The Besicovitch-Hausdorff dimension of the Residual Set of Packings of Convex Bodies in R^n

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ABSTRACT

I undertake a study of the Besicovitch-Hausdorff dimension of the residual set of arbitrary packings of convex bodies in $\mathbf{R}^n$.

In my second chapter, I consider packings of convex bodies of bounded radius of curvature and of fixed orientation into the unit plane square. I show that the Besicovitch-Hausdorff dimension, s, of the residual set of an arbitrary packing satisfies

$$s > 1 + \frac{1}{\log r_\theta}$$

where r_θ is the bound for the radius of curvature.

In chapter 3, I construct a packing which demonstrates that this bound is of the correct order.

I generalise the 2-dimensional result to higher dimensions in chapter 4. I use a slicing arguement to prove this.

In the final chapter, I tackle the disk packing problem. Using Dirichlet cells, I

Abstract

improve the bound obtained in [1] to 1.033.

ACKNOWLEDGEMENTS

I would like to thank Professor D.G. Larman, my supervisor, for his encouragement and guidance throughout the research of this work. I would like to thank him for his patience throughout its duration.

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Finally I would like to thank my parents for their encouragement and support throughout all of my studies. I would like to dedicate this thesis to them.

Contents

1. INTRODUCTION

In this thesis I study the Besicovitch-Hausdorff dimension of the residual set of packings of convex bodies in $\mathbf{R}^n$. When studying packings of $\mathbf{R}^n$ we restrict our attention to the unit n-cube in $\mathbf{R}^n$; it is clear that doing so is not detremental to the generality of the problem.

Let

$$I_n = \left\{ \underline{x} \in \mathbf{R}^n \colon \|\underline{x}\|_\infty \leq \frac{1}{2} \right\}$$

where $\|\underline{x}\|_\infty = \max_{1 \leq k \leq n} |x_k|$, for $\underline{x} = (x_1, \ldots, x_n)$.

A packing of I_n is the union of disjoint open bodies, θ_i, for $i \in \mathcal{I}$, some index set, such that

$$\bigcup_{i \in \mathcal{I}} \theta_i \subset I_n.$$

For a given packing of I_n, we define the residual set

$$R = I_n \backslash \bigcup_{i \in \mathcal{I}} \theta_i.$$

In this thesis I consider arbitrary packings of I_n by reduced copies of some general convex body θ. We assume that the orientation of the reduced copies are fixed. The results in this thesis give bounds on the Besicovitch-Hausdorff dimension of R, for an arbitrary packing $\{\theta_i\}_{i\in\mathcal{I}}$, dependent on the convex body θ.

1.1. Besicovitch-Hausdorff dimension

In this section we give the definition of the Besicovitch-Hausdorff dimension of a set $E \subset \mathbf{R}^n$. Firstly we define the $s-$dimensional Besicovitch-Hausdorff measure of E. Let $0 < s < \infty$ and $\delta \geq 0$, then we define

$$\begin{aligned}\mathcal{H}^s_\delta(E) = \; & \inf \{\textstyle\sum_i \operatorname{diam}(b_i)^s \colon b_i \text{ are balls in } \mathbf{R}^n \text{ such that} \\ & E \subset \textstyle\bigcup_i b_i, \text{ and } \operatorname{diam}(b_i) \leq \delta\}.\end{aligned}$$

It is easy to see that $\mathcal{H}^s_\delta(E)$ is non-increasing, as a function of δ decreasing. Thus the limit

$$\mathcal{H}^s(E) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(E)$$

exists. This limit is the s-dimensional Besicovitch-Hausdorff measure of the set E. We can now define the Besicovitch-Hausdorff dimension of the set E.

Definition 1. *The Besicovitch-Hausdorff dimension of a set $E \subset \mathbf{R}^n$ is*

$$\begin{aligned} \dim E &= \sup\{s: \mathcal{H}^s(E) > 0\} = \sup\{s: \mathcal{H}^s(E) = \infty\} \\ &= \inf\{t: \mathcal{H}^t(E) < \infty\} = \inf\{t: \mathcal{H}^t(E) = 0\}. \end{aligned}$$

1.2. Results

A considerable amount of work has been undertaken concerning packings of open discs in the plane. In [1] D.G. Larman showed that a lower bound for the Besicovitch-Hausdorff dimension of the residual set of arbitrary packing of disks in the plane is 1.03. Previously, in [5], K. Hirst had considered the Apollonian packing of circles using different methods. P. Gruber, in [3], considered packings of general convex bodies, and dealt with packings where variation of orientation was permitted.

In chapter 2 we consider packings of convex bodies of bounded radius of curvature, and of fixed orientation. The methods we use are based on those of D.G. Larman in [1] for the disc packing problem.

1. Introduction

We pack a collection of open, strictly convex bodies $\{\theta_i\}_{i=1}^{\infty}$ of bounded radius of curvature r_θ and of fixed orientation into the unit plane square I_2. Then the residual set $R = I_2 \backslash \{\theta_i\}_{i=1}^{\infty}$ is compact and has Besicovitch-Hausdorff dimension at least 1. We show that for packings of this type that the Besicovitch-Hausdorff dimension $s_2(r_\theta)$ of the residual set R is at least

$$1 + \epsilon(r_\theta) \quad \text{where } \epsilon(r_\theta) = O\left(\frac{1}{\log r_\theta}\right) \tag{1.1}$$

$$0 < \epsilon(r_\theta) < 1,$$

and where r_θ is a bound for the radius of curvature of the convex body θ. In chapter 3, we continue our study of 2-dimensional packings of convex bodies and proceed to construct, for θ sufficiently large, a packing for which the Besicovitch-Hausdorff dimension of the residual set is of the same order as our lower bound,

$$1 + \frac{1}{\log r_\theta},$$

thereby demonstrating that the bound we obtain in chapter 2 is of the correct order.

In chaper 4, we consider packings in higher dimensions, where θ is an n-

dimensional convex body with radius of curvature bounded above by r_θ. We extend the 2-dimensional result, using an inductive slicing arguement, to show that in higher dimensions the Besicovitch-Hausdorff dimension $s_n(r_\theta)$ of the residual set R is at least

$$s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1. \tag{1.2}$$

Here $s_n(r_\theta)$ is defined by

$$\begin{aligned} s_n(r_\theta) = \inf\{ \quad & s\colon s \text{ is the dimension of the residual set} \\ & \text{of the packing } (\theta_m)_{m=1}^{\infty} \text{ in } I_n\} \end{aligned}$$

where the infimum is taken over all packings of convex n-bodies with radius of curvature bounded by r_θ. This will lead to the result

$$s_n(r_\theta) \geq (n-1) + \epsilon(r_\theta) \tag{1.3}$$

where, by combining (1.1) and (1.2)

$$\epsilon(r_\theta) = O\left(\frac{1}{\log r_\theta}\right). \tag{1.4}$$

In chapter 5 we turn our attention to arbitrary packings of disks $\{\theta_m\}_{m=1}^{\infty}$ into the plane square I_2. A lower bound for the Besicovitch-Hausdorff dimension of the residual set R is greater than 1.03. This was shown by D.G.Larman in [1]. We improve this bound by developing the methods used to attempt to obtain the best possible result in 2-dimensions. We do this using Dirichlet cell methods.

2. Convex bodies of bounded radius of curvature

Theorem 1. *Suppose that $\{\theta_n\}_{n=1}^{\infty}$ forms a packing within the unit plane square I_2 of strictly convex bodies whose radius of curvature is bounded above by r_θ. Then the residual set $R = I_2 \setminus \bigcup_{n=1}^{\infty} \theta_n$ has Besicovitch - Hausdorff dimension s where*

$$s > 1 + \epsilon \qquad \text{where } \epsilon \sim \frac{1}{\log r_\theta}$$

.

Proof. We may suppose without loss of generality that the θ_n are open sets, and that diam $(\theta_{n+1}) \leq$ diam (θ_n) for $n = 1, 2, \ldots$. This gives us an order to our packing. The largest copy being θ_1, and the θ_i's decreasing in size as i increases.

Let $s = 1 + \epsilon$ where $0 < \epsilon < 1$, and define the n^{th} stage residual set,

$$R_n = I_2 \backslash \bigcup_{m=1}^{n} \theta_m$$

and the residual set, R, for the packing, by

$$R = \bigcap_{n=1}^{\infty} R_n = I_2 \backslash \bigcup_{m=1}^{\infty} \theta_m.$$

Since R is compact we need only consider coverings of R by finite collections of open squares $\{C_j\}_{j=1}^{q}$ when determining whether the $(1+\epsilon)$-dimensional Hausdorff measure of R is positive.

Let $\{C_j\}_{n=1}^{p}$ be a cover of R by a finite number of open squares. We may suppose that all of the C_j's are necessary, i.e. none of them are contained within any of the θ_n, $n = 1, 2, \ldots$, nor contained in each other. Let the diameter of C_j be $\sqrt{2}\Delta_j$, for $j = 1, 2, \ldots, p$.

Each R_n is compact, and $R_{n+1} \subset R_n$ for each n and hence $\{R_n\}_{n=1}^{\infty}$ is a nested sequence of non-empty compact sets. This implies there exists some $m \in \mathbf{N}$ such that

$$R_m \subset \bigcup_{j=1}^{p} C_j$$

and from this point on we shall fix such a sufficiently large m.

We shall find a condition on ϵ such that under this condition $\sum_j \Delta_j^{1+\epsilon}$ cannot be close to zero which will in turn imply $\dim(\mathrm{R}) \geq 1+\epsilon = \mathrm{s}$

Now suppose that the x-axis is the horizontal axis, and the y-axis is the vertical axis. Let $0 \leq x \leq 1$, and let l_x denote the vertical line through the point $(x, 0)$, i.e. $l_x\{(X, Y), X = x\}$.

Let $l'(x, j)$ be the open interval equal to $l_x \cap C_j$, and let $l_{(x,j)}$ represent its length. Then, either we have

$$l_{(x,j)} = \frac{1}{\sqrt{2}}\mathrm{Diam}(\mathrm{C_j}) = \Delta_{\mathrm{j}},$$

or

$$l_{(x,j)} = 0.$$

Let us define a function $f(x)$ to be

$$f(x) = \sum_{j=1}^{p} l_{(x,j)}^{\epsilon}.$$

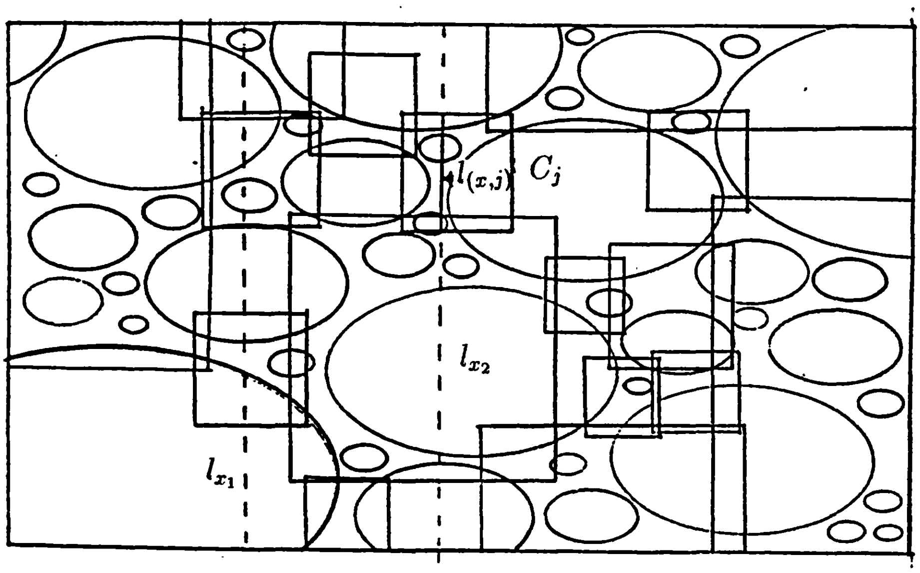

Figure 2.1: **If $x = x_1$ then $l_{(x,j)} = 0$. If $x = x_2$ then $l_{(x,j)} = \Delta_j$.**

Integrating from 0 to 1 gives,

$$\int_0^1 f(x)\, dx = \int_0^1 \sum_{j=1}^{p} l^{\epsilon}_{(x,j)} dx = \sum_{j=1}^{p} \Delta_j^{1+\epsilon} \qquad (2.1).$$

In time we will define another function g_m such that $\int_0^1 f(x)dx \geq \int_0^1 g_m(x)dx$ and then proceed to show $\int_0^1 g_0(x)dx > 0$ and deduce our result.

So let us consider $\cup_{j=1}^{p} C_j$, then l_x intersects it in a collection of non overlapping

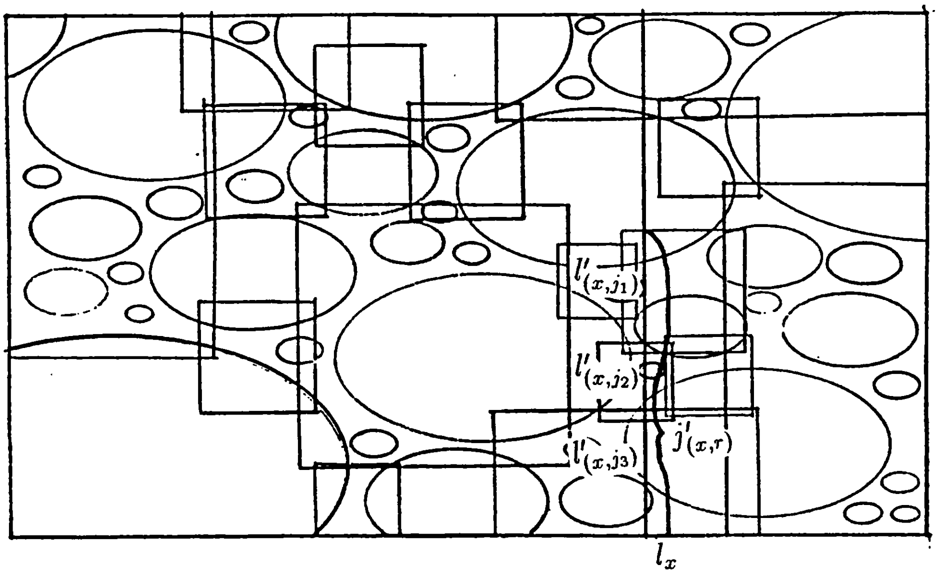

Figure 2.2: **For example** $j'_{(x,r)} = \bigcup_{i=1}^{3} l_{(x,j_i)}$.

open intervals $\left\{j'_{(x,r)}\right\}_{r=1}^{v}$ of lengths $\left\{j_{(x,r)}\right\}_{r=1}^{v}$.

Now each of these intervals can be expressed as the union of $\left\{l'_{(x,j_i)}\right\}_{i=1}^{v(r)}$ chosen from $\left\{l'_{(x,j)}\right\}_{j=1}^{p}$, so $j'_{(x,r)} = \cup_{i=1}^{r} l'_{(x,j)}$.

Now $j_{(x,r)} \leq \sum l_{(x,j)}$, and since $0 < \epsilon < 1$,

$$j^{\epsilon}_{(x,r)} \leq \sum_{i=1}^{v(r)} l^{\epsilon}_{(x,j_i)} \qquad r = 1, \ldots, v$$

Summing over r gives,

$$\sum_{r=1}^{v} j^{\epsilon}_{(x,r)} \leq \sum_{r=1}^{p} \sum_{i=1}^{v(r)} l^{\epsilon}_{(x,j)} \leq \sum_{j=1}^{p} l^{\epsilon}_{(x,j)}. \tag{2.2}$$

Let us define our function $g_m(x)$, by

$$g_m(x) = \sum_{r=1}^{v} j^{\epsilon}_{(x,r)}.$$

So, if we integrate $g_m(x)$ from 0 to 1 and use (2.1), we then have

$$\begin{aligned} \int_0^1 g_m(x)\, dx &= \int_0^1 \sum_{r=1}^{v} j^{\epsilon}_{(x,r)}\, dx \\ &\leq \int_0^1 \sum_{j=1}^{p} l^{\epsilon}_{(x,j)}\, dx \\ &= \sum_{j=1}^{p} \Delta_j^{1+\epsilon}. \end{aligned} \tag{2.3}$$

If we show for any cover $\{C_k\}_{k=1}^{p}$, that

$$\int_0^1 g_m(x)\, dx > 1,$$

then the residual set R will have Besicovitch-Hausdorff dimension of at least $1+\epsilon$.

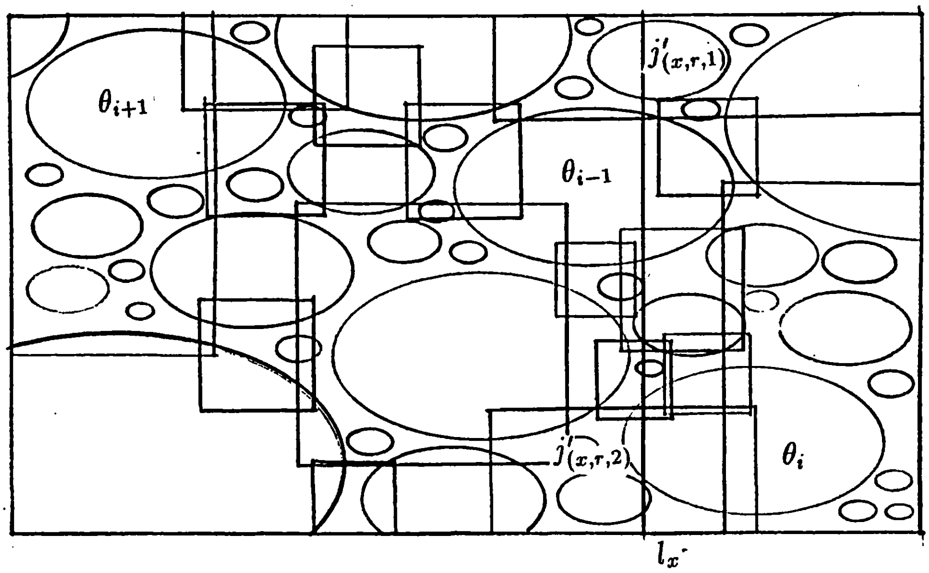

Figure 2.3: l_x **intersects** $\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k$ **in a collection of disjoint intervals.**

Let i be some integer, $0 \leq i \leq m-1$ and consider

$$\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k.$$

Then l_x intersects this union in a collection of disjoint intervals $\left\{j'_{(x,r,i)}\right\}_{r=1}^{v(i,x)}$ of lengths $\left\{j_{(x,r,i)}\right\}_{r=1}^{v(i,x)}$ respectively.

Define

$$g_i(x) = \sum_{r=1}^{v(i,x)} j^{\epsilon}_{(x,r_i,i)}$$

If i is one of $1,\ldots,m$, then

$$\int_0^1 g_i(x)\,dx = \int_0^1 g_{i-1}(x)\,dx + \int_0^1 \left(g_i(x) - g_{i-1}(x)\right)\,dx \tag{2.4}$$

We shall show that for any i

$$\int_0^1 (g_i(x) - g_{i-1}(x))dx \geq 0$$

and since we clearly have $\int_0^1 g_0(x)dx = 1$ the result follows, since this implies

$$\int_0^1 g_m(x)dx \geq \int_0^1 g_0(x)dx = 1.$$

For $0 \leq x \leq 1$, l_x meets θ_i in an interval, possibly empty, of length α say.

Let $l'_{(x,\theta_i)}$ be the interval $l_x \cap \theta_i$, and $l_{(x,\theta_i)}$ be it's length. Then either

$$l_{(x,\theta_i)} = \alpha, \qquad \text{if } l_x \text{ meets } \theta_i,$$

or

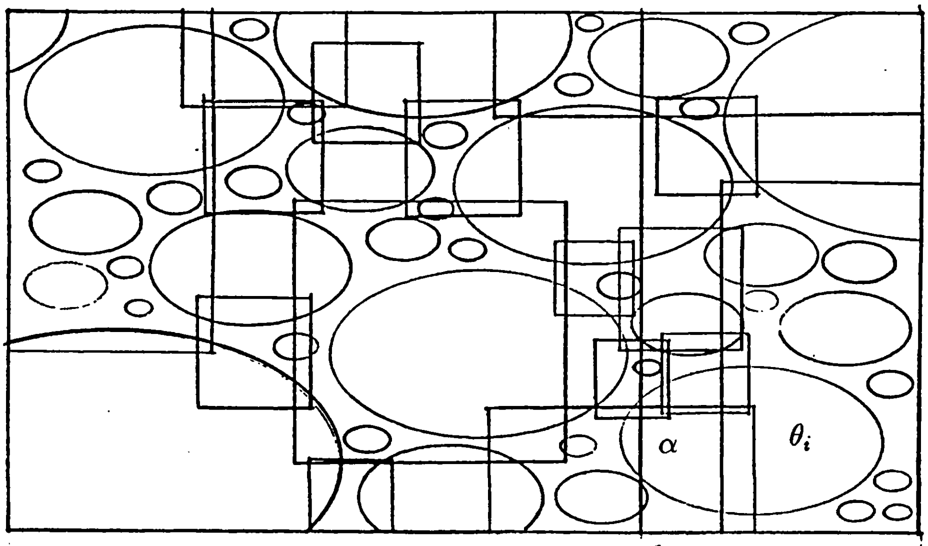

Figure 2.4:

$$l_{(x,\theta_i)} = 0 \qquad \text{otherwise.}$$

Now $g_i(x) = g_{i-1}(x)$ for those x such that l_x does not meet θ_i. As there are no extra intervals to consider, so we need only worry about those x where l_x meets θ_i. The segment of l_x which lies in θ_i meets $\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k$ in a collection of non-overlapping intervals $\{r'_j\}_{j=2}^{w-1}$ of lengths $\{r_j\}_{j=2}^{w-1}$, whose closures do not meet

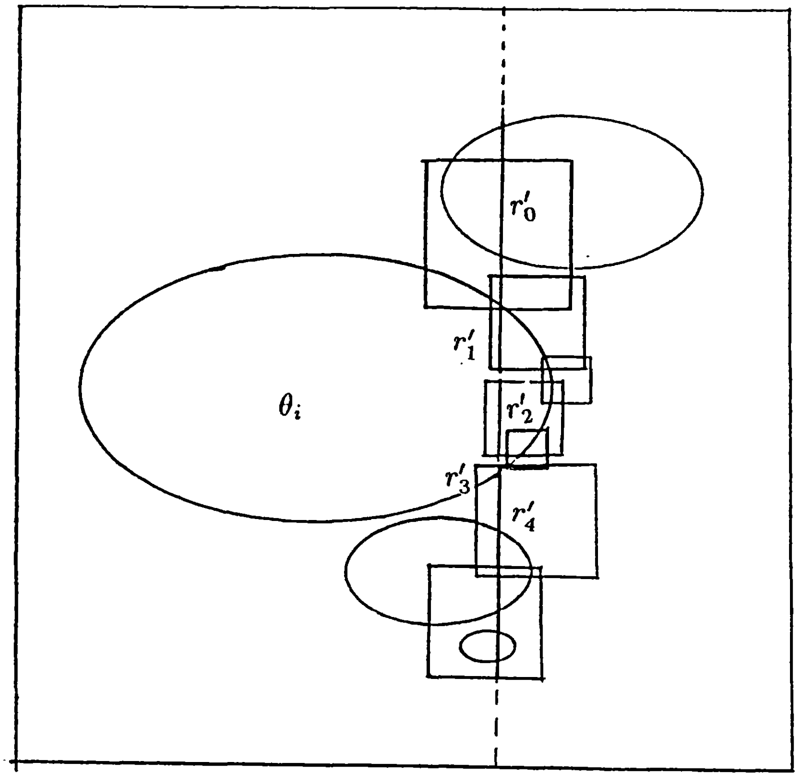

Figure 2.5:

the boundary of θ_i, and two intervals r_1 and r_w , (r_w below r_1) , whose closures meet θ_i, in fact they may coincide.

The line l_x also meets $\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k$ in two intervals r'_0, r'_{w+1} of lengths r_0, r_{w+1} , immediately above and below θ_i respectively.

Let us define a function $T(x)$ such that

$$T(x) = \begin{cases} 1 & \text{if } r_1' \text{ and } r_w' \text{ do not coincide} \\ 0 & \text{otherwise} \end{cases}$$

Then the difference is

$$\begin{aligned} g_i(x) - g_{i-1}(x) = T(x)[(r_0 &+ r_1)^\epsilon + r_2^\epsilon + \cdots \\ &+ r_{w-1}^\epsilon + (r_w + r_{w+1})^\epsilon - (r_0 + \alpha + r_{w+1})^\epsilon]. \quad (2.5) \end{aligned}$$

Suppose that $0 \leq \mu \leq r_0$ and $0 \leq \lambda \leq r_{w+1}$.

Let

$$\begin{aligned} h_i(x, \mu, \lambda) = T(x)[(\mu + r_1)^\epsilon + r_2^\epsilon + \cdots + r_{w-1}^\epsilon &+ (r_w + \lambda)^\epsilon \\ &- (\mu + \alpha + \lambda)^\epsilon]. \quad (2.6) \end{aligned}$$

Then $g_i(x) - g_{i-1}(x) = h_i(x, r_0, r_{w+1})$. We show that $g_i(x) - g_{i-1}(x) \geq 0$ by first differentiating $h_i(x, r_0, r_{w+1}) \geq h_i(x, 0, 0)$ and then that $h_i(x, 0, 0) \geq 0$.

Then, since both $r_1, r_w \leq \alpha$ we have,

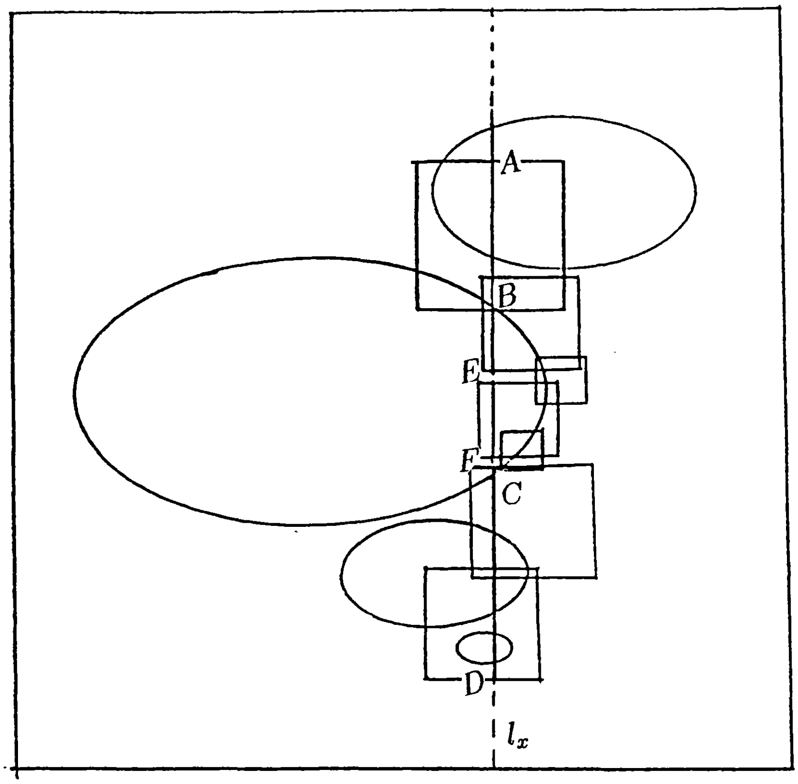

Figure 2.6:

$$\frac{d}{d\mu}h_i(x,\mu,\lambda) \geq 0 \text{ and } \frac{d}{d\lambda}h_i(x,\mu,\lambda) \geq 0 \qquad (2.7)$$

Let the following intervals be labelled in the following way,

$$r_0' \equiv AB, \;\; r_{w+1}' \equiv CD \;\; , r_0' \cup r_1' \equiv ABE, \;\; r_w' \cup r_{w+1}' \equiv CDF.$$

Then, if G_1, H_1 are points of A_1B_1, C_1D_1 respectively, we may define, for all x such that l_x meets θ_i, $g_i'(x)$ to be the same as $g_i(x)$ except that $A_1B_1E_1$, $C_1D_1F_1$ are replaced by $G_1B_1E_1$, $F_1C_1H_1$, and $g_{i-1}'(x)$ defined as $g_{i-1}(x)$ with A_1D_1 replaced with G_1H_1. Let $g_i'(x) = g_i(x)$, and $g_{i-1}'(x) = g_{i-1}(x)$ for all other x. Then we have for all x, using (2.4), (2.6) and (2.7), that

$$\begin{aligned}\int_0^1 g_i(x)\,dx &= \int_0^1 g_{i-1}(x)\,dx + \int_0^1 (g_i(x) - g_{i-1}(x))\,dx \\ &\geq \int_0^1 g_{i-1}(x)\,dx + \int_0^1 \left(g_i'(x) - g_{i-1}'(x)\right)\,dx \qquad (2.8)\end{aligned}$$

We use this to simplify our problem.

We examine θ_m , and find a polygon P_m which encloses it.

Lemma 1. *Let* $\left\{\bigcup_{n=1}^{\infty} \theta_n\right\}$ *form an ordered packing of the unit plane square* I_2*. Then there exists a convex polygon* P_m *which encloses* θ_m *and whose interior does not intersect* θ_i *for* $i < m$*;* P_m *having no more than 15 sides.*

Proof. Consider θ_m and those θ_i, $1 \leq i < m$, surrounding it.

Let Ψ_1 be the maximal ellipsoid which is contained within θ_m, and Ψ_2 be the minimal ellipsoid which contains θ_m. Then Ψ_1 has no less than twice the area of Ψ_2. This is easily shown to be true for the worst case an equilateral triangle and a circle, all other cases for triangles can be reduced via an affine transformation of the equilateral case.

The number of sides of our polygon P_m will depend on all those θ_i, $1 \leq i < m$ within some neighbourhood of θ_m. Let Ψ_3 be the ellipse with the same centre as Ψ_1 but with radius three times as large. We look at the largest $(m-1)$ copies of θ which intersect the ellipse Ψ_3. Since we only need an upper bound for the number of sides of P_m we may replace those θ_i, $1 \leq i < m$ by the ellipse Ψ_1 within Ψ_3. Taking the Dirichlet cell of $\{\theta_i : i \leq m\}$ produces a polygon P'_m which has at most 15 sides.

$$\text{The number of sides of } P'_m = \frac{\text{Area}\,\Psi_3 - \text{Area}\,\Psi_1}{\text{Area}\,\Psi_1} = \frac{16\pi ab - \pi ab}{\pi ab} = 15.$$

The polygon P'_m can be defined as the intersection of some finite number of half-planes, H_i. Let $H(\underline{u}, \alpha) = \{\underline{x} \in \mathbf{R}^2 :< \underline{x}, \underline{u} > \leq \alpha\}$. Then

$$P'_m = \bigcap_{i=1}^{I'} H_i = \bigcap_{i=1}^{I'} H(\underline{u}_i, \alpha'_i), \qquad I' \leq 15,$$

where $\underline{u}_i$ is the normal to the half-planes H_i chosen to pass through the centre of Ψ_1. We translate each half-plane in turn along their normal until they become

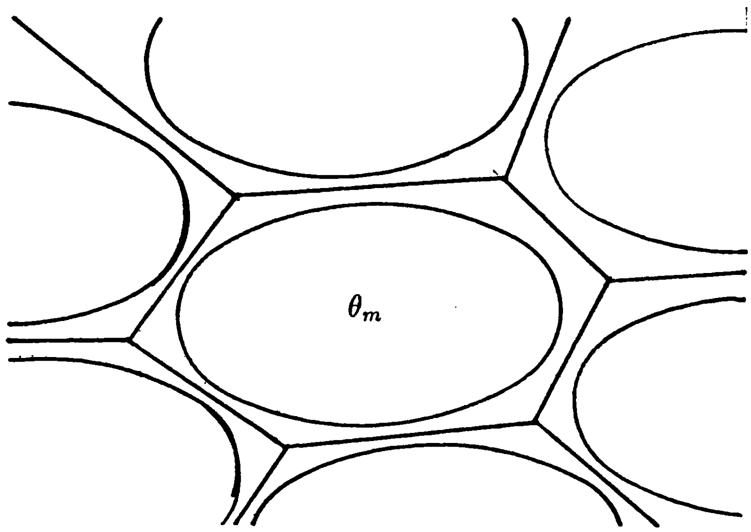

Figure 2.7:

supporting half-spaces of θ_m; that is until the boundary of the half-plane touches the boundary of θ_m. This produces a new polygon P_m which contains θ_m.

Then P_m may have fewer sides than P'_m, since some may be lost during the process of translating the half-planes. So

$$P_m = \bigcap_{i=1}^{I} H_i = \bigcap_{i=1}^{I} H(\underline{u}_i, \alpha_i), \qquad I \leq 15$$

$\square$

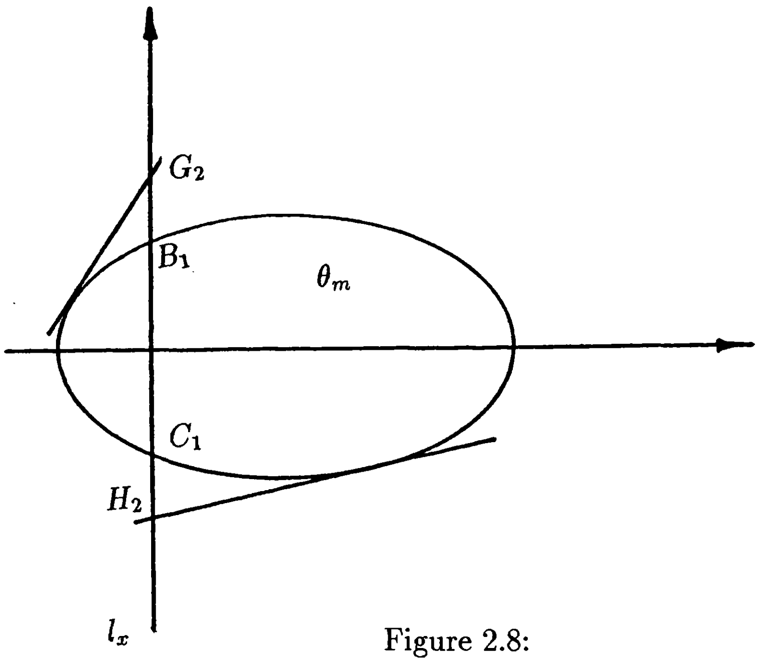

Figure 2.8:

If x is in the interval $[0,1]$, and l_x meets θ_m then l_x meets $P_m \backslash \theta_m$ in two intervals G_2B_1 of length l_1 and C_1H_2 of length l_2. By construction of P_m, these two intervals do not meet $\{\theta_i\}_{i=1}^{m-1}$.

Let $G \equiv G_2$ and $H \equiv H_2$ then we may define $g_i''(x)$, $g_{i-1}''(x)$, as $g_i'(x)$, $g_{i-1}'(x)$, respectively with these choices of G and H, and deduce from (2.8) that

$$\int_0^1 g_i(x)\,dx \geq \int_0^1 g_{i-1}(x)\,dx + \int_0^1 \left(g_i''(x) - g_{i-1}''(x)\right)\,dx. \tag{2.9}$$

If we show that,

$$\int_0^1 \left(g_i''(x) - g_{i-1}''(x)\right)\,dx \geq 0, \tag{2.10}$$

then we are done. So in view of (2.5) we have to show that,

$$\int_0^1 (l_x')^\epsilon\,dx \leq \int_0^1 l_1^\epsilon\,dx + \int_0^1 l_2^\epsilon\,dx, \qquad l_x' = l_x \cap P_n \tag{2.11}$$

The question remains, how do we know there exists such an epsilon?

Lemma 2. *Suppose θ_m is our convex body and is contained within some polygon, Q_m say, then there exists $s = (1+\epsilon)$ where $0 < \epsilon < 1$ such that*

$$\int_0^1 (l_x')^\epsilon\,dx \leq \int_0^1 l_1^\epsilon\,dx + \int_0^1 l_2^\epsilon\,dx, \qquad l_x' = l_x \cap P_n.$$

Proof. Suppose that (2.11) is false, then in particular there exists an at most 15 sided polygon Q_m containing θ_m such that,

$$\int_0^1 (l_x')^{\frac{1}{m}}\, dx \leq \int_0^1 l_1^{\frac{1}{m}}\, dx + \int_0^1 l_2^{\frac{1}{m}}\, dx.$$

This would be true for every m, so letting $m \to \infty$ we produce a contradiction, i.e $1 \geq 2$. Therefore there exists some s.□

Let us now cover θ_m by the intersection of at most 15 discs, whose radii are bounded by kr_θ , where k is the reduction factor from θ to θ_m. We will now calculate explicitly a value of ϵ for which

$$\int_0^1 (l_x')^{\epsilon}\, dx \leq \int_0^1 l_1^{\epsilon}\, dx + \int_0^1 l_2^{\epsilon}\, dx,$$

which means removing θ_m decreases the integral in (2.8).

At least one of the edges of P_m will be an interval, $[a, b]$, of length greater than $\frac{kr_\theta}{15}$. Let $[a, b]$ be the projection of this interval onto the x-axis. We concentrate on one of these intervals, as they will contribute most to the integral of $g_i(x)$. We integrate along this interval $[a, b]$, doubling up the contribution from the shallowest arc, and ignoring the other side. Without loss of generality we may assume it is the top arc; the radius of curvature in this interval is kr_θ , and the centre of this

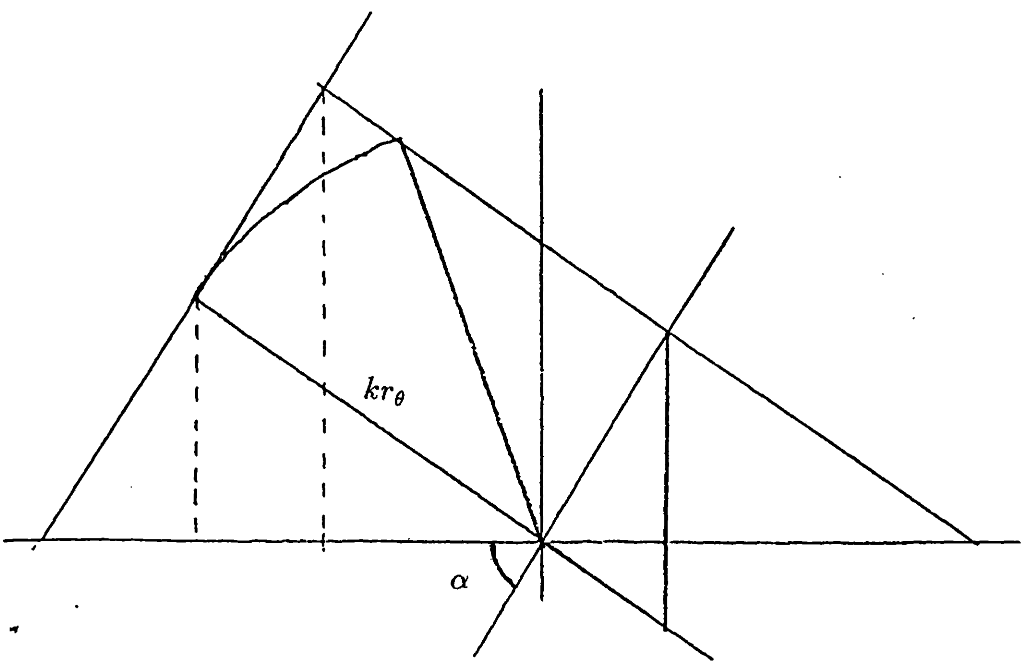

Figure 2.9:

arc is the origin.

To prove our hypothesis we must show

$$\int_a^b (l_x')^\epsilon \, dx \leq 2 \int_a^b l_1^\epsilon \, dx.$$

Then,

$$\begin{aligned}
2\int_a^b l_1^\epsilon dx &= 2\int_{-kr_\theta \sin\alpha}^{-kr_\theta \sin\alpha + \frac{kr_\theta}{15}\cos\alpha} \left(kr_\theta - \left(k^2{r_\theta}^2 - x^2\right)^{\frac{1}{2}}\right)^\epsilon \sec^\epsilon \alpha dx \quad (2.12)\\
&= 2\int_{-\alpha}^{-\delta} (kr_\theta)^{1+\epsilon} \cos\phi(1-\cos\phi)^\epsilon \sec^\epsilon \alpha \, d\phi \quad (2.13)\\
&= (2kr_\theta)^{1+\epsilon} \sec^\epsilon \alpha \int_{-\alpha}^{-\delta} \sin^{2\epsilon}\left(\frac{\phi}{2}\right) \cos\phi \, d\phi \quad (2.14)\\
&\geq \int_{-kr_\theta \sin\alpha}^{-kr_\theta \sin\alpha + \frac{kr_\theta}{15}\cos\alpha} k^\epsilon \, dx \quad (2.15)\\
&= \int_{-\alpha}^{-\delta} k^{1+\epsilon} r_\theta \cos\phi \, d\phi. \quad (2.16)
\end{aligned}$$

So,

$$\int_{-\alpha}^{-\delta} \left(2^{1+\epsilon} \sec^\epsilon \alpha {r_\theta}^\epsilon \sin^{2\epsilon}\left(\frac{\phi}{2}\right) - 1\right) \cos\phi \, d\phi \geq 0$$

cancelling the other terms. Now we split up our range of integration into the intervals $[-\alpha, -\beta]$,$[-\beta, -\gamma]$,$[-\gamma, -\delta]$ so that

$$\begin{aligned}
\sin^{2\epsilon}\left(-\frac{\beta}{2}\right) &\geq 2^{-1-\epsilon} (r_\theta \sec\alpha)^{-\epsilon} \quad (2.17)\\
r_\theta \sin(-\delta) &\geq \frac{1}{15} \quad (2.18)
\end{aligned}$$

We then have,

$$\int_{-\alpha}^{-\delta}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi$$

$$\begin{aligned}
&\geq \int_{-\alpha}^{-\beta}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi\\
&+ \int_{-\beta}^{-\gamma}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi\\
&+ \int_{-\gamma}^{-\delta}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi
\end{aligned} \tag{2.19}$$

The first integral will provide a negative contribution, the second will be small and positive but insufficient to compensate for the first. The third integral is also positive and sufficiently large to compensate. Therefore

$$\int_{-\alpha}^{-\delta}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi$$

$$\geq \int_{-\alpha}^{-\beta}-\cos\phi\,d\phi+\int_{-\gamma}^{-\delta}\left(2^{1+\epsilon}{r_\theta}^{\epsilon}\sec^{\epsilon}(\alpha)\sin^{2\epsilon}\left(\frac{\phi}{2}\right)-1\right)\cos\phi\,d\phi$$

We choose $-\gamma$ so that

$$2^{1+\epsilon} r_\theta{}^\epsilon \sec^\epsilon(\alpha) \sin^{2\epsilon}\left(\frac{\phi}{2}\right) - 1 \simeq \frac{1}{2}$$

We take

$$-\gamma = -\frac{\delta}{2}$$

so we have

$$\int_{-\alpha}^{-\delta} \left(2^{1+\epsilon} r_\theta{}^\epsilon \sec^\epsilon \alpha \sin^{2\epsilon}\left(\frac{\phi}{2}\right) - 1\right) \cos\phi \, d\phi$$

$$\begin{aligned}
&\geq \int_{-\alpha}^{-\beta} -\cos\phi \, d\phi + \int_{-\gamma}^{-\delta} \left(2^{1+\epsilon} \sec^\epsilon \alpha r_\theta{}^\epsilon \sin^{2\epsilon}\left(\frac{\phi}{2}\right) - 1\right) \cos\phi \, d\phi \\
&\geq \sin(-\alpha) - \sin(-\beta) + \frac{1}{2}\int_{-\gamma}^{-\delta} \cos\phi \, d\phi \\
&\geq 0
\end{aligned}$$

We now take $\sin\alpha = 0$, the worst possible case. Therefore we have,

$$\frac{1}{4}\sin(-\delta) \geq \sin(-\beta) \tag{2.20}$$

$$\sin^{2\epsilon}\left(-\frac{\beta}{2}\right) \geq 2^{-1-\epsilon}\left(r_\theta \sec\alpha\right)^{-\epsilon} \tag{2.21}$$

$$8^{-2\epsilon}\sin^{2\epsilon}(-\delta) \geq 2^{-1-\epsilon}\left(r_\theta \sec\alpha\right)^{-\epsilon} \tag{2.22}$$

Using (2.18)

$$(120r_\theta)^{-2\epsilon} \geq 2^{-1-\epsilon}(r_\theta \sec\alpha)^{-\epsilon}$$

Taking logarithms,

$$2\epsilon\log(120r_\theta) \leq (1+\epsilon)\log 2 + \epsilon\log(r_\theta \sec\alpha)$$

Rearranging this gives,

$$\epsilon \leq \frac{\log 2}{\log r_\theta + 2\log 120 - \log\sec\alpha}.$$

Therefore, provided $\epsilon \leq \frac{\log 2}{\log r_\theta + 2\log 120 - \log\sec\alpha}$,

$$\int_0^1 (l'_x)^\epsilon \, dx \geq \int_0^1 l_1^\epsilon \, dx + \int_0^1 l_2^\epsilon \, dx.$$

That is

$$\int_0^1 \left(g_i''(x) - g_{i-1}''(x)\right) \, dx \geq 0$$

which implies,

$$\int_0^1 g_i(x) \, dx \geq \int_0^1 g_{i-1}(x) \, dx \geq, \ldots, \geq \int_0^1 g_0(x) \, dx = 1$$

and hence,

$$\sum_{j=1}^{p} \Delta_j^{1+\epsilon} > \int_0^1 g_n(x)\,dx \geq 1$$

as required. $\square$

3. Construction of a 2-Dimesnional Packing

Let I_2 be the unit plane square whose vertices are at $[\pm\frac{1}{2}, \pm\frac{1}{2}]$. Then our convex body θ is the intersection of four discs of radius r_θ $(\infty > r_\theta \gg 1)$ of centres $[0, \pm(r_\theta - \frac{1}{2})]$, $[\pm(r_\theta - \frac{1}{2}), 0]$.

By construction, at all but four points of the boundary, θ has radius of curvature bounded by r_θ. The construction of θ can be considered to be the act of slicing off the sides of I_2 using shallow arcs. This idea of transforming a square into a copy of θ will be used later.

Let ${}_k\Gamma$ be the boundary arcs of θ, $k = 1, ..., 8$.

Let $p = (x, y)$ be the intersection of the two arcs ${}_1\Gamma$ and ${}_2\Gamma$ as indicated. This is where both ${}_1\Gamma$ and ${}_2\Gamma$ meet the line $x = y$. We find $p = (x, y)$, enabling us to evaluate the horizontal distance $\hat{x}$ from θ to I_2, in fact $\hat{x} = \frac{1}{2} - x$.

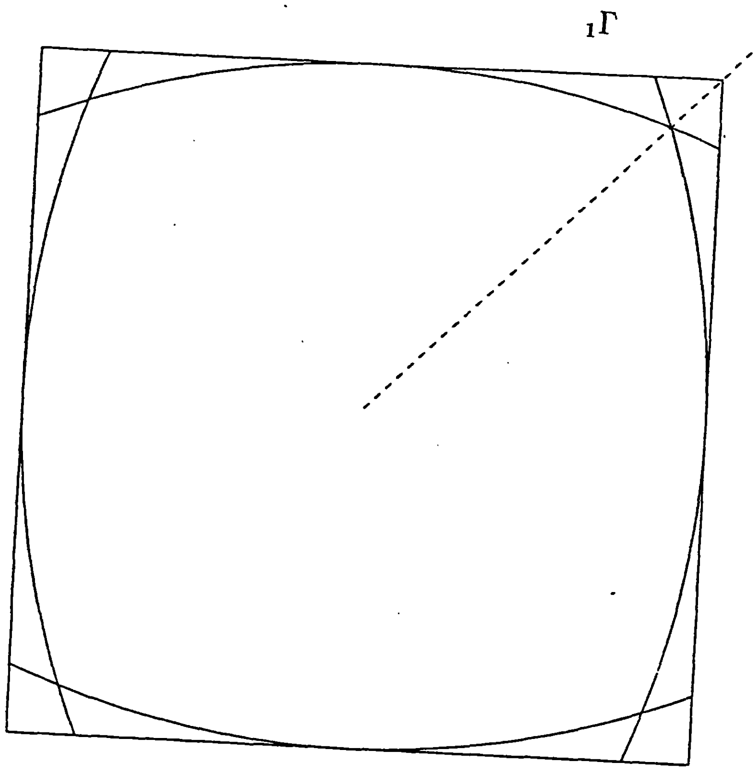

Figure 3.1:

The equation of the top arc $_1\Gamma$ is

$$x^2 + \left(y + r_\theta - \frac{1}{2}\right)^2 = r_\theta^2.$$

We want to find $p = (x, y)$ and hence $\hat{x}$ so we substitute in $x = y$. This gives

$$x^2 + \left(x + r_\theta - \frac{1}{2}\right)^2 = r_\theta^2.$$

$$2x^2 + 2\left(r_\theta - \frac{1}{2}\right)x + \left(r_\theta - \frac{1}{2}\right)^2 - r_\theta^2 = 0.$$

Solutions are of the form

$$\begin{aligned}
x &= \frac{1}{4}\left[-2(r_\theta - \frac{1}{2}) \pm \sqrt{4\left(r_\theta - \frac{1}{2}\right) - 8\left(\left(r_\theta - \frac{1}{2}\right)^2 - r_\theta^2\right)}\right] \\
&= -\frac{1}{2}\left(r_\theta - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{2r_\theta^2 - \left(r_\theta - \frac{1}{2}\right)^2} \\
&= \frac{1}{4} - \frac{r_\theta}{2} \pm \frac{1}{2}\sqrt{r_\theta^2 + r_\theta - \frac{1}{4}}
\end{aligned}$$

Take the positive root as we want $x > 0$.

$$\begin{aligned}
x &= \frac{1}{4} - \frac{r_\theta}{2} + \frac{1}{2}\sqrt{r_\theta^2 + r_\theta - \frac{1}{4}}, \\
\hat{x} &= \frac{1}{2} - x \quad = \quad \frac{1}{4} + \frac{r_\theta}{2} - \frac{1}{2}\sqrt{r_\theta^2 + r_\theta - \frac{1}{4}}.
\end{aligned}$$

For simplicity we use an approximation to x.

$$
\begin{aligned}
x &= \frac{\left(\left(\frac{1}{4}-\frac{r_\theta}{2}\right)+\frac{1}{2}\sqrt{r_\theta^2+r_\theta-\frac{1}{4}}\right)\left(\left(\frac{1}{4}-\frac{r_\theta}{2}\right)-\frac{1}{2}\sqrt{r_\theta^2+r_\theta-\frac{1}{4}}\right)}{\left(\left(\frac{1}{4}-\frac{r_\theta}{2}\right)-\frac{1}{2}\sqrt{r_\theta^2+r_\theta-\frac{1}{4}}\right)} \\
&= \frac{1}{2}\frac{\left(\left(\frac{1}{2}-r_\theta\right)^2-r_\theta^2+r_\theta-\frac{1}{4}\right)}{\left(\frac{1}{2}-r_\theta\right)-\sqrt{r_\theta^2+r_\theta-\frac{1}{4}}} \\
&= \frac{1}{2}\frac{2r_\theta-\frac{1}{2}}{\sqrt{r_\theta^2+r_\theta-\frac{1}{4}}+r_\theta-\frac{1}{2}} \\
&= \frac{1}{2}\frac{1-\frac{1}{4r_\theta}}{\sqrt{\frac{1}{4}+\frac{1}{4r_\theta}-\frac{1}{16r_\theta^2}}+\frac{1}{2}-\frac{1}{4r_\theta}} \\
&= \frac{1-\frac{1}{4r_\theta}}{\sqrt{1+\frac{1}{r_\theta}-\frac{1}{4r_\theta^2}}+1-\frac{1}{2r_\theta}} \\
&\simeq \frac{1}{2}\left(1-\frac{1}{4r_\theta}\right)
\end{aligned}
$$

$$
\text{So, } \hat{x} \simeq \frac{1}{8r_\theta}.
$$

3.1. The Packing and its Notation

At each stage we pack countably many reduced copies of θ into I_2, building up the packing progressively. Let R_n be the n^{th} stage residual set. A cover of R_n is given at every stage.

The packing is built up by repeatedly applying a transformation T to I_2, this transformation acts upon all squares at this stage in the following way:-

Let S_Y^X be one such square, and let X and Y be previous labels. Let Δ_Y^X denote $\frac{1}{\sqrt{2}}$ diam (S_Y^X); in fact Δ_Y^X is the side length of the square S_Y^X.

T replaces S_Y^X by a reduced copy θ by a factor Δ_Y^X and countably many smaller squares, which together cover S_Y^X apart from a set of small measure. (Along the arcs of the newly created θ labeled θ_Y^X, ${}_k\Gamma_Y^X$.)

T covers $S_Y^X \backslash \theta_Y^X$ with squares by splitting it up into twelve regions.

Four corner squares of side $\frac{\Delta_Y^X}{8r} \equiv \Delta_{Y_0}^{X_k}$, $k = 1, ..., 4$ labeled $S_{Y_0}^{X_k}$.This leaves eight similar regions ${}_k\Omega_Y^X$ $k = 1, ..., 8$. These are approximately triangular; their perpendicular sides are of lengths $\frac{\Delta_Y^X}{8r_\theta}, \frac{\Delta_Y^X}{2}(1 - \frac{1}{4r_\theta})$ and their other sides are the arcs ${}_k\Gamma_Y^X$.

Let $\Delta_{Y_i}^{X_0} = \frac{1}{\sqrt{2}}\text{diam}\, S_{Y_i}^{X_0}$. The $\Delta_{Y_i}^{X_0}$ are found by solving $L_{Y_i}^{X_0}$ and ${}_K\Gamma_Y^X$ to find

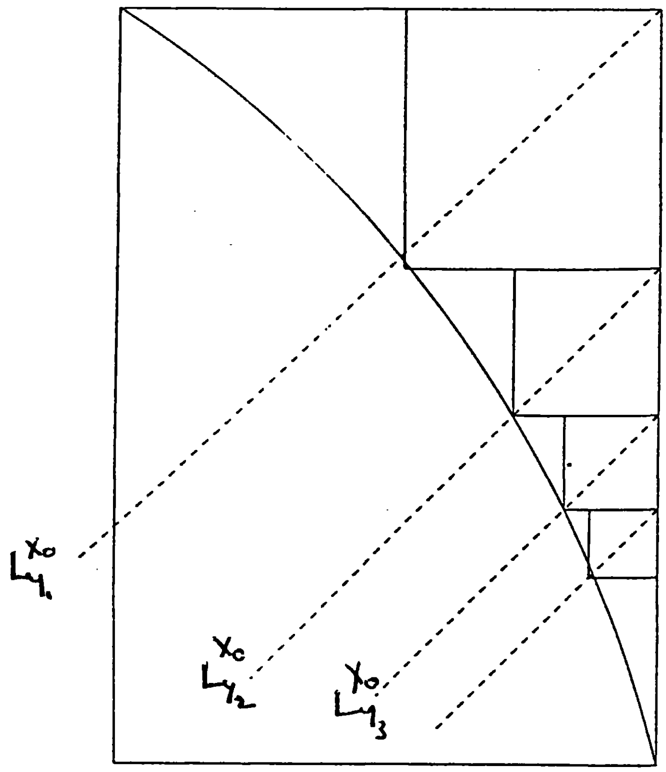

Figure 3.2:

their points of intersection.

$$L_{Y_i}^{X_0} \quad : \quad \tilde{y}_i = \tilde{x}_i - \sum_{j=0}^{i-1} \Delta_{Y_i}^{X_0}.$$

Since we sum over subscripts and not superscripts we write this as

$$L_{Y_i} \quad : \quad \tilde{y}_i = \tilde{x}_i - \sum_{j=0}^{i-1} \Delta_{Y_i}.$$

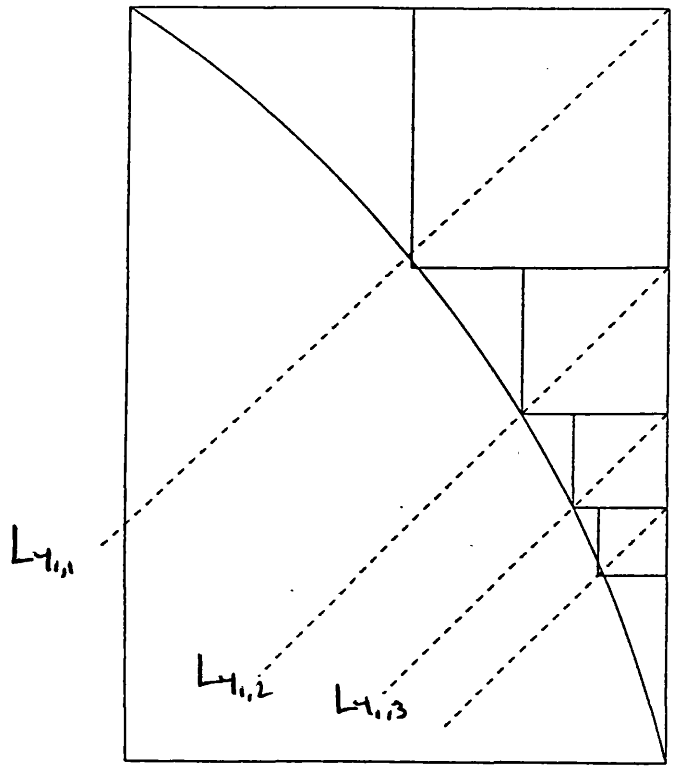

Figure 3.3:

Let $\Delta_{Y_0}^{X_K} \equiv \Delta_{Y_0}^{X_0} \equiv \Delta_{Y_0}$.

So at the n^{th} stage when we pack sideways we increase the length of the n^{th} subscript.

Find $\Delta_{Y_{i,j}}$ by solving for $j = 1$:-

$$L_{Y_{i,1}} \quad : \quad \tilde{y}_{i,1} = \tilde{x}_{i,1} - \delta_{Y_i},$$

for $j > 1$

$$L_{Y_{i,j}} \quad : \quad \tilde{y}_{i,j} = \tilde{x}_{i,j} - \delta_{Y_i} - \sum_{k=1}^{j-1} \Delta_{Y_{i,k}},$$

with equation of the relevant arc, in this case

$$\tilde{x}^2 + (\tilde{y} + (r_\theta - \frac{1}{2}))^2 = r_\theta^2$$

Let

$$\delta_{Y_i} = \Delta_{Y_{i-1}} - \Delta_{Y_i}$$

So by construction

$$\sum_{j=1}^{\infty} \Delta_{Y_{i,j}} = \Delta_{Y_i},$$

and further

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \Delta_{Y_{i,j}} = \frac{\Delta_Y}{2}.$$

This leaves us with countably many approximately triangular sections $\left\{I_{Y_i}^{X_0}\right\}_{i=1}^{\infty}$. For simplicity again knowing we are in ${}_k\Omega_Y^X$ we can drop the superscript. The perpendicular sides of the $\left\{I_{Y_i}^{X_0}\right\}_{i=1}^{\infty}$ are of lengths $\delta_{Y_i} = \Delta_{Y_{i-1}} - \Delta_{Y_i}$, and Δ_{Y_i} for $i = 1, 2, \ldots$

$\left(\delta_{Y_i}^{X_0} = \Delta_{Y_{i-1}}^{X_0} - \Delta_{Y_i}^{X_0}\right)$.

We then pack the I_{Y_i} similarly for each i. From now on we drop all the superscripts since they are all X_0.

Our left over regions are the $\{I_{Y_i}\}_{i=1}^{\infty}$ which are approximately triangular sections with perpendicular sides $\delta_{Y_{i,j}} = \Delta_{Y_{i,j-1}} - \Delta_{Y_{i,j}}$, and $\Delta_{Y_{i,j}}$ for $j = 1, 2, \ldots$ Note when $j = 1$, $\delta_{Y_{i,j}} = \delta_{Y_i} - \Delta_{Y_{i,1}}$ so we let $\Delta_{Y_{i,0}} \equiv \delta_{Y_i}$ for simplicity. So we repeat the layering process, packing the largest possible square into the corner, then the next largest possible underneath it, and so on, labeling as we pack.

Let $\delta_{Y_{i_1,\ldots,i_n}} = \Delta_{Y_{i_1,\ldots,i_n-1}} - \Delta_{Y_{i,\ldots,i_n}}$, the $i_n > 0$ with $\Delta_{Y_{i_1,\ldots,i_{n-1},0}} \equiv \delta_{Y_{i_1,\ldots,i_{n-1}}}$. So the approximately triangular region $I_{Y_{i_1,\ldots,i_n}}$ has perpendicular sides $\delta_{Y_{i_1,\ldots,i_n}}$ and $\Delta_{Y_{i_1,\ldots,i_n}}$. The $\Delta_{Y_{i_1,\ldots,i_{n+1}}} \equiv \frac{1}{\sqrt{2}}\mathrm{diam}\left(C_{Y_{i_1,\ldots,i_{n+1}}}\right)$ which are found by solving, for $i_{n+1} = 1$

$$L_{Y_{i_1,\ldots,i_n,1}} \quad : \quad \tilde{y}_{i_1,\ldots,i_n,1} = \tilde{x}_{i_1,\ldots,i_n,1} - \delta_{Y_{i_1,\ldots,i_n}},$$

for $i_{n+1} > 1$

$$L_{Y_{i_1,\ldots,i_n,i_{n+1}}} \quad : \quad \tilde{y}_{i_1,\ldots,i_n,i_{n+1}} = \tilde{x}_{i_1,\ldots,i_n,i_{n+1}} - \delta_{i_1,\ldots,i_n} - \sum_{k=1}^{i_{n+1}-1} \Delta_{Y_{i_1,\ldots,i_n,k}},$$

with equation of the relevant arc, in this case

$$\tilde{x}^2 + (\tilde{y} + (r_\theta - \frac{1}{2}))^2 = r_\theta^2.$$

So by construction we have,

$$\sum_{i_{n+i}=1}^{\infty} \Delta_{Y_{i_1,\ldots,i_{n+1}}} = \frac{\Delta_{Y_{i_1,\ldots,i_n}}}{2}$$

$$\sum_{i_1=0}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_{n+1}=1}^{\infty} \Delta_{Y_{i_1,\ldots,i_{n+1}}} = \frac{\Delta_Y}{2}$$

We sum over all subscripts along one side of our initial square S_1^0. Let I be this indexing set, then this gives,

$$\sum_I \Delta_Y = \frac{1}{2}.$$

We have given I_2 the initial label S_1^0, then at each stage we apply T to all squares created at the previous stage.

Given any θ_Y^X it is possible to identify where it entered the packing by the depth of the subscript Y.

Define the depth of Y, denoted by dep(Y), to be n if the subscript Y is of the form,

$$Y = [L_1]_{[L_2]_{\cdot_{\cdot_{\cdot_{[L_n]}}}}}$$

Where each $[L_i]$ is a string of integers of countable length. If the subscript Y has depth n then θ_Y^X entered the packing at the n^{th} stage. So at first we apply the transformation T to I_2, which have labeled S_1^0, this produces a single copy of θ labeled θ_1^0, and countably many squares. On the next application of T all those newly created squares have their sides sliced off, and become

$$\theta_{1_0}^{0_k} \text{ for } k = 1, \ldots, 4$$

and

$$_k\left\{\theta_{1_{i_1}}\right\}_{i_1=1}^{\infty} \quad _k\left\{\left\{\theta_{1_{i_1,i_2}}\right\}_{i_1=1}^{\infty}\right\}_{i_2=1}^{\infty}$$

$$_k\left\{\left\{\left\{\theta_{1_{i_1,i_2,i_3}}\right\}_{i_1=1}^{\infty}\right\}_{i_2=1}^{\infty}\right\}_{i_3=1}^{\infty}.$$

For each of these θ there are also countably many squares.

3.2. The Cover

Our cover for the n^{th} stage residual set R_n consists of two subcovers;

(1) Covering the sets of small measure on the arcs of the θ_Y^X

for which $\operatorname{dep}(Y) < n$,

(2) Covering the part of the residual set which is the countable union of sets of the form $\left(S_Y^X \backslash \theta_Y^X\right)$ for $\operatorname{dep}(Y) = n$.

3.2.1. The cover of type (1).

For each i in turn we cover the arcs of those θ_Y^X with $\operatorname{dep}(Y) = i$, by choosing sets of squares $W_1^0 = \{{}_k w_1^0\}_{k=1}^{m_1}$ so that,

$$\sum_{k=1}^{m_1} \operatorname{diam}\left({}_k w_1^0\right)^{1+\epsilon} \leq \frac{\left(\sqrt{2}\right)^{1+\epsilon}}{2^n}.$$

This is possible since we can choose our squares arbitrarily small and by construction our θ's will go under the cover.

For each θ_Y^X with $\operatorname{dep}(Y) < i$ we cover its arcs ${}_k\Gamma_Y^X$ for $k = 1, \ldots, 8$ by sets of the form $W_Y^X = \left\{{}_k w_Y^X\right\}_{k=1}^{m_Y^X}$ such that for each θ_Y^X

$$\sum_{k=1}^{m_Y^X} \operatorname{diam}\left({}_k w_Y^X\right)^{1+\epsilon} \leq \frac{\left(\sqrt{2}\Delta_Y^X\right)^{1+\epsilon}}{2^n}.$$

Summing over all θ_Y^X with $\operatorname{dep}(y) = i$ we have,

$$\begin{aligned}
\underbrace{\sum\sum,\ldots,\sum}_{Y:\operatorname{dep}(Y)=i}\sum_k \left(\operatorname{diam}\left({}_k w_Y^X\right)\right)^{1+\epsilon} &\leq \underbrace{\sum\sum,\ldots,\sum}_{Y:\operatorname{dep}(Y)=i} \frac{\left(\sqrt{2}\Delta_Y^X\right)^{1+\epsilon}}{2^n} \\
&\leq \underbrace{\sum\sum,\ldots,\sum}_{Y:\operatorname{dep}(Y)\leq n-1} \frac{\left(\sqrt{2}\Delta_Y^X\right)^{1+\epsilon}}{2^n} \\
&\leq \frac{\left(\sqrt{2}\right)^{1+\epsilon}}{2^n}
\end{aligned}$$

We obtain this for each $i = 1, \ldots, n-1$, and so,

$$\underbrace{\sum\sum,\ldots,\sum}_{Y:\operatorname{dep}(Y)\leq n-1}\sum_k \left(\operatorname{diam}\left({}_k w_Y^X\right)\right)^{1+\epsilon} \leq \frac{\left(\sqrt{2}\right)^{1+\epsilon}(n-1)}{2^n} < 1.$$

3.2.2. The cover of type (2).

We are concerned with those θ_Y^X which have most recently entered the packing, so those θ_Y^X with $\operatorname{dep}(Y) = n$. We cover each section of R_n of the form $S_Y^X \backslash \theta_Y^X$ using four strips of $8r_\theta$ squares of diameter $\frac{(\sqrt{2}\Delta_Y^X)}{8r_\theta}$. We call these sets the V_Y^X. The set V_Y^X covers $S_Y^X \backslash \theta_Y^X$ ($\operatorname{dep}(Y) = n$), and is of the form $V_Y^X = \left\{{}_j v_Y^X\right\}_{j=1}^{32r_\theta}$. We choose ϵ so that,

$$\sum_{j=1}^{32r_\theta} \left(\text{diam}\left({}_jv_Y^X\right)\right)^{1+\epsilon} \leq \left(\sqrt{2}\Delta_Y^X\right)^{1+\epsilon}$$

$$32r_\theta \left(\frac{\Delta_Y^X}{8r_\theta}\right)^{1+\epsilon} \leq \left(\Delta_Y^X\right)^{1+\epsilon}$$

$$\frac{4}{(8r_\theta)^\epsilon} \leq 1$$

$$\epsilon \geq \frac{2\log 2}{3\log 2 + \log r_\theta}$$

And hence,

$$\underbrace{\sum\sum,\ldots,\sum}_{Y:\text{dep}(Y)=n}\sum_{j=1}^{32r} \left(\text{diam}\left({}_jv_Y^X\right)\right)^{1+\epsilon} \leq \underbrace{\sum\sum,\ldots,\sum}_{Y:\text{dep}(Y)=n} \left(\sqrt{2}\Delta_Y^X\right)^{1+\epsilon}$$

$$\leq \left(\sqrt{2}\right)^{1+\epsilon}$$

3.2.3. Justification for the use of this cover.

For each piece of the residual set of the form $S_Y^X \backslash \theta_Y^X$ we choose a cover $\left\{ {}_j v_Y^X \right\}_j$ to maximise the sum of the diameters of the covering sets to the power $1+\epsilon$. Let us drop the subscripts and superscripts for simplicity.So the $\{v_j\}_{j\epsilon J}$ cover $S \backslash \theta$ and let $\operatorname{diam}(S) = \sqrt{2}\Delta$, and $\operatorname{diam}(v_j) = \sqrt{2}\Delta_j$. Then we maximise the following over all covers,

$$\sum_{j \in J} (\Delta_j)^{1+\epsilon} \leq (\Delta_j)^{1+\epsilon}$$

subject to,

$$\begin{aligned} \sum_{j \in J} (\Delta_j) &= \Delta \\ \Delta_j &\leq \frac{\Delta}{8r_\theta} \\ \Delta_{j+1} &< \Delta_j \end{aligned}$$

Suppose that $\{v_j\}_{j=1}^{\infty}$ cover the residual set $S \backslash \theta$ and maximise the sum subject to the constraints.

We cover the cover using squares of diameter $\sqrt{2}\frac{\Delta}{8r_\theta}$. This increases the sum, but we ensure that the sum is $\leq \Delta^{1+\epsilon}$. Eventually we will show that this over estimate is finite, and hence so is our original.

Suppose without loss of generality, that $\Delta_1 < \frac{\Delta}{8r_\theta}$ and not equal to it. The consider,

$$(\Delta_1+\delta)^{1+\epsilon}+(\Delta_2-\delta)^{1+\epsilon}+\sum_{j=3}^{\infty}\Delta_j=\sum_1$$

We have borrowed from Δ_2 to increase Δ_1. Let $\sum_{j=1}^{\infty}\Delta_j=\sum_2$. Then consider,

$$\begin{aligned}
\sum_1-\sum_2 &= (\Delta_1+\delta)^{1+\epsilon}-\Delta_1^{1+\epsilon}+(\Delta_2-\delta)^{1+\epsilon}-\Delta_2^{1+\epsilon}\\
&= (1+\epsilon)\left(\Delta_1^{\epsilon}-\Delta_2^{\epsilon}\right)\delta+\frac{(1+\epsilon)\,\epsilon}{2}\left(\Delta_1^{\epsilon-1}-\Delta_2^{\epsilon-1}\right)\delta^2\\
&+ \frac{(1+\epsilon)\,\epsilon\,(\epsilon-1)}{6}\left(\Delta_1^{\epsilon-2}-\Delta_2^{\epsilon-2}\right)\delta^3+\ldots>0
\end{aligned}$$

Since $\Delta_1 > \Delta_2$

So we take $\Delta_1 = \frac{\Delta}{8r_\theta}$, and then we can repeat the procedure borrowing from the smaller Δ_j's to increase the early Δ_j's in the same manner, until we have squares of diameter $\frac{\sqrt{2}\Delta}{8r_\theta}$ as required.

Now R, the residual set for our packing is $\bigcap_{n=1}^{\infty} R_n$ and so $R \subset R_n$, for all n. Therefore the $(1+\epsilon)$ -dimensional measure of R, denoted by $m^{1+\epsilon}(R)$ is less than or equal to the $(1+\epsilon)$-dimensional measure of the R_n, for all n. So,

$$m^{1+\epsilon}(R) \leq m^{1+\epsilon} R_n \qquad \text{for all } n.$$

If we show that $m^{1+\epsilon} R_n$ is finite for all n then we have shown that so is $m^{1+\epsilon}(R)$, and we are done. We proceed in the following way. At the n^{th} stage we have,

for $n = 1$,

$$m(R_1) \leq \sum_{j=1}^{32r_\theta} d({}_j v_1^0)^{1+\epsilon} \leq \sqrt{2}^{1+\epsilon} < 4$$

for $n > 1$,

$$\begin{aligned} m(R_n) &\leq 4\left[\sum_{Y:\text{dep}(Y)=n} \sum_{j=1}^{32r_\theta} d({}_j v_1^0)^{1+\epsilon} + \sum_{Y:\text{dep}(Y)=n} \sum_k d({}_k w_Y^X)^{1+\epsilon}\right] \\ &\leq 4\left[\sqrt{2}^{1+\epsilon} + \sqrt{2}^{1+\epsilon}\left(\frac{n-1}{2^n}\right)\right] \\ &= 4\sqrt{2}^{1+\epsilon}\left(1 + \frac{n-1}{2^n}\right) \\ &< 10 \end{aligned}$$

Therefore the $(1+\epsilon)$-dimensional measure of R is at most 10 and hence the Hausdorff dimension of R is at most $1+\epsilon$, where

$$\epsilon = \frac{2\log 2}{3\log 2 + \log r_\theta} \qquad \square.$$

4. Higher Dimensional results on the Besicovitch-Hausdorff dimension of Packings of Convex bodies of Bounded Radius of Curvature.

In two dimensions we have shown for packings of this type that the Besicovitch-Hausdorff dimension $s_2(r_\theta)$ of the residual set R is at least

$$1 + \epsilon(r_\theta) \quad \text{where } \epsilon(\mathrm{r}_\theta) \sim \frac{1}{\log \mathrm{r}_\theta} \tag{4.1}$$

$$0 < \epsilon(r_\theta) < 1$$

We will assume that all convex bodies mentioned within this chapter are of bounded radius of curvature.

We will, using an inductive slicing arguement, show that in higher dimensions the dimension $s_n(r_\theta)$ of the residual set R is at least

$$s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1$$

where $s_n(r_\theta)$ is defined by

$$s_n(r_\theta) = \inf\{s\colon s \text{ is the Besicovitch-Hausdorff dimension of } R\}$$

where the infimum is taken over all packings of bodies with radius of curvature bounded by r_θ. This will lead to the result

$$s_n(r_\theta) \geq (n-1) + \epsilon(r_\theta) \tag{4.2}$$

where

$$\epsilon(r_\theta) \propto \frac{1}{\log r_\theta} \tag{4.3}$$

by combining (4.2) and (4.3). Hence

$$s_n(r_\theta) > n - 1 \ \text{ for n} = 2, 3, \ldots. \tag{4.4}$$

In the proceeding paragraphs we will need the following notation: let C be a set in $\mathbf{R}^n$, let $C(y)$ denote the vertical slice of C, y units along the X_n axis, i.e. $C(y)$ is the subset of C which lies within the hyperplane $X_n = y$.

Theorem 2. *Let $\{\theta_m\}_{m=1}^{\infty}$ be a solid packing of homothetic copies of the convex n-body θ into the unit cube I_n; θ having radius of curvature bounded above by r_θ.*

Then with $s_n(r_\theta)$ defined as above we have

$$s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1.$$

Proof. Let $\{\theta_m\}_{m=1}^{\infty}$ be a packing of convex n-bodies, with radius of curvature of θ bounded above by $r_\theta < \infty$, into the unit n-cube I_n.

We may assume without loss of generality that the $\{\theta_m\}$ are open since the set

$$\bigcup_{m=1}^{\infty} \partial\theta_m$$

has Besicovitch-Hausdorff dimension $n-1$, and we will show

$$s_n > n-1$$

Then the residual set R is compact and hence it is sufficient to consider finite coverings of R by open sets.

We proceed by defining a function on the reals. For $\delta > 0$, $0 < s \leq s_{n-1}(r_\theta)$ we define

$$f(z) = m_\delta^s(R(z)), \qquad \forall z > 0.$$

This is the s-dimensional δ-measure of the slice of R which is contained in the hyperplane $x_n = z$.

We integrate this measurable function in the x_n direction to produce the required result.

We have

$$f(z) > 0 \ \ \forall z \in [0,1] \tag{4.5}$$

To show f is measurable we use the sufficient condition that f is a measurable function if and only if $\{z : f(z) < c\}$ for any real c is a measurable set.

Consider $R(z)$, this is the $(n-1)$-dimensional slice of R which is contained in the hyperplane $x_n = z$. $R(z)$ is compact for all real z, and so given real λ, z, we can find a finite δ-cover $\{E_i\}_{i=1}^{p}$ of $R(z)$ such that

$$\sum_{i=1}^{p} \operatorname{diam}^s(E_i) < m_\delta^s(R(z)) + \lambda = f(z) + \lambda \tag{4.6}$$

$$\operatorname{diam}^s(E_i) < \delta, \quad i = 1, ..., p$$

Note that the E_i have dimension $n-1$.

Let us now define sets $E_i(\mu)$ for $i = 1, \ldots, p$ by

$$E_i(\mu) = \{\underline{x} \in \mathbf{R}^n : \|\underline{x} - \underline{y}\| \leq \mu,\ \underline{y} \in E_i\}.$$

This can be viewed as giving thickness to the E_i which are $(n-1)$-dimensional sets sitting in n dimensions. So the $E_i(\mu)$ have dimension n. They are open sets and have diameter

$$\operatorname{diam}(E_i) < \operatorname{diam}(E_i) + 2\mu.$$

It is easy to see that if we choose μ correctly we are able to ensure that:

$$\operatorname{diam}(E_i(\mu)) < \operatorname{diam}(E_i) + 2\mu < \delta. \tag{4.7}$$

Let h be a positive real number such that $h < \mu$.

We choose h in such a way as to ensure that if for some real number $z' \in (z - h, z + h)$ our cover $\{E_i(\mu)\}_{i=1}^{p}$ is also a cover for $R(z')$. We find that a sufficient condition on the size of h is

$$2(h + h^2) < \mu. \tag{4.8}$$

So let $z' \in (z - h, z + h)$ and let $\underline{a} \in \mathbf{R}^n$ such that

$$\underline{a} = (0, \ldots, 0, z - z').$$

Suppose $\underline{x} \in R(z')$ and consider $(R(z') + \underline{a}) \cap R(z)$. We have two cases

<u>Case 1</u>

$\underline{x} \in R(z')$ and $(R(z') + \underline{a}) \cap R(z) \neq \phi$. Then $\{E_i(\mu)\}_{i=1}^{p}$ is also a cover of $R(z')$ since $h < \mu$.

<u>Case 2</u>

$\underline{x} \in R(z')$ and $(R(z') + \underline{a}) \cap R(z) = \phi$. Then there is a θ_m such that $\underline{x} \in \theta_m(z')$. But $\underline{x} + \underline{a} \in \theta_m(z)$ and hence

$$\operatorname{diam}(\theta_m(z)) - \operatorname{diam}(\theta_i(z')) \leq \sqrt{2(h + h^2) - h^2}$$

$$= \quad (2h + h^2)^{\frac{1}{2}}.$$

Let

$$d = \inf\{\|\underline{x} + \underline{a} - \underline{y}\| : \underline{y} \in \bigcup_{i=1}^{p} E_i\}$$

then

$$d \leq (2h + h^2)^{\frac{1}{2}} < (2h + 2h^2)^{\frac{1}{2}} < \mu.$$

Hence we have that $\underline{x} + \underline{a}$ is less than a distance μ from the set $\bigcup_{i=1}^{p} E_i$. So we have that $\{E_i\}_{i=1}^{p}$ is also a cover for $R(z')$.

Now we have

$$f(z) = m_\delta^s(R(z))$$

and

$$\sum_{i=1}^{p} \operatorname{diam}^s(E_i(\mu)) < m_\delta^s(R(z)) + \lambda$$

$$\operatorname{diam}(E_i(\mu)) < \delta \quad i = 1, ..., p$$

with $\{E_i(\mu)\}_{i=1}^{p}$ also a cover for $R(z')$. So this implies

$$m_\delta^s(R(z')) \leq \sum_{i=1}^{p} \operatorname{diam}^s(E_i(\mu)) < m_\delta^s(R(z)) + \lambda, \qquad \forall z' \in (z - h, z + h).$$

Writing this with respect to our function f we have

$$f(z') \le f(z) + \epsilon, \quad \forall z' \in (z-h, z+h). \tag{4.9}$$

Let c be some positive real number then define the set

$$Z(c) = \{z : f(z) \le c\}.$$

Then it follows from (4.8) that $Z(c)$ is open and hence measurable.

Therefore f is a measurable function. Hence f is Lebesgue integrable and we have

$$\int_0^1 f(z)dz > 0.$$

Now let us consider a finite open δ-cover of R by n-cubes $\{C_j\}_{j=1}^q$, with edges of same orientation as the co-ordinate axis.

Then, if we take an $n-1$ dimensional slice of R_n at some real number z we will, for some given j, have either

1. A face of C_j, i.e. an (n-1) dimensional cube, of diameter

$$\frac{\sqrt{n-1}}{\sqrt{n}} d(C_j)$$

, or

2. No intersection with C_j.

Let us define

$$g(z) = \sum_{j=1}^{q} \operatorname{diam}^{s}(C_j(z)).$$

Then $g(z)$ is integrable with $f(z) \leq g(z), \ \ \forall z \in [0,1]$, and we have

$$0 < \int_0^1 f(z)dz \leq \int_0^1 g(z)dz = \sum_{j=1}^{q} \operatorname{diam}^{1+s}(C_j(z))$$

and, since $s \leq s_{n-1}(r_\theta)$, the result follows. That is

$$s_n(r_\theta) \geq 1 + s_{n-1}(r_\theta) \qquad \square$$

A corollary to this result is

$$s_n(r_\theta) > n - 1$$

This result follows inductively from our two dimensional result and our theorem.

5. An improved bound on the Besicovitch-Hausdorff dimension of the residual set of arbitrarily packed disks in the plane

In this chapter we turn our attention to arbitrary packings of disks into the unit plane square, I_2. A lower bound for the Besicovitch-Hausdorff dimension of the residual set R was shown by D.G. Larman in [1] to be greater than 1.03. We improve this bound by developing the methods used in chapter 2.

Theorem 3. *Suppose that $\{\theta_n\}_{n=1}^{\infty}$ forms a packing of disks within the unit plane square I_2. Then the residual set $R = I_2 \setminus \cup_{n=1}^{\infty} \theta_n$ has Besicovitch - Hausdorff dimension s and*

$$s > 1.033.$$

Proof. We may suppose without loss of generality that each disk θ_n is open, and that diam $(\theta_{n+1}) \leq$ diam (θ_n) for $n = 1, 2, \ldots$. This gives us an order to our packing. The largest copy being θ_1, and the θ_i's decreasing in size as i increases. We define the n^{th} stage residual set to be

$$R_n = I_2 \backslash \bigcup_{m=1}^{n} \theta_m$$

and the residual set for the packing,

$$R = \bigcap_{n=1}^{\infty} R_n = I_2 \backslash \bigcup_{m=1}^{\infty} \theta_m.$$

Since R is compact we need only consider coverings of R by finite collections of open squares $\{C_j\}_{j=1}^{\infty}$ when determining whether the s-dimensional Hausdorff measure of R is positive. Let the diameter of C_j be $\sqrt{2}\Delta_j$.

Let $\{C_j\}_{n=1}^{p}$ be a minimal cover of R by open squares, so all of the C_j's are necessary, i.e. none of them are contained within any of the θ_n, $n = 1, 2, \ldots$.

Note that each R_n is compact, and that for each $n > 0$, $R_{n+1} \subset R_n$. Hence $\{R_n\}_{n=1}^{\infty}$ is a nested sequence of compact sets. This implies there exists some $m \in \mathbf{N}$ such that $R_m \subset \cup_{j=1}^{p} C_j$. Let us now fix such a sufficiently large m.

To prove our result we find a condition on s such that under this condition $\sum_j \Delta_j^s$ cannot be close to zero which will in turn imply $\dim(\mathrm{R}) \geq \mathrm{s}$.

Suppose that the x-axis is the horizontal axis, and the y-axis is the vertical axis. Let $0 \leq x \leq 1$, and let l_x denote the vertical line through the point $(x, 0)$, therefore $l_x = \{(X, Y) \in \mathbf{R}^2 : X = x\}$.

Let $l'_{(x,j)}$ be the open interval equal to $l_x \cap C_j$, and let the length of $l'_{(x,j)}$ be $l_{(x,j)}$. Then either

$$l_{(x,j)} = \frac{1}{\sqrt{2}}\mathrm{Diam}(\mathrm{C_j}) = \Delta_\mathrm{j}$$

or

$$l_{(x,j)} = 0.$$

We now define a function $f(x)$ by

$$f(x) = \sum_{j=1}^{p} l_{(x,j)}^{s-1}.$$

Then we deduce that,

$$\int_0^1 f(x)\, dx = \int_0^1 \sum_{j=1}^{p} l_{(x,j)}^{s-1} dx = \sum_{j=1}^{p} \Delta_j^s. \tag{5.1}$$

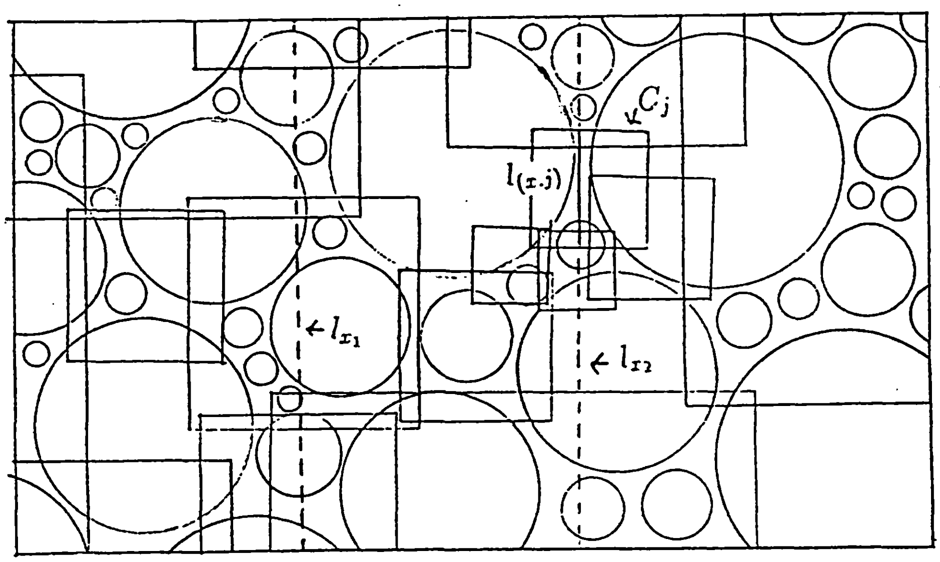

Figure 5.1:

We will define another function g_m such that $\int_0^1 f \geq \int_0^1 g_m$ and then proceed to show $\int_0^1 g_0 > 0$ and deduce our result.

Let us consider $\bigcup_{j=1}^{p} C_j$, then l_x intersects it in a collection of non overlapping open intervals $\left\{j'_{(x,r)}\right\}_{r=1}^{v}$ of lengths $\left\{j_{(x,r)}\right\}_{r=1}^{v}$. Now each of these intervals can be expressed as the union of $\left\{l'_{(x,j_i)}\right\}_{i=1}^{v(r)}$ chosen from $\left\{l'_{(x,j)}\right\}_{j=1}^{p}$, so $j'_{(x,r)} = \bigcup_{i=1}^{v(r)} l'_{(x,j_i)}$.

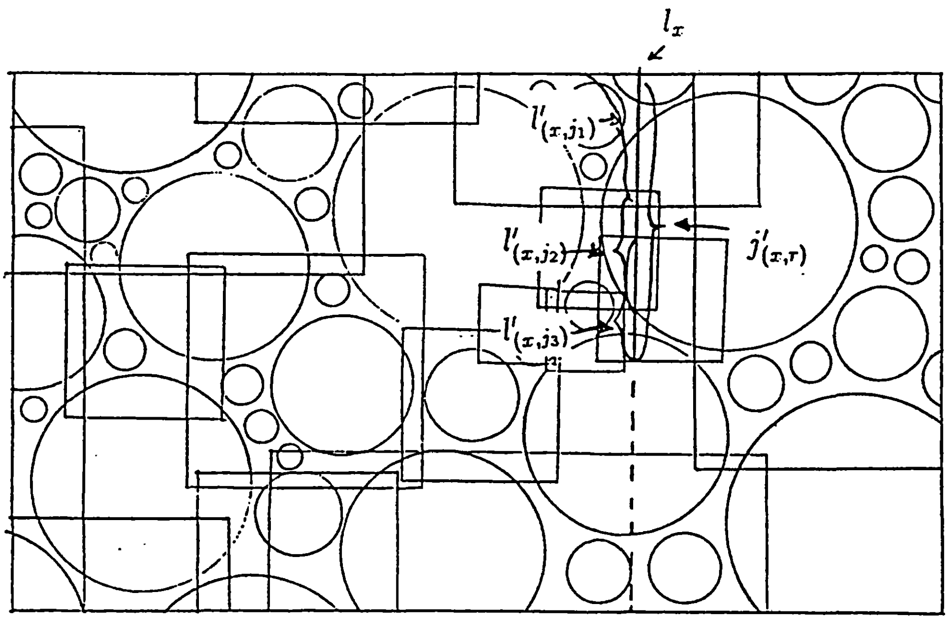

Figure 5.2:

Now we have

$$j_{(x,r)} \leq \sum_{i=1}^{v(r)} l_{(x,j_i)} \leq \sum_{j=1}^{p} l_{(x,j)} \qquad r = 1, \ldots, v$$

and since $0 < s-1 < 1$,

$$j_{(x,r)}^{s-1} \leq \sum_{i=1}^{v(r)} l_{(x,j_i)}^{s-1} \qquad r = 1, \ldots, v$$

So summing over r we have,

$$\sum_{r=1}^{v} j_{(x,r)}^{s-1} \leq \sum_{r=1}^{p} \sum_{i=1}^{v(r)} l_{(x,j)}^{s-1} \leq \sum_{j=1}^{p} l_{(x,j)}^{s-1} \tag{5.2}$$

Let us define our function $g_m(x)$, m fixed

$$g_m(x) = \sum_{r=1}^{v} j_{(x,r)}^{s-1}$$

So, from (1) and (2) we have

$$\begin{aligned} \int_0^1 g_m(x)\, dx &= \int_0^1 \sum_{r=1}^{v} j_{(x,r)}^{s-1}\, dx \\ &\leq \int_0^1 \sum_{j=1}^{p} l_{(x,j)}^{s-1}\, dx \\ &= \sum_{j=1}^{p} \Delta_j^s. \end{aligned} \tag{5.3}$$

If we show that $\int_0^1 g_m(x)\, dx > 1$, for any cover $\{C_k\}_{k=1}^{p}$,then the residual set R will have Besicovitch-Hausdorff dimension of at least s.

Let i be some integer, $0 \leq i \leq m-1$ and consider

$$\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k.$$

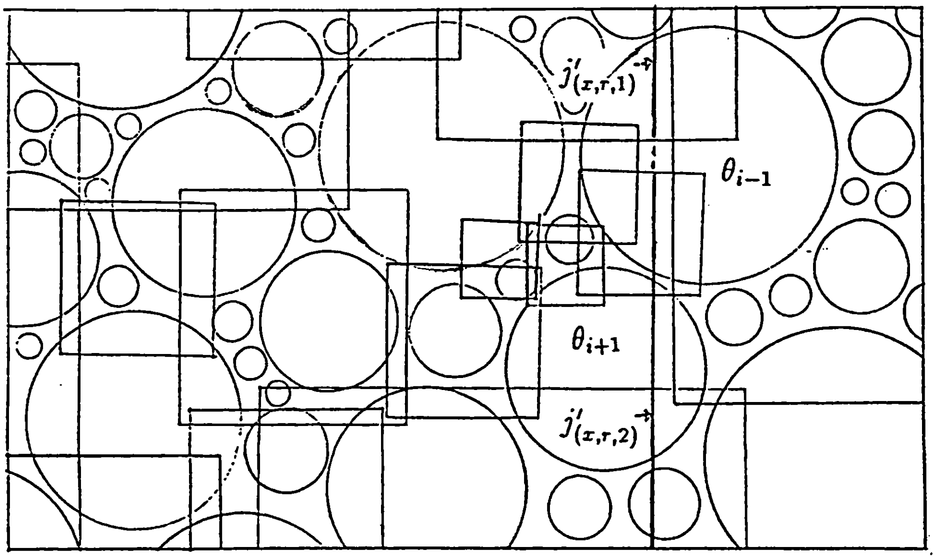

Figure 5.3:

This is $\{C_j\}_{i=1}^{p}$ union those disks larger than θ_{m+1}, and smaller than θ_i.

l_x intersects this union in a collection of disjoint intervals $\left\{j'_{(x,r,i)}\right\}_{r=1}^{v(i,x)}$ of lengths $\left\{j_{(x,r,i)}\right\}_{r=1}^{v(i,x)}$ respectively.

Define

$$g_i(x) = \sum_{r=1}^{v(i,x)} j_{(x,r_i,i)}^{s-1}$$

So suppose i is one of $1, \ldots, m$, then,

$$\int_0^1 g_i(x)\, dx = \int_0^1 g_{i-1}(x)\, dx + \int_0^1 (g_i(x) - g_{i-1}(x))\, dx \tag{5.4}$$

We shall show that for any i

$$\int_0^1 (g_i(x) - g_{i-1}(x))dx \geq 0$$

which gives,

$$\int_0^1 g_m(x)\, dx \geq \int_0^1 g_{m-1}(x)\, dx \geq \ldots \geq \int_0^1 g_0(x)\, dx$$

and since we clearly have $\int_0^1 g_0(x)dx = 1$ the result follows, since this implies

$$\int_0^1 g_m(x)dx \geq \int_0^1 g_0(x)dx = 1$$

For $0 \leq x \leq 1$, l_x meets θ_i in an interval of length 2α say, or does not meet θ_i at all. Let $l'_{(x,\theta_i)}$ be the open interval equal to $l_x \cap \theta_i$, which has length $l_{(x,\theta_i)}$ then either

$$l'_{(x,\theta_i)} = \phi$$

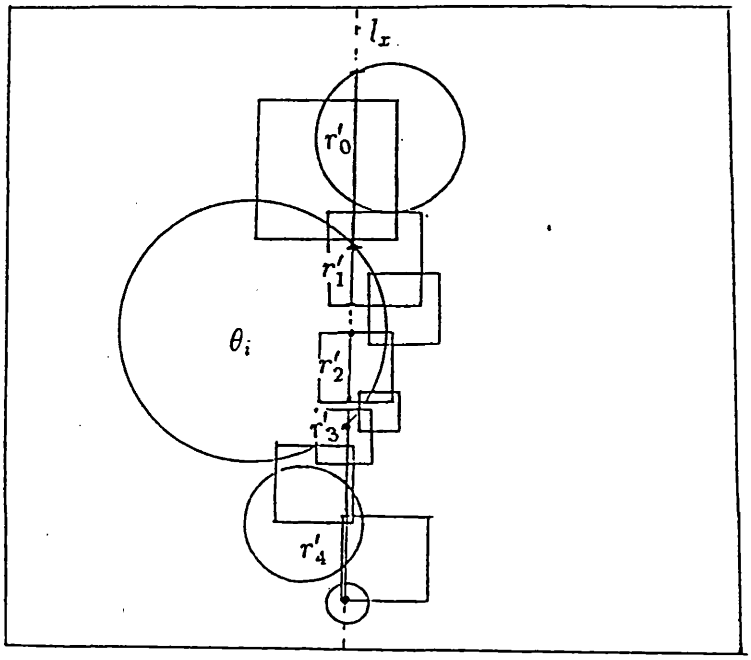

Figure 5.4:

or

$$l_{(x,\theta_i)} = 2\alpha$$

Now $g_i(x) = g_{i-1}(x)$ for x such that l_x does not meet θ_i, as there are no extra intervals to consider, so we need only worry about those x where l_x meets θ_i.

The segment of l_x which lies in θ_i meets $\left(\bigcup_{j=1}^{p} C_j\right) \cup \left(\bigcup_{k=i+1}^{m} \theta_k\right)$ in a collection of non overlapping intervals $\{r'_j\}_{j=2}^{w-1}$ of lengths $\{r_j\}_{j=2}^{w-1}$, whose closures do not meet the boundary of θ_i, and two intervals r'_1 and r'_w, (r_w below r_1), whose closures meet θ_i, in fact they may coincide. The lengths of r'_1 and r'_w being r_1 and r_w respectively. l_x also meets $\left(\bigcup_{j=1}^{p} C_j\right) \cup \left(\bigcup_{k=i+1}^{m} \theta_k\right)$ in two intervals r'_0, r'_{w+1} of lengths r_0, r_{w+1}, immediately above and below θ_i respectively.

Let us define a function $T(x)$ such that

$$T(x) = \begin{cases} 1 & \text{if } r'_1 \text{ and } r'_w \text{ do not coincide} \\ 0 & \text{otherwise.} \end{cases}$$

Then the difference is,

$$\begin{aligned} g_i(x) - g_{i-1}(x) &= T(x)[(r_0 + r_1)^{s-1} + r_2^{s-1} + \dots \\ &+ r_{w-1}^{s-1} + (r_w + r_{w+1})^{s-1} - (r_0 + 2\alpha + r_{w+1})^{s-1}] \end{aligned} \tag{5.5}$$

Suppose that $0 \leq \mu \leq r_0$ and $0 \leq \lambda \leq r_{w+1}$.

Let

$$\begin{aligned} h_i(x, \mu, \lambda) = T(x)[(\mu + r_1)^{s-1} + r_2^{s-1} &+ ,\dots, + r_{w-1}^{s-1} + (r_w + \lambda)^{s-1} \\ &- (\mu + 2\alpha + \lambda)^{s-1}] \end{aligned}$$

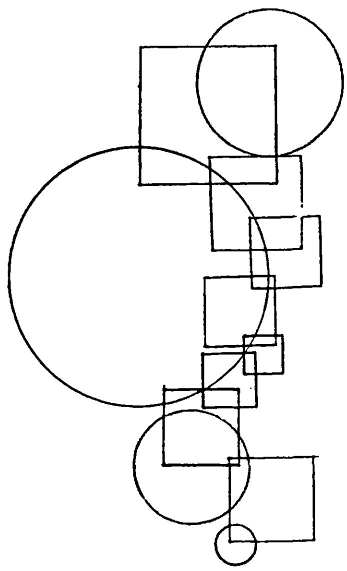

Figure 5.5:

Then $g_i(x)-g_{i-1}(x) = h_i(x, r_0, r_{w+1})$. We show that $g_i(x)-g_{i-1}(x) \geq 0$ by first differentiating $h_i(x, r_0, r_{w+1}) \geq h_i(x, 0, 0)$ and then demonstrating that $h_i(x, 0, 0) \geq 0$.

Now,

$$\frac{d}{d\mu} h_i(x, \mu, \lambda) = (s-1)T(x)[(\mu + r_1)^{s-2} - (\mu + 2\alpha + \lambda)^{s-2}]$$

and,

$$\frac{d}{d\lambda}h_i(x,\mu,\lambda) = (s-1)T(x)[(r_w+\lambda)^{s-2} - (\mu+2\alpha+\lambda)^{s-2}]$$

.

Then since both $r_1.r_w \leq 2\alpha$ we have,

$$\frac{d}{d\mu}h_i(x,\mu,\lambda) \geq 0 \text{ and } \frac{d}{d\lambda}h_i(x,\mu,\lambda) \geq 0 \tag{5.6}$$

Let the following intervals be labelled in the following way,

$$r_0' \equiv A_1B_1, \;\; r_{w+1}' \equiv C_1D_1, \;\; r_0' \cup r_1' \equiv A_1B_1E_1, \;\; r_w' \cup r_{w+1}' \equiv C_1D_1F_1.$$

Then if G_1, H_1 are points of A_1B_1, C_1D_1 respectively, we may define for all x such that l_x meets θ_i, $g_i'(x)$ the same as $g_i(x)$ except that $A_1B_1E_1$, $C_1D_1F_1$ are replaced by $G_1B_1E_1$, $F_1C_1H_1$, and $g_{i-1}'(x)$ defined as $g_{i-1}(x)$ with A_1D_1 replaced with G_1H_1. Let $g_i'(x) = g_i(x)$, and $g_{i-1}'(x) = g_{i-1}(x)$ for all other x. Then we have, for all x, using (5.4), (5.5) and (5.6), that

$$\begin{aligned}\int_0^1 g_i(x)\,dx &= \int_0^1 g_{i-1}(x)\,dx + \int_0^1 (g_i(x) - g_{i-1}(x))\,dx \\ &\geq \int_0^1 g_{i-1}(x)\,dx + \int_0^1 \left(g_i'(x) - g_{i-1}'(x)\right)\,dx \qquad (5.7)\end{aligned}$$

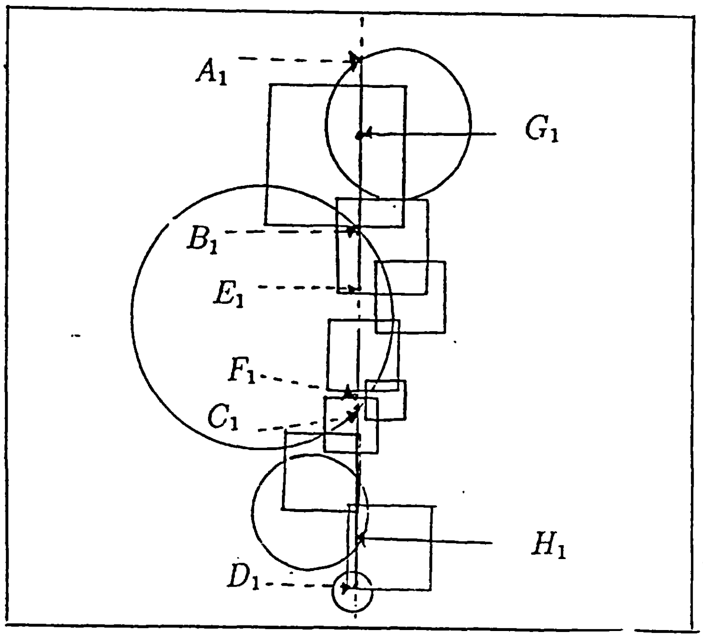

Figure 5.6:

We use this to simplify our problem.

We examine θ_i, and find a polygon $\mathcal{H}_i$ which encloses it. Suppose that the disk θ_i has diameter $2t$ and is centred at (x_i, y_i), which we label O_i.

Let $L_{(y_i)}$ be the line which passes horizontally throught the centre of θ_i.

$$L_{(y_i)} = \{(x, y) \in \mathbf{R}^2 \ : \ y = y_i\}$$

Now θ_i can be contained within a minimal square centred at O_i of diameter $2\sqrt{2}\,t$. Let S_i denote this square.

Now let $\mathcal{H}$ be the regular hexagon centred at O_i of diameter $3\sqrt{2}\,\mathrm{Diam}(\theta_i) = 6\sqrt{2}\,t$, which contains θ_i. Let the orientation of $\mathcal{H}$ be such that 2 of its sides are parallel to the $x-$axis.

Let $\theta'_{i+1}, \ldots, \theta'_m$ be disks centred at $O_{i+1}, \ldots, O_m$ respectively; each of diameter $2\,t$. Then there is a subset, say $\{\theta'_k\}_{k=1}^{n(i)}$ of these, congruent to θ_i, which are contained in $\mathcal{H}$, having centres $\{O'_k\}_{k=1}^{n(i)}$: some subset of $\{O_j\}_{j=i+1}^{m}$.

Let $\mathcal{H}_k$, $k = 1, \ldots, n(i)$, denote the set of points of $\mathcal{H}$ which are at least as close to O'_k as to any other O'_j.

$$\mathcal{H}_k = \left\{ p \in \mathcal{H} \; : \; \min_{1 \leq j \leq n(i)} |p - O'_j| = |p - O'_k| \right\}$$

$\mathcal{H}_k$ is the *Dirichlet Cell* (or *Voronoi Region*) of O'_k, and is a convex polygon. We now appeal to the following lemma which can be found in [6], page 47.

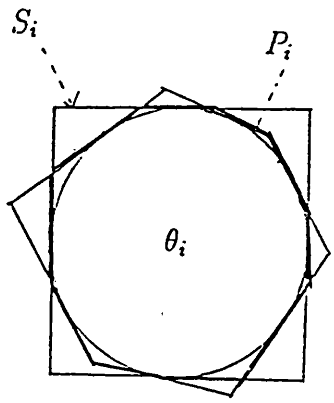

Figure 5.7:

Lemma 3. *If $\mathcal{H}$ is a convex hexagon, $\{\theta_i\}_{i=1}^n$ is a packing of circles in $\mathcal{H}$, O_i denoting the centre of θ_i. Let h_i be the number of sides of $\mathcal{H}_i$, then*

$$\sum_{i=1}^{n} h_i \leq 6n$$

Hence if n is sufficiently large, we may assume that the $\mathcal{H}_k$ are convex hexagons. Let $\mathcal{H}_i'$ be the hexagon formed by pushing the facets of $\mathcal{H}_i$ towards θ_i until they touch its boundary.

Now consider the following polygon:

$$P_i = \mathcal{H}_i' \cap S_i$$

Then P_i is a polygon that contains θ_i and which has at most 8 non-vertical sides.

Now if $x \in [0,1]$ and l_x meets θ_i then l_x meets $P_i - \theta_i$ in two intervals: G_2B_1, C_1H_2 immediately above and below θ_i respectively.

By construction, G_2B_1, C_1H_2 do not meet any of the θ_j, $j = 1, \ldots, i-1$.

We shall now define $g_i''(x)$ as $g_i'(x)$ except that $G_1B_1E_1$, $F_1C_1H_1$ are replaced by $G_2B_1E_1$, $F_1C_1H_2$; and $g_{i-1}''(x)$ as $g_{i-1}'(x)$ except G_1H_1 is replaced by G_2H_2. We then have that

$$g_i'(x) - g_{i-1}'(x) \geq g_i''(x) - g_{i-1}''(x)$$

and hence from (7),

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}'(x)dx + \int_0^1 (g_i''(x) - g_{i-1}''(x))dx \tag{5.8}$$

Suppose that H_2, G_2 lie on the segments Q_1R_1, Q_2R_2 of the polygon P_i respectively. Let V_k be the point of intersection of Q_kR_k and the boundary of θ_i.

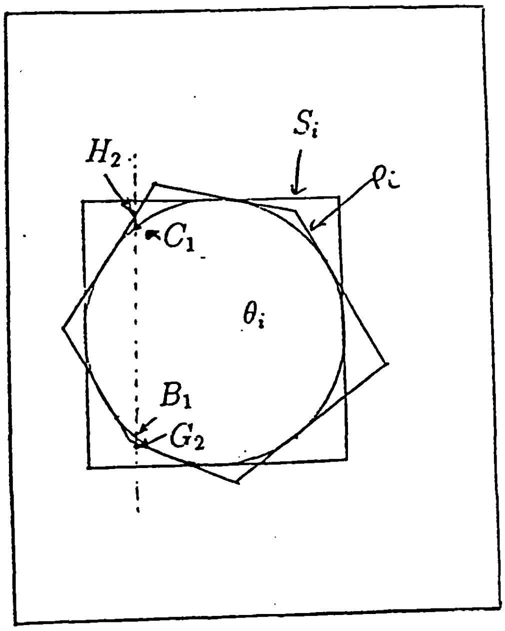

Figure 5.8:

Let U denote the intersection of l_x and $L_{(y_i)}$, and the horizontal distances from H_2, G_2 to V_1, V_2 be y' and y'' respectively.

Now

$$(V_1H_2)^2 = H_2C_1 \cdot H_2B_1 \tag{5.9}$$

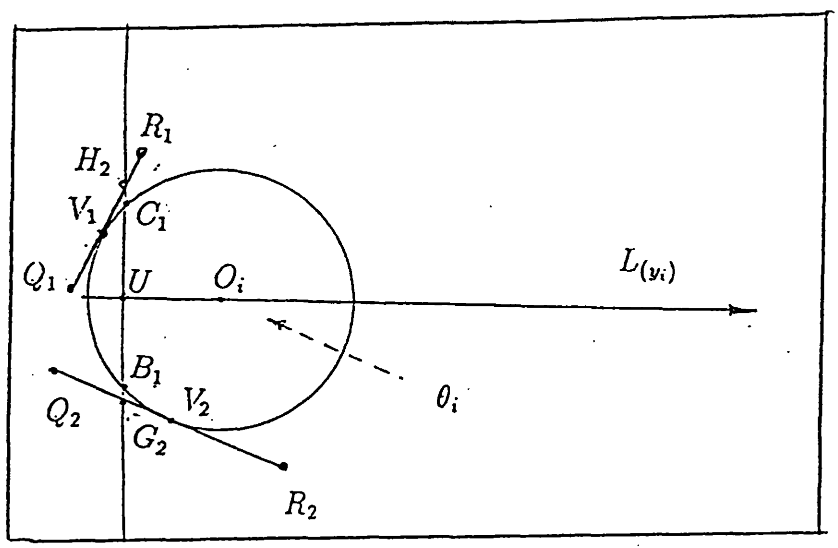

Figure 5.9:

H_2B_1 has length at most the diameter of θ_i, hence

$$H_2B_1 \leq 2t \tag{5.10}$$

Let ψ be the acute angle that Q_1R_1 makes with $L_{(y_i)}$, then

$$y' = V_1H_2 \cos\psi \tag{5.11}$$

Combining these we get

$$
\begin{aligned}
H_2C_1 = \frac{(V_1H_2)^2}{H_2B_1} &\geq \frac{(V_1H_2)^2}{2t} \\
&= \frac{(y')^2 \sec^2 \psi}{2t} \\
&\geq \frac{y'^2}{2t}
\end{aligned}
$$

Since UH_2 is bounded by the radius of θ_i, ($UH_2 \leq t$), it follows that

$$
\frac{H_2C_1}{UH_2} \geq \frac{y'^2}{2t^2} \tag{5.12}
$$

Similarly we have

$$
(V_2G_2)^2 = G_2B_1 \cdot G_2C_1
$$

And using

$$
G_2C_1 \leq 2t \qquad y'' = V_2G_2 \cos \eta
$$

Where η is the acute angle which Q_2R_2 makes with $L_{(y_i)}$ we have

$$
\frac{G_2B_1}{UG_2} \geq \frac{(y'')^2}{2t^2} \tag{5.13}
$$

since $UG_2 \leq t$.

Given some $\delta \in [0,1]$, and providing that

$$y' \geq \delta t, \qquad y'' \geq \delta t$$

then it follows from (5.12) and (5.13) that

$$\frac{H_2C_1}{UH_2} \geq \frac{1}{2}\delta^2 \text{ and } \frac{G_2B_1}{UG_2} \geq \frac{1}{2}\delta^2. \tag{5.14}$$

Let s, σ, δ be positive real numbers which satisfy the following:

$$2^{3-2s}\delta^{2(s-1)} \geq 1+\sigma \tag{5.15}$$

and

$$\frac{\sigma}{27}\left(\frac{2\sqrt{53}}{27}\right)^{s-1} \geq 8 \cdot 2^s\delta \tag{5.16}$$

where $\sigma < 1$ and $\delta \leq 1/81$.

We shall verify that an allowable set of values is

$$s = 1.033 \qquad \sigma = (2\sqrt{53})^{-1} \qquad \delta = \frac{\left(\frac{\sqrt{53}}{27}\right)^{1.033}}{1696} \tag{5.17}$$

Now using (5.14) and (5.15) we have

$$2(H_2C_1)^{s-1} - (2UH_2)^{s-1} \geq \sigma(2UH_2)^{s-1} \tag{5.18}$$

$$2(G_2B_1)^{s-1} - (2UG_2)^{s-1} \geq \sigma(2UG_2)^{s-1} \tag{5.19}$$

Except possibly if x belongs to at most 8 intervals whose union is Q. Each interval of this type having length $2\delta t$. The centre of such an interval being a horizontal projection of a point of contact of P_i with the boundary of θ_i.

Let us suppose that the length of H_2C_1 is greater than G_2B_1 by some length denoted r. Then if $x \notin Q$ we have using (18) and (19)

$$\begin{aligned}
& (H_2C_1)^{s-1} + (G_2B_1)^{s-1} - (H_2G_2)^{s-1} \\
= \ & (G_2B_1 + r)^{s-1} + (G_2B_1)^{s-1} - (2UG_2 + r)^{s-1} \\
\geq \ & 2(G_2B_1)^{s-1} - (2UG_2)^{s-1} \\
\geq \ & \sigma(2UG_2)^{s-1}
\end{aligned}$$

Hence

$$\begin{aligned}(H_2C_1)^{s-1} &+ (G_2B_1)^{s-1} - (H_2G_2)^{s-1} \\ &\geq \sigma \min[(2UG_2)^{s-1}, (2UH_2)^{s-1}] \qquad (5.20)\end{aligned}$$

If, conversely, G_2B_1 is greater than H_2C_1 by r, we have

$$\begin{aligned}& (H_2C_1)^{s-1} + (G_2B_1)^{s-1} - (H_2G_2)^{s-1} \\ = \;& (H_2C_1)^{s-1} + (H_2C_1 + r)^{s-1} - (2UH_2 + r)^{s-1} \\ \geq \;& 2(H_2C_1)^{s-1} - (2UH_2)^{s-1} \\ \geq \;& \sigma(2UH_2)^{s-1}.\end{aligned}$$

If r'_{11} coincides with r'_{w1} then we have

$$g''_i(x) = g''_{i-1}(x). \qquad (5.21)$$

Otherwise, if $x \notin Q$, we deduce from (5.20)

$$g''_i(x) \geq g''_{i-1}(x) + \sigma \min[(2UG_2)^{s-1}, (2UH_2)^{s-1}], \qquad (5.22)$$

If $x \in Q$, then, since $P_i \subset S_i$, it follows that

$$\begin{aligned}
\int_Q (g_i''(x) - g_{i-1}''(x))dx &\geq -\int_Q (H_2G_2)^{s-1}dx \\
&\geq -8\int_0^{2\delta t}(2t)^{s-1}dx \\
&= -8 \cdot 2^s \delta t^s. \qquad (5.23)
\end{aligned}$$

Let θ_i'' be a disk of radius $(1-\delta^{1/s})t$, centred at O_i. The remainder of this proof then splits into three cases as follows:

Case 1 A side of C_q, one of the $\{C_j\}_{j=1}^p$ meets θ_i''.

Case 2 Some C_q contains θ_i'' entirely.

Case 3 The disk, θ_i'', does not intersect $\cup_{i=1}^p C_j$.

Case 1. If C_q, say, has a side which meets θ_i, then there is a portion of length $\rho(q,i)$ which is entirely contained within θ_i. As $\{C_j\}_{j=1}^p$ is a minimal cover, we know that none of the squares $\{C_j\}_{j=1}^p$ lie entirely within θ_i. Then

$$\rho(q,i) \geq \delta^{1/s}t \qquad (5.24)$$

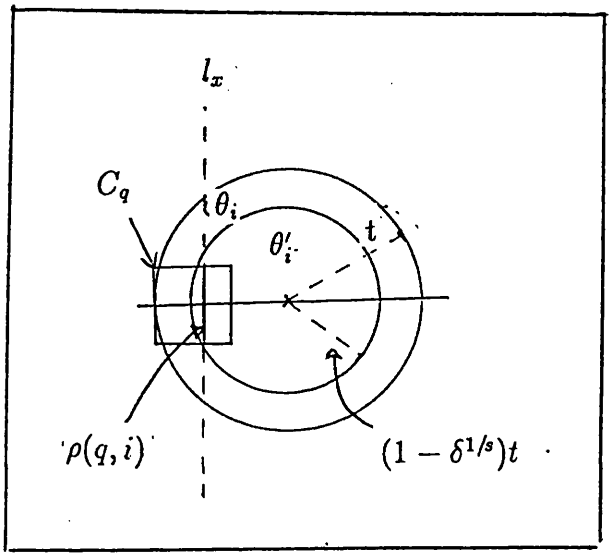

Figure 5.10:

From (5.24) we have

$$-8 \cdot 2^s \delta t^s \geq -8 \cdot 2^s \rho^s(q,i)$$

Since $s < 2$ we have

$$-8 \cdot 2^s \rho^s(q,i) \geq -32\rho^s(q,i) \tag{5.25}$$

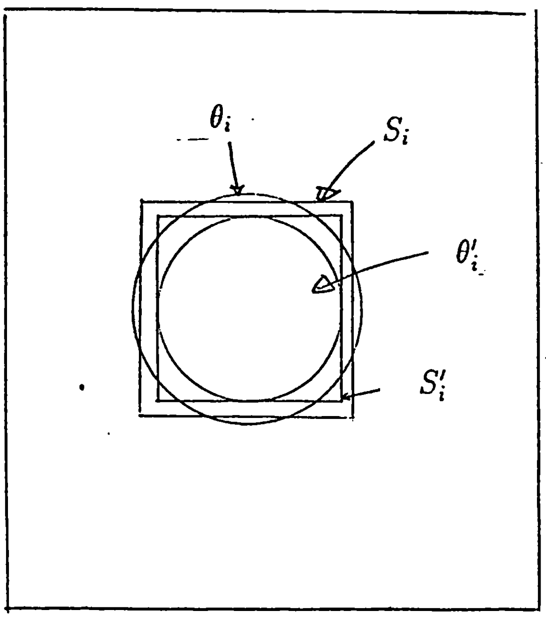

Figure 5.11:

which gives, with (5.8):

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx - 32\rho^s(q,i) \tag{5.26}$$

Case 2. Let S'_i be the square that circumscribes θ''_i which is the same orientation as S_i. δ is sufficiently small to ensure that all four corners of S'_i are contained in the compliment of θ_i. Coverings of this type are not economical.

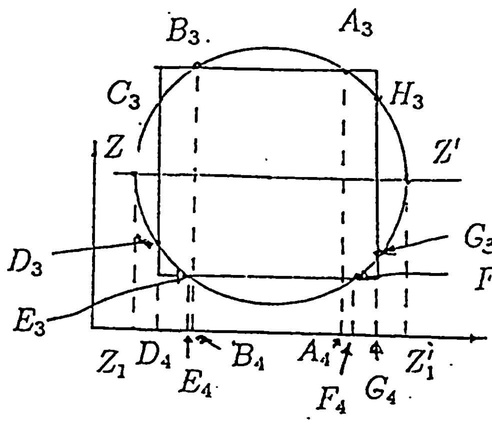

Figure 5.12:

Now C_q must contain S_i'. Then C_q must also have all four corners in the compliment of θ_i.

Suppose that the horizontal intervals where the edges of C_q meet θ_i are A_3B_3, E_3F_3, and the vertical intervals are C_3D_3, G_3H_3 shown in Fig . Some of these intervals may not exist.

Suppose that A_3B_3 and E_3F_3 both exist, and that

$$A_3B_3 \leq E_3F_3.$$

Let $g_i'''(x)$, $g_{i-1}'''(x)$ be defined as $g_i'(x)$, $g_{i-1}'(x)$ by taking $G_1 \equiv B_1$ and $H_1 \equiv C_1$. This gives

$$g_i'(x) - g_{i-1}'(x) \geq g_i'''(x) - g_{i-1}'''(x).$$

Then using (5.7),

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx + \int_0^1 (g_i'''(x) - g_{i-1}'''(x))dx. \tag{5.27}$$

Let the points of intersection of θ_i and C_q be as indicated and note their horizontal projections. If $0 \leq x \leq 1$ then l_x meets

(i) $C_q \cap \theta_i$ in an interval of length $\beta(x)$.

(ii) θ_i in an interval of length $2\alpha(x)$.

If Q' denotes the union of the intervals $Z'D_4$, E_4F_4, G_4Z_1' then it follows that

$$\int_0^1 (g_i'''(x) - g_{i-1}'''(x))dx \geq \int_{Q'} ((\beta(x))^{s-1} - (2\alpha(x))^{s-1})dx \tag{5.28}$$

Now $2\alpha(x) \leq 2t$ and $\beta = (t^2 - (\frac{1}{2}A_3B_3)^2)^{1/2} + (t^2 - (\frac{1}{2}E_3F_3)^2)^{1/2}$ Hence

$$\begin{aligned}
\beta(x)^{s-1} - (2\alpha(x))^{s-1} &\geq -(2\alpha(x) - \beta(x))^{s-1} \\
&\geq -(2t - \beta(x))^{s-1} \\
&= -[(t^2)^{1/2} - (t^2 - (\frac{1}{2}A_3B_3)^2)^{1/2} \\
&\qquad +(t^2)^{1/2} - (t^2 - (\frac{1}{2}E_3F_3)^2)^{1/2}]^{s-1} \\
&\geq -(\frac{1}{2}A_3B_3 + \frac{1}{2}E_3F_3)^{s-1} \\
&\geq -(E_3F_3)^{s-1} \qquad (5.29)
\end{aligned}$$

Similarly if x belongs to E_4B_4 or to A_4F_4

$$(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -(\frac{1}{2}E_3F_3)^{s-1} \qquad (5.30)$$

If x belongs to $Z'D_4$

$$(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -(C_3D_3)^{s-1} \qquad (5.31)$$

If x belongs to G_4Z_1'

$$(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -(H_3G_3)^{s-1} \tag{5.32}$$

Combining (5.28) - (5.32) and noting

$$C_3D_3 \geq Z'D_4, \qquad H_3C_3 \geq G_4Z_1',$$

we find that

$$\int_0^1 (g_i'''(x) - g_{i-1}'''(x))dx \geq -((A_3B_3)^s + (C_3D_3)^s + (E_3F_3)^s + (G_3H_3)^s). \tag{5.33}$$

This also holds if $A_3B_3 \geq E_3F_3$ and therefore combining this with (5.27) we obtain, for Case 2,

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx - \sum_{j=1}^{4}(\gamma(i,q,j))^s, \tag{5.34}$$

where $\{\gamma(i,q,j)\}_{j=1}^4$ is the disjoint intervals of boundary of C_q which lie in θ_i.

Case 3. If L denotes the maximal line segment of $L_{(y_i)} \cap \theta_i$ which contains O_i, but does not meet $\bigcup_{j=1}^{p} C_j$, then the length of L is at least $(2 - 2\delta^{1/2})t$, which,

using (5.16), is at least equal to $\frac{16}{9}t$. As there are at most 8 non-vertical sides of P_i there is an interval w of length $\frac{1}{9}t$ on the x-axis which is contained in the horizontal projection of L but which does not contain the horizontal vertex of P_i or horizontal of projection of a point of contact of P_i with the boundary of θ_i.
Hence, there is an interval W' of W, which has length $\frac{1}{27}t$ which is at least a distance $\frac{1}{27}t$ from the horizontal projection of either a vertex of P_i or a point of contact of P_i with the boundary of θ_i.
Then, if x is a point of W', let H_2, C_1, U, B_1, G_2 be as indicated and let the points of intersection of Q_1R_1 and Q_2R_2 and the boundary of θ_i be V_1, V_2 respectively.
As U is at least a distance $\frac{t}{27}$ from the complement of θ_i, we have

$$\min(H_2U,\, UG_2) \geq \left(\sqrt{1-\left(\frac{26}{27}\right)^2}\right) t = \frac{\sqrt{53}}{27}t. \tag{5.35}$$

Now suppose that Q_1R_1, Q_2R_2 are sides of P_i which lie above and below $L_{(y_i)}$ respectively. Suppose also that both of their horizontal projections onto the x-axis contain the interval W.
Let IJ be the segment of $L_{(y_i)}$ which projects horizontaly onto W'. Note that the x co-ordinate of J is greater than that of I.

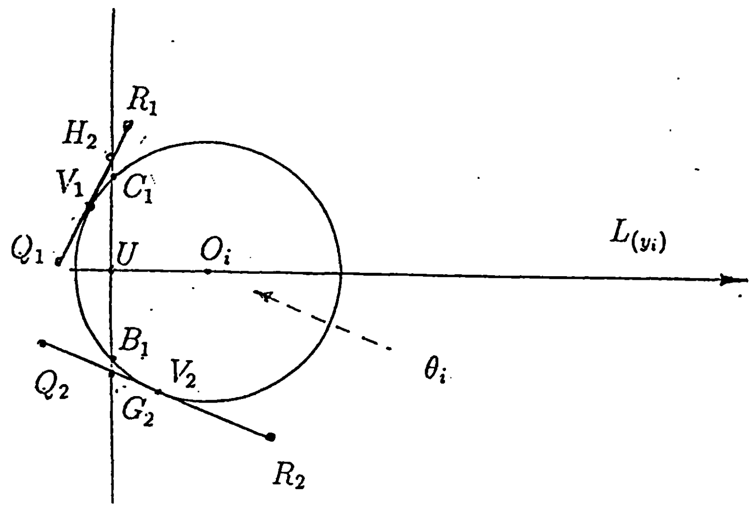

Figure 5.13:

We have for this case that θ_i does not meet $\cup_{j=1}^{p} C_j$ and hence none of the squares meet IJ. It follows that

$$g_i''(x) - g_{i-1}''(x) \geq (H_2C_1)^{s-1} + (G_2B_1)^{s-1} - (H_2G_2)^{s-1} \tag{5.36}$$

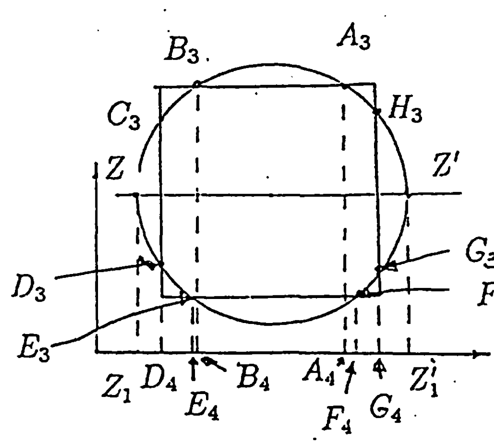

Figure 5.14:

Which, using (5.20), gives

$$g_i''(x) - g_{i-1}''(x) \geq \sigma \min[(2UG_2)^{s-1}, (2UH_2)^{s-1}].$$

So, using (5.35),

$$g_i''(x) - g_{i-1}''(x) \geq \sigma \left(\frac{2\sqrt{53}}{27} t \right)^{s-1}.$$

And hence

$$\begin{aligned}\int_{W'}(g_i''(x) - g_{i-1}''(x))dx &\geq \sigma\left(\frac{2\sqrt{53}}{27}t\right)^{s-1}\frac{t}{27} \\ &\geq 8\cdot 2^s\delta t^s, \end{aligned} \qquad (5.37)$$

using (5.16) and $\frac{1}{27} > \delta$.

Given (5.8), (5.23) and (5.37), we have

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx. \qquad (5.38)$$

We now combine cases 1,2 and 3, using (5.26), (5.34) and (5.38), to deduce

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx - 32\sum_{j=1}^{4}(\gamma(i,q,j))^s \qquad (5.39)$$

where $\{\gamma(i,q,j)\}_{j=1}^{4}$are the lengths of the disjoint portions of sides of C_q which lie entirely within θ_i.

Now repeating this argument for $i = m, m-1, \ldots, 1$ we deduce

$$\int_0^1 g_m(x)dx \geq \int_0^1 1^{s-1}dx - 32\sum_{i=1}^{m}\sum_{j=1}^{4}(\gamma(i,q,j))^s, \qquad (5.40)$$

where $\{\gamma(i, q(i), j)\}_{j=1}^{4}$ are the lengths of disjoint portions of the sides of a square, C_q, of $\{C_j\}_{j=1}^{p}$, which lie entirely within θ_i.

Let $\{\{\gamma(k,j)\}_{k=1}^{w(j)}\}_{j=1}^{p}$ be a rearrangement of $\{\{\gamma(i, q(i), j)\}_{j=1}^{4}\}_{i=1}^{n}$ so that $\{\gamma(k,j)\}_{k=1}^{w(j)}$ is all the lengths of those intervals which belong to the boundary of C_j. Since $\{\theta_i\}_{i=1}^{m}$ are disjoint, and $s > 1$, we have

$$\sum_{k=1}^{w(j)} \gamma(k,j)^s \leq 4\Delta_j^s. \tag{5.41}$$

So

$$-32\sum_{j=1}^{p}\sum_{k=1}^{w(j)} \gamma(k,j)^s \geq -128\sum_{j=1}^{p}\Delta_j^s.$$

From (5.3), (5.40) and (5.41), we have

$$\sum_{j=1}^{p}\Delta_j^s \geq \int_0^1 g_n(x)dx \geq 1 - 128\sum \Delta_j.$$

Hence we obtain

$$\sum_{j=1}^{p}\Delta_j^s \geq \frac{1}{124} \tag{5.42}$$

where s is a real number, $1 < s < 2$, such that, with σ, δ, satisfy (5.15) and (5.16).

So R has Besicovitch-Hausdorff dimension at least s.

To show $s = 1.033$ is an allowable value for s, let

$$\sigma = (2\sqrt{53})^{-1} \tag{5.43}$$

$$2^{3-2s}\delta^{2(s-1)} = 1 + \sigma \tag{5.44}$$

$$\frac{\sigma}{27}\left(\frac{2\sqrt{53}}{27}\right)^{s-1} = 8 \cdot 2^s \delta. \tag{5.45}$$

Now substituting (5.43) and (5.45) into (5.44) we have

$$\frac{2^{13-12s}53^{s^2-3s+2}}{27^{2s^2-2s}} = 1 + \frac{1}{2\sqrt{53}} \tag{5.46}$$

Taking log's produces the following quadratic

$$\begin{aligned} (\log 53 \quad &- \quad 2\log 27)s^2 \\ &+ \quad (-12\log 2 - 3\log 53 + 2\log 27)s \\ &+ \quad (13\log 2 + 2\log 53 - \log(1 + 2\sqrt{53})^{-1}) = 0 \end{aligned}$$

calculation of the coefficients produces:

$$a = -1.1384517$$

$$b = -5.92246$$

$$c = 7.3330939$$

The value of s is the larges root of this quadratic, which is 1.033; and this completes the proof. □

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