattice has at least three vertices. This follows from Lemma 4.2.1, since by $(P4)$, a 2-polytope has at least two edges, and different edges do not contain the same pair of vertices.

Lemma 4.2.4 *If G is a proper face of an n-polytope lattice $\mathcal{P}$, then the graph of $\mathcal{P}!G$ is connected.*

Proof. For $n = 2$, $\mathcal{P}$ is a cycle with $k \geqslant 3$ vertices; the removal of an edge leaves a path of length $k-3$ for $k > 3$, or a single vertex for $k = 3$. For $n \geqslant 3$, let F be a facet of $\mathcal{P}$ such that $G \leqslant F$. For each facet $F' \neq F$, the graph of $F'!(F' \wedge G)$ is connected, by induction on n. If $F', F'' \neq F$ are facets of $\mathcal{P}$, and $F' \wedge F''$ is a ridge, then $(F' \wedge F'') \nleqslant F$, by $(P4)$. Hence there is a vertex of $F' \wedge F''$, which is not a vertex of F; the 1-strata of $F'!(F' \wedge G)$ and $F''!(F'' \wedge G)$ have this vertex in common. The dual of Lemma 4.2.3 implies that the $(n-1)$-stratum of $\mathcal{P}\backslash F$ is connected, and so the 1-strata of the remaining facets join up as required. $\square$

Lemma 4.2.5 *If G is a proper face of an n-polytope $\mathcal{P}$, then the graph of $P\backslash G$ is connected.*

Proof. Note that $P\backslash G$ has at least one vertex; we remove only vertices and edges that lie in G. The lemma is true for $n = 2$, since $\mathcal{P}$ is a cycle. We assume by induction that the lemma holds for $n-1$. Thus $F\backslash G$ has a connected graph for each facet F of $\mathcal{P}$. The $(n-1)$-strata of $\mathcal{P}$ and $\mathcal{P}\backslash G$ are identical, unless G is a facet. In any case, the dual of Lemma 4.2.3 implies that the $(n-1)$-stratum of $\mathcal{P}\backslash G$ is connected, and the 1-strata of the facets of $\mathcal{P}\backslash G$ join up as required. $\square$

We now come to our main result. Perles and Prabhu [29] proved the $k = 1$ case for convex polytopes using a geometric argument. Their proof may be modified slightly to give a proof for general k, but it relies on convexity. Our proof is purely combinatorial, and holds for abstract polytope lattices.

Theorem 4.2.6 *If $\mathcal{X}$ is a subset of an n-polytope lattice $\mathcal{P}$ that satisfies*

$$\sum_{X \in \mathcal{X}} (r(X) + 1) \leqslant n - k$$

for $k \geqslant 1$, then the k-stratum of $\mathcal{P}!\mathcal{X}$ is connected.

Proof. For $n = 1$, there is nothing to prove. We assume that the theorem holds for $n-1$. Let Y be the supremum of all elements of $\mathcal{X}$. If a facet F satisfies $Y \not\leqslant F$, then

$$\sum_{X \in \mathcal{X}} (r(X \wedge F) + 1) \leqslant n - k - 1.$$

The k-stratum of $F!\mathcal{X}$ is connected, by induction on n. If another facet F' also satisfies $Y \not\leqslant F$, and $F \wedge F'$ is a ridge, then $(F \wedge F')!\mathcal{X}$ contains at least one face of rank $k-1$, by Theorem 4.2.2. Each proper face G of $\mathcal{P}!\mathcal{X}$ lies in a facet that does not contain Y, since otherwise $G \geqslant Y \geqslant X$ for some $X \in \mathcal{X}$, contradicting the definition of $\mathcal{P}!\mathcal{X}$.

It remains to show that, for any two facets F, F' of $\mathcal{P}$ which do not contain Y, there exists a sequence $F = F_0, \ldots, F_m = F'$ of facets of $\mathcal{P}$ which also do not contain Y such that $F_i \wedge F_{i+1}$ is a ridge for $i = 0, \ldots, m-1$. An equivalent statement is that the $(n-1)$-stratum of $\mathcal{P} \backslash (F_n/Y)$ is connected. This will imply that each pair of k-faces of $\mathcal{P}!\mathcal{X}$ is connected by the k-strata of those facets which do not contain Y.

If $Y = F_n$, and so $Y \not\leqslant F$ for every facet F, then the result follows from the fact that the $(n-1)$-stratum of $\mathcal{P}$ is connected. Otherwise, Y is a proper face of $\mathcal{P}$, and the result follows from the dual of Lemma 4.2.5: the graph of $\mathcal{P}^* \backslash Z$ is connected, where Z is the face of $\mathcal{P}^*$ which corresponds to Y. □

A special case of Theorem 4.2.6 is when $k = 1$, and $\mathcal{X}$ is a set of $n-1$ vertices. This gives a generalization of Balinski's Theorem to abstract polytope lattices.

Theorem 4.2.7 *If an n-polytope $\mathcal{P}$ is a lattice, then its graph is n-connected.*

4.3 On a question of Lockeberg

Define a *strong k-chain* to be an alternating sequence of k-faces and $(k-1)$-faces such that consecutive faces are incident. A strong 1-chain, in which no face appears more than once, is a called a *path.* This is consistent with the usual graph theoretic definition (see [3]). A well-known theorem of Menger states that the minimum size of a set of vertices which separates vertices u and v is the maximum number of independent paths from u to v. By *independent*, we mean that the paths do not meet except at their endpoints. With Theorem 4.2.6 in mind, the following conjecture makes the link between connectivity and strong chains.

Conjecture 4.3.1 Lockeberg [15]) *If u and v are vertices of a d-polytope lattice $\mathcal{P}$, and $d_1, \ldots, d_n \in \mathbb{N}$ are such that $\sum_{i=1}^{n} d_i = d$, then there exists a set $\{C_1, \ldots, C_n\}$ such that C_j is a strong d_j-chain from u to v and $C_j \cap C_k = \{u, v\}$ if $j \neq k$.*

Actually, this conjecture was made by Lockeberg for convex polytopes only. If $d_1 = \cdots = d_n = 1$, then the connectivity condition is just Balinski's Theorem, and we may appeal to Menger's Theorem. Hence, for a proof, we might try to generalize Menger's Theorem. We first replace "d-connected" with a more general notion, replacing paths with strong chains. Let u and v be vertices of a lattice $\mathcal{P}$ that satisfies $(P1)$, $(P2)$ and $(P3)$, and let $d_1, \ldots, d_k \geqslant 0$. If, for every subset $\mathcal{X} = \{X_1, \ldots, X_k\}$ of $\mathcal{P}$, with $r(X_i) = d_i$ for $i = 1, \ldots, k$, that does not contain u or v, there is a strong (d_i+1)-chain from u to v in $\mathcal{P}!(\mathcal{X} \backslash X_i)$, then we say that $\mathcal{P}$ is $(d_1, \ldots, d_k)$-connected between u and v. The following is a simple corollary of Theorem 4.2.6.

Corollary 4.3.2 *If $\mathcal{P}$ is an n-polytope lattice, and*

$$\sum_{i=1}^{k}(d_i+1) \leqslant n,$$

then $\mathcal{P}$ is $(d_1,\ldots,d_k)$-connected between any pair of vertices of $\mathcal{P}$.

Proof. Let $\mathcal{X} = \{X_1,\ldots,X_k\}$ be such that $r(X_i) = d_i$ for $i = 1,\ldots,k$, and let $\mathcal{X}_j = \mathcal{X} \setminus \{X_j\}$. Then

$$\sum_{i\neq j}(d_i+1) \leqslant n-(d_j+1).$$

Hence, by Theorem 4.2.6, the (d_j+1)-stratum of $\mathcal{P}!\mathcal{X}$ is connected, and so between each pair of vertices of $\mathcal{P}!\mathcal{X}$ there is a strong (d_j+1)-chain. □

Menger's Theorem holds for graphs in general, and so we should perhaps not restrict ourselves to looking at polytopes only, but to subsets of polytopes. The proofs of Menger's Theorem concentrate attention on a fixed pair of vertices, and so we formulate the problem in a similar way.

Conjecture 4.3.3 *If a lattice $\mathcal{P}$ satisfies $(P1)$, $(P2)$, and $(P3)$, and $\mathcal{P}$ is $(d_1,\ldots,d_k)$-connected between vertices u and v, then there is a set $\{C_1,\ldots,C_k\}$, such that C_i is a strong (d_i+1)-chain, and $C_i \cap C_j = \{u,v\}$ for $i \neq j$.*

If true, Conjecture 4.3.3 would imply Conjecture 4.3.1 (the conditions of the latter are stronger). Also, a connected graph G may be regarded as a lattice of rank 2, which satisfies $(P1)$, $(P2)$, and $(P3)$. If G is k-connected between two vertices u and v of G, then $d_1 = \cdots = d_k = 0$. Conjecture 4.3.3 says that there are k independent paths from u to v, which is Menger's Theorem.

4.4 A problem in graph theory

The following lies entirely within graph theory, although its relevance to our previous discussion should be clear. If u and v are vertices of a graph G, and the removal of any set X of vertices of size $n-1$ leaves a path from u to v in $G\backslash X$, then we say that u is *n-connected* to v. Consider the following simple problem.

Proposition 4.4.1 *Let $u, v_1, \ldots, v_n$ be vertices of a graph G, where u is n-connected to v_i for $i = 1, \ldots, n$. If G' is obtained from G by introducing a new vertex w and joining it to v_i for $i = 1, \ldots, n$, then u is n-connected to w.*

Proof. If we remove a set X of size $n-1$ from G', then we leave at least one of $v_1, \ldots, v_n$. There is a path from u to this vertex and then on to w. □

Now, Menger's Theorem states that u is n-connected to v if and only if there are n independent paths from u to v. Thus the following proposition is equivalent to the previous one.

Proposition 4.4.2 *Suppose that $u, v_1, \ldots, v_n$ are vertices of a graph G and that there are n independent paths from u to v_i for $i = 1, \ldots, n$. If G' is obtained from G by introducing a new vertex w and joining it to v_i for $i = 1, \ldots, n$, then there are n independent paths from u to w.*

From a constructive point of view, the latter form is much more useful. We would, ideally, like an analogue of this theorem for higher dimensional strong chains, which motivates an alternative proof.

Theorem 4.4.3 *If u and $v_1, \ldots, v_k$ are vertices of a graph G, and there are $\sum_{j=1}^{i} d_j$ independent paths from u to v_i that do not contain v_j for $j = i+1, \ldots, k$, for $d_1 \ldots, d_k \geqslant 0$, then there are independent paths, with d_i of them from u to v_i.*

Proof. For $k = 1$, we have nothing to prove. We proceed by induction: assume that the theorem holds for $k-1$. There are independent paths in G with d_i from u to v_i for $i = 1, \ldots, k-1$, none of which contains v_k, and $\sum_{i=1}^{k} d_i$ paths from u to v_k. We construct a simpler graph G' with these same properties as follows.

We remove an edge which does not lie in any path, or, if a vertex other than $u, v_1, \ldots, v_k$ has degree 2, then we may replace it with an edge between its two neighbours. We repeat these two reductions until every edge lies in at least one path, and there are no vertices of degree 2, with the possible exceptions $u, v_1, \ldots, v_k$. Thus there are independent paths in G' with d_i from u to v_i for $i = 1, \ldots, k-1$, none of which contains v_k, and $\sum_{i=1}^{k} d_i$ paths from u to v_k. We assume that the paths are of minimum combined length, so that if a path intersects both vertices of an edge then it also contains that edge. In particular, each path contains exactly one neighbour of u. At most $\sum_{i=1}^{k-1}$ of the paths from u to v_k contain a neighbour of u that lies in a path from u to one of $v_1, \ldots, v_{k-1}$. Thus d_k paths from u to v_k do not contain such a neighbour. But every vertex apart from v_k lies in a path from u to one of $v_1, \ldots, v_{k-1}$, and so these d_k paths contain no vertices other than u and v_k: there are d_k edges from u to v_k. $\square$

Note that this theorem implies Proposition 4.4.2. For if there are n independent paths from u to v_i for $i = 1, \ldots, n$, then there are i independent paths from u to v_i that do not contain v_j for $j = i + 1, \dots, n$.

Chapter 5

Weights

In this chapter, we shall describe the weight algebra. McMullen's proof of the g-theorem in [24] used the *polytope algebra* first defined in [23]. McMullen then used the weight algebra, defined in [25], to simplify further his proof of the g-theorem (as well as the Hodge-Riemann-Minkowski and Generalized Aleksandrov-Fenchel inequalities) found in [24]. We shall begin the chapter by outlining some of the main results from [24] and [25].

Weights share many of the geometric properties of volume, which makes many of the constructions easy to visualize. Weights are easily shown to be equivalent to linear stresses on the polar polytope as described by Lee [14]. In Lee's paper, the connexion between stresses and the Stanley-Reisner face-ring is discussed. It is easy to see that stresses may be defined on rather general complexes and not just convex polytopes. We shall therefore translate much of this work into the language of weights and generalize the notion of a weight to these more general complexes. We shall then find that many results proved for weights generalize easily.

The local structure of weights is described, and is shown to have a clear link with weight multiplication. This provides the basis for the comparison with the face-ring.

The algebra of weights on a polytope P has many useful properties, when P is simple. When P is not simple, very little is known. Timorin recently presented a generalization of the g-theorem to polytopes with infrequent singularities [40]; each facet of such a polytope P has at most one non-simple vertex. Not only does the weight algebra of P have interesting properties, but the dimensions of some of its grades have combinatorial significance. We shall improve this result to give a generalization of the Hodge Riemann Minkowski inequalities, and the Lefschetz Decomposition (proved in [24]). This allows us to prove Stanley's conjecture of the unimodality of the h-vector (in the dual) for polytopes with at most one non-simple vertex in each $\lfloor (d+1)/2 \rfloor$-face. This combinatorial result has recently been proved for general polytopes by Karu [11]. His proof does not use weights, but rather the intersection cohomology of the normal fan used in Stanley's original proof (in [36]) of the g-theorem. The proof however follows the inductive approach found in [24].

We shall then describe a geometric construction which identifies certain multiplicative maps on the weight-spaces of a polytope P with weights on a pyramid over P. In order to prove the g-theorem, it is sufficient to show that the weights on the pyramid restrict to P in the correct way. We describe a construction which goes some way to proving this, and the difficulties which arise.

The multiplication of a weight-space by a 1-weight is shown only to be singular for a small set of 1-weights. In the final part of this chapter, we shall examine some examples of these sets, and make some generalizations.

5.1 A combinatorial motivation

McMullen defines the f-polynomial $f(P,\sigma,\tau)$ of a d-polyhedron P to be

$$\sum_{r=0}^{d} f_r(P)\sigma^{d-r}\tau^r,$$

where $f_r(P)$ is the number of faces of P of dimension r. It is often useful to instead look at the h-polynomial of P, which is obtained by a change of variables:

$$h(P,\sigma,\tau) = \sum_{r=0}^{d} h_r(P)\sigma^{d-r}\tau^r := f(P,\sigma,\tau-\sigma).$$

For simple polytopes, the h-polynomial is symmetric in σ and τ. This theorem is a restatement of the well-known *Dehn-Sommerville equations*, which hold for more general objects than polytopes. Define the g-polynomial of P by

$$(\sigma-\tau)h(P,\sigma,\tau) = g(P,\sigma,\tau) = \sum_{r=0}^{d+1} g_r(P)\sigma^{d-r+1}\tau^r,$$

so that

$$g_r(P) = \begin{cases} 1, & \text{for } r = 0, \\ h_r(P) - h_{r-1}(P), & \text{for } 1 \leqslant r \leqslant d+1. \end{cases}$$

We often refer to the f-vector, defined by $f(P) = (f_0(P),\dots,f_d(P))$, with the h- and g-vectors defined analogously.

The g-theorem, (stated below), was conjectured in 1970 by McMullen [19] and completely describes the possible g-vectors of simple polytopes. The conditions of the g-theorem are often called *McMullen's conditions*. The necessity of these conditions was first proved by Billera and Lee [4, 5] by constructing a polytope with a given M-sequence as its g-vector. Stanley proved sufficiency in [36] using deep tools of algebraic geometry. After describing the polytope algebra in [23], McMullen gave a much simpler proof in [24]. He simplified this proof further in [25] by using the weight algebra, dispensing with the polytope algebra entirely.

Theorem 5.1.1 (g-theorem) *For $(g_0, \ldots, g_{d+1})$ to be the g-vector of some simple d-polytope, it is necessary and sufficient that*

i) $g_r = -g_{d+1-r}$ *for each* r,

ii) $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-sequence.*

We can define an M-sequence purely combinatorially. If a, r are positive integers, then there is a unique sequence $a_r, a_{r-1}, \ldots, a_i$ such that

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_i}{i}$$

and $a_r > a_{r-1} > \cdots > a_i \geqslant i \geqslant 1$. If s is another positive integer then we define the *partial power* $a^{\langle s|r \rangle}$ to be

$$a^{\langle s|r \rangle} = \binom{a_r + s - r}{s} + \binom{a_{r-1} + s - r}{s-1} + \cdots + \binom{a_i + s - r}{i + s - r}.$$

We also define $0^{\langle r|s \rangle} = 0$ for all r and s. A sequence $(h_0, h_1, \ldots)$ is an *M-sequence* if $h_0 = 1$, $h_1 > 0$, and $0 \leqslant h_{r+1} \leqslant h_r^{\langle r+1|r \rangle}$ for $r \geqslant 1$.

The following result forms the starting point for every proof of the g-theorem so far.

Theorem 5.1.2 (Macaulay [16]) *There is a graded (commutative) algebra* $R = \bigoplus_{r\geqslant 0} R_r$ *over a field* $\mathbb{F}$ *generated by the finite dimensional* R_1*, with* $R_0 \cong \mathbb{F}$*, if and only if the sequence given by* $g_r = \dim(R_r)$ *is an* M*-sequence.*

Thus the necessity of the conditions follows from constructing a polynomial algebra

$$R = \bigoplus_{r=0}^{\lfloor d/2 \rfloor} R_r,$$

generated by R_1, with $\dim R_r = g_r(P)$ for $0 \leqslant r \leqslant d/2$.

5.2 Weights

We now define the structure in which we shall work. Let P be a d-polyhedron (not necessarily bounded) and let $\mathcal{F}_r(P)$ denote the set of r-faces of P for $r = 0, \ldots, d$. An *r-weight* on P is a real-valued function a on $\mathcal{F}_r(P)$ which satisfies the *Minkowski relation* on each $G \in \mathcal{F}_{r+1}(P)$, namely,

$$\sum_{F \in \mathcal{F}_r(G)} a(F) u(F, G) = o,$$

where the vector $u(F, G)$ is the unit outer normal vector to F parallel to $\operatorname{aff}(G)$.

When P is a polytope and $a(F) = \operatorname{Vol}_r(F)$, this relation is satisfied (this is Minkowski's Theorem – see [8] for a proof); we refer to this particular weight as the *r-class* of P. It is easy to see that the r-weights on P form a vector space, which we shall denote $\Omega_r(P)$. We write

$$\Omega(P) = \bigoplus_{r=0}^{d} \Omega_r(P).$$

Notice that $\Omega_r(P)$ could have been defined equivalently on the type-cone of P, since strongly isomorphic polytopes have parallel faces. Indeed, if $P \preccurlyeq Q$, then $a \in \Omega_r(P)$ may be regarded as an r-weight on Q thus:

$$a(F(Q,u)) = \begin{cases} a(F(P,u)), & \text{if } \dim F(P,u) = r, \\ 0, & \text{if } \dim F(P,u) < r. \end{cases}$$

An important property of the space $\Omega_1(P)$ of 1-weights on P is its relationship with the type cone $\mathcal{K}(P)$. The 1-classes of elements in $\mathcal{K}(P)$ are 1-weights on P itself, and take positive values on every edge of P. Indeed, these 1-classes actually generate $\Omega_1(P)$.

Theorem 5.2.1 (McMullen [25]) *The 1-weights on a polytope P which take positive values on every edge are in one-to-one correspondence with elements of $\mathcal{P}$.*

If P is a simple polytope, then the space $\Omega_r(P)$ has dimension $h_r(P)$; see [25] for details. The restriction $a|_F$ of a weight a on P to a face F of P is clearly a weight on F. Moreover, if P is simple, then to any small perturbation of the facets of F there corresponds a small perturbation of those of P, and it immediately follows that $\Omega_1(F) = \Omega_1(P)|_F$ (an observation in [25]). It is also shown in [25] that $\Omega_r(P)$ is generated by the r-classes of polytopes in $\mathcal{K}(P)$ for $r = 0, \ldots, d$, generalizing Theorem 5.2.1 for simple polytopes. Thus, when talking about weights, we are in fact talking about linear combinations of volumes. This observation makes our constructions very natural and easy to visualize. The fact that $\Omega_r(P)$ is generated by r-classes of polytopes also allows us to generalize $\Omega_1(F) = \Omega_1(P)|_F$ to the following, using the same argument.

Lemma 5.2.2 *If P is a simple polytope and $F \in \mathcal{F}(P)$, then*

$$\Omega_r(F) = \Omega_r(P)|_F$$

for $r = 0, \ldots, d-1$.

This does not hold for non-simple P in general. For example, if P is a square based pyramid, with the triangular faces equilateral, then a 1-weight on P must take equal values on every edge of each traingle. But every edge of P is an edge of such a triangle, and so the 1-weight must take equal values on every edge: the space of 1-weights on P is therefore 1-dimensional. The square base F, however, has a 2-dimensional space of 1-weights: a 1-weight on F must only take equal values on opposite edges.

As we shall see in the next section, a multiplication may be defined on $\Omega(P)$, which is preserved by restriction to a face of P. This multiplication has some very useful features, and provides a link between the algebra and combinatorics of the area. The algebra $\Omega(P)$ is graded and the dimensions of those grades have combinatorial significance. Stanley [36] proved the following important theorem, which provides a Lefschetz decomposition of $\Omega(P)$, and hence proves the g-theorem.

Theorem 5.2.3 (Stanley [36]) *If P is a simple d-polytope, then there exists $\omega \in \Omega_1(P)$ such that, for $r \leqslant d/2$,*

$$\omega^{d-2r}\Omega_r(P) = \Omega_{d-r}(P).$$

McMullen [24] showed that that the 1-class p of P was a suitable choice for ω. In particular, for $r \leqslant (d-1)/2$, the map induced by multiplication by ω is injective. The quotient algebra

$$\Omega(P)/\langle\omega\rangle = \bigoplus_{r=0}^{(d-1)/2} \Omega_r(P)/\omega\Omega_{r-1}(P)$$

is thus generated by its first grade, and

$$\dim(\Omega_r(P)/\omega\Omega_{r-1}(P)) = h_r(P) - h_{r-1}(P) = g_r(P).$$

This is the condition, described in Section 5.1, for $g_0, \ldots, g_{(d-1)/2}$ to be an M-sequence, and the g-theorem follows.

This result was originally proved in the dual in [36], but it was then re-proved (and greatly simplified) by McMullen [24]. Define the *rth primitive space* of $\Omega(P)$ to be

$$\tilde{\Omega}_r(P) = \left\{x \in \Omega_r(P) : p^{d-2r+1}x = o\right\}.$$

Then Theorem 5.2.3 proves that $\Omega(P)$ admits a *Lefschetz decomposition*:

Theorem 5.2.4 (Lefschetz Decomposition) *If P is a simple d-polytope, then*

$$\Omega_s(P) = \bigoplus_{r=0}^{s} p^{s-r}\tilde{\Omega}_r(P).$$

In [24], the stronger Hodge-Riemann-Minkowski inequalities are also proved. Indeed they form part of the proof of Theorem 5.2.4.

Theorem 5.2.5 (Hodge-Riemann-Minkowski inequalities) *Let $0 \leqslant r \leqslant \frac{1}{2}d$. Then the quadratic form*

$$(-1)^r p^{d-2r} x^2$$

is positive definite on the primitive space $\tilde{\Omega}_r(P)$.

McMullen's proof of Theorem 5.2.5 includes an important step which deduces Theorem 5.2.3 from Theorem 5.2.5 in dimension $d-1$. It is in this step that convexity is crucial. We reproduce it here.

Theorem 5.2.6 *Theorem 5.2.5 in dimension $d-1$ implies Theorem 5.2.3.*

Proof. Let $x \in \Omega_r(P)$ be such that $p^{d-2r}x = o$. We shall deduce that $x = o$. The restriction of this equation to a facet F (with 1-class f) is

$$p|_F^{d-2r} x|_F = f^{d-2r} x|_F = o,$$

which is the primitivity condition for $x|_F$. Thus, by Theorem 5.2.5 applied to F,

$$(-1)^r p|_F^{d-2r-1} x|_F^2 \geqslant 0,$$

with equality if and only if $x|_F = o$. The multiplication of weights has the property that

$$(-1)^r p^{d-2r} x^2 = \sum_F (-1)^r p|_F^{d-2r-1} x|_F^2 \eta_F,$$

where η_F is the support parameter of F in P. (This is explained in the next section). The convexity condition says that all of the support parameters for the facets are positive (after a suitable translation), and so

$$(-1)^r p^{d-2r} x^2 \geqslant 0,$$

with equality if and only if $x_| F = o$ for every facet F, that is, if and only if $x = o$. □

5.3 Multiplication and local weights

Multiplication of weights is described by McMullen [25] with definitions that rely heavily on convexity; we shall first reproduce most of that description here. Later we shall present an alternative construction that tells us something about the local behaviour of weights. It has the advantage that convexity plays no major role.

Our first step shall be to multiply a weight a on P with a weight b on an orthogonal polytope Q to obtain a weight $a \times b$ on $P \times Q$. We shall then show that a weight on a polytope induces a weight on its image under a linear projection. Since $P + Q$ is such an image of $P \times Q$, a product $ab \in \Omega(P + Q)$ is obtained.

Lemma 5.3.1 *Weights $a \in \Omega_r(P)$ and $b \in \Omega_s(Q)$ induce a weight $a \times b \in \Omega_{r+s}(P \times Q)$.*

Proof. Define $a \times b$ by

$$(a \times b)(F \times G) = a(F)b(G),$$

for $F \in \mathcal{F}_r(P)$ and $G \in \mathcal{F}_s(Q)$. Thus $a \times b$ is non-zero only on faces of this type, and the Minkowski relations need only be verified on a product of an $(r+1)$-face of P with an s-face of Q, or a product of an r-face of P with an $(s+1)$-face of Q. By symmetry, we need only consider the former case. If F' is an $(r+1)$-face of P and G is an s-face of Q, then

$$u(F \times G, F' \times G) = (u(F, F'), o).$$

The Minkowski relation

$$\sum_{F \in \mathcal{F}_r(F')} a(F)u(F, F') = o$$

implies that

$$\sum_{F\in\mathcal{F}_r(F')} (a\times b)(F\times G)u(F\times G, F'\times G) = \sum_{F\in\mathcal{F}_r(F')} a(F)b(G)u(F,F',o) = (o,o),$$

and hence $(a\times b)\in\Omega_{r+s}(P\times Q)$. □

The second part is a little more technical. We shall only sketch the ideas, which are geometric, but we shall include some useful results along the way.

Lemma 5.3.2 *If $P\subset\mathbb{X}$ and $Q\subset\mathbb{Y}$ are d-polytopes with $Q=P\phi$, for some linear map $\phi:\mathbb{X}\longrightarrow\mathbb{Y}$, then ϕ induces an isomorphism between $\Omega_r(P)$ and $\Omega_r(Q)$.*

Proof. Let $\phi^*:\mathbb{Y}^*\longrightarrow\mathbb{X}^*$ be the dual map between the dual spaces. The vector $u(F,G)\phi^*$ is normal to the corresponding face $F'\in\mathcal{F}_r(P)$ in the affine hull of the corresponding face $G\in\mathcal{F}_{r+1}(P)$. The length of $u(F,G)\phi^*$ is $\gamma(F)$, given by

$$\gamma(F)\operatorname{Vol}_r(F)=\operatorname{Vol}_r(F\phi).$$

Thus if

$$\sum_{F'\in\mathcal{F}_{r+1}(G'} a(F)u(F',G')=o$$

for each $(r+1)$-face G' of P, then

$$\sum_{F\in\mathcal{F}_{r+1}(G} a(F')\gamma(F)u(F,G)=o.$$

Thus we define $a\phi\in\Omega_r(Q)$ by

$$(a\phi)(F)=a(F')\gamma(F).$$

□

It remains to deal with the case when ϕ is singular on $\operatorname{aff}(Q)$.

Lemma 5.3.3 *If a is a weight on Q, and $P = Q\phi$, then the image $a\phi$ is a weight on P.*

Proof. By the previous lemma, it is enough to consider projections with 1-dimensional kernel. Let a be an r-weight on a d-polytope Q and let P be the image of Q, under a projection ϕ, onto a hyperplane below it. If we "view" the pre-image of an $(r+1)$-face F of P from above, we see some faces of $F_1, \ldots, F_n$ of Q inducing a subdivision of F. But r-faces which are not mapped into r-faces of P are mapped to the interior of P and form the boundary between two cells $F_i\phi$ and $F_j\phi$ of the subdivision of F. The outer normals $u(G, F_i\phi)$ and $u(G, F_j\phi)$ are equal and opposite, where $G = F_i \cap F_j$. Therefore, if we sum the Minkowski relations over $F_1\phi, \ldots, F_n\phi$, then we obtain a linear dependence of the outer unit normals of F. The value of $a\phi$ on a face G of F is the coefficient of $u(G, F)$ in this dependence. This is given explicitly by

$$(a\phi)(G) = \sum_{i=1}^{m} (a\phi)(G_i\phi),$$

where $G_1, \ldots, G_m$ are the faces of $F_1, \ldots, F_n$ which are mapped to cells of the induced subdivision of G. □

By using the words *above* and *below*, we overlooked the fact that this involves a choice. Indeed, the construction would be identical if we projected Q in the opposite direction, although we would induce a different subdivision of P. It is not immediately obvious that this subdivision induces the same weight on P, and in order to prove that it indeed does, we would be forced to introduce more concepts and definitions. Without wishing to be distracted, we instead refer the reader to [25]. For more detail on subdivisions induced by linear maps we refer the reader to [42]. We have proved

Theorem 5.3.4 (McMullen [25]) *A linear map between polytopes induces a corresponding linear map between their weight spaces.*

We now have enough to define multiplication of weights. Let $P, Q \subset \mathbb{E}^d$ be polytopes which contain the origin in their relative interior (not necessarily full dimensional), and let $\phi : \mathrm{aff}(P) \times \mathrm{aff}(Q) \longrightarrow \mathrm{aff}(P \cup Q)$ be the linear map given by

$$(x, y)\phi = x + y.$$

Thus the image of the product $P \times Q$ under ϕ is the Minkowski sum $P + Q$. For weights $a \in \Omega_r(P)$ and $b \in \Omega_s(Q)$, define their product $ab \in \Omega_{r+s}(P + Q)$ by

$$ab = (a \times b)\phi.$$

The multiplication is associative and commutative. Notice that, by multiplying an r-weight a on P by a 0-weight b on Q, we obtain an r-weight ab on $P + Q$. If this 0-weight takes a value of 1 on every vertex, then we may identify ab with a and regard a as a weight on $P+Q$. Since every polytope is a summand of a simple polytope, we often assume that we are working with a simple polytope.

It is easy to verify that multiplication of weights restricts to faces in a natural way:

Theorem 5.3.5 (McMullen [25]) *If $F = F(P, u)$ and $G = F(Q, u)$, for some $u \in \mathrm{aff}(P \cup Q)$, and if $a \in \Omega_r(P)$ and $b \in \Omega_s(Q)$, then the restriction of ab to $F + G = F(P + Q, u)$ is the product $a|_F b|_G$ of the restrictions of a and b to F and G respectively.*

We state an alternative formulation of multiplication of two weights, in a special case.

The equivalence of this with the previous definition is proved in [25], but we shall prove it again later in a different way.

Proposition 5.3.6 *If p is the 1-class of a polytope*

$$P = \{x \in \mathbb{E} : \langle x, u_i \rangle \leqslant \eta_i \text{ for } i = 1, \ldots, n\},$$

and $b \in \Omega_{d-1}(P)$, then

$$pb = \sum_{j=1}^{n} \eta_j b(F_j),$$

where η_j is the support parameter of F_j.

Note that, for some $t \in \mathbb{E}^d$, the translate $P + t$ has support parameters

$$\eta_1 + \langle t, u_1 \rangle, \ldots, \eta_n + \langle t, u_n \rangle,$$

and

$$\sum_{j=1}^{n} (\eta_j + \langle t, u_j \rangle) b(F_j) = \sum_{j=1}^{n} \eta_j b(F_j) + \sum_{j=1}^{n} \langle t, u_j \rangle b(F_j).$$

The final term vanishes since b is a weight, and so pb is well defined.

Note that, since the 1-classes of polytopes in $\mathcal{K}(P)$ generate the whole space of 1-weights on P, every 1-weight on P may be expressed as the difference of 1-classes of polytopes in $\mathcal{K}(P)$. This construction therefore allows us to multiply a $(d-1)$-weight by any 1-weight.

This completes our overview of McMullen's work on weights from [23, 24, 25]. As promised, we shall now give our second presentation of the definition of multiplication of weights. This approach is essentially the same as the construction of the face ring and a certain quotient of it; we shall discuss this later. If F is a face of a polyhedron P, and a is a weight on P, then we say a is *local* to F if and only if a vanishes on every face which does not intersect F. We say that a facet F of P is *simply situated* if, for each $G \in \mathcal{F}_{r-1}(F)$, there exists a unique r-face $\overline{G} \in \mathcal{F}_r(P)$ such that $\overline{G} \cap F = G$.

Theorem 5.3.7 *If Q is a polytope in V, then the space of r-weights on Q which are local to a simply situated facet P is isomorphic to $\Omega_{r-1}(P)$. Moreover, the restriction of this space of weights to P is $(p-p')\Omega_{r-1}(P) \subseteq \Omega_r(P)$, where p is the 1-class of P and p' is the 1-class of the intersection P' of Q with a hyperplane H parallel to and close to P.*

Proof. Note that P' is strongly isomorphic to P: if $F \in \mathcal{F}_{r-1}(P)$, then $\overline{F} \cap H$ is the corresponding (parallel) $(r-1)$-face of P'. Thus the 1-weight $p - p'$ is defined up to a non-zero multiple, and hence $(p-p')\Omega_{r-1}(P)$ is well defined.

For each $(r-1)$-face F of P denote by $\overline{F}$ the r-face of Q whose intersection with P, is F. Let $G \in \mathcal{F}_r(P)$. Recall that $(\operatorname{aff} G)^{\perp}$ denotes the orthogonal complement of $\operatorname{aff} G$. Thus, for each $a \in \Omega_{r-1}(P)$,

$$\begin{aligned}
\sum_{F \in \mathcal{F}_{r-1}(G)} a(F)u(F,G) = o \quad &\Leftrightarrow \quad \sum_{F \in \mathcal{F}_{r-1}(G)} \frac{a(F)}{\langle u(F,G), u(\overline{F},\overline{G})\rangle} u(\overline{F},\overline{G}) \in (\operatorname{aff} G)^{\perp} \\
&\Leftrightarrow \quad \sum_{F \in \mathcal{F}_{r-1}(G)} \frac{a(F)s_F}{s_G} u(\overline{F},\overline{G}) \in (\operatorname{aff} G)^{\perp} \\
&\Leftrightarrow \quad \sum_{F \in \mathcal{F}_{r-1}(G)} a(F)(s_F)u(\overline{F},\overline{G}) \in (\operatorname{aff} G)^{\perp},
\end{aligned}$$

where s_F is the distance between F and the corresponding face F' in P', and s_G is the distance between G and G'. This argument is illustrated in Figure 5.1, where the reader should regard the plane of the page as $\operatorname{aff} \overline{G}$.

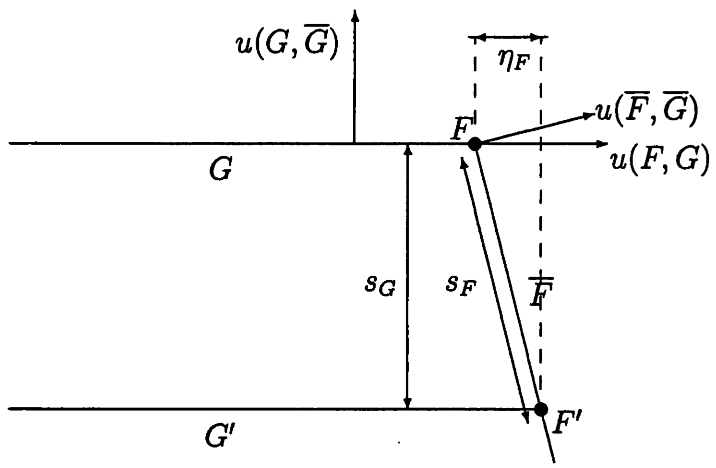


Figure 5.1: Scaling factors

If, for each $F \in \mathcal{F}_{r-1}(G)$, we set

$$b(\overline{F}) = a(F)s_F,$$

and

$$\begin{aligned} b(G) &= -\sum_{F\in\mathcal{F}_{r-1}(G)} a(F)s_F\left\langle u(\overline{F},\overline{G}), u(G,\overline{G})\right\rangle \\ &= -\sum_{F\in\mathcal{F}_{r-1}(G)} a(F)\eta_F \end{aligned}$$

then $b \in \Omega_r(Q)$. Note that η_F is the change of support parameter of F in G, and so $b = -(p'-p)a = (p-p')a$, by Proposition 5.3.6.

We have constructed a map from $\Omega_{r-1}(P)$ into $\Omega_r(Q)$. The map is injective since s_F is non-zero for each $F \in \mathcal{F}_{r-1}(G)$, so that $a(F)$ may be recovered from $b(\overline{F})$.

If $b \in \Omega_r(Q)$ is local to P, then set $a(F) = b(\overline{F})s_F^{-1}$, so that $b(\overline{F}) = a(F)s_F$. The previous argument is reversible, and we deduce that $a \in \Omega_{r-1}(P)$. The map from $\Omega_{r-1}(P)$ to the weights on Q that are local to P is therefore an isomorphism. □

Let Q and H be as above, and let Q' be the intersection of Q with the halfspace the other side of H from P: Q' is obtained by "slicing" the facet P off, introducing a strongly isomorphic facet P'. Let q and q' be the 1-classes of Q and Q' respectively. The construction of the map from $\Omega_{r-1}(P)$ to $\Omega_r(Q)$ in Theorem 5.3.7 is equivalent to demonstrating that, for each $\omega \in \Omega_{r-1}(Q)$, the product $(q - q')\omega$ depends only on the restriction $a = \omega_P$ of ω to P, and the restriction of $(q - q')\omega$ to P is then $(p - p')a$. Note that the definition $b(\overline{F}) = a(F)s_F$ may be regarded as the product of the restrictions of $q - q'$ and a to $\overline{F}$, since the only facet of $\overline{F}$ to have moved is F.

Theorem 5.3.8 *If Q is a simple d-polytope and $a \in \Omega_r(Q)$, and $b_1, \ldots, b_s$ are non-zero 1-weights which are local to distinct facets $F_1, \ldots, F_s$ respectively, then $ab_1 \cdots b_s = o$ if and only if $a|_F = o$, where*

$$F = \bigcap_{i=1}^{s} F_i.$$

Moreover, $ab_1 \cdots b_s$ is local to F.

Proof. This follows from repeating the above argument s times. □

This has important consequences. The first result is from [24, 25], but we have come at it from a different angle:

Theorem 5.3.9 (Separation theorem) *If Q is simple, then, for $r = 0, \ldots, d$, if $o \neq a \in \Omega_r(Q)$, then there exists $b \in (\Omega_1(Q))^{d-r} \subset \Omega_{d-r}(Q)$ such that $ab \neq 0$.*

Proof. Let F be an r-face such that $a(F) \neq 0$, and use the previous theorem. □

We say that $(\Omega_1(Q))^r$ *separates* $\Omega_{d-r}(Q)$. By reversing the positions of r and $d-r$, we may show that $(\Omega_1(Q))^{d-r}$ separates $\Omega_r(Q)$. The dimension of a space cannot exceed that of one that separates it, and so Theorem 5.3.9 applied twice, once with r and $d-r$ interchanged yields

$$\dim \Omega_{d-r}(Q) \leqslant (\Omega_1(Q))^r \leqslant \Omega_r(Q) \leqslant \Omega_{d-r}(Q).$$

Thus we must have equality throughout, and $(\Omega_1(Q))^r = \Omega_r(Q)$. This is shown in [24].

Each point in the representation space of P may be expressed as the difference of two representative vectors of polytopes in $\mathcal{K}(P)$. Hence there is a natural identification of a point in the representation space with a 1-weight.

Proposition 5.3.10 *Let F be a facet of a polytope P, and let the point $\overline{u}$ in the representation space correspond to F. A non-zero 1-weight on P which is local to F is identified with a non-zero multiple of the point $\overline{u}$.*

Proof. If P' is obtained by slightly perturbing the facet F of P by a distance ε, then the difference of the 1-classes of P and P' is non-zero, and is local to F. The difference in the support parameters of the facets of P and P' is non-zero only for F. □

Theorem 5.3.11 *For distinct facets $F_1, \ldots, F_r$,*

$$F = \bigcap_{i=1}^{r} F_i \neq \emptyset$$

if and only if $m = \overline{u}_1 \cdots \overline{u}_r \in \Omega_r(P)$ is non-zero.

Proof. If $F \neq \emptyset$, then there exists a weight $a \in \Omega_{d-r}(P)$ such that $a(F) \neq 0$ — for example the $(d-r)$-class of P. Thus $ma \neq 0$, by the same argument used to prove Theorem 5.3.9, and so $m \neq o$. For the converse, since $m \neq o$, then $m' = \overline{u}_1 \cdots \overline{u}_{r-1} \neq o$. By induction,

$$F' = \bigcap_{i=1}^{r-1} F_i \neq \emptyset.$$

If $F_r \cap F' = \emptyset$, then $m'|_{F_r} = o$, and so $m'\overline{u}_r = o$. □

We can now present our alternative definition of weight multiplication for simple polytopes. We regard an r-weight as a homogeneous polynomial of degree r in $\overline{u}_1, \ldots, \overline{u}_n$; in particular, a 1-weight is a linear combination of $\overline{u}_1, \ldots, \overline{u}_n$. We first show that we need only consider linear combinations of square-free monomials.

Lemma 5.3.12 *If P is a simple d-polytope, then, for $r = 1, \ldots, d$, $\Omega_r(P) = (\Omega_1(P))^r$ is generated by square-free monomials of degree r in $\overline{u}_1, \ldots, \overline{u}_n$.*

Proof. There is nothing to prove for $r = 1$. We assume that the result is true for $s < r$, and proceed by induction. Let a be a monomial in $\overline{u}_1, \ldots, \overline{u}_n$ of degree $r-1$. Without loss of generality, let $a = \overline{u}_1 \cdots \overline{u}_s$. If $a \neq o$, then

$$F = \bigcap_{i=1}^{s} F_i \neq \emptyset$$

by Theorem 5.3.11. We now recall Theorem 3.2.1, which implies that the 1-class p of P lies in the positive hull of $\{\overline{u}_{s+1}, \ldots, \overline{u}_n\}$. Since P is simple, then the type-cone of P is $(n-d)$-dimensional, and so the positive hull of $\{\overline{u}_{s+1}, \ldots, \overline{u}_n\}$ is $(n-d)$-dimensional. In particular, any 1-weight b may be expressed as a linear combination of $\overline{u}_{s+1}, \ldots, \overline{u}_n$, and the product ab may be expressed as a linear combination of square-free monomials of degree r. □

By linearity, Lemma 5.3.12 allows us to calculate the product of an $(r-1)$-weight (expressed as a linear combination of square-free monomials) and a 1-weight, without even mentioning outer normal vectors.

5.4 Ideals and Stanley's Conjecture

In [37], Stanley generalized the h-polynomial, and gave a definition which is valid for all polytopes. Rather than generalizing the h-polynomial for simple polytopes, Stanley's starting point was the h-polynomial for a simplicial polytope P, given by

$$h(P,\sigma,\tau) = \sum_{F\in\mathcal{F}(P)\setminus P} (\tau-\sigma)^{\dim P-\dim F-1}\sigma^{\dim F+1}.$$

We shall follow Stanley's lead, as the definition is a little more natural, and the definition is now widely used. We therefore warn the reader that, in the rest of this section, the h-numbers of a simplicial polytope P are what we refer to elsewhere as h-numbers of the simple polytope P^Δ. The generating polynomials $h(P,\sigma,\tau) = \sum_{r=0}^{d} h_r(P)\sigma^{\dim P-r}\tau^r$ and $g(P,\sigma,\tau) = \sum_{r=0}^{\lfloor \dim F/2\rfloor} g_r(P)$ are defined inductively by

$$\begin{aligned}
g(\emptyset,\sigma,\tau) &= 1,\\
h(P,\sigma,\tau) &= \sum_{F\in\mathcal{F}(P)\setminus P} (\tau-\sigma)^{\dim P-\dim F-1} g(F,\sigma,\tau), \text{ and}\\
g(P,\sigma,\tau) &= [(\sigma-\tau)h(P,\sigma,\tau)]_{\lfloor \dim P/2\rfloor},
\end{aligned}$$

where $[\cdots]_{\lfloor \dim P/2 \rfloor}$ denotes that we ignore terms within the brackets which are divisible by $\tau^{\lfloor \dim P/2 \rfloor + 1}$. As the following theorem shows, the g-polynomial of an r-simplex is σ^{r+1}, and so this definition is consistent with the one for simplicial polytopes.

Theorem 5.4.1 *For $d \geqslant 0$, if P is a d-simplex, then*

$$h(P, \sigma, \tau) = \sigma^d + \sigma^{d-1}\tau + \cdots + \sigma\tau^{d-1} + \tau^d.$$

and

$$g(P, \sigma, \tau) = \sigma^{d+1}.$$

Proof. If $d = 0$, then $h(P, \sigma, \tau) = 1$, and $g(P, \sigma, \tau) = [\sigma - \tau]_0 = \sigma$. Let P be a d-simplex, and assume that the theorem holds for all $r \leqslant d - 1$. Then

$$\begin{aligned} h(P, \sigma, \tau) &= \sum_{F \in \mathcal{F}(P)} (\tau - \sigma)^{d - \dim F - 1} \sigma^{\dim F + 1} \\ &= \sum_{r=-1}^{d-1} \binom{d+1}{r+1} (\tau - \sigma)^{d-r-1} \sigma^{r+1} \\ &= \sum_{r=0}^{d} \binom{d+1}{r} (\tau - \sigma)^{d-r} \sigma^r \\ &= \frac{\tau^{d+1} - \sigma^{d+1}}{\tau - \sigma} \\ &= \sigma^d + \sigma^{d-1}\tau + \cdots + \sigma\tau^{d-1} + \tau^d. \end{aligned}$$

This proves the first part, and so

$$g(P, \sigma, \tau) = [(\sigma - \tau) h(P, \sigma, \tau)]_{\lfloor d/2 \rfloor} = [\sigma^{d+1} - \tau^{d+1}]_{\lfloor d/2 \rfloor} = \sigma^{d+1},$$

which completes the induction. □

Stanley [37] showed that the h-polynomial is symmetric in σ and τ for general polytopes, and made the following conjecture.

Conjecture 5.4.2 (Stanley's conjecture,[37]) *The h-numbers of a general d-polytope P satisfy*

$$h_0(P) \leqslant h_1(P) \leqslant \cdots \leqslant h_{\lfloor d/2 \rfloor}(P).$$

We shall prove this result for a d-polytope P that has simplicial ridges. By gluing a pyramid $F * v$ with apex v onto each non-simplicial facet F of P, we may obtain a simplicial polytope Q. We must ensure that the apex of each pyramids is sufficiently close to the base that Q is convex. Note that Q is not unique, but any choice of Q will have the same f-vector.

The polytope P is the polar of a polytope P^Δ which may be made into a simple polytope Q^Δ by cutting off those vertices which are contained in more than d facets.

We shall show that the dimensions of some of the weight-spaces of P^Δ give the h-numbers of P. We may express the h-polynomial of P in terms of the h-polynomials of the facets $F_1, \ldots, F_n$, which are not simplices, and $h(Q, \sigma, \tau)$.

Theorem 5.4.3 *If P is a polytope with simplicial ridges, and Q is a polytope obtained by gluing pyramids onto the non-simplicial facets $F_1, \ldots, F_n$ of P, then*

$$h(P, \sigma, \tau) = h(Q, \sigma, \tau) - \sigma \sum_{i=1}^{n} h(F_i, \sigma, \tau) + \sum_{i=1}^{n} g(F_i, \sigma, \tau).$$

Proof. We shall make no assumption that Q is simplicial, and so it suffices to prove the case $n = 1$; Q is obtained from P by gluing a pyramid with apex v to one of its facets

F. By the formula,

$$\begin{aligned}
h(Q,\sigma,\tau) &= \sum_{G\in\mathcal{F}(Q)} (\tau-\sigma)^{d-\dim G-1} g(G,\sigma,\tau) \\
&= \sum_{G\in\mathcal{F}(P)} (\tau-\sigma)^{d-\dim G-1} g(G,\sigma,\tau) - (\tau-\sigma)^{d-\dim F-1} g(F,\sigma,\tau) \\
&\quad + \sum_{G\in\mathcal{F}(P),\, v\in G} (\tau-\sigma)^{d-\dim G-1} g(G,\sigma,\tau) \\
&= h(P,\sigma,\tau) - g(F,\sigma,\tau) + \sum_{G\in\mathcal{F}(F)} (\tau-\sigma)^{d-(\dim G+1)-1} g(G*v,\sigma,\tau).
\end{aligned}$$

Since F has simplicial ridges, then $G*v$ is a simplex for each $G\in\mathcal{F}(F)$, and so $g(G*v,\sigma,\tau)=\sigma^{\dim G+2}$, by Theorem 5.4.1. Hence

$$\begin{aligned}
h(Q,\sigma,\tau) &= h(P,\sigma,\tau) - g(F,\sigma,\tau) + \sum_{G\in\mathcal{F}(F)} (\tau-\sigma)^{d-(\dim G+1)-1}\sigma^{\dim G+2} \\
&= h(P,\sigma,\tau) - g(F,\sigma,\tau) + \sigma \sum_{G\in\mathcal{F}(F)} (\tau-\sigma)^{(d-1)-\dim G-1}\sigma^{\dim G+1} \\
&= h(P,\sigma,\tau) - g(F,\sigma,\tau) + \sigma h(F,\sigma,\tau).
\end{aligned}$$

□

Since no term of $g(F_i,\sigma,\tau)$ is divisible by $\tau^{\lfloor \dim F/2\rfloor+1}=\tau^{\lceil d/2\rceil}$, we have an immediate corollary.

Corollary 5.4.4 *For $r\geqslant \lceil d/2\rceil$,*

$$h_r(Q)=h_r(P)+\sum_{r=0}^{n} h_{r-1}(F_i).$$

We have already encountered ideals of the weight algebra on a simple polytope Q. We shall now examine those ideals which are themselves weight algebras of polytopes. We show that such an ideal is the weight algebra of a polytope P with simple edges (there are $d-1$ facets incident at each edge of P). For a special class of these polytopes, we show that the Hodge-Riemann-Minkowski Inequalities generalize.

We shall denote the rth grade of an ideal by a subscript r. Let F be a facet of a polytope P and let $I_r = \{a \in \Omega_r(P) : a|_F = o\}$. By Theorem 5.3.5, it follows that $I = \bigoplus_{r \geqslant 1} I_r$ is an ideal of $\Omega(P)$. It is easy to see that, if P is an unbounded polyhedron obtained by ignoring the facet hyperplane of F, then $I = \Omega(P)$. More generally, if I is the subset of $\Omega(P)$ which vanish on a fixed set $\mathcal{A}$ of faces, or, equivalently, there is a set M of monomials such that

$$I = \{a \in \Omega(P) : ma = o \text{ for each } m \in M\},$$

then $I = \bigoplus_{r=0}^{d} I_r$ is an ideal of $\Omega(P)$.

Let us introduce a special class of polytopes. A polytope is said to be *simple at the edges* if each edge is contained in exactly $d-1$ facets, or, equivalently, it becomes simple if we cut off every non-simple vertex. Polytopes which are simple at the edges are relevant here because of the following theorem. Let $\Omega^+(P) = \bigoplus_{r=1}^{d} \Omega_r(P)$.

Theorem 5.4.5 *If P is a summand of a simple polytope Q, then $\Omega^+(P)$ is an ideal of $\Omega(Q)$ if and only if P is simple at the edges.*

Recall that, if P and Q are d-polytopes, then $P \preccurlyeq Q$ if and only if

$$\dim F(P, u) \leqslant \dim F(Q, u)$$

for each $u \in \mathbb{E}^d$. We have a stronger result for a special family of polytopes.

Lemma 5.4.6 *If P is a summand of a simple polytope Q and* $\dim F(P,u) = \dim F(Q,u)$ *or* $\dim F(P,u) = 0$ *for each $u \in \mathbb{E}^d$, then P has simple edges.*

Proof. Let E be an edge of P, with normal cone N. By our assumption, N is contained in the normal cone N' of an edge E' of Q. Since Q is simple, N' contains exactly $d-2$ facet normals. Thus N also contains $d-2$ facet normals, and E is simple. □

Proof of theorem. Suppose that $\Omega^+(P)$ is an ideal of $\Omega(Q)$, and that $o \neq a \in \Omega_r(P) \leqslant \Omega_r(Q)$. Let $F = F(Q,u)$ be a k-face of Q such that $a|_F \neq o$. By the separation property of $\Omega(F)$, there is a $(k-r)$-weight on F which extends to some $b \in \Omega_{k-r}(Q)$ such that $(ab)|_F \neq o$. Since $\Omega(P)$ is an ideal of $\Omega(Q)$, the product ab lies in $\Omega(P)$, and $F(P,u)$ has dimension k also. By the lemma, P has simple edges. Using the lemma again, the converse follows from the previous construction of an ideal: an element of $\Omega(P)$ is an element of $\Omega(Q)$ that vanishes on the set

$$\{F(Q,u) : \dim F(P,u) \neq \dim F(Q,u)\}$$

of faces of Q. □

For the rest of the section, let P be a polytope with simple edges and let Q be a simple polytope obtained from P by cutting off every non-simple vertex of P. Thus $\Omega^+(P)$ is an ideal of $\Omega(Q)$. The new facets shall be called *inserted* facets. Note that the inserted facets do not meet each other.

Theorem 5.4.7 *If P has simple edges, then*

$$\dim \Omega_r(P) = h_r(P^\Delta)$$

for $r \geqslant \lceil d/2 \rceil$.

Proof. By Theorem 5.3.7, the space of r-weights on Q that are local to an inserted facet F is isomorphic to $\Omega_{r-1}(F)$, and the restriction of these local weights to F is $f\Omega_{r-1}(F)$, where f is the 1-class of F. This is because the intersection of Q with a hyperplane parallel and close to F is not only strongly isomorphic to F, but actually homothetic to F.

We use Theorem 5.2.3, or rather a weaker corollary that says that $f\Omega_{r-1}(F) = \Omega_r(P)$ for $r \geqslant d/2$. The restriction to F of the weights that are local to F is the whole of $\Omega_r(F)$: each r-weight on an inserted facet F extends locally to one on Q. Thus a weight $a \in \Omega_r(Q)$ may be expressed as a sum of weights local to inserted facets and a weight which vanishes on all of them. The latter may be regarded as a weight on P, and so

$$\dim \Omega_r(Q) = \dim \Omega_r(P) + \sum_F \dim \Omega_r(F),$$

where the sum ranges over all inserted facets F. Since Q and each inserted facet are simple we may make the substitutions $\dim \Omega_r(Q) = h_r(Q^\Delta)$ and $\dim \Omega_r(F) = h_r(F^\Delta)$ to give

$$h_r(Q^\Delta) = \dim \Omega_r(P) + \sum_F \dim h_r(F^\Delta).$$

Rearranging and applying Corollary 5.4.4 gives the required result. □

Let $I = \bigoplus_{r=1}^{d} I_r$ be the ideal in $\Omega(Q)$ given by

$$I = \langle x_F : \ F \text{ is an inserted facet} \rangle,$$

where x_F is the 1-weight on P which is local to F (unique up to scalar multiple).

Lemma 5.4.8 *The product of the ideals $\Omega^+(P)$ and I is zero.*

Proof. The restriction of $a \in \Omega^+(P)$ to any inserted facet F is zero and hence the product ax_F is also zero. Hence $aI = o$. □

In particular, if p and q are the 1-classes of P and Q respectively, then their difference $q - p$ lies in I_1, and hence

$$(q-p)\Omega(P) = 0.$$

We have the following useful decomposition of $\Omega_r(Q)$ for $r \leqslant d/2$.

Theorem 5.4.9 *For $r \leqslant d/2$,*

$$\Omega_r(Q) = \Omega_r(P) \oplus I_r \oplus X_r,$$

where X_r is a subspace of minimal dimension such that the restriction of X_r to an inserted facet F is the primitive space $\tilde{\Omega}_r(F)$.

Proof. Since Q is simple, the restriction of $\Omega_r(Q)$ to F is the whole of $\Omega_r(F)$. Let $Y \leqslant \Omega_r(Q)$ be a subspace of minimal dimension such that

$$Y|_F = \Omega_r(F)$$

for each inserted facet F; the elements of Y are uniquely identified by their restrictions to inserted facets. We may choose Y so that $x_F\Omega_{r-1}(Q)$ is a subspace of Y for each inserted facet F. Thus

$$Y = I_r \oplus X$$

for some complementary subspace X_r. Let F be an inserted facet. Then, by projecting X in the direction $x_F\Omega_{r-1}(Q)$ if necessary, we may assume that

$$X|_F = \tilde{\Omega}_r(F).$$

Such a projection does not affect the restriction to other inserted facets, since $x_F\Omega_{r-1}(Q)$ is local to F, and so we may assume that this property holds for all inserted facets, as required. It is easy to see that Y and $\Omega_r(P)$ are complementary spaces, since $\Omega_r(P)$ is the space of r-weights which vanish on all inserted facets. $\square$

We shall now extend two theorems, which will prove Stanley's Conjecture for a more general class of polytopes. Recall the Hodge-Riemann-Minkowski inequalities (Theorem 5.2.5), of which Theorem 5.2.3 is a corollary. We now give a generalized form of these inequalities for polytopes with not too many non-simple vertices, and similarly deduce an analogue to Theorem 5.2.4. We mimic the proof of Theorem 5.2.6, and then use continuity arguments to complete the proof. The bound on the number of non-simple vertices seems to be a technicality. It is conjectured that this bound may be dispensed with altogether, to extend the result to all polytopes with simple edges.

Theorem 5.4.10 (Generalized Hodge-Riemann-Minkowski inequalities) *Let P be a d-polytope with simple edges, such that no face of dimension $\lceil d/2 \rceil$ contains more that one non-simple vertex. Obtain a simple polytope Q by cutting off the non-simple vertices of P. Then the quadratic form*

$$(-1)^r p^{d-2r} x^2$$

is positive definite on the space

$$\tilde{\Omega}_r(Q)/I_r$$

for $r \leqslant \frac{1}{2}(d-1)$.

Since $pI_r = o$, the form is well defined. This means that, for $x \in \tilde{\Omega}_r(Q)$, the quadratic form $(-1)^r p^{d-2r} x^2 \geqslant 0$, with equality if and only if $x \in I_r$. We also have an extension of Theorem 5.2.3.

Theorem 5.4.11 *Let P and Q be as defined in Theorem 5.4.10. Then, for $r \leqslant \frac{1}{2}(d-1)$, the map*

$$p^{d-2r} : \Omega_r(Q)/I_r \longrightarrow \Omega_{d-r}(P)$$

is an isomorphism.

Both theorems follow from the following lemma. We conjecture that the bound on the number of non-simple vertices may be dispensed with.

Lemma 5.4.12 *Let $r \leqslant \frac{1}{2}(d-1)$. If no $(r+1)$-face of P contains more than one non-simple vertex, and $a \in \Omega_r(Q)$ is such that*

$$a|_F \in I_r|_F$$

for each non-inserted facet F, then

$$a \in I_r.$$

Proof. Observe that, for each $(r+1)$-face G, the restriction $a|_G$ lies in $I_r|_G$. In particular, if G does not intersect an inserted facet, then $a|_G = o$, and if it does intersect some inserted facet F', then $a|_G$ is local to F'.

For each inserted facet F, let $\omega_F : \mathcal{F}_r(Q) \longrightarrow \mathbb{R}$ be the map which is equal to $a(G)$ on each r-face G which intersects F and zero elsewhere. By our observation, ω_F is equal to a on every $(r+1)$-face which intersects F, and zero elsewhere. Hence ω_F satisfies the Minkowski relations and is local to F. Therefore $\omega_F \in I_r$.

It is now easy to see that $a = \sum_F \omega_F$, where the sum is taken over all inserted facets.

□

Proof of theorem. We begin by showing that Theorem 5.4.10 in dimension $d-1$ implies Theorem 5.4.11 in dimension d. We need only show that $\ker p^{d-2r} \leqslant I_r$, since the reverse inequality follows from Lemma 5.4.8. We assume that Theorem 5.4.10 holds for every facet of P, and that the corresponding facet of Q is obtained by cutting off all of its non-simple vertices (and perhaps some simple vertices). If

$$p^{d-2r}x = o$$

for some $x \in \Omega_r(Q)$, then

$$f^{d-2r}x|_F = o$$

for each facet F of Q, where $f = p|_F$. Hence, by Theorem 5.4.10 in dimension $d-1$,

$$(-1)^r f^{d-2r-1}x|_F^2 \geqslant 0,$$

with equality if and only if $x|_F = o$. Hence, by summing over all facets, which all have positive support parameters, we obtain

$$(-1)^r p^{d-2r}x^2 \geqslant 0,$$

with equality only if $x|_F \in I_r|_F$ for every facet F. But we do have equality by our initial assumption, and by the lemma, $x \in I_r$.

If $q_\lambda = (1-\lambda)p + \lambda q$, then $q_\lambda \longrightarrow p$ as $\lambda \longrightarrow 0$. We may identify the quotient space $\Omega_r(Q)/I_r$ with a subspace of $\Omega_r(Q)$, and so the quadratic form $(-1)^r q_\lambda^{d-2r} x^2$ tends to $(-1)^r p^{d-2r} x^2$ on this subspace. By Theorem 5.4.11, the latter is non-singular on this subspace, and hence has the same signature as $(-1)^r q_\lambda^{d-2r} x^2$, which proves Theorem 5.4.10 in dimension d, and the induction is complete. □

Combining Theorems 5.4.11 and 5.4.7, we have proved a special case of Stanley's conjecture (recently proved for general polytopes by Karu [11]).

Theorem 5.4.13 *If P is as defined in Theorem 5.4.10, then*

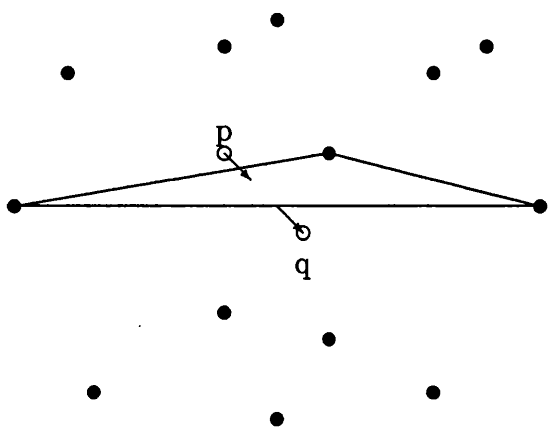
$$h_{\lceil d/2 \rceil}(P^\Delta) \geqslant h_{\lceil d/2 \rceil + 1}(P^\Delta) \geqslant \cdots \geqslant h_d(P^\Delta).$$

5.5 Flips

In order to compare weight spaces on different simple polytopes, we need some way of moving between strong isomorphism classes. A natural way of doing this is by the notion of a *flip*.

Let $U = (u_1, \ldots, u_n)$ span $\mathbb{E}^d$ and let $P, Q \in \mathcal{P}(U)$ be simple polytopes such that, in the representation space, $\mathcal{K}(P)$ and $\mathcal{K}(Q)$ are adjacent type-cones, and the intersection of their closures

$$\overline{\mathcal{K}(P)} \cap \overline{\mathcal{K}(Q)}$$



contains a subset of U of size $n-d-1$. This last condition is slightly weaker than insisting on general position for all of $u_1, \ldots, u_n$, as required by the description in [24].

If $\mathcal{K}(P) \cap \mathcal{K}(Q)$ is spanned by a set $\{\overline{u}_{d+2}, \ldots, \overline{u}_n\}$ say, then the remaining vertices $\{\overline{u}_1, \ldots, \overline{u}_{d+1}\}$ are divided into two disjoint sets, $\{\overline{u}_1, \ldots, \overline{u}_m\}$ and $\{\overline{u}_{m+1}, \ldots, \overline{u}_{d+1}\}$ say, which lie either side of $\operatorname{lin}\{\overline{u}_{d+2}, \ldots, \overline{u}_n\}$. If there are m points on the same side as P, then we say that Q is obtained from P by an *m-flip*.

Let P be obtained from Q by a flip, as above. Let $F_1, \ldots, F_{d+1}$ be the facets of P involved in the flip. Incidence relations not involving these facets are not affected by the flip. In P, there is a face $F = F_{m+1} \cap \cdots \cap F_{d+1}$ (say) which is an $(m-1)$-simplex bounded in its affine hull by the facets $F_1, \ldots, F_m$. In Q, we get the reversed pattern: there is a face $G = G_1 \cap \cdots \cap G_m$ which is a $(d-m)$-simplex bounded in its affine hull by $G_{m+1}, \ldots, G_{d+1}$, where G_i is the facet of Q parallel to F_i.

There is a corresponding pattern in the representation space. Here, the hyperplane $\operatorname{lin}\{\overline{u}_{d+2}, \ldots, \overline{u}_n\}$ separates $\mathcal{K}(P)$ from $\mathcal{K}(Q)$. Any subset of $\{\overline{u}_1, \ldots, \overline{u}_{d+1}\}$ which contains

p in its positive hull also contains q. Hence incidence relations which do not involve these facets are preserved. The removal of $\{\overline{u}_{m+1}, \ldots, \overline{u}_{d+1}\}$ leaves p in the positive hull of the remainder. Similarly, the removal of $\{\overline{u}_1, \ldots, \overline{u}_m\}$ leaves q in the positive hull of the remainder.

We remark that the inverse of an m-flip is a $(d+1-m)$-flip. We also note that, while a $(d+1)/2$-flip does not change the f-vector for d odd, it may change the combinatorial type. We now describe how the weight spaces are affected by an m-flip. We shall do this in two ways. The first is as described in [39] and [24], and describes everything in the space in which the polytopes lie. The second way shows how this relates to local weights and Theorem 5.3.11 by keeping the description entirely within the representation space.

The hyperplanes $H_j = \operatorname{aff}(G_j)$ bound a d-simplex $\overline{S}$. If u_j is the outer facet normal of G_j, then the outer facet normals of $\overline{S}$ are $-u_1, \ldots, -u_m, u_{m+1}, \ldots, u_{d+1}$. (We thus think of $\overline{S}$ as sharing the $(d-m)$-face G with Q.) Each face of $\overline{S}$ is parallel to a unique face of P which meets F or of Q which meets G (possibly both).

The sign changes of the facet normals cause sign changes of the face normals $u(J, K)$. If $\overline{J}$ is an r-face of $\overline{S}$ and k of the $d-r$ hyperplanes which intersect in $\overline{J}$ are from the set $H_1, \ldots, H_m$, then we say that $\overline{J}$ is of *kind* k. Suppose that $\overline{J}$ is an r-face of some $(r+1)$-face $\overline{K}$ of $\overline{S}$. If $\overline{J}$ is of kind k and J and K are the corresponding parallel faces of G, then

$$u(\overline{J}, \overline{K}) = (-1)^k u(J, K).$$

Since $\overline{S}$ is a simplex, the weight space $\Omega_r(\overline{S})$ is one-dimensional for $r = 0, \ldots, d$. It is therefore generated by the r-class $\overline{s}^r$ of $\overline{S}$, where $\overline{s}$ is the 1-class of $\overline{S}$. A weight on $\overline{S}$ may be transformed into a weight s_r on Q by multiplying the value of $\overline{s}^r$ on each face of kind k by $(-1)^k$. We call s_r the *r-evert*.

Let us now describe the evert in terms of local weights, and monomials. Let $a_i \neq o$ be a 1-weight on P which is local to the facet F_i of P. Similarly let $b_i \neq o$ be a 1-weight local to the facet G_i of Q. Recall Theorem 5.3.11, which states that a square-free monomial $a_{n(1)} \cdots a_{n(r)} \in \Omega_r(P)$ is equal to zero if and only if

$$\bigcap_{i=1}^{r} F_{n(i)} = \emptyset,$$

or, equivalently

$$p \in \operatorname{pos}\{\overline{u}_j : j \neq n(i) \text{ for } i = 1, \ldots, r\}.$$

Keeping our notation, let Q be obtained from P by an m-flip, such that

$$\overline{\mathcal{K}(P)} \cap \overline{\mathcal{K}(Q)} \subset \operatorname{lin}\{\overline{u}_{d+2}, \ldots, \overline{u}_n\},$$

with the sets $\{\overline{u}_1, \ldots, \overline{u}_m\}$ and $\{\overline{u}_{m+1}, \ldots, \overline{u}_{d+1}\}$ on the same sides as p and q respectively.

Proposition 5.5.1 *If $r \geqslant m$, then $\langle s_r \rangle$ is generated by the rth grade of the ideal generated by $a_1 \cdots a_m$, that is,*

$$\langle s_r \rangle = \langle a_1 \cdots a_m \rangle_r .$$

Proof. Since s_r is local to the face

$$\bigcap_{i=1}^{m} F_i,$$

then the result follows from Theorem 5.3.8. □

Note that, if $r \geqslant m$ and $r \geqslant d + 1 - m$, then

$$\langle s_r \rangle = \langle \overline{u}_1 \cdots \overline{u}_m \rangle_r = \langle \overline{u}_{m+1} \cdots \overline{u}_{d+1} \rangle_r ,$$

and the evert lies in $\Omega_r(P) \cap \Omega_r(Q)$. If $r < m$ and $r < d+1-m$, then the evert does not lie in either space. Otherwise, the evert lies in one, but not both of $\Omega_r(P)$ and $\Omega_r(Q)$.

Theorem 5.5.2 *Let P and Q be simple d-polytopes such that Q is obtained from P by an m-flip, with $m \leqslant \frac{1}{2}(d+1)$. Then*

i) $\Omega_r(Q) \cong \Omega_r(P)$, *for* $0 \leqslant r < m$,

ii) $\Omega_r(Q) = \Omega_r(P) \oplus \langle s_r \rangle$, *for* $m \leqslant r \leqslant d-m$,

iii) $\Omega_r(Q) = \Omega_r(P)$, *for* $d-m < r \leqslant d$.

5.6 A geometric description of the g-theorem

The following construction gives a purely geometric description of a condition which implies the surjectivity of the maps from $\Omega_{r-1}(P) \longrightarrow \Omega_r(P)$ induced by multiplication by p for $r = \lfloor (d+1)/2 \rfloor, \ldots, d$. Indeed, we show that the two are equivalent. Proving this condition directly would give a much simplified proof of the g-theorem.

The aim is to prove the following directly, without using the Theorem 5.2.3. It is trivial to show that it is indeed a corollary of Theorem 5.2.3.

Proposition 5.6.1 *If P is a simple d-polytope, then the map $p\colon \Omega_{r-1}(P) \longrightarrow \Omega_r(P)$ is surjective for $r \geqslant (d+1)/2$.*

We first reduce this to a single case, by induction. Since

$$p\Omega_{s-1}(P) = p\Omega_{r-1}(P)\Omega_{s-r}(P)$$

for $s > r$, then $p\Omega_{s-1}(P) = \Omega_s(P)$ follows, if we can prove the proposition for $r < s$. Also, if we prove surjectivity for a d-polytope, we may deduce it for its facets. Hence it suffices to prove the proposition for d odd, and $r = (d+1)/2$. Note that, in this case, the two spaces $\Omega_{r-1}(P)$ and $\Omega_r(P)$ have equal dimension, and the map is an isomorphism.

Theorem 5.3.7 implies that, if Q is a pyramid over P, then

$$\Omega_r(Q)|_P = p\Omega_{r-1}(P).$$

Hence the following conditions are equivalent:

i) $p\Omega_{r-1}(P) = \Omega_r(P)$,

ii) $\Omega_r(Q)|_P = \Omega_r(P)$.

Let U be some fixed set of normal vecotrs. Then, for a polytope $Q \in \mathcal{P}(U)$, we say that a polytope $Q' \in \mathcal{P}(U)$ is a *strong approximant* to Q if $Q \preccurlyeq Q'$. Note that the type-cone of Q is the intersection of the closures of the type-cones of its strong approximants. More is true.

Theorem 5.6.2 (McMullen, Sturmfels—private communication) *There exists a family* $\{Q_0, \ldots, Q_n\}$ *of simple strong approximants to* Q *such that*

$$\Omega(Q) = \bigcap_{i=0}^{n} \Omega(Q_i).$$

We shall prove a stronger form, using Theorem 3.2.1.

Theorem 5.6.3 *Let Q and $Q_1, \ldots, Q_k$ be polytopes in $\mathcal{P}(U)$. If q_i is the representative point of Q_i for $i = 1, \ldots, k$, and $q \in \operatorname{pos}\{q_1, \ldots, q_k\}$ is the representative point of Q, then*

$$\bigcap_{i=1}^{k} \Omega(Q_i) \leqslant \Omega(Q).$$

In particular, if $Q \preccurlyeq Q_i$ for $i = 1, \ldots, k$, then

$$\bigcap_{i=1}^{k} \Omega(Q_i) = \Omega(Q).$$

Proof. Let $I \subset \{1, \ldots, n\}$ be such that, for $j = 1, \ldots, k$, a face of Q_j exists which lies in F_i if and only if $i \notin I$. By Theorem 3.2.1, this is equivalent to

$$q_j \in \operatorname{relint} \operatorname{pos}\{u_i : i \in I\}$$

for $j = 1, \ldots, k$. Since $q \in \operatorname{pos}\{q_1, \ldots, q_k\}$, then

$$q \in \operatorname{relint} \operatorname{pos}\{u_i : i \in I\},$$

and a face of Q exists, which lies in F_i if and only if $i \notin I$. This face is therefore parallel to, and of the same dimension as faces of $Q_1, \ldots, Q_k$. This is enough to prove the first part. The second part follows immediately: if $Q \preccurlyeq Q_i$ for $i = 1, \ldots, k$, then $\Omega(Q) \leqslant \Omega(Q_i)$. □

We wish to apply this theorem to a special kind of polytope. Let d be odd, and let P be a simple polytope, whose facets are in general position. Let Q be a pyramid over P. The representation space of Q is obtained from that of P by appending a new point at p (see [21] for a proof). This point becomes the representative q of Q. Let q' and q'' be representatives of simple strong approximants Q' and Q'' of Q such that q lies on the line segment between q' and q''. Then by Theorem 5.6.3,

$$\Omega(Q) = \Omega(Q') \cap \Omega(Q'').$$

We may move q' and q'' within their respective type-cones, and preserve this property. If we move them into general position, then the line segment between q' and q'' will intersect other type-cones, and, since the facets of P are in general position, we will obtain a sequence $Q' = Q_1, \dots, Q_k = Q''$ of simple polytopes with the properties:

i) $\Omega_r(Q) = \Omega_r(Q_1) \cap \Omega_r(Q_k)$,

ii) Q_{i+1} is obtained from Q_i by a m_i-flip for some m_i,

iii) $q_1, \dots, q_k$ lie on a line, in order,

iv) Q_i has a facet P_i which is strongly isomorphic to P.

For each $i = 1, \dots, k-1$, the flip to take Q_i to Q_{i+1} does not involve the facets P_i or P_{i+1}. Hence the r-evert e_i vanishes on these facets. In addition, when $r = (d+1)/2$, the evert lies in $\Omega_r(Q_i)$ or $\Omega_r(Q_{i+1})$. For the rest of this section, we set $r = (d+1)/2$. We make the observations that $\Omega_r(Q_i)$ separates itself (by Theorem 5.3.9), and $\Omega_r(Q) = q\Omega_{r-1}(Q_i)$, for $i = 1, \dots, k$.

Lemma 5.6.4 *For $i = 1, \dots, k-1$,*

$$\Omega_r(Q_i) \cap \Omega_r(Q_{i+1}) = \{x \in \langle \Omega_r(Q_i), \Omega_r(Q_{i+1}) \rangle : xe_i = o\}.$$

In particular, $e_i^2 \neq o$.

Proof. By symmetry, we may assume that $\Omega_r(Q_i) = \Omega_r(Q_{i+1}) \oplus \langle e_i \rangle$. Since $\Omega_r(Q_i)$ is self-dual,

$$e_i \Omega_r(Q_i) = e_i \left(\Omega_r(Q_{i+1}) \oplus \langle e_i \rangle \right) \neq 0,$$

because $e_i \neq o$. But since $e_i \Omega_r(Q_{i+1}) = 0$, it follows that $e_i^2 \neq o$, and so

$$\Omega_r(Q_i) \cap \Omega_r(Q_{i+1}) = \Omega_r(Q_{i+1}) = \{x \in \Omega_r(Q_i) : x e_i = o\}.$$

□

This may be applied repeatedly to prove the following.

Corollary 5.6.5 *An r-weight $a \in \langle \Omega_r(Q_1), \ldots, \Omega_r(Q_k) \rangle$ is an r-weight on Q if and only if $a e_i = o$ for $i = 1, \ldots, k$.*

This may be refined, so that we may work in $\Omega_r(Q_1)$ rather than $\langle \Omega_r(Q_1), \ldots, \Omega_r(Q_k) \rangle$:

Proposition 5.6.6 *A weight $a \in \Omega_r(Q_1)$ is an r-weight on Q if and only if*

$$a(\langle e_1, \ldots, e_{k-1} \rangle \cap \Omega_r(Q_1)) = 0.$$

Proof. If $a \in \Omega_r(Q)$, then $a(\langle e_1, \ldots, e_{k-1} \rangle \cap \Omega_r(Q_1)) = o$ by the previous result. For the converse, we proceed by induction. If $e_1 \in \Omega_r(Q_1)$, then $a e_1 = o$ and so $a \in \Omega_r(Q_2)$ also. Assume that $a e_1 = \cdots = a e_{i-1} = o$, and hence that

$$a \in \bigcap_{j=1}^{i} \Omega_r(Q_j).$$

As we discussed in Section 5.5, we may obtain a weight on Q_i from one on Q_{i+1} by adding some multiple of the appropriate evert (in this case e_i). Thus, if $e_i \in \Omega_r(Q_i)$, then

$$e_i + \sum_{j=1}^{i-1} \lambda_j e_j \in \Omega_r(Q_1)$$

for some $\lambda_1, \dots, \lambda_{i-1}$. Since $a(\langle e_1, \dots, e_{k-1}\rangle \cap \Omega_r(Q_1)) = 0$, then $ae_i = o$, and $a \in \Omega_r(Q_i)$. If on the other hand $e_i \in \Omega_r(Q_{i+1})$, then $a \in \Omega_r(Q_i) \leqslant \Omega_r(Q_{i+1})$, and so $ae_i = o$ by Lemma 5.6.4. This completes the induction. □

This brings the discussion entirely into the weight space $\Omega_r(Q_1)$. However, the subspace $\langle e_1, \dots, e_{k-1}\rangle \cap \Omega_r(Q_1)$ is defined in terms of $e_1, \dots, e_{k-1}$ which are weights on other polytopes. It is of greater aesthetic appeal to describe $\langle e_1, \dots, e_{k-1}\rangle \cap \Omega_r(Q_1)$ in terms of monomials (that is, local weights) in $\Omega_r(Q_1)$. In Section 5.5, we showed that, if Q_{i+1} is obtained from Q_i by a flip, then corresponding r-monomials of Q_i and Q_{i+1} differ by a multiple of the r-evert e_i. With this in mind, we define f_i to be the r-monomial in $\Omega_r(Q_1)$ which corresponds to the monomial e_i. Note that the monomial f_i does not necessarily correspond to a face of Q_1; in this case $f_i = o$.

Proposition 5.6.7 *For $i = 1, \dots, k-1$*

$$\langle e_1, \dots, e_i\rangle \cap \Omega_r(Q_1) = \langle f_1, \dots, f_i\rangle .$$

Proof. By the observation that corresponding monomials differ by multiples of the everts, we may obtain f_j from e_j by adding some linear combination of $e_1, \dots, e_{j-1}$ for $j = 1, \dots, i$. Thus

$$\langle f_1, \dots, f_i\rangle \leqslant \langle e_1, \dots, e_i\rangle \cap \Omega_r(Q_1).$$

For the reverse inequality, there are two cases to consider: first, when $e_i \in \Omega_r(Q_i)$; second, when $e_i \in \Omega_r(Q_{i+1})$. If $e_i \in \Omega_r(Q_i)$, then by our previous observation there exist

$\lambda_1, \ldots, \lambda_{i-1}$ such that

$$f_i = e_i + \sum_{j=1}^{i-1} \lambda_j e_j,$$

and hence $e_i \in \langle e_1, \ldots, e_{i-1}, f_i \rangle$. By induction,

$$\langle e_1, \ldots, e_{i-1}, f_i \rangle \cap \Omega_r(Q_1) = \langle \langle e_1, \ldots, e_{i-1} \rangle \cap \Omega_r(Q_1), f_i \rangle = \langle f_1, \ldots, f_i \rangle .$$

If $e_i \in \Omega_r(Q_{i+1})$, then we use the fact that $q_1, \ldots, q_{k-1}$ lie on a line: the face of Q_{i+1} which corresponds to the monomial e_i is non-empty, but the corresponding intersections of facets of $Q_1, \ldots, Q_{i-1}$ are all empty. In particular, $f_i = o$ since it corresponds to an empty intersection. Hence $\langle e_1, \ldots, e_{i-1} \rangle$ vanishes on this face of Q_{i+1}, but e_i does not, and so a linear combination $\lambda_1 e_1 + \cdots + \lambda_i e_i$, which lies in $\Omega_r(Q_1)$, must have $\lambda_i = 0$. Thus

$$\Omega_r(Q_1) \cap \langle e_1, \ldots, e_i \rangle = \Omega_r(Q_1 \cap \langle e_1, \ldots, e_{i-1} \rangle = \langle f_1, \ldots, f_{i-1} \rangle ,$$

and since $f_i = o$ the proof is complete. □

Combining these results, we have proved the following theorem.

Theorem 5.6.8 *A weight $a \in \Omega_r(Q_1)$ is a weight on Q if and only if*

$$af_1 = \cdots = af_{k-1} = 0.$$

Essentially, this theorem picks out a set of faces of Q_1 (those which correspond to the monomials $f_1, \ldots, f_{k-1}$) such that, if a vanishes on all of them, then a must be local to P_1.

We observed earlier that the space $\Omega_r(Q_1)$ is self-dual. The following general results describe the self-dual subspaces of such a space. The proofs are quite trivial, but are included here for completeness.

Theorem 5.6.9 *Let X be a self-dual space, and let Y be a subspace of X. Then the following are equivalent:*

i) Y is self-dual,

ii) $X = Y \oplus Y^{\perp}$,

iii) $Y^{\perp}$ is self-dual, and $Y = (Y^{\perp})^{\perp}$,

where $Y^{\perp} = \{x \in X : xY = 0\}$.

Proof. First, we show that *i)* implies *ii)*. Since Y is the largest subspace of X which is separated by Y, then some complementary subspace Y' is annihilated by Y: that is, $Y'Y = 0$. Hence $Y' = Y^{\perp}$, which implies *ii)*.

Let $o \neq a \in Y^{\perp} \leqslant X$. Since X is self-dual, there exists $b \in X$ such that $ab \neq 0$. By *ii)*, $b = b_1 + b_2$ for some $b_1 \in Y$ and $b_2 \in Y^{\perp}$. By definition, $ab_1 = 0$, and so $ab_2 \neq 0$. Hence $Y^{\perp}$ separates itself, and $Y^{\perp}$ is self-dual.

Let $a \in Y$ and $b \in Y^{\perp}$ be such that $(a + b)Y^{\perp} = 0$. Since $YY^{\perp} = 0$, then $bY^{\perp} = 0$ and so $b = o$. Hence by *ii)*, if $x \in X$ satisfies $xY^{\perp} = 0$ (that is, $x \in (Y^{\perp})^{\perp}$), then $x \in Y$, which proves that *ii)* implies *iii)*.

Finally, to show that *iii)* implies *i)* and *ii)*, we repeat the same arguments with Y and $Y^\perp$ interchanged. □

We make the following conjecture, the solution of which would give a proof of the slightly weaker Theorem 5.6.1.

Conjecture 5.6.10 *The subspace $\langle f_1, \ldots, f_{k-1}\rangle$ is self dual.*

By the previous two theorems, the conjecture implies that

$$\Omega_r(Q_1) = \langle f_1, \ldots, f_{k-1}\rangle \oplus \Omega_r(Q).$$

Since each monomial f_i vanishes on P_1, then the restriction of both sides to P_1 are equal, and

$$\Omega_r(Q)|_{P_1} = \Omega_r(P_1),$$

and Theorem 5.6.1 follows. The conjecture is perhaps not far from being equivalent to Theorem 5.6.1:

Proposition 5.6.11 *Theorem 5.6.1 implies that $\Omega_r(Q) = \langle f_1, \ldots, f_{k-1}\rangle^\perp$ is self-dual.*

Proof. Since $\Omega_r(Q) = q\Omega_{r-1}(Q_1)$, each weight in $\Omega_r(Q)$ is uniquely identified by its restriction to P. Hence the map

$$q : \Omega_r(Q) \longrightarrow \Omega_{r+1}(Q)$$

is injective. If $o \neq a \in \Omega_r(Q)$, then $qa \neq o$. Since $\Omega_{r-1}(Q_1)$ separates $\Omega_{r+1}(Q_1) \geqslant \Omega_{r+1}(Q)$, then there exists some $b \in \Omega_{r-1}(Q_1)$ such that $qab = (qb)a \neq 0$. But $qb \in \Omega_r(Q)$

and so $\Omega_r(Q)$ separates itself, that is, $\Omega_r(Q)$ is self dual. □

One way to try to prove the conjecture might be to find linear combinations $g_1, \ldots, g_{k-1}$ of $f_1, \ldots, f_{k-1}$ such that $\langle g_1, \ldots, g_{k-1} \rangle = \langle f_1, \ldots, f_{k-1} \rangle$, and such that $g_i f_j = 0$ if and only if either $i \neq j$ or $f_j = o$.

This is not true in general with $g_i = f_i$ for $i = 1, \ldots, k-1$, but perhaps some more subtle construction might achieve this.

5.7 Matrices and stress

As we have described, an r-weight allocates a number to each r-face in such a way that certain linear relations (the Minkowski relations) are satisfied. We may therefore identify $\Omega_r(P)$ with the right-kernel of the *r-weight matrix* whose rows are indexed by the r-faces, and whose 'columns' are indexed by the $(r+1)$-faces. The entry in the F, G position is the $1 \times d$ block $u(F, G)$, with the convention that $u(F, G) = o$ if $F \not< G$.

Working with matrices makes some arguments rather clearer, and we may identify patterns more easily.

Theorem 5.7.1 *Let P be a d-polytope, and let ϕ be a linear map whose restriction to the affine hull of any $(r+1)$-face is non-singular. Then, for each $a \in \Omega_r(P)$, there is a map*

$$a\phi : \mathcal{F}_r(P) \longrightarrow \mathbb{R}$$

which satisfies the Minkowski relations on $G\phi$ for each $(r+1)$-face G. Moreover, the *vector space*

$$\{a\phi : a \in \Omega_r(P)\}$$

has the same dimension as $\Omega_r(P)$.

Proof. The construction of $a\phi$ is the same as in Lemma 5.3.2. The scaling factor $\gamma(F)$ depends only on the r-face F, and so $a\phi$ is well defined. If ϕ_i is the restriction of ϕ to the $(r+1)$-face G_i, and we represent ϕ_i by a matrix, then the effect of ϕ on the r-weight matrix is left-multiplication by the matrix

$$\begin{pmatrix} \phi_1 & o & \cdots & o \\ o & \phi_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ o & \cdots & o & \phi_k \end{pmatrix},$$

where $k = f_{r+1}(P)$. Since each ϕ_i is non-singular, this matrix is also non-singular, and the second part of the lemma follows. □

Theorem 5.7.1 may be used to project the $(r+1)$-skeleton into an $(r+1)$-dimensional space, so that the affine hull of an r-face is a hyperplane. In this case, the space normal to an r-face F is 1-dimensional, and the unit outer normals are one of only two vectors u_F and $-u_F$.

Much of the work referred to in this chapter was originally done in the dual, or described in a rather different way (see [14, 41] for details). In order to go though all the material in this area, it is useful to be able to translate between various concepts and their duals. For example, the face ring assigns a monomial to a face according to the vertices contained in it, whereas the dual face ring uses the facets which contain a given face. This section may be a useful guide to translating some of the results found in the literature into the language used in this work. Of course, the different approaches are all valid, but by translating them into a common language, we avoid repetition, and enable comparison of different results.

Using vectors $u(F,G)$ obtained in a similar way, we define an *affine r-stress* as a map $a : \mathcal{F}_r \to \mathbb{R}$ which satisfies the relations

$$\sum_{G:F\in\mathcal{F}(G)} a(G)u(F,G) = o,$$

for each $F \in \mathcal{F}_{r-1}$. We denote the vector space of affine r-stresses on P by $Stress_r^{\mathrm{A}}(P)$. Similarly, $Stress_r^{\mathrm{A}}(P)$ is the kernel of the *affine r-stress matrix* which is obtained from the r-weight matrix and replacing $u(F,G)^T$ with $u(F,G)$, and transposing; an element of the left kernel of this matrix satisfies vector relations indexed by r-faces.

For a simplicial d-polytope P, and for $r \leqslant \frac{1}{2}(d+1)$, the space of affine r-stresses has dimension $g_r(P)$. This result is dual to the surjectivity of the map induced on weight spaces by multiplication by the 1-class.

Let $v(F,G) \in \operatorname{lin}(G) \cap (\operatorname{lin}(F))^{\perp}$ be a unit vector such that $\langle u(F,G), v(F,G)\rangle > 0$. A *linear r-stress* is a mapping $a : \mathcal{F}_r \to \mathbb{R}$ which satisfies the relations

$$\sum_{G>F} a(G)v(F,G) = o,$$

for all $F \in \mathcal{F}_{r-1}$. Notice that, for a polytope P, a linear r-stress on P is a $(d-r-1)$-weight on its polar P^{Δ}. We denote the vector space of linear r-stresses on P by $Stress_r(P)$. It is the kernel of the *linear r-stress matrix*, obtained from the affine r-weight matrix by replacing $u(F, G)$ by $v(F, G)$.

Thus defined, affine and linear stresses are consistent with definitions in other literature ([12, 13, 41]), apart from their indices. An r-stress is often defined as a map from the $(r-1)$-faces to the reals which satisfy certain relations indexed by the $(r-2)$-faces. We call such a stress an $(r-1)$*-stress*; we shall always index stresses by the dimensions of the faces on which they are defined.

There appears to be a gap in the literature. We now introduce the concept dual to affine stresses. Using the vectors $v(F, G)$ again, we define an *r-A-weight* as a mapping $a : \mathcal{F}_r \to \mathbb{R}$ which satisfies

$$\sum_{F<G} a(F)v(F, G) = o,$$

for all $G \in \mathcal{F}_{r+1}$.

We denote the vector space of r-A-weights by Ω_r^{A}. This is the kernel of the *r-A-weight matrix* obtained from the linear r-stress matrix by replacing $v(F, G)$ with $v(F, G)^*$ and transposing. An r-A-weight on a polytope P is an affine $(d-r-1)$-stress on P^{Δ}.

So far, we have only looked at the left-kernels of matrices. We denote the right-kernels of the affine $(r+1)$-stress matrix, the linear $(r+1)$-stress matrix, the $(r-1)$-weight matrix and the $(r-1)$-A-weight matrix by $Motion_r$, $Motion_r^{\mathrm{A}}$, $Circ_r$ and $Circ_r^{\mathrm{A}}$ respectively.

An *r-motion* is an element of $Motion_r$. In the case $r = 0$, this has a geometric interpretation. A 0-motion may be thought of as attaching a velocity vector to each vertex, under the constraint that each edge has fixed length. There are 0-motions which correspond to rigid body motions of the whole polytope. They are the 0-motions which remain when we insist that the distance between any pair of vertices is constant. In other words, we add in extra edges so that there is an edge between each pair of vertices. Such a 0-motion is called *trivial.* This idea generalizes: we add in extra r-faces so that every set of $r+1$ affinely independent vertices are those of an r-face; a *trivial r-motion* of a polytope P is an r-motion of P which satisfies the extra constraints imposed by these extra faces. The subspace of $Motion_r(P)$ of trivial r-motions is denoted $Triv_r(P)$ and its orthogonal complement by $NonTriv_r(P)$. We say that P is *r-rigid* if all of its r-motions are trivial. If P is 0-rigid, then the 1-skeleton, as regarded as a framework of bars and universal joints, is rigid in an intuitive sense.

Theorem 5.7.2 *All simplicial polytopes are 0-rigid.*

The notions of linear and affine stress have been generalized to more general complexes. Tay, White and Whiteley [41] have conjectured the following, which, if proved for some class of (shellable) simplicial complexes, would imply the g-theorem for that class.

Conjecture 5.7.3 *All simplicial polytopes are r-rigid for $0 \leqslant r \leqslant d/2$.*

The space $Circ_r(P)$ of r-circulations also has a geometric interpretation. An r-circulation may be thought of as a flow through the r-skeleton of P, which is linear on each face, and the total flow entering each $(r-1)$-face is zero.

5.8 Singular weight multiplication

To prove the g-theorem, we do not need the whole Lefschetz decomposition used by Stanley. As discussed earlier (with slightly different indices), it suffices to show the existence of ω such that the map induced by multiplication by ω from $\Omega_r(P)$ to $\Omega_{r+1}(P)$ is an isomorphism, for d odd and $r = (d-1)/2$.

Let $r = (d-1)/2$. The spaces $\Omega_r(P)$ and $\Omega_{r+1}(P)$ have the same dimension. Thus the map

$$\omega : \Omega_r(P) \longrightarrow \Omega_{r+1}(P).$$

is a linear map between isomorphic spaces, and, by fixing bases for each, we may define its determinant $\zeta(\omega)$. Thus ζ is a map from $\Omega_1(P)$ to $\mathbb{R}$, and the zero set

$$Z(P) = \{x \in \Omega_1(P) : \zeta(x) = 0\}$$

is an algebraic set, which does not depend on the choice of bases for $\Omega_r(P)$ and $\Omega_{r+1}(P)$. The zero set is the set of elements of $\Omega_1(P)$ for which multiplication of $\Omega_r(P)$ does not induce an isomorphism.

By Theorem 5.2.3, there exists ω such that $\zeta(\omega) \neq 0$, and so $Z(P)$ is not the whole space $\Omega_1(P)$. It must be the union of a family of algebraic surfaces of combined degree at most $n-d-1$. The exact structure of $Z(P)$ is of interest. Since $\zeta(\lambda\omega) = 0$ implies that $\zeta(\omega) = 0$, for $\lambda \neq 0$, the structure of $Z(P)$ may be illustrated on a slice through the representation space (a dual Gale diagram).

Let P be a regular 3-cube, with opposite pairs of facets F_1, F_2, F_3, F_4, and F_5, F_6. Let x_i be the 1-weight local to F_i for $i = 1, \ldots, 6$. Then $x_1 = x_2$, $x_3 = x_4$ and $x_5 = x_6$.

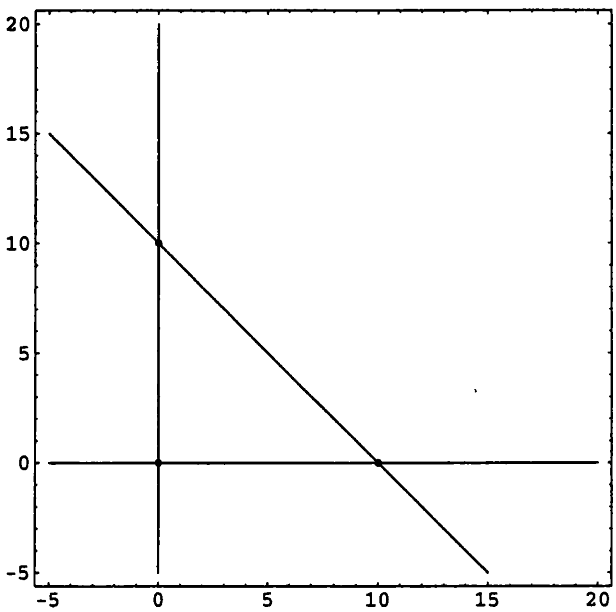
The product x_1x_2 is local to the intersection $F_1 \cap F_2 = \emptyset$ and hence vanishes. Similarly, $x_3x_4 = x_5x_6 = o$. Note that $x_i^2 = 0$ for $i = 1, \ldots, 6$. The regular cube is a rather special case, where x_i is local to F_i but vanishes on F_i. The reader may verify that the space of 2-weights is given by

$$\Omega_2(P) = \langle x_1x_3, x_1x_5, x_3x_5 \rangle .$$

If $\omega = ax_1 + bx_3$, then

$$\begin{aligned} \omega(ax_1 - bx_3) &= a^2x_1^2 - abx_1x_3 + abx_1x_3 + b^2x_3^2 \\ &= o. \end{aligned}$$

Hence multiplication by ω does not induce an isomorphism between $\Omega_1(P)$ and $\Omega_2(P)$ and so $\omega \in Z(P)$. Similarly, if $\omega = ax_1 + bx_5$ or if $\omega = ax_3 + bx_5$, then $\omega \in Z(P)$. The following diagram shows a slice through the representation space of P, which may be considered to be a dual Gale diagram of P. Also marked is the slice through $Z(P)$ as described above.



The regular cube is a rather special example, and so now we move our facet normals (and hence dual Gale diagram vertices) into general position. We may choose a basis for $\Omega_1(P)$ so that $x = (1,0,0)$ and $y = (0,1,0)$, and $z = (0,0,1)$ with the other three vertices in general positions $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$. We shall now calculate the zero set $Z(P)$. Using the following relations:

$$\begin{aligned} a &= a_1x + a_2y + a_3z, \\ b &= b_1x + b_2y + b_3z, \\ c &= c_1x + c_2y + c_3z, \end{aligned}$$

we obtain

$$\begin{aligned} x^2 &= x(a - a_2y - a_3z)a_1^{-1} \\ &= -\frac{a_2}{a_1}xy - \frac{a_3}{a_1}xz, \end{aligned}$$

noting that $a_1 \neq 0$, since $\mathrm{pos}\,\{a, y, z\} = \Omega_1(P)$. Similarly, we obtain expressions for y^2 and z^2. Since $F(b) \cap F(b) \cap F(c)$ is a vertex, then

$$\langle a, y, z\rangle = \Omega_1(P),$$

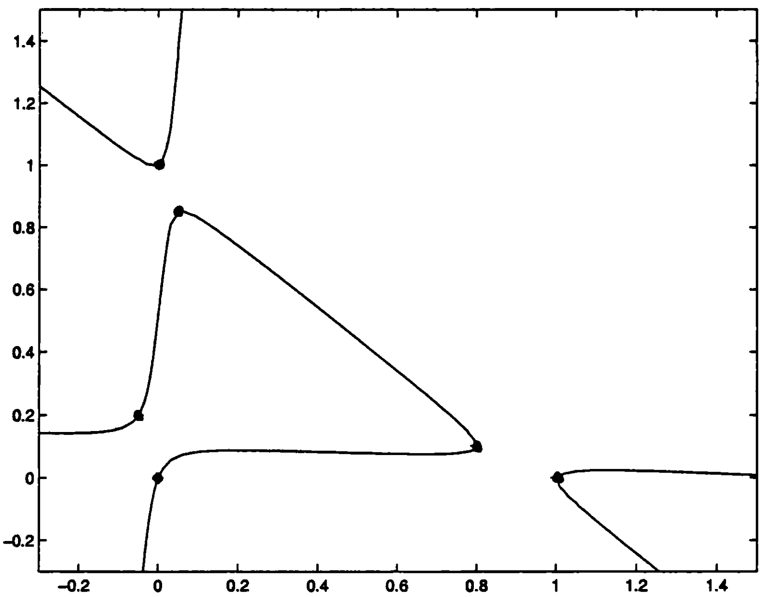
by Theorem 3.2.1. The products ax, by and cz are zero, since the corresponding facets do not intersect. Thus (yz, xz, xy) is a basis for $\Omega_2(P)$. Hence multiplication by $\omega = \omega_1x + \omega_2y + \omega_3z$ is singular only if and only if

$$\begin{pmatrix} (\omega_1x + \omega_2y + \omega_3z)x \\ (\omega_1x + \omega_2y + \omega_3z)y \\ (\omega_1x + \omega_2y + \omega_3z)z \end{pmatrix} = \begin{pmatrix} 0 & (\omega_3 - \omega_1\frac{a_3}{a_1}) & (\omega_2 - \omega_1\frac{a_2}{a_1}) \\ (\omega_1 - \omega_1\frac{b_1}{b_2}) & 0 & (\omega_3 - \omega_1\frac{b_3}{b_2}) \\ (\omega_2 - \omega_1\frac{c_2}{c_3}) & (\omega_1 - \omega_1\frac{c_1}{c_3}) & 0 \end{pmatrix}$$

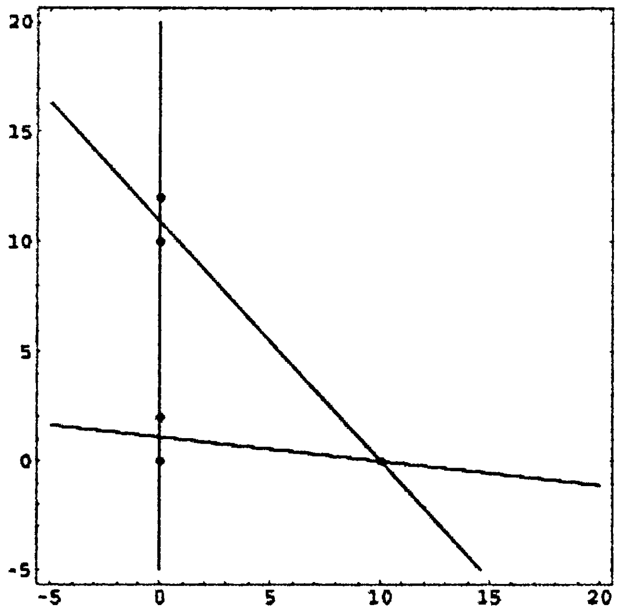
has zero determinant, and so $Z(P)$ is the zero set of the polynomial

$$(a_1\omega_2 - a_2\omega_1)(c_3\omega_1 - c_1\omega_3)(b_2\omega_3 - b_3\omega_2) + (a_1\omega_3 - a_3\omega_1)(b_2\omega_1 - b_1\omega_2)(c_3\omega_2 - c_2\omega_3).$$

We provide two examples of this general case. We depict a section of the representation space (a dual Gale diagram) with the appropriate section of $Z(P)$. As we described earlier, this includes all the information. The first example has $a = (0.8, 0.1, 0.1)$, $b = (0.05, 0.85, 0.1)$ and $c = (-0.05, 0.2, 0.85)$:



The second example has four collinear points and a pair of coincident points. The coordinates are $a = (1, 0, 0.1)$, $b = (0, 1.2, -0.2)$ and $c = (0, 0.2, 0.8)$.



Multiplication by an element of the type-cone $\mathcal{K}(P)$ induces an isomorphism from $\Omega_1(P)$ to $\Omega_2(P)$ and this can be seen here. The zero set $Z(P)$ in the regular case is quite simple: it is the union of linear subspaces. However, in the first of these more general cases, $Z(P)$ is an algebraic surface of degree 3. The space of 2-weights which are local to a facet F has dimension 2, and so $Z(P)$ must pass through all of the vertices of the diagram. However, it is not clear if there is any property of the other 1-weights in $Z(P)$ which is geometrically significant. In the second of the examples, $Z(P)$ is the union of linear subspaces as in the regular case, but this time two of the linear factors pass through only one vertex

The calculation of these surfaces can be rather more difficult than these examples. However, we may simplify things with the following results.

Theorem 5.8.1 *If Q is obtained from P by an m-flip, with $m < (d+1)/2$, then*

$$Z(Q) = Z(P) \cup H,$$

where H is the hyperplane in the representation space which corresponds to 1-weights on the transition polytope.

Proof. Let $r = (d+1)/2$, and let s_1 and s_r be the evert elements so that

$$\Omega_r(Q) = \Omega_r(P) \oplus \langle s_r \rangle ,$$

and each 1-weight on P differs from one on Q by a unique multiple of s_1. If $\omega \in \Omega_1(P)$, then there is a unique 1-weight $\omega' \in \Omega_1(Q)$ such that

$$\omega' = \omega + \lambda s_1,$$

for some λ. Then, for each $a \in \Omega_r(Q)$, with $a = b + \mu s_r$ for $b \in \Omega_r(P)$,

$$\begin{aligned} \omega' a &= (\omega + \lambda s_1)(b + \mu s_r) = \omega b + \mu \omega s_r \lambda s_1 b + \lambda \mu s_1 s_r \\ &= \omega b + \lambda \mu s_1 s_r. \end{aligned}$$

But $\omega b \in \Omega_{r+1}(P)$ and

$$\Omega_{r+1}(Q) = \Omega_{r+1}(P) \oplus \langle s_1 s_r \rangle .$$

Thus $\omega' a = o$ if and only if $\omega b = o$ and $\lambda\mu = 0$. If $\lambda = 0$, that is, $\omega \in H$, then $\omega' s_r = \omega s_r = o$ and so $\omega' \in Z(Q)$. If $\omega b = o$ for some $b \neq o$, and $\lambda \neq 0$, then

$$\omega' b = (\omega + \lambda s_1) b = \omega b = o.$$

and $\omega' \in Z(Q)$ if and only if $\omega \in Z(P)$. □

An application of this result is the following, which is proved by induction.

Corollary 5.8.2 *If $P_0, \ldots, P_n = P$ is a sequence of polytopes, where P_0 is a simplex, P_i is obtained from P_{i-1} by an m_i-flip, and $m_i < \frac{1}{2}(d+1)$ for $i = 1, \ldots, n$, then $Z(P)$ is a union of hyperplanes.*

It is tempting to conjecture that, only when P satisfies this condition is $Z(P)$ the union of hyperplanes. However, as we have seen, the regular cube provides a counterexample, as does the second more general example above. Both of these counterexamples rely on the rather special position of the vertices, which suggests the following conjecture.

Conjecture 5.8.3 *If the dual diagram of P has vertices in general position and $Z(P)$ is the union of hyperplanes, then there is a sequence $P_0, \ldots, P_n = P$ of polytopes, where P_0 a simplex, P_i is obtained from P_{i-1} by an m_i-flip, and $m_i < \frac{1}{2}(d+1)$ for $i = 1, \ldots, n$.*

Notice that, if P is obtained in this way, then $g_r(P) = 0$ for some $r < \frac{1}{2}(d+1)$. Compare this conjecture to the Generalized Lower Bound Conjecture (GLBC).

Conjecture 5.8.4 (GLBC) *If $g_r(P) = 0$ for $r < \frac{1}{2}(d+1)$, then there exists a sequence $P_0, \ldots, P_n = P$ of polytopes, where P_0 a simplex, and P_i is obtained from P_{i-1} by an m_i-flip, and $m_i < \frac{1}{2}(d+1)$ for $i = 1, \ldots, n$.*

A consequence of the GLBC is that this sequence of flips induces a triangulation of the polar polytope P^Δ. If the GLBC holds, then to prove it we would hope that this subdivision is regular; that is, it arises from a projection of the upper surface of a $(d+1)$-polytope. In this case, the sequence $P_0, \ldots, P_n = P$ arises from moving in a single direction, in the representation space, from infinity to the type-cone of P through a set of hyperplanes spanned by vertices of the dual diagram, each of which lies in $Z(P)$.

The polytopes covered by the above constructions are somewhat special. The calculation of $Z(P)$ is quite easy - it may be done by hand. However, even for what may be described as the next most simple example, the calculations are laborious. Consider the polytope P obtained by moving the facets of a product of two triangles and a line segment into somewhat general position. The outer normals of P are

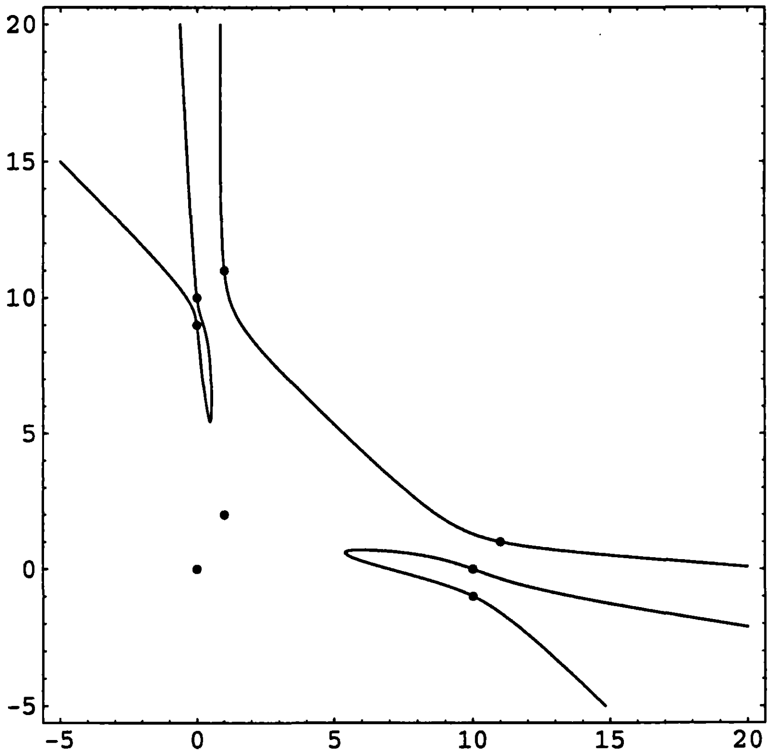
$$U = (u_1, \ldots, u_8) = \begin{pmatrix} 11 & 1 & -2 & -10 & 0 & 0 & 0 & 0 \\ 10 & -1 & 1 & 0 & -10 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 & -10 & 0 & 0 \\ 1 & 11 & -2 & 0 & 0 & 0 & -10 & 0 \\ 1 & 2 & 7 & 0 & 0 & 0 & 0 & -10 \end{pmatrix},$$

The rows of the above matrix give the linear dependences between the vectors $(\overline{u}_1, \ldots, \overline{u}_8)$. We (that is, the reader if (s)he wishes!) may calculate the weight spaces by hand as before, using the linear dependences and the *non-face* relations. We may even find bases for $\Omega_2(P)$ and $\Omega_3(P)$. Expressing a general product of a 2-weight and a 1-weight in terms of the basis elements for $\Omega_3(P)$ involves extremely time-consuming linear reduction, and it is infeasible to do this by hand. However, this linear reduction may be done on a computer (we used Mathematica for convenience), to obtain a 5×5 matrix with terms that are linear in x, y, and z. We can then find the determinant of this matrix (again we used Mathematica) to obtain an expression for $Z(P)$ as anirreducible polynomial in x, y, and

z of degree 5:

$$
\begin{aligned}
Z(P) = 1890x^4y \;&+\; 4550x^3y^2 + 3430x^2y^3 + 770xy^4 - 1089x^4z - 1819x^3yz \cdots \\
\cdots \;&+\; 178079x^2y^2z - 12101xy^3z - 110y^4z + 462x^3z^2 - 51080x^2yz^2 \cdots \\
\cdots \;&-\; 20782xy^2z^2 + 880y^3z^2 + 4851x^2z^3 + 1989xyz^3 + 990y^2z^3.
\end{aligned}
$$

The slice ($z = 1 - x - y$) through our diagram is as follows.



Chapter 6

The face ring and generalized weights

6.1 The face ring

The *face ring* is a general construction which attaches a ring structure to the face lattice of a simplicial polytope P. Its usual definition identifies a face F of P with the vertex set of F. We shall instead give a dual presentation which identifies a face F with the set of facets which contain it. We shall then see a natural ring isomorphism between a quotient of the face ring of P and the weight algebra of P.

The *dual face ring* $A(P)$ of a simple polytope (or simple abstract polytope lattice) P with facets $F_1, \ldots, F_n$ over a field $\mathbb{F}$ is the quotient ring

$$\mathbb{F}[x_1, \ldots, x_n]/I,$$

where I is the ideal generated by monomials $x_{i_1} \cdots x_{i_s}$ such that

$$\bigcap_{i=1}^{s} F_{i_s} = \emptyset.$$

We shall often use this association between monomials and faces. If $m = x_{i_1} \cdots x_{i_r}$ is a monomial, then the face associated with m is

$$F(m) = \bigcap_{i=1}^{s} F_{i_s}.$$

Conversely, if F is a face of P, then the associated monomial is m_F. The face ring is graded by degree in a natural way. There are no non-empty faces contained in more than d facets, and so

$$A(P) = \bigoplus_{i=0}^{d} A_i(P).$$

We define the quotient $B(P)$ of $A(P)$ over the ideal generated by $\theta_1, \ldots, \theta_d \in A_1(P)$ by

$$B_i(P) = A_i(P) / \langle \theta_1, \ldots, \theta_d \rangle_i,$$

and

$$B(P) = \bigoplus_{i=1}^{d} B_i(P) = A(P) / \langle \theta_1, \ldots, \theta_d \rangle.$$

In the dual, Stanley proved [34] that, if $\theta_1, \ldots, \theta_d$ are suitably general, then $\dim B_i(P) = h_i(P)$, for $i = 0, \ldots, d$. In [36], he then uses a connexion with the cohomology of an associated toric variety, with the Hard Lefschetz Theorem for such varieties, to prove the following (also in the dual).

Theorem 6.1.1 *If $A(P)$ is the dual face ring of a simple polytope P, then there exist $\theta_1, \ldots, \theta_d \in A_1(P)$ and $\omega \in B_1(P)$ such that*

$$\dim B_i(P) = h_i(P)$$

for $i = 0, \ldots, d$, and multiplication by ω^{d-2r} induces an isomorphism between $B_r(P)$ and $B_{d-r}(P)$ for $r = 0, \ldots, \lfloor d/2 \rfloor$.

An immediate consequence is that

$$C(P) = \bigoplus_{i=0}^{d} = B(P)/\langle \omega \rangle$$

is a graded ring with $\dim C_i(P) = g_i(P)$ for $i = 0, \ldots, \lfloor d/2 \rfloor$. This is the condition described in Section 5.1 that proves the g-theorem.

We shall now illustrate the relationship between the weight algebra $\Omega(P)$ on P and the quotient $B(P)$ of the face ring. It is appropriate to work in the representation space $\operatorname{lin} \overline{U}$ of P, where $\overline{U}$ is a linear transform of the set U of outer facet normals of P. We identify $\overline{u}_i$ with the 1-weight obtained by perturbing the facet F_i, as described in Proposition 5.3.10.

Theorem 6.1.2 *The weights on P which are local to a facet F form an ideal of $\Omega(P)$.*

Proof. If $\overline{u}$ is the (essentially unique) 1-weight local to a facet F, then the restriction of $\overline{u}$ to a facet G is local to $F \cap G$. By induction, and using Theorem 5.3.5, the restriction to G of the product ua of u with an r-weight a is local to $F \cap G$. Since a product is determined by its restrictions to facets, then ua is local to F. $\square$

Theorem 6.1.3 *If $\overline{u}_{i(1)}, \ldots, \overline{u}_{i(r)}$ are distinct, then their product $a = \overline{u}_{i(1)} \cdots \overline{u}_{i(r)}$ is non-zero if and only if*

$$F = \bigcap_{j=1}^{r} F_{i_j} \neq \emptyset;$$

moreover, a is local to F.

Proof. Let G be a k-face such that $F \cap G = \emptyset$. If $F \neq \emptyset$, then G is contained in a facet, and, by induction, the restriction of a to this facet is local to F. Thus a vanishes on G. The restriction of $\overline{u}_i \cdot \cdots \cdot \overline{u}_{i_{r-1}}$ to F_r is non-zero by induction. By Theorem 5.3.7, there is an isomorphism between $(r-1)$-weights on F_r and r-weights local to F_r. The latter are obtained by multiplying $(r-1)$-weights on P by $\overline{u}_{i_r}$ and hence a non-zero restriction to F_r implies that $a \neq o$. Conversely, if $F = \emptyset$, then by induction the restriction of $\overline{u}_i \cdots \overline{u}_{i_{r-1}}$ to F_r is zero, and $a = o$. □

The connexion between the quotient $B(P)$ of the face ring and $\Omega(P)$ should now be clear. Theorem 6.1.3 corresponds to factoring out the ideal generated by non-faces, to obtain $A(P)$, and then, to obtain $B(P)$, we use the linear relations between $\overline{u}_i, \ldots, \overline{u}_{i_r}$, which are elements of $\Omega_1(P)$. Thus the dimension of $\Omega_r(P)$ is $h_r(P)$, and $\Omega_r(P)$ is generated by monomials in $\overline{u}_1, \ldots, \overline{u}_n$. Thus we may identify $\overline{u}_i$ with the indeterminate x_i. We make a useful observation.

Lemma 6.1.4 *If $B(P) = \Omega(P)$, and $x_{i_1} \cdots x_{i_r} \neq o$ is a square-free monomial, then*

$$B_1 = \langle x_j : j \neq i_r \textit{ for } i = 1, \ldots, r \rangle .$$

Proof. Since P is simple, there is an open set in the representation space which contains only representatives of polytopes strongly isomorphic to P. By Theorem 3.2.1, $\text{pos}\,\{\overline{u}_j : j \neq i_r \text{ for } i = 1, \ldots, r\}$ contains all of these points. Hence $\Omega_1(P) = B_1(P)$ is generated by

$$\text{pos}\,\{x_j : j \neq i_r \text{ for } i = 1, \ldots, r\}\,.$$

$\square$

Note that this property is also possessed by $B(P)$ if

$$B_1(P) = \left\langle x_{i_1}, \ldots, x_{i_{n-d}} \right\rangle$$

for each set $\left\{x_{i_1}, \ldots, x_{i_{n-d}}\right\}$ of $n - d$ distinct indeterminates, that is, if $x_1, \ldots, x_n$ are in general position.

Lemma 6.1.4 allows us to do a number of useful things. The *restriction* of an element of $B(P)$ to a facet F need not be confined to the language of weights. Suppose that a facet $F = F_{m+1}$ intersects facets $F_1, \ldots, F_m$. Let

$$\phi : B(P) \longrightarrow B(F)$$

be the linear projection given by

$$\begin{cases} x_j\phi = x_j & \text{for } j = 1, \ldots, m, \\ x_j\phi = o & \text{for } j = m+2, \ldots, n, \\ \theta_i\phi = o & \text{for } i = 1, \ldots, d, \end{cases}$$

where $\theta_1, \ldots, \theta_d$ are as before. Since F has at least one vertex, then, by Lemma 6.1.4, the points $x_{m+2}, \ldots, x_n$ are linearly independent, and so

$$\dim \ker \phi = n - m - 1.$$

Since there are vertices of P outside F, then, by Lemma 6.1.4, there is a linear relation $\theta \in \langle \theta_1\phi, \ldots, \theta_d\phi \rangle$ whose coefficient of $x_{m+1}\phi$ is not zero, and hence

$$x_{m+1}\phi \in \langle x_1\phi, \ldots, x_m\phi \rangle .$$

Therefore,

$$\langle x_1\phi, \ldots, x_m\phi \rangle = B_1(F).$$

When $B(P) = \Omega(P)$, this process simply identifies weights which are equal on F. In other words, $\Omega(P)\phi$ is the restriction of $\Omega(P)$ to F. Thus, for $a \in B(P)$, we refer to $a\phi$ (or $a|_F$) as the *restriction* of a to F, even though $B(P)$ is not really defined on F. We define the restriction of a to a face of smaller dimension recursively. It is not difficult to see that the restriction of a product ab to a face is the product of their restrictions.

Theorem 6.1.5 *The spaces $m_F B_r(P)$ and $B_r(P)|_F$ are isomorphic for each face F of P.*

Proof. We need only consider the case when F is a facet and then apply the result recursively to obtain the general case. But multiplication by m_F induces the same linear map as ϕ, and

$$m_F B_r(P) \cong B_r(P)|_F = B_r(F).$$

□

This result only really has combinatorial use when Theorem 6.1.1 holds for the ideal

$\langle\theta_1\phi,\ldots,\theta_d\phi\rangle$, and recursively for lower dimensional faces, in other words, when

$$\dim B_r(P)|_F = h_r(F)$$

for each face F of P. This holds for suitably general choices of $\theta_1,\ldots,\theta_d$ and when $B_1(P) = \Omega_1(P)$. For the rest of the section we shall assume that this result does hold.

Lemma 6.1.4 has another important use. We shall now show that $B(P)$ has useful separation properties, and is generated by square-free monomials.

Theorem 6.1.6 *For $r = 0,\ldots,d$, the space $B_1(P)^{d-r}$ separates $B_r(P)$.*

Proof. This means that, if $o \neq a \in B_r(P)$, then there exists $b \in B_1(P)$ such that $ab \neq o$. We prove that b may be chosen to be one of the vectors x_i for some $1 \leqslant i \leqslant n$. It is not difficult to see that this is an equivalent assertion. By Theorem 6.1.5, we must show that, if $o \neq a \in B_r(P)$, then $a|_{F_i} \neq o$ for some $1 \leqslant i \leqslant n$. In the context of weights, we need to verify that a non-zero weight may not vanish on every facet. If $a|_{F_i} = o$, then a is an element of the ideal

$$X_i = \langle x_j : F_j \cap F_i = \emptyset\rangle\,.$$

Thus, to prove the theorem, we must show that

$$\bigcap_{i=1}^{n} X_i = \{o\}\,.$$

Since $F_j \cap F_i = \emptyset$ implies that $F_j \cap E = \emptyset$ for any edge E of F_i, then

$$X_i \leqslant \langle x_j : F_j \cap E = \emptyset\rangle\,,$$

and

$$\bigcap_{i=1}^{n} X_i \leqslant \bigcap_{E\in\mathcal{F}_1(P)} \langle x_j : F_j \cap E = \emptyset \rangle \,.$$

If

$$v = \bigcap_{i=1}^{d} F_i$$

is a vertex, then by Lemma 6.1.4 each of $x_1, \ldots, x_d$ may be written as a linear combination of $x_{d+1}, \ldots, x_n$. Thus, if a lies in the intersection

$$\begin{aligned} \bigcap_{E\in\mathcal{F}_1(P)} \langle x_j : F_j \cap E = \emptyset \rangle \;&\leqslant\; \bigcap_{v\in\mathcal{F}_0(E)} \langle x_j : F_j \cap E = \emptyset \rangle \\ &= \bigcap_{i=d+1}^{n} \langle x_j : j \neq 1, \ldots, d, i \rangle \,, \end{aligned}$$

then, expressed as a polynomial in $x_{d+1}, \ldots, x_n$, no term of a is divisible by x_i, for $i = d+1, \ldots, n$; that is, $a = o$. We have shown that, for $o \neq a \in B_r(P)$, there exists $x \in B_1(P)$ such that $ax \neq o$, and by induction $B_1(P)^{d-r} \leqslant B_{d-r}(P)$ separates $B_r(P)$. $\square$

Since the spaces $B_r(P)$ and $B_{d-r}(P)$ separate each other, they are *dual* to each other. The map

$$\omega : B_r(P) \longrightarrow B_{r+1}(P)$$

induced by multiplication by $\omega \in B_1(P)$ has a dual map

$$\omega^* : B_{d-r-1}(P) \longrightarrow B_{d-r}(P).$$

Corollary 6.1.7 *For $r = 1, \ldots, d$*

$$B_r(P) = B_1(P)^r.$$

Proof. Theorem 6.1.6 (with the roles of r and $d-r$ reversed) says that $B_1(P)^r$ separates $B_{d-r}(P)$. The dimension of a space cannot exceed that of one which separates it. Thus

$$\dim B_r(P) \leqslant \dim B_{d-r}(P) \leqslant \dim B_1(P)^r \leqslant \dim B_r(P)$$

which implies that we must have equality throughout. $\square$

Thus $B(P)$ is generated as an algebra by $B_1(P)$, and so we have proved the following theorem.

Theorem 6.1.8 *If P is a simple polytope, then the sequence*

$$h_0(P), \ldots, h_d(P)$$

is an M-sequence.

A corollary is the Upper Bound Theorem for simple polytopes. The following observation shows that, to define multiplication of weights, we need only consider square-free monomials.

Proposition 6.1.9 *The space $B_r(P)$ is generated by square-free monomials.*

Proof. For $r = 0, 1$, there is nothing to prove. We proceed by induction, assuming that the result holds for $r - 1$. Thus it suffices to show that the product of $a = x_{i_1} \cdots x_{i_{r-1}}$ and x_j may be expressed as a linear combination of square-free monomials. If $a = o$, then $ax_j = o$ and we are done. If $a \neq o$, then

$$F = \bigcap_{j=1}^{r-1} F_{i_j} \neq \emptyset,$$

and

$$\langle x_k : k \neq i_r \text{ for } i = 1, \ldots, r-1 \rangle = B_1(P),$$

by Lemma 6.1.4. Thus $x_j \in \langle x_k : k \neq i_r \text{ for } i = 1, \ldots, r-1 \rangle$, and ax_j is a linear combination of square-free monomials. □

We actually have a slightly stronger result.

Proposition 6.1.10 *If $B(P)$ satisfies Lemma 6.1.4, then the space $B_r(P)$ is generated by square-free monomials in $x_1, \ldots, x_{n-1}$*

Proof. This is trivial for $r \leqslant 1$. For the inductive step, we need only show that, the product of $a \in B_1(P)$ with a square-free monomial $o \neq b = x_{i_1} \cdots x_{i_{r-1}}$, with $i_j \leqslant n-1$, may be expressed as a linear combination of square-free monomials in $x_1, \ldots, x_{n-1}$. If $bx_n = o$, then, since $b \neq o$, there is a face

$$F = \bigcap_{j=1}^{r} F_{i_j} \neq \emptyset,$$

and so

$$B_1 = \langle x_j : j \neq i_1, \ldots, i_{r-1} \rangle .$$

Thus

$$ab = \sum_{j \neq i_1, \ldots, i_{r-1}} \lambda_j x_j b = \sum_{j \neq i_1, \ldots, i_{r-1}, n} \lambda_j x_j b$$

for some $\lambda_j \in \mathbb{R}$. If $bx_n \neq o$, then F intersects F_n and so

$$B_1 = \langle x_j : j \neq i_1, \ldots, i_{r-1}, n \rangle ,$$

and

$$ab = \sum_{j \neq i_1, \ldots, i_{r-1}, n} \lambda_j x_j b$$

for some $\lambda_j \in \mathbb{R}$. □

It would be desirable to extend results such as the Hodge-Riemann-Minkowski inequalities, and the Lefschetz decomposition to Cohen-Macaulay complexes. The arguments for proving the former would have to be altered considerably, since the inductive step relies on the support parameters of P being positive. The latter is perhaps more accessible, since it makes no assertion as to the sign of the image of maps. Even more promising is the prospect of generalizing the proof of the g-theorem from the previous chapter. This has technical difficulties, but we shall at least show that the tools that we might need are available.

We have extended the concept of restriction from $\Omega(P)$ to $B(P)$. Theorem 6.1.6 actually shows that the restriction of a non-zero element of $B_r(P)$ to some r-face is non-zero. Since $B_r(P)$ and $B_{d-r}(P)$ are dual spaces, and since $B_{d-r}(P)$ is generated by square-free monomials, an element of $B_r(P)$ is determined by its restrictions to r-faces. Thus, in the same way as weights are defined, we may think of an element of $B_r(P)$ as a map from $\mathcal{F}_r(P)$ to $\mathbb{R}$ which satisfies certain linear relations determined by $B_1(P)$.

Lemma 6.1.11 *If*

$$a = \sum_{i=1}^{n} a_i x_i \in B_1(P),$$

and $b \in B_{d-1}(P)$, *then*

$$ab = \sum_{i=1}^{n} a_i b|_{F_i}.$$

Proof. We may assume that $a = x_i$ and extend by linearity. Thus we must show that

$$x_i b = x_i b|_{F_i}.$$

But, by the definition of restriction, b_{F_i} differs from b by some linear combination of monomials whose product with x_i is zero. □

This proves Proposition 5.3.6 in a more general setting. We identify the matrices $\Theta^T = (\theta_1, \ldots, \theta_d)^T$ and $U = (u_1, \ldots, u_n)$, for vectors $u_1, \ldots, u_n$, where $\theta_1, \ldots, \theta_d$ are as described in the definition of $B(P)$. Note that U is a linear transform of Θ, since $\Theta U = 0$, and Θ has full rank.

Theorem 6.1.12 *If $G = \bigcap_{i=n-d+r}^{n} F_i$ is an $(r+1)$-face of P with r-faces $J_i = G \cap F_i$ for $i = 1, \ldots, k$, and $a \in B_r(P)$, then*

$$\sum_{i=1}^{k} a|_{J_i} u_i \in \langle u_{d-r}, \ldots, u_n \rangle .$$

Proof. First, suppose that $r = d - 1$. By the lemma, if

$$\theta = \sum_{i=1}^{n} \eta_i x_i \in \operatorname{lin} \Theta,$$

then

$$o = a\theta = \sum_{i=1}^{n} a|_{F_i} \eta_i.$$

Hence the vector $(a|_{F_1}, \ldots, s|_{F_n})$ lies in the left kernel of $\Theta^T = U$, and so

$$\sum_{i=1}^{n} a|_{F_i} u_i = o.$$

For smaller r, the argument is the same. For $i = 1, \ldots, n$ the restriction $a|_{J_i}$ is the restriction of am_G to the facet F_i, and so

$$\sum_{i=1}^{n} a|_{J_i} u_i = \sum_{i=1}^{n} (am_G)|_{F_i} u_i = o.$$

But, for $i = k+1, \ldots, d-r-1$, the product am_G vanishes to give

$$\sum_{i=1}^{k} (am_G)|_{F_i} u_i + \sum_{i=d-r}^{n} (am_G)|_{F_i} u_i = o,$$

as required. □

As a converse to Theorem 6.1.12, we shall now give an alternative definition of $B(P)$, which is consistent with that of $\Omega(P)$. Let $U = (u_1, \ldots, u_n)$ be a set of n vectors. Then $B_r(P)$ is (or is identified with) the space of all functions

$$a : \mathcal{F}_r(P) \longrightarrow \mathbb{R}$$

such that, if $G = \bigcap_{i=d-r}^{n} F_i$ is an $(r+1)$-face of P with r-faces $J_i = G \cap F_i$ for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} a(J_i) u_i \in \langle u_{d-r}, \ldots, u_n \rangle .$$

In the language of stresses, $B(P^\Delta)$ is the space of linear stresses on a simplicial complex with the facial lattice $\mathcal{F}(P^\Delta)$ with vertices $u_1, \ldots, u_n$. (In particular the two definitions of $B_r(P)$ have the same dimension and hence they are equal).

6.2 Matrices

In order to learn more about the structure of $B(P)$, we describe matrices whose kernels give $B_r(P)$ for $r = 0, \ldots, d$. Although this description is not the shortest or necessarily most efficient, it is easy to input our matrices into algorithms to calculate their ranks.

We define three matrices and then combine them into one. Let $U_1 = U = (u_1, \ldots, u_n) = \Theta^T$, be the matrix obtained by transposing the matrix $\Theta = (\theta_1, \ldots, \theta_d)$. Define U_{i+1} recursively as the matrix with n copies of U_i along the diagonal. Thus U_i has n^i columns indexed by monomials of degree i in non-commutative variables $y_1, \ldots, y_n$, and dn^{i-1} rows.

Let C_i be a matrix with n^i rows, indexed in the same way as the columns of U_i. The columns are indexed by monomials of degree i in commutative variables $z_1, \ldots, z_n$. The column indexed by the monomial

$$z_{m(1)} \cdots z_{m(i)}$$

has an entry 1 for each monomial $y_{m'(1)} \cdots y_{m'(i)}$ such that $(m'(1), \ldots, m'(i))$ is a permutation of $(m(1), \ldots, m(i))$, and zero otherwise. The linear map induced by right multiplication by C_i is equivalent to assigning a commutivity law to the monomials in $y_1, \ldots, y_n$ of degree i.

We shall now define our last matrix. Let $N_i = D(\delta_1, \ldots, \delta_k)$ be the diagonal matrix whose rows and columns are indexed by monomials in $z_1, \ldots, z_n$ in lexicographical order such that the entry indexed by $z_{m(1)} \cdots z_{m(i)}$ is 1 if

$$\bigcap_{j=1^i} F_{m(j)} \neq \emptyset,$$

and zero otherwise.

Let M_i be the product of these three matrices given by

$$M_i = U_i C_i N_i.$$

Theorem 6.2.1 *The nullspace of the matrix M_r is isomorphic to $B_r(P)$.*

Proof. The columns of M_r are indexed by the r-faces of P, and the rows of M_r are the Minkowski-like relations of Theorem 6.1.12. A function

$$a : \mathcal{F}_r(P) \longrightarrow \mathbb{R}$$

is an element of the kernel of M_r if and only if it satisfies all of these relations, that is, $a \in B_r(P)$. $\square$

When d is odd, we may use the matrix $M_{(d+1)/2}$ to calculate the zero-set $Z(P)$ defined earlier. Recall that $Z(P)$ identifies the elements of $\Omega_1(P)$ for which multiplication induces a singular map between $\Omega_{(d-1)/2}(P)$ and $\Omega_{(d+1)/2}(P)$. We may generalize the definition of $Z(P)$ to deal with $B(P)$ rather than $\Omega(P)$:

$$Z(P) = \left\{\omega \in \; B_1(P) : \omega B_{(d-1)/2}(P) \neq B_{(d+1)/2}(P)\right\}.$$

As before, unless $Z(P)$ is zero everywhere, it is an algebraic surface of order $h_{(d-1)/2}(P)$. If we could properly understand these surfaces, and how $Z(P)$ relates to $Z(Q)$ when Q is obtained from P by a flip, then we may be able to lessen the reliance on convexity in a proof that these surfaces are not identically zero.

Chapter 7

Straight line graphs

We shall now introduce a method to estimate the dimension of the type-cone of a convex polytope. We regard the 1-skeleton of a polytope as a graph, realised in a vector space. Indeed our definitions are general enough that we could apply them to the 1-skeletons of more general structures such as piece-wise linear complexes.

7.1 Graph theory

We need to describe the notion of a *cycle space*, in the context of *oriented graphs*. A (directed) graph $G = (V, E)$ is a set $V = V(G)$ called its *vertices*, together with a set $E = E(G)$ of (ordered) pairs of distinct vertices called its *edges*. An *oriented* graph is a directed graph obtained from a graph by assigning an orientation to each edge of a graph: if $u, v \in V(G)$, then at most one of uv and vu is an edge of an oriented graph. If $e = uv$ is an edge of a directed graph, then we say that e is *directed from u to v*.

An *oriented path* W in a graph G is an alternating sequence of distinct vertices and edges $v_0, e_1, v_1, e_2, \ldots, e_{k-1}, v_{k-1}$ such that v_{i-1}, v_i are the vertices of the edge e_i for $i = 1, \ldots, k-1$. We use the shorthand $W = v_0 v_1 \cdots v_k$; since the definition of a graph prohibits multiple edges, the edges of W are uniquely specified by its vertices.

An *oriented cycle* is a path $v_0 v_1 \cdots v_k$ such that v_0 and v_k are the vertices of an edge e_k. We denote this cycle by $v_0 \cdots v_k$. Although this notation appears confusing, since it also denoted the path $v_0 v_1 \ldots v_k$, it will be clear from the context whether $v_0 v_1 \ldots v_k$ is to be regarded as a path or a cycle. We make no distinction between cyclic permutations of a cycle, that is, $v_0 \cdots v_k = v_i \cdots v_k v_0 \cdots v_{i-1}$.

Let $W = v_0 v_1 \cdots v_k$ be an oriented path (or cycle) in an oriented graph G. If $e_i = v_{i-1} v_i \in E(G)$, then we say that e_i is *oriented as* W (otherwise, $e_i = v_i v_{i-1} \in E(G)$).

The *edge space* $C_1(G)$ of a graph G is the vector space (over $\mathcal{F}$) of formal sums of edges of G. If $E(G) = (e_1, \ldots, e_n)$, then an element of $C_1(G)$ may be written in the form

$$x = \sum_{i=1}^{n} x_i e_i$$

for $x_1, \ldots, x_n \in \mathbb{F}$. If G is an oriented graph, and W is an oriented walk (or cycle), then we may identify W with an element $z_W = (z_1, \ldots, z_n)$ of the edge space.

$$z_i = \begin{cases} 1 & \text{if } e_i \text{ is an edge of } L\text{, and is oriented as } W, \\ -1 & \text{if } e_i \text{ is an edge of } L\text{, and is not oriented as } W, \\ 0 & \text{if } e_i \text{ is not an edge of } W. \end{cases}$$

The subspace $Z(G)$ spanned by elements identified with cycles is called the *cycle space* of G.

7.2 Straight line graphs

A *straight line graph* is an oriented connected graph G whose vertices lie in a normed vector space $\mathbb{V}$. The *dimension* of G is the dimension of the affine hull of its vertices.

If G_1 and G_2 are straight line graphs that are isomorphic (as oriented graphs), then we say that G_1 is *strongly isomorphic* to G_2 if, for each pair of corresponding edges $e_1 = (u_1, v_1)$ of G_1 and $e_2 - (u_2, v_2)$ of G_2, the equation $v_1 - u_1 = \lambda(v_2 - u_2)$ is satisfied for some $\lambda > 0$. It is not difficult to verify that strong isomorphism is an equivalence relation.

If G_1 and G_2 are strongly isomorphic, then we may define their *sum* $G_1 + G_2$ to be the strongly isomorphic graph whose vertices are the sums of corresponding pairs of vertices of G_1 and G_2. Thus if (v, w) is an edge of G_1, and $(\lambda v + t, \lambda w + t)$ is the corresponding edge of G_2 for some $\lambda > 0$, $t \in \mathbb{V}$, then $((1+\lambda)v + t, (1+\lambda)w + t)$ is the corresponding edge of $G_1 + G_2$. Note that

$$((1+\lambda)w + t) - ((1+\lambda)v + t) = (1+\lambda)(w - v),$$

and hence $G_1 + G_2$ is strongly isomorphic to G_1 and G_2.

We define the *dilatate* of a straight line graph G by $\lambda > 0$ to be the isomorphic oriented graph on $\lambda V(G)$. The *translate* of G by $t \in \mathcal{V}$ is the isomorphic oriented graph on $V(G) + t$. Clearly a straight line graph G is strongly isomorphic to a straight line graph of the form $G' = \lambda G + t$.

The set $\mathcal{K}(G)$ of graphs which are strongly isomorphic to a graph G is called the *type-cone* of G. The type-cone of G is closed under dilatation and addition, as shown above, and is therefore isomorphic to a cone. We denote the dimension of $\mathcal{K}(G)$ by $\lambda(G)$. The type cone of a straight line graph contains a subspace isomorphic to $\mathbb{V}$, which corresponds to translations.

Let G be a straight line graph with $E(G) = \{e_1, \ldots, e_n\}$. If $e_i = (v, w) \in E(G)$, then let u_i be the unit vector $\frac{w - v}{||w - v||}$. We define a 1-weight on G to be a map $a : E(G) \longrightarrow \mathbb{R}$ such that

$$\sum_{i=1}^{n} (a(e_i)z_i)u_i = 0$$

for each $(z_1, \ldots, z_n) \in Z(G)$. By linearity, we need only consider the equations when $(z_1, \ldots, z_n) = z_L$ for some oriented cycle L. It is easy to see that 1-weights on G form a vector space under addition: $(a + b)(e) = a(e) + b(e)$ and scaling: $(\lambda a)(e) = \lambda(a(e))$. We denote the space of 1-weights on G by $\Omega_1(G)$. If G' is strongly isomorphic to G, and $a \in \Omega_1(G)$, then we can obtain a 1-weight a' on G' by assigning $a'(e') = a(e)$, for each pair of corresponding edges $e \in E(G)$, $e' \in E(G')$. We shall in fact identify a and a': a 1-weight on G is a 1-weight on any element of $\mathcal{K}(G)$.

Note that, if $a(e_i)$ is the length of e_i for each i, then a is a 1-weight on G; we call a the *1-class* of G. Recall that the function which takes each edge of a polytope P to its length is a 1-weight on P (the 1-class of P). Indeed, if F is a 2-face of P, then the associated Minkowski relation is

$$\sum_{e_i \in \mathcal{F}_1(F)} a(e_i)u(e_i, F) = 0.$$

If we rotate $u(e_i, F)$ through a quarter turn in aff F, then $u(e_i, F)$ lies parallel with u_i. Thus the Minkowski relation can be expressed in a linearly equivalent way replacing

$u(e, F)$ with $\pm u_e$ for each edge e of F. We may express every Minkowski relation in this way. The choice of u_i in $\operatorname{aff} e$ is an assignment of an orientation of e. The cycle space of the graph of a polytope is spanned by its 2-faces, and the following theorem follows immediately.

Theorem 7.2.1 *If G is a straight line graph obtained from the 1-skeleton of a polytope P by assigning an orientation to each edge, then*

$$\Omega_1(G) \cong \Omega_1(P).$$

As with polytopes, the space of 1-weights on a straight line graph is closely related to its type-cone. We have already observed that the 1-classes of straight line graphs in $\mathcal{K}(G)$ are 1-weights on G. Generalizing Theorem 5.2.1, we prove that $\Omega_1(G)$ is actually spanned by these 1-classes.

Theorem 7.2.2 *A 1-weight on a straight line graph G which takes positive values on every edge corresponds to an element of $\mathcal{K}(G)$.*

Proof. The proof follows that of Theorem 5.2.1. Let $a \in \Omega_1(G)$ be such that a takes positive values on every edge. We shall show that a is the 1-class of some graph $G' \in \mathcal{K}(G)$; that is, if vw is an edge of G', then $a(vw) = ||w - v||$. We define the position of each vertex of G' by fixing some vertex v at the origin, and, if W is a path from v to w, then the position of w is given by

$$\sum_{i=1}^{n} a(e_i) x_i u_i,$$

where $(x_1, \ldots, x_n) = z_W$ is the element of the edge space identified with W. So long as each position is well defined, the length of an edge e is then $a(e)$.

We must check that the position of w is well defined: two different paths give the same position for w. It suffices to consider paths which meet only at v and w. Let W and W' be paths from v to w that meet at v and w only, and let $-W'$ be the path W' but in the opposite direction, that is, from w to v. Let $(x'_1, \ldots, x'_n) = z_{-W'} \in C_1(G)$ be the element identified with $-W'$. By joining the W and $-W'$ together at v and w to form a cycle L, we have

$$\begin{aligned} z_W - z_{W'} &= z_W + z_{-W'} \\ &= \sum_{i=1}^{n} a(e_i)x_i u_i + \sum_{i=1}^{n} a(e_i)x'_i u_i \\ &= \sum_{i=1}^{n} a(e_i)z_i u_i \end{aligned}$$

where $(z_1, \ldots, z_n) = z_L \in Z(G)$. By definition of a weight, this is equal to zero, and $z_W = z_{W'}$. The position of w is therefore well defined. □

Theorem 7.2.2 identifies a positive weight with an element of the type-cone which is unique up to translation, and hence

$$\dim \Omega_1(G) = \dim \mathcal{K}(G) + \dim \mathbb{V}.$$

For polytopes, in particular, the following theorem is rather useful. It allows us to project the 1-skeleton onto a plane without changing the dimension of the type-cone of the graph. If $G = (V, E)$ is a straight line graph in $\mathbb{V}$, and $\Phi\colon \mathbb{V} \longrightarrow \mathbb{W}$ is a linear map, then define $G\Phi$ to be the straight line graph on $V\Phi$, with edges

$$E(G\Phi) = \{(v\Phi, w\Phi) : (v, w) \in E(G)\}.$$

If Φ is singular, then we require that the vertices of each edge are mapped to different points; that is, $u_i\Phi \neq o$ for $i = 1, \ldots, n$. Note that the graphs G and $G\Phi$ are combinatorially equivalent, and in particular $Z(G) = Z(G\Phi)$. Define the *dimension* of a cycle of G, and of its corresponding element in $Z(G)$, to be the dimension of the affine hull of its edges.

Theorem 7.2.3 *If ϕ is a linear map which preserves the dimension of each element in a basis B of $C_1(G)$, then*

$$\Omega_1(G) \cong \Omega_1(G\Phi).$$

Proof. Let $a \in \Omega_1(G)$ so that

$$\sum_{i=1}^{n}(a(e_i)z_i)u_i = 0$$

for each $(z_1, \ldots, z_n) \in B$. For each edge e_i with associated unit vector u_i, define γ_i to be the length of $u_i\Phi$, so that $\gamma_i^{-1}u_i\Phi$ is a unit vector. Note that the length of $e_i\Phi$ is $\gamma_i||e_i||$. Define b: $E(G\Phi) \longrightarrow \mathbb{R}$ by $b(e_i\Phi) = a(e_i)\gamma_i$. Then

$$\begin{aligned}\sum_{i=1}^{n}(b(e_i\Phi)z_i)\gamma_i^{-1}u_i\Phi &= \sum_{i=1}^{n}a(e_i)z_iu_i\Phi \\ &= (\sum_{i=1}^{n}a(e_i)z_iu_i)\Phi \\ &= 0.\end{aligned}$$

Thus b is a weight on $G\Phi$, and we have an injective map from $\Omega_1(G)$ to $\Omega_1(G\Phi)$.

For the reverse map, let $b \in \Omega_1(G\Phi)$ and define $a\colon E(G) \longrightarrow \mathbb{R}$ by $a(e_i) = b(e_i\Phi)\gamma_i^{-1}$. Then

$$\begin{aligned}\left(\sum_{i=1}^{n} a(e_i)z_i)u_i\right)\Phi &= \sum_{i=1}^{n} b(e_i\Phi)z_iu_i\Phi \\ &= o\end{aligned}$$

for each $(z_1, \ldots, z_n) \in B$. But the dimension of $(z_1, \ldots, z_n)$ is preserved by Φ, that is, Φ is non-singular on the span of $(z_1u_1\, dots, z_nu_n)$. Hence

$$\sum_{i=1}^{n} a(e_i)z_i)u_i = 0,$$

and $a \in \Omega_1(G)$. We therefore have an injective map in the reverse direction and the isomorphism of $\Omega_1(G)$ and $\Omega_1(G\Phi)$ is proved. □

If G is a straight line graph obtained from the 1-skeleton of a convex polytope P in $\mathbb{V}$, then the cycles corresponding to the 2-faces of P span $Z(G)$. Thus if Φ is a linear projection from $\mathbb{V}$ to a 2-plane, which does not map any cycle into a line, then the space of 1-weights on the resulting graph $G\Phi$ is isomorphic to $\Omega_1(P)$.

7.3 Applications in polytope theory

We have already mentioned the most useful application of straight-line graphs, that is, we may construct a straight line graph from the 1-skeleton of a polytope. They allow us to prove results about polytopes in an elementary way, without appealing to convexity.

Theorem 7.3.1 *Let P be a polytope, and let G_P be a straight line graph obtained from the 1-skeleton of P. If G is a subgraph of G_P which touches every facet of P. Then*

$$\dim \mathcal{K}(P) \leqslant \lambda(G).$$

Proof. Given an element of $\mathcal{K}(G)$ (suitably close to G), we determine the positions of all the vertices of G and hence the support parameters of each facet of P. This gives us the corresponding element of $\mathcal{K}(P)$. $\square$

Theorem 7.3.1 becomes particularly interesting when $\lambda(G) = d+1$, forcing $\dim \mathcal{K}(P) = d+1$. If P is a convex polytope, then $\dim \mathcal{K}(P) = d+1$ indicates that every member of $\mathcal{K}(P)$ is homothetic to P. If $\dim \mathcal{K}(P) > d+1$, then P can be expressed as the sum of two polytopes neither of which is homothetic to P. In the former case, P is called *indecomposable* and, in the latter, *decomposable.* We extend this terminology to straight-line graphs for convenience. We then obtain the following as a corollary of Theorem 7.3.1.

Corollary 7.3.2 *A polytope P is indecomposable if and only if the 1-skeleton of P has an indecomposable subgraph which touches every facet.*

The following results allow us to build large indecomposable graphs from smaller ones, and often establish the indecomposability of polytopes.

Theorem 7.3.3 *Let G_1, G_2 be indecomposable graphs, which share two common vertices. Then $G = G_1 \cup G_2$ is indecomposable.*

Proof. Since G_1 is indecomposable, an element of $\mathcal{K}(G_1)$ may be identified by the distance between these two intersection vertices, and likewise for G_2. Since in G these two distances must be the same, the element of $\mathcal{K}(G)$ can be identified by this one distance, and so G is indecomposable. □

Theorem 7.3.4 *Let $G = \bigcup_{i=1}^{k} G_i$, where the G_i are indecomposable graphs whose only pairwise intersections are $v_i = G_i \cap G_{i+1}$ for $i = 1, \ldots, k-1$ and $v_k = G_k \cap G_1$. Then*

$$\lambda(G) = k - \dim \operatorname{aff} \{v_1, \ldots v_k\} = k - \dim(\operatorname{lin} \{v_2 - v_1, \ldots, v_k - v_{k-1}\}).$$

Proof. An element of $\mathcal{K}(G)$ may be identified by the k lengths $|v_i - v_{i+1}|$. However, there is a vector relation between these lengths of rank $\dim \operatorname{aff}(v_1, \ldots v_k))$, and so it may be identified by just $k - \dim(\operatorname{aff}(v_1, \ldots v_k))$ of them. □

This construction may be used to build up large indecomposable subgraphs of the edge skeleton of a polytope, from smaller ones. For example, let P be a polytope with indecomposable faces F and G such that each vertex of P lies in either F or G. Then $\dim \mathcal{P} = 2$, if and only if the edges which do not lie in F or G are parallel, or the lines which contain them meet at a single point. Otherwise P is indecomposable.

The following result was proved by Smilansky [33] by considering the freedom one had to move facets through parallel displacements whilst satisfying the equations needed to preserve facet intersections. We end by proving it again using a rather more combinatorial argument.

Theorem 7.3.5 *If P is a 3-polytope with more vertices than facets, then P is decompos-*

able.

Proof. We use Theorem 7.2.3, and consider the projection of the 1-skeleton straight line graph G of P to a plane such that each facet is mapped to a polygon in the plane. It suffices to show that

$$\lambda(G) \geqslant f_0(P) - f_2(P) + 1 = 2f_0(P) - f_1(P) - 1,$$

where the equality is derived from Euler's theorem:

$$f_0(P) - f_1(P) + f_2(P) = 2.$$

Let T be a spanning tree of G; thus

$$\lambda(T) = f_1(T) = f_0(P) - 1.$$

Now let us add the remaining edges of G one at a time. Let $T = G_0, \ldots, G_k = G$ be graphs such that G_{i+1} is obtained from G_i by adding an edge, with $k = f_1(G) - f_1(T) = f_1(P) - f_0(P) + 1$. Then

$$\lambda(G_{i+1}) \geqslant \lambda(G_i) - 1,$$

since a space of codimension 1 in $\mathcal{K}(G_i)$ has the vertices on the inserted edge in line with that edge.

The subgraph G_{k-1} of G is the image of a subgraph H of the 1-skeleton of P. The minimal cycles of H are all two dimensional, except the one which contains the two end-points of the final edge (x, y). But this cycle may be expressed as the sum of all other minimal cycles that are all 2-dimensional. Hence

$$\lambda(G_{k-1}) = \lambda(H).$$

The edge (x, y) may be expressed as a sum of other edges of P from each of the two facets which contain it. Hence the difference $x - y$ may be expressed as a sum of edges from two sets of edges of H, each of whose affine hulls are hyperplanes. The affine hull of x and y is therefore determined as the (1-dimensional) intersection of these hyperplanes, and hence

$$\lambda(H) = \lambda(P).$$

Putting these equations together, we obtain

$$\begin{aligned}
\lambda(P) &= \lambda(G_{k-1}) \\
&\geqslant \lambda(T) - (k-1) \\
&= (f_0(P) - 1) - (f_1(P) - f_0(P)) \\
&= 2f_0(P) - f_1(P) - 1,
\end{aligned}$$

as required. □

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Bibliography

[1] M. L. Balinski, On the graph structure of convex polyhedra in n-space. *Pacific J. Math.* **11** (1961), 431–434.

[2] D. W. Barnette, A short proof of the d-connectedness of d-polytopes. *Discrete Math.* **137** (1995), 351–352.

[3] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1998.

[4] L. J. Billera and C. W. Lee, Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes, *Bull. Amer. Math. Soc.* **2** (1980), 181–185.

[5] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes, *J. Combinat. Theory* A **31** (1981), 237–255.

[6] A. Brønsted and G. Maxwell, A new proof of the d-connectedness of d-polytopes. *Canadian Math. Bull.* **32** (1989), 252–254.

[7] D. Gale, Neighbouring vertices on a convex polyhedron. In Linear Inequalities and Related Systems (ed. H. W. Kuhn and A. W. Tucker), Princeton (1956).

[8] B. Grünbaum, *Convex Polytopes*, Wiley Interscience, (1967).

[9] B. Grünbaum, *Convex Polytopes*, Second edition, edited by V. L. Klee, V. Kaibel and G. M. Ziegler, Springer, (2003).

[10] M. Kallay, Indecomposable polytopes. *Israel J. Math.* **41** (1982), 235–243.

[11] K. Karu, Hard Lefschetz Theorem for non-rational polytopes. To be published.

[12] C.W. Lee, Some recent results on convex polytopes. *Contemp. Math.* **114** (1990), 3–19.

[13] C.W. Lee, Generalized stress and motions, in *Polytopes: Abstract, Convex, and Computational*, T. Bistriczsky, P. McMullen, R. Schneider, and A. I. Weiss, eds., NATO ASI Series C, Vol. 440, Kluwer, Dordrecht, 1994, 249–271.

[14] C. W. Lee, P.L. spheres, convex polytopes and stress. *Discrete Comput. Geom.* **15** (1996), 389–421.

[15] E. R. Lockeberg, *Refinements in Boundary Complexes of Polytopes.* Ph.D. Thesis, University of London (1977).

[16] F. S. Macaulay, Some properties of enumeration in the theory of modular systems. *Proc. Lond. Math. Soc.*, I. Ser. **26** (1927), 531–555.

[17] K. Menger, Zur allgemeinen Kurventheorie. *Fund. Math.* **10** (1927), 95–115.

[18] P. McMullen, The maximum numbers of faces of a convex polytope. *Mathematika* **17** (1970), 179–184

[19] P. McMullen, The numbers of faces of simplicial polytopes. *Israel J. Math.* **9** (1971), 559–570.

[20] P. McMullen, Representations of polytopes and polyhedral sets. *Geom. Ded.* **2** (1973), 83–99.

[21] P. McMullen, Transforms, diagrams and represenations. In: (eds.) Tölke j., Wills, J. M. Constributions to Geometry, Boston Stuttgart Basel Birkhäuser (1979), 92–130.

[22] P. McMullen, Indecomposable convex polytopes. *Israel J. Math.* **58** (1987), 321–323.

[23] P. McMullen, The Polytope Algebra. *Advances Math.* **78** (1989), 76–130.

[24] P. McMullen, On simple polytopes. *Invent. Math.* **113** (1993), 419–444.

[25] P. McMullen, Weights on polytopes. *Discrete Comput. Geom.* **15** (1996), 363–388.

[26] P. McMullen and E. Schulte, *Abstract Regular Polytopes, (Encyclopedia of Mathematics and its Applications* **92**) Cambridge University Press (2002).

[27] W. Meyer, Indecomposable polytopes. *Trans. Amer. Math. Soc.* **190** (1974), 77–86.

[28] T. S. Motzkin, Beiträge zur Theorie der linearen Ungleichungen, Doktorarbeit, Univeristät Basel 1936; translation: Contributions to the theory of linear inequalities, RAND Corporation Translations **22**, Santa Monica 1952, 86 pages; also in: T. S. Motzkin, Selected papers (D. Cantor, B. Gordon, and B. Rothschild, eds.), Birkhäuser, Boston 1983, 1–80.

[29] M. Perles and N. Prabhu, A property of graphs of convex polytopes, Note, *J. Combinat. Theory A* **62** (1993), 155–157.

[30] G. C. Shephard, Decomposable convex polyhedra. *Mathematika* **10** (1963), 89–95.

[31] Z. Smilansky, An indecomposable polytope, all of whose facets are decomposable. *Mathematika* **33** (1986), 192–196.

[32] Z. Smilansky, *Decomposability of Polytopes and Polyhedra*, Ph.D. Thesis, Hebrew University of Jerusalem (1986) (Hebrew with English summary).

[33] Z. Smilansky, Decomposability of polytopes and polyhedra. *Geom. Ded.* **24** (1987), 29–49.

[34] R. P. Stanley, The upper bound conjecture and Cohen-Macaulay rings, *Stud. Appl. Math.,* **54** (1975), 135–142.

[35] R. P. Stanley, Hilbert functions of graded algebras. *Advances Math.* **28**, (1978), 57–83.

[36] R. P. Stanley, The numbers of faces of a simplicial convex polytope. *Advances Math.* **35**, (1980), 236–238.

[37] R. P. Stanley, Generalized h-vectors, intersection cohomology of toric varieties, and related results. *Adv. Stud. Pure Math.*, **11**, (1987), 187–213.

[38] E. Steinitz, Polyeder und Raumeinteilungen. *Enzykl. math Wiss.*, **3** (Geometrie), Part 3AB12, (1922) 1–139.

[39] V. A. Timorin, An analogue of the Hodge-Riemann relations for simple convex polytopes. *Russ. Math. Surveys* **54**, No. 2, (1999), 381–426.

[40] V. A. Timorin, On polytopes that are simple at the edges. (English. Russian original). *Funct. Anal. Appl.* **35**, No. 3, (2001) 189–198.; translation from *Funkts. Anal. Prilozh.* **35**, No. 3, (2001), 36–47.

[41] T-S. Tay, N. White and W. Whiteley, Skeletal rigidity of simplicial complexes, *Europ. J. Combinatorics* **16** (1995), I:381–403, II:503–523.

[42] G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1995.