Σ-MATRIX IDEALS AND Σ-INVERTING HOMOMORPHISMS

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Abstract

In Free Rings and their Relations, P.M.Cohn constructed a skew field from a prime matrix ideal using admissible matrices. When Σ is a lower multiplicative set of matrices over any non-commutative ring this method can be generalised to construct any epic Σ-inverting homomorphism upto isomorphism. This depends on the introduction of the concept of a Σ-matrix ideal. Every Σ-inverting homomorphism gives rise to a Σ-matrix ideal and conversely our main theorem shows that, given a Σ-matrix ideal, an epic Σ-inverting homomorphism can be constructed and that the matrices which are admissible for zero are precisely those lying in the Σ-matrix ideal.

It is shown that the least Σ-matrix ideal induces the universal Σ-inverting homomorphism. A description of the least Σ-matrix ideal is then obtained yielding a new description of the kernel of the Σ-inverting homomorphism and a criterion for it to be an embedding; Malcolmson and Gerasimovs' respective descriptions of the kernel are also proved.

When Σ is taken to be the complement of a prime matrix ideal the construction reduces to that used by Cohn to construct a skew field.

It is further shown that the definition of a prime matrix ideal $\mathcal{P}$ can be simplified by restricting the class of matrices necessarily lying in $\mathcal{P}$ to be the hollow and degenerate matrices. The condition that $\mathcal{P}$ be closed with respect to row determinantal sums can be dropped completely. As a consequence, Cohn's criterion for the existence of a homomorphism from a ring to a field and the criterion for the existence of a field of fractions for a ring can be refined somewhat. For completeness Dicks and Sontag's result

that Sylvester domains form the precise class of rings which have a universal field of fractions inverting all full matrices is also included.

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Contents

Chapter 1 Σ-Inverting Homomorphisms

We shall be concerned with inverting certain sets of matrices over a non-commutative ring or more generally homomorphisms which invert certain matrices. In the commutative case this reduces to inverting certain sets of elements of the ring since we can invert any square matrix A by adjoining an inverse of detA. However for a non-commutative ring we shall have to adjoin all the entries in the matrix's inverse. In this chapter we consider general Σ-inverting homomorphisms and in particular the universal localisation at a set of matrices. We conclude with a description of the universal localisation in terms of the universal right localisation.

Definitions If R is any ring and Σ a set of matrices over R then a ring homomorphism $f:R\to S$ is said to be Σ-inverting if every matrix in Σ is mapped to an invertible matrix over S.

Similar notions of f being Σ-right (left) inverting are clear.

If $f:R\to S$ is a ring homomorphism and Σ a set of matrices over R then we define $R_{\Sigma,f}(S)$ as the set of all final components to solutions u over S of equations,

$$Au=A_o \text{ where } A\in\Sigma f \text{ and } A_o \text{ is a column over } \mathrm{im} f. \quad (1)$$

When the context is clear we write $R_\Sigma(S)$ for $R_{\Sigma,f}(S)$.

We will have to impose some restrictions on Σ so we also make the following definition:

A set of matrices Σ over a ring is said to be **lower multiplicative** if $A,B\in\Sigma$ implies that $\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}\in\Sigma$ where C is any matrix of the appropriate size and $1\in\Sigma$.

Lemma 1.1 *Let $f:R \longrightarrow S$ be a ring homomorphism and Σ a lower multiplicative set of matrices over R. Then $R_\Sigma(S)$ is a subring of S containing $\operatorname{im} f$.*

Proof. If $c \in \operatorname{im} f$ then $c=af$ for some $a \in R$ and c satisfies the equation $1.u=af$. Hence $\operatorname{im} f \subseteq R_\Sigma(S)$, so $R_\Sigma(S)$ is non-empty containing in particular 0 and 1.

Let p be the final component of the solution $u=(u_* \; p)^T$ of (1) where $A=(A_* A_\infty)$ and A_∞ is the final column of A and also let q be the final component of the solution v of $Bv=B_o$ where $B \in \Sigma f$ etc. Then $p-q$ is the final component of a solution w of the equation,

$$\begin{bmatrix} B_* & B_\infty & 0 & 0 \\ 0 & A_\infty & A_* & A_\infty \end{bmatrix} w = \begin{bmatrix} B_o \\ A_o \end{bmatrix}$$

Namely $w=(v^T \; u_* \; p-q)^T$ is a solution. Similarly pq is the final component of a solution w of the equation,

$$\begin{bmatrix} B_* & B_\infty & 0 & 0 \\ 0 & -A_o & A_* & A_\infty \end{bmatrix} w = \begin{bmatrix} B_o \\ 0 \end{bmatrix}$$

That is to say $w=(v^T \; u_*q \; pq)^T$ is a solution. This shows that $R_\Sigma(S)$ is closed under subtraction and multiplication and since we have already shown that $1 \in R_\Sigma(S)$ we have that $R_\Sigma(S)$ is a subring of S as claimed. ■

Now any given equation (1) need not necessarily have any solution in S. However if f is Σ-right inverting then every equation has a (not necessarily unique) solution. In this case we have the following characterisation of $R_\Sigma(S)$.

<u>Theorem</u> <u>1.2</u> *If R is any ring, Σ a lower multiplicative set of matrices over R and let $f:R\longrightarrow S$ be a Σ-right inverting ring homomorphism. Then $x\in R_\Sigma(S)$ if and only if x is an entry in the right inverse of the image of a matrix in Σ.*

<u>Proof</u>. ($\Leftarrow$) x is (say) the ith entry in a column $u=(u_* u_n)^T$ which satisfies the equation,

$$Au = e_j \qquad A\in\Sigma_n f;\ n\geq j\in\mathbb{N} .$$

[e_j is the column which is zero except in the jth entry which is one and likewise e_∞ is zero except in the final entry.] If i=n then the conclusion follows by the definition of $R_\Sigma(S)$. Otherwise we note that the following holds;

$$\begin{bmatrix} A_* & A_\infty & 0 & 0 \\ C & A_\infty & A_* & A_\infty \end{bmatrix} \begin{bmatrix} u_*^T \\ u_n \\ u_*^T \\ x \end{bmatrix} = \begin{bmatrix} e_j \\ e_j \end{bmatrix} ,$$

where C has as ith column $-A_\infty$ and the rest zero. Since the matrix on the L.H.S. lies in Σf and e_j has entries in imf the conclusion follows.

($\Rightarrow$) x is the final component of the solution to (1). Now A has a right inverse over S, say B. Then

$$\begin{bmatrix} 1 & 0 \\ -A_o & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & B \end{bmatrix} = I \qquad .$$

Hence x is an entry in the right inverse of an element of Σf. ∎

This result highlights the one-sided nature of the definition of

$R_\Sigma(S)$. We shall return to this matter at the end of Chapter 2. When $R_\Sigma(S)=S$ for a Σ-inverting homomorphism $f:R\to S$ then we shall call f **epic Σ-inverting**.

If we have an equation (1) then we have an important analogue of Cramer's Rule;

<u>Proposition 1.3</u> *Let R be any ring, Σ a lower multiplicative set of matrices over R and $f:R\to S$ a Σ-inverting homomorphism. If we have an equation $Au=A_o$ over S where $u=(u_* \ p)^T$ has final component p, $A=(A_*A_\infty)\in\Sigma f$ and A_o is a column over imf then p is stably associated to (A_oA_*) over S.*

<u>Proof</u>. If so we have

$$(A_oA_*) = (A_*A_\infty)\begin{bmatrix} u_*^T & I \\ p & 0 \end{bmatrix}, \text{ and } (A_*A_\infty) \text{ is invertible over } S. \quad \blacksquare$$

<u>Definition</u> For a set Σ of matrices over a ring R a universal Σ-inverting homomorphism is a homomorphism $\lambda_\Sigma:R\to R_\Sigma$ such that any Σ-inverting homomorphism $f:R\to S$ factors uniquely through λ via a homomorphism $f':R_\Sigma\to S$. The ring R_Σ is determined up to isomorphism by these conditions and is called the universal Σ-inverting ring or the universal localisation of R. This was first considered by P.M.Cohn [1] and below is his proof of its existence. Similarly there is a universal Σ-right inverting homomorphism $\lambda_{(\Sigma}:R\to R_{(\Sigma}$.

<u>Theorem 1.4</u> (Cohn) *Let R be any ring and Σ any set of matrices over R. Then there is a universal [right] localisation R_Σ $[R_{(\Sigma}]$, unique up to isomorphism, with a universal Σ-[right] inverting homomorphism*

$$\lambda_\Sigma:R\to R_\Sigma \qquad [\ \lambda_{(\Sigma}:R\to R_{(\Sigma}\ ]\ .$$

Moreover, λ_Σ [$\lambda_{(\Sigma}$] is injective if and only if R can be embedded in a ring over which all the matrices of Σ have [right] inverses.

Proof. For each mxn matrix $A=(a_{ij})$ in Σ we take a set of mn symbols, arranged as an nxm matrix $A'=(a'_{ji})$ and take a ring presentation of R_Σ [$R_{(\Sigma}$] consisting of all the elements of R, as well as all the a'_{ji} as generators, and as defining relations take all the relations holding in R, together with the relations, in matrix form,

$$AA'=A'A=I \quad \text{for each } A\in\Sigma. \quad [\ AA'=I \text{ each } A\in\Sigma]$$

The mapping taking each element of R to the corresponding element of R_Σ [$R_{(\Sigma}$] is clearly a homomorphism $\lambda_\Sigma:R\longrightarrow R_\Sigma$ [$\lambda_{(\Sigma}:R\longrightarrow R_{(\Sigma}$], which is Σ-[right] inverting by construction. If $f:R\longrightarrow S$ is any Σ-[right] inverting homomorphism, we define a homomorphism $f':R_\Sigma\longrightarrow S$ [$f':R_{(\Sigma}\longrightarrow S$] by putting $xf'=xf$ for all $x\in R$ and for any matrix $A\in\Sigma$ defining f' on $(A\lambda)^{-1}$ by putting $(A\lambda)^{-1}f'=(Af)^{-1}$. This gives a well defined homomorphism f', because any relation in R_Σ [$R_{(\Sigma}$] is a consequence of the defining relations in R and the relations above and all these relations also hold in S. If there is a Σ-[right] inverting homomorphism f which is injective then it must factor through λ_Σ [$\lambda_{(\Sigma}$] which then must also be injective. ∎

It is now possible to show how R_Σ is related to $R_{(\Sigma}$ when Σ is a lower multiplicative set of matrices over R.

Lemma 1.5 *If $f:R\longrightarrow S$ is a ring homomorphism and Σ a lower multiplicative set of matrices over R then the set $J_{\Sigma,f}(S)$ of final components of solutions u to equations,*

$$Au = 0 \quad \text{where} \quad A \in \Sigma f \qquad (2)$$

is a two sided ideal of $R_{\Sigma,f}(S)$.

<u>Proof</u>. If p,q are the final components of solutions u,v respectively to equations Au=0 and Bv=0 where $A,B\in\Sigma f$ then p-q is the final component to the solution w of

$$\begin{bmatrix} B_* & B_\infty & 0 & 0 \\ 0 & A_\infty & A_* & A_\infty \end{bmatrix} w = 0 \quad .$$

$J_\Sigma(S)$ is non-empty since zero is easily seen to be a member. If $r\in R_\Sigma(S)$ and (2) holds then Aur=0. It remains to show closure under multiplication by elements of $R_\Sigma(S)$ on the left. By the definition of $R_\Sigma(S)$ r is the final component of the solution v to say $Cv=C_o$ where $C\in\Sigma f$ and C_o has entries in imf. Then rp is the final component of a solution w of,

$$\begin{bmatrix} A_* & A_\infty & 0 & 0 \\ 0 & -C_o & C_* & C_\infty \end{bmatrix} w = 0.$$

Hence $rp\in J_\Sigma(S)$ and $J_\Sigma(S)$ is a two sided ideal of $R_\Sigma(S)$. ∎

When Σ is lower multiplicative then by Lemma 1.1 we have that $R_{\Sigma,\lambda_{(\Sigma}}(R_{(\Sigma})$ is a subring of $R_{(\Sigma}$ and from the ring presentation construction of the universal Σ-right inverting homomorphism and Theorem 1.2 it is clear that $R_\Sigma(R_{(\Sigma})$ contains all of the generators of $R_{(\Sigma}$ hence $R_\Sigma(R_{(\Sigma})=R_{(\Sigma}$. Consequently $R_{(\Sigma}$ has a two sided ideal J where every element of J is the final component of the solution u to an equation Au=0 where $A\in\Sigma\lambda_{(\Sigma}$.

Theorem 1.6 *If R is any ring, Σ a lower multiplicative set of matrices over R then the universal Σ-inverting homomorphism $\lambda_\Sigma: R \to R_\Sigma$ factors through the universal Σ-right inverting homomorphism $\lambda_{(\Sigma}: R \to R_{(\Sigma}$ as follows;*

$$R \xrightarrow{\lambda_{(\Sigma}} R_{(\Sigma} \xrightarrow{\nu} R_{(\Sigma}/J \cong R_\Sigma$$

where J is the two sided ideal of $R_{(\Sigma}$ consisting of final components of solutions to equations $Au=0$, $A \in \Sigma\lambda_{(\Sigma}$ and ν is the natural homomorphism . Also $\ker\lambda_\Sigma = \lambda_{(\Sigma}^{-1}(J)$.

Proof. Write $\lambda=\lambda_{(\Sigma}.\nu$ then λ is Σ-right inverting since $\lambda_{(\Sigma}$ is Σ-right inverting.

Now we shall denote by $\bar{r}$ the equivalence class of $R_{(\Sigma}/J$ which has r as a representative. If $\bar{A}\bar{u}=0$ in $R_{(\Sigma}/J$ where $A \in \Sigma\lambda_{(\Sigma}$ then we have an equation $(A+U)(u+u_2)=u_3$ in $R_{(\Sigma}$ where U, u_2 and u_3 all have entries in J. Expanding and rearranging we get

$$Au = u_3 - Au_2 - Uu - Uu_2 .$$

All the terms on the right hand side are over J since J is a two sided ideal of $R_{(\Sigma}$. Say $Au=u_4 \in J$. Since A lies in $\Sigma\lambda_{(\Sigma}$ it has a right inverse, B say. Then $Au=ABu_4$ and so $A(u-Bu_4)=0$ in $R_{(\Sigma}$. Hence by the definition of J the final component of $u-Bu_4$ lies in J. But again Bu_4 has entries in J since u_4 is over J, so the final component of u lies in J also. Therefore the final component of $\bar{u}$ is zero. It is straightforward to show that the other entries in $\bar{u}$ are zero since we can use the trick employed in Theorem 1.2 to write an equation in which they occur as final components. We have shown therefore that $\bar{A}\bar{u}=0$ implies that $\bar{u}=0$. Since

$$\bar{A}(\bar{B}\bar{A} - I) = (\bar{A}\bar{B})\bar{A} - \bar{A} = I\bar{A} - \bar{A} = 0,$$

we must have $\bar{B}\bar{A}-I=0$ and we see that λ is Σ-inverting.

Any Σ-inverting homomorphism $f:R\longrightarrow S$ factors through the universal Σ-right inverting homomorphism via a unique homomorphism f' say.

$$\begin{tikzcd} & R_{(\Sigma} \arrow[dr, "f'"] & \\ R \arrow[ur, "\lambda_{(\Sigma}"] \arrow[rr, "f"'] & & S \end{tikzcd}$$

Now $J\subseteq\ker f'$ since otherwise there is an equation in which a member of Σf is a left zero divisor. Hence f' factors through ν and the following diagram commutes.

$$\begin{tikzcd} R_{(\Sigma} \arrow[r, "\nu"] \arrow[dr, "f'"] & R_{(\Sigma}/J \arrow[d, "f''"] \\ R \arrow[u, "\lambda_{(\Sigma}"] \arrow[r, "f"'] & S \end{tikzcd}$$

So f factors through λ via f'' and this is unique by the uniqueness of f' so since λ is Σ-inverting we see that λ is the universal Σ-inverting homomorphism. Also it is immediate that $\ker\lambda_{\Sigma}=\lambda_{(\Sigma}^{-1}(J)$. ∎

In fact P.M.Cohn has shown in [2] that any ring R can be embedded in a fully right inversive ring S such that every left regular matrix over R is mapped to a right invertible matrix over S. What this implies is the following:

<u>Theorem</u> <u>1.7</u> (Cohn [2]) *If R is any ring and Σ any set of matrices over R then the universal Σ-right inverting homomorphism is an*

embedding if and only if Σ does not contain any right zero divisors. ∎

Chapter 2 Σ-Matrix Ideals

This chapter contains our main construction. From the definition of a Σ-matrix ideal given later we generalise Cohn's construction using admissible matrices (Chap.7 [**1**]) to show that every Σ-matrix ideal gives rise to an epic Σ-inverting ring homomorphism and conversely every Σ-inverting homomorphism factors through such a map.

From Theorem 1.2 we see that when Σ is lower multiplicative then formally attaching to a ring R the entries in right inverses of elements of Σ is equivalent to attaching the final components of solutions u to equations $Au=A_o$, where $A\in\Sigma$. This latter system is more conveniently written $(A_o\ A_*\ A_\infty)$ where $A=(A_*A_\infty)$, A_∞ is the final column of A and A_* the remaining columns of A. The matrix $(A_*A_\infty)\in\Sigma$ which we aim to invert will be called the **denominator** of the **admissible matrix** $(A_oA_*A_\infty)$; the matrix (A_*) will be called the **core**; and noting Cramer's rule (Proposition 1.3), the **numerator** of the admissible matrix is the matrix (A_oA_*).

<u>Definition</u> If R is any ring and Σ a lower multiplicative set of matrices over R then we define M_Σ to be the set of all admissible matrices, i.e. all matrices A such that $A=(A_oA_*A_\infty)$ with $(A_*A_\infty)\in\Sigma$ and A_o a column over R. For any two admissible matrices A,B we will write $A\to B$ if they are obtainable from each other by a series of the following invertible **permissible operations**;

(a) row operations, i.e multiplication on the left by an invertible matrix over R.

(b) column operations within the core A_* over the characteristic ring and we also permit the addition of a right multiple (over the characteristic ring of R) of a column of the core to A_∞ i.e. this amounts to multiplication on the right by a **half bordered** matrix over the characteristic ring of R where a half bordered matrix is an invertible matrix of the form $1\oplus Q$ where the bottom row of Q is e_∞^T.

The above relation is easily shown to be an equivalence and by M_Σ we shall understand $M_\Sigma/\rightarrow$. Any application of the above operations will be denoted by an arrow.

<u>Definition.</u> If $f:R\longrightarrow S$ is a Σ-inverting homomorphism then $(A_0 A_* A_\infty)\in M_\Sigma$ is said to be **admissible for zero** under f if $e_\infty^T(A_* f A_\infty f)^{-1}A_0 f=0$. Similarly a subset of M_Σ is said to be admissible for zero under f if every member of the set has that property.

We now proceed by defining a commutative semigroup structure on M_Σ as follows;

$$A \,\square\, B = \begin{bmatrix} B_0 & B_* & B_\infty & 0 & 0 \\ A_0 & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} \qquad (1)$$

Firstly this is compatible with the permissible operations (a) and (b), i.e. if $A\rightarrow\hat{A}$ then $A\square B\rightarrow\hat{A}\square B$ etc., and hence is well defined.

Also $A\square B \in M_\Sigma$ since Σ is lower multiplicative and A and B are admissible. Now we verify that $\square$ is an associative operation:

$$(A\square B)\square C = \begin{bmatrix} C_0 & C_* & C_\infty & 0 & 0 & 0 & 0 \\ B_0 & 0 & 0 & B_* & B_\infty & 0 & 0 \\ A_0 & 0 & -A_\infty & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} C_o & C_* & C_\infty & 0 & 0 & 0 & 0 \\ B_o & 0 & -B_\infty & B_* & B_\infty & 0 & 0 \\ A_o & 0 & 0 & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} = A\square(B\square C)$$

So M_Σ forms a semigroup with respect to this operation. Commutativity follows as easily;

$$A \square B = \begin{bmatrix} B_o & B_* & B_\infty & 0 & 0 \\ A_o & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} \to \begin{bmatrix} A_o & 0 & -A_\infty & A_* & A_\infty \\ B_o & B_* & B_\infty & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} A_o & A_* & A_\infty & 0 & 0 \\ B_o & 0 & -B_\infty & B_* & B_\infty \end{bmatrix} = B \square A \quad .$$

It is also possible to define another binary operation on M_Σ, multiplication, which is also associative.

$$A.B = \begin{bmatrix} B_o & B_* & B_\infty & 0 & 0 \\ 0 & 0 & -A_o & A_* & A_\infty \end{bmatrix} \qquad (2)$$

Again this is compatible with the permissible operations and so is a well defined binary operation. Also since Σ is lower multiplicative A.B lies in M_Σ if A and B lie in M_Σ. The operation is immediately seen to be associative. Hence we have shown;

<u>Proposition</u> <u>2.1</u> *If R is any ring and Σ a lower multiplicative set of matrices over R then the set of admissible matrices M_Σ is a*

commutative semigroup with respect to addition '□' defined by (1) and is a semigroup with respect to multiplication '.' defined by (2). ■

We now investigate when M_Σ can have a ring structure imposed on it. First we need a lemma on commutative semigroups. If H is a subset of a commutative semigroup then H is said to be **unitary** if $a \in H$ and $a+b \in H$ imply that $b \in H$. The following is a commutative version of a result from Clifford and Preston [3].

<u>Lemma 2.2</u> *Let S be a non-empty commutative semigroup, U a unitary subsemigroup of S and let there exist a unary operation $^{o}: S \longrightarrow S$ such that $m+m^{o} \in U$ for any $m \in S$. Then the relation '$\sim$' defined by*

$$a \sim b \quad \text{iff } \exists\ x \in S;\ a+x,\ b+x \in U \qquad (3)$$

is a congruence on S, the quotient $S/\sim$ is an abelian group and moreover if e is a representative of the equivalence class which is the identity in $S/\sim$ then $a \sim e$ if and only if $a \in U$.

<u>Proof</u>. It is clear that '$\sim$' is symmetric by definition and $a+a^{o} \in U$ for any $a \in S$ hence $a \sim a$. If $a \sim b$, $b \sim c$ with $a+x$, $b+x$, $b+y$, $c+y \in U$ then $a+x+b+y=(a+y)+(b+x) \in U$. Since U is unitary and $b+x \in U$ then $a+y \in U$ and $a \sim c$. Also for any $d \in S$ we have $(a+x)+(d+d^{o})$, $(b+x)+(d+d^{o}) \in U$ hence $a+d \sim b+d$ and we have established that (3) defines a congruence on S.

Let S/U denote $S/\sim$ the quotient semigroup which is commutative since S is commutative. Since U is closed under addition it is clear that U is contained in a single equivalence class. Conversely if $b \sim u \in U$ then $b+x$, $u+x \in U$ so $u \in U$ implies that $x \in U$ which in turn implies that $b \in U$. We denote the equivalence class of U by 0. Now $b+0=b$ in S/U since $e, b+e+x \in U$ imply that $b+x \in U$ and $x=(b+e)^{o}$ gives

$b+e+x \in U$. So 0 is the identity element in S/U. That S/U is a group follows since $b+b^{o} \in U$ for any $b \in S$. ∎

If Σ is a lower multiplicative set of matrices over a ring R then we consider the case when S is taken to be M_{Σ}; the commutative semigroup of admissible matrices with respect to addition '$\square$'. In this case U will be the unitary subsemigroup of admissible matrices which are going to represent zero in the quotient semigroup. We know that the unary operation $^{o}: M_{\Sigma} \rightarrow M_{\Sigma}$ corresponds to finding an additive inverse for each $m \in M_{\Sigma}/U$. Within our construction we define such an operation by

$$(A_o \; A_* \; A_\infty)^{o} = (-A_o \; A_* \; A_\infty) \; . \qquad (4)$$

Since if $(A_* \; A_\infty)$ is invertible (under some homomorphism to a ring) then, $e_\infty^T (A_* A_\infty)^{-1}(-A_o) = -e_\infty^T (A_* A_\infty)^{-1} A_o$. Taking (4) as the definition of the unary operation we see that U must contain all matrices of the form $(A_o A_* A_\infty) \square (-A_o A_* A_\infty)$. This sum can written as follows;

$$\begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 \\ A_o & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 \\ 0 & 0 & 0 & A_* & A_\infty \end{bmatrix}$$

In fact we shall impose a stronger condition on U. Define $\mathcal{L}$ to be the set of admissible matrices which after a sequence of permissible operations can be written;

$$\begin{bmatrix} K_o & K & 0 \\ 0 & 0 & L \end{bmatrix} \quad \text{where } K, L \in \Sigma \text{ and } K_o \text{ is a column.}$$

Certainly it is easy to see that any matrix in $\mathcal{L}$ must be admissible for zero under any Σ-inverting homomorphism. For this reason $\mathcal{L}$ will be called the **minimal set of matrices admissible for zero**. It is important to realise that $\mathscr{L}$ need not contain **all** the matrices admissible for zero. We are now in a position to define a Σ-matrix ideal.

Definition Let R be any ring and Σ a lower multiplicative set of matrices over R. Then a **Σ-matrix ideal** $\mathfrak{U}$ of R is a well defined subset of M_Σ with the following properties:

(i) $\mathfrak{U}$ contains $\mathscr{L}$, the minimal set of matrices admissible for zero.

(ii) $\mathfrak{U}$ is an additive unitary subsemigroup of M_Σ.

(iii) $\mathfrak{U}$ is a multiplicative semigroup ideal of M_Σ.

We note that the intersection of any family of Σ-matrix ideals is also a Σ-matrix ideal and hence we can talk of the least Σ-matrix ideal which we shall write $\mathfrak{U}_\Sigma$. This Σ-matrix ideal has important properties which are taken up in the next chapter. Now we investigate the properties of a general Σ-matrix ideal. Firstly we note that it is straightforward to prove that a Σ-inverting homomorphism gives rise to a Σ-matrix ideal.

Lemma 2.3 *Let R be a ring, Σ a lower multiplicative set of matrices over R and $f:R\longrightarrow S$ a Σ-inverting homomorphism then* $\Sigma\text{-Ker}f=\{(A_oA_*A_\infty)\in M_\Sigma \mid e_\infty^T(A_*fA_\infty f)^{-1}A_of=0\}$ *is a Σ-matrix ideal.* ∎

Given a Σ-matrix ideal for a ring R we can construct an R-ring as follows;

Theorem 2.4 *Let R be any ring and let Σ be a set of lower multiplicative matrices over R. If $\mathfrak{U}$ is a Σ-matrix ideal of R then $M_\Sigma/\mathfrak{U}$ has a ring structure and the map $\lambda_\mathfrak{U}:R\longrightarrow M_\Sigma/\mathfrak{U}$ given by $\lambda_\mathfrak{U}(r)$ =*

(r 1) is a ring homomorphism from R to $M_\Sigma/\mathfrak{U}$. Moreover $M_\Sigma/\mathfrak{U}$ is the zero ring if and only if $\mathfrak{U}=M_\Sigma$.

<u>Proof</u>. From the above remarks $\mathfrak{U}$ satisfies the conditions of Lemma 2.2 with respect to the unary operation defined in (4). Applying Lemma 2.2, $M_\Sigma/\mathfrak{U}$ is an abelian group under '$\square$' and we have $A\sim B$ if and only if $A\square B^{o}\in\mathfrak{U}$. Since M_Σ has a multiplication defined on it we define multiplication on $M_\Sigma/\mathfrak{U}$ in the obvious way. It remains to be shown that this operation is well defined. However we can prove the following identities in $M_\Sigma/\mathfrak{U}$ for any $(A_*A_\infty),P\in\Sigma$ by noting that $\mathfrak{U}$ contains $\mathscr{L}$.

$$\begin{bmatrix} 0 & P & 0 & 0 \\ A_o & Q & A_* & A_\infty \end{bmatrix} \sim (A_o\ A_*\ A_\infty) \qquad \ldots\ldots(5)$$

$$\begin{bmatrix} B_o & P & B_* & B_\infty \\ A_o & 0 & A_* & A_\infty \end{bmatrix} \sim (A_o\ A_*\ A_\infty) \qquad \ldots\ldots(6)$$

whenever the denominator of the l.h.s of (6) lies in Σ. We prove these by showing $(\text{L.H.S.})\square(\text{R.H.S})^{o}\in\mathscr{L}$, so we have for (5);

$$\begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & P & 0 & 0 \\ A_o & 0 & -A_\infty & Q & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & Q & A_* & A_\infty \end{bmatrix} \in\mathscr{L}$$

and similarly for (6);

$$\begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 & 0 \\ B_o & 0 & -B_\infty & P & B_* & B_\infty \\ A_o & 0 & -A_\infty & 0 & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 & 0 \\ B_o & -B_* & -B_\infty & P & B_* & B_\infty \\ 0 & 0 & 0 & 0 & A_* & A_\infty \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} -A_o & A_* & A_\infty & 0 & A_* & A_\infty \\ B_o & -B_* & -B_\infty & P & 0 & 0 \\ 0 & 0 & 0 & 0 & A_* & A_\infty \end{bmatrix} \to \begin{bmatrix} -A_o & A_* & A_\infty & 0 & 0 & 0 \\ B_o & -B_* & -B_\infty & P & 0 & 0 \\ 0 & 0 & 0 & 0 & A_* & A_\infty \end{bmatrix} \in \mathscr{L}$$

These identities can in turn be used to show that the distributive identities hold in $M_\Sigma/\mathfrak{U}$.

$$A.C \,\square\, B.C \underset{\sim}{} (A \,\square\, B).C \qquad \ldots\ldots\ldots(7)$$

$$C.A \,\square\, C.B \underset{\sim}{} C.(A \,\square\, B) \qquad \ldots\ldots\ldots(8)$$

[We note for future use that we actually show here that $A.C \,\square\, B.C \,\square\, \{(A \,\square\, B).C\}^o \in \mathscr{L}$ and similarly for (8).] The l.h.s. of (7) is, (with a dot representing a zero);

$$\begin{bmatrix} C_o & C_* & C_\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -B_o & B_* & B_\infty & \cdot & \cdot & \cdot & \cdot \\ C_o & \cdot & \cdot & \cdot & \cdot & C_* & C_\infty & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -A_\infty & \cdot & -A_o & A_* & A_\infty \end{bmatrix} \to$$

$$\begin{bmatrix} C_o & C_* & C_\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -B_o & B_* & B_\infty & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_* & C_\infty & \cdot & \cdot \\ \cdot & \cdot & -A_o & \cdot & -A_\infty & \cdot & -A_o & A_* & A_\infty \end{bmatrix}$$

which by (5),

$$\sim \begin{bmatrix} C_o & C_* & C_\infty & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -B_o & B_* & B_\infty & \cdot & \cdot \\ \cdot & \cdot & -A_o & \cdot & -A_\infty & A_* & A_\infty \end{bmatrix} = (A \,\square\, B).C$$

And again the l.h.s. of (8) is,

$$\begin{bmatrix} B_o & B_* & B_\infty & . & . & . & . & . & . \\ . & . & -C_o & C_* & C_\infty & . & . & . & . \\ A_o & . & . & . & . & A_* & A_\infty & . & . \\ . & . & . & . & -C_\infty & . & -C_o & C_* & C_\infty \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} B_o & B_* & B_\infty & . & . & . & . & . & . \\ . & . & -C_o & C_* & C_\infty & . & . & . & . \\ A_o & . & -A_\infty & . & . & A_* & A_\infty & . & . \\ . & . & C_o & -C_* & -C_\infty & . & -C_o & C_* & C_\infty \end{bmatrix}$$

After adding the second row block to the fourth row block we can apply (6) to get;

$$\sim \begin{bmatrix} B_o & B_* & B_\infty & . & . & . & . \\ A_o & . & -A_\infty & A_* & A_\infty & . & . \\ . & . & . & . & -C_o & C_* & C_\infty \end{bmatrix} = C.(A \,\square\, B)$$

We can also use (5) and (6) to establish the existence of a multiplicative identity element (1 1) since we have;

$$(A_oA_*A_\infty).(1\ 1) = \begin{bmatrix} 1 & 1 & . & . \\ . & -A_o & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & . & . \\ A_o & . & A_* & A_\infty \end{bmatrix} \sim (A_oA_*A_\infty)$$

Also we note that similarly;

$$(1\ 1).(A_oA_*A_\infty) = \begin{bmatrix} A_o & A_* & A_\infty & . \\ . & . & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} A_o & A_* & . & A_\infty \\ . & . & -1 & 1 \end{bmatrix} \sim (A_oA_*A_\infty)$$

We can now show that multiplication is well defined. Suppose that $A\sim B$ i.e. that $A\square B^{\circ}\in\mathfrak{U}$. Then we require that $A.C\sim B.C$ and $C.A\sim C.B$ for any $C\in M_{\Sigma}$. Now $A.C\sim B.C$ if and only if $A.C\square(B.C)^{\circ}\sim 0$. It is an easy check that $(B.C)^{\circ}=B.C^{\circ}=B^{\circ}.C$ in M_{Σ}. Hence $A.C\square(B.C)^{\circ}\sim A.C\square B^{\circ}.C$ and using the distributive identity (7) we get $A.C\square(B.C)^{\circ}\sim(A\square B^{\circ}).C$ and $A\square B^{\circ}\sim 0$ so it suffices to show that if $A\sim 0$ then $A.C\sim 0$. Similarly on the l.h.s. we require that $C.A\sim 0$. But this is precisely the condition that $\mathfrak{U}$ is an ideal w.r.t. the multiplicative semigroup structure of M_{Σ}. Hence $M_{\Sigma}/\mathfrak{U}$ has a ring structure.

Now define a map $\lambda_{\mathfrak{U}}:R\longrightarrow M_{\Sigma}/\mathfrak{U}$ by $\lambda_{\mathfrak{U}}(r)=(r\ \ 1)$. We observe immediately that $\lambda_{\mathfrak{U}}$ preserves 1 and the following identities show $\lambda_{\mathfrak{U}}$ to be a ring homomorphism.

$$(r\ \ 1).(s\ \ 1)=\begin{bmatrix} s & 1 & . \\ . & -r & 1\end{bmatrix}\to\begin{bmatrix} s & 1 & . \\ rs & . & 1\end{bmatrix}\sim(rs\ \ 1)$$

$$(r\ \ 1)\square(s\ \ 1)=\begin{bmatrix} s & 1 & . \\ r & -1 & 1\end{bmatrix}\to\begin{bmatrix} s & 1 & . \\ r+s & . & 1\end{bmatrix}\sim(r+s\ \ 1)$$

That $M_{\Sigma}/\mathfrak{U}$ is trivial precisely when $\mathfrak{U}=M_{\Sigma}$ follows from Lemma 2.2. ∎

From now on we will denote $M_{\Sigma}/\mathfrak{U}$ by $R_{\mathfrak{U}}$. In fact the above homomorphism is Σ-inverting but first we require a lemma.

<u>Lemma</u> <u>2.5</u> *Under the conditions of Theorem 2.4 we have the following identities in* $R_{\mathfrak{U}}$;

(i) $(A_0\ \ A_*A_\infty)\square(B_0A_*A_\infty)=(A_0+B_0A_*A_\infty)$

(ii) If $p\in R_{\mathfrak{U}}$ *has as an admissible matrix* $(A_\infty A_*A_\infty)$ *then* $p=1$ *in* $R_{\mathfrak{U}}$.

(iii) $(A_0\ \ A_*\ \ A_\infty)=(A_0+A_*q\ \ A_*\ \ A_\infty)$ *when* q *is a column of the appropriate size over the characteristic ring of* R.

Proof. (i) Writing out the l.h.s. we have

$$\begin{bmatrix} B_o & A_* & A_\infty & 0 & 0 \\ A_o & 0 & -A_\infty & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} B_o & A_* & A_\infty & 0 & 0 \\ A_o+B_o & 0 & 0 & A_* & A_\infty \end{bmatrix}$$

which by (6) is equivalent to $(A_o+B_o\ A_*\ A_\infty)$.

(ii) $$(A_\infty A_* A_\infty)\square(1\ 1)^o = \begin{bmatrix} -1 & 1 & 0 & 0 \\ A_\infty & -A_\infty & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & A_* & A_\infty \end{bmatrix}$$

The final matrix on the r.h.s. lies in $\mathcal{L}$ and hence is equivalent to zero.

(iii) $$(A_o A_* A_\infty) \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ A_o & 0 & A_* & A_\infty \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ A_o+A_*q & A_*q & A_* & A_\infty \end{bmatrix}$$

(The second operation here is acheived by row operations.)

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ A_o+A_*q & 0 & A_* & A_\infty \end{bmatrix} \sim (A_o+A_*q\ A_*\ A_\infty)$$

(The first operation is permitted by column operations within the core.) ∎

[n.b. when two matrices differ in one column only e.g. $A=(A_o A_*)$, $B=(B_o A_*)$ then the **determinantal sum** of the two matrices is defined w.r.t. that column and is written $A\nabla B=(A_o+B_o\ A_*)$. This is developed later in Chapter 5. In (i) above we have shown that in $R_{\mathfrak{U}}$ the determinantal sum w.r.t. the initial column is admissible for the sum of the two elements represented by the original two matrices.]

<u>Theorem 2.6</u> *Under the same hypotheses as Theorem 2.4 the homomorphism* $\lambda_{\mathfrak{U}}: R \rightarrow R_{\mathfrak{U}}$ *is epic* Σ*-inverting.*

Proof. Let $A=(A_* A_\infty)\in\Sigma_n$. We shall denote the (i,j)th element of A by a_{ij} and the kth column of A by A_k. Also if $i<n$ we will write $A_\infty^{(i)}$ for the square matrix whose columns are all zero except the ith which is $-A_\infty$ and the nth which is A_∞; for $i=n$ define $A_\infty^{(i)}=0$.

Let B be the square matrix over $R_{\mathfrak{U}}$ whose (i,j)th element, b_{ij} has as an admissible matrix

$$\begin{bmatrix} e_j & A & 0 \\ e_j & A_\infty^{(i)} & A \end{bmatrix} .$$

Then we can show that B is a left inverse for $\lambda_{\mathfrak{U}}(A)$ over $R_{\mathfrak{U}}$. If Σ admitted column permutations then we could (as in Cohn [1] for his skew field construction) define a simpler admissible matrix for b_{ij} but we shall use the above. Then we have

$$(B\lambda_{\mathfrak{U}}(A))_{ik} = \textstyle\sum_j b_{ij}\lambda_{\mathfrak{U}}(A)_{jk} ,$$

where the r.h.s. has an admissible matrix;

$$\boxed{\textstyle\sum_j} \begin{bmatrix} a_{jk} & 1 & . & . \\ . & -e_j & A & . \\ . & -e_j & A_\infty^{(i)} & A \end{bmatrix}$$

Considering the jth individual summand we see that after adding the top row to the $j+1$ th and $n+j+1$ th row we have;

$$\begin{bmatrix} a_{jk} & 1 & . & . \\ a_{jk}e_j & . & A & . \\ a_{jk}e_j & . & A_\infty^{(i)} & A \end{bmatrix}$$

which simplifies by (6) hence,

$$(B\lambda_{\mathfrak{U}}(A))_{ik} = \boxed{\Sigma_j} \begin{bmatrix} a_{jk}e_j & A & . \\ a_{jk}e_j & A_\infty^{(i)} & A \end{bmatrix} .$$

We notice that each summand is only dependent on j in the first column so we can apply Lemma 2.5(i) inductively to obtain;

$$(B\lambda_{\mathfrak{U}}(A))_{ik} = \begin{bmatrix} A_k & A & . \\ A_k & A_\infty^{(i)} & A \end{bmatrix}$$

If $i \neq k$ then by simply subtracting the relevant two columns ($k \neq n$) or relevant column ($k=n$) from the first column we get a zero initial column hence;

$$(B\lambda_{\mathfrak{U}}(A))_{ik} = 0 \quad \text{for } i \neq k.$$

For $i=k$ there are two cases. Firstly if $i=k=n$ then $A_\infty^{(i)}=0$ and subtracting the $(n+1)$th column from the first we see that the initial and final columns agree since $A_n = A_\infty$. By Lemma 2.5 (ii) we have an admissible matrix for 1. Secondly if $i=k\neq n$ we have;

$$\begin{bmatrix} A_i & A_1 & A_2 & ..A_i.... & A_\infty & 0 \\ A_i & 0 & 0 & ..-A_\infty & 0..A_\infty & A \end{bmatrix} \rightarrow \begin{bmatrix} 0 & A & 0 \\ A_i+A_\infty & A_\infty^{(i)} & A \end{bmatrix}$$

and subtracting the $(n+i+1)$th column from the first we can again apply Lemma 2.5 (ii) to obtain a matrix that is admissible for 1. Hence we have proved that $(B\lambda_{\mathfrak{U}}(A))_{ik} = \delta_{ik}$ and B is a left invese for the image of A under $\lambda_{\mathfrak{U}}$.

If $v_n \in R_{\mathfrak{U}}$ has as admissible matrix $(A_o A_* A_\infty)$ where $(A_* A_\infty) \in \Sigma_n$ then we can show that there is an equation in $R_{\mathfrak{U}}$;

$$\lambda_{\mathfrak{U}}(A_* A_\infty)(v_1\ v_2\ \ldots\ldots v_n)^T - \lambda_{\mathfrak{U}}(A_o) = 0 \qquad (9)$$

where each v_i has as an admissible matrix

$$\begin{bmatrix} A_o & A & 0 \\ A_o & A_\infty^{(i)} & A \end{bmatrix}$$

To prove this we shall take n=2 for simplicity. The proof for larger n is entirely analogous. Then the ith row of the l.h.s. of (9) has as an admissible matrix;

$$(-a_{io}\ 1) \,\square\, (a_{i1}\ 1).\begin{bmatrix} A_o & A & 0 \\ A_o & A_\infty^{(1)} & A \end{bmatrix} \square\, (a_{i2}\ 1).\begin{bmatrix} A_o & A & 0 \\ A_o & A_\infty^{(2)} & A \end{bmatrix}$$

However since n=2 $A_\infty^{(2)}=0$ and $\begin{bmatrix} A_o & A & 0 \\ A_o & 0 & A \end{bmatrix} \sim (A_o A)$ by (6).

So we have,

$$(-a_{io}\ 1) \,\square\, \begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 \\ A_o & -A_2 & A_2 & A_1 & A_2 & 0 \\ 0 & 0 & 0 & 0 & -a_{i1} & 1 \end{bmatrix} \square \begin{bmatrix} A_o & A_1 & A_2 & 0 \\ 0 & 0 & -a_{i2} & 1 \end{bmatrix} =$$

$$\begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{i2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_o & 0 & 0 & 0 & A_1 & A_2 & 0 & 0 & 0 & 0 \\ A_o & 0 & 0 & 0 & -A_2 & A_2 & A_1 & A_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -a_{i1} & 1 & 0 \\ -a_{io} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

After adding the seventh and the eight columns to the fifth and then subtracting the third row block from the fourth we get;

$$\begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{i2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_o & 0 & 0 & 0 & A_1 & A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1 & A_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -a_{i1} & 0 & 0 & -a_{i1} & 1 & 0 \\ -a_{io} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

which by (5) is equivalent to,

$$\begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{i2} & 1 & 0 & 0 & 0 & 0 \\ A_o & 0 & 0 & 0 & A_1 & A_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -a_{i1} & 0 & 1 & 0 \\ -a_{io} & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Similarly by repeating this process we get

$$\begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 \\ 0 & 0 & -a_{i2} & 1 & 0 & 0 \\ 0 & -a_{i1} & 0 & -1 & 1 & 0 \\ -a_{io} & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

After several row operations we can reduce this to

$$\begin{bmatrix} A_o & A_1 & A_2 & 0 & 0 & 0 \\ 0 & 0 & -a_{i2} & 1 & 0 & 0 \\ 0 & -a_{i1} & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which we see lies in $\mathscr{L}$. Hence we have shown that (9) holds and taking $A_o=e_j$ (j=1 to n) we have that the image of A has a right inverse over $R_{\mathfrak{U}}$. Hence by Theorem 1.2 every element of $R_{\mathfrak{U}}$ is an entry in the inverse of the image of an element of Σ and $\lambda_{\mathfrak{U}}:R\longrightarrow R_{\mathfrak{U}}$ is epic Σ-inverting. ∎

<u>Corollary</u> *If R is any ring, Σ a lower multiplicative set of matrices over R and $\mathfrak{U}$ a Σ-matrix ideal then* $\Sigma\text{-Ker}\lambda_{\mathfrak{U}}=\mathfrak{U}$.

<u>Proof</u>. If we have an equation (9) in $R_{\mathfrak{U}}$ then v_n is unique since $(A_*A_\infty)\lambda_{\mathfrak{U}}$ is invertible. Hence if v_n is zero then $(A_oA_*A_\infty)\in\mathfrak{U}$ or else it is admissible for a non-zero element v_n and we have the same equation by what precedes (9), which is a contradiction. ∎

We have the following result which shows that a general Σ-inverting homomorphism will factor through certain homomorphisms $\lambda_{\mathfrak{U}}:R\longrightarrow R_{\mathfrak{U}}$.

<u>Lemma</u> <u>2.7</u> *If R is a ring, Σ a set of lower multiplicative matrices over R, $f:R\longrightarrow S$ a Σ-inverting homomorphism and $\mathfrak{U}$ a Σ matrix ideal such that* $\mathfrak{U}\subseteq\Sigma\text{-Ker}f$ *then f factors uniquely through* $\lambda_{\mathfrak{U}}:R\longrightarrow R_{\mathfrak{U}}$.

<u>Proof</u>. Define a map $f':R_{\mathfrak{U}}\longrightarrow S$ by $(A_oA_*A_\infty)\longmapsto e_\infty^T(A_*fA_\infty f)^{-1}A_of$. We know that this map is linear and multiplicative by construction (e.g. Lemma 1.1, 2.3). Hence to show f' is well defined it suffices to show that if $(A_oA_*A_\infty)\sim 0$ in $R_{\mathfrak{U}}$ then $(A_oA_*A_\infty)f'=0$, but this is precisely the property that $\mathfrak{U}$ has since $\mathfrak{U}$ is admissible for zero under f. If $r\in R$ then $r\lambda_{\mathfrak{U}}f'=(r\ 1)f'=1.rf=rf$. Hence f factors through $\lambda_{\mathfrak{U}}$. Suppose f also factors through $\lambda_{\mathfrak{U}}$ via f''. If $v_n\in R_{\mathfrak{U}}$ with admissible matrix $(A_oA_*A_\infty)$ then we have an equation (9) as in Theorem 2.6 taking the image of this equation under f' and f'' we

see that $v_n f' = v_n f''$ since $f = \lambda_{u} f' = \lambda_{u} f''$ is Σ-inverting. ■

There is an alternative way of doing this whole construction using rows as opposed to columns. The set of matrices Σ will then have to be upper multiplicative and our set of admissible matrices will be given by $A = \begin{bmatrix} a_o \\ a_* \\ a_\infty \end{bmatrix}$ where a_o is a row and the remaining matrix A^∞ lies in Σ with final row a_∞.

The two binary operations are then;

$$A \square B = \begin{bmatrix} b_o & a_o \\ b_* & . \\ b_\infty & -a_\infty \\ . & a_* \\ . & a_\infty \end{bmatrix} \qquad A.B = \begin{bmatrix} a_o & . \\ a_* & . \\ a_\infty & -b_o \\ . & b_* \\ . & b_\infty \end{bmatrix}$$

The minimal set of admissible matrices for zero $\mathcal{L}$ consists of all matrices which after a series of permissible operations (these differ also) can be written in the form below;

$$\begin{bmatrix} k_o & . \\ K & . \\ . & L \end{bmatrix}, \text{ where } K, L \in \Sigma \text{ and } k_o \text{ is a row..}$$

This we shall call the dual construction. We do not go into detail here but much is analogous to the proofs used above. In the next chapter we shall quote without proof the dual of our results on universal localisation.

Chapter 3 Universal Localisation

We start by showing that the universal Σ-inverting homomorphism is the R-ring constructed from the least Σ-matrix ideal $\mathfrak{U}_\Sigma$. We then proceed to describe $\mathfrak{U}_\Sigma$ in terms of $\mathcal{L}$ the minimal set of matrices admissible for zero and once this is achieved we have a description of the kernel. Having determined $\ker\lambda_\Sigma$ we give a necessary and sufficient condition for the universal Σ-inverting homomorphism to be an embedding.

<u>Theorem</u> <u>3.1</u> *If R is any ring, Σ a lower multiplicative set of matrices over R and $\mathfrak{U}_\Sigma$ the least Σ-matrix ideal then the universal Σ-inverting homomorphism is given by* $\lambda_{\mathfrak{U}_\Sigma}:R\longrightarrow R_{\mathfrak{U}_\Sigma}$.

<u>Proof</u>. Let σ be the collection of all Σ-inverting homomorphisms then $\mathfrak{U}=\bigcap_{\{f\in\sigma\}}\Sigma\text{-Ker}f$ is a Σ-matrix ideal which is admissible for zero under <u>all</u> Σ-inverting homomorphisms. Hence by Lemma 2.7 every $f\in\sigma$ factors uniquely through $\lambda_{\mathfrak{U}}:R\longrightarrow R_{\mathfrak{U}}$ which is itself a Σ-inverting homomorphism by Theorem 2.6 i.e. this is the universal Σ-inverting homomorphism. Since $\mathfrak{U}_\Sigma$ is the least Σ-matrix ideal we have $\mathfrak{U}\supseteq\mathfrak{U}_\Sigma$ but since $\Sigma\text{-Ker}\lambda_{\mathfrak{U}_\Sigma}=\mathfrak{U}_\Sigma$ by the corollary to Theorem 2.6 we have $\mathfrak{U}_\Sigma\supseteq\mathfrak{U}$ and we have proved the result. ∎

Now we proceed to describe $\mathfrak{U}_\Sigma$.

<u>Lemma</u> <u>3.2</u> *Let R be a ring, Σ a lower multiplicative set of matrices and $\mathcal{L}$ the minimal set of admissible matrices for zero. Then $\mathcal{L}$ is an additive subsemigroup of M_Σ and a multiplicative ideal of M_Σ.*

<u>Proof</u> If L_1 and L_2 lie in $\mathcal{L}$ then by definition we have;

$$L_1 \rightarrow \begin{bmatrix} A_o & A & . & . \\ . & . & B_* & B_\infty \end{bmatrix} \qquad L_2 \rightarrow \begin{bmatrix} C_o & C & . & . \\ . & . & D_* & D_\infty \end{bmatrix}$$

where A, (B_*B_∞), C and (D_*D_∞) lie in Σ; A_o and C_o are columns over R. Then we have;

$$L_1 \mathbin{\square} L_2 \rightarrow \begin{bmatrix} C_o & C & . & . & . & . & . \\ . & . & D_* & D_\infty & . & . & . \\ A_o & . & . & . & A & . & . \\ . & . & . & -B_\infty & . & B_* & B_\infty \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} C_o & C & . & . & . & . & . \\ A_o & . & . & . & A & . & . \\ . & . & D_* & D_\infty & . & . & . \\ . & . & . & -B_\infty & . & B_* & B_\infty \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} C_o & C & . & . & . & . & . \\ A_o & . & A & . & . & . & . \\ . & . & . & D_* & D_\infty & . & . \\ . & . & . & . & -B_\infty & B_* & B_\infty \end{bmatrix} \in \mathcal{L}$$

Hence $L_1 \mathbin{\square} L_2 \in \mathcal{L}$ so $\mathcal{L}$ is an additive subsemigroup of M_Σ.

If now $(E_oE_*E_\infty)=E$ is any admissible matrix then we have the following;

$$L_1.E \rightarrow \begin{bmatrix} E_o & E_* & E_\infty & . & . & . \\ . & . & -A_o & A & . & . \\ . & . & . & . & B_* & B_\infty \end{bmatrix} \in \mathcal{L}$$

and also;

$$E.L_1 \rightarrow \begin{bmatrix} A_0 & A & . & . & . & . \\ . & . & B_* & B_\infty & . & . \\ . & . & . & -E_0 & E_* & E_\infty \end{bmatrix} \in \mathcal{L}$$

So $\mathcal{L}$ is also a multiplicative ideal of M_Σ. ∎

Now we require a lemma on commutative semigroups.

Lemma 3.3 *If H is a subsemigroup of a commutative semigroup S then $H:H = \{a \in S \mid \exists b \in H; a+b \in H\}$ is the least unitary subsemigroup of S containing H.*

Proof Firstly H:H is a subsemigroup of S. For if $a,c \in H:H$ then there are $b,d \in S$ such that $b, a+b, d, c+d \in H$. Then $(a+b)+(c+d) =(a+c)+(b+d) \in H$ and $b+d \in H$, hence $a+c \in H:H$.

H:H is unitary since if $a+b, b \in H:H$ then there exist $c,d \in H$ with $(a+b)+c, b+d \in H$. So $b+c+d \in H$ and $a+b+c+d \in H$ hence $a \in H:H$.

If K is a unitary subsemigroup of S containing H then $a \in H:H$ implies that there exists $b \in H$ with $a+b \in H$. So $b, a+b \in K$ and since K is unitary $a \in K$. Hence $K \supseteq H:H$. ∎

Applying this to $\mathfrak{U}$, an additive subsemigroup of M_Σ we see that $\mathfrak{U}:\mathfrak{U}$ is the least unitary subsemigroup containing $\mathfrak{U}$.

Lemma 3.4 *If R is any ring, Σ a lower multiplicative set of matrices and $\mathfrak{U} \supseteq \mathcal{L}$ is an additive subsemigroup and a multiplicative ideal of M_Σ then $\mathfrak{U}:\mathfrak{U}$ is the least Σ-matrix ideal containing $\mathfrak{U}$.*

Proof From the above remark the result will follow if we can show that $\mathfrak{U}:\mathfrak{U}$ is a multiplicative ideal of M_Σ.

If $A\in \mathfrak{U}:\mathfrak{U}$ then there exists $B\in\mathfrak{U}$ such that $A\square B\in\mathfrak{U}$. Now let C be any admissible matrix in M_{Σ} then from the remark succeeding (7) and (8) in Theorem 2.4 we have;

$$\{C.(A\square B)\}^{o} \ \square\ C.A \ \square\ C.B \quad \in \mathcal{L} \subseteq \mathfrak{U}:\mathfrak{U}$$

Since $B\in\mathfrak{U}$ and $\mathfrak{U}$ is an ideal we have that $C.B\in\mathfrak{U}$. So since $\mathfrak{U}:\mathfrak{U}$ is unitary the following holds;

$$\{C.(A\square B)\}^{o} \ \square\ \ C.A \quad \in \mathfrak{U}:\mathfrak{U}$$

Now $C^{o}.(A\square B)\in\mathfrak{U}$ by the same argument that holds for C.B. And in M_{Σ} we have $C^{o}.X=(C.X)^{o}$ so again since $\mathfrak{U}:\mathfrak{U}$ is unitary we have that $C.A\in\mathfrak{U}:\mathfrak{U}$.

The argument showing $A.C\in\mathfrak{U}:\mathfrak{U}$ follows in exactly the same manner using the distributive law on the other side. ∎

By applying this to $\mathcal{L}$ we have;

<u>Theorem 3.5</u> *If R is any ring and Σ a lower multiplicative set of matrices over R then the least Σ-matrix ideal $\mathfrak{U}_{\Sigma}$ is $\mathcal{L}:\mathcal{L}$ where $\mathcal{L}$ is the minimal set of matrices admissible for zero.* ∎

<u>Corollary</u> *If $\mathcal{M}$ is an additive unitary subset of M_{Σ} which contains $\mathcal{L}$, the minimal set of admissible matrices for zero, and every admissible matrix in $\mathcal{M}$ necessarily represents zero under any Σ-inverting homomorphism then $\mathcal{M}$ is the least Σ-matrix ideal.*

<u>Proof</u> $\mathcal{M}$ contains $\mathcal{L}$ so $\mathcal{M}:\mathcal{M}$ contains $\mathcal{L}:\mathcal{L}$ which is the least Σ-matrix ideal from the above. However $\mathcal{M}$ is unitary so $\mathcal{M}:\mathcal{M}$ is just $\mathcal{M}$ and hence we have that $\mathcal{M}$ contains $\mathfrak{U}_{\Sigma}$. Conversely since all admissible matrices in $\mathcal{M}$ represent zero under any Σ-inverting homomorphism $\mathcal{M}$ must be contained in the least Σ-matrix ideal. ∎

Now we have our description of the kernel of the universal Σ-inverting homomorphism.

Theorem 3.6 *If R is any ring and Σ a lower multiplicative set of matrices over R then $r\in R$ lies in the kernel of the universal Σ-inverting homomorphism if and only if there exists an equation:*

$$A_* Q = (A_\infty r \ *)\Sigma \qquad (1)$$

Where $(A_* A_\infty)$ and Σ lie in Σ, Q has entries in the characteristic ring, and $*$ is some matrix of the appropriate size.

Proof An element r of R lies in the kernel if and only if (r 1) lies in $\mathfrak{U}_\Sigma$, which we have shown is $\mathscr{L}:\mathscr{L}$. Then by the definition of $\mathscr{L}:\mathscr{L}$ we must have an equation; $\{(U_0 U)\oplus V\}\square(r1)\in\mathscr{L}$ where $U,V\in\Sigma$. Hence we get a matrix equation as below.

$$\begin{bmatrix} r & 1 & . & . & . \\ U_0 & . & U & . & . \\ . & -V_\infty & . & V_* & V_\infty \end{bmatrix}\begin{bmatrix} 1 & . & . \\ . & Q_{11} & Q_{12} \\ . & Q_{21} & Q_{22} \\ . & Q_{31} & Q_{32} \\ . & . & e_\infty^T \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \\ P_{31} & P_{32} \end{bmatrix}\begin{bmatrix} L_0 & L & . \\ . & . & M \end{bmatrix}$$

Where U_0, L_0 are columns; $U, (V_* V_\infty), L, M\in\Sigma$. The half bordered matrix together with the other invertible matrix (with entries in the P_{ij}) have been blocked appropriately. From this we can obtain directly the following matrix equation;

$$\begin{bmatrix} r & 1 & . \\ . & -V_\infty & V_* \end{bmatrix}\begin{bmatrix} 1 & . \\ . & Q_{11} \\ . & Q_{31} \end{bmatrix} = \begin{bmatrix} P_{11} \\ P_{31} \end{bmatrix}\begin{bmatrix} L_0 & L \end{bmatrix} \qquad (2)$$

After eliminating V_∞ by row operations this gives us the equations,

$V_\infty r = P'_{31} L_o$ and $V_* Q_{31} = P'_{31} L$, where P'_{31} is the transformed P_{31} (after eliminating V_∞ by row operations). Rewriting these in the following form we have proved the necessity of the result.

$$V_*(0\ Q_{31}) = (V_\infty r\ P'_{31}) \begin{bmatrix} 1 & 0 \\ -L_o & L \end{bmatrix}$$

It is a straightforward calculation to show that the condition is also sufficient and this is discussed below. ∎

We also have the dual result:

Theorem 3.6* *If R is any ring and Σ an upper multiplicative set of matrices over R then r∈R lies in the kernel of the universal Σ-inverting homomorphism if and only if there exists an equation:*

$$Q1_* = \Sigma \begin{bmatrix} r1_\infty \\ * \end{bmatrix} \qquad (3)$$

where Σ and $\begin{bmatrix} 1_* \\ 1_\infty \end{bmatrix} \in \Sigma$, *Q has entries in the characteristic ring and* * *is some matrix of the appropriate size.* ∎

From this description of the kernel we can state a necessary and sufficient condition on a ring for the universal Σ-inverting homomorphism to be an embedding. If we have an equation (1) for $(A_* A_\infty) \in \Sigma$ where r is non-zero we shall say that $(A_* A_\infty)$ is a **column Σ-zero divisor** (and for an equation (3) we say **row Σ-zero divisor**) since if Σ is right-inverted $(A_* A_\infty)$ becomes a left zero divisor via the first column in the expression below;

$$A_* Q \Sigma^{-1} = (A_\infty r\ *).$$

Hence we have the following.

Theorem 3.7 *If R is any ring, Σ a lower multiplicative set of matrices over R then the universal Σ-inverting homomorphism is an embedding if and only if Σ contains no column Σ-zero divisors.* ■

And the dual result:

Theorem 3.7* *If R is any ring, Σ an upper multiplicative set of matrices over R then the universal Σ-inverting homomorphism is an embedding if and only if Σ contains no row Σ-zero divisors.* ■

There is a special case where this condition simplifies.

Lemma 3.8 *If R is any ring, Σ a lower multiplicative set of matrices and if $\mathcal{L}$ the minimal set of admissible matrices for zero is closed w.r.t. first column determinantal sums then the universal Σ-inverting homomorphism is an embedding if and only if Σ does not contain a left zero divisor.*

Proof If $\Sigma_1 u=0$ in R where $\Sigma_1 \in \Sigma$ and u is a non-zero column over R then in R_Σ we can cancel the image of Σ_1 to show that the components of u must lie in the kernel. Conversely we have;

$$\begin{bmatrix} r & 1 & . & . & . \\ U_o & . & U & . & . \\ . & -V_\infty & . & V_* & V_\infty \end{bmatrix} \nabla \begin{bmatrix} . & 1 & . & . & . \\ -U_o & . & U & . & . \\ . & -V_\infty & . & V_* & V_\infty \end{bmatrix} = \begin{bmatrix} r & 1 & . & . & . \\ . & . & U & . & . \\ . & -V_\infty & . & V_* & V_\infty \end{bmatrix}$$

If r lies in the kernel then by Theorem 3.5 there exist $U, (V_* V_\infty) \in \Sigma$ and U_o a column such that the first matrix on the l.h.s. lies in $\mathcal{L}$.

After permuting some rows and columns of the second matrix on the l.h.s. we see that it lies in $\mathcal{L}$. Hence by hypothesis the r.h.s. must lie in $\mathcal{L}$ also. After adding left multiples of the top row to rows within the bottom row block and then a column operation we see that there exist invertible matrices P and Q where Q is half bordered such that there is a factorisation,

$$P\begin{bmatrix} r & 1 & . & . & 1 \\ . & . & U & . & . \\ V_\infty r & . & . & V_* & V_\infty \end{bmatrix} = \begin{bmatrix} L_o & L & . & . \\ . & . & M_* & M_\infty \end{bmatrix} Q$$

Equating first and last columns on both sides and noting that Q is half bordered we see that,

$$P\begin{bmatrix} 1 \\ . \\ V_\infty \end{bmatrix} r = \begin{bmatrix} L_o \\ . \end{bmatrix} \quad \text{and} \quad P\begin{bmatrix} 1 \\ . \\ V_\infty \end{bmatrix} = \begin{bmatrix} * \\ M_* q + M_\infty \end{bmatrix}$$

where an asterisk denotes a non-specific column and q is also a column. Hence if r is a non-zero member of the kernel then we have a non-trivial column u such that $(M_* M_\infty)u=0$ as required. ∎

Chapter 4 The Criteria of Malcolmson and Gerasimov

The results of the previous chapter now enable us to prove the results of Malcolmson and Gerasimov respectively on the kernel of the universal Σ-inverting homomorphism. Of course this can be achieved directly by using the equations in (2) of Theorem 3.6 in the previous chapter but we shall verify what amounts to their description of the least Σ-matrix ideal by using the corollary to Theorem 3.5.

Malcolmson's construction, using a so called "zigzag" method, constructs R_{Σ} as a set of equivalence classes of triples (b,A,c) where b is a row, c a column and A lies in Σ. The equivalence class of (b,A,c) is to be interpreted as the element $bA^{-1}c$ of R_{Σ}. In the present construction we are considering elements $e_{\infty}^{T}A^{-1}A_{o}$. Hence by substituting $b=e_{\infty}^{T}$ and $c=A_{o}$ we can determine Malcolmson's description of the elements (e_{∞}^{T},A,A_{o}) which lie in the equivalence class of $(1,1,0)$ i.e. those admissible matrices which represent zero and therefore lie in $\mathfrak{U}_{\Sigma}$.

In Malcolmson [4] we see that $(e_{\infty}^{T},A,A_{o})\sim(1,1,0)$ if there exist $L,M,P,Q\in\Sigma$, rows j and u the sizes of L and P, respectively, and columns w and v the sizes of M and Q, respectively, such that

$$\begin{bmatrix} A & . & . & . & A_{o} \\ . & 1 & . & . & . \\ . & . & L & . & . \\ . & . & . & M & w \\ e_{\infty}^{T} & 1 & j & . & . \end{bmatrix} = \begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix}$$

In fact we can simplify this slightly. We shall prove the following theorem which is essentially Malcolmson's description of $\mathfrak{U}_{\Sigma}$ the least Σ-matrix ideal. Then we will gain a description of the kernel of the universal Σ-inverting homomorphism which is Malcolmson's criterion without having used his "zigzag" construction. However our result will have to be slightly less general.

<u>Theorem</u> <u>4.1</u> *If R is a ring and Σ a lower multiplicative set of matrices closed w.r.t. multiplication on the left and right by elements of $GL(R)$ then $\mathfrak{U}_\Sigma$ the least Σ-matrix ideal consists precisely of those admissible matrices $(A_o A_* A_\infty)$ for which there exist matrices $L,M,P,Q\in\Sigma$; rows j and u the sizes of L and P respectively; and columns w and v the sizes of M and Q respectively, such that there exists a factorisation (1) below.*

$$\begin{bmatrix} A & . & . & A_o \\ . & L & . & . \\ . & . & M & w \\ e_\infty^T & j & . & . \end{bmatrix} = \begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix} \qquad (1)$$

Where $A=(A_* A_\infty)$.

<u>Proof</u> Such a factorisation will be called an <u>allowable</u> <u>factorisation</u> for $(A_o A_* A_\infty)$.

Given the R.H.S. of (1) we note that we can perform certain elementary operations viz. if $S,T\in GL(R)$ and t is any column of the appropriate size then,

$$\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix} \begin{bmatrix} T & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} SP \\ u \end{bmatrix} \begin{bmatrix} QT & Qt+v \end{bmatrix}.$$

Consequently we can apply these <u>restricted</u> <u>operations</u> to both sides of (1) and maintain the form of the factorisation on the R.H.S.. Any application of these restricted operations will be denoted by an arrow. We note that given a factorisation (1) for an admissible matrix A then by using these restricted operations we can obtain an allowable factorisation for any matrix obtainable from A by a series of permissible operations. Hence $\mathfrak{U}$ is a well defined subset of M_Σ. We can now proceed to show that $\mathfrak{U}$ is the least Σ-matrix ideal by using the corollary to Theorem 3.5. Observing the following factorisation;

$$\begin{bmatrix} K & . & . & K_o \\ . & L & . & . \\ . & . & I & . \\ . & e_\infty^T & . & . \end{bmatrix} = \begin{bmatrix} I & . & . \\ . & L & . \\ . & . & I \\ . & e_\infty^T & . \end{bmatrix} \begin{bmatrix} K & . & . & K_o \\ . & I & . & . \\ . & . & I & . \end{bmatrix}$$

where $K, L \in \Sigma$ and K_o is any column we see that $\mathfrak{U}$ contins $\mathcal{L}$. Now suppose that $(A_oA_*A_\infty)\square(B_oB_*B_\infty) \in \mathfrak{U}$ with an allowable factorisation (1) and also $(B_oB_*B_\infty) \in \mathfrak{U}$ with a similar factorisation in which P is replaced by P_1 etc. then,

$$\begin{bmatrix} P & . \\ . & P_1 \\ u & -u_1 \end{bmatrix} \begin{bmatrix} Q & . & v \\ . & Q_1 & v_1 \end{bmatrix} = \begin{bmatrix} A_* & A_\infty & . & . & . & . & . & . & . & . & A_o \\ . & -B_\infty & B_* & B_\infty & . & . & . & . & . & . & B_o \\ . & . & . & . & L & . & . & . & . & . & . \\ . & . & . & . & . & M & . & . & . & . & w \\ . & . & . & . & . & . & B_* & B_\infty & . & . & B_o \\ . & . & . & . & . & . & . & . & L_1 & . & . \\ . & . & . & . & . & . & . & . & . & M_1 & w_1 \\ . & . & . & 1 & j & . & . & -1 & -j_1 & . & . \end{bmatrix}$$

After adding the fourth column block to the second and to the eighth column blocks, and the third to the seventh we have;

$$\rightarrow \quad \begin{bmatrix} A_* & A_\infty & . & . & . & . & . & . & . & . & A_o \\ . & . & B_* & B_\infty & . & . & B_* & B_\infty & . & . & B_o \\ . & . & . & . & L & . & . & . & . & . & . \\ . & . & . & . & . & M & . & . & . & . & w \\ . & . & . & . & . & . & B_* & B_\infty & . & . & B_o \\ . & . & . & . & . & . & . & . & L_1 & . & . \\ . & . & . & . & . & . & . & . & . & M_1 & w_1 \\ . & 1 & . & 1 & j & . & . & . & -j_1 & . & . \end{bmatrix}$$

Then subtracting the fifth row block from the second and permuting allowable rows and columns we get an allowable factorisation for $(A_o A_* A_\infty)$ so that $\mathfrak{U}$ is unitary.

It is clear that any admissible matrix with an allowable factorisation (1) represents zero under any Σ-inverting homomorphism for then we have;

$$0 = uv = uQ(PQ)^{-1}Pv = e_\infty^T A^{-1} A_o + jL^{-1}0 + 0M^{-1}w = e_\infty^T A^{-1} A_o$$

Indeed this was Malcolmson's initial motivation for defining his equivalence. So consequently by the corollary to Theorem 3.5 we deduce that $\mathfrak{U}$ is the least Σ-matrix ideal. ∎

Malcolmson's Description of the Kernel

Now $r \in \ker(\lambda)$ if and only if $-r \in \ker(\lambda)$ which is if and only if $(-r \;\; 1) \in \mathfrak{U}_\Sigma$ and this is if and only if there is an allowable factorisation for $(-r \;\; 1)$. Suppose so then;

$$\begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix} = \begin{bmatrix} 1 & . & . & -r \\ . & L & . & . \\ . & . & M & w \\ 1 & j & . & . \end{bmatrix} \rightarrow \begin{bmatrix} 1 & . & . & . \\ . & L & . & . \\ . & . & M & w \\ 1 & j & . & r \end{bmatrix}$$

This is an instance of a factorisation (2) below.

$$\begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix} = \begin{bmatrix} L & . & . \\ . & M & w \\ j & . & r \end{bmatrix} \qquad (2)$$

where P,Q,... etc. satisfy the conditions of (1). If we have such a factorisation then we have $r \in \ker(\lambda_\Sigma)$ by the following allowable factorisation for (-r 1).

$$\begin{bmatrix} 1 & . \\ . & P \\ 1 & u \end{bmatrix} \begin{bmatrix} 1 & . & -r \\ . & Q & v \end{bmatrix} = \begin{bmatrix} 1 & . & -r \\ . & PQ & . \\ 1 & uQ & . \end{bmatrix}$$

We have proved the following description of the kernel due to Malcolmson [**4**].

<u>Theorem</u> <u>4.2</u> *If R is any ring, Σ a set of lower multiplicative matrices closed w.r.t. multiplication on the left and right by elements of GL(R) then the kernel of the universal Σ-inverting homomorphism consists precisely of those elements r∈R for which there exist matrices L,M,P,Q∈Σ; rows j and u the sizes of L and P respectively; and columns w and v the sizes of M and Q respectively, such that*

$$\begin{bmatrix} L & 0 & 0 \\ 0 & M & w \\ j & 0 & r \end{bmatrix} = \begin{bmatrix} P \\ u \end{bmatrix} \begin{bmatrix} Q & v \end{bmatrix} .$$

■

Gerasimov's Criterion

Gerasimov constructed the universal Σ-inverting ring by considering not elements of R_Σ but all matrices over R_Σ. There is an account of the construction in Chapter 7 [**1**]. As before we will take the description of those admissible matrices over R which represent zero and prove that they form the least Σ-matrix ideal.

Gerasimov constructed all matrices over R_Σ by considering 4-block matrices over R of the form

$$A = \left[\begin{array}{c|c} A' & \tilde{A} \\ \hline A^{o} & {}^{\backslash}A \end{array}\right]$$

where $A^{o} \in \Sigma$. Then A was to be interpreted as the matrix over R_Σ given by $\tilde{A} - A'(A^{o})^{-1}\,{}^{\backslash}A$. Hence in this construction an admissible matrix $(A_o A_* A_\infty)$ is represented by the 4-block matrix,

$$A = \left[\begin{array}{c|c} \cdot \ \ -1 & \cdot \\ \hline A_* A_\infty & A_o \end{array}\right] .$$

From the account of Gerasimov's construction in 7.11 [**1**] we see that this matrix represents zero in R_Σ if and only if there exists a factorisation

$$\begin{bmatrix} \cdot & -1 & * & * \\ A_* & A_\infty & * & * \\ \cdot & \cdot & \Sigma & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \\ \Sigma & * \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & A_o \\ \cdot & \cdot \end{bmatrix}$$

where Σ denotes any matrix lying in Σ and an asterisk denotes any matrix of the appropriate size. In fact we shall simplify this slightly and prove the following result.

Theorem 4.3 *If R is any ring and Σ a lower multiplicative set of matrices over R then the least Σ-matrix ideal consists precisely of those admissible matrices* $(A_o A_* A_\infty)$ *for which there exists a factorisation (3) below.*

$$\begin{bmatrix} A_* & * & * \\ 0 & \Sigma & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ \Sigma & * \end{bmatrix} = \begin{bmatrix} 0 & A_o \\ 0 & 0 \end{bmatrix} \qquad (3)$$

Where an asterisk denotes any matrix over R of the appropriate size and Σ denotes any matrix lying in Σ.

Proof Let $\mathfrak{U}$ be all those admissible matrices for which there exists such a factorisation. Firstly we show that $\mathfrak{U}$ is indeed a well defined subset of M_Σ. If P is any matrix over R such that PA_* is defined then multiplying both sides of (3) on the left by $P \oplus I$ gives us,

$$\begin{bmatrix} PA_* & P* & P* \\ 0 & \Sigma & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ \Sigma & * \end{bmatrix} = \begin{bmatrix} 0 & PA_o \\ 0 & 0 \end{bmatrix}$$

If Q is an invertible matrix and A_*Q is defined then we note;

$$\begin{bmatrix} A_*Q & * & * \\ 0 & \Sigma & * \end{bmatrix} \begin{bmatrix} Q^{-1}* & Q^{-1}* \\ * & * \\ \Sigma & * \end{bmatrix} = \begin{bmatrix} 0 & A_o \\ 0 & 0 \end{bmatrix}$$

We have shown that if $(A_o A_* A_\infty) \to (B_o B_* B_\infty)$ then $(A_o A_* A_\infty) \in \mathfrak{U}$ implies that $(B_o B_* B_\infty) \in \mathfrak{U}$. Hence $\mathfrak{U}$ is a well defined subset of M_Σ. We can now show that $\mathfrak{U}$ is the least Σ-matrix ideal of R by using the corollary to Theorem 3.5.

Firstly $\mathfrak{U}$ contains $\mathscr{L}$ since if $K, L \in \Sigma$ and K_o is any column over R then we have a factorisation for $(K_o K) \oplus L_*$;

$$\begin{bmatrix} K & 0 & 0 & I \\ 0 & L_* & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & 0 \\ 0 & 0 \\ K & K_o \end{bmatrix} = \begin{bmatrix} 0 & K_o \\ 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

To show that $\mathfrak{U}$ is a unitary subset of the additive semigroup M_Σ we suppose that $(A_o A_* A_\infty) \square (B_o B_* B_\infty)$ has an allowable factorisation below;

$$\begin{bmatrix} A_* & A_\infty & . & U & V \\ . & -B_\infty & B_* & \bar{U} & \bar{V} \\ . & . & . & \Sigma_1 & W \end{bmatrix} \begin{bmatrix} M & m \\ T & t \\ Q & q \\ N & n \\ \Sigma_2 & s \end{bmatrix} = \begin{bmatrix} . & A_o \\ . & B_o \\ . & . \end{bmatrix}$$

;and also that $(B_o B_* B_\infty)$ lies in $\mathfrak{U}$ with a factorisation given below;

$$\begin{bmatrix} B_* & U^+ & V^+ \\ . & \Sigma_1^+ & W^+ \end{bmatrix} \begin{bmatrix} M^+ & m^+ \\ N^+ & n^+ \\ \Sigma^+ & s^+ \end{bmatrix} = \begin{bmatrix} . & B_o \\ . & . \end{bmatrix}$$

Then we see from the following factorisation that $(A_o A_* A_\infty) \in \mathfrak{U}$.

$$\begin{bmatrix} A_* & \cdot & -A_\infty & U & \cdot & V & \cdot \\ \cdot & \Sigma_1^+ & \cdot & \cdot & \cdot & \cdot & W^+ \\ \cdot & \cdot & \Sigma_1 & \cdot & \cdot & W & \cdot \\ \cdot & U^+ & \bar{U} & B_* & B_\infty & \bar{V} & V^+ \end{bmatrix} \begin{bmatrix} M & \cdot & m \\ \cdot & N^+ & -n^+ \\ N & \cdot & n \\ Q & M^+ & q-m^+ \\ -T & \cdot & -t \\ \Sigma_2 & \cdot & s \\ \cdot & \Sigma_2^+ & -s^+ \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & A_o \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

By the corollary to Theorem 3.5 it is now possible to show that $\mathfrak{U}$ is the least Σ-matrix ideal by showing that every admissible matrix in $\mathfrak{U}$ represents zero under any Σ-inverting homomorphism.

Suppose that we have a factorisation below for an admissible matrix $(A_oA_*A_\infty)\in\mathfrak{U}$.

$$\begin{bmatrix} A_* & U & V \\ \cdot & \Sigma_1 & W \end{bmatrix} \begin{bmatrix} M & m \\ N & n \\ \Sigma_2 & s \end{bmatrix} = \begin{bmatrix} \cdot & A_o \\ \cdot & \cdot \end{bmatrix} \qquad (4)$$

Then upon expansion we get;

$$\Sigma_1 N + W\Sigma_2 = 0 \qquad (5)$$

$$\Sigma_1 n + Ws = 0 \qquad (6)$$

$$A_*M + UN + V\Sigma_2 = 0 \qquad (7)$$

$$A_*m + Un + Vs = A_o \qquad (8)$$

From (5) and (6) we get;

$$UN=-U\Sigma_1^{-1}W\Sigma_2 \quad \text{and} \quad Un=-U\Sigma_1^{-1}Ws \quad .$$

Substituting into (7) and (8) respectively we get;

$$A_*M + (V-U\Sigma_1^{-1}W)\Sigma_2= 0$$

$$A_*m + (V-U\Sigma_1^{-1}W)s = A_o$$

Thus we see that $A_*(m-M\Sigma_2^{-1}s)=A_o$. After multiplying both sides on the left by $(A_*A_\infty)^{-1}$ we conclude that the last component of

$(A_* A_\infty)^{-1} A_o$ is zero. Hence $\mathfrak{U}$ is the least Σ-matrix ideal. ∎

Now the following theorem is Gerasimov's description of the kernel of the universal Σ-inverting homomorphism.

Theorem 4.4 *If R is any ring and Σ a lower multiplicative set of matrices over R then the kernel of the universal Σ-inverting homomorphism consists precisely of those elements r of R for which there exists a factorisation below.*

$$\begin{bmatrix} . & r \\ . & . \end{bmatrix} = \begin{bmatrix} * & * \\ \Sigma & * \end{bmatrix} \begin{bmatrix} * & * \\ \Sigma & * \end{bmatrix}$$

Where Σ represents any matrix lying in Σ and $*$ denotes any matrix over R of the appropriate size.

Proof For any $r \in R$, r lies in the kernel if and only if $(r\ 1) \in \mathfrak{U}$. Then we have a factorisation (4) for $(r\ 1)$;

$$\begin{bmatrix} u & v \\ \Sigma_1 & W \end{bmatrix} \begin{bmatrix} N & n \\ \Sigma_2 & s \end{bmatrix} = \begin{bmatrix} . & r \\ . & . \end{bmatrix}$$

Since the core of $(r\ 1)$ is null u and v are rows, M and m are null. ∎

By comparison it is straightforward to show the equations in (2) Theorem 3.6 give the following:

Theorem 4.5 *Under the same conditions as Theorem 4.4 then $r \in \ker \lambda_\Sigma$ if and only if there exists a factorisation below.*

$$\begin{bmatrix} . & r \\ . & . \end{bmatrix} = \begin{bmatrix} e_\infty^T & * \\ \Sigma & * \end{bmatrix} \begin{bmatrix} Q & 0 \\ \Sigma & * \end{bmatrix} \quad \Sigma \in \Sigma \textit{ etc.}$$ ∎

Chapter 5 **Prime Matrix Ideals**

The concept of a prime matrix ideal $\mathcal{P}$ was introduced by P.M.Cohn to construct a skew field using admissible matrices whose numerators lie in $\mathcal{P}$. This construction was the basis of the more general construction in Chapter 2. Here we define, slightly more simply than Cohn, a prime matrix ideal going on to determine conditions for a ring to have a prime matrix ideal. This criterion will be shown to refine Cohn's result for a ring R to have a homomorphism to a skew field.

Prime matrix ideals are collections of square matrices over a ring R with certain defined properties. We will show that given a prime matrix ideal $\mathcal{P}$ the set of admissible matrices whose numerators lie in $\mathcal{P}$ is a Σ-matrix ideal where Σ is the complement of $\mathcal{P}$. This can then be shown to lead directly to the construction of a skew field using Theorem 2.6.

Definitions. The **inner rank** of an $r \times s$ matrix A over a ring R is defined as the least n such that A=PQ where P is $r \times n$ and Q is $n \times s$ over R. We shall write $\rho(A)=n$. A square matrix is called **full** if its inner rank equals its order and **non-full** otherwise.

Let R be any ring, $f:R \longrightarrow F$ a homomorphism from R to a field F; then we can consider $\mathcal{P}$, the set of all square matrices over R which are mapped to singular matrices over F. This set is called the **singular kernel** of f and is denoted by Kerf. We make the following observations;

N1. All non-full matrices lie in $\mathcal{P}$. This is simply because f preserves the property of being non-full and since a field has UGN only full matrices can be invertible.

N2. If the determinantal sum of $A,B \in \mathcal{P}$ is defined w.r.t. some column then $A \nabla B \in \mathcal{P}$. This is since $f(A \nabla B)=f(A) \nabla f(B)$ and the expression on the right is the determinantal sum of two singular matrices over a field which must be singular.

N3. If $A \in \mathcal{P}$ then for any square matrix B over R, $A \oplus B \in \mathcal{P}$.

N4. If $1 \oplus A \in \mathcal{P}$ then $A \in \mathcal{P}$.

N5. $1 \notin \mathcal{P}$ and if A,B are square matrices with $A \oplus B \in \mathcal{P}$ then at least one of A or B must lie in $\mathcal{P}$.

A square matrix is said to be **hollow** if it has an $r \times s$ block of zeros (maybe after column permutations) and $r+s>n$ where n is the order of the matrix. A square matrix is said to be **degenerate** if one column is a right multiple of another (distinct) column. Both hollow and degenerate matrices can easily be shown to be non-full. With the above definitions and observations in mind we now introduce the notions of matrix pre-ideal, matrix ideal and prime matrix ideal and aim to derive a condition for the existence of the latter.

If $\mathcal{P}$ is a set of square matrices over a ring R then $\mathcal{P}$ is said to be a **matrix pre-ideal** if the following conditions hold.

M1. All square matrices which are hollow or degenerate lie in $\mathcal{P}$. (Such matrices will be said to have property M1.)

M2. $\mathcal{P}$ is closed w.r.t. column determinantal sums (when defined).

M3. If $A \in \mathcal{P}$ and B is any square matrix over R then $A \oplus B \in \mathcal{P}$.

A **matrix ideal** is a matrix pre-ideal which also satisfies,

M4. If $1 \oplus A \in \mathcal{P}$ then $A \in \mathcal{P}$.

A matrix ideal is said to be **proper** if $1 \notin \mathcal{P}$. A **prime matrix ideal** is a proper matrix ideal which additionally satisfies, M5. If $A \oplus B \in \mathcal{P}$ where both A and B are square then at least one of A or B lies in $\mathcal{P}$.

It is clear from what has been said above that,

<u>Proposition</u> <u>5.1</u> *If $f: R \longrightarrow F$ is a homomorphism from a ring R to a field F then the singular kernel of f is a prime matrix ideal of R.* ■

The following hold for a matrix pre-ideal.

(a) If $C = A \nabla B$ and B has property M1 then $C \in \mathcal{P}$ if and only if $A \in \mathcal{P}$.

For if $A \in \mathcal{P}$ then by M2 $C \in \mathcal{P}$. Conversely changing the sign of a column of B preserves property M1, so taking B to the other side of the equation the result follows.

(b) If $A \in \mathcal{P}$ then the result of adding a right multiple of one column of A to another also lies in $\mathcal{P}$.

Since if $A = (A_1, A_2, \ldots . A_n)$ where A_i is the ith column of A then $(A_1 + A_2c, A_2, A_3, \ldots A_n) = A \nabla (A_2c, A_2, A_3, \ldots A_n)$ and the second matrix on the r.h.s. is in $\mathcal{P}$ by M1. So the result follows by M2.

(c) If A and B are square and C,D are matrices of the appropriate size then the following hold.

(i) $$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{P} \overset{(*)}{\Leftrightarrow} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} A & 0 \\ D & B \end{bmatrix} \in \mathcal{P} .$$

(ii) $$\begin{bmatrix} 0 & B \\ A & C \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} D & B \\ A & 0 \end{bmatrix} \in \mathcal{P} .$$

We shall prove (i)(*). The other results follow in an entirely analogous way.

If $B = (B_1, B')$ where B_1 is the first column of B and similarly $C = (C_1, C')$ then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C_1 & C' \\ 0 & 0 & B' \end{bmatrix} \nabla \begin{bmatrix} A & 0 & C' \\ 0 & B & B' \end{bmatrix}$$

The first matrix on the R.H.S. is hollow and hence has property M1. Thus by (a) we have,

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} A & 0 & C' \\ 0 & B_1 & B' \end{bmatrix} \in \mathcal{P} .$$

In a similar way the columns of C' can be varied and the assertion proved.

(d) If A and B are square and of the same size (although not necessarily in $\mathcal{P}$) then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} 0 & -A \\ B & 0 \end{bmatrix} \in \mathcal{P}.$$

Since by column operations as in (b) we get

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} A & 0 \\ B & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow$$

$$\begin{bmatrix} A & -A \\ B & 0 \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} 0 & -A \\ B & 0 \end{bmatrix} \in \mathcal{P} .$$

Now if further $\mathcal{P}$ is a matrix ideal then we have

(e) If A and B are square and of the same size and if at least one of them lies in $\mathcal{P}$ then AB lies in $\mathcal{P}$.

Firstly let $A \in \mathcal{P}$ then by M3 $A \oplus B \in \mathcal{P}$ and so by (c) we obtain,

$$\begin{bmatrix} A & -I \\ 0 & B \end{bmatrix} \in \mathcal{P}.$$

By succesive column operations as in (b) we can get,

$$\begin{bmatrix} 0 & -I \\ BA & B \end{bmatrix} \in \mathcal{P} .$$

So by (c) $\begin{bmatrix} 0 & -I \\ BA & 0 \end{bmatrix} \in \mathcal{P}$, and by (d) this gives us $\begin{bmatrix} I & 0 \\ 0 & BA \end{bmatrix} \in \mathcal{P}$.

Now M4 implies that $BA \in \mathcal{P}$. Hence $\mathcal{P}$ admits left multiplication by square matrices. In particular $\mathcal{P}$ allows the corresponding row operations to (b). As a consequence

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} 0 & -A \\ B & 0 \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} B & 0 \\ 0 & -A \end{bmatrix} \in \mathcal{P} \Leftrightarrow \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \in \mathcal{P}.$$

By an identical argument to above we then get $AB \in \mathcal{P}$.

(f) If A belongs to a matrix ideal $\mathcal{P}$ then the result of permuting the rows or columns of A in any way belongs to $\mathcal{P}$.

For we can achieve any permutation by multiplying by an appropriate permutation matrix on the left or right.

(g) A matrix ideal $\mathcal{P}$ is proper if and only if $\mathcal{P} \neq \mathfrak{M}(R)$.
If $\mathcal{P}$ is proper then $1 \notin \mathcal{P}$ by definition. If $\mathcal{P}$ is not proper then $I \in \mathcal{P}$ and hence by (e) $A = A.I \in \mathcal{P}$ for any $A \in \mathfrak{M}(R)$.

We now proceed to establish a necessary and sufficient condition on a ring R for a proper matrix ideal to exist.

Let (P_λ) be any family of matrix ideals; then it is clear that

$P=\bigcap P_\lambda$ is again a matrix ideal. We can therefore speak of the 'least' matrix ideal containing a given subset $\mathcal{X}$ of $\mathfrak{M}(R)$. This least matrix ideal is also called the matrix ideal generated by $\mathcal{X}$. Similarly we can define the matrix pre-ideal generated by $\mathcal{X}$.

Let $\mathfrak{L}$ be the matrix pre-ideal generated by the empty set. Clearly this is the least matrix pre-ideal and it consists precisely of all column determinantal sums of square matrices which are hollow or degenerate.

We note that $\mathfrak{L}$ has an additional property apart from M1-M3. If $B\in\mathfrak{M}(R)$ and $A\in\mathfrak{L}$ then $B\oplus A\in\mathfrak{L}$. This is similar to but not equivalent to M3 (although it is in the prescence of M1, M2 and M4). In particular we shall use $I\oplus A\in\mathfrak{L}$ for all $A\in\mathfrak{L}$, where I is any unit matrix. We shall denote by ℓ the set of all unit matrices.

Now define $\mathfrak{L}/\ell=\{A\in\mathfrak{M}(R)\mid I\oplus A\in\mathfrak{L}\}$. From the above remark we see that $\mathfrak{L}/\ell\supseteq\mathfrak{L}$ and hence $\mathfrak{L}/\ell$ satisfies M1. If $A_i\in\mathfrak{L}/\ell$ $(i=1,2)$, then $I_{n_i}\oplus A_i\in\mathfrak{L}$. By the above remark $I_{n_1}\oplus I_{n_2}\oplus A_i\in\mathfrak{L}$ and if $A_1\nabla A_2$ is defined then,

$$(I_{n_1}\oplus I_{n_2}\oplus A_1)\ \nabla\ (I_{n_1}\oplus I_{n_2}\oplus A_2)=I_{n_1}\oplus I_{n_2}\oplus(A_1\nabla A_2)\ .$$

The L.H.S. lies in $\mathfrak{L}$ hence $A_1\nabla A_2\in\mathfrak{L}/\ell$ and so we have M2.

If B is square over R and $A\in\mathfrak{L}/\ell$ then $I\oplus A\in\mathfrak{L}$ so $I\oplus A\oplus B\in\mathfrak{L}$ and hence $A\oplus B\in\mathfrak{L}/\ell$. Therefore M3 holds. Also if $1\oplus A\in\mathfrak{L}/\ell$ then $I_n\oplus 1\oplus A\in\mathfrak{L}$. But $I_n\oplus 1=I_{n+1}$ so $A\in\mathfrak{L}/\ell$. Consequently M4 holds and we have shown that $\mathfrak{L}/\ell$ is a matrix ideal. If $\mathcal{P}$ is a matrix ideal containing $\mathfrak{L}$ then;

$$A\in\mathfrak{L}/\ell \Rightarrow I\oplus A\in\mathfrak{L} \Rightarrow I\oplus A\in\mathcal{P} \Rightarrow A\in\mathcal{P} \Rightarrow \mathcal{P}\supseteq\mathfrak{L}/\ell\ .$$

Hence $\mathfrak{L}/\ell$ is the least matrix ideal containing $\mathfrak{L}$. Now $\mathfrak{L}/\ell$ is proper if and only if $\mathfrak{L}\cap\ell=\emptyset$. When $\mathcal{P}$ is a matrix ideal and Σ is either a singleton set or a set of square matrices closed under diagonal sums and containing 1 then in a similar fashion we can show that $\mathcal{P}/\Sigma=\{B\in\mathfrak{M}(R)\,|\,B\oplus A\in\mathcal{P}, A\in\Sigma\}$ is also a matrix ideal containing $\mathcal{P}$. Summarizing these results we see,

<u>Proposition</u> <u>5.2</u> *$\mathfrak{L}/\ell$ is the least matrix ideal containing $\mathfrak{L}$ and $\mathfrak{L}/\ell$ is proper if and only if $\mathfrak{L}\cap\ell=\emptyset$.* ∎

Since every matrix ideal contains $\mathfrak{L}$ we see $\mathfrak{L}/\ell$ is the least matrix ideal (written as $\mathcal{S}_0$) and we have,

<u>Proposition</u> <u>5.3</u> *Let R be any ring. Then R has a proper matrix ideal if and only if no unit matrix can be written as a column determinantal sum of matrices which are hollow or degenerate.* ∎

It is now possible to consider when a ring R has a prime matrix ideal. The following is taken directly from Cohn [**1**].

<u>Definition.</u> Given two matrix ideals $\mathcal{P}_1,\mathcal{P}_2$ in a ring R, their **product**, denoted by $\mathcal{P}_1\mathcal{P}_2$, is defined as the matrix ideal generated by all $A_1\oplus A_2$ with $A_i\in\mathcal{P}_i$ (i=1,2).

The product so defined is easily seen to be associative and from property (f) it follows that the product is commutative.

<u>Lemma</u> <u>5.4</u> *In any ring R, let $\mathcal{X}_i\subseteq\mathfrak{M}(R)$ (i=1,2), $\mathcal{X}$ the set of matrices $A_1\oplus A_2$ ($A_i\in\mathcal{X}_i$) and $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}$ the matrix ideals generated by $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}$, respectively. Then $\mathcal{P}=\mathcal{P}_1\mathcal{P}_2$.*

<u>Proof</u>. Clearly $\mathcal{X}\subseteq\mathcal{P}_1\mathcal{P}_2$, hence $\mathcal{P}\subseteq\mathcal{P}_1\mathcal{P}_2$. To establish equality, let $A_i\in\mathcal{X}_i$, then $A_1\oplus A_2\in\mathcal{P}$ by definition, hence $\mathcal{X}_1\subseteq\mathcal{P}/A_2$ and so $\mathcal{P}_1\subseteq\mathcal{P}/A_2$. It

follows that $B_1 \oplus A_2 \in \mathcal{P}$ for all $B_1 \in \mathcal{P}_1$, so fixing B_1 we have $\mathcal{X}_2 \subseteq \mathcal{P}/B_1$, hence $\mathcal{P}_2 \subseteq \mathcal{P}/B_1$ and so $B_1 \oplus B_2 \in \mathcal{P}$ for all $B_1 \in \mathcal{P}_1$, and it follows that $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P}$. ∎

<u>Proposition</u> <u>5.5</u> *for any matrix ideal $\mathcal{P}$ in a ring R the following three properties are equivalent.*

(a) $\mathcal{P}$ is prime,

(b) $\mathcal{P}$ is proper and for any matrix ideals $\mathcal{P}_1, \mathcal{P}_2$ we have $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P} \Rightarrow \mathcal{P}_1 \subseteq \mathcal{P}$ or $\mathcal{P}_2 \subseteq \mathcal{P}$,

(c) $\mathcal{P}$ is proper and for any matrix ideals $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P}_i \supseteq \mathcal{P}$ we have $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P} \Rightarrow \mathcal{P}_1 = \mathcal{P}$ or $\mathcal{P}_2 = \mathcal{P}$.

<u>Proof</u>. (a)⇒(b). Let $\mathcal{P}$ be prime and $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P}$ but $\mathcal{P}_i \not\subseteq \mathcal{P}$ (i=1,2). Then there exist $A_i \in \mathcal{P}_i$ but $A_i \notin \mathcal{P}$. Since $\mathcal{P}$ is prime, $A_1 \oplus A_2 \notin \mathcal{P}$, but $A_1 \oplus A_2 \in \mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P}$, a contradiction. Clearly $\mathcal{P}$ is also proper.

(b)⇒(c) is clear; to prove (c)⇒(a), suppose that $A_1 \oplus A_2 \in \mathcal{P}$. Consider the matrix ideal $\mathcal{P}_i$ generated by $\mathcal{P}$ and A_i; for any $B_i \in \mathcal{P} \cup \{A_i\}$, $B_1 \oplus B_2 \in \mathcal{P}$, hence $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P}$. By hypothesis, $\mathcal{P}_1$ or $\mathcal{P}_2$ must equal $\mathcal{P}$, so A_1 or A_2 lies in $\mathcal{P}$, i.e. (a). ∎

The usual method of constructing prime ideals also works for prime matrix ideals:

<u>Theorem</u> <u>5.6</u> *Let R be any ring, Σ a non-empty subset of $\mathfrak{M}(R)$ closed under diagonal sums and $\mathcal{A}$ any matrix ideal such that $\mathcal{A} \cap \Sigma = \emptyset$. Then there exists a matrix ideal $\mathcal{P}$ which is maximal subject to the conditions $\mathcal{P} \supseteq \mathcal{A}$, $\mathcal{P} \cap \Sigma = \emptyset$, and any such matrix ideal is prime.*

<u>Proof</u>. The collection $\mathcal{C}$ of all matrix ideals containing $\mathcal{A}$ and disjoint from Σ is clearly inductive, so by Zorn's lemma it has a

maximal member $\mathcal{P}$, and this satisfies the conditions of the theorem. Any such $\mathcal{P}$ is proper, because Σ is non-empty; now let $\mathcal{P}_i \supseteq \mathcal{P}$ be matrix ideals such that $\mathcal{P}_1\mathcal{P}_2 \subseteq \mathcal{P}$. If $\mathcal{P}_i \neq \mathcal{P}$ (i=1,2), then $\mathcal{P}_i \supset \mathcal{P}$, so by maximality of $\mathcal{P}$, $\mathcal{P}_i \cap \Sigma \neq \emptyset$. Take $A_i \in \mathcal{P}_i \cap \Sigma$, then $A_1 \oplus A_2 \in \mathcal{P} \cap \Sigma$, which is a contradiction. Hence $\mathcal{P}_1$ or $\mathcal{P}_2$ equals $\mathcal{P}$, and so $\mathcal{P}$ is prime by the previous proposition. ∎

This theorem shows for example that every maximal proper matrix ideal is prime; we need only take $\Sigma = \ell$.

Corollary. *Any ring R has a prime matrix ideal if and only if R has a proper matrix ideal.* ∎

Hence by Proposition 5.3 we have;

Theorem 5.7 *Any ring R has a prime matrix ideal if and only if no unit matrix can be written as a column determinantal sum of matrices which are either hollow or degenerate.* ∎

Localisation at a Prime Pair.

We can now proceed to show that when a ring R does indeed have a prime matrix ideal then there exists a homomorphism from R to a field. We shall first consider the concept of localisation at a prime pair introduced by Malcolmson [**4**]. If Σ is a lower multiplicative set of matrices over a ring R and $\mathcal{S}$ is a matrix ideal of R then a **pair** $Q=(\Sigma,\mathcal{S})$ is said to be a **prime pair** if $A \oplus B \in \mathcal{S}$ and $A \in \Sigma$ implies that $B \in \mathcal{S}$. The idea is to form a localisation where the image of Σ becomes invertible while somehow setting the elements of $\mathcal{S}$ to become 'singular'. If $(\Sigma,\mathcal{S})$ is a pair Malcolmson showed that $(\Sigma,\mathcal{S}/\Sigma)$ is in fact prime and $\mathcal{S}/\Sigma$ is proper iff $\mathcal{S} \cap \Sigma = \emptyset$.

Given a prime pair $Q=(\Sigma,\mathscr{P})$ we can form the well defined **associated subset of** M_Σ as $\Omega=\{(A_o A_* A_\infty)\in M_\Sigma \mid (A_o A_*)\in\mathscr{P}\}$. Below we show that Ω is a Σ-matrix ideal and then we call the homomorphism $\lambda_\Omega : R \longrightarrow R_\Omega$ the **localisation at** Ω.

Proposition 5.8 *If $Q=(\Sigma,\mathscr{P})$ is a prime pair then the associated subset of M_Σ, Ω, is a Σ-matrix ideal, and the localisation at Ω is Σ-inverting.*

Proof. Since $\mathscr{P}$ is a matrix ideal it contains all hollow matrices and so $\Omega \supseteq \mathscr{L}$. If $A,B\in\Omega$ then the numerator of $A\square B$ can be written;

$$\begin{bmatrix} B_o & B_* & B_\infty & 0 \\ 0 & 0 & -A_\infty & A_* \end{bmatrix} \quad \nabla \quad \begin{bmatrix} 0 & B_* & B_\infty & 0 \\ A_o & 0 & -A_\infty & A_* \end{bmatrix}$$

and the matrix on the left is in $\mathscr{P}$ since $(B_o B_*)\in\mathscr{P}$ and $\mathscr{P}$ is a matrix ideal. After some row and column permutations the matrix on the right is similarly seen to lie in $\mathscr{P}$ since $(A_o A_*)\in\mathscr{P}$. Hence since $\mathscr{P}$ is closed w.r.t. determinantal sums we see that $A\square B\in\Omega$ and Ω is an additive subsemigroup of M_Σ. We now show that Ω is unitary. Assume that $A\square B, B\in\Omega$. Then the numerator of $A\square B$,

$$\begin{bmatrix} B_o & B_* & B_\infty & 0 \\ A_o & 0 & -A_\infty & A_* \end{bmatrix} \text{ lies in } \mathscr{P}.$$

If $(B_o B_*)\in\mathscr{P}$ then by the properties of a matrix ideal,

$$\begin{bmatrix} -B_o & B_* & B_\infty & 0 \\ 0 & 0 & -A_\infty & A_* \end{bmatrix} \in \mathscr{P}.$$

The determinantal sum of the above two matrices is defined w.r.t. the first column and hence,

$$\begin{bmatrix} 0 & B_* & B_\infty & 0 \\ A_o & 0 & -A_\infty & A_* \end{bmatrix} \in \mathcal{S} \Rightarrow \begin{bmatrix} B_* & B_\infty & 0 & 0 \\ 0 & 0 & A_o & A_* \end{bmatrix} \in \mathcal{S}.$$

Since Q is a prime pair we get $(A_oA_*)\in\mathcal{S}$ and $A\in\mathfrak{Q}$. So $\mathfrak{Q}$ is unitary and it just remains to show that $\mathfrak{Q}$ is an ideal w.r.t. the multiplicative semigroup structure of M_Σ. If $A\in\mathfrak{Q}$ and $B\in M_\Sigma$ then the numerator of both A.B and B.A is immediately seen to lie in $\mathcal{S}$ by the properties of a matrix ideal. Consequently by Theorem 2.6 $\lambda_\mathfrak{Q}:R\longrightarrow R_\mathfrak{Q}$ is Σ-inverting. ∎

Malcolmson used this result to get a bound on the kernel of the universal Σ-inverting homomorphism by taking a 'minimal' prime pair $(\Sigma,\mathcal{S}_o/\Sigma)$ where $\mathcal{S}_o$ is the least matrix ideal. This is no longer necessary as we have a complete description of the kernel.

When a ring R has a prime matrix ideal $\mathcal{P}$ then the complement of $\mathcal{P}$, $\backslash\mathcal{P}$, is easily seen to be a lower multiplicative set of square matrices over R and $(\backslash\mathcal{P},\mathcal{P})$ is a prime pair. With the present notation the localisation at this prime pair is written $\lambda_\mathfrak{P}:R\longrightarrow R_\mathfrak{P}$.

<u>Theorem</u> <u>5.9</u> (Cohn) *If R is any ring with a prime matrix ideal $\mathcal{P}$ then the localisation at $\mathfrak{P}$ (the associated $\backslash\mathcal{P}$-matrix ideal) is a homomorphism from R to a skew field whose singular kernel is precisely $\mathcal{P}$.*

<u>Proof</u> $\mathfrak{P}$ is the associated $\backslash\mathcal{P}$-matrix ideal and consists of all those admissible matrices with numerators lying in $\mathcal{P}$ so $(A_oA_*A_\infty)\notin\mathfrak{P}$ implies that $(A_oA_*)\notin\mathcal{P}$ and $(A_\infty A_*A_o)$ is an admissible matrix. We observe the following;

$$(A_\infty\ A_*\ A_o).(A_o\ A_*\ A_\infty) = \begin{bmatrix} A_o & A_* & A_\infty & 0 & 0 \\ 0 & 0 & -A_\infty & A_* & A_o \end{bmatrix} \to$$

$$\begin{bmatrix} A_o & A_* & A_\infty & 0 & 0 \\ 0 & -A_*-A_\infty & & A_* & A_o \end{bmatrix} \to \begin{bmatrix} A_o & A_* & A_\infty & 0 & 0 \\ 0 & 0 & -A_\infty & A_* & A_o \end{bmatrix} \sim (A_o\ A_*\ A_o) \sim 1$$

Similarly we can get $(A_o\ \ A_*\ \ A_\infty).(A_\infty\ \ A_*\ \ A_o)\sim 1$, and hence all non-zero elements in $R_{\mathfrak{p}}$ have multiplicative inverses and $R_{\mathfrak{p}}$ is seen to be a field.

We already have that $\mathrm{Ker}\lambda_{\mathfrak{p}} \subseteq \mathcal{P}$ since $\lambda_{\mathfrak{p}}$ inverts the complement of $\mathcal{P}$. Now we proceed to show that $\mathcal{P} \subseteq \mathrm{Ker}\lambda_{\mathfrak{p}}$. Let $P \in \mathcal{P}$ and denote by $P^{(i)}$ the matrix obtained from P by omitting the first column and the ith row. If $P^{(i)} \in \mathcal{P}$ for all i, then by induction and the use of column determinantal sums w.r.t. the initial column we get that $P \in \mathrm{Ker}\lambda_{\mathfrak{p}}$. So w.l.o.g. we can assume that $P^{(1)} \notin \mathcal{P}$. Then since $\backslash\mathcal{P}$ is lower multiplicative and admits row permutations we have that (Pe_1) is admissible, and moreover it is admissible for zero since the numerator lies in $\mathcal{P}$. Hence from Theorem 2.6 there is an equation in $R_{\mathfrak{p}}$; $P_o=(P_* e_1)(u_* 0)^T$ i.e P is a zero divisor in $R_{\mathfrak{p}}$ and so as required we have $P \in \mathrm{Ker}\lambda_{\mathfrak{p}}$. ∎

(Cohn actually showed that $R_{\mathcal{P}}$ the universal $\backslash\mathcal{P}$-inverting ring is local with residue class field $R_{\mathfrak{p}}$.)

<u>Corollary</u> *For any ring R there exists a homomorphism from R to a field if and only if no unit matrix can be written as a column determinantal sum of hollow or degenerate matrices.* ∎

Chapter 6 Fields of Fractions

With the results of the previous chapter we are now in a position to determine a criterion for any ring to have a field of fractions. This is a refinement of Cohn's criterion (a reference is given after the proof). For completeness we then go on to prove Dicks and Sontag's result classifying all rings having a universal field of fractions inverting all full matrices over that ring.

Theorem 6.1 *Any non-zero ring R has a field of fractions if and only if no diagonal matrix over R with non-zero diagonal entries can be expressed as the column determinantal sum of matrices that are either hollow or degenerate.*

Proof If R has a field of fractions then R is embeddable in a field over which every diagonal matrix with non-zero diagonal entries is invertible. However over a field the column determinantal sum of non-full matrices is singular, so we have proved the necessity of the condition.

To prove sufficiency we note that Σ, the set of diagonal matrices over R with non-zero diagonal entries is closed w.r.t. diagonal sums and non-empty. Then the condition states that Σ is disjoint from the least matrix pre-ideal. From the description of the least matrix ideal in Proposition 5.2 we can see that Σ is also disjoint from the least matrix ideal. Hence by applying Theorem 5.6 we have a prime matrix ideal $\mathcal{P}$ disjoint from Σ which by Theorem 5.9 gives rise to a homomorphism from R to a field F with singular kernel precisely $\mathcal{P}$ i.e. Σ is inverted which is sufficient to prove that we have an embedding in a field of fractions. ∎

(*cf.* Cohn [**1**], Th. 7.5.6.). We note that this condition ensures that R is an integral domain for if ab=0, then

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & b \end{bmatrix} \nabla \begin{bmatrix} 0 & 0 \\ -1 & b \end{bmatrix}$$

where the second matrix on the right is clearly hollow, and the first matrix on the right is degenerate since the first column multiplied by b on the right is just the second column.

If there is an homomorphism from a ring R to a field then no non-full matrix over R can be inverted since a non-full matrix over a field is singular. Hence the most that can be hoped for is that all full matrices over R be inverted and this is only possible if the singular kernel of the homomorphism consists precisely of the non-full matrices over R. In this case we must have an embedding since any non-zero 1x1 matrix is full and so maps to an invertible element. A ring homomorphism which keeps all full matrices full is called **honest** so by the above we have that there is an honest homomorphism from a ring to a field if and only if there is a fully inverting homomorphism from the ring to a field. Let $\Phi=\Phi(R)$ denote the set of all full matrices over R, if this set is lower multiplicative then we can consider R_Φ the universal localisation at Φ.

Proposition 6.2 (Cohn) *Let R be any ring. If Φ, the set of full matrices over R, is lower multiplicative and $R_\Phi \neq 0$ then R_Φ is a field.*

Proof Let p be a non-unit in R_Φ with an admissible matrix

$A=(A_0A_*A_\infty)$. By Cramer's rule the image of the numerator over R_Φ is stably associated to p and hence must be non-full over R or else p is invertible. So $(A_0A_*)=PQ$ where $P\in{}^nR^{n-1}$ and $Q\in{}^{n-1}R^n$ and then we can write A as follows;

$$A = (A_0A_*A_\infty) = (PQ\ A_\infty) = (P\ A_\infty)\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $(P\ A_\infty)$ must be full for it is a left factor of the denominator of A which is full by definition and so the image of $(P\ A_\infty)$ is invertible over R_Φ. Since A is an admissible matrix for p then we have an equation Au=0 over R_Φ where the last component of u is p, but from above we can cancel the left factor $(P\ A_\infty)$ in the equation to deduce that p=0 as required. ∎

We are now in a position to determine exactly when R_Φ is a field.

Theorem 6.3 (Cohn) *If R is any ring then the two following conditions are equivalent:*

(i) The set of non-full matrices over R is a prime matrix ideal.

(ii) R has a universal field of fractions inverting all full matrices over R.

Proof If (i) holds then the set Φ of full matrices is lower multiplicative and so by Proposition 6.2 R_Φ is either a field or the zero ring. If $R_\Phi=0$ then the admissible matrix (1 1) represents zero. However from Theorem 5.7 we know that the least Φ-matrix ideal has numerators which are non-full because (i) ensures that those admissible matrices with full denominators and non-full numerators form a Φ-matrix ideal which necessarily contains the least Φ-matrix ideal. Hence if $R_{\Phi}=0$ then 1 is non-full; a contradiction, so R_{Φ} is a field.

Conversely if R is embeddable in such a field then we have a fully inverting homomorphism from R to a field. Hence the singular kernel is a prime matrix ideal and from an earlier comment we know that the singular kernel must consist precisely of the non-full matrices over R. ■

We are now in a position to classify all rings with a universal field of fractions inverting all full matrices but first we shall have to state (without proof) a couple of results from Cohn [1] Chapter 5.

<u>Theorem</u> <u>6.4</u> *Every honest homomorphism of a ring preserves the inner rank.* ■

Let R be a ring. If PQ=0 where $P\in {}^{r}R^{n}$, $Q\in {}^{n}R^{s}$ and r+s>n implies that,

$$\rho(P)+\rho(Q) \leq n$$

then R is said to be a **Sylvester domain**.

<u>Lemma</u> <u>6.5</u> *Let R be a Sylvester domain. Then*

(i) if A,B are full matrices over R then $A\oplus B$ is also full.

(ii) if A,B,C are any matrices over R with the same number of rows and if $\rho(A,B)=\rho(A,C)=\rho(A)$ and A is right full, then

$$\rho(A\ B\ C\ ) = \rho(A)\ . \qquad ■$$

With these two results we can prove the main theorem of this chapter.

Theorem 6.6 (Dicks and Sontag [6]) *A ring R has a universal field of fractions inverting all full matrices over R if and only if R is a Sylvester domain.*

Proof If R_Φ is a field then $\lambda: R \longrightarrow R_\Phi$ is an honest homomorphism from R to a field, hence by Theorem 6.4 we have an inner rank preserving homomorphism from R to a field. Any field necessarily satisfies Sylvester's law of nullity and so is seen to be a Sylvester domain. Therefore R must also be a Sylvester domain as the inner rank is preserved by the homomorphism.

Conversely by Theorem 6.3 it suffices to show that the set of non-full matrices over a Sylvester domain form a prime matrix ideal. Using the basic properties of non-full matrices this amounts to showing that the set of non-full matrices is closed under column determinantal sums and that the set of full matrices is closed under diagonal sums. Lemma 6.5 (i) shows that the set of full matrices over a Sylvester domain is closed w.r.t. diagonal sums. To prove that the set of non-full matrices is closed w.r.t. column determinantal sums suppose that $A=(A_*,a)$, $B=(A_*,b)$, $C=(A_*,a+b)$ where $A_* \in {}^nR^{n-1}$; a,b are columns over R and A,B are not full. If A_* is right full then $n-1= \rho(A_*)= \rho(A_*,a)= \rho(A_*,b)$, so by Lemma 6.5 (ii) $n-1= \rho(A_*,a,b)= \rho(A_*,a+b,b) \geq \rho(A_*,a+b)= \rho(C)$. If A_* is not right full then $\rho(A_*)<n-1$ and then $\rho(C)<n$. In either case C is non-full as required. ∎

In particular, since a semifir is a Sylvester domain we have the following corollary.

Corollary 6.7 (Cohn) *Any semifir R has a universal field of fractions K and any full matrix over R is invertible over K.* ∎

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