The Besicovitch-Hausdorff dimension of the Residual Set of Packings of Convex Bodies in $\mathbb{R}^n$

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Submitted in partial fulfilment towards the degree of Doctor of Philosophy
I undertake a study of the Besicovitch-Hausdorff dimension of the residual set of arbitrary packings of convex bodies in $\mathbb{R}^n$.

In my second chapter, I consider packings of convex bodies of bounded radius of curvature and of fixed orientation into the unit plane square. I show that the Besicovitch-Hausdorff dimension, $s$, of the residual set of an arbitrary packing satisfies

$$s > 1 + \frac{1}{\log r_{\theta}}$$

where $r_{\theta}$ is the bound for the radius of curvature.

In chapter 3, I construct a packing which demonstrates that this bound is of the correct order.

I generalise the 2-dimensional result to higher dimensions in chapter 4. I use a slicing argument to prove this.

In the final chapter, I tackle the disk packing problem. Using Dirichlet cells, I
Abstract

improve the bound obtained in [1] to 1.033.
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1. INTRODUCTION

In this thesis I study the Besicovitch-Hausdorff dimension of the residual set of packings of convex bodies in \( \mathbb{R}^n \). When studying packings of \( \mathbb{R}^n \) we restrict our attention to the unit \( n \)-cube in \( \mathbb{R}^n \); it is clear that doing so is not detrimental to the generality of the problem.

Let

\[
I_n = \left\{ \xi \in \mathbb{R}^n : \|\xi\|_\infty \leq \frac{1}{2} \right\}
\]

where \( \|\xi\|_\infty = \max_{1 \leq k \leq n} |x_k| \), for \( \xi = (x_1, \ldots, x_n) \).

A packing of \( I_n \) is the union of disjoint open bodies, \( \theta_i \), for \( i \in \mathcal{I} \), some index set, such that

\[
\bigcup_{i \in \mathcal{I}} \theta_i \subset I_n.
\]

For a given packing of \( I_n \), we define the residual set

\[
R = I_n \setminus \bigcup_{i \in \mathcal{I}} \theta_i.
\]
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In this thesis I consider arbitrary packings of $I_n$ by reduced copies of some general convex body $\theta$. We assume that the orientation of the reduced copies are fixed. The results in this thesis give bounds on the Besicovitch-Hausdorff dimension of $R$, for an arbitrary packing $\{\theta_i\}_{i \in \tau}$, dependent on the convex body $\theta$.

1.1. Besicovitch-Hausdorff dimension

In this section we give the definition of the Besicovitch-Hausdorff dimension of a set $E \subset \mathbb{R}^n$. Firstly we define the $s$-dimensional Besicovitch-Hausdorff measure of $E$. Let $0 < s < \infty$ and $\delta \geq 0$, then we define

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_i \text{diam}(b_i)^s : b_i \text{ are balls in } \mathbb{R}^n \text{ such that} \right. \left. E \subset \bigcup_i b_i, \text{ and diam}(b_i) \leq \delta \right\}.$$  

It is easy to see that $\mathcal{H}_\delta^s(E)$ is non-increasing, as a function of $\delta$ decreasing. Thus the limit

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(E)$$
exists. This limit is the $s$-dimensional Besicovitch-Hausdorff measure of the set $E$. We can now define the Besicovitch-Hausdorff dimension of the set $E$.

**Definition 1.** The Besicovitch-Hausdorff dimension of a set $E \subset \mathbb{R}^n$ is

$$
\dim E \equiv \sup\{s: \mathcal{H}^s(E) > 0\} = \sup\{s: \mathcal{H}^s(E) = \infty\}
= \inf\{t: \mathcal{H}^t(E) < \infty\} = \inf\{t: \mathcal{H}^t(E) = 0\}.
$$

1.2. Results

A considerable amount of work has been undertaken concerning packings of open discs in the plane. In [1] D.G. Larman showed that a lower bound for the Besicovitch-Hausdorff dimension of the residual set of arbitrary packing of disks in the plane is 1.03. Previously, in [5], K. Hirst had considered the Apollonian packing of circles using different methods. P. Gruber, in [3], considered packings of general convex bodies, and dealt with packings where variation of orientation was permitted.

In chapter 2 we consider packings of convex bodies of bounded radius of curvature, and of fixed orientation. The methods we use are based on those of D.G. Larman in [1] for the disc packing problem.
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We pack a collection of open, strictly convex bodies \( \{ \theta_i \}_{i=1}^{\infty} \) of bounded radius of curvature \( r_\theta \) and of fixed orientation into the unit plane square \( I_2 \). Then the residual set \( R = I_2 \setminus \{ \theta_i \}_{i=1}^{\infty} \) is compact and has Besicovitch-Hausdorff dimension at least 1. We show that for packings of this type that the Besicovitch-Hausdorff dimension \( s_2(r_\theta) \) of the residual set \( R \) is at least

\[
1 + \epsilon(r_\theta) \quad \text{where} \quad \epsilon(r_\theta) = O\left(\frac{1}{\log r_\theta}\right)
\]

(1.1)

\[0 < \epsilon(r_\theta) < 1\]

and where \( r_\theta \) is a bound for the radius of curvature of the convex body \( \theta \). In chapter 3, we continue our study of 2-dimensional packings of convex bodies and proceed to construct, for \( \theta \) sufficiently large, a packing for which the Besicovitch-Hausdorff dimension of the residual set is of the same order as our lower bound,

\[1 + \frac{1}{\log r_\theta}\]

thereby demonstrating that the bound we obtain in chapter 2 is of the correct order.

In chapter 4, we consider packings in higher dimensions, where \( \theta \) is an \( n\)-
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dimensional convex body with radius of curvature bounded above by \( r_\theta \). We extend the 2-dimensional result, using an inductive slicing argument, to show that in higher dimensions the Besicovitch-Hausdorff dimension \( s_n(r_\theta) \) of the residual set \( R \) is at least

\[
s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1. \quad (1.2)
\]

Here \( s_n(r_\theta) \) is defined by

\[
s_n(r_\theta) = \inf \{ s : s \text{ is the dimension of the residual set of the packing } (\theta_m)_{m=1}^{\infty} \text{ in } I_n \}
\]

where the infimum is taken over all packings of convex \( n \)-bodies with radius of curvature bounded by \( r_\theta \). This will lead to the result

\[
s_n(r_\theta) \geq (n - 1) + \epsilon(r_\theta) \quad (1.3)
\]

where, by combining (1.1) and (1.2)

\[
\epsilon(r_\theta) = O \left( \frac{1}{\log r_\theta} \right). \quad (1.4)
\]
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In chapter 5 we turn our attention to arbitrary packings of disks \( \{\theta_m\}_{m=1}^{\infty} \) into the plane square \( I_2 \). A lower bound for the Besicovitch-Hausdorff dimension of the residual set \( R \) is greater than 1.03. This was shown by D.G. Larman in [1]. We improve this bound by developing the methods used to attempt to obtain the best possible result in 2-dimensions. We do this using Dirichlet cell methods.
2. **Convex bodies of bounded radius of curvature**

**Theorem 1.** Suppose that \( \{ \theta_n \}_{n=1}^{\infty} \) forms a packing within the unit plane square \( I_2 \) of strictly convex bodies whose radius of curvature is bounded above by \( r_\theta \).

Then the residual set \( R = I_2 \setminus \bigcup_{n=1}^{\infty} \theta_n \) has Besicovitch - Hausdorff dimension \( s \) where

\[
s > 1 + \epsilon \quad \text{where} \quad \epsilon \sim \frac{1}{\log r_\theta}
\]

**Proof.** We may suppose without loss of generality that the \( \theta_n \) are open sets, and that \( \text{diam } (\theta_{n+1}) \leq \text{diam } (\theta_n) \) for \( n = 1, 2, \ldots \). This gives us an order to our packing. The largest copy being \( \theta_1 \), and the \( \theta_i \)'s decreasing in size as \( i \) increases.
Let $s = 1 + \epsilon$ where $0 < \epsilon < 1$, and define the $n^{\text{th}}$ stage residual set,

$$R_n = I_2 \setminus \bigcup_{m=1}^{n} \theta_m$$

and the residual set, $R$, for the packing, by

$$R = \bigcap_{n=1}^{\infty} R_n = I_2 \setminus \bigcup_{m=1}^{\infty} \theta_m.$$ 

Since $R$ is compact we need only consider coverings of $R$ by finite collections of open squares $\{C_j\}_{j=1}^{p}$ when determining whether the $(1+\epsilon)$-dimensional Hausdorff measure of $R$ is positive.

Let $\{C_j\}_{n=1}^{p}$ be a cover of $R$ by a finite number of open squares. We may suppose that all of the $C_j$'s are necessary, i.e. none of them are contained within any of the $\theta_n$, $n = 1, 2, \ldots$, nor contained in each other. Let the diameter of $C_j$ be $\sqrt{2} \Delta_j$, for $j = 1, 2, \ldots, p$.

Each $R_n$ is compact, and $R_{n+1} \subset R_n$ for each $n$ and hence $\{R_n\}_{n=1}^{\infty}$ is a nested sequence of non-empty compact sets. This implies there exists some $m \in \mathbb{N}$ such that

$$R_m \subset \bigcup_{j=1}^{p} C_j$$

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and from this point on we shall fix such a sufficiently large $m$.

We shall find a condition on $\epsilon$ such that under this condition $\sum_j \Delta_j^{1+\epsilon}$ cannot be close to zero which will in turn imply $\dim(R) \geq 1 + \epsilon = s$

Now suppose that the $x$-axis is the horizontal axis, and the $y$-axis is the vertical axis. Let $0 \leq x \leq 1$, and let $l_x$ denote the vertical line through the point $(x,0)$, i.e. $l_x\{(X,Y), X = x\}$.

Let $l'(x,j)$ be the open interval equal to $l_x \cap C_j$, and let $l_{(x,j)}$ represent its length. Then, either we have

$$l_{(x,j)} = \frac{1}{\sqrt{2}} \text{Diam}(C_j) = \Delta_j,$$

or

$$l_{(x,j)} = 0.$$

Let us define a function $f(x)$ to be

$$f(x) = \sum_{j=1}^{p} l_{(x,j)}.$$
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Figure 2.1: If $x = x_1$ then $l_{(x,j)} = 0$. If $x = x_2$ then $l_{(x,j)} = \Delta_j$.

Integrating from 0 to 1 gives,

$$
\int_0^1 f(x) \, dx = \int_0^1 \sum_{j=1}^{p} t_{(x,j)} \, dx = \sum_{j=1}^{p} \Delta_j^1 + \epsilon
$$

(2.1).

In time we will define another function $g_m$ such that $\int_0^1 f(x) \, dx \geq \int_0^1 g_m(x) \, dx$ and then proceed to show $\int_0^1 g_0(x) \, dx > 0$ and deduce our result.

So let us consider $\cup_{j=1}^{p} C_j$, then $l_x$ intersects it in a collection of non overlapping
Figure 2.2: For example $j_{(x,r)} = \bigcup_{i=1}^{v} l_{(x,j_i)}$.

open intervals $\{j_{(x,r)}\}_{r=1}^{v}$ of lengths $\{j_{(x,r)}\}_{r=1}^{v}$.

Now each of these intervals can be expressed as the union of $\{l_{(x,j_i)}\}_{i=1}^{v(r)}$ chosen

from $\{l_{(x,j)}\}_{j=1}^{p}$, so $j_{(x,r)} = \bigcup_{i=1}^{r} l_{(x,j_i)}$.

Now $j_{(x,r)} \leq \sum l_{(x,j_i)}$, and since $0 < \epsilon < 1$,

$$j_{(x,r)} \leq \sum_{i=1}^{v(r)} l_{(x,j_i)} \quad r = 1, \ldots, v$$
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Summing over \( r \) gives,

\[
\sum_{r=1}^{v} j^{(x,r)} \leq \sum_{r=1}^{p} \sum_{i=1}^{v(r)} t^{(x,i)} \leq \sum_{j=1}^{p} t^{(x,j)}.
\] (2.2)

Let us define our function \( g_{m}(x) \), by

\[
g_{m}(x) = \sum_{r=1}^{v} j^{(x,r)}.\]

So, if we integrate \( g_{m}(x) \) from 0 to 1 and use (2.1), we then have

\[
\int_{0}^{1} g_{m}(x) \, dx = \int_{0}^{1} \sum_{r=1}^{v} j^{(x,r)} \, dx \leq \int_{0}^{1} \sum_{j=1}^{p} t^{(x,j)} \, dx = \sum_{j=1}^{p} \Delta_{j}^{1+\varepsilon}. \] (2.3)

If we show for any cover \( \{C_{k}\}_{k=1}^{p} \), that

\[
\int_{0}^{1} g_{m}(x) \, dx > 1,
\]

then the residual set \( R \) will have Besicovitch-Hausdorff dimension of at least \( 1 + \varepsilon \).
Figure 2.3: \( l_x \) intersects \( \bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k \) in a collection of disjoint intervals.

Let \( i \) be some integer, \( 0 \leq i \leq m - 1 \) and consider

\[
\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k.
\]

Then \( l_x \) intersects this union in a collection of disjoint intervals \( \{j_{(x,r,i)}^{u(i,x)}\}_{r=1}^{v(i,x)} \) of lengths \( \{j_{(x,r,i)}^{u(i,x)}\}_{r=1}^{v(i,x)} \) respectively.
Define

\[ g_i(x) = \sum_{r=1}^{u(i,x)} j(x,r,i) \]

If \( i \) is one of 1, \ldots, \( m \), then

\[
\int_0^1 g_i(x) \, dx = \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g_i(x) - g_{i-1}(x)) \, dx \tag{2.4}
\]

We shall show that for any \( i \)

\[
\int_0^1 (g_i(x) - g_{i-1}(x)) \, dx \geq 0
\]

and since we clearly have \( \int_0^1 g_0(x) \, dx = 1 \) the result follows, since this implies

\[
\int_0^1 g_m(x) \, dx \geq \int_0^1 g_0(x) \, dx = 1.
\]

For \( 0 \leq x \leq 1 \), \( l_x \) meets \( \theta_i \) in an interval, possibly empty, of length \( \alpha \) say.

Let \( l'_{(x,\theta_i)} \) be the interval \( l_x \cap \theta_i \), and \( l_{(x,\theta_i)} \) be its length. Then either

\[
l_{(x,\theta_i)} = \alpha, \quad \text{if } l_x \text{ meets } \theta_i,
\]

or
Now $g_i(x) = g_{i-1}(x)$ for those $x$ such that $l_x$ does not meet $\theta_i$. As there are no extra intervals to consider, so we need only worry about those $x$ where $l_x$ meets $\theta_i$. The segment of $l_x$ which lies in $\theta_i$ meets $\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k$ in a collection of non-overlapping intervals $\{r'_j\}_{j=2}^{w-1}$ of lengths $\{r_j\}_{j=2}^{w-1}$, whose closures do not meet
Figure 2.5:

the boundary of $\theta_i$, and two intervals $r_1$ and $r_w$, ($r_w$ below $r_1$), whose closures meet $\theta_i$, in fact they may coincide.

The line $l_x$ also meets $\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k$ in two intervals $r'_0, r'_{w+1}$ of lengths $r_0, r_{w+1}$, immediately above and below $\theta_i$ respectively.
Let us define a function $T(x)$ such that

$$T(x) = \begin{cases} 
1 & \text{if } r'_1 \text{ and } r'_w \text{ do not coincide} \\
0 & \text{otherwise}
\end{cases}$$

Then the difference is

$$g_i(x) - g_{i-1}(x) = T(x)[(r_0 + r_1) \epsilon + r'_2 + \cdots + r'_{w-1} + (r_w + r_{w+1}) \epsilon - (r_0 + \alpha + r_{w+1}) \epsilon]. \tag{2.5}$$

Suppose that $0 < \mu < \rho$ and $0 < \lambda < r_{w+1}$.

Let

$$h_i(x, \mu, \lambda) = T(x)[(\mu + r_1) \epsilon + r'_2 + \cdots + r'_{w-1} + (r_w + \lambda) \epsilon - (\mu + \alpha + \lambda) \epsilon]. \tag{2.6}$$

Then $g_i(x) - g_{i-1}(x) = h_i(x, r_0, r_{w+1})$. We show that $g_i(x) - g_{i-1}(x) \geq 0$ by first differentiating $h_i(x, r_0, r_{w+1}) \geq h_i(x, 0, 0)$ and then that $h_i(x, 0, 0) \geq 0$.

Then, since both $r_1, r_w \leq \alpha$ we have,
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Let the following intervals be labelled in the following way,

\[ r'_0 \equiv AB, \quad r'_{w+1} \equiv CD, \quad r'_0 \cup r'_1 \equiv ABE, \quad r'_w \cup r'_{w+1} \equiv CDF. \]

Figure 2.6:
Then, if $G_1, H_1$ are points of $A_1B_1, C_1D_1$ respectively, we may define, for all $x$ such that $l_x$ meets $\theta_i$, $g_i'(x)$ to be the same as $g_i(x)$ except that $A_1B_1E_1, C_1D_1F_1$ are replaced by $G_1B_1E_1, F_1C_1H_1$, and $g_{i-1}'(x)$ defined as $g_{i-1}(x)$ with $A_1D_1$ replaced with $G_1H_1$. Let $g_i'(x) = g_i(x)$, and $g_{i-1}'(x) = g_{i-1}(x)$ for all other $x$. Then we have for all $x$, using (2.4), (2.6) and (2.7), that

$$\int_0^1 g_i(x) \, dx = \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g_i(x) - g_{i-1}(x)) \, dx$$

$$\geq \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g_i'(x) - g_{i-1}'(x)) \, dx$$

(2.8)

We use this to simplify our problem.

We examine $\theta_m$, and find a polygon $P_m$ which encloses it.

Lemma 1. Let $\left\{ \bigcup_{n=1}^{\infty} \theta_n \right\}$ form an ordered packing of the unit plane square $I_2$. Then there exists a convex polygon $P_m$ which encloses $\theta_m$ and whose interior does not intersect $\theta_i$ for $i < m$; $P_m$ having no more than 15 sides.

Proof. Consider $\theta_m$ and those $\theta_i$, $1 \leq i < m$, surrounding it.

Let $\Psi_1$ be the maximal ellipsoid which is contained within $\theta_m$, and $\Psi_2$ be the minimal ellipsoid which contains $\theta_m$. Then $\Psi_1$ has no less than twice the area of $\Psi_2$. This is easily shown to be true for the worst case an equilateral triangle and
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a circle, all other cases for triangles can be reduced via an affine transformation of the equilateral case.

The number of sides of our polygon $P_m$ will depend on all those $\theta_i$, $1 \leq i < m$ within some neighbourhood of $\theta_m$. Let $\Psi_3$ be the ellipse with the same centre as $\Psi_1$ but with radius three times as large. We look at the largest $(m - 1)$ copies of $\theta$ which intersect the ellipse $\Psi_3$. Since we only need an upper bound for the number of sides of $P_m$ we may replace those $\theta_i$, $1 \leq i < m$ by the ellipse $\Psi_1$ within $\Psi_3$. Taking the Dirichlet cell of $\{\theta_i : i \leq m\}$ produces a polygon $P'_m$ which has at most 15 sides.

The number of sides of $P'_m = \frac{\text{Area } \Psi_3 - \text{Area } \Psi_1}{\text{Area } \Psi_1} = \frac{16\pi ab - \pi ab}{\pi ab} = 15.$

The polygon $P'_m$ can be defined as the intersection of some finite number of half-planes, $H_i$. Let $H(\underline{u}, \alpha) = \{x \in \mathbb{R}^2 : \langle x, \underline{u} \rangle \leq \alpha\}$. Then

$$P'_m = \bigcap_{i=1}^{I'} H_i = \bigcap_{i=1}^{I'} H(\underline{u}_i, \alpha'_i), \quad I' \leq 15,$$

where $\underline{u}_i$ is the normal to the half-planes $H_i$ chosen to pass through the centre of $\Psi_1$. We translate each half-plane in turn along their normal until they become
supporting half-spaces of $\theta_m$; that is until the boundary of the half-plane touches the boundary of $\theta_m$. This produces a new polygon $P_m$ which contains $\theta_m$.

Then $P_m$ may have fewer sides than $P'_m$, since some may be lost during the process of translating the half-planes. So

$$P_m = \bigcap_{i=1}^{I} H_i = \bigcap_{i=1}^{I} H(u_i, \alpha_i), \quad I \leq 15$$
If $x$ is in the interval $[0,1]$ and $l_x$ meets $\theta_m$ then $l_x$ meets $P_m \setminus \theta_m$ in two intervals $G_2B_1$ of length $l_1$ and $C_1H_2$ of length $l_2$. By construction of $P_m$, these two intervals do not meet $\{\theta_i\}_{i=1}^{m-1}$.

Let $G \equiv G_2$ and $H \equiv H_2$ then we may define $g_i''(x), g_{i-1}''(x)$, as $g_i'(x), g_{i-1}'(x)$,
respectively with these choices of $G$ and $H$, and deduce from (2.8) that

$$
\int_0^1 g_i(x) \, dx \geq \int_0^1 g_{i-1}(x) \, dx + \int_0^1 \left( g''_i(x) - g''_{i-1}(x) \right) \, dx. \quad (2.9)
$$

If we show that,

$$
\int_0^1 \left( g''_i(x) - g''_{i-1}(x) \right) \, dx \geq 0, \quad (2.10)
$$

then we are done. So in view of (2.5) we have to show that,

$$
\int_0^1 (l''_x) \, dx \leq \int_0^1 l_1' \, dx + \int_0^1 l_2' \, dx, \quad l'_x = l_x \cap P_n \quad (2.11)
$$

The question remains, how do we know there exists such an epsilon?

**Lemma 2.** Suppose $\theta_m$ is our convex body and is contained within some polygon, $Q_m$ say, then there exists $s = (1 + \epsilon)$ where $0 < \epsilon < 1$ such that

$$
\int_0^1 (l''_x) \, dx \leq \int_0^1 l_1' \, dx + \int_0^1 l_2' \, dx, \quad l'_x = l_x \cap P_n.
$$

*Proof.* Suppose that (2.11) is false, then in particular there exists an at most 15
sided polygon $Q_m$ containing $\theta_m$ such that,

$$
\int_0^1 (l'_x)^{\frac{1}{m}} \, dx \leq \int_0^1 l_1^{\frac{1}{m}} \, dx + \int_0^1 l_2^{\frac{1}{m}} \, dx.
$$

This would be true for every $m$, so letting $m \to \infty$ we produce a contradiction, i.e $1 \geq 2$. Therefore there exists some $s$.\(\square\)

Let us now cover $\theta_m$ by the intersection of at most 15 discs, whose radii are bounded by $kr_\theta$, where $k$ is the reduction factor from $\theta$ to $\theta_m$. We will now calculate explicitly a value of $\epsilon$ for which

$$
\int_0^1 (l'_x)^\epsilon \, dx \leq \int_0^1 l_1 \, dx + \int_0^1 l_2 \, dx,
$$

which means removing $\theta_m$ decreases the integral in (2.8).

At least one of the edges of $P_m$ will be an interval, $[a, b]$, of length greater than $\frac{kr_\theta}{15}$. Let $[a, b]$ be the projection of this interval onto the $x$-axis. We concentrate on one of these intervals, as they will contribute most to the integral of $g_i(x)$. We integrate along this interval $[a, b]$, doubling up the contribution from the shallowest arc, and ignoring the other side. Without loss of generality we may assume it is the top arc; the radius of curvature in this interval is $kr_\theta$, and the centre of this
arc is the origin.

To prove our hypothesis we must show

\[ \int_a^b (t_2) c \, dx \leq 2 \int_a^b l_1^t \, dx. \]
Then,

\[ 2 \int_{\alpha}^{b} l_{1} dx = 2 \int_{-kr_{\theta} \sin \alpha}^{-kr_{\theta} \sin \alpha + kr_{\theta} \cos \alpha} \left( k_{r_{\theta}} - \left( k^{2} r_{\theta}^{2} - x^{2}\right)^{\frac{1}{2}} \right)^{\varepsilon} \sec \alpha \ dx \]  \hspace{1cm} (2.12)

\[ = 2 \int_{-\alpha}^{\delta} (k r_{\theta})^{1+\varepsilon} \cos \phi (1 - \cos \phi)^{\varepsilon} \sec \alpha \ d\phi \]  \hspace{1cm} (2.13)

\[ = (2 k r_{\theta})^{1+\varepsilon} \sec \alpha \sin^{2\varepsilon} \left( \frac{\phi}{2} \right) \cos \phi \ d\phi \]  \hspace{1cm} (2.14)

\[ \geq \int_{-kr_{\theta} \sin \alpha}^{-kr_{\theta} \sin \alpha + kr_{\theta} \cos \alpha} k^{\varepsilon} \ dx \]  \hspace{1cm} (2.15)

\[ = \int_{-\alpha}^{\delta} k^{1+\varepsilon} r_{\theta} \cos \phi \ d\phi . \]  \hspace{1cm} (2.16)

So,

\[ \int_{-\alpha}^{\delta} \left( 2^{1+\varepsilon} \sec \alpha r_{\theta} \sin^{2\varepsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \ d\phi \geq 0 \]

cancelling the other terms. Now we split up our range of integration into the intervals \([-\alpha, -\beta],[-\beta, -\gamma],[-\gamma, -\delta]\) so that

\[ \sin^{2\varepsilon} \left( \frac{-\beta}{2} \right) \geq 2^{-1-\varepsilon} (r_{\theta} \sec \alpha)^{-\varepsilon} \]  \hspace{1cm} (2.17)

\[ r_{\theta} \sin(-\delta) \geq \frac{1}{15} \]  \hspace{1cm} (2.18)
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We then have,

\[ \int_{-\alpha}^{\beta} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]

\[ \geq \int_{-\alpha}^{\beta} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]

\[ + \int_{-\beta}^{-\gamma} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]

\[ + \int_{-\gamma}^{0} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]

(2.19)

The first integral will provide a negative contribution, the second will be small and positive but insufficient to compensate for the first. The third integral is also positive and sufficiently large to compensate. Therefore

\[ \int_{-\alpha}^{-\delta} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]

\[ \geq \int_{-\alpha}^{\beta} - \cos \phi \, d\phi + \int_{-\gamma}^{-\delta} \left( 2^{1+\epsilon} \sec^c(\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi \]
We choose $-\gamma$ so that

$$2^{1+\epsilon} \sec^\epsilon (\alpha) \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \approx \frac{1}{2}$$

We take

$$-\gamma = -\frac{\delta}{2}$$

so we have

$$\int_{-\alpha}^{-\delta} \left( 2^{1+\epsilon} \sec^\epsilon \alpha \sin^{2\epsilon} \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi$$

$$\geq \int_{-\alpha}^{-\beta} - \cos \phi \, d\phi + \int_{-\gamma}^{-\delta} \left( 2^{1+\epsilon} \sec^\epsilon \alpha \sec^\epsilon \left( \frac{\phi}{2} \right) - 1 \right) \cos \phi \, d\phi$$

$$\geq \sin(-\alpha) - \sin(-\beta) + \frac{1}{2} \int_{-\gamma}^{-\delta} \cos \phi \, d\phi$$

$$\geq 0$$

We now take $\sin \alpha = 0$, the worst possible case. Therefore we have,

$$\frac{1}{4} \sin(-\delta) \geq \sin(-\beta) \quad (2.20)$$

$$\sin^{2\epsilon} \left( -\frac{\beta}{2} \right) \geq 2^{-1-\epsilon} (r_{g} \sec \alpha)^{-\epsilon} \quad (2.21)$$

$$8^{-2\epsilon} \sin^{2\epsilon} (-\delta) \geq 2^{-1-\epsilon} (r_{g} \sec \alpha)^{-\epsilon} \quad (2.22)$$
2. Convex Bodies

Using (2.18)

\[(120r_\theta)^{-2\epsilon} \geq 2^{-1-\epsilon} (r_\theta \sec \alpha)^{-\epsilon}\]

Taking logarithms,

\[2\epsilon \log (120r_\theta) \leq (1 + \epsilon) \log 2 + \epsilon \log (r_\theta \sec \alpha)\]

Rearranging this gives,

\[\epsilon \leq \frac{\log 2}{\log r_\theta + 2 \log 120 - \log \sec \alpha}.\]

Therefore, provided \(\epsilon \leq \frac{\log 2}{\log r_\theta + 2 \log 120 - \log \sec \alpha}\),

\[\int_0^1 (l'_2)^t \, dx \geq \int_0^1 l'_1 \, dx + \int_0^1 l'_2 \, dx.\]

That is

\[\int_0^1 (g''_i(x) - g''_{i-1}(x)) \, dx \geq 0\]

which implies,

\[\int_0^1 g_i(x) \, dx \geq \int_0^1 g_{i-1}(x) \, dx \geq \ldots \geq \int_0^1 g_0(x) \, dx = 1\]
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and hence,

$$\sum_{j=1}^{P} \Delta_{j}^{1+\epsilon} > \int_{0}^{1} g_{n}(x) \, dx \geq 1$$

as required. \(\square\)
3. CONSTRUCTION OF A 2-DIMENSIONAL PACKING

Let $I_2$ be the unit plane square whose vertices are at $[\pm \frac{1}{2}, \pm \frac{1}{2}]$. Then our convex body $\theta$ is the intersection of four discs of radius $r_\theta \left( \infty > r_\theta > 1 \right)$ of centres $[0, \pm (r_\theta - \frac{1}{2})], [\pm (r_\theta - \frac{1}{2}), 0]$.

By construction, at all but four points of the boundary, $\theta$ has radius of curvature bounded by $r_\theta$. The construction of $\theta$ can be considered to be the act of slicing off the sides of $I_2$ using shallow arcs. This idea of transforming a square into a copy of $\theta$ will be used later.

Let $k \Gamma$ be the boundary arcs of $\theta, k = 1, \ldots, 8$.

Let $p = (x, y)$ be the intersection of the two arcs $1 \Gamma$ and $2 \Gamma$ as indicated. This is where both $1 \Gamma$ and $2 \Gamma$ meet the line $x = y$. We find $p = (x, y)$, enabling us to evaluate the horizontal distance $\hat{x}$ from $\theta$ to $I_2$, in fact $\hat{x} = \frac{1}{2} - x$. 

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The equation of the top arc $\Gamma$ is

$$x^2 + \left( y + r_\theta - \frac{1}{2} \right)^2 = r_\theta^2.$$ 

We want to find $p = (x, y)$ and hence $\hat{x}$ so we substitute in $x = y$. This gives

$$x^2 + \left( x + r_\theta - \frac{1}{2} \right)^2 = r_\theta^2.$$
2-dimensional Packing Construction

\[ 2x^2 + 2 \left( r_\theta - \frac{1}{2} \right) x + \left( r_\theta - \frac{1}{2} \right)^2 - r_\theta^2 = 0. \]

Solutions are of the form

\[
x = \frac{1}{4} \left[ -2(r_\theta - \frac{1}{2}) \pm \sqrt{4 \left( r_\theta - \frac{1}{2} \right)^2 - 8 \left( \left( r_\theta - \frac{1}{2} \right)^2 - r_\theta^2 \right)} \right]
\]

\[
= -\frac{1}{2} \left( r_\theta - \frac{1}{2} \right) \pm \frac{1}{2} \sqrt{2r_\theta^2 - \left( r_\theta - \frac{1}{2} \right)^2}
\]

\[
= \frac{1}{4} \left( \frac{1}{2} - r_\theta \pm \frac{1}{2} \sqrt{r_\theta^2 + r_\theta - \frac{1}{4}} \right)
\]

Take the positive root as we want \( x > 0 \).

\[
x = \frac{1}{4} - \frac{r_\theta}{2} + \frac{1}{2} \sqrt{r_\theta^2 + r_\theta - \frac{1}{4}},
\]

\[
\hat{x} = \frac{1}{2} - x = \frac{1}{4} + \frac{r_\theta}{2} - \frac{1}{2} \sqrt{r_\theta^2 + r_\theta - \frac{1}{4}}.
\]
2-dimensional Packing Construction

For simplicity we use an approximation to $x$.

$$x = \frac{\left(\left(\frac{1}{4} - \frac{r_θ}{2}\right) + \frac{1}{2}\sqrt{r_θ^2 + r_θ - \frac{1}{4}}\right) \left(\frac{1}{4} - \frac{r_θ}{2}\right) - \frac{1}{2}\sqrt{r_θ^2 + r_θ - \frac{1}{4}}}{\left(\frac{1}{4} - \frac{r_θ}{2}\right) - \frac{1}{2}\sqrt{r_θ^2 + r_θ - \frac{1}{4}}}$$

$$= \frac{1}{2} \left(\left(\frac{1}{2} - r_θ\right)^2 - r_θ^2 + r_θ - \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{2r_θ - \frac{1}{2}}{\sqrt{r_θ^2 + r_θ - \frac{1}{4} + r_θ - \frac{1}{2}}}$$

$$= \frac{1}{2} \frac{1 - \frac{1}{4r_θ}}{\sqrt{\frac{1}{4} + \frac{1}{4r_θ} - \frac{1}{16r_θ^2} + \frac{1}{2} - \frac{1}{4r_θ}}}$$

$$\approx \frac{1}{2} \left(1 - \frac{1}{4r_θ}\right)$$

So, $\hat{x} \approx \frac{1}{8r_θ}$.  

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3.1. The Packing and its Notation

At each stage we pack countably many reduced copies of $\theta$ into $I_2$, building up the packing progressively. Let $R_n$ be the $n^{th}$ stage residual set. A cover of $R_n$ is given at every stage.

The packing is built up by repeatedly applying a transformation $T$ to $I_2$, this transformation acts upon all squares at this stage in the following way:

Let $S^X_Y$ be one such square, and let $X$ and $Y$ be previous labels. Let $\Delta^X_Y$ denote $\frac{1}{\sqrt{2}}$ diam ($S^X_Y$); in fact $\Delta^X_Y$ is the side length of the square $S^X_Y$.

$T$ replaces $S^X_Y$ by a reduced copy $\theta$ by a factor $\Delta^X_Y$ and countably many smaller squares, which together cover $S^X_Y$ apart from a set of small measure. (Along the arcs of the newly created $\theta$ labeled $\theta^X_Y$, $k\Gamma^X_Y$.)

$T$ covers $S^X_Y \setminus \theta^X_Y$ with squares by splitting it up into twelve regions.

Four corner squares of side $\frac{\Delta^X_Y}{8r} \equiv \Delta^X_{Y^k}$, $k = 1, \ldots, 4$ labeled $S^X_{Y^k}$. This leaves eight similar regions $k\Omega^X_Y$ $k = 1, \ldots, 8$. These are approximately triangular; their perpendicular sides are of lengths $\frac{\Delta^X_Y}{8r}$, $\frac{\Delta^X_Y}{2} (1 - \frac{1}{4r})$ and their other sides are the arcs $k\Gamma^X_Y$.

Let $\Delta^X_{Y^k} = \frac{1}{\sqrt{2}}$ diam $S^X_{Y^k}$. The $\Delta^X_{Y^k}$ are found by solving $L^X_{Y^k}$ and $k\Gamma^X_Y$ to find
2-dimensional Packing Construction

Their points of intersection.

\[ L_{Y_i}^{X_0} : \quad \tilde{y}_i = \tilde{x}_i - \sum_{j=0}^{i-1} \Delta_{Y_i}^{X_0}. \]

Since we sum over subscripts and not superscripts we write this as

\[ L_{Y_i} : \quad \tilde{y}_i = \tilde{x}_i - \sum_{j=0}^{i-1} \Delta_{Y_i}. \]
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Let \( \Delta_{Y_0} = \Delta_{X_0} = \Delta_{Y_0} \).

So at the \( n \)th stage when we pack sideways we increase the length of the \( n \)th subscript.

Find \( \Delta_{Y_{i,j}} \) by solving for \( j = 1 \):

\[
L_{Y_{i,1}} : \quad \tilde{y}_{i,1} = \tilde{x}_{i,1} - \delta_{Y_i},
\]
2-dimensional Packing Construction

for \( j > 1 \)

\[ L_{y_{i,j}} : \tilde{y}_{i,j} = \tilde{x}_{i,j} - \delta_{Y_i} - \sum_{k=1}^{j-1} \Delta_{Y_{i,k}}, \]

with equation of the relevant arc, in this case

\[ \tilde{x}^2 + (\tilde{y} + (r_\theta - \frac{1}{2}))^2 = r_\theta^2 \]

Let

\[ \delta_{Y_i} = \Delta_{Y_{i-1}} - \Delta_{Y_i} \]

So by construction

\[ \sum_{j=1}^{\infty} \Delta_{Y_{i,j}} = \Delta_{Y_i}, \]

and further

\[ \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \Delta_{Y_{i,j}} = \frac{\Delta_{Y}}{2}. \]

This leaves us with countably many approximately triangular sections \( \{I_{Y_i}^{X_0}\}_{i=1}^{\infty} \).

For simplicity again knowing we are in \( k\Omega_{x}^{X} \) we can drop the superscript. The perpendicular sides of the \( \{I_{Y_i}^{X_0}\}_{i=1}^{\infty} \) are of lengths \( \delta_{Y_i} = \Delta_{Y_{i-1}} - \Delta_{Y_i} \), and \( \Delta_{Y_i} \) for \( i = 1, 2, \ldots \)
2-dimensional Packing Construction

\((\delta Y_i = \Delta Y_i - \Delta Y_{i-1})\).

We then pack the \(I_i\) similarly for each \(i\). From now on we drop all the superscripts since they are all \(X_0\).

Our left over regions are the \(\{I_i\}_{i=1}^\infty\) which are approximately triangular sections with perpendicular sides \(\delta Y_{i,j} = \Delta Y_{i,j-1} - \Delta Y_{i,j}\), and \(\Delta Y_{i,j}\) for \(j = 1, 2, \ldots\)

Note when \(j = 1\), \(\delta Y_{i,1} = \delta Y_i - \Delta Y_{i,1}\) so we let \(\Delta Y_{i,0} = \delta Y_i\) for simplicity. So we repeat the layering process, packing the largest possible square into the corner, then the next largest possible underneath it, and so on, labeling as we pack.

Let \(\delta Y_{i_1, \ldots, i_n} = \Delta Y_{i_1, \ldots, i_n-1} - \Delta Y_{i_1, \ldots, i_n}\); the \(i_n > 0\) with \(\Delta Y_{i_1, \ldots, i_n-1,0} = \delta Y_{i_1, \ldots, i_n-1}\).

So the approximately triangular region \(I_{i_1, \ldots, i_n}\) has perpendicular sides \(\delta Y_{i_1, \ldots, i_n}\) and \(\Delta Y_{i_1, \ldots, i_n}\). The \(\Delta Y_{i_1, \ldots, i_{n+1}} = \frac{1}{\sqrt{2}} \text{diam} \left(C_{Y_{i_1, \ldots, i_{n+1}}}\right)\) which are found by solving, for \(i_{n+1} = 1\)

\[
L_{i_1, \ldots, i_{n+1}} : \quad \bar{y}_{i_1, \ldots, i_{n+1}} = \bar{x}_{i_1, \ldots, i_{n+1}} - \delta Y_{i_1, \ldots, i_{n+1}},
\]

for \(i_{n+1} > 1\)

\[
L_{Y_{i_1, \ldots, i_n, i_{n+1}}} : \quad \bar{y}_{i_1, \ldots, i_n, i_{n+1}} = \bar{x}_{i_1, \ldots, i_n, i_{n+1}} - \delta Y_{i_1, \ldots, i_n} - \sum_{k=1}^{i_{n+1}-1} \Delta Y_{i_1, \ldots, i_n, k},
\]
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with equation of the relevant arc, in this case

\[ \tilde{x}^2 + (\tilde{y} + (r_0 - \frac{1}{2}))^2 = r_0^2. \]

So by construction we have,

\[ \sum_{i_{n+1}=1}^{\infty} \Delta Y_{1, \ldots, i_{n+1}} = \frac{\Delta Y_{1, \ldots, n}}{2} \]

\[ \sum_{i_1=0}^{\infty} \sum_{i_2=1}^{\infty} \ldots \sum_{i_{n+1}=1}^{\infty} \Delta Y_{1, \ldots, i_{n+1}} = \frac{\Delta Y}{2} \]

We sum over all subscripts along one side of our initial square \( S_1^0 \). Let \( I \) be this indexing set, then this gives,

\[ \sum_I \Delta Y = \frac{1}{2}. \]

We have given \( I_2 \) the initial label \( S_1^0 \), then at each stage we apply \( T \) to all squares created at the previous stage.

Given any \( \theta^X_Y \) it is possible to identify where it entered the packing by the depth of the subscript \( Y \).

Define the depth of \( Y \), denoted by \( \text{dep}(Y) \), to be \( n \) if the subscript \( Y \) is of the
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form,

\[ Y = [L_1][L_2] \ldots [L_n] \]

Where each \([L_i]\) is a string of integers of countable length. If the subscript \(Y\) has depth \(n\) then \(\theta^Y_i\) entered the packing at the \(n^{th}\) stage. So at first we apply the transformation \(T\) to \(I_2\), which have labeled \(S^0_1\), this produces a single copy of \(\theta\) labeled \(\theta^0_1\), and countably many squares. On the next application of \(T\) all those newly created squares have their sides sliced off, and become

\[ \theta^0_{1k} \text{ for } k = 1, \ldots, 4 \]

and

\[
\begin{align*}
&k \left\{ \theta_{i_1} \right\}_{i_1=1}^\infty \quad k \left\{ \left\{ \theta_{i_1,i_2} \right\}_{i_1=1}^\infty \right\}_{i_2=1}^\infty \\
&k \left\{ \left\{ \left\{ \theta_{i_1,i_2,i_3} \right\}_{i_1=1}^\infty \right\}_{i_2=1}^\infty \right\}_{i_3=1}^\infty
\end{align*}
\]

For each of these \(\theta\) there are also countably many squares.

### 3.2. The Cover

Our cover for the \(n^{th}\) stage residual set \(R_n\) consists of two subcovers;

1. Covering the sets of small measure on the arcs of the \(\theta^Y\)
for which $\text{dep}(Y) < n$,

(2) Covering the part of the residual set which is the countable union

of sets of the form $(S_Y^X \setminus \theta_Y^X)$ for $\text{dep}(Y) = n$.

3.2.1. The cover of type (1).

For each $i$ in turn we cover the arcs of those $\theta_Y^X$ with $\text{dep}(Y) = i$, by choosing

sets of squares $W_i^0 = \{k w_i^0\}_{k=1}^{m_i}$ so that,

$$\sum_{k=1}^{m_i} \text{diam} (k w_i^0)^{1+\epsilon} \leq \frac{\left(\sqrt{2}\right)^{1+\epsilon}}{2n}.$$ 

This is possible since we can choose our squares arbitrarily small and by construc­tion our $\theta$'s will go under the cover.

For each $\theta_Y^X$ with $\text{dep}(Y) < i$ we cover its arcs $k \Gamma_Y^X$ for $k = 1, \ldots, 8$ by sets of

the form $W_Y^X = \{k w_Y^X\}_{k=1}^{m_Y^X}$ such that for each $\theta_Y^X$

$$\sum_{k=1}^{m_Y^X} \text{diam} (k w_Y^X)^{1+\epsilon} \leq \frac{\left(\sqrt{2} \Delta Y\right)^{1+\epsilon}}{2n}. $$
Summing over all $\theta^X_Y$ with $\text{dep}(y) = i$ we have,

$$\sum_{Y : \text{dep}(Y) = i} \sum_k \sum (\text{diam}(k w^X_Y))^{1+\epsilon} \leq \sum_{Y : \text{dep}(Y) = i} \sum \frac{(\sqrt{2}\Delta^X_{Y})^{1+\epsilon}}{2^n}$$

$$\leq \sum_{Y : \text{dep}(Y) \leq n-1} \sum (\sqrt{2}\Delta^X_{Y})^{1+\epsilon}$$

$$\leq \frac{(\sqrt{2})^{1+\epsilon}}{2^n}$$

We obtain this for each $i = 1, \ldots, n - 1$, and so,

$$\sum_{Y : \text{dep}(Y) \leq n-1} \sum_k \sum (\text{diam}(k w^X_Y))^{1+\epsilon} \leq \frac{(\sqrt{2})^{1+\epsilon}(n-1)}{2^n} < 1.$$  

3.2.2. The cover of type (2).

We are concerned with those $\theta^X_Y$ which have most recently entered the packing, so those $\theta^X_Y$ with $\text{dep}(Y) = n$. We cover each section of $R_n$ of the form $S^X_Y \setminus \theta^X_Y$ using four strips of $8r_\theta$ squares of diameter $\frac{\sqrt{2}\Delta^X_{Y}}{8r_\theta}$. We call these sets the $V^X_Y$.

The set $V^X_Y$ covers $S^X_Y \setminus \theta^X_Y$ ($\text{dep}(Y) = n$), and is of the form $V^X_Y = \{j w^X_Y\}_{j=1}^{32r_\theta}$. We choose $\epsilon$ so that,
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\[
\sum_{j=1}^{32r_\theta} \left( \text{diam} \left( x^j \right) \right)^{1+\epsilon} \leq (\sqrt{2}\Delta Y)^{1+\epsilon}
\]

\[
32r_\theta \left( \frac{\Delta Y}{8r_\theta} \right)^{1+\epsilon} \leq (\Delta Y)^{1+\epsilon}
\]

\[
\frac{4}{(8r_\theta)^\epsilon} \leq 1
\]

\[
\epsilon \geq \frac{2 \log 2}{3 \log 2 + \log r_\theta}
\]

And hence,

\[
\sum_{Y : \text{dep}(Y) = n} \sum_{j=1}^{32r_\theta} \left( \text{diam} \left( x^j \right) \right)^{1+\epsilon} \leq \sum_{Y : \text{dep}(Y) = n} \sum_{j=1}^{32r_\theta} (\sqrt{2} \Delta Y)^{1+\epsilon}
\]

\[
\leq (\sqrt{2})^{1+\epsilon}
\]
3.2.3. Justification for the use of this cover.

For each piece of the residual set of the form $S^y_i \setminus \theta_i^y$ we choose a cover $\{v_j^y\}_j$ to maximise the sum of the diameters of the covering sets to the power $1 + \epsilon$. Let us drop the subscripts and superscripts for simplicity. So the $\{v_j\}_{j \in J}$ cover $S' \setminus \theta$ and let $\text{diam}(S) = \sqrt{2}\Delta$, and $\text{diam}(v_j) = \sqrt{2}\Delta_j$. Then we maximise the following over all covers,

$$\sum_{j \in J} (\Delta_j)^{1+\epsilon} \leq (\Delta_j)^{1+\epsilon}$$

subject to,

$$\sum_{j \in J} (\Delta_j) = \Delta$$

$$\Delta_j \leq \frac{\Delta}{8r_{\theta}}$$

$$\Delta_{j+1} < \Delta_j$$

Suppose that $\{v_j\}_{j=1}^{\infty}$ cover the residual set $S' \setminus \theta$ and maximise the sum subject to the constraints.

We cover the cover using squares of diameter $\sqrt{2}\frac{\Delta}{8r_{\theta}}$. This increases the sum, but we ensure that the sum is $\leq \Delta^{1+\epsilon}$. Eventually we will show that this over
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estimate is finite, and hence so is our original.

Suppose without loss of generality, that $\Delta_1 < \frac{A}{8r}$ and not equal to it. The consider,

$$(\Delta_1 + \delta)^{1+\varepsilon} + (\Delta_2 - \delta)^{1+\varepsilon} + \sum_{j=3}^{\infty} \Delta_j = \sum_1^1$$

We have borrowed from $\Delta_2$ to increase $\Delta_1$. Let $\sum_{j=1}^{\infty} \Delta_j = \sum_2$. Then consider,

$$\sum_1 - \sum_2 = (\Delta_1 + \delta)^{1+\varepsilon} - \Delta_1^{1+\varepsilon} + (\Delta_2 - \delta)^{1+\varepsilon} - \Delta_2^{1+\varepsilon}$$

$$= (1 + \varepsilon) (\Delta_1^{\varepsilon} - \Delta_2^{\varepsilon}) \delta + \frac{(1 + \varepsilon) \varepsilon}{2} (\Delta_1^{\varepsilon-1} - \Delta_2^{\varepsilon-1}) \delta^2$$

$$+ \frac{(1 + \varepsilon) \varepsilon (\varepsilon - 1)}{6} (\Delta_1^{\varepsilon-2} - \Delta_2^{\varepsilon-2}) \delta^3 + \ldots > 0$$

Since $\Delta_1 > \Delta_2$

So we take $\Delta_1 = \frac{A}{8r}$, and then we can repeat the procedure borrowing from the smaller $\Delta_j$'s to increase the early $\Delta_j$'s in the same manner, until we have squares of diameter $\frac{\sqrt{2A}}{8r}$ as required.

Now $R$, the residual set for our packing is $\cap_{n=1}^{\infty} R_n$ and so $R \subset R_n$, for all $n$. Therefore the $(1 + \varepsilon)$-dimensional measure of $R$, denoted by $m^{1+\varepsilon}(R)$ is less than
2-dimensional Packing Construction

or equal to the $(1 + \epsilon)$-dimensional measure of the $R_n$, for all $n$. So,

$$m^{1+\epsilon}(R) \leq m^{1+\epsilon}R_n$$

for all $n$.

If we show that $m^{1+\epsilon}R_n$ is finite for all $n$ then we have shown that so is $m^{1+\epsilon}(R)$, and we are done. We proceed in the following way. At the $n^{th}$ stage we have,

for $n = 1$,

$$m(R_1) \leq \sum_{j=1}^{32r_0} d(jv_1^0)^{1+\epsilon} \leq \sqrt{2^{1+\epsilon}} < 4$$

for $n > 1$,

$$m(R_n) \leq 4 \left[ \sum_{Y : \text{dep}(Y) = n} \sum_{j=1}^{32r_0} d(jv_1^0)^{1+\epsilon} + \sum_{Y : \text{dep}(Y) = n} \sum_{k} d(kw_Y^X)^{1+\epsilon} \right]$$

$$\leq 4 \left[ \sqrt{2^{1+\epsilon}} + \sqrt{2^{1+\epsilon}} \left( \frac{n-1}{2^n} \right) \right]$$

$$= 4\sqrt{2^{1+\epsilon}} \left( 1 + \frac{n-1}{2^n} \right)$$

$$< 10$$

Therefore the $(1 + \epsilon)$-dimensional measure of $R$ is at most 10 and hence the Hausdorff dimension of $R$ is at most $1 + \epsilon$, where

$$\epsilon = \frac{2\log 2}{3\log 2 + \log r_0} \quad \square.$$

In two dimensions we have shown for packings of this type that the Besicovitch-Hausdorff dimension \( s_2(r_\theta) \) of the residual set \( R \) is at least

\[
1 + \epsilon(r_\theta) \quad \text{where} \quad \epsilon(r_\theta) \sim \frac{1}{\log r_\theta} \quad (4.1)
\]

\[0 < \epsilon(r_\theta) < 1\]

We will assume that all convex bodies mentioned within this chapter are of bounded radius of curvature.
4. Higher Dimensional Results

We will, using an inductive slicing argument, show that in higher dimensions the dimension $s_n(r_\theta)$ of the residual set $R$ is at least

$$s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1$$

where $s_n(r_\theta)$ is defined by

$$s_n(r_\theta) = \inf\{s: s \text{ is the Besicovitch-Hausdorff dimension of } R\}$$

where the infimum is taken over all packings of bodies with radius of curvature bounded by $r_\theta$. This will lead to the result

$$s_n(r_\theta) \geq (n - 1) + \epsilon(r_\theta) \quad (4.2)$$

where

$$\epsilon(r_\theta) \propto \frac{1}{\log r_\theta} \quad (4.3)$$
4. Higher Dimensional Results

by combining (4.2) and (4.3). Hence

$$s_n(r_\theta) > n - 1 \text{ for } n = 2, 3, \ldots \quad (4.4)$$

In the proceeding paragraphs we will need the following notation: let $C$ be a set in $\mathbb{R}^n$, let $C(y)$ denote the vertical slice of $C$, $y$ units along the $X_n$ axis, i.e. $C(y)$ is the subset of $C$ which lies within the hyperplane $X_n = y$.

**Theorem 2.** Let $\{\theta_m\}_{m=1}^\infty$ be a solid packing of homothetic copies of the convex $n$-body $\theta$ into the unit cube $I_n$; $\theta$ having radius of curvature bounded above by $r_\theta$.

Then with $s_n(r_\theta)$ defined as above we have

$$s_n(r_\theta) \geq s_{n-1}(r_\theta) + 1.$$

**Proof.** Let $\{\theta_m\}_{m=1}^\infty$ be a packing of convex $n$-bodies, with radius of curvature of $\theta$ bounded above by $r_\theta < \infty$, into the unit $n$-cube $I_n$.

We may assume without loss of generality that the $\{\theta_m\}$ are open since the set

$$\bigcup_{m=1}^\infty \partial \theta_m$$
4. Higher Dimensional Results

has Besicovitch-Hausdorff dimension $n - 1$, and we will show

$$s_n > n - 1$$

Then the residual set $R$ is compact and hence it is sufficient to consider finite coverings of $R$ by open sets.

We proceed by defining a function on the reals. For $\delta > 0$, $0 < s \leq s_{n-1}(r_\delta)$ we define

$$f(z) = m_\delta^s(R(z)), \quad \forall z > 0.$$ 

This is the $s$-dimensional $\delta$-measure of the slice of $R$ which is contained in the hyperplane $x_n = z$.

We integrate this measurable function in the $x_n$ direction to produce the required result.

We have

$$f(z) > 0 \quad \forall z \in [0, 1] \quad (4.5)$$

To show $f$ is measurable we use the sufficient condition that $f$ is a measurable function if and only if $\{z : f(z) < c\}$ for any real $c$ is a measurable set.

Consider $R(z)$, this is the $(n - 1)$-dimensional slice of $R$ which is contained in the
4. Higher Dimensional Results

hyperplane $x_n = z$. $R(z)$ is compact for all real $z$, and so given real $\lambda, z$, we can find a finite $\delta$-cover $\{E_i\}_{i=1}^{p}$ of $R(z)$ such that

$$\sum_{i=1}^{p} \text{diam}^*(E_i) < m^*_R(R(z)) + \lambda = f(z) + \lambda$$

(4.6)

$$\text{diam}^*(E_i) < \delta, \quad i = 1, \ldots, p$$

Note that the $E_i$ have dimension $n - 1$.

Let us now define sets $E_i(\mu)$ for $i = 1, \ldots, p$ by

$$E_i(\mu) = \{x \in \mathbb{R}^n : \|x - y\| \leq \mu, \ y \in E_i\}.$$  

This can be viewed as giving thickness to the $E_i$ which are $(n - 1)$-dimensional sets sitting in $n$ dimensions. So the $E_i(\mu)$ have dimension $n$. They are open sets and have diameter

$$\text{diam} (E_i) < \text{diam} (E_i) + 2\mu.$$  

It is easy to see that if we choose $\mu$ correctly we are able to ensure that:

$$\text{diam} (E_i(\mu)) < \text{diam} (E_i) + 2\mu < \delta.$$  

(4.7)
4. Higher Dimensional Results

Let $h$ be a positive real number such that $h < \mu$.

We choose $h$ in such a way as to ensure that if for some real number $z' \in (z - h, z + h)$ our cover $\{E_i(\mu)\}_{i=1}^{\infty}$ is also a cover for $R(z')$. We find that a sufficient condition on the size of $h$ is

$$2(h + h^2) < \mu. \quad (4.8)$$

So let $z' \in (z - h, z + h)$ and let $g \in \mathbb{R}^n$ such that

$$g = (0, \ldots, 0, z - z').$$

Suppose $x \in R(z')$ and consider $(R(z') + g) \cap R(z)$. We have two cases

Case 1

$x \in R(z')$ and $(R(z') + g) \cap R(z) \neq \emptyset$. Then $\{E_i(\mu)\}_{i=1}^{\infty}$ is also a cover of $R(z')$ since $h < \mu$.

Case 2

$x \in R(z')$ and $(R(z') + g) \cap R(z) = \emptyset$. Then there is a $\theta_m$ such that $x \in \theta_m(z')$.

But $x + g \in \theta_m(z)$ and hence

$$\text{diam} (\theta_m(z)) - \text{diam} (\theta_i(z')) \leq \sqrt{2(h + h^2) - h^2}$$
4. Higher Dimensional Results

\[ (2h + h^2)^{\frac{1}{2}}. \]

Let

\[ d = \inf \{ \|x + \alpha - y\| : y \in \bigcup_{i=1}^{p} E_i \} \]

then

\[ d \leq (2h + h^2)^{\frac{1}{2}} < (2h + 2h^2)^{\frac{1}{2}} < \mu. \]

Hence we have that \( x + \alpha \) is less than a distance \( \mu \) from the set \( \bigcup_{i=1}^{p} E_i \). So we have that \( \{E_i\}_{i=1}^{p} \) is also a cover for \( R(z') \).

Now we have

\[ f(z) = m_{z'}^*(R(z)) \]

and

\[ \sum_{i=1}^{p} \text{diam}^*(E_i(\mu)) < m_{z'}^*(R(z)) + \lambda \]

\[ \text{diam}(E_i(\mu)) < \delta \quad i = 1, \ldots, p \]

with \( \{E_i(\mu)\}_{i=1}^{p} \) also a cover for \( R(z') \). So this implies

\[ m_{z'}^*(R(z')) \leq \sum_{i=1}^{p} \text{diam}^*(E_i(\mu)) < m_{z'}^*(R(z)) + \lambda, \quad \forall z' \in (z - h, z + h). \]
Writing this with respect to our function \( f \) we have

\[
f(z') \leq f(z) + \epsilon, \quad \forall z' \in (z - h, z + h).
\] (4.9)

Let \( c \) be some positive real number then define the set

\[
Z(c) = \{ z : f(z) \leq c \}.
\]

Then it follows from (4.8) that \( Z(c) \) is open and hence measurable.

Therefore \( f \) is a measurable function. Hence \( f \) is Lebesgue integrable and we have

\[
\int_0^1 f(z) \, dz > 0.
\]

Now let us consider a finite open \( \delta \)-cover of \( R \) by \( n \)-cubes \( \{ C_j \}_{j=1}^\delta \), with edges of same orientation as the co-ordinate axis.

Then, if we take an \( n - 1 \) dimensional slice of \( R_n \) at some real number \( z \) we will, for some given \( j \), have either

1. A face of \( C_j \), i.e. an \((n-1)\) dimensional cube, of diameter

\[
\frac{\sqrt{n-1}}{\sqrt{n}} d(C_j)
\]
4. Higher Dimensional Results

2. No intersection with $C_j$.

Let us define

$$g(z) = \sum_{j=1}^{q} \text{diam}^s(C_j(z)),$$

Then $g(z)$ is integrable with $f(z) \leq g(z)$, $\forall z \in [0,1]$, and we have

$$0 < \int_0^1 f(z)dz \leq \int_0^1 g(z)dz = \sum_{j=1}^{q} \text{diam}^{1+s}(C_j(z))$$

and, since $s \leq s_{n-1}(r_\theta)$, the result follows. That is

$$s_n(r_\theta) \geq 1 + s_{n-1}(r_\theta)$$

A corollary to this result is

$$s_n(r_\theta) > n - 1$$

This result follows inductively from our two dimensional result and our theorem.
5. AN IMPROVED BOUND ON THE 

BESICOVITCH-HAUSSDORFF DIMENSION OF 

THE RESIDUAL SET OF ARBITRARILY PACKED 

DISKS IN THE PLANE

In this chapter we turn our attention to arbitrary packings of disks into the unit plane square, $I_2$. A lower bound for the Besicovitch-Hausdorff dimension of the residual set $R$ was shown by D.G. Larman in [1] to be greater than 1.03. We improve this bound by developing the methods used in chapter 2.

**Theorem 3.** Suppose that $\{\theta_n\}_{n=1}^\infty$ forms a packing of disks within the unit plane square $I_2$. Then the residual set $R = I_2 \setminus \bigcup_{n=1}^\infty \theta_n$ has Besicovitch - Hausdorff dimension $s$ and

$$s > 1.033.$$
5. An Improved Bound

Proof. We may suppose without loss of generality that each disk $\theta_n$ is open, and that $\text{diam}(\theta_{n+1}) \leq \text{diam}(\theta_n)$ for $n = 1, 2, \ldots$. This gives us an order to our packing. The largest copy being $\theta_1$, and the $\theta_i$'s decreasing in size as $i$ increases.

We define the $n$th stage residual set to be

$$R_n = I_2 \setminus \bigcup_{m=1}^{n} \theta_m$$

and the residual set for the packing,

$$R = \bigcap_{n=1}^{\infty} R_n = I_2 \setminus \bigcup_{m=1}^{\infty} \theta_m.$$

Since $R$ is compact we need only consider coverings of $R$ by finite collections of open squares $\{C_j\}_{j=1}^{\infty}$ when determining whether the $s$-dimensional Hausdorff measure of $R$ is positive. Let the diameter of $C_j$ be $\sqrt{2}\Delta_j$.

Let $\{C_j\}_{n=1}^{p}$ be a minimal cover of $R$ by open squares, so all of the $C_j$'s are necessary, i.e. none of them are contained within any of the $\theta_n$, $n = 1, 2, \ldots$.

Note that each $R_n$ is compact, and that for each $n > 0$, $R_{n+1} \subset R_n$. Hence $\{R_n\}_{n=1}^{\infty}$ is a nested sequence of compact sets. This implies there exists some $m \in \mathbb{N}$ such that $R_m \subset \bigcup_{j=1}^{p} C_j$. Let us now fix such a sufficiently large $m$. 

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To prove our result we find a condition on \( s \) such that under this condition \( \sum_j \Delta_j^s \) cannot be close to zero which will in turn imply \( \dim(R) \geq s \).

Suppose that the \( x \)-axis is the horizontal axis, and the \( y \)-axis is the vertical axis. Let \( 0 \leq x \leq 1 \), and let \( l_x \) denote the vertical line through the point \((x,0)\), therefore \( l_x = \{ (X,Y) \in \mathbb{R}^2 : X = x \} \).

Let \( l'_{(x,j)} \) be the open interval equal to \( l_x \cap C_j \), and let the length of \( l'_{(x,j)} \) be \( l_{(x,j)} \). Then either

\[
l_{(x,j)} = \frac{1}{\sqrt{2}} \text{Diam}(C_j) = \Delta_j
\]

or

\[
l_{(x,j)} = 0.
\]

We now define a function \( f(x) \) by

\[
f(x) = \sum_{j=1}^{p} l'_{(x,j)}^{x-1}.
\]

Then we deduce that,

\[
\int_0^1 f(x) \, dx = \sum_{j=1}^{p} \int_0^1 l'_{(x,j)}^{x-1} \, dx = \sum_{j=1}^{p} \Delta_j^s. \tag{5.1}
\]
We will define another function $g_m$ such that $\int_0^1 f \geq \int_0^1 g_m$ and then proceed to show $\int_0^1 g_0 > 0$ and deduce our result.

Let us consider $\bigcup_{j=1}^{p} C_j$, then $l_x$ intersects it in a collection of non overlapping open intervals $\{j'_{(x,r)}\}_{r=1}^{v}$ of lengths $\{j_{(x,r)}\}_{r=1}^{v}$. Now each of these intervals can be expressed as the union of $\{l'_{(x,i_{(r)})}\}_{i_{(r)}}^{u(r)}$ chosen from $\{l'_{(x,i)}\}_{j=1}^{p}$, so $j'_{(x,r)} = \bigcup_{i_{(r)}}^{u(r)} l'_{(x,i_{(r)})}$. 

5. An Improved Bound
Now we have

$$j_{(x,r)}(r) \leq \sum_{i=1}^{\nu(r)} l_{(x,i)} \leq \sum_{j=1}^{p} l_{(x,j)} \quad r = 1, \ldots, v$$

and since $0 < s - 1 < 1$,

$$j_{(x,r)}^{s-1} \leq \sum_{i=1}^{\nu(r)} l_{(x,i)}^{s-1} \quad r = 1, \ldots, v$$
5. An Improved Bound

So summing over \( r \) we have,

\[
\sum_{r=1}^{v} j^{*,-1}_{(x,r)} \leq \sum_{r=1}^{p} \sum_{i=1}^{u(r)} l^{*,-1}_{(x,j)} \leq \sum_{j=1}^{p} l^{*,-1}_{(x,j)} \tag{5.2}
\]

Let us define our function \( g_m(x), m \) fixed

\[
g_m(x) = \sum_{r=1}^{v} j^{*,-1}_{(x,r)}
\]

So, from (1) and (2) we have

\[
\int_0^1 g_m(x) \, dx = \int_0^1 \sum_{r=1}^{v} j^{*,-1}_{(x,r)} \, dx \\
\leq \int_0^1 \sum_{j=1}^{p} l^{*,-1}_{(x,j)} \, dx \\
= \sum_{j=1}^{p} \Delta_j. \tag{5.3}
\]

If we show that \( \int_0^1 g_m(x) \, dx > 1 \), for any cover \( \{C_k\}_{k=1}^{p} \), then the residual set \( R \) will have Besicovitch-Hausdorff dimension of at least \( s \).

Let \( i \) be some integer, \( 0 \leq i \leq m - 1 \) and consider

\[
\bigcup_{j=1}^{p} C_j \cup \bigcup_{k=i+1}^{m} \theta_k.
\]
5. An Improved Bound

This is \( \{C_j\}_{i=1}^n \) union those disks larger than \( \theta_{m+1} \), and smaller than \( \theta_i \).

\( l_x \) intersects this union in a collection of disjoint intervals \( \{j_{(x,r,i)} \}_{r=1}^{v(i,x)} \) of lengths \( \{j_{(x,r,i)} \}_{r=1}^{v(i,x)} \) respectively.

Define

\[
g_i(x) = \sum_{r=1}^{v(i,x)} j_{(x,r,i)}^{\sigma-1}
\]
5. An Improved Bound

So suppose \( i \) is one of 1, \ldots, \( m \), then,

\[
\int_0^1 g_i(x) \, dx = \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g_i(x) - g_{i-1}(x)) \, dx
\]  
(5.4)

We shall show that for any \( i \)

\[
\int_0^1 (g_i(x) - g_{i-1}(x)) \, dx \geq 0
\]

which gives,

\[
\int_0^1 g_m(x) \, dx \geq \int_0^1 g_{m-1}(x) \, dx \geq \ldots \geq \int_0^1 g_0(x) \, dx
\]

and since we clearly have \( \int_0^1 g_0(x) \, dx = 1 \) the result follows, since this implies

\[
\int_0^1 g_m(x) \, dx \geq \int_0^1 g_0(x) \, dx = 1
\]

For \( 0 \leq x \leq 1 \), \( l_x \) meets \( \theta_i \) in an interval of length \( 2\alpha \) say, or does not meet \( \theta_i \) at all. Let \( l'_{(x, \theta_i)} \) be the open interval equal to \( l_x \cap \theta_i \), which has length \( l'_{(x, \theta_i)} \) then either

\[
l'_{(x, \theta_i)} = \phi
\]
5. An Improved Bound

Figure 5.4:

or

\[ l_{(x, \theta_i)} = 2\alpha \]

Now \( g_i(x) = g_{i-1}(x) \) for \( x \) such that \( l_x \) does not meet \( \theta_i \), as there are no extra intervals to consider, so we need only worry about those \( x \) where \( l_x \) meets \( \theta_i \).

The segment of \( l_x \) which lies in \( \theta_i \) meets \( \left( \bigcup_{j=1}^{p} C_j \right) \cup \left( \bigcup_{k=i+1}^{m} \theta_k \right) \) in a collection of non-overlapping intervals \( \{r_j'\}_{j=2}^{w-1} \) of lengths \( \{r_j\}_{j=2}^{w-1} \), whose closures do not
meet the boundary of \( \theta_i \), and two intervals \( r'_1 \) and \( r'_w \), \((r_w \text{ below } r_1)\), whose closures meet \( \theta_i \), in fact they may coincide. The lengths of \( r'_1 \) and \( r'_w \) being \( r_1 \) and \( r_w \) respectively. \( I_x \) also meets \( \bigcup_{j=1}^{p} C_j \bigcup \bigcup_{k=i+1}^{m} \theta_k \) in two intervals \( r'_0, r'_{w+1} \) of lengths \( r_0, r_{w+1} \), immediately above and below \( \theta_i \) respectively.

Let us define a function \( T(x) \) such that

\[
T(x) = \begin{cases} 
1 & \text{if } r'_1 \text{ and } r'_w \text{ do not coincide} \\
0 & \text{otherwise.}
\end{cases}
\]

Then the difference is,

\[
g_i(x) - g_{i-1}(x) = T(x) \left[ (r_0 + r_1)^{s-1} + r_2^{s-1} + \ldots \right. \\
+ \left. r_{w-1}^{s-1} + (r_w + r_{w+1})^{s-1} - (r_0 + 2\alpha + r_{w+1})^{s-1} \right] \quad (5.5)
\]

Suppose that \( 0 \leq \mu \leq r_0 \) and \( 0 \leq \lambda \leq r_{w+1} \).

Let

\[
h_i(x, \mu, \lambda) = T(x) \left[ (\mu + r_1)^{s-1} + r_2^{s-1} + \ldots + r_{w-1}^{s-1} + (r_w + \lambda)^{s-1} \right. \\
- \left. (\mu + 2\alpha + \lambda)^{s-1} \right]
\]
Then $g_i(x) - g_{i-1}(x) = h_i(x, r_0, r_{w+1})$. We show that $g_i(x) - g_{i-1}(x) \geq 0$ by first differentiating $h_i(x, r_0, r_{w+1}) \geq h_i(x, 0, 0)$ and then demonstrating that $h_i(x, 0, 0) \geq 0$.

Now,

$$\frac{d}{d\mu} h_i(x, \mu, \lambda) = (s - 1)T(x)[(\mu + r_1)^{s-2} - (\mu + 2\alpha + \lambda)^{s-2}]$$
5. An Improved Bound

and,

\[
\frac{d}{d\lambda} h_i(x, \mu, \lambda) = (s - 1)T(x)[(r_w + \lambda)^{s-2} - (\mu + 2\alpha + \lambda)^{s-2}]
\]

Then since both \( r_1, r_w \leq 2\alpha \) we have,

\[
\frac{d}{d\mu} h_i(x, \mu, \lambda) \geq 0 \quad \text{and} \quad \frac{d}{d\lambda} h_i(x, \mu, \lambda) \geq 0 \quad (5.6)
\]

Let the following intervals be labelled in the following way,

\[
r'_0 \equiv A_1B_1, \quad r'_0 \cup r'_1 \equiv C_1D_1, \quad r'_w \cup r'_{w+1} \equiv C_1D_1E_1, \quad r'_w \cup r'_{w+1} \equiv C_1F_1.
\]

Then if \( G_1, H_1 \) are points of \( A_1B_1, C_1D_1 \) respectively, we may define for all \( x \) such that \( l_x \) meets \( \theta_i, g'_i(x) \) the same as \( g_i(x) \) except that \( A_1B_1E_1, C_1D_1F_1 \) are replaced by \( G_1B_1E_1, F_1C_1H_1, \) and \( g'_{i-1}(x) \) defined as \( g_{i-1}(x) \) with \( A_1D_1 \) replaced with \( G_1H_1 \). Let \( g'_i(x) = g_i(x) \), and \( g'_{i-1}(x) = g_{i-1}(x) \) for all other \( x \). Then we have, for all \( x \), using (5.4), (5.5) and (5.6), that.

\[
\int_0^1 g_i(x) \, dx = \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g_i(x) - g_{i-1}(x)) \, dx \\
\geq \int_0^1 g_{i-1}(x) \, dx + \int_0^1 (g'_i(x) - g'_{i-1}(x)) \, dx \quad (5.7)
\]
We use this to simplify our problem.

We examine $\theta_i$, and find a polygon $\mathcal{H}_i$ which encloses it. Suppose that the disk $\theta_i$ has diameter $2t$ and is centred at $(x_i, y_i)$, which we label $O_i$.

Let $L_{(y_i)}$ be the line which passes horizontally through the centre of $\theta_i$.

\[ L_{(y_i)} = \{(x, y) \in \mathbb{R}^2 : y = y_i\} \]
5. An Improved Bound

Now $\theta_i$ can be contained within a minimal square centred at $O_i$ of diameter $2\sqrt{2} t$.

Let $S_i$ denote this square.

Now let $H$ be the regular hexagon centred at $O_i$ of diameter $3\sqrt{2} \text{Diam}(\theta_i) = 6\sqrt{2} t$, which contains $\theta_i$. Let the orientation of $H$ be such that 2 of its sides are parallel to the $x$-axis.

Let $\theta_1', \ldots, \theta_m'$ be disks centred at $O_{i+1}, \ldots, O_m$ respectively; each of diameter $2t$.

Then there is a subset, say $\{\theta_k')\}_{k=1}^{n(i)}$ of these, congruent to $\theta_i$, which are contained in $H$, having centres $\{O_k')\}_{k=1}^{n(i)}$: some subset of $\{O_j\}_{j=i+1}^m$.

Let $H_k, k = 1, \ldots, n(i)$, denote the set of points of $H$ which are at least as close to $O_k'$ as to any other $O_j'$.

$$H_k = \left\{ p \in H : \min_{1 \leq j \leq n(i)} |p - O_j'| = |p - O_k'| \right\}$$

$H_k$ is the Dirichlet Cell (or Voronoi Region) of $O_k'$, and is a convex polygon. We now appeal to the following lemma which can be found in [6], page 47.
Lemma 3. If $\mathcal{H}$ is a convex hexagon, $\{\theta_i\}_{i=1}^n$ is a packing of circles in $\mathcal{H}$, $O_i$ denoting the centre of $\theta_i$. Let $h_i$ be the number of sides of $\mathcal{H}_i$, then

$$\sum_{i=1}^n h_i \leq 6n$$

Hence if $n$ is sufficiently large, we may assume that the $\mathcal{H}_k$ are convex hexagons.

Let $\mathcal{H}_i'$ be the hexagon formed by pushing the facets of $\mathcal{H}_i$ towards $\theta_i$ until they
5. An Improved Bound

touch its boundary.

Now consider the following polygon:

\[ P_i = \mathcal{H}_i \cap S_i \]

Then \( P_i \) is a polygon that contains \( \theta_i \) and which has at most 8 non-vertical sides.

Now if \( x \in [0,1] \) and \( l_x \) meets \( \theta_i \) then \( l_x \) meets \( P_i - \theta_i \) in two intervals: \( G_2B_1, C_1H_2 \) immediately above and below \( \theta_i \) respectively.

By construction, \( G_2B_1, C_1H_2 \) do not meet any of the \( \theta_j \), \( j = 1, \ldots, i - 1 \).

We shall now define \( g''_i(x) \) as \( g'_i(x) \) except that \( G_1B_1E_1, F_1C_1H_1 \) are replaced by \( G_2B_1E_1, F_1C_1H_2 \); and \( g''_{i-1}(x) \) as \( g'_{i-1}(x) \) except \( G_1H_1 \) is replaced by \( G_2H_2 \). We then have that

\[ g'_i(x) - g'_{i-1}(x) \geq g''_i(x) - g''_{i-1}(x) \]

and hence from (7),

\[ \int_0^1 g_i(x)\,dx \geq \int_0^1 g'_{i-1}(x)\,dx + \int_0^1 (g''_i(x) - g''_{i-1}(x))\,dx \quad (5.8) \]

Suppose that \( H_2, G_2 \) lie on the segments \( Q_1R_1, Q_2R_2 \) of the polygon \( P_i \) respectively. Let \( V_k \) be the point of intersection of \( Q_kR_k \) and the boundary of \( \theta_i \).
Let $U$ denote the intersection of $l_z$ and $L_{(y_t)}$, and the horizontal distances from $H_2, G_2$ to $V_1, V_2$ be $y'$ and $y''$ respectively.

Now

$$ (V_1 H_2)^2 = H_2 C_1 \cdot H_2 B_1 $$

(5.9)
5. An Improved Bound

Figure 5.9:

$H_2B_1$ has length at most the diameter of $\theta_i$, hence

$$H_2B_1 \leq 2t$$  \hspace{1cm} (5.10)

Let $\psi$ be the acute angle that $Q_1R_1$ makes with $L(u)$, then

$$y' = V_1H_2 \cos \psi$$  \hspace{1cm} (5.11)
5. An Improved Bound

Combining these we get

\[ H_2C_1 = \frac{(V_1H_2)^2}{H_2B_1} \geq \frac{(V_1H_2)^2}{2t} = \frac{(y')^2 \sec^2 \psi}{2t} \geq \frac{y'^2}{2t} \]

Since \( UH_2 \) is bounded by the radius of \( \theta_i \), \( (UH_2 \leq t) \), it follows that

\[ \frac{H_2C_1}{UH_2} \geq \frac{y'^2}{2t^2} \quad (5.12) \]

Similarly we have

\[ (V_2G_2)^2 = G_2B_1 \cdot G_2C_1 \]

And using

\[ G_2C_1 \leq 2t \quad y'' = V_2G_2 \cos \eta \]

Where \( \eta \) is the acute angle which \( Q_2R_2 \) makes with \( L_{(\psi_i)} \) we have

\[ \frac{G_2B_1}{UG_2} \geq \frac{(y'')^2}{2t^2} \quad (5.13) \]

since \( UG_2 \leq t \).
5. An Improved Bound

Given some \( \delta \in [0, 1] \), and providing that

\[
y' \geq \delta t, \quad y'' \geq \delta t
\]

then it follows from (5.12) and (5.13) that

\[
\frac{H_2 C_1}{U H_2} \geq \frac{1}{2} \delta^2 \quad \text{and} \quad \frac{G_2 B_1}{U G_2} \geq \frac{1}{2} \delta^2.
\]  
(5.14)

Let \( s, \sigma, \delta \) be positive real numbers which satisfy the following:

\[
2^{3-2s} \delta^{2(s-1)} \geq 1 + \sigma
\]  
(5.15)

and

\[
\frac{\sigma}{27} \left( \frac{2\sqrt{53}}{27} \right)^{s-1} \geq 8 \cdot 2^s \delta
\]  
(5.16)

where \( \sigma < 1 \) and \( \delta \leq 1/81 \).

We shall verify that an allowable set of values is

\[
s = 1.033 \quad \sigma = (2\sqrt{53})^{-1} \quad \delta = \left( \frac{\sqrt{53}}{27} \right)^{1.033} \frac{1.033}{1696}
\]  
(5.17)
5. An Improved Bound

Now using (5.14) and (5.15) we have

\[ 2(H_2C_1)^{s-1} - (2UH_2)^{s-1} \geq \sigma(2UH_2)^{s-1} \]  \hspace{1cm} (5.18)

\[ 2(G_2B_1)^{s-1} - (2UG_2)^{s-1} \geq \sigma(2UG_2)^{s-1} \]  \hspace{1cm} (5.19)

Except possibly if \( x \) belongs to at most 8 intervals whose union is \( Q \). Each interval of this type having length \( 2\delta t \). The centre of such an interval being a horizontal projection of a point of contact of \( P_i \) with the boundary of \( \theta_i \).

Let us suppose that the length of \( H_2C_1 \) is greater than \( G_2B_1 \) by some length denoted \( r \). Then if \( x \notin Q \) we have using (18) and (19)

\[
(H_2C_1)^{s-1} + (G_2B_1)^{s-1} - (H_2G_2)^{s-1} \\
= (G_2B_1 + r)^{s-1} + (G_2B_1)^{s-1} - (2UG_2 + r)^{s-1} \\
\geq 2(G_2B_1)^{s-1} - (2UG_2)^{s-1} \\
\geq \sigma(2UG_2)^{s-1}
\]
5. An Improved Bound

Hence

\[(H_2 C_1)^{s-1} + (G_2 B_1)^{s-1} - (H_2 G_2)^{s-1}\]
\[\geq \sigma \min[(2U G_2)^{s-1}, (2U H_2)^{s-1}]\]  \hspace{1cm} (5.20)

If, conversely, \(G_2 B_1\) is greater than \(H_2 C_1\) by \(r\), we have

\[(H_2 C_1)^{s-1} + (G_2 B_1)^{s-1} - (H_2 G_2)^{s-1}\]
\[= (H_2 C_1)^{s-1} + (H_2 C_1 + r)^{s-1} - (2U H_2 + r)^{s-1}\]
\[\geq 2(H_2 C_1)^{s-1} - (2U H_2)^{s-1}\]
\[\geq \sigma (2U H_2)^{s-1}.\]

If \(r'_{11}\) coincides with \(r'_{u1}\) then we have

\[g_i''(x) = g_i''(x).\] \hspace{1cm} (5.21)

Otherwise, if \(x \not\in Q\), we deduce from (5.20)

\[g_i''(x) \geq g_i''(x) + \sigma \min[(2U G_2)^{s-1}, (2U H_2)^{s-1}],\] \hspace{1cm} (5.22)
5. An Improved Bound

If \( x \in Q \), then, since \( P_i \subset S_i \), it follows that

\[
\int_Q (g_i''(x) - g_i'''(x)) dx \geq -\int_Q (H_2 G_2)^{s-1} dx \\
\geq -8 \int_0^{2t} (2t)^{s-1} dx \\
= -8 \cdot 2^s t^s. \tag{5.23}
\]

Let \( \theta_i'' \) be a disk of radius \( (1 - \delta^{1/4})t \), centred at \( O_i \). The remainder of this proof then splits into three cases as follows:

Case 1 A side of \( C_q \), one of the \( \{C_j\}_{j=1}^p \) meets \( \theta_i'' \).

Case 2 Some \( C_q \) contains \( \theta_i'' \) entirely.

Case 3 The disk, \( \theta_i'' \), does not intersect \( \cup_{i=1}^p C_j \).

Case 1. If \( C_q \), say, has a side which meets \( \theta_i \), then there is a portion of length \( \rho(q, i) \) which is entirely contained within \( \theta_i \). As \( \{C_j\}_{j=1}^p \) is a minimal cover, we know that none of the squares \( \{C_j\}_{j=1}^p \) lie entirely within \( \theta_i \). Then

\[
\rho(q, i) \geq \delta^{1/4} t \tag{5.24}
\]
From (5.24) we have

$$-8 \cdot 2^s \delta t^* \geq -8 \cdot 2^s \rho^s(q,i)$$

Since $s < 2$ we have

$$-8 \cdot 2^s \rho^s(q,i) \geq -32 \rho^s(q,i) \quad (5.25)$$
5. An Improved Bound

which gives, with (5.8):

\[
\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx - 32\rho^*(q, i)
\]  

(5.26)

**Case 2.** Let \( S'_i \) be the square that circumscribes \( \theta'_i \) which is the same orientation as \( S_i \). \( \delta \) is sufficiently small to ensure that all four corners of \( S'_i \) are contained in the compliment of \( \theta_i \). Coverings of this type are not economical.
5. An Improved Bound

Now $C_q$ must contain $S'_i$. Then $C_q$ must also have all four corners in the complement of $\theta_i$.

Suppose that the horizontal intervals where the edges of $C_q$ meet $\theta_i$ are $A_3B_3$, $E_3F_3$, and the vertical intervals are $C_3D_3$, $G_3H_3$ shown in Fig. Some of these intervals may not exist.
5. An Improved Bound

Suppose that $A_3B_3$ and $E_3F_3$ both exist, and that

$$A_3B_3 \leq E_3F_3.$$  

Let $g''_i(x), g''_{i-1}(x)$ be defined as $g'_i(x), g'_{i-1}(x)$ by taking $G_1 \equiv B_1$ and $H_1 \equiv C_1$. This gives

$$g'_i(x) - g'_{i-1}(x) \geq g''_i(x) - g''_{i-1}(x).$$

Then using (5.7),

$$\int_0^1 g_i(x)dx \geq \int_0^1 g_{i-1}(x)dx + \int_0^1 (g''_i(x) - g''_{i-1}(x))dx. \quad (5.27)$$

Let the points of intersection of $\theta_i$ and $C_q$ be as indicated and note their horizontal projections. If $0 \leq x \leq 1$ then $l_x$ meets

(i) $C_q \cap \theta_i$ in an interval of length $\beta(x)$.

(ii) $\theta_i$ in an interval of length $2\alpha(x)$.

If $Q'$ denotes the union of the intervals $Z'D_4$, $E_4F_4$, $G_4Z'_1$ then it follows that

$$\int_0^1 (g''_i(x) - g''_{i-1}(x))dx \geq \int_{Q'} ((\beta(x))^{*1} - (2\alpha(x))^{*1})dx \quad (5.28)$$
5. An Improved Bound

Now $2\alpha(x) \leq 2t$ and $\beta = (t^2 - (\frac{1}{2}A_3B_3)^2)^{1/2} + (t^2 - (\frac{1}{2}E_3F_3)^2)^{1/2}$ Hence

$$\beta(x)^{s-1} - (\alpha(x))^s \geq -(2\alpha(x) - \beta(x))^{s-1}$$

$$\geq -(2t - \beta(x))^{s-1}$$

$$= -[(t^2)^{1/2} - (t^2 - (\frac{1}{2}A_3B_3)^2)^{1/2} + (t^2)^{1/2} - (t^2 - (\frac{1}{2}E_3F_3)^2)^{1/2}]^{s-1}$$

$$\geq -\left(\frac{1}{2}A_3B_3 + \frac{1}{2}E_3F_3\right)^{s-1}$$

$$\geq -(E_3F_3)^{s-1} \tag{5.29}$$

Similarly if $x$ belongs to $E_4B_4$ or to $A_4F_4$

$$(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -\left(\frac{1}{2}E_3F_3\right)^{s-1} \tag{5.30}$$

If $x$ belongs to $Z'D_4$

$$(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -(C_3D_3)^{s-1} \tag{5.31}$$
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If \( x \) belongs to \( G_4Z'_1 \)

\[
(\beta(x))^{s-1} - (2\alpha(x))^{s-1} \geq -(H_3G_3)^{s-1} \tag{5.32}
\]

Combining (5.28) - (5.32) and noting

\[
\int g''(x)dx \geq \int g''_{-1}(x)dx = ((A_3B_3)^* + (C_3D_3)^* + (E_3F_3)^* + (G_3H_3)^*).
\tag{5.33}
\]

This also holds if \( A_3B_3 \geq E_3F_3 \) and therefore combining this with (5.27) we obtain, for Case 2,

\[
\int_0^1 g_i^*(x)dx \geq \int_0^1 g_{i-1}(x)dx - \sum_{j=1}^4 (\gamma(i,q,j))^*, \tag{5.34}
\]

where \( \{\gamma(i,q,j)\}_{j=1}^4 \) is the disjoint intervals of boundary of \( C_q \) which lie in \( \theta_i \).

Case 3. If \( L \) denotes the maximal line segment of \( L_{(y_i)} \cap \theta_i \) which contains \( O_i \), but does not meet \( \bigcup_{j=1}^p C_j \), then the length of \( L \) is at least \( (2 - 2\delta^{1/2})t \), which,
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using (5.16), is at least equal to $\frac{16}{9}t$. As there are at most 8 non-vertical sides of $P_i$ there is an interval $w$ of length $\frac{1}{9}t$ on the $x$-axis which is contained in the horizontal projection of $L$ but which does not contain the horizontal vertex of $P_i$ or horizontal of projection of a point of contact of $P_i$ with the boundary of $\theta_i$.

Hence, there is an interval $W'$ of $W$, which has length $\frac{1}{27}t$ which is at least a distance $\frac{1}{27}t$ from the horizontal projection of either a vertex of $P_i$ or a point of contact of $P_i$ with the boundary of $\theta_i$.

Then, if $x$ is a point of $W'$, let $H_2, C_1, U, B_1, G_2$ be as indicated and let the points of intersection of $Q_1R_1$ and $Q_2R_2$ and the boundary of $\theta_i$ be $V_1, V_2$ respectively.

As $U$ is at least a distance $\frac{1}{27}$ from the complement of $\theta_i$, we have

$$\min(H_2U, UG_2) \geq \left(\sqrt{1 - \left(\frac{26}{27}\right)^2}\right) t = \frac{\sqrt{53}}{27}t.$$ (5.35)

Now suppose that $Q_1R_1, Q_2R_2$ are sides of $P_i$ which lie above and below $L_{(y_i)}$ respectively. Suppose also that both of their horizontal projections onto the $x$-axis contain the interval $W$.

Let $IJ$ be the segment of $L_{(y_i)}$ which projects horizontally onto $W'$. Note that the $x$-co-ordinate of $J$ is greater than that of $I$. 

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We have for this case that $\theta_i$ does not meet $\cup_{j=1}^n C_j$ and hence none of the squares meet $IJ$. It follows that

$$g''_i(x) - g''_{i-1}(x) \geq (H_2 C_1)^{s-1} + (G_2 B_1)^{s-1} - (H_2 G_2)^{s-1}$$

(5.36)
Which, using (5.20), gives

\[ g_i''(x) - g_i''(x) \geq \sigma \min[(2UG_2)^{s-1}, (2UH_2)^{s-1}]. \]

So, using (5.35),

\[ g_i''(x) - g_i''(x) \geq \sigma \left( \frac{2\sqrt{53}}{27} t \right)^{s-1}. \]
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And hence

$$\int_{W,} (g''_i(x) - g''_{i-1})(x) dx \geq \sigma \left( \frac{2\sqrt{53}}{27} t \right)^{s-1} \frac{t}{27}$$

$$\geq 8 \cdot 2^s \delta t^s,$$  \hspace{1cm} (5.37)

using (5.16) and $\frac{1}{27} > \delta$.

Given (5.8), (5.23) and (5.37), we have

$$\int_0^1 g_i(x) dx \geq \int_0^1 g_{i-1}(x) dx.$$  \hspace{1cm} (5.38)

We now combine cases 1, 2 and 3, using (5.26), (5.34) and (5.38), to deduce

$$\int_0^1 g_i(x) dx \geq \int_0^1 g_{i-1}(x) dx - 32 \sum_{j=1}^4 (\gamma(i,q,j))^s \hspace{1cm} (5.39)$$

where $\{\gamma(i,q,j)\}_{j=1}^4$ are the lengths of the disjoint portions of sides of $C_q$ which lie entirely within $\theta_i$.

Now repeating this argument for $i = m, m-1, \ldots, 1$ we deduce

$$\int_0^1 g_m(x) dx \geq \int_0^1 1^{s-1} dx - 32 \sum_{i=1}^m \sum_{j=1}^4 (\gamma(i,q,j))^s,$$  \hspace{1cm} (5.40)
where \( \{\gamma(i, q(i), j)\}_{j=1}^4 \) are the lengths of disjoint portions of the sides of a square, \( C_q \), of \( \{C_j\}_{j=1}^p \), which lie entirely within \( \theta_i \).

Let \( \{\gamma(k, j)^{w(j)}\}_{j=1}^p \) be a rearrangement of \( \{\gamma(i, q(i), j)\}_{j=1}^4 \) so that \( \{\gamma(k, j)^{w(j)}\} \) is all the lengths of those intervals which belong to the boundary of \( C_j \). Since \( \{\theta_i\}_{i=1}^m \) are disjoint, and \( s > 1 \), we have

\[
\sum_{k=1}^{w(j)} \gamma(k, j)^s \leq 4\Delta_j^s. \tag{5.41}
\]

So

\[
-32 \sum_{j=1}^{p} \sum_{k=1}^{w(j)} \gamma(k, j)^s \geq -128 \sum_{j=1}^{p} \Delta_j^s.
\]

From (5.3), (5.40) and (5.41), we have

\[
\sum_{j=1}^{p} \Delta_j^s \geq \int_0^1 g_\alpha(x)dx \geq 1 - 128 \sum \Delta_j.
\]

Hence we obtain

\[
\sum_{j=1}^{p} \Delta_j^s \geq \frac{1}{124} \tag{5.42}
\]

where \( s \) is a real number, \( 1 < s < 2 \), such that, with \( \sigma, \delta \), satisfy (5.15) and (5.16).

So \( R \) has Besicovitch-Hausdorff dimension at least \( s \).
To show $s = 1.033$ is an allowable value for $s$, let

$$\sigma = (2\sqrt{53})^{-1} \quad (5.43)$$

$$2^{3-2s}2^{2(s-1)} = 1 + \sigma \quad (5.44)$$

$$\frac{\sigma}{27} \left( \frac{2\sqrt{53}}{27} \right)^{s-1} = 8 \cdot 2^{s}\delta. \quad (5.45)$$

Now substituting (5.43) and (5.45) into (5.44) we have

$$\frac{2^{13-12s}53^2-3s+2}{272s^2-2s} = 1 + \frac{1}{2\sqrt{53}} \quad (5.46)$$

Taking log's produces the following quadratic

$$(\log 53 - 2\log 27)s^2$$

$$+ (-12 \log 2 - 3 \log 53 + 2 \log 27)s$$

$$+ (13 \log 2 + 2 \log 53 - \log(1 + 2\sqrt{53})^{-1}) = 0$$

calculation of the coefficients produces:

$$a = -1.1384517$$
5. An Improved Bound

\[ b = -5.92246 \]

\[ c = 7.3330939 \]

The value of \( s \) is the largest root of this quadratic, which is 1.033; and this completes the proof. \( \square \)
BIBLIOGRAPHY


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