Rectifiability Results for $l^3_\infty$

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Abstract

As a first step to generalising Rectifiability and Density Results, Radon measures with density properties with respect to the cube are studied. For reasons of isometric immersion of metric spaces into $l_\infty$ and the extremal nature of $l_\infty^3$ among finite dimensional normed vector spaces, the question of rectifiability of such measures is the simplest unknown case of any generalisation.

It is proved that locally 2-uniform measures in $l_\infty^3$ have rectifiable subsets in all neighbourhoods of all points of their support. By a well known theorem on tangent measures an immediate Corollary to this is that measures with positive finite 2-density almost everywhere have weak tangents.
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Introduction

This thesis is the study of a model problem for generalising Rectifiability and Density results to Non-Euclidean spaces. Given the involved nature of the subject I will start with an informal introduction to Rectifiability and Densities in Euclidean spaces.

I would like to make completely clear that it is not my intention to give a rigorous development of this subject, my intention is to explain and hopefully arouse interest in the results of this field. The readers I am primarily addressing this to, are students or others thinking about going in this direction. A more formal and perhaps better account can be found in [1]. What follows is a personal rendition written at an absolute minimum level of formality and rigor.

Hausdorff Measure

Hausdorff measure is the most natural way to measure lower dimensional objects in higher dimensional space.

Let \( \alpha (m) = L^m (B_1 (0)) \) so \( \alpha (1) = 2, \alpha (2) = \pi. \)

Given a set \( S \subset \mathbb{R}^n \) we define Hausdorff \( m \)-measure in two stages as follows:

Given \( \delta > 0 \) define

\[
H^m_\delta (S) = \inf \left\{ \alpha (m) \sum_{k=1}^{\infty} \left( \frac{d(C_k)}{2} \right)^m : \right. \\
\left. S \subset \bigcup_{k=1}^{\infty} C_k, \text{ where } C_k \text{ are convex sets in } \mathbb{R}^n \text{ with the property that } d (C_k) < \delta \right\}
\]

Hausdorff \( m \)-measure, \( H^m \) is defined as

\[
H^m (S) = \lim_{\delta \to 0} H^m_\delta (S)
\]
Here are some examples of Hausdorff 1-measure in action:

The diagrams show why we force the diameter of the covering sets to go down to zero, in the definition of Hausdorff measure.

As we can see Hausdorff 1-measure in these nice examples, is exactly length. Similarly Hausdorff 2-measure is exactly surface area, Hausdorff 3-measure is volume, etc.
In the 20's Besicovitch became interested in sets of positive finite Hausdorff 1-measure in the plane (what I will from now on refer to as 1-sets). He found it was possible to construct examples of strange sets, sets with the following properties:

**Bad Sets**

* Measure zero intersection with all Lipshitz graphs

* Cone density at almost all points from almost all directions

* No shadow in almost all directions
Writing things out slightly more formally:

- Measure zero intersection with Lipschitz graphs means for every one dimensional subspace $V$ in the plane and every Lipschitz function $f : V \to V^\perp$ the Hausdorff 1-measure of the intersection of our set with the graph of $f$ is zero.

- Cone density at almost all points from almost all directions means for $H^1$ almost all point $x$ in our set for almost all directions $v \in S^1$ if we take a cone of arbitrary thinness in direction $v$, centered on $x$, then for some sequence of radii $r_n \to 0$ the Hausdorff measure of our set inside the cone intersected with a ball radius $r_n$ will be bigger than some small factor of $r_n$. Geometrically speaking the set approaches almost all its points from almost all directions.

- No shadow in almost all directions means that if we take almost any direction $v \in S^1$ and project our set down onto the subspace $v^\perp$, the Lebesgue measure of this set will be zero. So despite this set having positive 'length' it manages to be so dispersed that if we shine a light through it from almost any direction all the light will pass through the set and so the set casts no shadow.

Sets such as these can arise quite easily, the easiest example I know is as follows:

Consider the numbers in $[0,1)$ written out in decimal expansion of base 4, so for every $x \in [0,1)$ we have an infinite sequence of coefficients $(c_n)$ where $c_n \in \{0,1,2,3\}$ such that

$$x = \sum_{n=1}^{\infty} c_n \frac{1}{4^n}$$

the base 4 decimal expansion is the sequence $(c_n)$.

Now for any $x \in [0,1)$ define $f(x)$ to be the number whose base 4 decimal expansion is given by the sequence $([5 - c_n]_4)$. 


Now it is easy to see that the graph of \( f \) is contained in the boxes on the diagram below and inside each box the same pattern is repeated.

So we know that the graph of \( f \) has positive Hausdorff measure because its projection onto the \( y \) axis has Lebesgue measure 1. We also know its Hausdorff measure is bounded by \( \sqrt{2} \) because for every \( k \in \mathbb{N} \) the graph is contained in \( 4^k \) boxes of side length \( \frac{1}{4^k} \).

The graph of this function is an example of a ‘bad’ set. For example inspection of the diagram shows that the projections onto the \( y = x \) or \( y = -x \) diagonals will have zero Lebesgue measure, more careful inspection shows it has all the bad properties listed.
Good Sets

First let us consider what we want from a 'Good' $1$-set in the plane

- We want tangents. We want to know that the good set is locally well approximated by a line.
- Because we are measure theorists we don’t care about having tangents strictly everywhere we only care about having them $H^1$ almost everywhere.

The Answer

Lipschitz Graphs

By Rademacher’s theorem they are differentiable almost everywhere, and as having a differential is exactly having a tangent so Lipschitz graphs have tangents $H^1$ almost everywhere. In fact finite or even countable collections of Lipschitz graphs have just as good local properties as individual ones.

Definition

Rectifiable $1$-sets

Rectifiable $1$-sets are sets which up to a set of zero $H^1$ measure, can be contained in countable many Lipschitz graphs.

This definition seems at first slightly unnatural, what should be remembered is that all we care about is measure theoretically having good local geometric properties.

Purely unrectifiable $1$-sets

Purely unrectifiable sets are sets whose intersection with any Lipschitz graph has zero $H^1$ measure.

The first strange property of our 'bad' sets suffices for a definition.
I want to state what I mean by sets having measure theoretically good local properties.

A Rectifiable 1-set $S$, has measure theoretic tangents almost everywhere in the sense that for $H^1$ almost all $x \in S$ there exists a line $V$ in $\mathbb{R}^2$ such that for all $\epsilon > 0$ we have the following two properties

- Firstly we require that
  
  $$\lim_{r \to 0} \frac{H^1(B_r(x) \cap S \setminus N_\epsilon(V))}{2r} = 0.$$ 

- Also there exists $\lambda > 0$ such that for all small enough $r > 0$
  
  $$H^1(B_\epsilon(z) \cap S) > \lambda \epsilon r$$
  
  for all $z \in V \cap B_r(x)$.

So the first condition says the set is measure theoretically close to the line $V$, the second that it in some sense fills out the line $V$. The double condition makes things seem slightly complicated at first, however if the reader were to try and come up with a formal definition of 'measure theoretic tangent' for general 1-sets, my guess is that the definition would turn out to be almost exactly like the one given here.

Conversely any set that has measure theoretic tangents almost everywhere is a rectifiable set. So from the perspective of local geometric properties rectifiable sets are the natural concept.
Much of the utility of the theory of rectifiability comes from the characterization of sets of positive finite Hausdorff measure.

All 1-sets of positive finite Hausdorff measure can be decomposed uniquely up to set of zero $H^1$ measure into rectifiable sets and purely unrectifiable sets.

The same is true for sets of positive finite $H^m$ measure in $\mathbb{R}^n$, $m$-rectifiable sets in $\mathbb{R}^n$ are sets which are up to a set of zero $H^m$ measure are contained in countably many graphs of Lipschitz functions $f : V \to V^\perp$ where $V$ is some $m$-plane in $\mathbb{R}^n$.

Equivalently $m$-rectifiable sets are sets which are up to a set of zero $H^m$ measure contained in countably many $m$-dimensional $C^1$ submanifolds, this may seem a more natural definition.

In some sense rectifiable $m$-sets are the largest class of lower dimensional 'm-surfaces' in higher dimensional Euclidean space.

For this reason they have many applications.

**Applications**

- Higher dimensional minimal surface theory relies heavily on the theory of rectifiability, for reasons of compactness of a particular class of objects defined on rectifiable sets.

- Various free boundary variational problems, the most famous of which is the Mumford Shah functional, are studied with the theory of BV and SBV functions, these are functions whose distributional derivative can be represented as a measure, and for SBV functions the part of this measure that is singular with respect to Lebesgue measure, is supported on a rectifiable set, and this is the set that forms the 'free boundary' in applications.

- Recently the old problem of analytic capacity has been solved for sets of finite $H^1$ measure, a compact set $E \subset \mathbb{C}$ has analytic capacity zero (which means bounded analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ have analytic extension) if and only if it is purely unrectifiable.
Theorems that let you know if you have a rectifiable set

The most famous theorem of this type is Besicovitch-Federer's 1949 projection theorem which says:

**Theorem 1** The projection of a purely unrectifiable $m$-set onto almost all $m$-dimensional subspaces has zero $L^m$ measure.

This theorem was proved fifty years ago and partly for this reason it is the most commonly used.

In practice if you have an unknown $m$-set and you assume the worst, that it is purely unrectifiable, then you know from this theorem it has the much stranger property of casting no shadow in almost all directions. It is much easier to get a contradiction from this property than the original definition of vanishing on all Lipschitz graphs.

**Rectifiability and Density Theorems**

The early results are by Besicovitch for 1-sets in the plane.

**Theorem 2** If $S$ is a 1-set with the property that for $H^1$ a.a. points $x\in S$

$$\lim_{r\to 0} \frac{H^1(B_r(x) \cap S)}{2r} = 1$$

then $S$ rectifiable.

This theorem is so easy to state that its significance can easily be missed. I will try to illustrate what it says.

Suppose we had a rectifiable set $S$ to start with, then, as I have stated, it has measure theoretic tangents almost everywhere. So for almost any $x \in S$ if we look at $S$ inside some small enough ball centered on $x$, we see the set forms something like a line going through $x$ and hence the Hausdorff measure of $S$ inside this ball will be approximately the diameter of the ball.

What Besicovitch’s theorem says is the converse. If we have an arbitrary 1-set with the property that almost always when we examine the set inside small balls (despite possibly seeing no geometry at all) we ascertain the ‘size’ of the set inside the ball is approximately its diameter, then the set is rectifiable and so, as we shrink the ball, the set inside must form a line going straight through.
This is the first instance of

**Metric knowledge of size in small balls ⇒ Geometry!**

Attractive though this result is, its proof disappoints. The starting point is a relatively easy trick establishing the existence of an extremely strong form of symmetry I call antipodal symmetry.

A set has antipodal symmetry if whenever we have two points $x$ and $y$ in the set we also have a point $x'$ of the set within some $\epsilon |x - y|$ neighborhood of the point $2y - x$, which is the antipodal point to $x$ on the sphere radius $|x - y|$ centered on $y$.

The trick to establish antipodal symmetry *entirely* depends on the fact that the density is equal to 1 and a particular spherical geometric property of the unit ball in Euclidean space. The proof gives the impression that there is no deep reason for this result, it appears to be just a trick.

For numerous reasons rectifiability in one dimension is very much easier than in higher dimension. In a major advance this theorem was generalised in 1961 by Marstrand [2], who proved

**Theorem 3** If $S$ is a 2-set in $\mathbb{R}^3$ with the property that for a. a. points $x \in S$

$$
\lim_{r \to 0} \frac{H^2(B_r(x) \cap S)}{\pi r^2} = 1
$$

then $S$ rectifiable.

This was further generalised to $m$-sets in $\mathbb{R}^n$ by Mattila in 1975, see [4].

Although there is massively more combinatorial subtlety in proving these results, the starting point is again the antipodal symmetry obtained via the same trick.

In 1964 Marstrand [3] found a completely new method. He realised knowledge of the size in small balls meant knowledge of the size in small annuli which meant knowledge of integrals of the form

$$
\int_{B_r(x) \cap S} \phi(|z - x|) \, dH^m z
$$

for any $x \in S$, for sufficiently small $r$. 
We will assume $0 \in S$. If we took $r$ very small so that we had very good knowledge and some smooth function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which decays outside $(0, r)$, and took points $x \in S$ very close to $0$ then we would have

$$\int_{B_r(x) \cap S} \phi(|z - x|) \, dH^m z - \int_{B_r(0) \cap S} \phi(|z|) \, dH^m z \approx 0$$

Since $\phi$ decays outside $r$ and the non-overlapping regions are contained in some thin annulus so the entire action of this cancellation must go on inside the overlap.

In the diagram $z$ represents an arbitrary point of $S \cap B_r(0) \cap B_r(x)$ and for this particular point $\phi(|z - x|) > \phi(|z|)$. As we integrate over $S \cap B_r(0) \cap B_r(x)$ the structure of the set must be such that $\phi(|z - x|) - \phi(|z|)$ ends up wiping itself out, so already we see that the set $S$ is forced to obey some sort of geometric restriction.
So let us take the simplest radial functional imaginable

$$\phi(p) = \begin{cases} 
    r^2 - p^2 & \text{for } p \in (0, r) \\
    0 & \text{for } p \in [r, \infty)
\end{cases}$$

Inside $B_r(0) \cap B_r(x)$ we have

$$\phi(|z - x|) - \phi(|z|) = |z - x|^2 - |z|^2$$

$$= |z|^2 - 2z \cdot x + |x|^2 - |z|^2$$

$$= -2z \cdot x + |x|^2$$

Making $x$ be as close to 0 as we like, the $|x|^2$ term becomes negligible, and, as $\phi$ is close to zero outside the overlap, we don’t care what happens there, so effectively we have

$$\int_{B_r(0) \cap S} z \cdot x \quad H^m z \approx o \left( |x|^2 \right)$$

As $|x|$ can be as small as we like we can forget about the error term and pretend we have

$$\int_{B_r(0) \cap S} z \cdot x \quad H^m z = 0 \quad !!$$

If $B_r(0) \cap S$ had the antipodal symmetry I talked about before, this would clearly be satisfied. Instead this is a kind of integral averaged antipodal symmetry.

Note: We made no assumption on what the density had to be and we get back from a completely different source almost exactly the same kind of symmetry we had to start with!

For trivial reasons we don’t even have to have the same density for different points, because we could just take a density point of the subset of $S$ which has density approximately $\alpha$. 
Using this, Marstrand was able to prove:

**Theorem 4** Given an $m$-set $S \subset \mathbb{R}^n$ with the property that

\[ 0 < \lim_{r \to 0} \frac{H^m(B_r(x) \cap S)}{r^m} < \infty \]

for $H^m$ a.a. $x \in S$ then for a.a. $y \in S$ we have the following property:

For every $\epsilon > 0$ we can find an $m$-plane $V$ such that

\[ \lim_{k \to \infty} \frac{H^m(B_{r_k}(y) \cap S \setminus N_{r_{2r_k}}(V))}{r_k^m} = 0 \]

for some sequence $r_k \to 0$.

To illustrate this, suppose you had an unknown 2-set with density positive and finite almost everywhere and a 3-dimensional microscope that allows you to zoom into the points of your set.

If you were zooming into some point of your set and you could see that the 2-dimensional 'size' of the picture in front of your eyes was somehow stabilizing to some number then necessarily the picture will start to form into something like a 2-plane. As you zoom into this rough 2-plane it will break up and the picture in front of your eyes will again become chaos, but as you keep zooming in, the picture will slowly form into an even tighter 2-plane, and then disperse, etc.

This was the main result needed to prove his spectacular theorem

**Theorem 5** Given a Radon measure $\mu$ on $\mathbb{R}^n$ with the property that

\[ 0 < \lim_{r \to 0} \frac{\mu(B_r(z))}{r^s} < \infty \]

for $\mu$ a.a. $z \in \text{Spt}\mu$ then $s$ in an integer.

So if you have any Radon measure in Euclidean space with density properties then you are forced into the integers. This is extraordinary because initially it would seem that density properties of Radon measures and the natural numbers have nothing to do with one another.
Finally, by greatly extending Marstrand methods, Preiss in 1986 solved the problem completely for Radon measures in Euclidean space, see [5]. A highly simplified version of his theorem is

**Theorem 6** Given an m-set $S \subset \mathbb{R}^n$ with the property that

$$0 < \lim_{r \to 0} \frac{H^m(B_r(x) \cap S)}{r^m} < \infty$$

for $H^m$ a.a. $x \in S$ then $S$ is rectifiable.

**Metric knowledge of the behavior of the size in small balls**

$\Rightarrow$ Geometry !!!

So if we were looking at our set through our microscope and we saw as we zoomed in that the 'm-size' of the picture was stabilizing then not only would we know the value the 'm-size' was stabilizing to (2 for $m = 1$, $\pi$ for $m = 2$ extra) but the picture forming in front of our eyes would be that of an $m$-plane.

As a further illustration, suppose we were zooming into a 2-set with our microscope and the '2-size' of the picture that appeared in front of our eyes seemed to be stabilizing to some number $\neq \pi$ then we would be mistaken. The picture cannot be stabilizing to this number. If it stabilizes then this picture must form into a 2-plane and so its 2-size would be $\pi$.

**Why this is true**

Suppose we have an m-set $S$ such that for some $\alpha > 0$, $S$ has the property that

$$\lim_{r \to 0} \frac{H^m(B(x, r) \cap S)}{r^m} = \alpha$$

for all $x \in S$.

I would like to illustrate the idea behind a particular very important tool called 'tangent measure' that was central to the proof of Preiss's theorem. This is a tool for studying the local properties of sets.

I would like the reader to imagine he lived on a 'world' which was a 2-set with the property of having density $\alpha$. The reader could then see the size of his world in small balls of radius $r$, centered in the world, was very much like $\alpha r^2$. If the reader were to get down on his knees and look closely at the ground, he would see its 'size' was very
well controlled in small balls. However if he were to stand up and look around him he would see a great fluctuation of the size of his world as he looked further and further out.

Now imagine the reader came across a phial of shrinking potion, and drank it. As the reader began shrinking if he were to look down at his feet he would see very good control of the size of his world in balls big enough such that he wouldn't have to get down on his knees to see inside them.

As he shrinks further and looks around him he would see that there was very good control of the size of the world in balls of diameter comparable to his diameter, and after a time very good control in ball much bigger than him. Finally he would have shrunk so much the world around him would look like a world with the property that for any centered ball of diameter $r$ the size of the world inside this ball would be $ar^2$.

Writing this out slightly more formally:

Take a point $a \in S$ and a map say $T_{a,q}(x) = \frac{x-a}{q}$ which blows up the ball radius $q$ centered at $a$ to a ball radius 1 centered at 0. Then the 'world' we would get after performing such a blow up would look more and more like a 'world' $T$ where

$$H^m(B(x,r) \cap T) = ar^m$$

for all $x \in T$, $r > 0$,

as $q \to 0$.

Any set $T$ with such a property we call an $m$-uniform set.

Now in this language having a measure theoretic tangent at a point $a \in S$ is the same as having the world we get when we blow up around that point, being flat.

So if we could prove $m$-uniform sets were flat this would prove Preiss's theorem.
Unfortunately this isn’t true

\[ S_1 = \{ x \in \mathbb{R}^4 : x_1^2 = x_2^2 + x_3^2 + x_4^2 \} \]

is a 3-uniform set, see [5].

However what Preiss was able to prove was that m-uniform sets are either m-planes or sets very far from being m-planes, they are sets that are ‘curved at infinity’ as \( S_1 \) is.

By a ‘time stopping’ type argument this is enough.

Below is a picture of the set \( \{ x_1^2 = x_2^2 + x_3^2 \} \), which is a slice through \( S_1 \) by the 3-plane given by \( x_4 = 0 \).
Results in $l_3^\infty$

Marstrand-Preiss methods rely *heavily* on the extremely good symmetry properties of the unit ball in Euclidean space as exhibited by the existence of the inner product.

The inner product introduces a kind of linearity into the problem which turns out to be extremely powerful, so much so that when finally pushed far enough the purely analytic problem of rectifiability of measures with density becomes a problem of multilinear algebra.

The problem is solved and the results of this are asymptotic expansions of the 'moments' of the measure. These asymptotic expansions are the engine of the proof.

Besicovitch-Marstrand-Mattila methods rely on a different property of the unit ball in Euclidean space, which is basically a kind of rotundness of the ball that allows the existence of objects of diameter $d$ that cannot be contained in a ball diameter $d$. This is essential in the trick used to prove antipodal symmetry.

Given that all the methods in the sixty year history of the subject have a strange reliance on particular properties of the unit ball in Euclidean space it may seem that the phenomenon of getting geometry from purely metric knowledge of the behavior in small balls is some sort of property of the Euclidean unit ball.

**Why $l_3^\infty$ ?**

Up to now, for simplicity, I have been talking about sets instead of measures. It is no longer meaningful do so. So I will explain how the results I have been describing are special cases of results for Radon measures.

Given any $m$-set $S \subset \mathbb{R}^n$ we can define a Radon measure $\mu$ by

$$\mu(A) = H^m(A \cap S) \quad \text{for any } A \subset \mathbb{R}^n. \quad (0.1)$$

Now recall the *support* of a measure $\mu$ is the set of points $x \in \mathbb{R}^n$ with the property

$$\mu(B(x, r)) > 0 \quad \text{for all } r > 0.$$ 

We will denote the *support* of measure $\mu$ as $\text{Spt}\mu$.

So in the case of the measure defined by (0.1) it should be clear that $\text{Spt}\mu = S$.

We say a Radon measure $\mu$ is rectifiable iff its support is rectifiable and it is absolutely continuous with respect to $H^m$ measure.
Preiss’s theorem is that Radon measures with positive finite density almost everywhere are rectifiable, clearly Theorem 6 is a special case of this.

There are two main conjectures generalising Euclidean Rectifiability and Density results.

**Conjecture 1** Given a metric space \( M \) with metric \( d \), let \( H^m_d \) denote Hausdorff \( m \)-measure where the diameter of the covering bodies is taken with respect to metric \( d \).

Suppose \( M \) has the property that

\[
\lim_{r \to 0} \frac{H^m_d(B(x, r))}{\alpha(m) r^m} = 1
\]

for \( H^m_d \) a.a. points \( x \in M \) then \( M \) is rectifiable in the sense that \( H^m_d \) almost all of \( M \) is contained in countably many Lipschitz images of subsets of \( m \)-dimensional Euclidean spaces.

This would be the ultimate rectifiability result for metric spaces, and would also provide a converse to the strong theorem of Kirchheim, see [8]. For case \( m = 1 \) this is known and has been proved by Preiss and Tiser see [6], using Besicovitch type methods.

Using Besicovitch-Marstrand-Mattila methods Chlebik (unpublished) proved Conjecture 1, for \( l_2 \).

As any metric space is isometric to some subset of \( l_\infty \), it seems clear that the first case of this conjecture that should be studied is the case when \( M = l^3_\infty \) and \( m = 2 \).

**Conjecture 2** Suppose \( S \) is a finite dimensional normed vector space and \( \mu \) is a Radon measure on \( S \) with the property that

\[
0 < \lim_{r \to 0} \frac{\mu(B_r(x))}{r^m} < \infty
\]

for \( \mu \) a.a. \( x \in \text{Spt}\mu \), then \( \text{Spt}\mu \) is rectifiable.

Rightly or wrongly my main interest is in Conjecture 2.

Given that \( l^3_\infty \) is an extremal finite dimensional normed vector space, results in this space could be considered as evidence for this conjecture.

Note, as I mentioned, the central result of Preiss’s proof is the classification of \( m \)-uniform measures in Euclidean space into two very different types, those flat at infinity and those curved at infinity. This result has no possible analogue in \( l^3_\infty \). Graphs of
lipschitz functions (with constant \( \leq 1 \)) from \( \ell^1 \) to \( \ell^1 \) support 2-uniform measure in \( l^3_\infty \) as do flat planes, so there is a continuum of 2-uniform measures in \( l^3_\infty \). So results in would \( l^3_\infty \) necessarily require very different ideas.

It should be noted that this conjecture is false for metric spaces.

Let \( S \) be the metric space given by the real line with metric \( d(x, y) = \sqrt{|x - y|} \).

Let \( \mu \) on \( S \) be defined by

\[
\mu(A) = H^2_d(A) \quad \text{for } A \subset \mathbb{R}
\]

where, by \( H^2_d \), I mean Hausdorff 2-measure defined in terms of metric \( d \).

So

\[
H^2_d(A) = \liminf_{\delta \to 0} \left\{ \sum_{n=1}^{\infty} \pi \left( \frac{d(C_n)}{2} \right)^2 : A \subset \bigcup_{n=1}^{\infty} C_n \text{ where } d(C_m) \leq \delta \right\}
\]

\[
= \liminf_{\delta \to 0} \left\{ \sum_{n=1}^{\infty} \frac{d_{\text{eucl}}(C_n)}{4} : A \subset \bigcup_{n=1}^{\infty} C_n \text{ where } d_{\text{eucl}}(C_m) \leq \delta^2 \right\}
\]

\[
= \frac{\pi}{4} H^1(A).
\]

Note that as the entire real line is in the support of \( \mu \) we have

\[
\mu(B_r(x)) = H^2_d(B_r(x)) = \frac{\pi}{4} H^1 \left( \left\{ z \in \mathbb{R} : \sqrt{|z - x|} < r \right\} \right)
\]

\[
= \frac{\pi}{2} r^2 \quad \forall \ x \in \text{Spt}\mu.
\]

So \( \mu \) is even 2-uniform, but the above equality shows the density of \( S \) with respect to \( H^m_d \) is everywhere equal to a half, i.e.

\[
\lim_{r \to 0} \frac{H^2_m(B_r(x))}{\pi r^2} = \frac{1}{2} \quad \forall \ x \in \text{Spt}\mu.
\]

However by Kirchheim's theorem if \( S \) was rectifiable then we would have that for \( H^2_m \), a.a. \( x \in S \) the density with respect to \( H^2_m \) would have to be 1, so \( S \) can not be rectifiable and Conjecture 2 is not true in metric spaces.
Results in $l^3_{\infty}$ are:

**Theorem 7** Given a Radon measure $\mu$ on $l^3_{\infty}$ with the property that

$$0 < \lim_{r \to 0} \frac{\mu(C_r(x))}{r^2} < \infty$$

for $\mu$ a.a. $x \in \text{Spt}\mu$ then for $\mu$ a.a. $y \in \text{Spt}\mu$ we have the following property:

For every $\epsilon > 0$ we can find an 2-plane $V$ such that

$$\lim_{k \to \infty} \frac{\mu(C_{r_k}(y) \cap N_{r_k}(V))}{r_k^2} = 0$$

for some sequence $r_k \to 0$.

Probably the above result is true for measures with $(n - 1)$-density in $l^n_{\infty}$.

There is a theorem about tangent measures which, for our purposes, basically says if you prove an $m$-uniform measure has a particular weak local geometric property (weak in the sense that this property exists in balls of radius $r_n$ for some sequence $r_n \to 0$) then the same weak local geometric property will be true for measures with positive finite $m$-density.

Informally: Anything you can say about $m$-uniform measures for one sequence going down to zero, you can say about measures with positive finite $m$-density.

So of course it was only necessary to prove Theorem 7 for uniform measures in $l^3_{\infty}$.

As can probably be guessed, uniform measures in $l^3_{\infty}$ are measures with the property that for some $\alpha > 0$,

$$\mu(C_r(x)) = \alpha r^2 \ \forall \ x \in \text{Spt}\mu, \ r > 0.$$  

Locally uniform measures in $l^3_{\infty}$ are ones with this property for all $r \in (0, \delta)$ for some small positive number $\delta$.

**Theorem 8** Given a locally 2-uniform measure $\mu$ in $l^3_{\infty}$ in every neighborhood of every point in the support, $\text{Spt}\mu$ has a rectifiable subset.

So, in other words locally 2-uniform measures in $l^3_{\infty}$ have dense rectifiable subsets. This result is weak in comparison to the results for Euclidean space; locally 2-uniform measures in $l^3_{\infty}$ will never have applications. However this is the first result that obtains rectifiability from initial conditions about density of a measure, in a completely non-Euclidean space, and as such is the first indication the unique phenomenon of metric knowledge of the behavior of the size in small balls implying geometry is not some fluke about the symmetry of the Euclidean unit ball.
Background

In accordance with historic convention I have, up to this point, been referring to measures with the property that for some $\alpha > 0$,

$$
\mu (B_r (x)) = ar^m \quad \forall \ x \in \text{Spt}\mu, \ r > 0
$$

as uniform measures. Since we will never need to know the measure on large balls in anything that follows, from now on uniform measures are measures with this property for all $0 < r < 1$.

Tangent Measures

We shall make use of some elementary results about tangent measures. Here precise definitions will be given, and the required results stated.

Given $a \in \mathbb{R}^3$ and $r > 0$ define $T_{a,r} (x) = \frac{(x-a)}{r}$.

Note that $T_{a,r} \mu (A) = \mu (rA + a), A \subset \mathbb{R}^3$.

Suppose $\mu$ is a Radon measure on $\mathbb{R}^3$. We say $\nu$ is a Tangent measure of $\mu$ at a point $a \in \mathbb{R}^3$ if $\nu$ is a non-zero Radon measure on $\mathbb{R}^3$ and there exists a sequence $(r_n)$ of positive numbers such that $r_n \to 0$ and $\frac{T_{a,r_n} \mu}{r_n^m} \to \nu$ as $n \to \infty$. Note that this is a more restrictive definition of tangent measure than is usually given.

We will denote by $\text{Tan} (\mu, x)$ the set of tangent measures to $\mu$ at $x$ and we will denote $\overline{\text{Tan}} (\mu, x)$, the set of supports of tangent measures to $\mu$ at $x$.

So using Tangent measure notation, a measure $\mu$ having a weak tangent $V$ at $x$ is equivalent to $\overline{\text{Tan}} (\mu, x) \cap G (2, 3) \neq \emptyset$, where $G (2, 3)$ denotes the space of 2-dimensional subspaces in $\mathbb{R}^3$.

We will now state two well known lemmas about tangent measures; both can be found in [1]. Although our definition of tangent measure is more restrictive than that
of [1] both lemmas still hold using the proofs given in [1].

**Lemma 1** Suppose $\mu$ measures $l_3^\infty$ with the property that for $\mu$ a.a. $x \in \text{Spt}\mu$

$$0 < \lim_{r \to 0} \frac{\mu(C_r(x))}{r^2} < \infty$$

then for $\mu$ a.a. $x \in \text{Spt}\mu$ every $\nu \in \text{Tan}(\mu, a)$ is 2-uniform with $0 \in \text{Spt}\nu$.

This is Corollary 14.9 [1].

**Lemma 2** Suppose $\mu$ measures $l_\infty^3$, then for $\mu$ a.a. $x \in \text{Spt}\mu$ every $\nu \in \text{Tan}(\mu, a)$ has the following two properties:

1. $T_{x,1}\nu \in \text{Tan}(\mu, x)$ for all $z \in \text{Spt}\mu$.
2. $\text{Tan}(\nu, z) \subseteq \text{Tan}(\mu, x)$ for all $z \in \text{Spt}\mu$.

This is a slightly weaker form of Theorem 14.16 [1].
List of notation

$\mathbb{N}$ the set of non-negative integers

$B_r(x)$ denote the open ball of radius $r > 0$ with respect to the Euclidean norm

$\mathbb{R}^n$ the set of real valued $n$-vectors

Given any subset $A \subseteq \mathbb{R}^n$ let $\tilde{A}$ denote the set of density points of $A$

i.e. $\tilde{A} = \{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{L^n(B_r(x) \cap A)}{r^n} = 0 \}$

$e_1, e_2, e_3$ be orthonormal vectors in $\mathbb{R}^3$

$e_{j+3} = -e_j$ for $j \in \{1, 2, 3\}$

$C_r(x)$ be the open cube of radius $r$ centered on $x$, whose sides are perpendicular to the orthonormal vectors

$\| || \|$ denotes the sup norm, so $\| x \| = \max \{ |e_1 \cdot x|, |e_2 \cdot x|, |e_3 \cdot x| \}$

$l^3_\infty$ denotes the space $\mathbb{R}^3$ with the sup norm

$\text{cl}(A) = \{ x \in \mathbb{R}^n : \text{There exists } z_n \in A \text{ s.t. } z_n \to x \text{ as } n \to \infty \}$ for any $A \subseteq \mathbb{R}^n$

$\partial A = \text{cl}(A) \cap \text{cl}(\mathbb{R}^n \setminus A)$ for any $A \subseteq \mathbb{R}^n$, i.e. the topological boundary

$\text{int}(A) = A \setminus \partial A$ for any $A \subseteq \mathbb{R}^n$

$A \Delta B = (A \setminus B) \cup (B \setminus A)$ for any $A \subseteq \mathbb{R}^3, B \subseteq \mathbb{R}^3$

$A(x, a, b) = C_b(x) \setminus \text{cl}(C_a(x))$

$< v_1, v_2, v_3 >$ is the linear span of a set of vectors $v_n \in \mathbb{R}^3, n = 1, 2, 3$

$P_\tau : \mathbb{R}^3 \to \tau$ is the orthogonal projection onto $\tau$ for any linear subspace $\tau \subseteq \mathbb{R}^3$
\[ K(x, v, s) = P_{v, \perp}^{s-1} (B_s(x) \cap v^\perp) \]

for any \( x \in \mathbb{R}^3, v \in S^2, s > 0 \)

\[ X(x, v, r) = \left\{ y \in \mathbb{R}^3 : |P_{v, \perp} (y - x)| \leq s |P_{< v, >} (y - x)| \right\} \]

for any \( x \in \mathbb{R}^3, v \in S^2, s > 0 \)

\[ G(m, n) = \{ \text{The set of m-dimensional linear subspaces of } \mathbb{R}^n \} \]

\( T_j^{(r)} \) be the side of \( \text{cl}(C_r(0)) \) perpendicular to \( e_j \) and which intersects the line \( <e_j> \)

for \( j = 1, 2 \ldots 6 \)

\[ S_j^{(0)} = \bigcup_{r \geq 0} T_j^{(r)} \text{ for } j = 1, 2 \ldots 6 \]

\[ S_j^{(0)} = S_j^{(0)} \text{ for } j = 1, 2 \ldots 6 \]

\[ S_j^{(x)} = S_j^{(0)} + x \text{ for } j = 1, 2 \ldots 6 \]

\[ d(A, B) = \inf \{||x_1 - x_2|| : x_1 \in A, x_2 \in B\} \]

\[ \hat{S}_{i,j} = \left\{ z \in S_j^{(y)} : (z - y) \cdot e_l \leq (1 - \rho)(z - y) \cdot e_j \right\} \text{ for } y \in \mathbb{R}^3, i \in \{1, 2, \ldots 6\}, \]

\( j \in \{i + 1, i + 2, i + 4, i + 5\} \) and \( l \in \{i + 1, i + 2, i + 4, i + 5\} \setminus \{j, j + 3\} \)

\[ \Psi^u(x, v, s, r) = \left\{ z \in C_r(x) : |P_{v, \perp} (z - x)| \leq s |P_{v} (z - (x + rv))| \right\} \]

for \( x \in \mathbb{R}^3, v \in \{e_1, e_2, \ldots e_6\}, s \in (0, 1) \) and \( r > 0 \)

\[ \Psi^d(x, v, s, r) = \left\{ z \in C_r(x) : |P_{v, \perp} (z - x)| \leq s |P_{v} (z - (x - rv))| \right\} \]

for \( x \in \mathbb{R}^3, v \in \{e_1, e_2, \ldots e_6\}, s \in (0, 1) \) and \( r > 0 \)

\[ \Psi(x, v, s, r) = \Psi^u(x, v, s, r) \cap \Psi^d(x, v, s, r) \]

A 2-uniform measure \( \mu \), is a measure with \( 0 \in \text{Spt}\mu \) and with the property that for every \( x \in \text{Spt}\mu, r \in (0, 1) \) we have that \( \mu(C_r(x)) = 4r^2 \)

The following notation is defined with respect to a 2-uniform measure \( \mu \), which remains fixed throughout our arguments.

\[ R_i^{(y)} = \partial S_i^{(y)} \cap \partial S_j^{(y)} \text{ for any } y \in \text{Spt}\mu \text{ for each pair } i, j \in \{1, 2, \ldots 6\} \text{ where } e_i \cdot e_j = 0 \]

\[ f_i^{(y)}(s) = \lim_{h \to 0} \frac{\mu(S_i^{(y)} \cap A(y, s-h, s+h))}{2h} \]

for all such \( s > 0 \) such that this limit exists, for any \( y \in \text{Spt}\mu \)

\[ G = \left\{ x \in \text{Spt}\mu : \forall \ \delta > 0, \ \psi \in S^2 \ \limsup \frac{\mu(C_r(x) \cap X(x, \psi, \delta))}{r^2} > 0 \right\}, \text{ the set of points with cone density} \]

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\( L(y) \) denotes the set of points \( s > 0 \) for which the derivatives of \( f^{(y)}_1, f^{(y)}_2 \) and \( f^{(y)}_3 \) all exist at \( s \)

\[
\Theta^{(y)}_i = \left\{ s > 0 : \partial C_s(y) \cap \left( \bigcup_{j_1, j_2 \in \{i+1, i+2, i+4, i+5\}, e_{j_1} e_{j_2} = 0} R^{(y)}_{j_1, j_2} \right) \cap \text{Spt} \mu = \emptyset \right\}
\]
Chapter 1

Statement of Theorem

**Theorem 9** If $\nu$ is a Radon measure on $\mathbb{R}^3$ with the following property:

There exist some $\delta > 0$, $\alpha > 0$ such that for all $x \in \text{Spt}\nu$ and all $r \in (0, \delta)$

$$\nu(C_r(x)) = \alpha r^2$$

then in every neighborhood of every point of $\text{Spt}\nu$ we can find a $C^1$ submanifold containing a subset of $\text{Spt}\nu$ of positive $\nu$ measure.
Chapter 2

Proof of Theorem

Proof

Suppose not and we have such a measure \( \nu \) with a point \( x \in \text{Spt}\nu \) such that \( C_{r_0}(x) \cap \text{Spt}\nu \) is purely unrectifiable for some small \( r_0 > 0 \).

Let \( T \) be the homothety defined by \( T(z) = \frac{2(z-x)}{r_0} \), and define \( \mu_0(A) = T_\sharp\nu(A) = \nu\left(\frac{1}{2}r_0A + x\right) \).

Clearly \( \text{Spt}\mu_0 = T(\text{Spt}\nu) \), and as \( T(C_{r_0}(x)) = C_2(0) \) we have that \( \text{Spt}\mu_0 \cap C_2(0) \) is purely unrectifiable.

Note also for any \( z \in \text{Spt}\mu_0 = T(\text{Spt}\nu) \), \( r \in (0,1) \) we have

\[
\mu_0(C_r(z)) = \nu\left(C_{\frac{r}{2}\cdot\left(\frac{r_0}{2}z + x\right)}\right) = \nu\left(C_{\frac{r}{2}\cdot\left(T^{-1}(z)\right)}\right) = \frac{r_0^2}{4}r^2.
\]

So let \( \mu = \frac{16}{r_0^2}\mu_0 \), \( \mu \) is a 2-uniform measure with the property that \( C_2(0) \cap \text{Spt}\mu \) is purely unrectifiable.

Now in all the coming Lemmas we will only need to use the uniformity of the measure on a finite number of points in \( C_1(0) \cap \text{Spt}\mu \), with cubes of diameter < 1. In fact everything is local and all the action will take place in \( C_2(0) \), with cubes centered in \( C_1(0) \) with radii less than 1.
2.1 Basic Estimate from shifting Radial functions

The first lemma is one of the classic inequalities obtained from shifting radial functions.

**Lemma 3** Let $\mu$ be a 2-uniform measure, given any two points $z, x \in \text{Spt}\mu \cap C_1(0)$, and any $r \in (0,1)$.

$$\left| \int_{C_r(z)} \|z - y\| - \|x - y\|d\mu y \right| \leq 24r\|x - z\|^2.$$  

**Proof**

Now

$$\int_{C_r(z)} \|z - y\| - \|x - y\|d\mu y = \int_{C_r(z)} r - \|x - y\|d\mu y - \int_{C_r(z)} r - \|z - y\|d\mu y.$$  

(2.1)

Now we note by the Fubini type theorem given by 1.15 [1]

$$\int_{C_r(z)} r - \|x - y\|d\mu y = \int_{t=0}^r \mu \{y \in C_r(x) : r - \|x - y\| \geq t\} dLt$$

$$= \int_{t=0}^r \mu \{C_{r-t}(x)\} dLt.$$  

And

$$\int_{C_r(z)} r - \|z - y\|d\mu y = \int_{t=0}^r \mu \{C_{r-t}(z)\} dLt.$$  

So

$$\int_{C_r(z)} r - \|z - y\|d\mu y - \int_{C_r(z)} r - \|x - y\|d\mu y = 0.$$  

As $x \in \text{Spt}\mu$ and $z \in \text{Spt}\mu$, so using this with equality (2.1)

$$\int_{C_r(z)} \|z - y\| - \|x - y\|d\mu y = \int_{C_r(z)} r - \|z - y\|d\mu y - \int_{C_r(z)} r - \|z - y\|d\mu y$$

$$\leq \int_{A(z,r-\|z-x\|,r+\|z-x\|)} |r - \|z - y\||\mu y$$

$$\leq \|z - x\|\mu(A(z,r - \|z - x\|,r + \|z - x\|))$$

$$\leq 16r\|z - x\|^2.$$  

Swapping the $x$ and the $z$ in the argument above, gives

$$\int_{C_r(z)} \|x - y\| - \|z - y\|d\mu y \leq 16r\|x - z\|^2.$$  

Now as

$$\int_{C_r(z)\triangle C_r(x)} \|x - y\| - \|z - y\|d\mu y \leq 8r\|x - z\|^2.$$  

Finally we have

$$\left| \int_{C_r(z)} \|z - y\| - \|x - y\|d\mu y \right| \leq 24r\|x - z\|^2.$$  

$\square$
2.1.1 Application: Measure Symmetry

This next lemma establishes what symmetry exists in $\mathbb{R}^3$ it can be considered an analogue of Lemma 4, p 374 in [3].

**Lemma 4** Let $\mu$ be a 2-uniform measure for which $\text{Spt} \mu \cap C_2 (0)$ is purely unrectifiable. For any $x \in \text{Spt} \mu \cap C_1 (0)$ such that $\text{Spt} \mu$ does not have a tangent at $x$ the following is true:

$$
\mu \left( S_j (x) \cap C_r (x) \right) = \mu \left( S_{j+3} (x) \cap C_r (x) \right)
$$

for every $j \in \{1, 2 \ldots 6\}, 0 < r < 1$.

**Proof**

We know $x$ doesn’t have a 2-plane as a tangent. So we can find $\gamma_k \in S^2$ where $\{\gamma_1, \gamma_2, \gamma_3\}$ are linearly independent and sequences $z^k_n \in \text{Spt} \mu$, such that $z^k_n \rightarrow x$ and

$$
\left\| \frac{z^k_n - x}{|z^k_n - x|} - \gamma_k \right\| \rightarrow 0 \text{ as } n \rightarrow \infty.
$$

Let $\vartheta^k_n = (z^k_n - x)$.

We now consider the behavior of $h(y) = \|z^k_n - y\| - \|x - y\|$ over $C_r (x)$. Now by pure unrectifiability we know that $\mu \left( \partial S_i (x) \cap C_r (x) \right) = \int_{h > 0} \left( S_i (x) \setminus S_i (x + h e_i) \right) \cap C_r (x)$ for each $i \in \{1, 2 \ldots 6\}$, we can find $h^m_i > 0$ such that $\mu \left( \left( S_i (x) \setminus S_i (x + h^m e_i) \right) \cap C_r (x) \right) < 2^{-m} r^2$.

Now

$$
\int_{C_r (x)} \|z^k_n - y\| - \|x - y\|d\mu y = \sum_{i=1}^{6} \int_{C_r (x) \setminus S_i (x)} \|z^k_n - y\| - \|x - y\|d\mu y.
$$

The lack of an inner product forces us to consider this integral by cutting it up into various pieces that we can understand. This will be a continuing theme of the work.

Note that there exists $\tau_m$ such that for $n > \tau_m$, we have $S_i (x) \setminus S_i (z^k_n) \subset S_i (x) \setminus S_i (x + h^m e_i)$ for $i = 1, 2 \ldots 6$, for each $k = 1, 2, 3$.

And $\|z^k_n - y\| - \|x - y\| = e_i \cdot (z^k_n - y) - e_i \cdot (x - y) = e_i \cdot (z^k_n - x)$ for all $y \in S_i (x) \cap S_i (x + h^m e_i)$, where $i = 1, 2 \ldots 6$, so

$$
\int_{C_r (x)} \|z^k_n - y\| - \|x - y\|d\mu y - \sum_{i=1}^{6} e_i \cdot (z^k_n - x) \mu \left( C_r (x) \cap S_i (x) \setminus S_i (x + h^m e_i) \right)
$$

$$
= \sum_{i=1}^{6} \int_{C_r (x) \setminus S_i (x)} \|z^k_n - y\| - \|x - y\|d\mu y
$$

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Now define $\psi \in \mathbb{R}^3$ by

$$\psi = \sum_{i=1}^{3} \left( \mu \left( \mathcal{C}_r(x) \cap S_i^{(x)} \right) - \mu \left( \mathcal{C}_r(x) \cap S_i^{(x+\epsilon_i)} \right) \right) e_i.$$ 

We have

$$\left| \sum_{i=1}^{6} e_i \cdot \left( z_n^k - x \right) \mu \left( \mathcal{C}_r(x) \cap S_i^{(x)} \cap S_i^{(x+\epsilon_i)} \right) - \psi \cdot \psi_n \right| \leq \sum_{i=1}^{6} \| z_n^k - x \| \mu \left( \mathcal{C}_r(x) \cap \left( S_i^{(x)} \setminus S_i^{(x+\epsilon_i)} \right) \right) \leq 2^{-m} 24 \| z_n^k - x \| r^2.$$

Now by Lemma 3 we have that

$$\left| \int_{\mathcal{C}_r(x)} \| z_n^k - y \| - \| x - y \| d\mu(y) \right| \leq 24r \| x - z_n^k \|^2.$$

So putting (2.2), (2.3) together with this we have

$$\left| \psi_n \cdot \psi \right| \leq 2^{-m} 48 \| z_n^k - x \| r^2 + 24r \| x - z_n^k \|^2.$$

Now as $\left| \psi_n \right| = \| z_n^k - x \| \geq b_0 \| z_n^k - x \|$ for some constant $b_0 > 0$.

Let $\gamma_n = \psi_n / \left| \psi_n \right|$, so for all $n \geq k_m$

$$\left| \gamma_n \cdot \psi \right| \leq \frac{48r^2}{2mb_0} + \frac{24r \| x - z_n^k \|}{b_0}.$$

By continuity of the Euclidean norm this means $\gamma_k \cdot \psi = 0$ for $k = 1, 2, 3$, as $\{\gamma_1, \gamma_2, \gamma_3\}$ are linearly independent so $\psi = 0$. \(\square\)

Remark

As in [1], p.139 we can induce a measure $\mu_r$ on $\partial \mathcal{C}_r(x)$ for $L^1$ a.a. $r > 0$ for any $x \in \text{Spt} \mu$, such that

$$\int_{a}^{b} \int_{\partial \mathcal{C}_r(x)} \phi(z) \ d\mu_r(z) \ dLr = \int_{A(x,a,b)} \phi(z) \ d\mu z$$

for all $\phi \in C_0(\mathbb{R}^n)$, $0 < a < b < \infty$. 

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Measure symmetry on boundaries

**Lemma 5** Let \( \mu \) be a 2-uniform measure where \( C_2(0) \cap \text{Spt}\mu \) is purely unrectifiable.

For any \( x \in C_1(0) \cap \text{Spt}\mu \) which doesn't have a tangent, and any \( j \in \{1, 2, \ldots, 6\} \), the following is true

\[
\mu_r \left( S_j^{(x)} \cap \partial C_r(x) \right) = \mu_r \left( S_{j+3}^{(x)} \cap \partial C_r(x) \right)
\]

for any \( j \in \{1, 2, \ldots, 6\} \), for \( L^1 \) a.a. \( r \in (0, 1) \).

**Proof** By Lemma 4, where \( \mu_r \) is defined we have

\[
\mu_r \left( S_j^{(x)} \cap \partial C_r(x) \right) = \lim_{h \to 0} \frac{\mu \left( A(x, r-h, r+h) \cap S_j^{(x)} \right)}{2h}
\]

\[
= \lim_{h \to 0} \frac{\mu \left( C_{r+h}(x) \cap S_j^{(x)} \right) - \mu \left( C_{r-h}(x) \cap S_j^{(x)} \right)}{2h}
\]

\[
= \lim_{h \to 0} \frac{\mu \left( C_{r+h}(x) \cap S_{j+3}^{(x)} \right) - \mu \left( C_{r-h}(x) \cap S_{j+3}^{(x)} \right)}{2h}
\]

\[
= \lim_{h \to 0} \frac{\mu \left( A(x, r-h, r+h) \cap S_{j+3}^{(x)} \right)}{2h}
\]

\[
= \mu_r \left( S_{j+3}^{(x)} \cap \partial C_r(x) \right). \quad \square
\]
2.2 Higher order shifted radial functions: Proof engine

**Lemma 6** Let \( \mu \) be a 2-uniform measure for which \( \text{Spt} \mu \cap C_2(0) \) is purely unrectifiable. Let \( y \in \text{Spt} \mu \cap C_1(0) \). Given \( x \in S_i^{(y)} \cap G \) for some \( i \in \{1, 2, \ldots, 6\} \), for any \( r \in (0, 1) \), \( h \in (0, \frac{r}{8}) \).

Define \( \sigma (z) = h^2 - (||z - y|| - r)^2 \), and denote \( A(y) = A(y, r, r + h) \).

Now for all such \( x \) sufficiently close to \( y \) we have

\[
\left| \int A(y) \cap S^{(x)}_k \frac{-4\sigma(z)}{||z - y|| - r} \, d\mu z + \int A(y) \cap S^{(x)}_{k+3} \frac{4\sigma(z)}{||z - y|| - r} \, d\mu z \right|
\]

\[
+ \int A(y) \cap (\overline{S^{(y)}_k \cup S^{(y)}_{k+3}}) \frac{4(h^2 - 3(||z - y|| - r)^2)}{||z - y|| - r} e_k \cdot (x - y) \, d\mu z
\]

\[
\leq c_0 ||x - y||^2 hr \quad \forall \ k \in \{i, i + 1, i + 2\}.
\]

Also for any \( a > 0 \) we have the bound

\[
\mu \left( A\left( y, \frac{7a}{8}, \frac{9a}{8} \right) \cap \left( \left( S^{(y)}_i \setminus S^{(x)}_i \right) \cup \left( S^{(x)}_{i+3} \setminus S^{(y)}_{i+3} \right) \right) \right) \leq c_0 ||x - y||^a.
\]

**Proof** For any \( r > 0, h \in (0, \frac{r}{8}) \) we can argue as follows.

Firstly we will assume \((x - y) \cdot e_{i+1} > 0 \) and \((x - y) \cdot e_{i+2} > 0 \).

Let \( \Psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be defined by

\[
\Psi(a) = \begin{cases} 
  h^4 & \text{for } a \in (0, r) \\
  (h^2 - (a - r)^2)^2 & \text{for } a \in [r, r + h] \\
  0 & \text{for } a \in (r + h, \infty)
\end{cases}
\]

So we know by Fubini that

\[
\int \Psi(||z - y||) \, d\mu z - \int \Psi(||z - x||) \, d\mu z = 0.
\]

For any \( \vec{x} \in \text{Spt} \mu \), let \( p_\vec{x}(z) = ||z - \vec{x}|| - ||z - y|| \) and let \( h_\vec{x}(z) = \Psi(||z - \vec{x}||) - \Psi(||z - y||) \).

Note

\[
(h^2 - (||z - \vec{x}|| - r)^2)^2 = \left( h^2 - (||z - y|| - r + p_\vec{x}(z))^2 \right)^2
\]

\[
= \left( h^2 - (||z - y|| - r)^2 - 2(||z - y|| - r)p_\vec{x}(z) - (p_\vec{x}(z))^2 \right)^2
\]

\[
= \left( h^2 - (||z - y|| - r)^2 \right)^2
\]

\[
+ 2 \left( h^2 - (||z - y|| - r)^2 \right) \left( -2(||z - y|| - r)p_\vec{x}(z) - (p_\vec{x}(z))^2 \right)
\]

\[
+ \left( -2(||z - y|| - r)p_\vec{x}(z) - (p_\vec{x}(z))^2 \right)^2.
\]

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So for \( z \in A(\tilde{x}) \cap A(y) \) we have

\[
h_x(z) = 2 \left( h^2 - (||z - y|| - r)^2 \right) \left( -2 (||z - y|| - r) p_{\tilde{x}}(z) - (p_{\tilde{x}}(z))^2 \right) \\
+ \left( -2 (||z - y|| - r) p_{\tilde{x}}(z) - (p_{\tilde{x}}(z))^2 \right)^2 \\
-4 \left( h^2 - (||z - y|| - r)^2 \right) (||z - y|| - r) p_{\tilde{x}}(z) \\
-2 \left( h^2 - (||z - y|| - r)^2 \right) (p_{\tilde{x}}(z))^2 \\
+4 (||z - y|| - r)^2 (p_{\tilde{x}}(z))^3 + 4 (||z - y|| - r) (p_{\tilde{x}}(z))^4 \\
= -4 \left( h^2 - (||z - y|| - r)^2 \right) (||z - y|| - r) p_{\tilde{x}}(z) \\
-2 \left( h^2 - 3 (||z - y|| - r)^2 \right) (p_{\tilde{x}}(z))^2 \\
+4 (||z - y|| - r) (p_{\tilde{x}}(z))^3 + (p_{\tilde{x}}(z))^4.
\]

Fig 2.1 below shows what \( h_x(z) \) looks like on \( C_r(y) \), the idea for this lemma comes from staring at this diagram for a long time until I realised what I wanted to do is perturb the \( x \), not only does this greatly simplify the picture it also reduces the action to something that goes on in the \( e_j \) direction, and so we get expressions only about the \( S_j^{(y)} \).
Let \( k \in \{i, i+1, i+2\} \).

Now since \( x \in G \) we can find some sequence \( x_n \in G \) where \( x_n \to x \) and \( \frac{|P_{x}^k(x_n-x)|}{|P_{x}^k(x_n-x)|} \to 0 \) as \( n \to \infty \).

And of course we have

\[
\int h_{x_n}(z) - h_x(z) \, d\mu z = 0. \tag{2.4}
\]

Now for \( z \in A(x_n) \cap A(x) \cap A(y) \) letting \( q_n(z) = \|z - x_n\| - \|z - x\| \) we have

\[
p_{x_n}(z) = \|z - x_n\| - \|z - y\| = \|z - x\| - \|z - y\| + q_n(z) = p_x(z) + q_n(z) .
\]

So

\[
(p_{x_n}(z))^2 = (p_x(z) + q_n(z))^2
= (p_x(z))^2 + 2p_x(z)q_n(z) + (q_n(z))^2.
\]

And so

\[
\begin{align*}
 h_{x_n}(z) - h_x(z) &= -4 \left(h^2 - (\|z - y\| - r)^2\right) (\|z - y\| - r) (p_{x_n}(z) - p_x(z)) \\
 &\quad -2 \left(h^2 - 3 (\|z - y\| - r)^2\right) (p_{x_n}(z))^2 - (p_x(z))^2 \\
 &\quad +4 (\|z - y\| - r) (p_{x_n}(z))^3 - (p_x(z))^3 \\
 &\quad + (p_{x_n}(z))^4 - (p_x(z))^4 \\
 &= -4 \left(h^2 - (\|z - y\| - r)^2\right) (\|z - y\| - r) q_n(z) \\
 &\quad -2 \left(h^2 - 3 (\|z - y\| - r)^2\right) (2p_x(z)q_n(z) + (q_n(z))^2) \\
 &\quad +4 (\|z - y\| - r) \left(3(p_x(z))^2 q_n(z) + 3p_x(z)(q_n(z))^2 + (q_n(z))^3\right) \\
 &\quad +4 (p_x(z))^3 q_n(z) + 6 (p_x(z))^2 (q_n(z))^2 + 4p_x(z)(q_n(z))^3 + (q_n(z))^4 \\
 &= -4 \left(h^2 - (\|z - y\| - r)^2\right) (\|z - y\| - r) q_n(z) \\
 &\quad -4 \left(h^2 - 3 (\|z - y\| - r)^2\right) p_x(z)q_n(z) \\
 &\quad +12 (\|z - y\| - r) (p_x(z))^2 q_n(z) + 4 (p_x(z))^3 q_n(z) + o(z), \tag{2.5}
\end{align*}
\]

where \( o(z) \) denotes functions bounded above and below by \( c_1\|x_n - x\|^2 \) for all \( z \).
We also should note that for any \( z \in A(x) \cap A(x_n) \) we have

\[
\begin{align*}
h_{x_n}(z) - h_x(z) &= \left( h^2 - (r - \|z - x_n\|) \right)^2 - \left( h^2 - (r - \|z - x\|) \right)^2 \\
&= \left( h^2 - (r - \|z - x\|) - q_n(z) \right)^2 - \left( h^2 - (r - \|z - x\|) \right)^2 \\
&= \left( h^2 - (r - \|z - x\|)^2 + 2 (r - \|z - x\|) q_n(z) - (q_n(z))^2 \right) \\
&\quad - \left( h^2 - (r - \|z - x\|) \right)^2 \\
&= 2 \left( h^2 - (r - \|z - x\|)^2 \right) \left( 2 (r - \|z - x\|) q_n(z) - (q_n(z))^2 \right) \\
&\quad + \left( 2 (r - \|z - x\|) q_n(z) - (q_n(z))^2 \right)^2 \\
&= 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) q_n(z) + o(z). \quad (2.6)
\end{align*}
\]

Now note that

\[
A(x) \setminus A(x_n) \subset A(x, r - \|x_n - x\|, r + \|x_n - x\|) \cup A(x, r + h - \|x_n - x\|, r + h + \|x_n - x\|).
\]

Now for \( z \in A(x, r - \|x_n - x\|, r + \|x_n - x\|) \)

\[
|h_{x_n}(z) - h_x(z)| = |\Psi(\|z - x_n\|) - \Psi(\|z - x\|)| \\
&\leq h^4 - \left( h^4 - (\|x_n - x\|) \right)^2 \\
&= h^4 - \left( h^4 - 2 h^2 \|x_n - x\|^2 + \|x_n - x\|^4 \right) \\
&\leq 2 h^2 \|x_n - x\|^2 - \|x_n - x\|^4. \quad (2.7)
\]

So

\[
\left| \int_{A(x, r - \|x_n - x\|, r + \|x_n - x\|)} h_{x_n}(z) - h_x(z) \, d\mu z \right| \\
\leq \mu(A(x, r - \|x_n - x\|, r + \|x_n - x\|)) 4 h^2 \|x_n - x\|^2 \\
\leq c_2 \|x_n - x\|^2.
\]

Arguing similarly we see that

\[
\left| \int_{A(x, r + h - \|x_n - x\|, r + h + \|x_n - x\|)} h_{x_n}(z) - h_x(z) \, d\mu z \right| \leq c_2 \|x_n - x\|^2.
\]

And so

\[
\left| \int_{A(x) \setminus A(x_n)} h_{x_n}(z) - h_x(z) \, d\mu z \right| \leq c_3 \|x_n - x\|^2. \quad (2.8)
\]

Now note for any \( z \in (A(x_n) \cup A(x))^c \) we have \( \Psi(\|z - x\|) = \Psi(\|z - x_n\|) \).
Now
\[
\int \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} \, d\mu z = \int_{(A(x_n) \cup A(x))} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} \, d\mu z \\
= \int_{A(x_n) \triangle A(x)} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} \, d\mu z + \int_{A(x_n) \cap A(x) \cap A(y)} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} \, d\mu z \\
+ \int_{A(x_n) \cap A(x) \setminus A(y)} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} \, d\mu z = 0. \tag{2.9}
\]

We can assume without loss of generality that our sequence \(x_n \to x\) has the property that \(x_n \cdot e_k > x \cdot e_k\) for all \(n\), because if this were not the case we would simply have to change the signs in the coming definition and everything works out the same.

Let \(I = A(x) \cap A(y)\).

We define the function \(\xi_k\) by
\[
\xi_k(z) = \begin{cases} 
-1 & \text{for } z \in I \cap S_k^{(x)} \\
1 & \text{for } z \in I \cap S_{k+3}^{(x)} \\
0 & \text{for } z \in I \cap \left( \bigcup_{j \in \{k+4, k+5\}} S_j^{(z)} \right) 
\end{cases}
\]

Since \(I\) is open we know by (2.5), (see fig 2.2) that the functions
\[
\Lambda_{A(x_n) \cap I}(z) \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|}
\]
converge pointwise in \(I\) to the function given by
\[
-4 \left( h^2 - (\|z - y\| - r)^2 \right) \left( \|z - y\| - r \right) \xi_k(z) \\
-4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_z(z) \xi_k(z) \\
+ 12 ((\|z - y\| - r) (p_z(z))^2 \xi_k(z) + 4 (p_z(z))^3 \xi_k(z).
\]

\(\text{fig 2.2}\)
We can also observe from (2.5) that the functions are uniformly bounded and so we can apply dominated convergence theorem to get that

$$\lim_{n \to \infty} \int_{A(x_n) \cap I} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} d\mu z$$

$$= \int_I -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z)$$

$$- 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x(z) \xi_k(z)$$

$$+ 12 (\|z - y\| - r) (p_x(z))^2 \xi_k(z) + 4 (p_x(z))^3 \xi_k(z) d\mu z. \quad (2.10)$$

In the same way we can see from (2.6) the functions

$$\chi_{A(x_n) \cap A(x) \setminus A(y)}(z) \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|}$$

converge pointwise in \( \text{int} (A(x) \setminus A(y)) \) to the function

$$4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k(z)$$

and are uniformly bounded so again by dominated convergence theorem we have

$$\lim_{n \to \infty} \int_{A(x_n) \cap A(x) \setminus A(y)} \frac{h_{x_n}(z) - h_x(z)}{\|x_n - x\|} d\mu z$$

$$= \int_{\text{int}(A(x) \setminus A(y))} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k(z) d\mu z$$

$$= \int_{A(x) \setminus A(y)} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k(z) d\mu z \quad (2.11)$$

where we are using pure unrectifiability for the last equality.

Now we will make some estimates that we will need later, note that

$$A(x) \setminus A(y) \subset A(x, r + \|x - y\|) \cup A(x, r + h - \|x - y\|, r + h),$$

and

$$\left| \int_{A(x, r + \|x - y\|)} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k(z) d\mu z \right|$$

$$\leq 4h^2 \mu((x, r + \|x - y\|))$$

$$\leq 16h^2 r \|x - y\|^2, \quad (2.12)$$

and

$$\left| \int_{A(x, r + h - \|x - y\|, r + h)} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k(z) d\mu z \right|$$
\[
\begin{align*}
&\leq 4 \left( h^2 - (h - \|x - y\|)^2 \right) h \mu (A (x, r + h - \|x - y\|, r + h)) \\
&\leq 4 h \left( h^2 - 2h \|x - y\| + \|x - y\|^2 \right) \mu (A (x, r + h - \|x - y\|, r + h)) \\
&\leq 16 h \left( 2h \|x - y\| - \|x - y\|^2 \right) \|x - y\| (r + h) \\
&\leq 64 h^2 r \|x - y\|^2,
\end{align*}
\]
also note that
\[
\left| \int_I 12 (\|z - y\| - r) (p_x (z))^2 \xi_k (z) + 4 (p_x (z))^3 \xi_k (z) \, d\mu z \right| \\
\leq c_4 h r^2 \|x - y\|^2.
\]
(2.13)

By (2.8) we have
\[
\lim_{n \to \infty} \int_{A(x_n) \triangle A(x)} \frac{h_x (z) - h_x (z)}{\|x_n - x\|} \, d\mu z = 0,
\]
so from (2.9) we get
\[
\lim_{n \to \infty} \int_{A(x_n) \cap I} \frac{h_x (z) - h_x (z)}{\|x_n - x\|} \, d\mu z + \lim_{n \to \infty} \int_{A(x_n) \cap A(z) \setminus A(y)} \frac{h_x (z) - h_x (z)}{\|x_n - x\|} \, d\mu z = 0,
\]
(2.15)
which gives
\[
\left| \lim_{n \to \infty} \int_{A(x_n) \cap I} \frac{h_x (z) - h_x (z)}{\|x_n - x\|} \, d\mu z \right| \\
= \left| \lim_{n \to \infty} \int_{A(x_n) \cap A(z) \setminus A(y)} \frac{h_x (z) - h_x (z)}{\|x_n - x\|} \, d\mu z \right|.
\]
So using (2.10) and (2.11) we get
\[
\left| \int_I -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k (z) \\
-4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x (z) \xi_k (z) \\
+12 (\|z - y\| - r) (p_x (z))^2 \xi_k (z) + 4 (p_x (z))^3 \xi_k (z) \, d\mu z \right| \\
\leq \left| \int_{A(x) \setminus A(y)} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k (z) \, d\mu z \right|,
\]
from which we get
\[
\left| \int_I -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k (z) \\
-4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x (z) \xi_k (z) \, d\mu z \right| \\
- \left| \int_I 12 (\|z - y\| - r) (p_x (z))^2 \xi_k (z) + 4 (p_x (z))^3 \xi_k (z) \, d\mu z \right| \\
\leq \left| \int_{A(x) \setminus A(y)} 4 \left( h^2 - (r - \|z - x\|)^2 \right) (r - \|z - x\|) \xi_k (z) \, d\mu z \right|.
\]

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So putting this together with (2.12), (2.13), (2.14) we have

\[
\left| \int_I -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z) -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x(z) \xi_k(z) \, d\mu z \right| \\
\leq c_6 hr \|x - y\|^2. \tag{2.16}
\]

Now because we prefer to deal with annuli rather than the strange shape \( I \) we will make the following easy estimates to give us a better shape.

Note that

\[
A(y) \triangle I \subset A(y, r - \|x - y\|, r + \|x - y\|) \cup A(y, r + h - \|x - y\|, r + h + \|x - y\|)
\]

Also note

\[
\int_{A(y, r - \|x - y\|, r + \|x - y\|)} -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z) \, d\mu z \\
\leq 4h^2 \|x - y\| \mu(A(y, r - \|x - y\|, r + \|x - y\|)) \\
\leq 16h^2 r \|x - y\|^2.
\]

Similarly

\[
\int_{A(y, r + h - \|x - y\|, r + h + \|x - y\|)} -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z) \, d\mu z \\
\leq 64h^2 r \|x - y\|^2.
\]

We can use these to estimate that the difference between the integral of

\[-4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z)\]

over \( I \) and over \( A(y) \) is less than \( 128h^2 r \|x - y\|^2 \) so putting this together with (2.16) we get that

\[
\left| \int_{A(y)} -4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \xi_k(z) \, d\mu z \\
+ \int_f -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x(z) \xi_k(z) \, d\mu z \right| \\
\leq c_6 hr \|x - y\|^2. \tag{2.17}
\]

Now we break off the flow of the argument to demonstrate the second part of the lemma.
Let $a > 0$, we can take $r = a$ and $h = \frac{a}{8}$ so that (2.17) gives us an estimate of the form

$$\left| \int_{A(y)} 4 \left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) (\|z - y\| - a) \xi_k (z) \, d\mu z \right|$$

$$\leq \int_I 4 \left( \frac{a^2}{16} - 3 (\|z - y\| - a)^2 \right) p_x (z) \xi_k (z) \, d\mu z + c_6 a^4 \|x - y\|^2$$

$$\leq 12a^2 \|x - y\| \mu (I) + c_6 a^4 \|x - y\|^2$$

$$\leq c_7 a^4 \|x - y\|.$$  (2.18)

Recall that we assumed that $x \in S_i^{(y)}$. Now by measure symmetry, Lemma 5 we know that

$$\int_{A(y) \cap S_i^{(y)}} 4 \left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) (\|z - y\| - a) \|x - y\| \, d\mu z$$

$$+ \int_{A(y) \cap S_{i+3}^{(y)}} 4 \left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) (\|z - y\| - a) \|x - y\| \, d\mu z = 0$$

Now by simply adding this to (2.18) we get

$$\left| \int_{A(y) \cap (S_i^{(y)} \setminus S_i^{(x)})} 4 \left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) \|x - y\| \, d\mu z \right|$$

$$+ \int_{A(y) \cap (S_{i+3}^{(y)} \setminus S_{i+3}^{(x)})} 4 \left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) \|x - y\| \, d\mu z \right|$$

$$\leq c_7 a^4 \|x - y\|.$$  (2.19)

Now for any $z \in A \left( y, \frac{7a}{8}, \frac{9a}{8} \right)$ we know

$$\left( \frac{a^2}{16} - (\|z - y\| - a)^2 \right) \|z - y\| \geq \left( \frac{a^2}{16} - \left( \frac{a}{8} \right)^2 \right) \frac{7a}{8} \geq 3a^3 \frac{128}{128}$$

So finally we get the bound

$$\mu \left( A \left( y, \frac{7a}{8}, \frac{9a}{8} \right) \cap \left( (S_i^{(y)} \setminus S_i^{(x)}) \cup (S_{i+3}^{(y)} \setminus S_{i+3}^{(x)}) \right) \right) \leq c_8 a \|x - y\|$$  (2.20)

and this establishes the second part of the lemma.

As $p_x (z) \xi_k (z) = e_k \cdot (x - y)$ for $z \in \left( (S_i^{(y)} \cap S_k^{(x)}) \cup (S_{k+3}^{(x)} \cap S_{k+3}^{(y)}) \right) \cap A(y)$ for each $k \in \{i, i + 1, i + 2\}$, this begins to look in the right form for the first part of the Lemma.

Now we continue the argument for the first part of the lemma, our intention is simply to clean up equation (2.17).
By inequality (2.20), we have that
\[
\int_I 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) p_x(z) \xi_k(z) \, d\mu z \\
- \int_{I \cap \left( S_k^{(y)} \cup S_{k+3}^{(y)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_k \cdot (x - y) \, d\mu z \\
\leq \|x - y\| \int_{I \cap \left( S_k^{(y)} \setminus S_k^{(z)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) \, d\mu z \\
+ \|x - y\| \int_{I \cap \left( S_{k+3}^{(z)} \setminus S_{k+3}^{(y)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) \, d\mu z \\
\leq c_r \|x - y\|^2 h^2 r
\]
for each \( k \in \{i, i + 1, i + 2\} \).

As it's clear that
\[
\int_{A(y) \setminus A(z)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_k \cdot (x - y) \, d\mu z \leq c_b \|x - y\|^2 r h^2.
\]

So putting (2.21), (2.22) together we can clean up (2.17) to get the result. □

**Remark** Here I will try to make clearer what Lemma 6 says. Firstly recall the notation
\[
f_k(r) = \mu_r \left( \partial C_r(y) \cap S_k^{(y)} \right)
\]
and also recall that by \( A(y) = A(y, r, r + h) \).

Now by measure symmetry Lemma 6 we see
\[
\int_{A(y) \setminus \left( S_k^{(y)} \cup S_{k+3}^{(y)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) \, d\mu z \leq c_b \|x - y\|^2 r h^2
\]
(2.23)

And as
\[
\int_{r}^{r+h} 4 \left( h^2 - 3 (p - r)^2 \right) f_k(r) \, dLp = 0
\]
so integral (2.23) is some expression of the growth \( f_k \) in \((r, r + h)\) and so informally I will refer to it as 'the 'growth' in \( A(y, r, r + h) \cap S_k^{(y)} \).
This in some sense explains one term in the statement of Lemma 6. Now for the other term. First note that if \( x \in S_i^{(y)} \) then using measure symmetry we have that

\[
\int_{A(y)} 4\sigma (z) (\|z - y\| - r) \xi_k (z) d\mu z
\]

\[
= \int_{A(y) \cap S_i^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z + \int_{A(y) \cap S_{i+3}^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z \tag{2.24}
\]

\[
= \int_{A(y) \cap S_i^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z + \int_{A(y) \cap S_{i+3}^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z
\]

\[
+ \left( \int_{A(y) \cap S_i^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z - \int_{A(y) \cap S_{i+3}^{(y)}} 4\sigma (z) (\|z - y\| - r) d\mu z \right)
\]

\[
= \int_{A(y) \cap \left( S_i^{(y)} \setminus S_i^{(z)} \right)} 4\sigma (z) (\|z - y\| - r) d\mu z + \int_{A(y) \cap \left( S_{i+3}^{(y)} \setminus S_{i+3}^{(z)} \right)} 4\sigma (z) (\|z - y\| - r) d\mu z
\]

and this last integral is an expression of the measure on the 'boundary'

\[
A(y) \cap \left( \left( S_i^{(y)} \setminus S_i^{(z)} \right) \cup \left( S_{i+3}^{(y)} \setminus S_{i+3}^{(z)} \right) \right)
\]

(as this is nothing like the actual boundary of \( A(y) \cap \left( S_i^{(y)} \cup S_{i+3}^{(y)} \right) \) I will call it the 'x-boundary') so when \( x \in S_i^{(y)} \) informally I will refer to integral (2.24) as 'the measure on the 'x-boundary' of \( A(y, r, r + h) \cap \left( S_i^{(y)} \cup S_{i+3}^{(y)} \right) \).

Unfortunately I will have to deal with cases where \( x \not\in S_i^{(y)} \) so I can't state Lemma 6 with an expression of the 'x-boundary' of \( A(y) \cap \left( S_i^{(y)} \cup S_{i+3}^{(y)} \right) \).

So informally we can express the statement of Lemma 6 as

\[
\text{Growth in } A(y, r, r + h) \cap S_i^{(y)} \approx \text{Measure of the } x \text{-boundary of } A(y, r, r + h) \cap \left( S_i^{(y)} \cup S_{i+3}^{(y)} \right)
\]

this is the main idea of this work.
2.2.1 The function $r \to \mu_r \left( C_r (y) \cap S_t^{(y)} \right)$ is a Lipschitz map

**Lemma 7** Let $\mu$ be 2-uniform measure for which $\text{Spt} \mu \cap C_2 (0)$ is purely unrectifiable. Given $y \in G \cap C_1 (0)$, $i \in \{1, 2, \ldots, 6\}$.

Let $f_i (r) = \mu_r \left( S_t^{(y)} \cap \partial C_r (y) \right)$ then $f_i$ is a Lipschitz map and $f_i (r) > 0$ for all $r \in (0, 1)$.

**Proof** We break the proof into 3 steps:

**Step 1:** Given $y \in G$, $i \in \{1, 2, \ldots, 6\}$ we will show that for any $s > 0$

$$\partial C_s (y) \cap S_t^{(y)} \cap \text{Spt} \mu \neq \emptyset.$$ 

Suppose not, then we can find interval $(a, b) \subset \mathbb{R}$ which contains $s$ and has the following properties,

- $\mu \left( A (y, a, b) \cap S_t^{(y)} \right) = 0$
- For every $\epsilon > 0$ we can find $s \in (a - \epsilon, a)$ such that
  $$\mu_s \left( \partial C_s (y) \cap S_t^{(y)} \right) > 0.$$ 

Note by measure symmetry, Lemma 5, we have that the interval $(a, b)$ has the same properties for $S_t^{(y)}$.

Let $\epsilon > 0$.

Now since $y \in G$ we must be able to find $z \in X (y, \epsilon_{i+3}, \epsilon) \cap C_{i+3 \epsilon} (y) \cap G$, by the above remark we lose no generality in assuming $z \in S_t^{(y)}$ because if we had $z \in S_t^{(y)}$ we could use the properties of $(a, b)$ with respect to $S_t^{(y)}$ and argue in exactly the same way to get, by measure symmetry, exactly the same conclusion. Fig 5.1 shows how the argument goes.

![Fig 5.1](image-url)
So pick \( s \in \left( a - \frac{\|y - z\|}{200}, a \right) \) such that

\[
\mu_s \left( \partial C_s(y) \cap S_i^{(y)} \right) > 0.
\]

So since \( \partial C_s(y) \cap S_i^{(y)} \subset \partial C_{s + \|y - z\|} (z) \cap S_i^{(z)} \) and using Lemma 5 we have that

\[
\mu_{s + \|y - z\|} \left( \partial C_{s + \|y - z\|} (z) \cap S_i^{(z)} \right) > 0.
\]

However \( \partial C_{s + \|y - z\|} (z) \cap S_i^{(z)} \subset A(y, a, b) \cap S_i^{(y)} \) and by our construction

\[
d \left( \partial C_{s + \|y - z\|} (z) \cap S_i^{(z)}, \partial \left( A(y, a, b) \cap S_i^{(y)} \right) \right) > 0.
\]

So we can find a point \( z_0 \in \text{int} \left( A(y, a, b) \cap S_i^{(y)} \right) \cap \text{Spt} \mu \) and so for some small \( \delta > 0 \) we have \( C_\delta(z_0) \subset A(y, a, b) \cap S_i^{(y)} \) thus \( \mu \left( A(y, a, b) \cap S_i^{(y)} \right) > 0 \), and by Lemma 4 this is a contradiction.

**Step 2:** \( f_i \) is monotone non-decreasing.

This follows from the same kind of reflection trick as Step 1.

Given \( r_1 < r_2 \) we can pick \( z \in S_i^{(y)} \cap \partial C_{r_2 - r_1} (y) \cap \text{Spt} \mu \).

Since \( r_1 + \|y - z\| = r_1 + \frac{r_2 - r_1}{2} = \frac{r_1 + r_2}{2} \) we have

\[
\partial C_{r_1} (y) \cap S_i^{(y)} \subset \partial C_{\frac{r_1 + r_2}{2}} (z) \cap S_i^{(z)}
\]

And by Lemma 5 we know

\[
\mu_{\frac{r_1 + r_2}{2}} \left( \partial C_{\frac{r_1 + r_2}{2}} (z) \cap S_i^{(z)} \right) = \mu_{\frac{r_1 + r_2}{2}} \left( \partial C_{\frac{r_1 + r_2}{2}} (z) \cap S_i^{(z)} \right).
\]

As

\[
\partial C_{\frac{r_1 + r_2}{2}} (z) \cap S_i^{(z)} \subset \partial C_{r_2} (y) \cap S_i^{(y)}
\]

So we have monotonicity because by Lemma 5

\[
\mu_{r_1} \left( \partial C_{r_1} (y) \cap S_i^{(y)} \right) \leq \mu_{r_2} \left( \partial C_{r_2} (y) \cap S_i^{(y)} \right) = \mu_{r_2} \left( \partial C_{r_2} (y) \cap S_i^{(y)} \right).
\]

Now since \( y \in G \) it is clear that \( \mu \left( S_i^{(y)} \cap C_\alpha (y) \right) > 0 \) for any \( \alpha > 0 \) and so we know that \( f_i (r) = \mu_r \left( \partial C_r (y) \cap S_i^{(y)} \right) > 0 \) for all \( r > 0 \).

**Step 3:** Lipschitzness.

We know that for a.a. \( s > 0 \)

\[
f_i (s) + f_{i+1} (s) + f_{i+2} (s) = 4s.
\]
So given \( r_1 < r_2 \)

\[
\begin{align*}
    f_i (r_2) - f_i (r_1) \\
    &= 4 (r_2 - r_1) - f_{i+1} (r_2) - f_{i+2} (r_2) + f_{i+1} (r_1) + f_{i+1} (r_1) \\
    &\leq 4 (r_2 - r_1)
\end{align*}
\]

by monotonicity. \( \square \)
2.2.2 Technical estimate of the 'growth' at points of differentiability

Lemma 8 Let $\mu$ be 2-uniform measure where $\text{Spt}\mu \cap C_2(0)$ is purely unrectifiable.

Let $y \in G \cap C_1(0)$, $i \in \{1, 2, \ldots, 6\}$.

Let $f_k(r) = \mu_r \left( S_k^{(y)} \cap \partial C_r(y) \right)$, for $k \in \{i, i + 1, i + 2\}$, $r \in (0, 1)$. Let $A(y) = A(y, r, r + h)$.

Given $\epsilon > 0$, if we have $h \in (0, 1)$ such that

$$f_k(p) \in (f_k(r) + (\lambda_k - \epsilon)(p-r), f_k(r) + (\lambda_k + \epsilon)(p-r))$$

for all $p \in (r, r + h)$, for each $k \in \{i, i + 1, i + 2\}$ then

$$\int_{A(y) \cap \left( S_k^{(y)} \cup S_{k+3}^{(y)} \right)} -4 \left( h^2 - 3 \left( \|z - y\| - r \right)^2 \right) d\mu z \in \left( 2(\lambda_k - \epsilon)h^4, 2(\lambda_k + \epsilon)h^4 \right).$$

Proof Note that

$$\int_{A(y) \cap \left( S_k^{(y)} \cup S_{k+3}^{(y)} \right)} -4 \left( h^2 - 3 \left( \|z - y\| - r \right)^2 \right) d\mu z = \int_r^{r+h} -8 \left( h^2 - 3 \left( p - r \right)^2 \right) f_k(p) dLp.$$

And note that

$$\int_r^{r+h} -8 \left( h^2 - 3 \left( p - r \right)^2 \right) f_k(p) dLp = \int_0^h -8 \left( h^2 - 3p^2 \right) f_k(r) dLp = 0.$$

Also note for any $\lambda > 0$

$$\int_r^{r+h} -8\lambda \left( h^2 - 3 \left( p - r \right)^2 \right) \left( p - r \right) dLp = \int_0^h -8\lambda \left( h^2 - 3p^2 \right) p dLp = 2\lambda h^4.$$

So we have

$$\int_{A(y) \cap \left( S_k^{(y)} \cup S_{k+3}^{(y)} \right)} -4 \left( h^2 - 3 \left( \|z - y\| - r \right)^2 \right) d\mu z \in \left( 2(\lambda_k - \epsilon)h^4, 2(\lambda_k + \epsilon)h^4 \right). \quad \square$$
2.3 $\Theta_i^{(y)}$ is the set of $r > 0$ with maximal 'growth'

**Lemma 9** Let $\mu$ be a 2-uniform measure where $\text{Spt} \mu \cap C_2(0)$ is purely unrectifiable. Let $y \in G \cap C_1(0)$. Recall that $f_k(r) = \mu_r \left( S_r^{(y)} \cap \partial C_r(y) \right)$ for any $k \in \{1, 2, \ldots 6\}$, $r \in (0, 1)$.

The following things hold true:

- $f_k'(r) \leq 2$ for all $r \in L^{(y)} \cap (0, 1)$, $k \in \{1, 2, \ldots 6\}$. And so for all $x \in \text{Spt} \mu \cap C_1(0)$, $\alpha \in (0, 1)$, $\mu \left( C_\alpha(x) \cap \left( S_j^{(x)} \cup S_j^{(x)} \right) \right) \geq \alpha^2$ for any $j, j_2 \in \{1, 2, \ldots 6\}$ such that $e_{j_1} \cdot e_{j_2} = 0$.

- If for some $r \in L^{(y)} \cap (0, 1)$ there exists some $s > 0$ such that either

  $\left( R_{i+1,i+2}^{(y)} \cup R_{i+4,i+5}^{(y)} \right) \cap \partial C_t(y) \cap \text{Spt} \mu \neq \emptyset$ for all $t \in (r - s, r + s)$,

  or

  $\left( R_{i+1,i+5}^{(y)} \cup R_{i+4,i+2}^{(y)} \right) \cap \partial C_t(y) \cap \text{Spt} \mu \neq \emptyset$ for all $t \in (r - s, r + s)$,

  then $f_i'(r) + \frac{1}{32} \leq f_{i+1}'(r) + f_{i+2}'(r)$.

- If for some $r \in L^{(y)} \cap (0, 1)$ either

  $\left( R_{i+1,i+2}^{(y)} \cup R_{i+4,i+5}^{(y)} \right) \cap \partial C_r(y) \cap \text{Spt} \mu = \emptyset$  

  or

  $\left( R_{i+1,i+5}^{(y)} \cup R_{i+4,i+2}^{(y)} \right) \cap \partial C_r(y) \cap \text{Spt} \mu = \emptyset$  

  then $f_i'(r) = 2$.

**Preproof** Despite the length of the proof, geometrically this lemma and its proof are very simple. I will try and give an informal explanation first, since all the ideas are contained in the proof of the third part of the lemma I will only talk about this.
First it's helpful to consider ourselves looking down on $A(y)$, from some point like $\lambda e_i + y$, as on the fig. 7.1.

Suppose we have the condition

$$\left( R_{i+1,i+5}^{(y)} \cup R_{i+4,i+2}^{(y)} \right) \cap A(y, r-a, r+a) \cap Spt\mu = \emptyset$$

(2.25)

for some $a > 0$.

Now imagine we were able to pick a point $x$ exactly on the line $y+ < e_i + e_{i+1} + e_{i+2}$

as shown in fig 7.1.

Now Lemma 6 tell us the 'growth' in $S_k(y) \cap A(y)$ is approximately the measure on the 'x-boundary' of $(S_k(y) \cup S_{k+3}^{(y)}) \cap A(y)$ (See the remark following the proof of Lemma 6 if this makes no sense) for each $k \in \{i, i+1, i+2\}$.

Roughly speaking (I will make clear what I mean by this afterwards I just want to state the idea first) the idea is that for $k \in \{i+1, i+2\}$ the 'x-boundary' of $(S_k(y) \cup S_{k+3}^{(y)}) \cap A(y)$ is made up of a 'sideways' part, and an 'upwards-downwards' part.

(2.25) ensures that the 'sideways' part is empty (has none of the support of $\mu$ in it) so for $k \in \{i+1, i+2\}$ we just have 'upwards-downwards' part. And the union of the 'upwards-downwards' parts of the 'x-boundaries' of $(S_k(y) \cup S_{k+3}^{(y)}) \cap A(y)$ for $k \in \{i+1, i+2\}$ is equal to the 'x-boundary' of $(S_i^{(y)} \cup S_{i+3}^{(y)}) \cap A(y)$.

So invoking the principle stated in the remark following Lemma 6 we now know that the 'growth' in $S_{i+1}^{(y)} \cap A(y)$ plus the 'growth' in $S_{i+2}^{(y)} \cap A(y)$ is equal to the 'growth' in $S_i^{(y)} \cap A(y)$, which must therefore be half the total 'growth'. And this gives our estimate.

Now I will say the same thing in a more precise way. For $k \in \{i+1, i+2\}$ the $x$-boundary of $(S_k^{(y)} \cup S_{k+3}^{(y)}) \cap A(y)$ is the set

$$\left( (S_k^{(y)} \setminus S_k^{(x)}) \cup (S_{k+3}^{(y)} \setminus S_{k+3}^{(x)}) \right) \cap A(y)$$

Now because of the exact positioning of $x$ on the line $y+ < e_i + e_{i+1} + e_{i+2}$ we know that $S_{i+1}^{(y)} \setminus S_{i+1}^{(x)}$ is decomposed into an 'downwards' part $S_{i+1}^{(y)} \cap S_{i+3}^{(x)}$ (direction $e_i$ is taken to be upwards) and a 'sideways' part $S_{i+1}^{(y)} \cap S_{i+5}^{(x)}$. It might be helpful to look at the diagrams to see this, the 'sideways' part is in-between the two parallel diagonal lines on fig 7.1. And the 'downwards' part is in-between the two parallel diagonal lines on fig 7.2.
Similarly \( S^{(x)}_{i+4} \setminus S^{(y)}_{i+4} \) is decomposed into an 'upwards' part \( S^{(x)}_{i+4} \cap S^{(y)}_i \) and a 'sideways' part \( S^{(x)}_{i+4} \cap S^{(y)}_{i+2} \).

Now the main point is that if we have the first condition in the statement of Lemma 9, see equation (2.25), then we have that both the 'sideways' parts in the decomposition of \( S^{(y)}_{i+1} \setminus S^{(x)}_i \) and of \( S^{(x)}_{i+4} \setminus S^{(y)}_{i+4} \) are empty (contain no points of \( \text{Spt} \mu \)) in the annulus \( A(y, r - a, r + a) \). So given any \( h \in (0, a) \) as we know by Lemma 6 that the 'growth' in \( S^{(y)}_{i+1} \cap A(y, r, r + h) \) is approximately the measure on the 'x-boundary' of \( (S^{(y)}_{i+4} \cup S^{(y)}_i) \cap A(y, r, r + h) \). And we have just seen that the measure on the 'x-boundary' only lives on the 'upwards', 'downwards' part of it. And the 'upwards', 'downwards' parts of it are exactly the parts that intersect the 'x-boundary' of \( (S^{(y)}_i \cup S^{(y)}_{i+3}) \cap A(y, r, r + h) \).

In fact the union of the 'upwards', 'downwards' parts of the 'x-boundaries' of \( (S^{(y)}_{i+1} \cup S^{(y)}_{i+4}) \cap A(y, r, r + h) \) and \( (S^{(y)}_{i+2} \cup S^{(y)}_{i+5}) \cap A(y, r, r + h) \) is the 'x-boundary' of \( (S^{(y)}_i \cup S^{(y)}_{i+3}) \cap A(y, r, r + h) \).

Therefore the 'growth' in \( S^{(y)}_{i+1} \cap A(y, r, r + h) \) + the 'growth' in \( S^{(y)}_{i+2} \cap A(y, r, r + h) \) = 'growth' in \( S^{(y)}_i \cap A(y, r, r + h) \).

Since we have a uniform measure, the induced measure on \( \partial C_r(y) \) is \( 8r \) and on \( \partial C_{r+h}(y) \) is \( 8(r + h) \), and so the 'growth' over \( S^{(y)}_i, S^{(y)}_{i+1}, S^{(y)}_{i+2} \) between \( r \) and \( r + h \) must be \( 4h \), we must therefore have the growth over \( S^{(y)}_i \) is equal to \( 2h \).

Taking the limit as \( h \to 0 \) gives us the result.

Of course this all depends on our being able to pick this point \( x \) exactly on the diagonal. But since we have a purely unrectifiable set and \( y \) is a point with cone density from almost all directions we pick a point just below the desired \( x \) to get an
upper bound, and a point just above to get a lower bound and this is how the proof works.

**Proof** Let $\epsilon > 0$, $k \in \{1, 2, \ldots, 6\}$.

Since $r \in L(y)$ so $f_k$ is differentiable at $r$, let $\lambda_k = f'_k(r)$ we can find some $h \in (0, a)$ such that

$$f_k(p) \in (f_k(r) + (\lambda_k - \epsilon)(p - r), f_k(r) + (\lambda_k + \epsilon)(p - r))$$

for all $p \in (r, r + h)$ and for each $k \in \{i, i + 1, i + 2\}$.

Let $A(z) = A(z, r, r + h)$ for any $z \in \mathbb{R}^3$.

By Lemma 8 we have

$$\int_{A(y) \cap (S_k^{(y)} \cup S_{i+3}^{(y)})} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) d\mu z \in \left( 2(\lambda_k - \epsilon) h^4, 2(\lambda_k + \epsilon) h^4 \right).$$

(2.26)

Also note that for a.a. $s > 0$

$$f_i(s) + f_{i+1}(s) + f_{i+2}(s) = 4s.$$  

$$f'_i(r) + f'_{i+1}(r) + f'_{i+2}(r) = \sum_{k \in \{i, i+1, i+2\}} \lim_{h \to 0} \frac{f_k(r + h) - f_k(r)}{h} = 4.$$  

(2.27)

Let $\psi_0 = e_i + e_{i+1} + e_{i+2}$ and let $\psi = <\psi_0>$. Let $\delta > 0$, $\alpha > 0$ be small numbers whose size will be determined later.

**Step 1:** We will show that $f'_i(r) \leq 2$, for any $r \in L(y)$.

Now $y \in G$ and we would like to find $x \in X(y, \psi, \delta) \setminus S_i^{(y)} \cap C_\alpha(y) \cap G$ with the following two properties

- $(x - y) \cdot e_{i+2} > (x - y) \cdot e_i$
- $x \in S_{i+1}^{(y)}$

if we take $x \in X(y, \psi_0 - \frac{\delta}{2}e_i + \frac{\delta}{4}e_{i+1}, \frac{\delta}{2\alpha})$ for arbitrary large $n$, then either $x$ would satisfy these properties or it would be in an antipodal position and would satisfy the properties

- $(x - y) \cdot e_{i+5} > (x - y) \cdot e_{i+3}$
- $x \in S_{i+4}^{(y)}$
And if this latter case occurred we could use these properties to argue (in exactly the same way) the required relation for

$$\left\{ A(y) \cap S_{i+3}^{(y)} \cap \text{Spt} \mu, \; A(y) \cap S_{i+4}^{(y)} \cap \text{Spt} \mu, \; A(y) \cap S_{i+5}^{(y)} \cap \text{Spt} \mu \right\}$$

to prove $f_{i+3}^1(r) \leq 2$ and by measure symmetry this is the same result. So for simplicity we will assume $x$ has our original set of properties.

Let $\sigma(z) = h^2 - (r - \|z - y\|)^2$, then by Lemma 6 we know that for each $k \in \{i, i+1, i+2\}$

$$\left( \int_{A(y) \cap S_k^{(y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z + \int_{A(y) \cap S_{k+3}^{(y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z \right)$$

$$+ \int_{A(y) \cap (S_k^{(y)} \cup S_{k+3}^{(y)})} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) c_k \cdot (x - y) \, d\mu z \leq 4\sigma(z) (\|z - y\| - r) ^2 c_k \cdot (x - y) \, d\mu z \leq c_0 \|x - y\|^2 hr. \quad (2.28)$$

We will need to split up the 'x-boundary' into various components, see fig 7.3.

Let $C_{i+1,j} = \left( \left( C_{i+1,j}^{(y)} \cap S_j^{(y)} \right) \cap A(y) \right)$ for $j \in \{i, i + 1, i + 2, i + 3\}$.

Let $C_{i+1,j} = \left( \left( C_{i+1,j}^{(y)} \cap S_j^{(y)} \right) \cap A(y) \right)$ for $j \in \{i, i + 1, i + 2, i + 3\}$.

Let $C_{i+1,j} = \left( \left( C_{i+1,j}^{(y)} \cap S_j^{(y)} \right) \cap A(y) \right)$ for $j \in \{i, i + 1, i + 2, i + 3\}$.

Let $C_{i+1,j} = \left( \left( C_{i+1,j}^{(y)} \cap S_j^{(y)} \right) \cap A(y) \right)$ for $j \in \{i, i + 1, i + 2, i + 3\}$.

Now as by Lemma 5 (measure symmetry) we know that for each $k \in \{i, i + 1, i + 2\}$

$$\int_{A(y) \cap S_k^{(y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z + \int_{A(y) \cap S_{k+3}^{(y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z = 0. \quad (2.29)$$

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Since \( x \in S_{i+1}^{(y)} \) adding this together with (2.28) and writing the resulting inequality in our new notation we get

\[
\left| \int_{\{i,i+3,i+5\}} \left( C_{i+1}^{(y,x)} \cup C_{i+5}^{(x,y)} \right) 4\sigma(z) (\|z - y\| - r) d\mu z \\
+ \int_{A(g) \cap \left( S_{i+1}^{(y)} \cup S_{i+3}^{(y)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) d\mu z \right| \\
\leq c_9 \|x - y\|^2 hr.
\] (2.30)

Now from fig 7.4 and fig 7.3 (ignoring the +, - signs on fig. 7.3 which are only relevant in the next inequality) we can see that for \( k = i + 2 \) inequality (2.28) can be written as

\[
\left| \int_{\bigcup_{j \in \{i,i+3,i+4,i+5\}} D_{i+2,j}^{(y,x)} \left( \bigcup_{j \in \{i,i+3,i+4,i+5\}} D_{i+2,j}^{(x,y)} \right) 4\sigma(z) (\|z - y\| - r) d\mu z \\
- \int_{C_{i+1,i+3}^{(y,x)} \cup C_{i+4,i+5}^{(x,y)}} 4\sigma(z) (\|z - y\| - r) d\mu z \\
+ \int_{A(g) \cap \left( S_{i+2}^{(y)} \cup S_{i+5}^{(y)} \right)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) d\mu z \right| \\
\leq c_9 \|x - y\|^2 hr.
\] (2.31)

Adding these two together we get
\[ \int \left( \bigcup_{j \in \{i, i+3, i+5\}} C^{(y,x)}_{i+1,j} \right) \cup \left( \bigcup_{j \in \{i, i+2, i+3\}} C^{(x,y)}_{i+4,j} \right) \ 4\sigma (z) (||z - y|| - r) \, d\mu z \]

\[ + \int \left( \bigcup_{j \in \{i, i+3, i+4\}} D^{(y,x)}_{i+1,j} \right) \cup \left( \bigcup_{j \in \{i, i+3, i+1\}} D^{(x,y)}_{i+5,j} \right) \ 4\sigma (z) (||z - y|| - r) \, d\mu z \]

\[ + \int_{A(y) \cap \left( S^{(y)}_{i+1} \cup S^{(y)}_{i+4} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i+1} \cdot (x - y) \, d\mu z \]

\[ + \int_{A(y) \cap \left( S^{(y)}_{i+2} \cup S^{(y)}_{i+5} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i+2} \cdot (x - y) \, d\mu z \]

\[ \leq 2c_9 ||x - y||^2 hr. \quad (2.32) \]

Now from fig 7.3 (now noting the +,- signs which are relevant) we can see that the

inequality (2.28) with \( k = i \) gives

\[ \int_{C^{(x,y)}_{i+4,1} \cup D^{(x,y)}_{i+3,1} \cup D^{(x,y)}_{i+4,1}} \ 4\sigma (z) (||z - y|| - r) \, d\mu z \]

\[ - \int_{C^{(x,y)}_{i+4,1} \cup D^{(x,y)}_{i+3,1} \cup D^{(x,y)}_{i+4,1}} \ 4\sigma (z) (||z - y|| - r) \, d\mu z \]

\[ + \int_{A(y) \cap \left( S^{(y)}_{i+1} \cup S^{(y)}_{i+4} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i} \cdot (x - y) \, d\mu z \]

\[ \leq c_9 ||x - y||^2 hr. \quad (2.33) \]

Now from inequality (2.32) and inequality (2.33) we have that

\[ \int_{A(y) \cap \left( S^{(y)}_{i+1} \cup S^{(y)}_{i+5} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i+1} \cdot (x - y) \, d\mu z \]

\[ + \int_{A(y) \cap \left( S^{(y)}_{i+2} \cup S^{(y)}_{i+4} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i+2} \cdot (x - y) \, d\mu z \]

\[ \geq \int_{C^{(x,y)}_{i+4,1} \cup D^{(x,y)}_{i+3,1} \cup D^{(x,y)}_{i+4,1}} \ 4\sigma (z) (||z - y|| - r) \, d\mu z \]

\[ + \int_{D^{(x,y)}_{i+3,1} \cup D^{(x,y)}_{i+4,1}} \ 4\sigma (z) (||z - y|| - r) \, d\mu z - 2c_9 ||x - y||^2 hr \]

\[ \geq \int_{A(y) \cap \left( S^{(y)}_{i+1} \cup S^{(y)}_{i+5} \right)} \ 4 \left( h^2 - 3 (||z - y|| - r)^2 \right) e_{i} \cdot (x - y) \, d\mu z \]

\[ + \int_{D^{(x,y)}_{i+3,1} \cup D^{(x,y)}_{i+4,1}} \ 4\sigma (z) (||z - y|| - r) \, d\mu z - 3c_9 ||x - y||^2 hr. \quad (2.34) \]
So immediately using (2.26) we get that
\[ 2 (\lambda_{i+1} + \epsilon) h^4 e_{i+1} \cdot (x - y) + 2 (\lambda_{i+2} + \epsilon) h^4 e_{i+2} \cdot (x - y) \]
\[ \geq 2 (\lambda_i - \epsilon) h^4 e_i \cdot (x - y) - 3c_9 \|x - y\|^2 hr. \]

Let \( a_k = \frac{\epsilon}{\|x - y\|} \) then we know that by choice of \( x \) that \( |a_{k1} - a_{k2}| \leq \delta \) for any \( k_1, k_2 \in \{i, i + 1, i + 2\} \).

So we have
\[ 2 (a_i + \delta) (\lambda_{i+1} + \epsilon) h^4 + 2 (a_i + \delta) (\lambda_{i+2} + \epsilon) h^4 \]
\[ = 2 \delta (\lambda_{i+1} + \epsilon) h^4 + 2 \delta (\lambda_{i+2} + \epsilon) h^4 \]
\[ + 2a_i (\lambda_{i+1} + \epsilon) h^4 + 2a_i (\lambda_{i+2} + \epsilon) h^4 \]
\[ \geq 2 (\lambda_i - \epsilon) h^4 a_i - 3c_9 \|x - y\|^2 hr. \]

Letting \( x \rightarrow y \) we have
\[ 2 \delta (\lambda_{i+1} + \lambda_{i+2}) h^4 + 4 \delta \epsilon h^4 + 4a_i \epsilon h^4 + 2a_i (\lambda_{i+1} + \lambda_{i+2}) h^4 \]
\[ \geq 2a_i \lambda_i h^4. \]

Now as \( \epsilon \) and \( \delta \) can be made arbitrarily small we have that
\[ \lambda_{i+1} + \lambda_{i+2} \geq \lambda_i. \]

And so by (2.27) we have that \( \lambda_i \leq 2 \). Using this we can argue as follows:

Note that for any \( z \in G, k \in \{1, 2, \ldots 6\} \) we have
\[ \mu_p \left( \partial C_p (z) \cap S_k^{(z)} \right) = f_k^{(z)} (p) = \int_0^p f_k^{(z)} (q) dLq \leq 2p. \]

So we have
\[ \mu \left( C_p (z) \cap S_k^{(z)} \right) = \int_0^p f_k^{(z)} (q) dLq \leq p^2. \]

So if we have \( j_1, j_2 \in \{i + 1, i + 2, i + 4, i + 5\} \) such that \( e_{j_1} \cdot e_{j_2} = 0 \) then let \( i \in \{1, 2, \ldots 6\} \) such that \( e_i < e_{j_1}, e_{j_2} >^\perp \).

So by measure symmetry, Lemma 4 we have that
\[ \mu \left( \left( S_i^{(z)} \cup S_{j_1}^{(z)} \cup S_{j_2}^{(z)} \right) \cap C_\alpha (z) \right) = \frac{\mu \left( C_\alpha (z) \right)}{2} = 2 \alpha^2. \]

So
\[ \mu \left( \left( S_{j_1}^{(z)} \cup S_{j_2}^{(z)} \right) \cap C_\alpha (z) \right) = \mu \left( \left( S_i^{(z)} \cup S_{j_1}^{(z)} \cup S_{j_2}^{(z)} \right) \cap C_\alpha (z) \right) - \mu \left( S_i^{(z)} \cap C_\alpha (z) \right) \]
\[ \geq 2 \alpha^2 - \alpha^2 \]
\[ = \alpha^2. \]

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Now by pure unrectifiability we have that for any $z \in \text{Spt} \mu \cap C_1(0)$ there exists a sequence $z_n \in G$, $z_n \to z$. For any $j_1, j_2 \in \{i + 1, i + 2, i + 4, i + 5\}$ such that $e_{j_1} e_{j_2} = 0$ we have

$$\mu \left( \left( S^{(z)}_{j_1} \cup S^{(z)}_{j_2} \right) \cap C_{\alpha}(z) \right) \geq \alpha^2$$

for all $z \in \text{Spt} \mu \cap C_1(0)$, $\alpha \in (0, 1)$, and this completes the proof of part 1.

**Step 2:** Now we will prove the second part of the lemma. For convenience, in order to make the equations we established in Step 1 works for us we will assume the we have $s > 0$ such that

$$\left( R^{(y)}_{i+1, i+5} \cup R^{(y)}_{i+4, i+2} \right) \cap \partial C_t(y) \cap \text{Spt} \mu \neq \emptyset$$

for all $t \in (r - s, r + s)$.

If we did not have this condition but had instead the other one we would simply have to chose an $x$ in the direction $e_t + e_{i+5} + e_{i+1}$ with the same properties as the one we chose in Step 1.

We'll assume that $h$, the width of our annuls (recall $A(y) = A(y, r, r + h)$) is less than $s$.

The idea is obviously to use (2.34) to show there is some number $c_{10} > 0$ such that

$$\lambda_{i+1} h^4 + \lambda_{i+2} h^4 > \lambda_i h^4 + c_{10} h^4$$

to do this we need to show

$$\int_{D^{(x,y)}_{i+5, i+1+1} \cup D^{(y,x)}_{i+2, i+4}} 4\sigma(z) \left( ||z - y|| - r \right) d\mu z \geq c_{10} h^4 ||x - y||.$$
So

\[
\int_{D(x, y) \cup D(x, z)} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
\geq \int_{(D(x, y) \cup D(x, z)) \cap A(y, r + \frac{h}{4}, r + \frac{3h}{4})} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
\geq 4 \left( h^2 - \left( \frac{3h}{4} \right)^2 \right) \frac{h}{4} \mu \left( (D(x, y) \cup D(x, z)) \cap (y, r + \frac{h}{4}, r + \frac{3h}{4}) \right) \\
\geq \frac{7}{16} h^3 (n - 1) 4 \left( \frac{\|x - y\|}{2} \right)^2 \\
\geq \frac{7}{32} h^3 n \|x - y\|^2.
\]

And note that \( \frac{7}{8} n \|x - y\| \geq \frac{h}{2} \) so

\[
\int_{D(x, y) \cup D(x, z)} 4\sigma(z) (\|z - y\| - r) \, d\mu z \geq \frac{h^4}{32} \|x - y\|.
\]

So using this with (2.34) we get

\[
\int_{A(y) \cap (s_{i+1}^y \cup s_{i+4}^y)} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) \, d\mu z \\
+ \int_{A(y) \cap (s_{i+1}^y \cup s_{i+4}^y)} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) \, d\mu z \\
\geq \int_{A(y) \cap (s_{i+1}^y \cup s_{i+4}^y)} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_i \cdot (x - y) \, d\mu z \\
+ \frac{h^4}{32} \|x - y\| - 3c_9 \|x - y\|^2 hr.
\]

So using (2.26) as before we turn this into a statement about derivatives

\[
2 (\lambda_{i+1} + \epsilon) h^4 e_{i+1} \cdot (x - y) + 2 (\lambda_{i+2} + \epsilon) h^4 e_{i+2} \cdot (x - y) \\
\geq 2 (\lambda_i - \epsilon) h^4 e_i \cdot (x - y) + \frac{h^4}{32} e_{i+2} \cdot (x - y) - 3c_9 \|x - y\|^2 hr.
\]

As before let \( a_k = \frac{e_k(x - y)}{\|x - y\|} \) so by choice of \( x \) we have \( |a_{k_1} - a_{k_2}| \leq \delta, \) so

\[
2 (\lambda_{i+1} + \epsilon) (a_i + \delta) h^4 + 2 (\lambda_{i+2} + \epsilon) (a_i + \delta) h^4 \\
\geq 2a_i (\lambda_i - \epsilon) h^4 + \frac{h^4}{16} a_i - 3c_9 \|x - y\|^2 hr
\]

as \( \delta \) and \( \epsilon \) and \( \|x - y\| \) can be made as small as we like dividing through by \( a_i h^4 \) gives

\[
\lambda_{i+1} + \lambda_{i+2} \geq \lambda_i + \frac{1}{32}
\]

thus establishing the second part of the lemma.
Step 3: We will show if \( r \in L^{(y)} \) for some \( a > 0 \)

\[
\left( R_{i+1,i+2}^{(y)} \cup R_{i+4,i+5}^{(y)} \right) \cap A(y, r - a, r + a) \cap \text{Spt} \mu = \emptyset \tag{2.35}
\]
or

\[
\left( R_{i+1,i+5}^{(y)} \cup R_{i+4,i+2}^{(y)} \right) \cap A(y, r - a, r + a) \cap \text{Spt} \mu = \emptyset \tag{2.36}
\]

then \( f_i'(r) = 2 \).

We will assume the later, (2.36), and argue \( f_i'(r) = 2 \) from it, in the latter case the argument is exactly the same.

Firstly as before since \( y \in G \) we can find \( x \in X(y, \psi, \delta) \cap C_\alpha(y) \cap G \), which we will assume to have the following properties:

- \( x \in S_i^{(y)} \)
- \( (x - y) \cdot e_{i+1} \geq (x - y) \cdot e_{i+2} \)

The reason we can assume this is that as before if we didn’t have these properties then \( x \) would be in an antipodal position to the one we want and instead we would have the properties:

- \( x \in S_{i+3}^{(y)} \)
- \( (x - y) \cdot e_{i+4} \geq (x - y) \cdot e_{i+5} \)

and by measure symmetry these would be just as good for our purposes, so we will assume we have \( x \) with our original set of properties.

Now let \( E_{i,j}^{(y,x)} = \left( \left( S_i^{(y)} \setminus S_i^{(x)} \right) \cap S_j^{(x)} \right) \cap A(y) \) for \( j \in \{i + 1, i + 2, i + 4, i + 5\} \).

And \( E_{i+3,j}^{(x,y)} = \left( \left( S_{i+3}^{(x)} \setminus S_{i+3}^{(y)} \right) \cap S_j^{(y)} \right) \cap A(y) \) for \( j \in \{i + 1, i + 2, i + 4, i + 5\} \).
Now from fig 7.5 we see that since \( x \in S_i \) we have \( C_i^{(y,x)} = \emptyset \) and \( C_i^{(x,y)} = \emptyset \) and for the same reason \( D_i^{(y,x)} = \emptyset \) and \( D_i^{(x,y)} = \emptyset \). From fig 7.6 we see that \( C_{i+1,i+5} \cap Spt\mu = D_i^{(x,y)} \cap Spt\mu = \emptyset \) and \( C_{i+2,i+4} \cap Spt\mu = D_i^{(y,x)} \cap Spt\mu = \emptyset \) by assumption (2.36).

So from fig 7.5 and fig 7.6 (ignoring the +,- signs in fig 7.6 which are only relevant in the next inequality) and by using measure symmetry in the same way as we did in inequality (2.30), writing out the statement of Lemma 6 (for \( k = i + 1 \)) in our new notation gives

\[
\left| \int_{C_i^{(y,x)} \cup C_i^{(x,y)} \cup C_{i+1}^{(y,x)} \cup C_{i+1}^{(x,y)} \cup C_{i+2,i+4}} 4\sigma (z) (\|z - y\| - r) d\mu z \\
- \int_{E_i^{(y,x)} \cup E_i^{(x,y)} \cup E_{i+3,i+4}} 4\sigma (z) (\|z - y\| - r) d\mu z \\
+ \int_{A(y) \cap (S_i^{(y,x)} \cup S_{i+4}^{(y,x)})} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) d\mu z \right| \\
\leq c_g \|x - y\|^2 \tau h. \\
(2.37)
\]

And in the same way we can see that from fig 7.6 and fig 7.7 (now noting the +,- signs in both cases) the statement of Lemma 6 (for \( k = i + 2 \)) becomes

\[
\left| \int_{D_i^{(y,x)} \cup D_i^{(x,y)} \cup D_{i+2,i+3}} 4\sigma (z) (\|z - y\| - r) d\mu z - \int_{C_i^{(y,x)} \cup C_i^{(x,y)} \cup C_{i+1,i+2} \cup C_{i+4,i+5}} 4\sigma (z) (\|z - y\| - r) d\mu z \\
- \int_{E_i^{(y,x)} \cup E_i^{(x,y)} \cup E_{i+3,i+5}} 4\sigma (z) (\|z - y\| - r) d\mu z \\
+ \int_{A(y) \cap (S_i^{(y,x)} \cup S_{i+4}^{(y,x)})} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) d\mu z \right| \\
\leq c_g \|x - y\|^2 \tau h. \\
(2.38)
\]
Now adding (2.37) and (2.38) together we get

\[ \int_{C(y,x)}^{C(x,y)} 4\sigma(z) (\|z - y\| - r) \, d\mu z + \int_{D(y,x)}^{D(x,y)} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
- \int_{E(y,x)}^{E(x,y)} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
+ \int_{A(y)n(S(y,x))}^{A(x,y)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) \, d\mu z \\
+ \int_{A(y)n(S(y,x))}^{A(x,y)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) \, d\mu z \\
\leq 2c_9 \|x - y\|^2 r h. \] (2.39)

Now from fig 7.5 we see that

\[ C_{i+1,i+3} = E_{i+3,i+1} \text{ and } C_{i+4,i} = E_{i+4,i} \]

And from fig 7.7 we see that

\[ D_{i+2,i+3} = E_{i+3,i+2} \text{ and } D_{i+5,i} = E_{i+5,i} \]

So from (2.39) we have that

\[ \int_{\bigcup_{j\in\{i+1,i+2,i+4,i+5\}} E_{i,j}^{(s,y)}}^{E_{i,j}^{(x,y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z + 2c_9 \|x - y\|^2 r h \\
= \int_{E_{i+3,i+1}^{(x,y)}}^{E_{i+3,i+1}^{(x,y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z + 2c_9 \|x - y\|^2 r h \\
- \int_{E_{i+3,i+5}^{(x,y)}}^{E_{i+3,i+5}^{(x,y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
+ \int_{A(y)n(S(y,x))}^{A(x,y)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) \, d\mu z \\
+ \int_{A(y)n(S(y,x))}^{A(x,y)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) \, d\mu z \] (2.40)

And of course we know that the statement of Lemma 6 (for \( k = i \)) can be written in the form

\[ \left| \int_{\bigcup_{j\in\{i+1,i+2,i+4,i+5\}} E_{i,j}^{(s,y)}}^{E_{i,j}^{(x,y)}} 4\sigma(z) (\|z - y\| - r) \, d\mu z \\
+ \int_{A(y)n(S(y,x))}^{A(x,y)} 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_i \cdot (x - y) \, d\mu z \right| \leq c_9 \|x - y\|^2 r h. \]
So

\[
\int_{A(y)}((s^{(y)}_{i} \cup s^{(y)}_{i+3}) - 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_i \cdot (x - y) d\mu z + c_9 \|x - y\|^2 hr \\
\geq \int_{\cup_{j \in \{i+1, i+2, i+3, i+4\}} E_{i_j}^{(y,x)} \cup E_{i,j+3}^{(x,y)}} 4\sigma (z) (\|z - y\| - r) d\mu z.
\]

So putting this together with (2.40) we have

\[
\int_{A(y)}((s^{(y)}_{i} \cup s^{(y)}_{i+3}) - 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_i \cdot (x - y) d\mu z + 3c_9 \|x - y\|^2 hr \\
\geq \int_{A(y)}((s^{(y)}_{i+1} \cup s^{(y)}_{i+4}) - 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+1} \cdot (x - y) d\mu z \\
+ \int_{A(y)}((s^{(y)}_{i+2} \cup s^{(y)}_{i+3}) - 4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) e_{i+2} \cdot (x - y) d\mu z. (2.41)
\]

Now as before we use (2.26) to turn this into a statement about derivatives of \( f_k \) at \( r \), so we have

\[
2 (\lambda_i + \epsilon) h^4 e_i \cdot (x - y) + 3c_9 \|x - y\|^2 hr \\
\geq 2 (\lambda_{i+1} - \epsilon) h^4 e_{i+1} \cdot (x - y) + 2 (\lambda_{i+2} - \epsilon) h^4 e_{i+2} \cdot (x - y). (2.42)
\]

So

\[
2 (\lambda_i + \epsilon) h^4 \frac{e_i \cdot (x - y)}{\|x - y\|} + 3c_9 \|x - y\| hr \\
\geq 2 (\lambda_{i+1} - \epsilon) h^4 \frac{e_{i+1} \cdot (x - y)}{\|x - y\|} + 2 (\lambda_{i+2} - \epsilon) h^4 \frac{e_{i+2} \cdot (x - y)}{\|x - y\|}.
\]

And as before we turn this into a statement about \( a_k = \frac{e_k \cdot (x - y)}{\|x - y\|} \) remembering that 

\[
|a_{k_1} - a_{k_2}| \leq \delta \text{ for any } k_1, k_2 \in \{i, i+1, i+2\}
\]

\[
2 (\lambda_i + \epsilon) h^4 a_i + 3c_9 \|x - y\| hr \\
\geq 2 (\lambda_{i+1} - \epsilon) h^4 (a_i - \delta) + 2 (\lambda_{i+2} - \epsilon) h^4 (a_i - \delta).
\]

\[
2 (\lambda_i + \epsilon) h^4 a_i + 3c_9 \|x - y\| hr \\
\geq 2 (\lambda_{i+1} - \epsilon) h^4 a_i + 2 (\lambda_{i+2} - \epsilon) h^4 a_i - 2\delta (\lambda_{i+1} - \epsilon) h^4 - 2\delta (\lambda_{i+2} - \epsilon) h^4.
\]

Letting \( x \to y \) and noting that \( \epsilon \) and \( \delta \) can be as small as we like we have

\[
\lambda_i \geq \lambda_{i+1} + \lambda_{i+2}.
\]

So as before by (2.27) we have that \( \lambda_i \geq 2 \) thus establishing the Lemma. □
2.4 Measure can’t stop itself crawling up a boundary

Remark This coming Lemma is crucial. It says we can not have points of the support of the measure going up a side of $S_i^{(x)}$ (in fig 9.0 side $R_{i,i+1}^{(y)}$) which suddenly stop at a point (say $z$) away from the edge of the side of $S_i^{(x)}$. This is fairly straight forward, on fig 9.0 we have a point $z$ away from the edge of $R_{i,i+1}^{(y)}$ with nothing above it, so using pure unrectifiability its not hard to see we can find a chain of points of Spt$\mu$ growing up the outside of $R_{i,i+1}^{(y)}$ which can be chosen as close to the side as we like. By closure of Spt$\mu$ this gives a contradiction.

\[ \text{fig 9.0} \]

The second part of the lemma is for the case where we know there are no points of the support of the measure on two touching sides in some annulus $A(y,a,b)$, \([R_{i,i+1}^{(y)} \text{ and } R_{i,i+2}^{(y)} \text{ for instance})\) then as we go down from a until we find a point of Spt$\mu$ in the two sides (i.e. find $\epsilon > 0$ so that $\partial C_{\alpha-\epsilon} \cap \text{int}(R_{i,i+1}^{(y)} \cup R_{i,i+2}^{(y)}) \cap \text{Spt}\mu \neq \emptyset$) we have that this point can’t be in the corner where the two sides meet.

The reason for this is as before, that we would then have a large expanse of nothing above our point, and so could find a chain of points crawling up the outside of $S_i^{(y)}$ gaining again a contradiction.

An immediate consequences of this is that if we have points of Spt$\mu$ on the boundary of $S_i^{(y)}$ they don’t stop crawling up the boundary of $S_i^{(y)}$, so there are always points on
the boundary of $S_{i}^{(y)}$. This is great for us because as I stated in the remark following Lemma 6, measure on the 'x-boundary' of $\left(S_{i}^{(y)} \cup S_{i+3}^{(y)}\right) \cap A (y, a, b)$ implies 'growth' in $S_{i}^{(y)} \cap A (y, a, b)$. So we always have growth! And so we always know that there is allot of measure in our segments $S_{j}^{(y)}$ for $j \in \{1, 2, \ldots , 6\}$. 
Lemma 10 Let \( \mu \) be a 2-uniform measure where \( \text{Spt}_2 \cap C_2(0) \) is purely unrectifiable. For any \( y \in \text{Spt}_2 \cap C_1(0), i \in \{1, 2, \ldots, 6\} \) if we have an interval \( (a, b) \subset (0, 1) \) for which \( R_{i,j}^{(y)} \cap A(y, a, b) \cap \text{Spt}_2 = \emptyset \) for some \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) then

\[
\partial C_a(y) \cap R_{i,j}^{(y)} \cap \text{Spt}_2 \subset \partial R_{i,j}^{(y)}.
\]

Also if we have interval \( (e, d) \subset (0, 1) \) such that for some \( j_1, j_2 \subset \{i + 1, i + 2, i + 4, i + 5\} \) with \( e_{j_1} \cdot e_{j_2} = 0 \) we have \( \int \left( R_{i,j_1}^{(y)} \cup R_{i,j_2}^{(y)} \right) \cap A(y, e, d) \cap \text{Spt}_2 = \emptyset \), then

\[
\partial C_e(y) \cap \left( R_{i,j_1}^{(y)} \cap R_{i,j_2}^{(y)} \right) \cap \text{Spt}_2 = \emptyset.
\]

Proof We will argue the first part of the lemma in detail, the second part can be argued in exactly the same way.

Suppose we can find interval \( (a, b) \subset (0, 1) \) for which \( R_{i,j}^{(y)} \cap A(y, a, b) \cap \text{Spt}_2 = \emptyset \) for some \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) and we can find some \( x \in \partial C_a(y) \cap R_{i,j}^{(y)} \cap \text{Spt}_2 \) such that \( d(x, \partial R_{i,j}^{(y)}) > 0 \).

Let \( d_0 = d(x, \partial R_{i,j}^{(y)}) \), let \( d = \min \left\{ \frac{d_0}{100}, \frac{b-a}{100} \right\} \).

We will define a function \( c \) that gives the distance of \( \text{Spt}_2 \) from \( R_{i,j}^{(y)} \cap \partial C_r(y) \) for \( r \in (a, b) \), how \( c \) works is shown in fig 9.1, and a formal definition follows.

\[
c(s) = d \left( \text{Spt}_2 \cap \left( y + (s + a) e_i + e_i^1 \right), R_{i,j}^{(y)} \cap \partial C_{a+s}(y) \right).
\]

So function \( c : (0, d) \rightarrow \mathbb{R}_+ \) is defined by

\[
c(s) = d \left( \text{Spt}_2 \cap \left( y + (s + a) e_i + e_i^1 \right), R_{i,j}^{(y)} \cap \partial C_{a+s}(y) \right).
\]
Now by closure of $\text{Spt}_\mu$ we have $c(s) > 0$ for every $s \in (0, d)$.

Now we have two possible cases, either $\mu \left( S_j^{(x)} \cap C_\delta (x) \right) = 0$ for some $\delta > 0$, or we can find some sequence $\delta_n \to 0$ such that $\mu \left( S_j^{(x)} \cap C_{\delta_n} (x) \right) > 0$. We will deal with the former case first because from it, it should be clear how to argue the latter.

Fig 9.2 shows how the argument goes.

So $\mu \left( S_j^{(x)} \cap C_\delta (x) \right) = 0$ for some $\delta > 0$ by Lemma 9 we have that $\mu \left( S_{i+3}^{(x)} \cap C_\delta (x) \right) > 0$ for every $\alpha \in (0, \delta)$ so we can find a point $z \in S_{i+3}^{(x)} \cap C_\delta (x)$ as close to $x$ as we like. We know by Lemma 7 that $f_j^{(z)} (p) = \mu_p \left( \partial C_p (z) \cap S_j^{(z)} \right) > 0$ for all $p > 0$. Let $n \in \mathbb{N}$, pick points $z_m \in \partial C_{n \delta} (z) \cap S_j^{(z)} \cap \text{Spt}_\mu$ for $m = 1, 2, \ldots \lceil \frac{n}{2} \rceil$. As indicated in fig 9.2. These points obviously lie outside of $S_j^{(x)} \cap C_\delta (x)$, but they can be chosen as close to the boundary as we like. So letting $n \to \infty$ and letting $x \to z$, by closure of $\text{Spt}_\mu$ we find that

$$\partial C_p (x) \cap \partial S_j^{(x)} \cap \text{Spt}_\mu \neq \emptyset$$

for all $p \in \left( 0, \frac{\delta}{2} \right)$.

Now the worst case for our purposes occurs when

$$\partial S_j^{(x)} \cap C_{\frac{\delta}{2}} (x) \cap \text{Spt}_\mu \subset R_j^{(x)}.$$

But even this does not present a problem (see fig 9.3 over-page) because for any $\epsilon_1 \in \left( 0, \frac{\delta}{2} \right)$ we can pick a point $z_0 \in R_j^{(x)} \cap C_{\frac{\epsilon_1}{2}} (x)$, now as $S_j^{(z_0)} \subset S_j^{(x)}$, so for any small $\epsilon_2 \in (0, \frac{\epsilon_1}{2})$ we have $\mu \left( S_j^{(z_0)} \cap C_{\epsilon_2} (z_0) \right) = 0$, as shown in fig 9.3. So by Lemma 9 we must have $\mu \left( S_j^{(z_0)} \cap C_{\epsilon_2} (z_0) \right) > \epsilon_2^2$. 

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So by pure unrectifiability we can pick some \( z_1 \in S_i^{(z_0)} \cap C_{\varepsilon_2} (z_0) \cap G \) for which

\[
d \left( z_1, \langle R_{j+3}^{(y)} \rangle \right) = t > 0.
\]

So note that

\[
\left( S_i^{(z_1)} \cap S_i^{(y)} \right) \subset C_{a+t} (y) ^c,
\]

and also note there must exist \( \kappa > 0 \) such that

\[
c (s) \geq \kappa \quad \forall \ s \in \left[ \frac{t}{2}, d \right),
\]

we'll assume \( \kappa \leq \frac{t}{4} \).

Now again by Lemma 7 we can iteratively find a sequence of points

\[
z_{n+1} \in A \left( z_n, \frac{\kappa}{16} \frac{\kappa}{8} \right) \cap S_i^{(z_n)} \cap G
\]

for \( n = 1, 2, \ldots \)

Of course we know that \( z_n \in S_i^{(z_1)} \) for all \( n \in \mathbb{N} \), and I claim that \( z_n \not\in S_i^{(y)} \) for any \( n \in \mathbb{N} \) such that \( z_n \in A (y, a, a + d) \).
Suppose not then, as $z_1 \not\in S_i^{(y)}$ we can find a $z_k$ which is the first to enter $S_i^{(y)}$, so $z_{k-1} \not\in S_i^{(y)}$. Now we know that $\|z_{k-1} - z_k\| \leq \frac{5}{6}$. So since $z_k \in S_i^{(y)} \cap S_i^{(z)}$ so we have that $\|z_k - y\| \geq a + t$ so $\|z_{k-1} - y\| \geq \|z_k - y\| - \|z_k - z_{k+1}\| \geq a + t - \frac{t}{2} = a + \frac{t}{2}$.

So there is some $s_1 \in \left[\frac{1}{2}, d\right)$ such that $z_{k-1} \in \text{Spt}_\mu \cap \left( y + (s_1 + a) e_i + e_i^\perp \right)$, yet $c(s_1) \leq d \left( z_{k-1}, R_{i,j}^{(y)} \cap \partial C_{a+s_1} \right) \leq d(z_{k-1}, z_k) \leq \frac{5}{6}$, contradiction.

So the claim is demonstrated, and we have a chain of points of $\text{Spt}_\mu$ crawling up the outside of $S_i^{(y)} \cap A(y, a, a + d)$. As this chain can be chosen to be as close to the sides of $S_i^{(y)} \cap A(y, a, a + d)$ as we want, so by closure of $\text{Spt}_\mu$ we have a contradiction.

Thus we can find no such $x \in \partial C_a(y) \cap R_{i,j}^{(y)} \cap \text{Spt}_\mu$ and so $\partial C_a(y) \cap R_{i,j}^{(y)} \cap \text{Spt}_\mu \subset \partial C_a(y) \cap \partial R_{i,j}^{(y)}$ and this proves the first part of the lemma.

The second part is if anything even easier.

Suppose we had an interval $(e, d) \subset \mathbb{R}$ such that for some $j_1, j_2 \in \{i + 1, i + 2, i + 4, i + 5\}$ where $e_{j_1} \cdot e_{j_2} = 0$ we have

$$\text{int} \left( R^{(y)}_{i,j_1} \cup R^{(y)}_{i,j_2} \right) \cap A(y, e, d) \cap \text{Spt}_\mu = \emptyset$$

but

$$\partial C_\epsilon(y) \cap \left( R^{(y)}_{i,j_1} \cup R^{(y)}_{i,j_2} \right) \cap \text{Spt}_\mu \neq \emptyset.$$ 

Let $i \in \{1, 2, \ldots, 6\}$ such that $e_i \leq e_{j_1}, e_{j_2}$ and let

$$z = \partial C_\epsilon(y) \cap \left( R^{(y)}_{i,j_1} \cup R^{(y)}_{i,j_2} \right) \cap \text{Spt}_\mu.$$

By Lemma 9 we know that

$$\mu \left( C_a(z) \cap \left( S_j^{(z)} \cup S_j^{(z)} \right) \right) \geq \alpha^2.$$
for all $\alpha > 0$.

So using pure unrectifiability we know that we can find some $z_1 \in C_\alpha (z) \cap (S_{j_1}^{(z)} \cup S_{j_2}^{(z)}) \cap G$ such that $d \left( z_1, \partial S_{i_1}^{(z)} \right) > 0$.

Without loss of generality assume $z_1 \in S_{j_1}^{(z)}$ as show on fig 9.35. Now as before using pure unrectifiability we can find a chain of points $z_n \in S_{i_1}^{(z_1)}$ crawling up the outside of $\partial S_{i_1}^{(z)} \cap A (y, e, d)$ and again by closure of $\text{Spt} \mu$ we have a contradiction. So this establishes the second part of the lemma. □.

2.4.1 Application: Always growing in every direction

Lemma 11 Let $\mu$ be a 2-uniform measure where $C_2 (0) \cap \text{Spt} \mu$ is purely unrectifiable set. For any $y \in G \cap C_1 (0), i \in \{1, 2, \ldots, 6\}$, $s \in L^{(y)} \cap (0, 1)$ we have that

$$f_i^{(y)} (s) \geq \frac{7}{960}.$$ And so for any $x \in \text{Spt} \mu \cap C_1 (0)$ for any $\alpha \in (0, 1), i \in \{1, 2, \ldots, 6\}$ we have

$$\mu \left( S_i^{(x)} \cap C_\alpha (x) \right) \geq \frac{7\alpha^2}{6160}.$$ 

Proof Step 1:

First we will show that for every $y \in G \cap C_1 (0), s \in (0, 1), i \in \{1, 2, \ldots, 6\}$

$$\partial C_s (y) \cap \partial S_i^{(y)} \cap \text{Spt} \mu \neq \emptyset.$$ 

Suppose not and we can find $y \in G \cap C_1 (0), i \in \{1, 2, \ldots, 6\}$ and $s > 0$ such that

$$\partial C_s (y) \cap \partial S_i^{(y)} \cap \text{Spt} \mu = \emptyset.$$ 

Let

$$a = \sup \left\{ e \in [0, s) : C_e (y) \cap \partial S_i^{(y)} \cap \text{Spt} \mu \neq \emptyset \right\}$$

by closure of $\text{Spt} \mu$ we know we can pick $z \in \partial C_a (y) \cap \partial S_i^{(y)} \cap \text{Spt} \mu$. So $z \in R_{i, j_1}^{(y)}$ for some $j_1 \in \{i + 1, i + 2, i + 4, i + 5\}$ and so we know by Lemma 10 that $z \in \partial R_{i, j_1}^{(y)} \cap \partial C_a (y)$ thus we can find $j_2 \in \{i + 1, i + 2, i + 4, i + 5\}$ where $e_{j_1} \cdot e_{j_2} = 0$ and $z \in R_{i, j_1}^{(y)} \cap R_{i, j_2}^{(y)} \cap \partial C_a (y)$.

However since

$$\text{int} \left( R_{i, j_1}^{(y)} \cup R_{i, j_2}^{(y)} \right) \cap A (y, a, s) \cap \text{Spt} \mu = \emptyset$$

by Lemma 10 we have a contradiction.

Step 2:
Step 1 established the existence of measure on the boundary of $S_i^{(y)}$, now we will invoke Lemma 6, which says that measure on the 'x-boundary' of \( (S_i^{(y)} \cup S_i^{(y)}) \cap A(y, s, s + h) \) is \( \approx \) 'growth' in $S_i^{(y)} \cap A(y, s, s + h)$.

So suppose we have $s \in L(y)$.

Let $\epsilon > 0$.

Take $h > 0$ such that

\[
 f_i^{(y)}(t) \in (\lambda_i - \epsilon) (t - s) + \frac{f_i^{(y)}(s)}{s} + (\lambda_i + \epsilon) (t - s) + f_i^{(y)}(s) \quad (2.43)
\]

for all $t \in (s, s + h)$.

Since $y \in G$ we can pick $x \in X(y, e_i, \epsilon) \cap G$, for reasons discussed before we can assume $x \in S_i^{(y)}$.

Recall $A(y) = A(y, r, r + h)$.

Now by Lemma 8 we have from (2.43) that

\[
 \int_{A(y) \cap (S_i^{(y)} \cup S_{i+3}^{(y)})} \frac{h^2 - 3 (\|z - y\| - r)^2}{d\mu} \in (2 \lambda_i - \epsilon) h^4, 2 (\lambda_i + \epsilon) h^4 \quad (2.44)
\]

Now since $x \in S_i^{(y)}$ as in the remark following Lemma 6, the statement of Lemma 6 can be written as

\[
 \left| \int_{A(y) \cap ((S_i^{(y)} \setminus S_i^{(x)}) \cup (S_{i+3}^{(y)} \setminus S_{i+3}^{(x)}))} (h^2 - (\|z - y\| - r)^2) (\|z - y\| - r) d\mu \right| \leq c_0 \|x - y\|^2 hr.
\]

So using (2.44) we get that

\[
 \int_{A(y) \cap (S_i^{(y)} \setminus S_i^{(x)})} (h^2 - (\|z - y\| - r)^2) (\|z - y\| - r) d\mu
\]

\[
 \quad + \int_{A(y) \cap (S_{i+3}^{(y)} \setminus S_{i+3}^{(x)})} (h^2 - (\|z - y\| - r)^2) (\|z - y\| - r) d\mu
\]

\[
 \leq 2 (\lambda_i + \epsilon) \|x - y\|^4 + c_0 \|x - y\|^2 hr. \quad (2.45)
\]

Now by the first part of this lemma we can pick points $z_s \in \partial C_s(x) \cap \partial S_i^{(x)} \cap \text{Spt}\mu$ for each $s \in (r, r + h)$, because we have chosen $x$ to be more or less straight above $y$ it makes no difference where on the sides of $\partial S_i^{(x)}$ these points are, but to make the diagram better we will assume that $z_s \in R_i^{(x)}$ for all $s \in (r, r + h)$. 

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So let $\alpha = \frac{||x-y||}{2}$ and note for every $s \in (r, r + h)$ we have that

$$C_\alpha (z_s) \cap \left( S_i^{(z_{i+3})} \cup S_i^{(z_{i+1})} \right) \subset S_i^{(y)} \setminus S_i^{(x)}.$$ 

Now by Theorem 2.1 [1] we can take a finite collection of points $s_k \in \left( r + \frac{h}{3}, r + \frac{2h}{3} \right)$ such that

$$\left\{ C_\alpha (z_{s_k}) \cap \left( S_i^{(z_{i+3})} \cup S_i^{(z_{i+1})} \right) : k = 1, 2, \ldots P \right\}$$

are disjoint, and $2P\alpha \geq \frac{h}{6}$.

Also we have

$$\bigcup_{k \in \{1,2,\ldots,P\}} C_\alpha (z_{s_k}) \cap \left( S_i^{(z_{i+3})} \cup S_i^{(z_{i+1})} \right) \subset \left( S_i^{(y)} \setminus S_i^{(x)} \right) \cap A \left( y, r + \frac{h}{4}, r + \frac{3h}{4} \right). \quad (2.46)$$

Using Lemma 9, from this we conclude

$$\mu \left( \left( S_i^{(y)} \setminus S_i^{(x)} \right) \cap A \left( y, r + \frac{h}{4}, r + \frac{3h}{4} \right) \right) \geq P\alpha^2 \geq \alpha \frac{h}{12}.$$
So using this

\[ \int_{A(y,r+r+h)\cap \left(S_i^{(y)}\setminus S_i^{(x)} \right)} \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu z \]

\[ \geq \int_{A(y,r+\frac{h}{4},r+\frac{3h}{8})\cap \left(S_i^{(y)}\setminus S_i^{(x)} \right)} \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu z \]

\[ \geq \left( h^2 - \left(\frac{3h}{4} \right)^2 \right) \frac{h}{4} \mu \left( \left(A(y,r+\frac{h}{4},r+\frac{3h}{4})\cap \left(S_i^{(y)}\setminus S_i^{(x)} \right) \right) \right) \]

\[ \geq \left( h^2 - \frac{9}{16} h^2 \right) \frac{h}{4} \alpha \frac{h}{12} \]

\[ = \left( \frac{7}{16} \frac{h^3}{48} \right) \frac{h \alpha}{12} \]

\[ \geq \frac{7h^4}{770} \alpha. \quad (2.47) \]

Now putting (2.45) and (2.47) together and remembering the definition of \( \alpha \) gives

\[ \frac{7h^4}{770} \|x - y\| \leq 2 (\lambda_i + \epsilon) \|x - y\| h^4 + c_0 \|x - y\|^2 h r. \]

Now dividing through by \( h^4 \|x - y\| \) gives

\[ \frac{7}{1540} (1 + \epsilon) \leq 2 (\lambda_i + \epsilon) + c_0 \|x - y\| \frac{r}{h^3}. \]

As \( \epsilon \) and \( \|x - y\| \) can be made as small as we want this gives the first part of the lemma.

**Step 3:**

The final part is of course obvious but I have included it rather than having to write it out a number of times in the rest of the proof. So as for any \( y \in G, i \in \{1, 2, \ldots, 6\} \) we have that

\[ f_i(t) = \int_{t=0}^{s} f_i'(s) \, dLs \geq \frac{7s}{3680} \]

and so

\[ \mu \left( S_i^{(y)} \cap C_\alpha(y) \right) \geq \int_{0}^{\alpha} f_i(t) \, dLt \geq \frac{7\alpha^2}{7360} . \]

Now since \( \text{Spt} \mu \cap C_1(0) \) is purely unrectifiable for any \( x \in \text{Spt} \mu \cap C_1(0) \) we can find a sequence \( \{y_n\} \in G \) where \( y_n \to x \) such that

\[ \mu \left( \left( S_i^{(x)} \cap C_\alpha(x) \right) \triangle \left( S_i^{(y_n)} \cap C_\alpha(y_n) \right) \right) \to 0 \]

as \( n \to \infty \), so this establishes the lemma. \( \square \)
2.5 Touching point arguments

First I will repeat some notation:

Let \( y \in \text{Spt} \mu \) and \( i \in \{1, 2, \ldots, 6\} \), \( j \in \{i + 1, i + 2, i + 4, i + 5\} \),
pick \( l \in \{i + 1, i + 2, i + 4, i + 5\} \setminus \{j, j + 3\} \).
Let \( S_{i,j}^{(y, \rho)} = \{z \in S_j^{(y)} : |(z - y) \cdot e_l| \leq (1 - \rho) |(z - y) \cdot e_j|\} \).

In words this is the set of points in \( S_j^{(y)} \) that are some angle away from the sides \( R_{j,l}^{(y)} \), \( R_{j,l+3}^{(y)} \), fig 11.1.

Given \( x \in \mathbb{R}^3 \), \( v \in \{e_1, e_2, \ldots, e_6\} \), \( s \in (0, 1) \) and \( r > 0 \) define
\[
\Psi^u(x, v, s, r) = \{z \in C_r(x) : |P_{(x \cdot v)}(z - x)| \leq s |P_{v}(z - (x + rv))|\}.
\]
\[
\Psi^d(x, v, s, r) = \{z \in C_r(x) : |P_{(x \cdot v)}(z - x)| \leq s |P_{v}(z - (x - rv))|\}.
\]

And define \( \Psi(x, v, s, r) = \Psi^u(x, v, s, r) \cap \Psi^d(x, v, s, r) \), fig 11.2.

**Remark** In all rectifiability and density results, one of the most basic methods is the so called 'touching point' method which allows us to deduce the existence of a tangent measure supported on one side of a plane, add to this the sort of integral averaged antipodal symmetry you get as the most basic application of shifting radial functions in Euclidean space, and you have Marstrand’s proof of weak tangents.

In \( \mathbb{R}^3_{\infty} \) what you would like to do, to get the kind of symmetry you have from shifting radial functions working for you, is to empty out one \( S_j^{(x)} \), and this you can’t do as has already been proved. But what you can do with basically standard touching point arguments and the projection theorem is empty out something a bit thinner,
Since $f_j$, $D_{Ca}(\rho)$ (for all $\alpha > 0$, this means for some subsequence the measure all gets piled up into the sides of $S_j \cap C_\alpha(y)$ so giving us the existence of specific weak tangents and this will prove very useful. So this is what the next three lemmas are for.
Lemma 12 Let $\mu$ be a 2-uniform measure where $C_2(0) \cap \text{Spt}\mu$ is purely unrectifiable. Let $i \in \{1, 2, \ldots, 6\}$.

Suppose for some $\rho > 0$, $\zeta \in (0, 1)$ we have a subset of positive $\mu$ measure $B_1 \subset \text{Spt}\mu \cap C_1(0)$ with the property that for all $x \in B_1$, $j \in \{i + 1, i + 2, i + 4, i + 5\}$, $r \in (0, \zeta)$
\[
\mu \left( \frac{S_{i,j}^{(2, \rho)}}{\rho^2} \cap C_r(x) \right) \geq \lambda_1,
\]
then we can find constants $\kappa_{\rho}^{\lambda_1} > 0$ and $\vartheta_{\rho}^{\lambda_1} > 0$ such that for $\mu$ a.a. $x \in B_1$ the following statement is true:

There exist $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1)$, $i \in \{1, 2 \ldots, 6\}$.

If $d \in (0, 1)$ is such that
\[
\frac{\mu \left( C_{2d}(x) \setminus B_1 \right)}{d^2} \leq \epsilon
\]
then for all $z \in \left( e_i^+ + x \right) \cap K(x, \epsilon_i, \kappa_{\rho}^{\lambda_1}d)$
\[
K \left( z, \epsilon_i, \sqrt{\epsilon \vartheta_{\rho}^{\lambda_1}d} \right) \cap C_d(x) \cap B_1 \neq \emptyset.
\]

Proof First note $\{e_{i+1}, e_{i+2}, e_{i+4}, e_{i+5}\} \subset e_i^+$.

Now note that there exist some constant $\xi_{\rho} > 0$ such that if $z \in \partial K(x, \epsilon_i, d)$ then for some $j \in \{i + 1, i + 2, i + 4, i + 5\}$ we have that $S_{i,j}^{(2, \rho)} \cap C_{d_{\xi_{\rho}}}(z) \subset K(x, \epsilon_i, d)$, as shown in fig 11.3
Also note that there exists constant $\zeta_p^{\lambda_1} > 0$ such that if $z \in \partial K(x, e_i, d) \cap B_1$
and $\mu \left( S_{i,j}^{(z,e)} \cap C_\delta \right) > \frac{\lambda_1}{d^2}$ for some $j \in \{i + 1, i + 2, i + 4, i + 5\}$ such that $S_{i,j}^{(z,e)} \cap C_\delta \subset K(x, e_i, d)$ then $K(x, e_i, (1 - \zeta_p^{\lambda_1})d) \cap \text{Spt} \mu \neq \emptyset$.

Let $\kappa_p^{\lambda_1} = \frac{\zeta_p^{\lambda_1}}{\xi_p}$.

Given $x \in B_1$ and $d > 0$ such that for some small $\epsilon_1 > 0$
\[
\frac{\mu \left( C_{2d} (x) \setminus B_1 \right)}{d^2} \leq \epsilon_1. \tag{2.48}
\]

Let $z \in e_i^+ + x$ be such that $\Psi \left( z, e_i, \kappa_p^{\lambda_1}, d \right) \subset C_d(x)$.

Define $H \left( z, e_i^+, \alpha \right) = \bigcup_{h \in (-\alpha, \alpha)} z + he_i + e_i^+$.

Let $\phi_k = \sum_{j=0}^k \xi_p \kappa_p^{\lambda_1} d \left( 1 - \zeta_p^{\lambda_1} \right)^j$ where we let $\phi_{-1} = 0$.

Let $\Gamma_k = H \left( z, e_i^+, \phi_k \right) \cap K \left( z, e_i, (1 - \zeta_p^{\lambda_1}) \kappa_p^{\lambda_1} d \right)$.

We will show
\[
\Psi \left( z, e_i, \kappa_p^{\lambda_1}, d \right) \subset \bigcup_{j=0}^\infty \Gamma_j. \tag{2.49}
\]
We argue inductively.

Firstly recall that $\phi_0 = \xi_0 \kappa^{\lambda_1}_\rho d$ and note that its obvious that

$$\Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right) \cap H \left( z, e_i^+, \xi_\rho \kappa^{\lambda_1}_\rho d \right) \subset \Gamma_0.$$  

Suppose we have that

$$\Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right) \cap H \left( z, e_i^+, \phi_k \right) \subset \bigcup_{j=0}^{k} \Gamma_j.$$  

Let $\vartheta_{k+1}$ be that radius of the circle given by $\partial \Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right) \cap H \left( z, e_i^+, \phi_k \right)$. 

So by definition of $\Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right)$ we know

$$\vartheta_{k+1} = \kappa^{\lambda_1}_\rho d - \kappa^{\lambda_1}_\rho \phi_k$$

$$= \kappa^{\lambda_1}_\rho d - \kappa^{\lambda_1}_\rho \left( \sum_{j=0}^{k} \xi_\rho \kappa^{\lambda_1}_\rho d \left( 1 - \zeta^{\lambda_1}_\rho \right)^j \right)$$

$$= \kappa^{\lambda_1}_\rho d - \xi_\rho \left( \kappa^{\lambda_1}_\rho \right)^2 d \left( \sum_{j=0}^{k} \left( 1 - \zeta^{\lambda_1}_\rho \right)^j \right).$$

And

$$\sum_{j=0}^{k} \left( 1 - \zeta^{\lambda_1}_\rho \right)^j = \frac{1 - \left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1}}{\zeta^{\lambda_1}_\rho}$$

$$= \frac{1}{\zeta^{\lambda_1}_\rho} - \frac{\left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1}}{\zeta^{\lambda_1}_\rho}$$

$$= \frac{1}{\zeta^{\lambda_1}_\rho} \left( 1 - \left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1} \right).$$

Recall $\kappa^{\lambda_1}_\rho = \frac{\vartheta_{k+1}}{\xi_\rho}$ so putting the above expression into (2.50) we get

$$\vartheta_{k+1} = \kappa^{\lambda_1}_\rho d - \xi_\rho \left( \kappa^{\lambda_1}_\rho \right)^2 d \frac{1 - \left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1}}{\zeta^{\lambda_1}_\rho}$$

$$= \kappa^{\lambda_1}_\rho d - \kappa^{\lambda_1}_\rho d \left( 1 - \left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1} \right)$$

$$= \kappa^{\lambda_1}_\rho d \left( 1 - \zeta^{\lambda_1}_\rho \right)^{k+1}$$

which is exactly the width of the cylinder $\Gamma_{k+1}$.

Now as $\vartheta_{k+1}$ is the biggest radius of circles in

$$\partial \Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right) \cap \left( H \left( z, e_i^+, \vartheta_{k+1} \right) \setminus H \left( z, e_i^+, \vartheta_k \right) \right).$$

So we know that

$$\Psi \left( z, e_i, \kappa^{\lambda_1}_\rho, d \right) \cap H \left( z, e_i^+, \vartheta_{k+1} \right) \subset \bigcup_{j=0}^{k+1} \Gamma_j.$$
This establishes (2.49).

Let \( k_1 = \max \{ k : \Gamma_k \cap \Psi(z, e_i, \kappa_\rho^{\lambda_1}, d) \cap B_1 \neq \emptyset \} \).

I will show \( k_1 \) is sufficiently big so that

\[
\lambda_1 \xi_\rho^2 \left( 1 - \zeta_\rho^{\lambda_1} \right)^{2(k_1+1)} \left( \kappa_\rho^{\lambda_1} d \right)^2 \leq 16\varepsilon_1 d^2.
\]

Suppose not and

\[
\lambda_1 \xi_\rho^2 \left( 1 - \zeta_\rho^{\lambda_1} \right)^{2(k_1+1)} \left( \kappa_\rho^{\lambda_1} d \right)^2 \geq 16\varepsilon_1 d^2. \tag{2.51}
\]

We know that \( \Gamma_{k_1+1} \cap \Psi(z, e_i, \kappa_\rho^{\lambda_1}, d) \cap B_1 = \emptyset \).

Let \( y_1 \in \Gamma_{k_1} \cap \Psi(z, e_i, \kappa_\rho^{\lambda_1}, d) \cap B_1 \), let \( h = \left| P_{e_1^+}(y_1 - z) \right| \) so we know that

\[
h \in \left( \left( 1 - \zeta_\rho^{\lambda_1} \right)^{k_1+1} \kappa_\rho^{\lambda_1} d, \left( 1 - \zeta_\rho^{\lambda_1} \right)^{k_1} \kappa_\rho^{\lambda_1} d \right).
\]

We also know that for some \( j \in \{ i + 1, i + 2, i + 4, i + 5 \} \) we have

\[
\hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \subset K(z, e_i, h)
\]
as \( h \xi_\rho \leq \xi_\rho \left( 1 - \zeta_\rho^{\lambda_1} \right)^{k_1} \kappa_\rho^{\lambda_1} d \)

and as

\[
\phi_{k_1} \leq \xi_\rho \kappa_\rho^{\lambda_1} d \left( \sum_{k=0}^{\infty} \left( 1 - \zeta_\rho^{\lambda_1} \right)^j \right) = d \frac{\zeta_\rho^{\lambda_1}}{\zeta_\rho^{\lambda_1}} = d
\]

so

\[
\hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \subset K(z, e_i, h) \cap H(z, e_i^+, \phi_{2k_1}) \subset C_{2d}(x).
\]

Now as \( h \xi_\rho \geq \left( 1 - \zeta_\rho^{\lambda_1} \right)^{k_1+1} \xi_\rho \kappa_\rho^{\lambda_1} d \) so

\[
\mu \left( \hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \right) \geq \lambda_1 (h \xi_\rho)^2
\]
\[
\geq \lambda_1 \xi_\rho^2 \left( 1 - \zeta_\rho^{\lambda_1} \right)^{2(k_1+1)} \left( \kappa_\rho^{\lambda_1} d \right)^2.
\]

By inequality (2.51) we have that

\[
\mu \left( \hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \right) \geq 16\varepsilon_1 d^2,
\]

so by inequality (2.48) we know that less than half the measure in \( \hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \) is outside \( B_1 \) so at least half of it is inside \( B_1 \), so writing things out formally

\[
\frac{1}{2} \mu \left( \hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \right) \geq 8\varepsilon_1 d^2
\]
\[
\geq \mu(C_{2d}(x) \setminus B_1)
\]
\[
\geq \mu \left( \hat{S}_{i,j}^{(y_1, \rho)} \cap C_{h\xi_\rho}(y_1) \setminus B_1 \right)
\]
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\[ \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \cap B_1 \right) = \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \right) - \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \setminus B_1 \right) \]
\[ \geq \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \right) - \frac{1}{2} \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \right) \]
\[ = \frac{1}{2} \mu \left( \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \right) \]
\[ \geq \frac{\lambda_1}{2} (h\xi_\rho)^2. \]

So we can pick \( y_2 \in \hat{S}_{i,j}^{(y_i,\rho)} \cap C_{h\xi_\rho} (y_1) \cap B_1 \) such that
\[ y_2 \in K \left( z, e_i, (1 - \zeta_\rho^{\lambda_1}) h \right) \subset K \left( z, e_i, (1 - \zeta_\rho^{\lambda_1})^{k_1+1} \kappa_\rho^{\lambda_1} d \right). \] (2.52)

Now \( y_1 \in \Psi^u \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \) which means
\[ h = \left| P_e^+ (y_1 - z) \right| \leq \kappa_\rho^{\lambda_1} \left| P_e (y_1 - (z + de_i)) \right|. \]

Now
\[ \left| P_e^+ (y_2 - z) \right| \leq (1 - \zeta_\rho^{\lambda_1}) h \leq (1 - \zeta_\rho^{\lambda_1}) \kappa_\rho^{\lambda_1} \left| P_e (y_1 - (z + de_i)) \right|. \]

And
\[ \left| P_e (y_2 - (z + de_i)) \right| \geq \left| P_e (y_1 - (z + de_i)) \right| - h\xi_\rho \]
\[ \geq \left| P_e (y_1 - (z + de_i)) \right| - \xi_\rho \kappa_\rho^{\lambda_1} \left| P_e (y_1 - (z + de_i)) \right| \]
\[ = \left| P_e (y_1 - (z + de_i)) \right| (1 - \zeta_\rho^{\lambda_1}). \]

Putting these two together we have
\[ \left| P_e^+ (y_2 - z) \right| \leq \kappa_\rho^{\lambda_1} \left| P_e (y_2 - (z + de_i)) \right|. \]

So \( y_2 \in \Psi^u \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \), similarly \( y_1 \in \Psi^d \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \) implies \( y_2 \in \Psi^d \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \) so we have \( y_2 \in \Psi \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \).

Now from (2.49) we know that \( y_2 - z \) is 'short' enough and from (2.52) we know \( y_2 - z \) 'thin' enough for
\[ y_2 \in \Psi \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \cap \left( \bigcup_{j=k_1+1}^{\infty} \Gamma_j \right) \cap B_1. \]

Contradicting the maximality of \( k_1 \).

So now we know that
\[ \lambda_1 \xi_\rho^2 \left( 1 - \zeta_\rho^{\lambda_1} \right)^{2(k_1+1)} \left( \kappa_\rho^{\lambda_1} d \right)^2 \leq 16\epsilon_1 d^2. \] (2.53)

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Let $\theta = \left(1 - \zeta_\rho^{\lambda_1}\right)^{k_1} \kappa_\rho^{\lambda_1} d$.

So

$$16\epsilon_1 d^2 \geq \lambda_1 \xi_\rho^2 \left(1 - \zeta_\rho^{\lambda_1}\right)^{2(k_1+1)} \left(\kappa_\rho^{\lambda_1} d \right)^2 = \lambda_1 \xi_\rho^2 \left(1 - \zeta_\rho^{\lambda_1}\right)^2 \theta^2.$$ 

And so

$$\theta \leq \frac{4\sqrt{\epsilon_1} d}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}}.$$ 

So for any $z \in e_i^\perp + x$ s.t. $\Psi \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \subset C_d (x)$ we have that if we know that

$$\Psi \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right) \cap B_1 \neq \emptyset$$

then

$$K \left( z, e_i, \frac{4\sqrt{\epsilon_1} d}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}} \right) \cap C_d (x) \cap B_1 \neq \emptyset.$$ 

As we know that for any $z \in (e_i^\perp + x) \cap K \left( x, e_i, \kappa_\rho^{\lambda_1} d \right)$ we have

$$x \in \Psi \left( z, e_i, \kappa_\rho^{\lambda_1}, d \right)$$

and this completes the proof. □
2.5.1 Application: Emptying cones

Lemma 13 Let $\mu$ be a 2-uniform measure where $Spt\mu \cap C_2(0)$ is purely unrectifiable. Let $i \in \{1, 2, \ldots, 6\}$.

For any $\rho > 0$ for $\mu$ a.a. $x \in Spt\mu \cap C_1(0)$

$$\liminf_{r \to 0} \frac{\mu \left( \hat{C}_{i,j}^{(x,\rho)} \cap C_r(x) \right)}{r^2} = 0$$

for some $j \in \{i + 1, i + 2, i + 4, i + 5\}$.

Preproof

The only purpose of the previous Lemma was to facilitate this one. The idea is simple:

It is possibly a good idea to look at fig 12.1 over-page before reading this.

If we knew that for some subset $B \subset Spt\mu$

$$\mu \left( \hat{C}_{i,j}^{(x,\rho)} \cap C_r(x) \right) \geq \lambda r^2$$

for all $j \in \{i + 1, i + 2, i + 4, i + 5\}$ and all $x \in B$ then as is standard we take density point of it, and get a cube $C$ with the vast majority of its measure in $B$. We know that the orthogonal projection down of $B \cap C$, onto some plane arbitrarily close to $e_i^\perp$, has zero Lebesgue measure. In the argument we can use any such plane so long as its close enough to $e_i^\perp$. For this reason we lose no generality in assuming the projection down of $B \cap C$ onto $e_i^\perp$ has zero Lebesgue measure.

So we have many empty cylinders (parallel to $e_i$) running up through our cube $C$. As is also standard we expand these cylinders until they touch the first point of our set $B$, by the property that points of $B$ have, we then know there is a $j \in \{i + 1, i + 2, i + 4, i + 5\}$ for which we have a $\hat{S}_{i,j}$ inside our cylinder, which has some measure in it (which must all be measure in $Spt\mu \setminus B$) so clearly we can not have many of these $\hat{S}_{i,j}$ contained inside $C$, and since they are centered in $C$, they must be intersecting the boundary of $C$. Now the purpose of the previous lemma was to show that these cylinders are thin and so we must have many points on the boundary of $C$, but as we know there can’t be too much measure on the boundary of $C$ because we have exact estimates for it, so we have a contradiction, fig 12.1 is probably clearer than anything I have said here.
As can be seen the union of all $\tilde{S}_{1,j}$ inside our cylinders will have measure bigger than some fraction of the surface area of the cube and this is much too much measure on the boundary.
Proof Suppose not, and we have \( i \in \{1, 2, \ldots, 6\} \) for which we can find a closed subset \( B_1 \subset \text{Spt}\mu \cap C_1 (0), \mu (B_1) > 0 \), and numbers \( \rho > 0, \varsigma > 0, \lambda_1 > 0 \), such that for all \( x \in B_1 \) and \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) and \( r \in (0, \varsigma) \) we have

\[
\frac{\mu (S_{i,j} (x, \rho) \cap C_r (x))}{r^2} \geq \lambda_1.
\]

Let \( \epsilon_1 > 0 \) be some very small number, to be determined later, then by Lemma 12, we have constant \( \kappa^\lambda_1 \rho \) such that if \( x \) is a density point of \( B_1 \), and we have \( d > 0 \) such that

\[
\mu (C_{2d} (x) \setminus B_1) \leq \epsilon_1
\]

and we let \( y \in P_{\epsilon_i} (C_d (x) \cap K \left( x, \epsilon_i, \kappa^\lambda_1 \rho \right) \setminus B_1)^c \) then defining \( r_y = \sup \{ r > 0 : K (y, e_i, rd) \subset C_d (x) \setminus B_1 = \emptyset \} \), we have \( r_y \leq \frac{4 \sqrt{\epsilon}}{\sqrt{\lambda_1} \xi_\rho \left( 1 - \varsigma \right)} d \).

Let \( U = P_{\epsilon_i} (C_d (x) \cap K \left( x, \epsilon_i, \kappa^\lambda_1 \rho \right) \setminus B_1)^c \). So \( U \subset \bigcup_{y \in U} (y, r_y) \) by Theorem 2.1, [1] we can find a finite or countable set \( \{y_n \in U : n \in \mathbb{N}\} \) such that

\[
\bigcup_{y \in U} B (y, r_y) \subset \bigcup_{n \in \mathbb{N}} B (y_n, 5r_{y_n}),
\]

and \( \{B (y_n, r_{y_n}) : n \in \mathbb{N}\} \) are disjoint.

As by Federer's Projection Theorem, [1] Theorem 18.1, and pure unrectifiability of \( \text{Spt}\mu \cap C_2 (0) \), we have \( L^2 (U) = \left( d \kappa^\lambda_1 \rho \right)^2 \pi \), so \( \sum_{n \in \mathbb{N}} (5r_{y_n})^2 \pi \geq \left( \kappa^\lambda_1 \rho \right)^2 d^2 \pi \).

So

\[
\sum_{n \in \mathbb{N}} r_{y_n}^2 \geq \frac{\left( \kappa^\lambda_1 \rho \right)^2 d^2}{25}. \tag{2.54}
\]

Now for each \( n \in \mathbb{N} \) we can find a point \( v_n \in \partial K (y_n, e_i, r_{y_n}) \cap C_d (x) \cap B_1 \) and \( j_n \in \{i + 1, i + 2, i + 4, i + 5\} \) such that

\[
S_{i,j_n} (v_n, \rho) \cap C_{\xi \rho r_{y_n}} (v_n) \subset K (y, e_i, r_{y_n}).
\]

Now we know that for each \( n \in \mathbb{N} \)

\[
\mu (S_{i,j_n} (v_n, \rho) \cap C_{\xi \rho r_{y_n}} (v_n)) \geq \lambda_1 (\xi \rho r_{y_n})^2.
\]

Let \( I_1 = \{n \in \mathbb{N} : S_{i,j_n} (v_n, \rho) \cap C_{\xi \rho r_{y_n}} (v_n) \subset C_d (x)\} \)

Let \( I_2 = \mathbb{N} \setminus I_1 \)

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Now firstly for any \( n \in I_1 \) as \( \tilde{S}_{i, j_n}^{(v_n, \rho)} \cap C_{\xi, r_{y_n}} (v_n) \subset K (y_n, \epsilon_1, r_{y_n}) \cap C_d (x) \) and so \( \tilde{S}_{i, j_n}^{(v_n, \rho)} \cap C_{\xi, r_{y_n}} (v_n) \cap \text{Spt} \mu \subset C_d (x) \cap (\text{Spt} \mu \setminus B_1) \), so that we have

\[
\mu \left( \bigcup_{n \in I_1} \tilde{S}_{i, j_n}^{(v_n, \rho)} \cap C_{\xi, r_{y_n}} (v_n) \right) \leq \epsilon_1 d^2.
\]

So

\[
\sum_{n \in I_1} (r_{y_n})^2 \leq \epsilon_1 \frac{d^2}{\lambda_1 \xi_\rho^2}.
\]

So given sufficient smallness of \( \epsilon_1 \)

\[
\sum_{n \in I_2} (r_{y_n})^2 \geq d^2 \left( \frac{\kappa_\rho^{\lambda_1}}{25} - \frac{\epsilon_1}{\lambda_1 \xi_\rho^2} \right) \geq d^2 \frac{\kappa_\rho^{\lambda_1}}{50}.
\]

Now as \( v_n \in C_d (x) \) for all \( n \in \mathbb{N} \) if \( n \in I_2 \), \( C_{r_{y_n}} (v_n) \cap \partial C_d (x) \neq \emptyset \).

As \( r_{y_n} \leq \frac{4 \sqrt{\epsilon_1 d}}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}} \) so we have

\[
\bigcup_{n \in I_2} C_{r_{y_n}} (v_n) \subset A \left( x, d - \frac{4 \sqrt{\epsilon_1 d}}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}}, d + \frac{4 \sqrt{\epsilon_1 d}}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}} \right).
\]

And so we have

\[
d^2 \frac{\kappa_\rho^{\lambda_1}}{50} \leq \sum_{n \in I_2} 4r_{y_n}^2 = \mu \left( \bigcup_{n \in I_2} C_{r_{y_n}} (v_n) \right) \leq 16d^2 \frac{4 \sqrt{\epsilon_1}}{\sqrt{\lambda_1 \xi_\rho \left(1 - \zeta_\rho^{\lambda_1}\right)}}.
\]

Assuming smallness of \( \epsilon_1 \) this is a contradiction. \( \square \)

### 2.5.2 Application: Existence of specific tangent measures

**Lemma 14** Let \( \mu \) be a 2-uniform measure where \( C_2 (0) \cap \text{Spt} \mu \) is purely unrectifiable.

Let \( i \in \{1, 2, \ldots, 6\} \).

For \( \mu \) a.a. \( y \in \text{Spt} \mu \cap C_1 (0) \) we can find a 2-plane \( V \in \{ < R^{(0)}_{i+1,i+2}, < R^{(0)}_{i+1,i+5} \} \) such that \( V \in \overline{\text{Tan}} (\mu, y) \).

**Proof** Now given \( i \in \{1, 2, \ldots, 6\} \).

Let \( y \in G, j \in \{i + 1, i + 2, i + 4, i + 5\}, r > 0 \).

By Lemma 11 we know

\[
\mu \left( C_r (y) \cap S_j^{(y)} \right) \geq \frac{7r^2}{6160}.
\]

By Lemma 13 for \( \mu \) a.a. \( y \in \text{Spt} \mu \) we can find a sequence \( r_n \to 0 \) such that for some \( j_0 \in \{i + 1, i + 2, i + 4, i + 5\} \)

\[
\lim_{n \to \infty} \frac{\mu \left( \hat{S}_{i, j_0}^{(y, \rho)} \cap C_{r_n} (y) \right)}{r_n^2} = 0.
\]

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Let $\nu \in \text{Tan}(\mu, y)$ be the tangent measure with respect to this sequence, i.e. $\nu = \lim_{n \to \infty} \frac{T_{x_2,2} \lambda}{r_2^n}$.

By (2.55) we know

$$\nu \left( S_{j_0}^{(0)} \cap C_1(0) \right) \geq \frac{7}{6160}$$

but for every $\rho > 0$ we have

$$\nu \left( \left( S_{j_0}^{(0)} \setminus \hat{S}_{j_0}^{(0,\rho)} \right) \cap C_1(0) \right) = 0.$$

So

$$\nu \left( \left( R_{j_0,l}^{(0)} \cup R_{j_0,l+3}^{(0)} \right) \cap C_1(0) \right) \geq \frac{7}{6160}$$

where $l \in \{i + 1, i + 2, i + 4, i + 5\} \setminus \{j, j + 3\}$ so let $z$ be a density point of $\left( R_{j_0,l}^{(0)} \cup R_{j_0,l+3}^{(0)} \right) \cap C_1(0) \cap \text{Spt}_z$, then for any $\nu_1 \in \text{Tan}(\nu, z)$ we have $\text{Spt}_z \nu_1 \subset V$ where

$$V \in \left\{ < R_{i+1,i+2}^{(0)}, < R_{i+1,i+5}^{(0)} > \right\}.$$

Now as by Lemma 2 we have that $\nu_1 \in \text{Tan}(\mu, y)$ this completes the proof. □
2.6 Lower bounds on 'growth' where measure exists on a diagonal part of the boundary

Lemma 15 Let \( \mu \) be 2-uniform measure where \( \text{Spt} \mu \cap C_2(0) \) is purely unrectifiable then for \( \mu \) a.a. \( y \in G \cap C_1(0) \) we have the following property:

If \( r \in \Theta_i(y) \cap L(y) \cap (0,1) \) for some \( i \in \{1,2,\ldots,6\} \) and there exists \( s > 0 \) such that for some \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) we have

\[
\partial C_i(y) \cap H_{i,j}^{(y)} \cap \text{Spt} \mu \neq \emptyset
\]

\[
\partial C_i(y) \cap H_{i+3,j+3}^{(y)} \cap \text{Spt} \mu \neq \emptyset
\]

for any \( t \in (r - s, r + s) \), then

\[
f_j^r(r) > 1 + \frac{1}{64}.
\]

Preproof

The idea for this Lemma comes from noticing that Lemma 6 gives a very strong relation between the 'growth' of the measure in \( S_j^{(y)} \cap A(y) \) and the measure on the 'x-boundary' of \( \left(S_j^{(y)} \cup S_{j+3}^{(y)}\right) \cap A(y) \).

By this I mean that both terms in the statement of the Lemma are of order \( \|x - y\| h^4 \) and the bound on the difference between them is of order \( \|x - y\|^2 r h \), since we can pick \( x \) as close to \( y \) as we like, the relation is as good as we like.

Now note that the only thing we used in establishing Lemma 6 other than uniformity of the measure was that we could find a sequence of support points \( x_n \to x \) along our coordinate axis \( \{e_i : i = 1,2,3\} \).

So we can run through large parts of the same argument with a standard flat measure like \( \mu_1 = H_{L_i}^2 e_i^{\perp} \). In this case if we took \( y = 0 \) and \( x = \lambda (e_i + e_{i+1}) \) for some small \( \lambda > 0 \) as show in fig 14.0, we have a sequence of support points \( x_n \to x \) where \( x_n - x \) is parallel to \( e_{i+1} \).

![fig 14.0](image_url)
So we can perturb in the direction $e_{i+1}$ with our sequence $x_n$ so we can obtain the statement of Lemma 6 for this direction.

In other words we know in this case that the 'x-boundary' of $\left( S^{(0)}_{i+1} \cup S^{(0)}_{i+4} \right) \cap A(0)$ is approximately the 'growth' in $S^{(0)}_{i+1} \cap A(0)$ for any annulus $A(0)$. And we clearly know that for $\mu_1$ the 'growth' in direction $e_{i+1}$ is the maximum possible growth 2, i.e. $f_{i+1}^{(0)}(s) = 2$ for all $s > 0$.

So we can use this example to estimate how much measure we need in the 'x-boundary' in order to get 'growth' $\geq 1$.

For instance if we had measure in the 'x-boundary' as indicated in fig 14.1 then as this is exactly half the measure in the 'x-boundary' of the first example we would have growth equal to 1.

![fig 14.1](image)

This is the motivating example of this lemma. The initial condition that

$$\partial C_t(y) \cap R_{i,j}^{(y)} \cap \text{Spt}\mu \neq \emptyset \quad \text{and} \quad \partial C_t(y) \cap R_{i+3,j+3}^{(y)} \cap \text{Spt}\mu \neq \emptyset$$

$t \in (s-r, s+r)$ gives us measure on the 'x-boundary' of $S_j^{(y)} \cup S_{j+3}^{(y)}$ for thin annuli as is depicted in fig 14.1 ($j = i + 4$ in this case). The condition that $r \in \Theta^{(y)}_t$ would imply $r \in \Theta^{(x)}_t$ if $x$ was close enough to $y$, and this ensures that the growth in direction $i$ is equal to 2, and so we must have that there is some measure on the diagonal $\left( R_{i,j}^{(x)} \cup R_{i+3,j+3}^{(x)} \right) \cap \partial C_s(x)$ for any $s$ close enough to $r$, for if not then by Lemma 9 we would have that the 'growth' in direction $e_l \in \langle e_i, e_j \rangle^\perp$ is equal to 2.
So as $f_i^{(x)}(s) + f_i^{(y)}(s) + f_j^{(y)}(s) = 4$ this means we must have $f_j'(s) = 0$ contradicting Lemma 11.

So $(R_{i,j}^{(x)} \cup R_{i+3,j+3}^{(x)}) \cap \partial C_s (x) \cap Spt\mu \neq \emptyset$ for all $s$ close enough to $r$ and so if we are careful we can fit in half cubes centered on points in the diagonal $(R_{i,j}^{(x)} \cup R_{i+3,j+3}^{(x)})$ in such a way as they are disjoint from those already on the diagonal $(R_{i,j}^{(y)} \cup R_{i+3,j+3}^{(y)})$ (which are depicted in fig 14.1), and so we will have that there is enough measure on the 'x-boundary' of $(S_{i+1}^{(y)} \cup S_{i+4}^{(y)}) \cap A(y)$ for there to be 'growth' in $S_{i+1}^{(y)} \cap A(y) \geq 1 + \frac{1}{64}$.

This is how the proof works.
**Proof** Given \( r \in \Theta_i^{(y)} \cap L_i^{(y)} \cap (0,1) \), as \( \Theta_i^{(y)} \) is an open set we can find \( s_0 > 0 \) such that \((r, r + s_0) \subset \Theta_i^{(y)} \) and by closure of \( \text{Spt}\mu \) we have that

\[
\tau = d\left( \text{Spt}\mu, A(y, r, r + s_0) \cap \bigcup_{j_1, j_2 \in \{i+1, i+2, i+4, i+5\}, e_{j_1}, e_{j_2} = 0} R_{j_1, j_2}^{(y)} \right) > 0.
\]

Let \( \epsilon > 0 \).

By Lemma 8 we can find some small \( h \in (0, s_0) \) such that

\[
\int_{A(y, r, r+h) \cap (S_j^{(y)} \cup S_{j+3}^{(y)})} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) d\mu z \in \left( (2\lambda_j - \epsilon) h^4, (2\lambda_j + \epsilon) h^4 \right).
\]

(2.56)

From now on let \( A(y) = A(y, r, r + h) \) and pick \( l \in \{i + 1, i + 2, i + 4, i + 5\} \backslash \{j, j + 3\} \).

We can assume \( y \in G \) is one of the a.a. points in \( G \) for which we have a weak tangent \( V \in \left\{ < R_{i,l}^{(0)}, < R_{i,l+3}^{(0)} > \right\} \), by Lemma 14. Assume without loss of generality that \( V = < R_{i,l}^{(0)} > \).

So we can find some \( \alpha \in (0, \frac{\tau}{100}) \) such that

\[
\mu( C_\alpha(y) \setminus N_\epsilon \alpha(V + y) ) \leq \epsilon \alpha^2
\]

and thus we are in a position to pick points \( x_2, x_1 \in C_\alpha(y) \cap N_\epsilon \alpha(V + y) \) which are in the \( e_i + e_l + e_{l+3} \) corner and the \( e_{i+3} + e_{l+3} + e_j \) corner of \( C_\alpha(y) \cap N_\epsilon \alpha(V + y) \), respectively.

So writing this out formally:

There exits \( x_2 \in S_{i+3}^{(y)} \cap C_\alpha(y) \cap G \) and \( x_1 \in S_j^{(y)} \cap C_\alpha(y) \cap G \) such that

\[
d(x_2, y + \alpha < e_i + e_l + e_{l+3} >) \leq \epsilon \alpha
\]

and

\[
d(x_1, y + \alpha < e_{i+3} + e_{l+3} + e_j >) \leq \epsilon \alpha.
\]

Let \( \delta = \min\{\|x_1 - y\|, \|x_2 - y\|\} \) and let \( n = \left\lceil \frac{\delta}{2\epsilon} \right\rceil \).

We can pick points

\[
z_m \in \partial C_{r+2m\delta} \cap \bar{R}_{i,j}^{(y)} \cap \text{Spt}\mu
\]

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for \( m = 1, 2, \ldots n - 1 \).

So clearly the set

\[
\{ C_{\delta} \left( z_m^k \right) : m = 1, 2, \ldots n - 1 \}
\]

is disjoint where \( k \in \{ 1, 2 \} \).

Equally clearly

\[
\bigcup_{k=1}^{2} \bigcup_{m=1}^{n-1} C_{\delta} \left( z_m^k \right) \subset A \left( y, r, r + h \right).
\]

Also from fig 14.4 we can see that

\[
\bigcup_{k=1}^{2} \left( \bigcup_{m=1}^{n-1} C_{\delta} \left( z_m^k \right) \right) \subset \left( S_{j}^{(x_j)} \setminus S_{j+3}^{(x)}, \right) \cup \left( S_{j+3}^{(x_j)} \setminus S_{j}^{(x)} \right).
\]

(2.57)

We will show that

\[
\int_{A(y) \cap \left( \bigcup_{k=1}^{2} \bigcup_{m=1}^{n-1} C_{\delta} \left( z_m^k \right) \right)} 4 \left( h^2 - \left( \|z - y\| - r\right)^2 \right) \left( \|z - y\| - r \right) d\mu z \\
\geq 4\delta h^4 - 384\delta^2 h^3.
\]

(2.58)

To see this first note that the function \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) given by

\[
g \left( p \right) = \begin{cases} 
0 & \text{for } p \in [0, r) \\
4 \left( h^2 - \left( p - r \right)^2 \right) \left( p - r \right) & \text{for } p \in [r, r + h] \\
0 & \text{for } p > r + h
\end{cases}
\]

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is a Lipschitz function with Lipschitz constant $8h^2$.

To see this we might as well work with the shifted function

$$\tilde{g}(s) = g(s - r)$$

and show its a Lipschitz function with constant $8h^2$.

Now note that for any $s \in (0, h)$

$$\tilde{g}'(s) = 4h^2 - 12s^2$$

and so $|\tilde{g}'(s)| \leq 8h^2$.

At 0 and $h$ the left and right hand-side derivatives exist and are bounded by $8h^2$ so $g$ is Lipschitz with constant $8h^2$.

Now given some small $\alpha > 0$ define

$$g_\alpha(s) = \inf\{g(t) : t \in [s - \alpha, s + \alpha]\}$$

this is Lipschitz, but as we don’t need to use this so we won’t prove it.

Now we will define the function

$$h(p) = \begin{cases} 
0 & \text{for } p \in [0, r + \delta) \\
\delta(r + 2m\delta) & \text{for } p \in [r + (2m - 1)\delta, r + (2m + 1)\delta] \quad m = 1, 2, \ldots, n - 1 \\
0 & \text{for } p > r + (2n - 1)\delta.
\end{cases}$$

If $x \in [r + (2m - 1)\delta, r + (2m + 1)\delta]$ for $m = 1, 2, \ldots, n - 1$ then

$$|g(x) - h(x)|$$
\[ = |g(x) - g_\delta(r + 2m\delta)| \]
\[ \leq |g(x) - g(r + 2m\delta)| + |g(r + 2m\delta) - g_\delta(r + 2m\delta)| \]
\[ \leq 32\delta h^2. \quad (2.59) \]

Now note that
\[
\int_{\mathcal{A}(y) \cap \left( \bigcup_{k=1}^{n-1} C_\delta(z_m^k) \right)} h(\|z - y\|) \, d\mu z
\]
\[ = \sum_{m=1}^{n-1} \int_{\mathcal{A}(y, r + (2m-1)\delta, r + (2m+1)\delta) \cap \left( \bigcup_{k=1}^{n-1} C_\delta(z_m^k) \right)} h(\|z - y\|) \, d\mu z
\]
\[ = \sum_{m=1}^{n-1} h(r + 2m\delta) \mu \left( \mathcal{A}(y, r + (2m-1)\delta, r + (2m+1)\delta) \cap \left( \bigcup_{k=1}^{n-1} C_\delta(z_m^k) \right) \right).
\]
As we picked \( z_m^k \in \partial C_{r + 2m\delta}(y) \cap \text{Spt} \mu \) for \( k \in \{1, 2\} \) and for each \( m \in \{1, 2, \ldots n - 1\} \),
so of course we have
\[ C_\delta(z_m^k) \subset \mathcal{A}(y, r + (2m-1)\delta, r + (2m+1)\delta) \]
for \( k \in \{1, 2\} \), for any \( m \in \{1, 2, \ldots n - 1\} \).
So
\[ \mu \left( \mathcal{A}(y, r + (2m-1)\delta, r + (2m+1)\delta) \cap \left( \bigcup_{k=1}^{n-1} C_\delta(z_m^k) \right) \right) \geq 8\delta^2. \]
for each \( m \in \{1, 2, \ldots n - 1\} \).
So that
\[
\int_{\mathcal{A}(y) \cap \left( \bigcup_{k=1}^{n-1} C_\delta(z_m^k) \right)} h(\|z - y\|) \, d\mu z
\]
\[ \geq \sum_{m=1}^{n-1} h(r + 2m\delta) 8\delta^2
\]
\[ = 4\delta \left( \sum_{m=1}^{n-1} 2h(r + 2m\delta) \delta \right)
\]
\[ = 4\delta \int_{\mathbb{R}} h(x) \, dx
\]
Now note that
\[
\int_{\mathbb{R}} |g(x) - h(x)| \, dx
\]
\[ = \int_{r}^{r+h} |g(x) - h(x)| \, dx
\]
\[ \leq 32\delta h^3. \quad (2.60) \]

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Now the integral of \( g(x) \) can be easily calculated so we will calculate it

\[
\int_{\mathbb{R}} g(x) \, dx = \int_{r}^{r+h} 4 \left( h^2 - (p - r)^2 \right) (p - r) \, dL_p
\]

\[
= \int_{0}^{h} 4 \left( h^2 - q^2 \right) q \, dL_q
\]

\[
= h^4.
\]

So we have that

\[
\int_{A(y) \cap \left( \bigcup_{k=1}^{n} \bigcup_{m=1}^{n-1} C_{\delta}(x_{m}^{k}) \right)} h ||z - y|| \, d\mu z
\]

\[
\geq 4\delta \left( h^4 - 32\delta h^3 \right).
\]

Now since \( n \leq \frac{h}{2\delta} \) so we have

\[
\int_{A(y) \cap \left( \bigcup_{k=1}^{n} \bigcup_{m=1}^{n-1} C_{\delta}(x_{m}^{k}) \right)} \left| h ||z - y|| - g ||z - y|| \right| \, d\mu z
\]

\[
\leq 32\delta h^2 \mu \left( A(y) \cap \left( \bigcup_{k=1}^{2} \bigcup_{m=1}^{n-1} C_{\delta}(x_{m}^{k}) \right) \right)
\]

\[
\leq 32\delta h^2 2 (n - 1) 4\delta^2
\]

\[
= 256\delta^3 h^2 n
\]

\[
\leq 256\delta^2 h^3.
\]

So finally we have that

\[
\int_{A(y) \cap \left( \bigcup_{k=1}^{n} \bigcup_{m=1}^{n-1} C_{\delta}(x_{m}^{k}) \right)} g ||z - y|| \, d\mu z
\]

\[
\geq 4\delta \left( h^4 - 32\delta h^3 \right) - 256\delta^2 h^3
\]

\[
= 4\delta h^4 - 128\delta^2 h^3 - 256\delta^2 h^3
\]

\[
\geq 4\delta h^4 - 384\delta^2 h^3.
\]

Now recall that

\[
A(y) \cap \left( \bigcup_{k=1}^{2} \bigcup_{m=1}^{n-1} C_{\delta}(x_{m}^{k}) \right) \subset A(y) \cap \left( \left( S_{j+3}(x_2) \setminus S_{j}(x_1) \right) \cup \left( S_{j+3}(x_1) \setminus S_{j+3}(x_2) \right) \right).
\]

As we know that

\[
A(y) \cap \left( \left( S_{j+3}(x_2) \setminus S_{j}(x_1) \right) \cup \left( S_{j+3}(x_1) \setminus S_{j+3}(x_2) \right) \right)
\]

is the union of

\[
B_1(y) = A(y) \cap \left( \left( S_{j+3}(x_2) \setminus S_{j}(y) \right) \cup \left( S_{j+3}(y) \setminus S_{j+3}(x_2) \right) \right)
\]

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and

\[ B_2(y) = A(y) \cap \left( (S_j^{(y)} \setminus S_j^{(x_1)}) \cup (S_j^{(x_1)} \setminus S_j^{(y)}) \right). \]

So we have that for some \( a \in \{1, 2\} \)

\[
\int_{\mathfrak{A}(y) \cap \left( \bigcup_{k=1}^{2} \bigcup_{n=1}^{n-1} \mathcal{C}_n(z_k) \right) \cap B_n(y)} g(\|z - y\|) \, d\mu_z \geq 2\delta h^4 - 192\delta^2 h^3, \tag{2.61}
\]

for simplicity we will assume \( a = 1 \).

Since \( h \in (0, s_0) \) we know that

\[ A(x_2, r, r + h) \cap \left( \bigcup_{j_1,j_2 \in \{i+1,i+2,i+4,i+5\}, e_{j_1} \cdot e_{j_2} = 0} R_{j_1,j_2}^{(x_2)} \right) \cap \text{Spt} \mu = \emptyset. \]

So we know that for every \( s \in (r, r + h) \) we have that

\[ \partial C_s(x_2) \cap \left( R_{i,j}^{(x_2)} \cup R_{i+3,j+3}^{(x_2)} \right) \cap \text{Spt} \mu \neq \emptyset \]

because if this was not so we would have \( f_i'(x_2)(s) = 2 \) by Lemma 9 (Recall \( l \) is such that \( e_i \prec e_j > 1 \)) so as \( f_i'(x_2)(s) + f_j'(x_2)(s) + f_i'(x_2)(s) = 4 \) we must have \( f_j'(s) = 0 \) contradicting Lemma 11.

So for each point \( s \in (r, r + h) \) we can pick a point \( z_s^3 \in \partial C_s(x_2) \cap \left( R_{i,j}^{(x_2)} \cup R_{i+3,j+3}^{(x_2)} \right) \cap \text{Spt} \mu. \)
Recall $n = \left[ \frac{h}{2^3} \right]$ so let $n_1 = \left[ \frac{h}{2^6} \right]$ and note that $\frac{7}{4} n_1 \geq n_1 + 1 > \frac{h}{8 \delta}$ so $14 \delta n_1 > h$.

As shown in diagram 14.6 we can pick

$$z_m^3 \in A \left( x_2, r + \frac{h}{3}, r + \frac{2h}{3} \right) \cap \left( R_{i,j}^{(x_2)} \cup R_{i+3,j+3}^{(x_2)} \right) \cap \text{Spt} \mu,$$

for $m = 1, 2, \ldots n_1$ in such a way that

- $\{C_\delta \left( z^3_m \right) : m = 1, 2, \ldots n_1 \}$ is disjoint
- $\{C_\delta \left( z^3_m \right) : m = 1, 2, \ldots n_1 \} \cap \{C_\delta \left( z^k_m \right) : k = 1, 2; m = 1, 2, \ldots n \} = \emptyset.$

Also as $\delta < \frac{r}{100}$ we know the points $\{z^3_m\}$ are sufficiently far from the boundary $\partial R_{i,j}^{(x_2)} \cup \partial R_{i+3,j+3}^{(x_2)}$ so that we know that

$$\left\{C_\delta \left( z^3_m \right) \cap \left( S_{i+3}^{(x_2)} \cup S_j^{(y)} \right) : m = 1, 2, \ldots n_1 \right\} \subset \left( S_{i+3}^{(x_2)} \setminus S_j^{(y)} \right) \cup \left( S_j^{(y)} \setminus S_{i+3}^{(x_2)} \right).$$

We also know by Lemma 9 that

$$\mu \left( C_\delta \left( z^3_m \right) \cap \left( S_{i+3}^{(x_2)} \cup S_j^{(y)} \right) \right) \geq \delta^2$$

for each $m = 1, 2, \ldots n_1$.

Now since for all $p \in \left[ r + \frac{h}{4}, r + \frac{3h}{4} \right]$ we have that

$$g\left( p \right) \geq 4 \left( h^2 - \left( \frac{3h}{4} \right)^2 \right) \frac{h}{4}$$

$$= 4 \left( h^2 - \frac{9}{16} h^2 \right) \frac{h}{4}$$

$$= \frac{7h^3}{16}.$$

(2.62)

$$\int \bigcup_{m=1,2,\ldots,n_1} C_\delta \left( z^3_m \right) \cap \left( S_{i+3}^{(x_2)} \cup S_j^{(y)} \right) g \left( \| z - y \| \right) d\mu z$$

$$\geq \frac{7h^3}{16} \mu \left( \bigcup_{m=1,2,\ldots,n_1} C_\delta \left( z^3_m \right) \cap \left( S_{i+3}^{(x_2)} \cup S_j^{(y)} \right) \right)$$

$$\geq \frac{7h^3}{16} n_1 \delta^2$$

$$= \frac{\delta h^3}{32} 14 \delta n_1$$

$$\geq \frac{\delta h^4}{32}.$$

Now recall inequality (2.61), as the two sets

$$A \left( y \right) \cap \left( \bigcup_{k=1}^{n-1} \bigcup_{m=1}^{n-1} C_\delta \left( z^3_m \right) \right) \cap B_1 \left( y \right)$$

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and

$$\bigcup_{m=1,2,...,n_1} C_\delta \left( z^m \right) \cap \left( S_{t+3}^{(x)} \cup S_j^{(x)} \right)$$

are disjoint and contained in $B_1(y)$. So we have that

$$\int_{A(y) \cap \left( \left( S_j^{(x)} \setminus S_j^{(y)} \right) \cup \left( S_{j+3}^{(y)} \setminus S_j^{(x)} \right) \right)} \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu z$$

$$\geq \frac{h^4}{32\delta} - 2\delta h^4 - 192\delta^2 h^3,$$

and this finally is exactly what we want.

Now as $x_2 \in S_j^{(y)}$ so as in the remark following Lemma 6 we have that

$$\int_{A(y) \cap \left( \left( S_j^{(x)} \setminus S_j^{(y)} \right) \cup \left( S_{j+3}^{(y)} \setminus S_j^{(x)} \right) \right)} \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu z$$

$$\leq - \int_{A(y) \cap \left( S_j^{(y)} \cup S_{j+3}^{(y)} \right)} \left( h^2 - 3 (\|z - y\| - r)^2 \right) \|x_2 - y\| d\mu z + c_3 h r \|x_2 - y\|^2.$$

So remembering (2.56) we have

$$\left( 2 + \frac{1}{32} \right) \delta h^4 - 192\delta^2 h^3$$

$$\leq 2 \|x_2 - y\| (\lambda_j + \epsilon) h^4 + c_3 h r \|x_2 - y\|^2. \tag{2.63}$$

Now $\delta = \min \left\{ \|x_1 - y\|, \|x_2 - y\| \right\}$ and since we have that

$\|x_2 - y\| \geq (1 - 2\epsilon) \|x_1 - y\|$ thus if we divide (2.63) by $\|x_2 - y\|$ we have

$$\left( 2 + \frac{1}{32} \right) (1 - 2\epsilon) h^4$$

$$\leq 2 (\lambda_j + \epsilon) h^4 + c_3 h r \|x_2 - y\| + 192 h^3 \|x_2 - y\|.$$

Now as $\epsilon$ and $\|x_2 - y\|$ can be made as small as we like, we finally have

$$1 + \frac{1}{64} \leq \lambda_j.$$
2.7 General lower bounds on 'growth'

**Lemma 16** Let $\mu$ be a 2-uniform measure where $C_2(0) \cap \text{Spt}\mu$ is purely unrectifiable, let $i \in \{1, 2, \ldots, 6\}$ and let $y \in G \cap C_1(0)$ then for any $r \in L^{(y)}$

$$f_i^{(y)}(r) \geq 1.$$ 

**Preproof** This lemma is extremely similar to the previous one, again the motivation comes from observing what Lemma 6 says about the specific measure given by $\mu_1 = H^2_{\mathbb{R}^2}e_i^+.$

![Diagram](image)

Notice the different position of $x$ in this case. Again the 'x-boundary' of $(S_i^{(0)} \cup S_i^{(0)}) \cap A(0)$ is approximately the 'growth' in $S_i^{(0)} \cap A(0)$ for any annulus $A(0),$ and again the growth is the maximal possible growth 2.

So if we have $x$ in the same position but had half the 'x-boundary' then we would have growth 1.

Take any $z \in \{\lambda x : \lambda \in (0, 1)\},$ in our measure $\mu_1$ we know that for every $r > 0$ the set given by

$$\partial C_r(z) \cap \left(\partial S_i^{(z)} \cup \partial S_i^{(z)}\right) \cap \text{Spt}\mu$$

consists of four points, one at each corner of the square $\partial C_r(z) \cap e_i^+.$

For our measure $\mu$ we know that for any $z \in \text{Spt}\mu$

$$\partial C_r(z) \cap \partial S_i^{(z)} \cap \text{Spt}\mu \neq \emptyset$$

$$\partial C_r(z) \cap \partial S_i^{(z)} \cap \text{Spt}\mu \neq \emptyset$$

by Lemma 11 so we know

$$\partial C_r(z) \cap \left(\partial S_i^{(z)} \cup \partial S_i^{(z)}\right) \cap \text{Spt}\mu$$

consists of at least two points, and so this gives some indication why we can in some sense do half as well.
By the existence of a weak tangent in 

\[ V \in \left\{ < R_{i+1, i+2}^{(0)} >, < R_{i+1, i+5}^{(0)} > \right\} \]

we can pick points \( \{x_n\} \subset \text{Spt}\mu \) going up in some sort of line parallel to \( e_i \) as indicated on fig 16.2, and so we can make as good an approximation to fig 16.2 as we like, then applying Lemma 6 we get growth \( \geq 1 \).

**Proof** Let \( \epsilon > 0 \), let \( q \in \mathbb{N} \).

By Lemma 14 we can find weak tangent \( V \in \{ < e_{i+1}, e_{i+2} >, < e_{i+1}, e_{i+5} > \} \) and so we can find some small \( \delta > 0 \) such that

\[ x_n \in X (y, e_i, \epsilon) \cap A (y, \delta (2p - \epsilon), \delta (2p + \epsilon)) \cap G \]

for each \( p = 1, 2, \ldots, q \).

Now since \( r \in L^{(y)} \) we can take some \( h > 0 \) such that

\[ \int A(y, r, r + h) \cap \left( S^{(y)} \cup S^{(y)}_{i+6} \right) \left( h^2 - 3 \left( \| z - y \|- r \right)^2 \right) \in \left( \left( \lambda_i - \epsilon \right) h^4, \left( \lambda_i + \epsilon \right) h^4 \right) \]

Let \( n = \left\lfloor \frac{h}{2} \right\rfloor \), and from now on \( A(y) = A(y, r, r + h) \).
Now let \( m \in \{1, 2, \ldots, n - 1\} \).
For any \( p \in \{1, 2, \ldots, q\} \) note that

\[
\|p_{e_i^+}(x_p - y)\| \leq \epsilon \|p_{e_i}(x_q - y)\| \\
\leq 4\epsilon \delta q.
\]  

(2.64)

Now as we can see from Fig. 16.3

\[
d\left(\partial S_t^{(x_{p+1})} \cap \partial C_{r+2m\delta}(y), \partial S_t^{(P_{e_i+y}(x_{p+1}))} \cap \partial C_{r+2m\delta}(y)\right) \leq \|P_{e_i^+}(x_{p+1} - y)\| 
\]  

(2.65)

and

\[
d\left(\partial S_t^{(x_p)} \cap \partial C_{r+2m\delta}(y), \partial S_t^{(P_{e_i+y}(x_p))} \cap \partial C_{r+2m\delta}(y)\right) \leq \|P_{e_i^+}(x_p - y)\|. 
\]  

(2.66)

And as

\[
d\left(\partial S_t^{(P_{e_i+y}(x_p))} \cap \partial C_{r+2m\delta}(y), \partial S_t^{(P_{e_i+y}(x_{p+1}))} \cap \partial C_{r+2m\delta}(y)\right) = \|P_{e_i+y}(x_p - x_{p+1})\| \\
= \|x_p - x_{p+1}\| \\
\geq 2\delta - 2\epsilon \delta \\
= 2\delta (1 - \epsilon).
\]
So by (2.64), (2.65), (2.66) we have

\[
d \left( \partial S_i^{(x_p)} \cap \partial C_{r + 2m \delta} (y) , \partial S_i^{(x_{p+1})} \cap \partial C_{r + 2m \delta} (y) \right) \geq 2 \delta (1 - \epsilon) - 8 \epsilon \delta q \\
\geq 2 \delta (1 - 5q\epsilon) . 
\]  

(2.67)

For exactly the same reasons

\[
d \left( \partial S_{i+3}^{(x_p)} \cap \partial C_{r + 2m \delta} (y) , \partial S_{i+3}^{(x_{p+1})} \cap \partial C_{r + 2m \delta} (y) \right) \geq 2 \delta (1 - 5q\epsilon) .
\]

(2.68)

So let \( \alpha = \delta (1 - 5q\epsilon) \).

For each \( p \in \{1, 2, \ldots q - 1\} \) we can pick

\[
z_p^1 \in \partial S_i^{(x_p)} \cap \partial C_{r + 2m \delta} (y) \cap \text{Spt}\mu \\
z_p^2 \in \partial S_{i+3}^{(x_p)} \cap \partial C_{r + 2m \delta} (y) \cap \text{Spt}\mu.
\]

So by (2.67) and (2.68) we have that the set

\[\{ C_\alpha \left(z_p^a \right) : p \in \{1, 2, \ldots q\} , a \in \{1, 2\} \}\]

is disjoint.

We know

\[
\bigcup_{k=1}^{q-1} \bigcup_{p=1}^2 C_\alpha \left(z_p^k \right) \subset \left( \left( S_i^{(y)} \backslash S_i^{(x_q)} \right) \cup \left( S_{i+3}^{(x_q)} \backslash S_{i+3}^{(y)} \right) \right) \cap A (y, r + (2m - 1) \delta, r + (2m + 1) \delta)
\]

and this of course gives us that

\[
\mu \left( \left( \left( S_i^{(y)} \backslash S_i^{(x_q)} \right) \cup \left( S_{i+3}^{(x_q)} \backslash S_{i+3}^{(y)} \right) \right) \cap A (y, r + (2m - 1) \delta, r + (2m + 1) \delta) \right) \\
\geq \sum_{a=1}^{2} \sum_{p=1}^{q-1} \mu \left( C_\alpha \left(z_p^a \right) \right) \\
\geq 8 (q - 1) \alpha^2 . 
\]  

(2.69)

As in Lemma 8 we would like to use this to gain the estimate on the integral

\[
\int_{A(y)} \left( \left( \left( S_i^{(y)} \backslash S_i^{(x_q)} \right) \cup \left( S_{i+3}^{(x_q)} \backslash S_{i+3}^{(y)} \right) \right) \right) 4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu z
\]

and we will do this in exactly the same way as before.

Once again we note that the function \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) given by

\[
g (p) = \begin{cases} 
0 & \text{for } p \in [0, r) \\
4 \left( h^2 - (p-r)^2 \right) (p-r) & \text{for } p \in [r, r + h] \\
0 & \text{for } p > r + h
\end{cases}
\]

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is a Lipschitz function with Lipschitz constant $8h^2$.

So it can be well approximated by a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$
h(x) = \begin{cases} 
0 & \text{for } x \in [0, r + \delta) \\
\inf \{ g(t) : t \in [r + (2m - 1) \delta, r + (2m + 1) \delta] \} & \text{for } x \in [r + (2m - 1) \delta, r + (2m + 1) \delta] \\
0 & \text{for } x \geq r + (2n - 1) \delta.
\end{cases}
$$

Now for $x \in [r + (2m - 1) \delta, r + (2m + 1) \delta]$, $m = 1, 2, \ldots n - 1$ we have that

$$
|g(x) - h(x)| = |g(x) - \inf \{ g(t) : t \in [r + (2m - 1) \delta, r + (2m + 1) \delta] \}|
\leq |g(x) - g(r + 2m\delta)|
+ |g(r + 2m\delta) - \inf \{ g(t) : t \in [r + (2m - 1) \delta, r + (2m + 1) \delta] \}|
\leq 32\delta h^2.
$$

Note that by (2.69) we have that

$$
\int_{A(y)} \left( \left( S_{i+1} \setminus S_{i} \right) \cup \left( S_{i+3} \setminus S_{i+2} \right) \right) h(\|z - y\|) \, d\mu z
= \sum_{m=1}^{n-1} \int_{A(y, r + (2m-1)\delta, r + (2m+1)\delta)} \left( \left( S_{i} \setminus S_{i+1} \right) \cup \left( S_{i+3} \setminus S_{i+2} \right) \right) h(\|z - y\|) \, d\mu z
= \sum_{m=1}^{n-1} h(r + 2m\delta) \mu \left( A(y, r + (2m - 1) \delta, r + (2m + 1) \delta) \cap \left( \left( S_{i} \setminus S_{i+1} \right) \cup \left( S_{i+3} \setminus S_{i+2} \right) \right) \right)
\geq \sum_{m=1}^{n-1} 8h(r + 2m\delta) (q - 1) \alpha^2
= \sum_{m=1}^{n-1} 8h(r + 2m\delta) (1 - 5q\varepsilon)^2 (q - 1) \delta^2.
$$

Now note that

$$
\sum_{m=1}^{n-1} 2\delta h(r + 2m\delta) = \int_{\mathbb{R}} h(x) \, dLx.
$$

So we have that

$$
\int_{A(y)} \left( \left( S_{i} \setminus S_{i+1} \right) \cup \left( S_{i+3} \setminus S_{i+2} \right) \right) h(\|z - y\|) \, d\mu z
\geq 4(1 - 5q\varepsilon)^2 (q - 1) \delta \int_{\mathbb{R}} h(x) \, dLx.
$$

Now we know that

$$
\left| \int_{\mathbb{R}} h(x) \, dLx - \int_{\mathbb{R}} q(x) \, dLx \right| \leq 32\delta h^3.
$$
As before we calculate that

\[
\int_{\mathbb{R}} g(x) \, dLx = \int_r^{r+h} 4 \left( h^2 - (p-r)^2 \right) (p-r) \, dLp \\
= \int_0^h 4 \left( h^2 - p^2 \right) \, p \, dLp \\
= 4 \left( \frac{h^2}{2} - \frac{h^4}{4} \right) \\
= h^4.
\]

So we have that

\[
\int_{A(y) \cap \left( S_{i+3}^{(y)} \setminus S_i^{(y)} \right)} g \left( \|z - y\| \right) \, d\mu_z. \\
\geq 4\delta \left( 1 - 5\epsilon \right)^2 \left( q - 1 \right) \delta \left( h^4 - 32\delta h^3 \right)
\]

Now as \( x_q \in S_i^{(y)} \) so as in the remark following Lemma 6 we have that

\[
\int_{A(y) \cap \left( S_i^{(y)} \setminus S_i^{(y)} \right)} 4 \left( h^2 - (\|z - y\| - r)^2 \right) (\|z - y\| - r) \, d\mu_z \\
\leq \int_{A(y) \cap \left( S_{i+3}^{(y)} \setminus S_i^{(y)} \right)} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) \|x_q - y\| \, d\mu_z + c_{ghr} \|x_q - y\|^2.
\]

Recall we chose \( h > 0 \) such that

\[
\int_{A(y) \cap \left( S_i^{(y)} \setminus S_i^{(y)} \right)} -4 \left( h^2 - 3 (\|z - y\| - r)^2 \right) \, d\mu_z \leq 2 (\lambda_i + \epsilon) h^4.
\]

Thus we have that

\[
4 \left( 1 - 5\epsilon \right)^2 \left( q - 1 \right) \delta \left( h^4 - 32\delta h^3 \right) \leq 2 (\lambda_i + \epsilon) \|x_q - y\| h^4 + c_{ghr} \|x_q - y\|^2.
\]  \( (2.71) \)

Note that

\[
4 \left( 1 - 5\epsilon \right)^2 \left( q - 1 \right) \delta \left( h^4 - 32\delta h^3 \right) = 4\delta \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right) - 4\delta \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right).
\]

Recall that \( 2\epsilon \delta \geq \|x_q - y\| - \epsilon \delta \) so that

\[
4 \left( 1 - 5\epsilon \right)^2 \left( q - 1 \right) \delta \left( h^4 - 32\delta h^3 \right) \geq 2 \left( \|x_q - y\| - \epsilon\delta \right) \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right) - 4\delta \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right).
\]

Thus we have

\[
2 \left( \|x_q - y\| - \epsilon\delta \right) \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right) \leq 2 (\lambda_i + \epsilon) \|x_q - y\| h^4 + c_{ghr} \|x_q - y\|^2 \]
\[+ 4\delta \left( 1 - 5\epsilon \right)^2 \left( h^4 - 32\delta h^3 \right).
\]
Now dividing through by $\|x_q - y\|$ 

$$2 \left( 1 - \frac{\epsilon \delta}{\|x_q - y\|} \right) (1 - 5q e)^2 \left( h^4 - 32\delta h^3 \right) \leq 2 (\lambda_i + \epsilon) h^4 + c \delta h^2 r^2 \|x_q - y\|$$

$$+ \frac{4\delta}{\|x_q - y\|} (1 - 5q e)^2 \left( h^4 - 32\delta h^3 \right).$$

As we let $q \to \infty$ so we have that $\frac{\delta}{\|x_q - y\|}$ becomes as small as we like, and note that no matter how big $q$ is we can always make $\epsilon$ sufficiently small so that $5q \epsilon$ is negligible, so we must have that

$$1 \leq \lambda_i$$

thus proving the lemma. $\square$
2.7.1 Application: Forcing gaps in parts of the boundary

**Lemma 17** Let \( \mu \) be a 2-uniform measure where \( \text{Spt} \mu \cap C_2(0) \) is purely unrectifiable. Let \( y \in G \cap C_1(0) \) and suppose \( r \in \Theta_i(y) \cap [0, 1) \) then for each \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) we must be able to find \( k \in \{i, i + 3\} \) such that for some \( r_0 \) arbitrarily close to \( r \)

\[
\partial C_{r_0}(y) \cap \left( R_{k,j_i}^{(y)} \cup R_{k,j_i+3}^{(y)} \right) \cap \text{Spt} \mu = \emptyset.
\]

**Proof**

First of all note that \( \Theta_i(y) \) is an open set so we can find \( s_0 > 0 \) such that

\[
(r - s_0, r + s_0) \subset \Theta_i(y).
\]

Now suppose for our particular choice of \( j \in \{i + 1, i + 2, i + 4, i + 5\} \) we have that for some \( k \in \{i, i + 3\} \) the following property holds:

For all \( s \in (r - s_0, r + s_0) \) we have that

\[
\partial C_s(y) \cap R_{j,k}^{(y)} \cap \text{Spt} \mu \neq \emptyset
\]

\[
\partial C_s(y) \cap R_{j+3,k+3}^{(y)} \cap \text{Spt} \mu \neq \emptyset
\]

then by Lemma 15 we have that for any \( s \in L^{(y)} \cap (r - s_0, r + s_0) \) so \( f_j^{(y)}(s) \geq 1 + \frac{1}{64} \) and so since we know that \( f_{j+1}^{(y)}(s) \geq 1 \) by Lemma 16 and \( f_{i}^{(y)}(s) = 2 \) by Lemma 9 so we would have that

\[
f_i^{(y)}(s) + f_j^{(y)}(s) + f_{j+1}^{(y)}(s) \geq 4 + \frac{1}{64}
\]

contradiction.

So we must be able to find \( r_1 \in (r - s_0, r + s_0) \) and \( s_1 \in (0, s_0) \) which has the property that

\[
A(y, r_1 - s_1, r_1 + s_1) \cap R_{j_1,k_1}^{(y)} \cap \text{Spt} \mu = \emptyset
\]

(2.72)

for some \( \{j_1, k_1\} \subseteq \{\{j, k\}, \{j + 3, k + 3\}\} \).

Note that by Lemma 11 we know that

\[
\partial C_s(y) \cap \partial S_{j_1}^{(y)} \cap \text{Spt} \mu \neq \emptyset
\]

(2.73)

for any \( s > 0 \).

We'll assume we took \( s_1 \) sufficiently small so that \( (r_1 - s_1, r_1 + s_1) \subset \Theta_i(y) \) so by definition we know there is no measure on \( R_{j_1,k_1+1}^{(y)} \cup R_{j_1,k_1+4}^{(y)} \) in the annulus \( A(y, r_1 - s_1, r_1 + s_1) \) as shown in fig 16.1 (over-page).
So necessarily we must have by (2.73)

$$\partial C_s(y) \cap R_{j_1,k_1}^{(y)} \cap \text{Spt} \mu \neq \emptyset$$

for any \( s \in (r_1 - s_1, r_1 + s_1) \).

Now if we had

$$\partial C_s(y) \cap R_{j_1+3,k_1}^{(y)} \cap \text{Spt} \mu \neq \emptyset$$

for every \( s \in (r_1 - s_1, r_1 + s_1) \), then as before we would have by Lemma 15 that any point \( s \in (r_1 - s_1, r_1 + s_1) \cap L^{(y)} \) is such that \( f_{j_1}'(s) > 1 + \frac{1}{64} \) and this is again a contradiction.

So we must be able to find \( r_2 \in (r_1 - s_1, r_1 + s_1) \) and \( s_2 \in (0, s_1) \) such that

$$A(y, r_2 - s_2, r_2 + s_2) \cap R_{j_1+3,k_1}^{(y)} \cap \text{Spt} \mu = \emptyset$$

Now by (2.72) we have

$$A(y, r_2 - s_2, r_2 + s_2) \cap \left( R_{j_1,k_1}^{(y)} \cup R_{j_1+3,k_1}^{(y)} \right) \cap \text{Spt} \mu$$

$$= A(y, r_2 - s_2, r_2 + s_2) \cap \left( R_{j,k_1}^{(y)} \cup R_{j+3,k_1}^{(y)} \right) \cap \text{Spt} \mu = \emptyset \quad \square$$
2.8 The final contradiction

In this section we finally prove the Theorem.

Pick \( i \in \{1, 2, \ldots, 6\} \) by Lemma 14 we know that for \( \mu \) a.a. \( z \in G \cap C_1(0) \) we can find a weak tangent \( V \in \{< R_{i+1,i+2}^{(z)}, < R_{i+1,i+5}^{(z)} >\} \).

So pick a point \( y \in G \cap C_1(0) \) which has a weak tangent \( V = < R_{i+1,i+2}^{(y)} > \).

Let \( \epsilon > 0 \).

We can find some \( \alpha > 0 \) such that

\[
\mu \left( C_\alpha (y) \setminus N_\epsilon (V) \right) < \epsilon \alpha^2. \tag{2.74}
\]

So necessarily we have that

\[
\partial C_{\frac{\alpha}{2}} (y) \cap \left( R_{i+1,i+5}^{(y)} \cup R_{i+2,i+4}^{(y)} \right) \cap \text{Spt}\mu = \emptyset
\]

since otherwise we could fix a cube of radius \( \frac{\alpha}{4} \) in \( C_\alpha (y) \setminus N_\epsilon (V) \) and this would contradict (2.74).

So by closure of \( \text{Spt}\mu \) we have that for some small \( s_0 > 0 \)

\[
A \left( y, \frac{\alpha}{2} - s_0, \frac{\alpha}{2} + s_0 \right) \cap \left( R_{i+1,i+5}^{(y)} \cup R_{i+2,i+4}^{(y)} \right) \cap \text{Spt}\mu = \emptyset
\]

and so by Lemma 9 part 3 we know that for all \( t \in \left( \frac{\alpha}{2} - s_0, \frac{\alpha}{2} + s_0 \right) \cap L^{(y)} \) we have \( f_t^{(y)}(t) = 2 \).
So we must be able to find some $t_1 \in \left( \frac{s}{2} - s_0, \frac{s}{2} + s_0 \right)$ such that

$$\partial C_{t_1}(z) \cap \left( R_{i+1,i+4}^{(y)} \cup R_{i+1,i+4,i+5}^{(y)} \right) \cap \text{Spt} \mu = \emptyset$$

since otherwise by Lemma 9 part 2 we would have that $f_{i+1}^{(y)}(t) + \frac{1}{32} \leq f_{i+1}^{(y)}(t) + f_{i+2}^{(y)}(t)$ for all $t \in \left( \frac{s}{2} - s_0, \frac{s}{2} + s_0 \right) \cap L^{(y)}$, and since we know that $f_{i+1}^{(y)}(t) = 2$ so we have that $f_{i+1}^{(y)}(t) + f_{i+2}^{(y)}(t) \geq 4 + \frac{1}{32}$ and this is a contradiction.

So by closure of Spt$\mu$ we have some $s_1 > 0$ such that

$$A(y, t_1 - s_1, t_1 + s_1) \cap \left( R_{i+1,i+4}^{(y)} \cup R_{i+1,i+4,i+5}^{(y)} \right) \cap \text{Spt} \mu = \emptyset$$

i.e. $(t_1 - s_1, t_1 + s_1) \subset \Theta_i^{(y)}$.

Now by Lemma 17 we have that there exists $k_1 \in \{ i, i + 3 \}$ such that for some $r_0 \in (t_1 - s_1, t_1 + s_1)$

$$\partial C_{r_0}^k(y) \cap \left( R_{k_1,i+4}^{(y)} \cup R_{k_1,i+4,i+5}^{(y)} \right) \cap \text{Spt} \mu = \emptyset$$

and again by closure of Spt$\mu$ we have that there is some $s_2 > 0$ such that

$$A(y, r_0 - s_2, r_0 + s_2) \cap \left( R_{k_1,i+1}^{(y)} \cup R_{k_1,i+4}^{(y)} \right) \cap \text{Spt} \mu = \emptyset$$

and $(r_0 - s_2, r_0 + s_2) \subset \Theta_i^{(y)}$.

By applying Lemma 17 again we have that there is some $k_2 \in \{ i, i + 3 \}$ such that for some $r_1 \in (r_0 - s_2, r_0 + s_2)$

$$\partial C_{r_1}^k(y) \cap \left( R_{k_2,i+2}^{(y)} \cup R_{k_2,i+5}^{(y)} \right) \cap \text{Spt} \mu = \emptyset.$$

Now $k_1 \neq k_2$ since otherwise we have that

$$\partial C_{r_1}^k(y) \cap S_{k_1}^{(y)} \cap \text{Spt} \mu = \emptyset$$

contradicting Lemma 11.

Now without loss of generality assume $k_1 = i$, $k_2 = i + 3$.

Let

$$B_1 = \left\{ s > 0 : A(y, s, r_1) \cap \left( \left( R_{i+1}^{(y)} \cup R_{i+4}^{(y)} \right) \cup \left( R_{i+3,i+4}^{(y)} \cup R_{i+3,i+5}^{(y)} \right) \right) \cap \text{Spt} \mu = \emptyset \right\}.$$

$$a = \inf \left\{ s \in [0, r_1] : (s, r_1) \subset \Theta_i^{(y)} \text{ and } (s, r_1) \subset B_1 \right\}.$$

Note that

$$\partial C_s^a(y) \cap \left( \bigcup_{j_1,j_2 \in \{i+1,i+2,i+4,i+5\}, e_{j_1}e_{j_2} = 0} \partial R_{j_1,j_2}^{(y)} \right)$$

$$= \partial C_s^a(y) \cap \left( \partial R_{i+1}^{(y)} \cup \partial R_{i+4}^{(y)} \cup \partial R_{i+3,i+2}^{(y)} \cup \partial R_{i+3,i+5}^{(y)} \right).$$
By closure of $\text{Spt}_\mu$ there is a point $z \in \partial C_a(y) \cap \text{Spt}_\mu$ and by Lemma 10

$$z \in \partial R_{i,i+1}^{(y)} \cup \partial R_{i,i+4}^{(y)} \cup \partial R_{i+3,i+2}^{(y)} \cup \partial R_{i+3,i+5}^{(y)}.$$ 

Now suppose $z \in \partial C_a(y) \cap \partial R_{i,i+1}^{(y)}$ then for some $j_1 \in \{i + 2, i + 5\}$ we have that

$$z \in \partial R_{i,i+1}^{(y)} \cap \partial R_{i+1,j_1}^{(y)}$$

and as

$$A(y,a,r_1) \cap \left( \text{int} \left( R_{i,i+1}^{(y)} \cup R_{i+1,j_1}^{(y)} \right) \right) \cap \text{Spt}_\mu = \emptyset$$

so our point $z$ has in the direction $e_{i+1}$ a great expanse of nothing above it as shown in fig 18.2.

This situation by Lemma 10 cannot happen. In all other cases, $z \in \partial R_{i,i+4}^{(y)}$, $z \in \partial R_{i+3,i+2}^{(y)}$ and $z \in \partial R_{i+3,i+5}^{(y)}$ we get exactly the same contradiction from Lemma 10.
This finally completes the proof. □
Open Problems

• Conjecture 3 Given a Radon measure \( \mu \) on \( l^3_\infty \) with the property that for some small \( \delta > 0 \) there exist \( \alpha > 0 \) such that

\[
\mu (C_r (x)) = \alpha r^2
\]

for all \( x \in \text{Spt}_\mu \) and all \( r \in (0, \delta) \) then \( \mu \) is rectifiable.

At this moment Conjecture 3 is still open, in order to push this subject further the first thing that must be done is to finally settle it. As I mentioned this would have no application outside Rectifiability and Densities in \( l^3_\infty \), however it has to be done. It should be noted that the best that is known about locally uniform measures in Euclidean space is that they are rectifiable and the only way to prove this is with half the machinery of Preiss’s proof, where as all the partial results in the subject by Besicovitch, Marstrand, Mattila and others, can be proved in a couple of pages with elementary parts of the machinery of this proof, plus some standard methods.

• Conjecture 4 Given a Radon measure \( \mu \) on \( l^3_\infty \) with the property that

\[
0 < \lim_{r \to 0} \frac{\mu (C_r (x))}{r^2} < \infty
\]

for \( \mu \) a.a. \( x \in \text{Spt}_\mu \) then \( \mu \) is rectifiable.

My guess is that the only way to get to this result is to prove.

• Conjecture 5 Let \( \{e_1, e_2, e_3\} \) be the orthonormal coordinates axes of \( l^3_\infty \).

Given a Radon measure \( \mu \) on \( l^3_\infty \) with the property that for some \( \alpha > 0 \)

\[
\mu (C_r (x)) = \alpha r^2
\]

for all \( x \in \text{Spt}_\mu \) and all \( r > 0 \) then \( \text{Spt}_\mu \) is either a 2-plane or the graph of a Lipschitz function \( f : e_i^* \to e_i \) with \( \text{Lip} (f) \leq 1 \), for some \( i \in \{1, 2, \ldots 6\} \).

• Proving measures with density in \( l^n_\infty \) have weak tangents is massively easier than proving any sort of rectifiability. By Lemma 2 the existence of rectifiable subsets for uniform measures implies weak tangents, I have however an elementary proof of this which is much shorter and requires none of the detailed calculation of the proof of Theorem 9. Unfortunately the elementary proof doesn’t seem to work at
all if the density is not taken with respect to \( r^2 \), it would have been nice to have shown that if the density is taken with respect to \( r^s \) then \( s \leq 2 \), this is still open. I haven't even tried to see what happen with the proof of Theorem 9 when we have a locally uniform measure with respect to \( r^s \).

Proving weak tangents for measures with density with respect to \( r^s \) where \( s < 2 \), is much harder because the support of the measure could be contained in a 2-plane cutting through the cube at any angle, and so the problem becomes one of density taken with a completely different convex body. So proving Marstrand theorem for the cube is probably just as hard as proving it for all convex bodies, however getting weak tangents is really not so hard and this might be an accessible problem, I certainly think its easier than Conjecture 5. I am not going to formulate this as a conjecture because I have spent very little time thinking about it, but I think its worth trying.

**Question 1** Given a Radon measure \( \mu \) on \( \mathbb{R}^n \) and a convex body \( C \) containing \( 0 \) where \( \mu \) has the property that

\[
0 < \lim_{r \to 0} \frac{\mu(x + rC)}{r^s} < \infty
\]

for \( \mu \) a.a. \( x \in \text{Spt}_\mu \) then is \( s \) is an integer?

- The biggest open problem is to find and prove the correct generalisation to Preiss's theorem. Conjectures 1 and 2 are attempts at this, but possibly the correct generalisation is of a very different form.
Bibliography


