Borel sets and σ-fragmentability of a Banach space

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Abstract

In this thesis we give a sufficient condition on a Banach space for it to have the same weak and norm Borel sets and to be a Borel subset of its bidual, when the latter is endowed with the weak* topology. We also deal with one-to-one maps between Banach spaces, say from X into Y, when Y has a countable cover by sets of small local diameter. Under these conditions we are able to characterize those maps which transfer that property to X. We use this kind of map to show that certain spaces have a countable cover by sets of small local diamter and to answer some questions on c_0-sums of Banach spaces and on topological invariants for the weak topology as well as some questions related to $C(K)$ spaces. We also study the inverses of some of these maps. Finally we construct injections of this type into $c_0(\Gamma)$ for spaces with Projectional Resolutions of Identity.

Contents

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Introduction

Throughout this thesis two notions will be constantly used. Both of them were introduced by Jayne, Namioka and Rogers in [12, 13].

Let (X, τ) be a Hausdorff space and let ϱ be a metric on X not necessarily related to the topology on X.

The space X is said to be *σ-fragmented* by the metric ϱ if, for each $\varepsilon > 0$, it is possible to write

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where each set X_i has the property that each non-empty subset of X_i has a non-empty relatively τ-open subset of ϱ-diameter less than ε.

We shall say that (X, τ) has a countable cover by sets of small local ϱ-diameter if, for each $\varepsilon > 0$, X can be expressed as a union

$$X = \bigcup_{i=1}^{\infty} X_i, \tag{0.1}$$

each non-empty X_i having the property that each of its points belongs to some relatively non-empty τ-open subset of ϱ-diameter less than ε.

Although our aim is to find results on Banach spaces we give the statements, when possible, in terms of topological spaces and metrics defined on them.

In Chapter 1 we begin by giving the two definitions above as well as several notions on discrete families in topological and metric spaces. We show that σ-fragmentability and having a countable cover by sets of small local diameter are equivalent concepts when the topology on the space is generated by a metric. We also prove that if a topological space (X, τ) has a countable cover by sets of small local diameter and the metric involved is lower semicontinuous for the topology τ, then the sets in equation (0.1) can be taken to be differences of τ-closed sets. When applied to a Banach space we obtain that if it has a countable cover by sets of small local diameter, then it has a countable cover by differences of weakly closed sets of small local diameter. This condition was shown to be sufficient, by the authors above, to have $Borel(X, \|\cdot\|) = Borel(X, weak)$, see [9], p. 215 and [11], Lemma 2.1. We prove something stronger, we show that if a Banach space X has a countable cover by sets of small local diameter, then both X and any $\|\cdot\|$-closed subset of X are Borel subsets of (X^{**}, w^*), and this implies the coincidence of the Borel subsets of X. We finish this Chapter characterizing both the property of having a countable cover by sets of small local diameter and the σ-fragmentability in terms of decompositions of $\|\cdot\|$-discrete families into a countable number of relatively weakly discrete families.

Chapter 2 is concerned with one-to-one maps between Banach spaces (or rather between topological spaces with metrics defined on them). If $T : X \rightarrow Y$ is a one-to-one map, with the Banach space Y having a countable cover by sets of small local diameter, we prove that X also has such a decomposition if and only if T maps discrete families from $(X, \|\cdot\|_X)$ into discretely σ-

decomposable families in $(Y, \|\cdot\|_Y)$. (Such a map will be said to be d. σ-d.). Moreover, if we denote by $T_{\|\cdot\|_Y}$ the metric given by

$$T_{\|\cdot\|_Y}(u,v) = \|Tu - Tv\|_Y, u, v \in X,$$

we show that $(X, T_{\|\cdot\|_Y})$ has a countable cover by sets of additive class α (for the topology $T_{\|\cdot\|_Y}$) of small local $\|\cdot\|_X$-diameter if and only if T^{-1} is of Borel class α and T is d. σ-d.

We also show that (i) σ-fragmentability (first proved in [19]), (ii) to have a countable cover by sets of small local diameter, and (iii) the coincidence of the weak and norm Borel sets on a Banach space are each topological invariants for the weak topology. We prove too that the c_0-sum of Banach spaces which have countable covers by sets of small local diameter (resp. σ-fragmentability) has a countable cover by sets of small local diameter (resp. σ-fragmentability), (the σ-fragmentability case is Theorem 6.1 in [9]), and apply this to the case of $C(K)$ spaces with their weak topologies with $K = \cup\{K_n : n \in N\}$, where $K, K_1, K_2, \ldots$ are compact spaces, when each $(C(K_n), weak)$ has either one of the properties, [15].

In Chapter 3 we give examples of spaces with countable covers by sets of small local diameter. We begin by giving a proof that $c_0(\Gamma)$, for any set Γ, has that property (first proved in [13], Corollary 6.3.1). Spaces with Projectional Resolutions of Identity are also shown to have this property through the construction of d. σ-d. maps from these spaces into $c_0(\Gamma)$. We apply this to three particular cases: WCD spaces, duals of Asplund spaces and $C(K)$ spaces with K being a Valdivia compact space. We finish the Chapter showing that spaces with Markushevich bases can be imbedded into $c_0(\Gamma)$ with an inverse map of the first Borel class.

Chapter 1

σ-fragmentability

1.1 Some definitions and remarks

We start by giving two definitions introduced by Jayne, Namioka and Rogers which can be found in [9].

Let (X, τ) be a Hausdorff space and let ϱ be a metric on X not necessarily related to the topology on X.

We define the *local ϱ-diameter* of a non-empty subset A of X to be the infimum of the positive numbers ε with the property that each point x of A belongs to a non-empty relatively open subset of A of ϱ-diameter less than ε.

Definition 1.1.1 *The space X is said to be σ-fragmented by the metric ϱ if, for each $\varepsilon > 0$, it is possible to write*

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where each set X_i has the property that each non-empty subset of X_i has a non-empty relatively open subset of ϱ-diameter less than ε.

Definition 1.1.2 *We shall say that the family $\mathcal{B}$ is a cover of X by sets of small local ϱ-diameter if each point of X belongs to sets of $\mathcal{B}$ having arbitrarily small local ϱ-diameter.*

We shall usually say that X has a countable cover by sets of small local diameter when both the original topology and the metric on X are clearly distinguished and it does not create any confusion.

Note that X has a countable cover by sets of small local diameter, if and only if, for each $\varepsilon > 0$, X can be expressed as a union

$$X = \bigcup_{i=1}^{\infty} X_i,$$

each non-empty X_i having the property that each of its points belongs to some non-empty relatively open subset of ϱ-diameter less than ε.

When X is a Banach space and the metric is the norm metric we shall talk of the local diameter of a set rather than its local norm-diameter.

We shall give some more definitions which will be of interest later on in this chapter. The definitions as well as some properties concerning them can be found in [5, 6].

Definition 1.1.3

i) For $\varepsilon > 0$ a family $\mathcal{A} = \{A_s\}_{s \in S}$ of subsets of a metric space (X, d) is called *ε-discrete* (or metrically discrete with separating distance ε) if

$$d(y_1, y_2) > \varepsilon \text{ whenever } y_1 \in A_s, y_2 \in A_t \text{ and } s \neq t.$$

ii) A family $\mathcal{A}=\{A_s\}_{s\in S}$ of subsets of a topological space X is called *discrete* if for each $x \in X$ there exists an open neighbourhood of x which intersects at most one element of the family $\mathcal{A}$.

iii) A family $\mathcal{A}=\{A_s\}_{s\in S}$ of subsets of a topological space X is called *σ-discrete* if it can be written as a countable union of families each of which is discrete.

iv) A family $\mathcal{A}=\{A_s\}_{s\in S}$ of subsets of a topological space X is called *discretely σ-decomposable* (*d. σ-d.*, for short) if for each $m \in N$ there exists a discrete family $\{B_s^{(m)}\}_{s\in S}$ such that

$$A_s = \bigcup_{m=1}^{\infty} B_s^{(m)}, \text{ for all } s \in S.$$

Remark 1.1.4

1) If $\mathcal{A}=\{A_s\}_{s\in S}$ is a discrete family of subsets of a Banach space X, we see that it is σ-decomposable into a countable set of metrically discrete families as follows. Define

$$A_s^{(n)} = \{x \in X : x \in A_s \text{ and } B(x, \frac{1}{n}) \cap A_t = \emptyset \text{ for all } t \neq s\}$$

for all $n \geq 1$ and $s \in S$. Then

$$A_s = \bigcup_{n=1}^{\infty} A_s^{(n)}, \text{ for all } s \in S.$$

And for each $n \in N$ the family $\mathcal{A}^{(n)} = \{A_s^{(n)} : s \in S\}$ is $(\frac{1}{n})$-discrete in X.

2) We will make use of the fact that in a metric space if a family is discrete in its union, then it is d. σ-d. in the whole space.

3) It is also well known that in a metric space any open cover of the space has a σ-discrete open refinement (see for example [16], p. 234).

1.2 σ-fragmentability of a metric space

Our first result shows that having a countable cover by sets of small local diameter, although apparently stronger than σ-fragmentability, turns out to be equivalent to the latter when the topology on the space is given by a metric. The ideas in the proof are from [9], Theorem 2.4. It reads as follows:

Proposition 1.2.1 *Let (X, d) be a metric space and ϱ be another metric defined on X. The following conditions are equivalent:*

i) (X, d) is σ-fragmented by ϱ;

ii) (X, d) has a countable cover by sets of small local ϱ-diameter.

When the sets in i) can be taken to be differences of d-closed sets (or more generally d-F_σ-sets), then the sets in ii) can be taken to be d-F_σ-sets.

Proof. ii)⇒ i) Is clear by definition.

i)⇒ ii) Given $\varepsilon > 0$ there is a decomposition of X given by the σ-fragmentability of the space

$$X = \bigcup_{i=1}^{\infty} C_i.$$

Fix $i \in N$. Because of the σ-fragmentability, there exists a family of d-open sets, $\{U_\alpha^i : 0 \leq \alpha < \mu\}$, covering C_i such that

$$C_i \cap U_\alpha^i \setminus \bigcup_{0 \leq \beta < \alpha} U_\beta^i \neq \emptyset, \text{ and } \varrho\text{-diam}(C_i \cap U_\alpha^i \setminus \bigcup_{0 \leq \beta < \alpha} U_\beta^i) < \varepsilon$$

for $0 \leq \alpha < \mu$.

Define

$$F_{\alpha,i}^n = \{x \in X : d(x, X \setminus U_\alpha^i) \geq \frac{1}{n}\} \text{ and } H_{\alpha,i}^n = (C_i \cap F_{\alpha,i}^n \setminus \bigcup_{0 \leq \beta < \alpha} U_\beta^i).$$

It is clear that ϱ-diam$(H_{\alpha,i}^n) < \varepsilon$. Now for $\alpha \neq \beta$ the sets $H_{\alpha,i}^n$ and $H_{\beta,i}^n$, when non-empty, are separated by d-distance at least $\frac{1}{n}$. So for each $n \in N$ the family $\{H_{\alpha,i}^n : 0 \leq \alpha < \mu\}$ is discrete in (X, d).

Set $H_i^n = \cup\{H_{\alpha,i}^n : 0 \leq \alpha < \mu\}$. We have

$$\bigcup_{n=1}^{\infty} H_i^n = \bigcup_{n=1}^{\infty} \bigcup_{0 \leq \alpha < \mu} H_{\alpha,i}^n = \bigcup_{n=1}^{\infty} \bigcup_{0 \leq \alpha < \mu} (C_i \cap F_{\alpha,i}^n \setminus \bigcup_{0 \leq \beta < \alpha} U_\beta^i) =$$

$$= \bigcup_{0 \leq \alpha < \mu} (C_i \cap U_\alpha^i \setminus \bigcup_{0 \leq \beta < \alpha} U_\beta^i) = C_i,$$

and therefore

$$X = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_i^n.$$

Let us see that for each $n, i \in N$ the set H_i^n has local ϱ-diameter less than ε. Take $x \in H_i^n$. We have that for some α, $x \in H_{\alpha,i}^n$. Since the family $\{H_{\alpha,i}^n : 0 \leq \alpha < \mu\}$ is discrete in (X, d) there must be a d-open neighbourhood V of x such that

$$V \cap H_{\beta,i}^n = \emptyset \text{ for } \beta \neq \alpha.$$

So

$$\varrho\text{-diam}(V \cap H_i^n) = \varrho\text{-diam}(V \cap H_{\alpha,i}^n) < \varepsilon.$$

Now if the C_i are differences of d-closed sets, and since the $F_{\alpha,i}^n$ are d-closed, we have that the $H_{\alpha,i}^n$ are differences of d-closed sets. So each H_i^n is the discrete union of sets which are differences of d-closed sets. Since d-open sets are F_σ-sets, it follows that the H_i^n's are also F_σ-sets. ∎

1.3 Countable cover by special sets of small local diameter

The decomposition of the space by means of arbitrary sets given in Definitions 1.1.1 and 1.1.2 can sometimes be improved. It is interesting to find conditions to impose on the metric so that we could take Borel sets, for example. As in [9] the condition will be that of the metric being *lower-semicontinuous* with respect to the original topology of the space, (i.e. the closed balls being also closed in the other topology).

Theorem 1.3.1 *Let (X,τ) be a topological space and ϱ be a lower semicontinuous metric on X. If (X,τ) has a countable cover by sets of small local ϱ-diameter, then (X,τ) has a countable cover by differences of τ-closed sets of small local ϱ-diameter. Moreover, if the ϱ-topology is stronger than τ, then the sets can be taken to be ϱ-closed.*

Proof. Let $\varepsilon > 0$ be any positive number. We shall show that X has a countable cover by differences of τ-closed sets of local diameter less than ε.

Let $\{U_\alpha^n : \alpha \in A, n \in N\}$ be a σ-discrete refinement of a cover of X by balls of diameter less than ε. We can suppose that each $\{U_\alpha^n : \alpha \in A\}$ is metrically discrete with separating distances $\delta_n > 0$.

For $n \in N$ let $\{C_m^n : m \in N\}$ be a countable cover of X with local diameter of each $C_m^n < \delta_n$. Then for each $n, m \in N$ the family $\{U_\alpha^n \cap C_m^n\}_{\alpha \in A}$ is discrete in (C_m^n, τ). (Each point in C_m^n has a non-empty relatively open neighbourhood with diameter less than δ_n. This open subset of C_m^n can intersect at most one of the $U_\alpha^n \cap C_m^n$'s.)

For each $x \in U_\alpha^n \cap C_m^n$ there exists a τ-open neighbourhood of x, say $U_{x,\alpha}^{n,m}$, such that

$$U_{x,\alpha}^{n,m} \cap (U_\alpha^n \cap C_m^n) \neq \emptyset,$$

$$U_{x,\alpha}^{n,m} \cap (U_\beta^n \cap C_m^n) = \emptyset, \text{ for } \beta \neq \alpha.$$

Set

$$G_\alpha^{n,m} = \bigcup_{x \in U_\alpha^n \cap C_m^n} U_{x,\alpha}^{n,m}.$$

$G_\alpha^{n,m}$ is a τ-open set with

$$G_\alpha^{n,m} \supset (U_\alpha^n \cap C_m^n)$$

and

$$G_\alpha^{n,m} \cap (U_\beta^n \cap C_m^n) = \emptyset \text{ for } \beta \neq \alpha.$$

Set $M_\alpha^{n,m} = \overline{(U_\alpha^n \cap C_m^n)}^{\tau} \cap G_\alpha^{n,m}$. It is clear that

$$X = \bigcup_m \bigcup_n (\bigcup_{\alpha \in A} M_\alpha^{n,m}).$$

We notice that, since ϱ is τ-lower semicontinuous and

$$\operatorname{diam}(U_\alpha^n \cap C_m^n) \leq \varepsilon,$$

we have

$$\operatorname{diam}(\overline{U_\alpha^n \cap C_m^n}^{\tau}) \leq \varepsilon.$$

We now show that

$$\bigcup_{\alpha \in A} (\overline{[U_\alpha^n \cap C_m^n]}^{\tau} \cap G_\alpha^{n,m}) = \overline{\bigcup_{\alpha \in A} (U_\alpha^n \cap C_m^n)}^{\tau} \cap \bigcup_{\alpha \in A} G_\alpha^{n,m}.$$

Since $G_\alpha^{n,m} \cap (U_\beta^n \cap C_m^n) = \emptyset$ for $\beta \neq \alpha$, it is clear that

$$\bigcup_{\alpha \in A} [\overline{(U_\alpha^n \cap C_m^n)}^{\tau} \cap G_\alpha^{n,m}] = \bigcup_{\alpha \in A} \overline{(U_\alpha^n \cap C_m^n)}^{\tau} \cap \bigcup_{\alpha \in A} G_\alpha^{n,m}.$$

Now, we always have

$$\bigcup_{\alpha\in A}\overline{(U_\alpha^n\cap C_m^n)}^{\tau}\subset\overline{\bigcup_{\alpha\in A}(U_\alpha^n\cap C_m^n)}^{\tau}$$

and so we must check that

$$\overline{\bigcup_{\alpha\in A}(U_\alpha^n\cap C_m^n)}^{\tau}\cap[\bigcup_{\alpha\in A}G_\alpha^{n,m}]\subset\bigcup_{\alpha\in A}\overline{(U_\alpha^n\cap C_m^n)}^{\tau}\cap[\bigcup_{\alpha\in A}G_\alpha^{n,m}].$$

Set $B_\alpha=U_\alpha^n\cap C_m^n$ and $G_\alpha=G_\alpha^{n,m}$ and suppose that there exists

$$x\in\overline{\bigcup_{\alpha\in A}B_\alpha}^{\tau}\cap[\bigcup_{\alpha\in A}G_\alpha],$$

such that

$$x\notin\bigcup_{\alpha\in A}\overline{B_\alpha}^{\tau}\cap[\bigcup_{\alpha\in A}G_\alpha].$$

Then, since

$$x\in\bigcup_{\alpha\in A}G_\alpha,$$

there must be U_x, a τ-open neighbourhood of x, such that $U_x\subset G_{\alpha_0}$ for some $\alpha_0\in A$ and

$$U_x\cap(\bigcup_{\alpha\in A}B_\alpha)\neq\emptyset.$$

Thus $U_x\cap B_{\alpha_0}\neq\emptyset$, since $G_{\alpha_0}\cap B_\beta=\emptyset$ for $\beta\neq\alpha_0$. Suppose now that there exists a τ-open neighbourhood V_x of x such that

$$V_x\cap B_{\alpha_0}=\emptyset.$$

Take $\emptyset\neq U_x\cap V_x\subset G_{\alpha_0}$. Since $U_x\cap V_x$ is a τ-open neighbourhood of x and

$$x\in\overline{\bigcup_{\alpha\in A}B_\alpha}^{\tau},$$

we have

$$(U_x \cap V_x) \cap (\cup_\alpha B_\alpha) \neq \emptyset,$$

which implies that $U_x \cap V_x \cap B_{\alpha_0} \neq \emptyset$. This is a contradiction.

Now we show that for each $n, m \in N$ the set

$$\overline{\bigcup_{\alpha \in A} B_\alpha}^{\tau} \cap (\bigcup_{\alpha \in A} G_\alpha)$$

has local diameter less than ε. Consider

$$x \in \overline{\bigcup_{\alpha \in A} B_\alpha}^{\tau} \cap (\bigcup_{\alpha \in A} G_\alpha) = \bigcup_{\alpha} (\overline{B_\alpha}^{\tau} \cap G_\alpha).$$

Then there exists α_0 such that $x \in G_{\alpha_0}$ and

$$\operatorname{diam}(\overline{\bigcup_{\alpha \in A} B_\alpha}^{\tau} \cap (\bigcup_{\alpha \in A} G_\alpha) \cap G_{\alpha_0}) = \operatorname{diam}(\bigcup_{\alpha \in A} (\overline{B_\alpha}^{\tau} \cap G_\alpha) \cap G_{\alpha_0}) =$$

$$= \operatorname{diam}(\overline{B_{\alpha_0}}^{\tau} \cap G_{\alpha_0}) \leq \operatorname{diam}(\overline{B_{\alpha_0}}^{\tau}) \leq \varepsilon.$$

So we have

$$X = \bigcup_{n \in N} \bigcup_{m \in N} (\bigcup_{\alpha \in A} M_\alpha^{n,m}),$$

where for each $n, m \in N$ the set

$$\bigcup_{\alpha \in A} M_\alpha^{n,m}$$

is a difference of two τ-closed sets and has local diameter less than ε. ∎

In [9], Theorem 2.4, the "moreover" part of our theorem was proved for σ-fragmentability. We do not know whether the whole statement in Theorem 1.3.1 holds for σ-fragmentable topological spaces or not.

As a corollary of the previous result we have:

Theorem 1.3.2 *Let X be a Banach space. If $(X, weak)$ has a countable cover by sets of small local norm-diameter, then $(X, weak)$ has a countable cover by differences of weakly closed sets of small local norm-diameter and also by norm closed sets of small local norm-diameter.*

Proof. Note that by Theorem 1.3.1, for each ε we have

$$X = \bigcup_{i=1}^{\infty}(F_i \cap G_i),$$

where the F_i are closed and the G_i are open in the weak topology, and therefore in the norm topology. Now each G_i is a F_σ-set, say $G_i = \cup\{H_{i,n} : n \in N\}$, with the $H_{i,n}$'s being closed sets. So we have

$$X = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} F_i \cap H_{i,n},$$

and clearly for each $i, n \in N$ the set $F_i \cap H_{i,n}$ has local diameter less than ε.■

1.4 Borel sets

Let $(X, \|\cdot\|)$ be a Banach space. The norm $\|\cdot\|$ is called a *Kadec* norm if the weak and norm topologies agree on $\{x \in X : \|x\| = 1\}$. It was shown by Edgar that if a Banach space X admits an equivalent Kadec norm then:

i) $Borel(X, \|\cdot\|) = Borel(X, weak)$, ([3], Theorem 1.1);

ii) $X \in Borel(X^{**}, weak^*)$, ([4], Corollary 2.3).

On the other hand, Jayne, Namioka and Rogers proved, ([9], Theorem 2.3), that a Banach space that admits an equivalent Kadec norm has a countable cover by sets that are differences of weakly closed sets of small local diameter.

We improve Edgar's result by showing that a Banach space with the JNR property verifies ii) above. Moreover, any norm-closed subset of the space is also a Borel subset of its bidual when considered with the weak* topology, from which we get i) above as a corollary.

Remark 1.4.0 It was proved in [9], p. 215, and [11] that a topological space (X,τ) which has a countable cover by differences of τ-closed sets of small local ϱ-diameter, for some metric ϱ, any ϱ-open set can be written as a countable union of differences of τ-closed sets and therefore the Borel structures for both the τ-topology and the ϱ-topology agree.

Theorem 1.4.1 *Let X be a Banach space and suppose that $(X, weak)$ has a countable cover by sets of small local diameter. Then X is the countable intersection of countable unions of differences of w^*-closed sets in X^{**} and therefore $X \in Borel(X^{**}, w^*)$. Moreover, any closed subset of X is of the same type.*

Proof. Let $\{U^p_{\alpha,n} : \alpha \in A, n \in N\}$ be a σ-discrete refinement of a cover of X by balls of diameter less than $\frac{1}{p}$. We can suppose that each $\{U^p_{\alpha,n} : \alpha \in A\}$ is metrically discrete with separating distances $\delta_{n,p} > 0$.

For $n, p \in N$ let $\{C^p_{n,m} : m \in N\}$ be a countable cover of X with the local diameter of each $C^p_{n,m} < \delta_{n,p}$. Then for each $n, m, p \in N$ the family $\{U^p_{\alpha,n} \cap C^p_{n,m}\}_{\alpha \in A}$ is weakly discrete in

$$\bigcup_{\alpha \in A} (U^p_{\alpha,n} \cap C^p_{n,m}),$$

in fact it is discrete in $C^p_{n,m}$. So for each $x \in U^p_{\alpha,n} \cap C^p_{n,m}$ there exists a w^*-open neighbourhood of x in X^{**}, say $U^{n,m,p}_{x,\alpha}$, such that

$$U^{n,m,p}_{x,\alpha} \cap (U^p_{\alpha,n} \cap C^p_{n,m}) \neq \emptyset,$$

$$U^{n,m,p}_{x,\alpha} \cap (U^p_{\beta,n} \cap C^p_{n,m}) = \emptyset, \text{ for } \beta \neq \alpha.$$

Set

$$G^{m,p}_{\alpha,n} = \bigcup_{x \in U^p_{\alpha,n} \cap C^p_{n,m}} U^{n,m,p}_{x,\alpha}.$$

$G^{m,p}_{\alpha,n}$ is clearly a w^*-open set with

$$G^{m,p}_{\alpha,n} \supset (U^p_{\alpha,n} \cap C^p_{n,m})$$

and

$$G^{m,p}_{\alpha,n} \cap (U^p_{\beta,n} \cap C^p_{n,m}) = \emptyset, \text{ for } \beta \neq \alpha.$$

Set $M_{\alpha,m,n,p} = \overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap G^{m,p}_{\alpha,n}$.

We show that

$$X = \bigcap_p \bigcup_m \bigcup_n \{ \bigcup_{\alpha \in A} M_{\alpha,m,n,p} \}.$$

Notice that, since $\operatorname{diam}(U^p_{\alpha,n} \cap C^p_{n,m}) \leq \frac{1}{p}$, we have

$$\operatorname{diam}(\overline{U^p_{\alpha,n} \cap C^p_{n,m}}^{w^*}) \leq \frac{1}{p}.$$

Let

$$x^{**} \in X^{**} \cap \bigcap_p \bigcup_m \bigcup_n \Big(\bigcup_{\alpha \in A} M_{\alpha,n,m,p} \Big).$$

For each $p \in N$ there exist $n, m \in N$ and $\alpha \in A$ such that

$$x^{**} \in \overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap G^p_{\alpha,n,m},$$

so there exists $x_p \in U^p_{\alpha,n} \cap C^p_{n,m}$ such that $\|x^{**} - x_p\| \leq \frac{1}{p}$. Thus we have

$$\|\cdot\|\text{-}\lim_{p \to \infty} x_p = x^{**}$$

and therefore $x^{**} \in X$.

We now show that

$$\bigcup_{\alpha \in A} \big(\overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap G^{m,p}_{\alpha,n} \big) = \overline{\bigcup_{\alpha \in A} (U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap \bigcup_{\alpha \in A} G^{m,p}_{\alpha,n}.$$

We have

$$\bigcup_{\alpha \in A} \big(\overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap G^{m,p}_{\alpha,n} \big) = \bigcup_{\alpha \in A} \overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap \bigcup_{\alpha \in A} G^{m,p}_{\alpha,n},$$

since $G^{m,p}_{\alpha,n} \cap (U^p_{\beta,n} \cap C^p_{n,m}) = \emptyset$, for $\beta \neq \alpha$. So we must check that

$$\overline{\bigcup_{\alpha \in A} (U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap [\bigcup_{\alpha \in A} G^{m,p}_{\alpha,n}] \subset \bigcup_{\alpha \in A} \overline{(U^p_{\alpha,n} \cap C^p_{n,m})}^{w^*} \cap [\bigcup_{\alpha \in A} G^{m,p}_{\alpha,n}].$$

Set $B_\alpha = U^p_{\alpha,n} \cap C^p_{n,m}$ and $G_\alpha = G^{m,p}_{\alpha,n}$, and suppose that there exists

$$x \in \overline{\bigcup_{\alpha \in A} B_\alpha}^{w^*} \cap [\bigcup_{\alpha \in A} G_\alpha]$$

such that

$$x \notin \bigcup_{\alpha \in A} \overline{B_\alpha}^{w^*} \cap [\bigcup_{\alpha \in A} G_\alpha]$$

There must be a w^*-open neighbourhood U_x of x, since

$$x \in \bigcup_{\alpha \in A} G_\alpha,$$

such that $U_x \subset G_{\alpha_0}$ for some $\alpha_0 \in A$ and

$$U_x \cap (\bigcup_{\alpha \in A} B_\alpha) \neq \emptyset.$$

Thus $U_x \cap B_{\alpha_0} \neq \emptyset$, since $G_{\alpha_0} \cap B_\beta = \emptyset$ for $\beta \neq \alpha_0$.

Suppose now that there exists a w^*-open neighbourhood V_x of x such that

$$V_x \cap B_{\alpha_0} = \emptyset.$$

Choose U_x with $\emptyset \neq U_x \cap V_x \subset G_{\alpha_0}$. Since $U_x \cap V_x$ is a w^*-open neighbourhood of x and

$$x \in \overline{\bigcup_{\alpha \in A} B_\alpha}^{w^*},$$

we have

$$(U_x \cap V_x) \cap (\bigcup_{\alpha \in A} B_\alpha) \neq \emptyset,$$

which implies again that $U_x \cap V_x \cap B_{\alpha_0} \neq \emptyset$. This is a contradiction.

If F is a norm closed subset of X, consider the families

$$\{F \cap U^p_{\alpha,n} \cap C^p_{n,m}\}_{\alpha \in A}$$

and follow the proof. In this case the vectors x_p's belong to F and, since F is closed, the limit also belongs to F. ∎

Our next result follows also from Theorem 1.3.2 and Remark 1.4.0 but we give a shorter proof here by using Theorem 1.4.1.

Corollary 1.4.1 *Let $(X, \|\cdot\|)$ be a Banach space with a countable cover by sets of small local diameter, then $Borel(X, \|\cdot\|) = Borel(X, weak)$.*

Proof. Let A be a norm-closed subset of X. By Theorem 1.4.1, $A \in Borel(X^{**}, weak^{*})$, i.e. there exists $B \in X^{**}$, $B \in Borel(X^{**}, weak^{*})$ and $A = B \cap X$. But the weak topology on X coincides with the restriction to X of the weak* topology on X^{**} and therefore A is a weak-Borel subset of X. ■

At this point we need to give another definition which can be found in [16], pages 345-346, as well as some basic properties.

Definition 1.4.1 Let F_0 be the family of closed sets of a metric space. Suppose that for an ordinal number α, we have defined the families F_ξ for $\xi < \alpha$. So the sets of the family F_α are countable intersections or unions of sets belonging to F_ξ with $\xi < \alpha$ according to whether α is even or odd (the limit ordinals are understood to be even). It is known that the transfinite union of this families gives us the family of all Borel sets.

We can also do the same using open sets. Set G_0 to be the family of open sets. Suppose that for an ordinal number α, we have defined the families G_ξ for $\xi < \alpha$. So the sets of the family G_α are countable unions or intersections of sets belonging to G_ξ with $\xi < \alpha$ according to whether α is even or odd.

The families F_α with even indices as well as the families G_α with odd indices are countably multiplicative, which means that, given a sequence of sets of the family, its intersection belongs to the same family. The sets belonging to such a family will be said to be of *multiplicative class* α. Similarly, the families F_α with odd indices as well as the families G_α with even indices are additive and form the *additive class* α.

Our last result of this section is a generalization of the one mentioned above from [11]. The ideas for the proof also come from there. It will be used later on in Chapter 2.

Lemma 1.4.1 *Let (X,τ) be a topological space such that any open set is an F_σ-set. Let ϱ be a metric on X and suppose that X has a countable cover by sets of additive class α of small local ϱ-diameter, then each ϱ-open subset of X is of additive class α.*

Proof. Let G be a ϱ-open subset of X. Let D_m, $m \geq 1$, be a countable cover of X by sets of additive class α of small local diameter.

For $n \geq 1$ set

$$M(n) = \{m \in N : m \geq 1 \text{ and } D_m \text{ having local diameter less than } \frac{1}{n}\}.$$

Then for each $n \geq 1$ the family $\{D_m : m \in M(n)\}$ covers X.

Write

$$G_n = \{x \in X : \{y \in X : \varrho(y,x) \subset G\}\}.$$

For each $n \geq 1$ and each $m \geq 1$ in $M(n)$, we consider the points $x \in G_n \cap D_m$. Since the local ϱ-diameter of D_m is less than $\frac{1}{n}$, we can choose a relatively open subset $U(x)$ of D_m, containing x and having ϱ-diameter less than $\frac{1}{n}$.

Thus $x \in U(x) \subset G$. Hence, for $m \geq 1$ and $m \in M(n)$ the set

$$U_{n,m} = \cup\{U(x) : x \in G_n \cap D_m\}$$

is a relatively open subset of D_m. So for every $n, m \in N$ there exist $V_{n,m}$, τ-open subsets of X, such that $U_{n,m} = D_m \cap V_{n,m}$. Now we have

$$V_{n,m} = \bigcup_{i=1}^{\infty} F_i^{n,m},$$

where the sets $F_i^{n,m}$ are τ-closed in X. Hence

$$U_{n,m} = \bigcup_{i=1}^{\infty} (F_i^{n,m} \cap D_m),$$

and therefore the sets $U_{n,m}$ are of additive class α. They contain $G_n \cap D_m$ and are contained in G. Hence

$$\cup\{U_{n,m} : n \geq 1,\ m \in M(n)\}$$

is a set of additive class α that coincides with G. ∎

1.5 Decomposition of discrete families

To finish this chapter, we will give characterizations of both the property of having a countable cover by sets of small local diameter and σ-fragmentability which will be of help in Chapter 2.

Proposition 1.5.1 *Let (X,τ) be a topological space and ϱ be a metric on X. Then the following conditions are equivalent:*

i) *(X,τ) has a countable cover by sets of small local ϱ-diameter;*

ii) *There exists a decomposition of X,*

$$X = \bigcup_{i=1}^{\infty} C_i$$

such that if $\mathcal{A}=\{A_s\}_{s\in S}$ is a ϱ-discrete family of subsets of X, then for each $s \in S$ there is a decomposition

$$A_s = \bigcup_{i=1}^{\infty} A_s^i,$$

with $A_s^i \subset C_i$ such that each family $\{A_s^i\}_{s\in s}$ is discrete in (C_i,τ).

Proof. i)$\Rightarrow$ ii) For each $n \in N$, consider

$$X = \bigcup_{i=1}^{\infty} C_i^n,$$

where the sets C_i^n have local ϱ-diameter less than $\frac{1}{n}$.

Let $\mathcal{A}=\{A_s\}_{s\in S}$ be a discrete family in (X,ϱ). Then there exist $\frac{1}{m}$-discrete families $\{A_s^m\}_{s\in S}$ such that

$$A_s = \bigcup_{m=1}^{\infty} A_s^m, \text{ for } s \in S.$$

Write $\{A_s^{i,m}\} = A_s^m \cap C_i^m$, for $m, i \in N$ and $s \in S$ and fix $i, m \in N$. Take $x \in C_i^m$. There exists a τ-open neighbourhood of x, say U, such that

$$\operatorname{diam}(U \cap C_i^m) < \frac{1}{m}.$$

Therefore U meets at most one element of the family $\{A_s^{i,m}\}_{s\in S}$. So $\{A_s^{i,m}\}_{s\in S}$ is discrete in (C_i^m, τ) and we have

$$A_s = \bigcup_{m=1}^{\infty} A_s^m = \bigcup_{m=1}^{\infty}\bigcup_{i=1}^{\infty} A_m^s \cap C_i^m = \bigcup_{i,m} A_s^{i,m}.$$

Set $C_i^m = F_n$ and $A_s^{i,m} = B_s^n$. So we have

$$A_s = \bigcup_{n=1}^{\infty} B_s^n, B_s^n \subset F_n,$$

and $\{B_s^n\}_{s\in S}$ is discrete in (F_n, τ).

ii)⇒ i) Given $\varepsilon > 0$, let $\{U_\alpha^n : \alpha \in \Gamma\}$ be a σ-discrete open refinement of an open cover of X by balls of radius less than $\frac{\varepsilon}{2}$. Let $\{C_m\}_{m=1}^{\infty}$ be a countable cover of X such that for $n, m \in N$,

$$U_\alpha^n = \bigcup_{m=1}^{\infty} B_\alpha^{n,m}, \quad B_\alpha^{n,m} \subset C_m$$

and $\{B_\alpha^{n,m}\}_{\alpha\in\Gamma}$ is discrete in (C_m, τ). Write

$$F_m^n = \bigcup_{\alpha\in\Gamma} B_\alpha^{n,m}.$$

Obviously,

$$X = \bigcup_{n,m} F_m^n.$$

We now show that for each $n, m \in N$, the set F_m^n has local diameter less than ε.

Take $x \in F_m^n$. Then $x \in B_{\alpha_0}^{n,m}$ and therefore there exists a τ-open neighbourhood U of x such that

$$U \cap C_m \cap B_\alpha^{n,m} = \emptyset \text{ for } \alpha \neq \alpha_0.$$

So $\text{diam}(U \cap F_m^n) = \text{diam}(U \cap B_{\alpha_0}^{n,m}) \leq \text{diam}(B_{\alpha_0}^{n,m}) < \varepsilon$. ∎

We give an analogous statement for the case of the space being σ-fragmented. We omit the proof because it follows the same line as the previous one.

Proposition 1.5.2 *Let (X, τ) be a topological space and ϱ be a metric on X. Then the following conditions are equivalent:*

i) *(X, τ) is σ-fragmented by ϱ;*

ii) *There exists a decomposition of X,*

$$X = \bigcup_{i=1}^{\infty} C_i,$$

such that if $\mathcal{A} = \{A_s\}_{s \in S}$ is a ϱ-discrete family of subsets of X, then for each $s \in S$ there is a decomposition

$$A_s = \bigcup_{i=1}^{\infty} A_s^i, \ A_s^i \subset C_i$$

such that each family $\{A_s^i\}_{s \in S}$ satisfies the following condition:

for any $i \in N$, if $\cup\{A_s^i : s \in S\} \neq \emptyset$, then for every non-empty subset A of $\cup\{A_s^i : s \in S\}$ there exists a τ-open subset U of X such that $U \cap A \subset A_s^i$ for exactly one $s \in S$.

Chapter 2

One-to-one maps

2.1 Introduction

Suppose we have two Banach spaces X and Y, and a one-to-one map $T : X \longrightarrow Y$. If we assume that Y has a countable cover by sets of small local diameter, what kind of condition do we have to impose on T for X to have such a cover as well? What can we say about T^{-1}? We shall give some answers to these questions in this chapter, but first of all we have to fix some notation.

If τ_1 and τ_2 are two topologies on a topological space H, we shall say that τ_1 is stronger than τ_2, denoted by $\tau_2 \preceq \tau_1$, if any τ_2-open subset of H is also τ_1-open.

Let (X, τ_1) and (Y, τ_2) be topological spaces and ϱ_1, ϱ_2 be metrics defined on X and Y, respectively. If $T : X \longrightarrow Y$ is a one-to-one map, we define: T_{τ_2}

to be the topology on X given by the family $\{T^{-1}(U) : U\ \tau_2\text{-open}\}$ and T_{ϱ_2} to be the topology on X associated to the metric $T \circ \varrho_2$, i.e, the family $\{T^{-1}(U) : U\ \varrho_2\text{-open}\}$. Note that if T is τ_1 -τ_2 continuous (respectively ϱ_1 -ϱ_2 continuous), then $T_{\tau_2} \preceq \tau_1$ (resp. $T_{\varrho_2} \preceq \varrho_1$).

Definition 2.1.1 *We shall say that a map T, as above, is discretely σ-decomposable, d. σ-d. for short, for the pair (ϱ_1, ϱ_2) if it is one-to-one and it maps ϱ_1-discrete families of subsets of X into ϱ_2-d. σ-d. families of subsets of Y.*

These maps were used by Hansell, see [6], where we refer for further properties, without imposing on the map the condition of being one-to-one. It is easy to see from Remarks 1.1.4 that ϱ_2-d. σ-d. families in $T(X)$ are also ϱ_2-d. σ-d. in Y.

2.2 d. σ-d. maps

We now prove a lemma which will be used in Theorem 2.2.2 below.

Lemma 2.2.1 *Let (X, τ_1) and (Y, τ_2) be topological spaces and ϱ_1, ϱ_2 be metrics defined on X and Y, respectively. Suppose that there exists a d. σ-d. map $T : X \longrightarrow Y$ for the pair (ϱ_1, ϱ_2). If (Y, τ_2) has a countable cover by sets of small local ϱ_2-diameter, then there exists a decomposition of X,*

$$X = \bigcup_{i=1}^{\infty} E_i,$$

such that if $\mathcal{A}= \{A_s\}_{s\in S}$ is a ϱ_1-discrete family of subsets of X, for each $s \in S$, *then there is a decomposition*

$$A_s = \bigcup_{i=1}^{\infty} A_s^i, \;\; A_s^i \subset E_i,$$

such that each family $\{A_s^i\}_{s\in S}$ *is discrete in the space* (E_i, T_{τ_2}).

Proof. Consider the map $T : X \longrightarrow Y$. Then the family $T\mathcal{A}$ is d. σ-d. Therefore there exit discrete families $\{B_s^p\}_{s\in S}$, $p \in N$, such that

$$TA_s = \bigcup_{p=1}^{\infty} B_s^p.$$

By Proposition 1.5.1 there exists a decomposition of Y,

$$Y = \bigcup_{n=1}^{\infty} C_n,$$

and discrete families $\{B_s^{n,p}\}_{s\in S}$ in the space (C_n, τ_2), with

$$B_s^p = \bigcup_{n=1}^{\infty} B_s^{n,p}, \text{ and } B_s^{n,p} \subset C_n \text{ for } p \in N, s \in S.$$

Since T is trivially T_{τ_2}-τ_2 continuous it follows that the family

$$T^{-1}(\{B_s^{n,p}\}_{s\in S})$$

is discrete in the space $(T^{-1}C_n, T\tau_2)$.

It is clear that

$$A_s = \bigcup_{p=1}^{\infty}\bigcup_{n=1}^{\infty} T^{-1}(B_s^{n,p}) = \bigcup_{i=1}^{\infty} A_s^i,$$

where $A_s^i = T^{-1}(B_s^{n,p})$, and $E_i = C_n$ for some $n, p \in N$. ■

We now give the analogous lemma for σ-fragmentability.

Lemma 2.2.2 *Let (X, τ_1) and (Y, τ_2) be topological spaces and let ϱ_1, ϱ_2 be metrics defined on X and Y, respectively. Suppose that there exists a d. σ-d. map $T : X \longrightarrow Y$ for the pair (ϱ_1, ϱ_2). If (Y, τ_2) is σ-fragmented by ϱ_2, then there exists a decomposition of X,*

$$X = \bigcup_{i=1}^{\infty} E_i,$$

such that if $\mathcal{A} = \{A_s\}_{s \in S}$ is a ϱ_1-discrete family of subsets of X, then for each $s \in S$, there is a decomposition

$$A_s = \bigcup_{i=1}^{\infty} A_s^i, \;\; A_s^i \subset E_i,$$

such that each family $\{A_s^i\}_{s \in S}$ satisfies the following condition:

> *for any $i \in N$, if $\cup\{A_s^i : s \in S\} \neq \emptyset$, then for every non-empty subset A of $\cup\{A_s^i : s \in S\}$ there exists a τ-open subset U of X such that $U \cap A \subset A_s^i$ for exactly one $s \in S$.*

Theorem 2.2.1 *Let (X, τ_1) and (Y, τ_2) be topological spaces and let ϱ_1, ϱ_2 be metrics defined on X and Y, respectively. Suppose that there exists a one-to-one map $T : X \longrightarrow Y$. Then the following conditions are equivalent:*

i) *T is d. σ-d. for the pair (ϱ_1, ϱ_2);*

ii) *(X, T_{ϱ_2}) has a countable cover by sets of small local ϱ_1-diameter;*

iii) (X, T_{ϱ_2}) *is σ-fragmented by* ϱ_1.

Proof. i)$\Rightarrow$ ii) Given $\varepsilon > 0$, let $\{U_\alpha^n : \alpha \in \Gamma\}$ be a σ-discrete open refinement of an open cover of X by balls of ϱ_1-radius less than $\frac{\varepsilon}{2}$.

For every $n \in N$ the family $\{T(U_\alpha^n)\}_{\alpha\in\Gamma}$ is d. σ-d. in Y. Thus we have

$$T(U_\alpha^n) = \bigcup_{m=1}^{\infty} B_\alpha^{n,m},$$

with $\{B_\alpha^{n,m}\}_{\alpha\in\Gamma}$ being ϱ_2-discrete.

Define $E_\alpha^{n,m} = T^{-1}(B_\alpha^{n,m}) \subset U_\alpha^n$. It is clear that

$$X = \bigcup_{n,m} \bigcup_{\alpha} E_\alpha^{n,m}.$$

We show that for any $n, m \in N$ the set

$$\bigcup_{\alpha\in\Gamma} E_\alpha^{n,m}$$

has local ϱ_1-diameter less than ε.

So take $x \in E_{\alpha_0}^{n,m}$. Then $Tx \in B_{\alpha_0}^{n,m}$, and so there exists a ϱ_2-open set, say V, such that $V \cap B_\alpha^{n,m} = \emptyset$ for $\alpha \neq \alpha_0$. Now take $G = T^{-1}(V)$, which is T_{ϱ_2}-open, and we have

$$\varrho_1\text{-diam}(G \cap (\bigcup_{\alpha\in\Gamma} E_\alpha^{n,m}) = \varrho_1\text{-diam}(G \cap E_{\alpha_0}^{n,m}) \leq \varrho_1\text{-diam}(U_{\alpha_0}^{n,m}) \leq \varepsilon.$$

ii)$\Leftrightarrow$ iii) Proposition 1.2.1.

ii)$\Rightarrow$ i) Because of Proposition 1.5.1, any ϱ_1-discrete family, $\mathcal{A}= \{A_s\}_{s\in S}$ can be decomposed in the following way:

$$A_s = \bigcup_{i=1}^{\infty} A_s^i,$$

with the family $\{A_s^i\}_{s\in S}$ being T_{ϱ_2}-discrete in its union. Since T_{ϱ_2} is a metric on X, it follows that these families are T_{ϱ_2}-d. σ-d. in X and so $\mathcal{A}$ is also T_{ϱ_2}-d. σ-d. Hence the map T is d. σ-d. for the pair (ϱ_1, ϱ_2). ∎

Theorem 2.2.2 *Let (X, τ_1) and (Y, τ_2) be topological spaces and let ϱ_1, ϱ_2 be metrics defined on X and Y, respectively, with $\tau_2 \preceq \varrho_2$. Suppose that there exists a one-to-one map $T : X \longrightarrow Y$ and that (Y, τ_2) has a countable cover by sets of small local ϱ_2-diameter. Then the following conditions are equivalent:*

i) *T is d. σ-d. for the pair (ϱ_1, ϱ_2);*

ii) *(X, T_{ϱ_2}) has a countable cover by sets of small local ϱ_1-diameter;*

iii) *(X, T_{ϱ_2}) is σ-fragmented by ϱ_1;*

iv) *(X, T_{τ_2}) has a countable cover by sets of small local ϱ_1-diameter;*

v) *(X, T_{τ_2}) is σ-fragmented by ϱ_1.*

Proof. i)⇒iv) For each $\varepsilon > 0$ and $p \in N$, let $\mathcal{A}_p = \{A_s^p\}_{s\in S}$ be discrete families in X such that $\cup\{A_p : p \in N\}$ is a refinement of a cover of X by balls of ϱ_1-radius less than $\frac{\varepsilon}{2}$.

By Lemma 2.2.1 there exists a number of sets C_j, with

$$X = \bigcup_{j=1}^{\infty} C_j,$$

and decompositions

$$A_s^p = \bigcup_{j=1}^{\infty} A_s^{p,j}\ , s \in S, p \in N,\ A_s^{p,j} \subset C_j$$

such that each family $\{A_s^{p,j}\}_{s\in S}$ is discrete in (C_j, T_{τ_2}).

Now define

$$B_j^p = \bigcup_{s\in S} A_s^{p,j}.$$

It is clear that

$$X = \bigcup_{j=1}^{\infty}\bigcup_{p=1}^{\infty} B_j^p.$$

We show that the family $\{B_j^p\}_{j,p}$ is a countable cover of X by sets of small local ϱ_1-diameter. Let $x \in B_j^p$. Since the family $\{A_s^{p,j}\}_{s\in S}$ is discrete in (C_j^p, T_{τ_2}) there exists a T_{τ_2} open neighbourhood V of x such that

$$(V \cap C_j) \cap A_s^{p,j} \neq \emptyset,$$

for some $s_0 \in S$ and

$$(V \cap C_j) \cap A_s^{p,j} = \emptyset, \text{ for } s \neq s_0.$$

If $V \cap C_j \cap A_s^{p,j} = \emptyset$ for every $s \in S$, we would have

$$V \cap C_j \cap (\bigcup_{s\in S} A_s^{p,j}) = \emptyset,$$

and this would imply that $x \notin B_j^p$, which is a contradiction.

So there exists $s_0 \in S$ such that

$$V \cap C_j \cap A_{s_0}^{p,j} \neq \emptyset \text{ and } V \cap C_j \cap A_s^{p,j} = \emptyset, \text{ for } s \neq s_0.$$

Now, since $A_{s_0}^{p,j}$ is contained in a ball of radius less than $\frac{\varepsilon}{2}$, ϱ_1-diam$(A_{s_0}^{p,j}) \leq \varepsilon$. Since $V \cap C_j \cap A_s^{p,j} = \emptyset$ for $s \neq s_0$, we have

$$\varrho_1\text{-diam}(V \cap B_j^p) = \varrho_1\text{-diam}(V \cap (\bigcup_{s \in S} A_s^{p,j})) = \varrho_1\text{-diam}(V \cap A_{s_0}^{p,j}) \leq \varepsilon.$$

iv)⇒v) Obvious.

v)⇒iii) Obvious. ■

Theorem 2.2.3 *Let (X, τ_1) and (Y, τ_2) be topological spaces and let ϱ_1, ϱ_2 be metrics defined on X and Y, respectively, with $\tau_2 \preceq \varrho_2$. Suppose that there exists a one-to-one map $T : X \longrightarrow Y$ and that (Y, τ_2) is σ-fragmented by ϱ_2. Then the following conditions are equivalent:*

i) *T is d. σ-d. for the pair (ϱ_1, ϱ_2);*

ii) *(X, T_{ϱ_2}) has a countable cover by sets of small local ϱ_1-diameter;*

iii) *(X, T_{ϱ_2}) is σ-fragmented by ϱ_1;*

iv) *(X, T_{τ_2}) is σ-fragmented by ϱ_1.*

Proof. i)⇒iv) The same proof as above but using Lemma 2.2.2.

iv)⇒iii) Obvious. ■

Remark 2.2.1 Note that if in Theorem 2.2.2 (Theorem 2.2.3) the map is, for instance, τ_1-τ_2 continuous, then we obtain that (X, τ_1) has a countable cover by sets of small local ϱ_1-diameter (resp. (X, τ_1) is σ-fragmented by ϱ_1).

2.3 Inverse mappings

We start by recalling a definition from [16].

Definition 2.3.1 *A mapping $f : X \longrightarrow Y$ is said to be of Borel class α if, for every closed subset $F \subset Y$, the set $f^{-1}(F)$ is Borel of multiplicative class α. (Equivalently, $f^{-1}(G)$ is of additive class α, for every open set G).*

Our next result improves part of a result in [14], Corollary 7.

Theorem 2.3.1 *Let X, Y be two Banach spaces and $f : X \longrightarrow Y$ be a continuous linear injection. Define $\varphi = f^{-1} : f(X) \longrightarrow X$. Then the following conditions are equivalent:*

i) *φ is of Borel class α and f is d. σ-d.;*

ii) *$(X, f_{\|\cdot\|_Y})$ has a countable cover by sets of additive class α (for the topology $f_{\|\cdot\|_Y}$) of small local $\|\cdot\|_X$-diameter.*

Proof. ii)$\Rightarrow$ i) By Theorem 2.2.2 we have that f is d. σ-d.

Now let G be a norm open subset of X. Since the topology $f_{\|\cdot\|_Y}$ in X verifies that any open set is an F_σ set, we apply Lemma 1.4.1 and obtain that G is of additive class α in $(X, f_{\|\cdot\|_Y})$. Hence the set $\varphi^{-1}(G)$ is of additive class α in $(f(X), \|\cdot\|_Y)$ and therefore φ is of Borel class α.

i)$\Rightarrow$ ii) For $n \in N$ let $\{U_\alpha^n : \alpha \in A\}$ be an open σ-discrete refinement of an open cover of X by balls of radius less than $\frac{1}{n}$. Consider $B_\alpha^n = f(U_\alpha^n)$. Then for each $n \in N$ the family $\{B_\alpha^n\}_{\alpha \in A}$ is d. σ-d., so for every $n, i \in N$ there exist discrete families $\{B_\alpha^{i,n}\}_{\alpha\in A}$ such that

$$B_\alpha^n = \bigcup_{i=1}^{\infty} B_\alpha^{i,n}.$$

Note that since the sets U_α^n are open in X and φ is of Borel class α, the sets $B_\alpha^n = \varphi^{-1}(U_\alpha^n)$ are of additive class α.

Now fix $i, n \in N$. The family $\{\overline{B_\alpha^{i,n}} \cap B_\alpha^n\}_{\alpha\in A}$ is discrete in $f(X)$ and its sets are of additive class α.

Set

$$\Xi_{i,n} = \bigcup_{\alpha\in A} (\overline{B_\alpha^{i,n}} \cap B_\alpha^n).$$

Then $\Xi_{i,n}$ is a discrete union of sets of additive class α and therefore is itself of additive class α. (For a proof of this fact see [16], page 358, Theorem 1). It is obvious that

$$f(X) = \bigcup_{i,n} \Xi_{i,n}.$$

Define $C_{i,n} = f^{-1}(\Xi_{i,n})$. The sets $C_{i,n}$'s are of additive class α and they form a countable cover of $(X, f_{\|\cdot\|_Y})$.

Now take $x \in C_{i,n}$ and $f(x) = y \in \Xi_{i,n}$. So $y \in \overline{B_{\alpha_0}^{i,n}} \cap B_{\alpha_0}^n$ for only one $\alpha_0 \in A$. Thus there exists an open neighbourhood U of y in Y such that $U \cap (\overline{B_\alpha^{i,n}} \cap B_\alpha^n) = \emptyset$ for $\alpha \neq \alpha_0$.

Write $V = f^{-1}(U)$. V is a $f_{\|\cdot\|_Y}$-open neighbourhood of x and

$$\operatorname{diam}(V \cap C_{i,n}) = \operatorname{diam}(f^{-1}(U \cap \Xi_{i,n})) = \operatorname{diam}(f^{-1}(U \cap \overline{B_{\alpha_0}^{i,n}} \cap B_{\alpha_0}^n)) \leq$$

$$\leq \operatorname{diam}(f^{-1}(B_\alpha^n)) = \operatorname{diam}(U_\alpha^n) < \varepsilon.$$

Therefore $C_{i,n}$ has local diameter less than ε. ∎

2.4 Some topological invariants

The following lemma provides us with a useful tool to check when some maps are d. σ-d. For a proof of it see [17].

Lemma 2.4.1 *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $T : X \longrightarrow Y$ be a one-to-one map such that for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X converging to some point x in the $T_{\|\cdot\|_Y}$ topology we have that x_n converges weakly (or pointwise in the case of X being a $C(K)$ space) to x. Then T is d. σ-d.*

Theorem 2.4.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Suppose that there exists an homeomorphism $\phi : (X, weak) \longrightarrow (Y, weak)$. Then the following results hold.*

i) *$(X, weak)$ has a countable cover by sets of small local $\|\cdot\|_X$-diameter if and only if $(Y, weak)$ has a countable cover by sets of small local $\|\cdot\|_Y$-diameter.*

ii) *$(X, weak)$ is σ-fragmented by $\|\cdot\|_X$ if and only if $(Y, weak)$ is σ-fragmented by $\|\cdot\|_Y$.*

iii) *$Borel(X, weak) = Borel(X, \|\cdot\|_X)$ if and only if $Borel(Y, weak) = Borel(Y, \|\cdot\|_Y)$.*

Proof. In this case it is clear that the conditions in Lemma 2.4.1 are fullfiled and so our map ϕ and its inverse are d. σ-d. Moreover, since there is weak to weak continuity, by Remark 2.2.1 and Theorems 2.2.2 and 2.2.3, i) and ii) hold. To prove iii) assume that $Borel(Y, weak) = Borel(Y, \|\cdot\|_Y)$.

Denote by $B_{(X,\|\cdot\|_X)}$ the closed unit ball of $(X, \|\cdot\|_X)$. $B_{(X,\|\cdot\|_X)}$ is a w-closed subset of X and, since ϕ is a homeomorphism, $\phi(B_{(X,\|\cdot\|_X)})$ is a w-closed subset of Y. Thus the norm $\|\cdot\|_X$ is lower semicontinuous on $(X, \phi_{\|\cdot\|_Y})$. Since ϕ is d. σ-d., $(X, \phi_{\|\cdot\|_Y})$ has a countable cover by differences of $\phi_{\|\cdot\|_Y}$-closed sets of small local $\|\cdot\|_X$-diameter. So if G is a $\|\cdot\|_X$-open subset of X, then, see Remark 1.4.0,

$$G = \bigcup_{i=1}^{\infty} C_i,$$

where C_i is the difference of two $\phi_{\|\cdot\|_Y}$-closed sets for every $i \in N$.

Set $B_i = \phi(C_i)$. Then the B_i's are differences of $\|\cdot\|_Y$-closed sets and therefore they are w-Borel sets. Thus $\phi^{-1}(B_i) = C_i$ are w-Borel in X. We conclude that G is a countable union of *weak*-Borel sets and therefore is itself a *weak*-Borel subet of X. ∎

2.5 The c_0-sum of some Banach spaces

Notation: *Let A be a set. We shall denote by $|A|$ the cardinality of the set A.*

Definition 2.5.1 *Let $\{X_n : n \in N\}$ be a sequence of Banach spaces with* *norms* $\{\|\cdot\|_n : n \in N\}$. *We denote by* $c_0\{X_n : n \in N\}$ *the Banach space of all sequences* $x = (x_n)_{n=1}^{\infty}$ *with* $x_n \in X_n$, *for* $n \in N$, *and such that given* $\varepsilon > 0$ *there exists* $m \in N$ *such that* $\|x_n\|_n \leq \varepsilon$ *for* $n \geq m$. *The norm in this space is*

$$\|x\|_\infty = sup\{\|x_n\|_n : n \in N\}.$$

In order to give our next result we need the next proposition. The proof of part i) can be found in [9], Theorem 6.1.

Theorem 2.5.1 *Let* $(X_n, \|\cdot\|_n)_{n=1}^{\infty}$ *be a sequence of Banach spaces. Set* $X = c_0\{X_n : n \in N\}$. *Then the following results hold.*

i) *If for each* $n \in N$ *the space* X_n *is* σ*-fragmented by its norm, then the space* X *is* σ*-fragmented by its norm.*

ii) *If for each* $n \in N$ *the space* X_n *has a countable cover by sets of small local* $\|\cdot\|_n$*-diameter, then* X *has a countable cover by sets of small local* $\|\cdot\|_\infty$*-diameter.*

Proof. For each $n \in N$, we can find a sequence $\mathcal{B}_n = (E_m^n)_{m=1}^{\infty}$ of subsets of X_n such that each point of X_n belongs to sets of $\mathcal{B}_n$ having arbitrarily small local $\|\cdot\|_n$-diameter. For $r, k \in N$ define

$$C_r^k = \{x = (x_n)_{n=1}^{\infty} \in X : |\{x_n : \|x_n\|_n > \frac{1}{k}\}| = r\}.$$

Denote by P_n the canonical projection from X onto X_n. For $n \in N$ denote by $\mathbf{m}_n$ an element of the form $(m_1, ..., m_n) \in \{1, ..., n\}^N$.

For $r \geq 1, k \geq 1$ define

$$A_{r,k}^{\mathbf{m}_n,\mathbf{d}_n} = C_r^k \cap P_{d_1}^{-1}(E_{m_1}^{d_1}) \cap ... \cap P_{d_r}^{-1}(E_{m_r}^{d_r}),$$

for $\mathbf{m}_r = (m_1, ..., m_r)$, $\mathbf{d}_r = (d_1, ..., d_r)$ and

$$A_{0,k} = \{x \in X : \|x\|_\infty \leq \frac{1}{k}\}.$$

It's clear that

$$X = \bigcup_{k=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcup_{\mathbf{d}_r \in \{1,...,r\}^N} \bigcup_{\mathbf{m}_r \in \{d_1,...,d_r\}^N} A_{r,k}^{\mathbf{m}_r,\mathbf{d}_r} \cup (\bigcup_{k=1}^{\infty} A_{0,k}).$$

So we have a countable collection of sets covering X and now we will show that given $x = (x_n)_{n=1}^{\infty} \in X$ and $\varepsilon > 0$ there exist $r, k, \mathbf{d}_r, \mathbf{m}_r$ such that

$$x \in A_{r,k}^{\mathbf{m}_r,\mathbf{d}_r}$$

and the local $\|\cdot\|_\infty$-diam$(A_{r,k}^{\mathbf{m}_r,\mathbf{d}_r}) \leq \varepsilon$.

Take $k \in N$ such that $\frac{1}{k} < \frac{\varepsilon}{4}$. Then either $\|x\|_\infty \leq \frac{1}{k}$ or there exists $r \in N$ such that $x \in C_r^k$. In the first case, $x \in A_{0,k}$ and obviously the local $\|\cdot\|_\infty$-diameter of $A_{0,k}$ is less or equal than ε.

So suppose that there exists $r \in N$ such that $x \in C_r^k$, i.e., $\|x_i\|_i > \frac{1}{k}$ for $i = d_1, ..., d_r$. Let

$$\delta = min\{\frac{\|x_i\|_i - \frac{1}{k}}{2}, \frac{\varepsilon}{2} : i = d_1, ..., d_r\}.$$

Now for $i \in \{d_1, ..., d_r\}$ we have that $x_i \in X_i$. Thus we can find $m_i \in N$ and $E_{m_i}^i \ni x$ such that the local $\|\cdot\|_i$-diam$(E_{m_i}^i) \leq \delta$. Hence there exists a weak open neighbourhood of x_i in X_i, say V_i, such that

$$\|\cdot\|_i\text{-diam}(V_i \cap E^i_{m_i}) \leq \delta.$$

Set

$$V = \bigcap_{i=1}^{r} P_i^{-1}(V_i).$$

V is a weak open neighbourhood of x in X. We show that

$$\|\cdot\|_\infty\text{-diam}(V \cap A^{\mathbf{m}_r,\mathbf{d}_r}_{r,k}) \leq \varepsilon.$$

Take $y \in V \cap A^{\mathbf{m}_r,\mathbf{d}_r}_{r,k}$. For $i = d_1, ..., d_r$,

$$\|y_i\|_i \geq |\|x_i\|_i - \|x_i - y_i\|_i| > \frac{1}{k},$$

so for $i \notin \{d_1, ..., d_r\}$, $\|y_i\|_i \leq \frac{1}{k}$. Therefore

$$\|x_i - y_i\|_i \leq \delta \leq \frac{\varepsilon}{2} \text{ for } i = d_1, ..., d_r,$$

and

$$\|x_i - y_i\|_i \leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \leq \frac{\varepsilon}{2} \text{ otherwise.}$$

Hence $\|\cdot\|_\infty$-diam$(V \cap A^{\mathbf{m}_r,\mathbf{d}_r}_{r,k}) \leq \varepsilon$ as required. ■

Following the notation in Theorem 2.5.2 below, in [15], Kenderov and Moors showed that if the spaces $(C(K_n), ptwise)$ are σ-fragmentable, then $(C(K), ptwise)$ is also σ-fragmentable. We prove it for the weak topology and also in the case of countable covers by sets of small local diameter.

Theorem 2.5.2 *Let $(K_n)_{n=1}^\infty$ be a sequence of closed subsets of a compact Hausdorff space K such that $K = \cup K_n$. Then the following results hold.*

i) *If for each $n \in N$ the space $(C(K_n), weak)$ is σ-fragmentable, then the space $(C(K), weak)$ is σ-fragmentable.*

ii) *If for each $n \in N$ the space $(C(K_n), weak)$ has a countable cover by sets of small local diameter, then the space $(C(K), weak)$ has a countable cover by sets of small local diameter.*

Proof. Define the map

$$T : C(K) \longrightarrow c_0(C(K_n), \|\cdot\|_\infty)$$

by the formula

$$T(f) = (\frac{1}{n} f|_{K_n})_{n=1}^{\infty}.$$

By Theorem 2.5.1 and Remark 2.2.1, since T is clearly weak to weak continuous, we only have to show that T is d. σ-d.

So take $(f_m)_{m=1}^{\infty}, f \in C(K)$ and suppose that $(T(f_m))$ converges to $T(f)$ in the norm of c_0, i.e., given $\varepsilon > 0$ there exists $m \in N$ such that for all $k \geq m$ we have

$$\|T(f_k) - T(f)\|_\infty \leq \varepsilon,$$

i.e.,

$$\|\frac{1}{n} f_k|_{K_n} - \frac{1}{n} f|_{K_n}\|_\infty \leq \varepsilon \text{ for all } n \in N.$$

Now, if $x \in K$, there exists $n \in N$ such that $x \in K_n$, and so

$$|\frac{1}{n} f_k(x) - \frac{1}{n} f(x)| \leq \varepsilon \text{ for all } k \geq m.$$

Thus f_m converges to f in the pointwise topology and by Lemma 2.4.1 we conclude that T is d. σ-d. ■

Chapter 3

Spaces with a countable cover by sets of small local diameter

We shall give some examples of spaces with countable covers by sets of small local diameter as well as some with well behaved injections into $c_0(\Gamma)$ for some set Γ.

3.1 The space $c_0(\Gamma)$

Definition 3.1.1 *Let* Γ *be a set. We define* $c_0(\Gamma)$ *to be the set*

$$c_0(\Gamma) = \{x \in R^\Gamma : \text{ for all } \varepsilon > 0 \quad |\{t \in \Gamma : |x(t)| > \varepsilon\}| < \infty\}.$$

When endowed with the supremum norm, $c_0(\Gamma)$ *is a Banach space.*

Our next proposition is based on Lemma 3.2 from [22]. It also follows from

[9], Theorem 2.1 b), since $c_0(\Gamma)$ has an equivalent locally uniformly convex norm and on bounded sets in $c_0(\Gamma)$ the weak and pointwise topologies coincide.

Proposition 3.1.2 *$(c_0(\Gamma), pointwise)$ has a countable cover by differences of pointwise closed sets of small local diameter.*

Proof. Given $\varepsilon > 0$, for $n \in N$ with $\frac{1}{n} < \frac{\varepsilon}{2}$ and $k \in N$ we put

$$B_k^n = \{y \in c_0(\Gamma) : |\{t \in \Gamma : |y_t| > \frac{1}{n}\}| = k\}.$$

We show that B_k^n is the intersection of an open and a closed set in the pointwise topology. Set

$$G^n(\alpha_1, \ldots, \alpha_m) = \{x \in c_0(\Gamma) : |x_{\alpha_i}| > \frac{1}{n}, i = 1, \ldots, m\}, \ \alpha_i \in \Gamma.$$

and

$$G_m^n = \bigcup_{(\alpha_1,\ldots,\alpha_m)\in\Gamma^m} G^n(\alpha_1, \ldots, \alpha_m).$$

G_m^n is open and clearly $B_k^n = G_k^n \backslash G_{k+1}^n$.

We now show that the sets B_k^n have local diameter less or equal than ε.

Let $y' \in B_k^n$ and let $\{t \in \Gamma : |y'_t| > \frac{1}{n}\} = \{t_1, \ldots, t_k\}$. Then there exists $0 < \delta < \frac{\varepsilon}{2}$ such that $|y'_{t_i}| - \delta > \frac{1}{n}$, $i = 1, \ldots, k$. The set

$$U = \{y \in B_k^n : |y'_{t_1} - y_{t_1}| < \delta, \ldots, |y'_{t_k} - y_{t_k}| < \delta\}$$

is an open neighbourhood of y' in B_k^n in the pointwise topology.

Take $x \in U$. Then for $i = 1, \ldots, k$

$$|x_{t_i}| = |y'_{t_i}| - (|y'_{t_i}| - |x_{t_i}|) \geq |y'_{t_i}| - \delta > |y'_{t_i}| - |y'_{t_i}| + \frac{1}{n} = \frac{1}{n}.$$

And since $x \in B_k^n$ and $|x_{t_i}| > \frac{1}{n}, i = 1, \dots, k$, we have $|x_t| < \frac{1}{n}$ for $t \neq t_i$.

So take $x, y \in U$. Let's show that $\|x - y\|_\infty \leq \varepsilon$. For $i \in \{1, \dots, k\}$,

$$|x_{t_i} - y_{t_i}| \leq |x_{t_i} - y'_{t_i}| + |y'_{t_i} - y_{t_i}| \leq \delta + \delta = 2\delta \leq \varepsilon.$$

And for $t \in \Gamma$ with $t \neq t_i$,

$$|x_t - y_t| \leq |x_t| + |y_t| \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \leq \varepsilon.$$

■

3.2 Projectional Resolutions of Identity

Definition 3.2.1 Let X be a Banach space. The *density character of* X, denoted by *dens(X)*, is the smallest cardinal number of a dense subset of X.

Definition 3.2.2 Let X be a Banach Space. We denote by μ the smallest ordinal such that its cardinality $|\mu| = dens(X)$. A *projectional resolution of identity, PRI* for short, on X is a collection $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ of projections from X into X that satisfy, for every α with $\omega_0 \leq \alpha \leq \mu$, the following conditions:

i) $\|P_\alpha\| = 1$;

ii) $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\alpha$ if $\omega_0 \leq \alpha \leq \beta \leq \mu$;

iii) $dens(P_\alpha(X)) \leq |\alpha|$;

iv) $\bigcup\{P_{\beta+1}(X) : \omega_0 \leq \beta < \alpha\}$ is norm dense in $P_\alpha(X)$;

v) $P_\mu = Id_X$.

The following lemma lists some properties of the *PRI*'s that we will need later on.

Lemma 3.2.1 *Let X be a Banach space and $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ be a PRI on X. We put $P_{\alpha+1} - P_\alpha = T_\alpha$, for $\omega_0 \leq \alpha < \mu$. Then the following results hold.*

i) *For every $x \in X$, if α is a limit ordinal, $\omega_0 \leq \alpha \leq \mu$, we have*

$$P_\alpha(x) = \|\cdot\| - \lim_{\beta<\alpha} P_\beta(x).$$

ii) *For every $x \in X$, $\{\|T_\alpha(x)\| : \alpha \in [\omega_0, \mu)\}$ belongs to $c_0([\omega_0,\mu))$.*

Proof. i) Let α be a limit ordinal, $\omega_0 < \alpha \leq \mu$, and let $x \in X$. If

$$x \in \bigcup_{\beta<\alpha} P_\beta(X),$$

say $x \in P_{\beta_0}(X)$, then $P_\beta(x) = x$ if $\beta \geq \beta_0$. Thus, in this case, we have

$$\|\cdot\| - \lim_{\beta\to\alpha} P_\beta(x) = x = P_\alpha(x). \qquad (1)$$

If $x \in P_\alpha(X)$, by (iv) in Definition 3.2.2, given $\varepsilon > 0$ there exists

$$y \in \bigcup_{\beta<\alpha} P_\beta(X)$$

such that $\|x - y\| < \frac{\varepsilon}{3}$. For this $\varepsilon > 0$ there must be $\beta_0 < \alpha$ such that for any β, $\beta_0 \leq \beta \leq \alpha$, we have

$$\|P_\beta(y) - P_\alpha(y)\| \leq \frac{\varepsilon}{3}.$$

Thus

$$\|P_\alpha(x) - P_\beta(x)\| \leq \|P_\alpha(x) - P_\alpha(y)\| + \|P_\alpha(y) - P_\beta(y)\| + \|P_\beta(y) - P_\beta(x)\| \leq$$

$$\leq \|P_\alpha\|\|x - y\| + \|P_\beta(y) - P_\alpha(y)\| + \|P_\beta\|\|x - y\| \leq \varepsilon,$$

whenever $\beta \geq \beta_0$.

Finally for $x \in X$ and $\beta < \alpha$ we have

$$P_\beta(x) - P_\alpha(x) = P_\beta(P_\alpha(x)) - P_\alpha(x)$$

and thus (1) holds for every $x \in X$.

ii) If the assertion is false, there exists $x_0 \in X$, $\varepsilon > 0$, and

$$\omega_0 < \alpha_1 < \alpha_2 < \ldots < \mu$$

such that for every $i \geq 1$, $\|T_{\alpha_i}(x_0)\| > \varepsilon$. Set

$$\alpha = \sup\{\alpha_i : i \geq 1\}.$$

It is clear that

$$\lim_{\beta \to \alpha} P_\beta(x_0) \neq P_\alpha(x_0),$$

which contradicts with (i). ■

3.2.1 d. σ-d. maps into $c_0(\Gamma)$ using PRI

The next theorem is a straightforward adapatation to the case of *PRI*'s from that in [22].

Theorem 3.2.1.1 *Let $\wp$ be a class of Banach spaces such that the following results hold:*

i) *for every $X \in \wp$ there exists a PRI on X, $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$.*

ii) $(P_{\alpha+1} - P_\alpha)(X) \in \wp$.

Then for every $X \in \wp$ there exists a one-to-one bounded linear map

$$T : X \longrightarrow c_0(\Gamma),$$

for some set Γ, such that for every discrete family $\mathcal{A}= \{A_s\}_{s\in S}$ of subsets of X, the family $T\mathcal{A}= \{TA_s\}_{s\in S}$ is d. σ-d. in $c_0(\Gamma)$.

Proof. Let $X \in \wp$. We proceed by induction on $dens(X)$.

When X is separable, we have that (B_{X^*}, w^*) is metrizable and separable. So let $\{f_n : n \geq 1\}$ be a dense subset of (B_{X^*}, w^*) and define

$$T : X \longrightarrow c_0(N) \text{ by } T(x) = \left(\frac{1}{n} f_n(x)\right)_{n=1}^{\infty}.$$

T is clearly a linear map, and it is one-to-one because $(f_n)_{n=1}^{\infty}$ is dense in B_{X^*}. In order to prove the d. σ-d. property we show first that discrete families in a separable Banach space are countable.

Let $\mathcal{A}= \{A_s\}_{s\in S}$ be a discrete family in X and let $\{B_n\}_{n=1}^{\infty}$ be an open base for the topology on X. For each $s \in S$, pick $a_s \in A_s$. Then there exists U_s, open neighbourhood of a_s, such that $U_s \cap A_t = \emptyset$, for $t \neq s$. Now for each $s \in S$ there exists $n_s \in N$ such that $a_s \in B_{n_s} \subset U_s$. Hence $|S| \leq |N|$ and therefore S must be countable.

So let $\mathcal{A}= \{A_n\}_{n\in N}$ be a discrete family and define

$$B_n^{(n)} = T(A_n) \text{ and } B_n^{(m)} = \emptyset, \text{ for } n \neq m$$

Then for every $m \in N$ the family $\left\{B_n^{(m)}\right\}_{n\in N}$ is discrete and

$$T(A_n) = \bigcup_{m=1}^{\infty} B_n^{(m)} \text{ for every } n \in N.$$

So the family $T\mathcal{A}$ is d. σ-d.

Let χ be an uncountable cardinal and $X \in \wp$ such that $dens(X) = \chi$. Suppose that the result is true for every $Y \in \wp$ with $dens(Y) < \chi$. Let μ be the smallest ordinal with cardinality $|\mu| = \chi$.

Let $\{P_\alpha : \omega_0 \leq \alpha < \mu\}$ be a *PRI* on X. For any α, $\omega_0 \leq \alpha < \mu$, we set $X_\alpha = (P_{\alpha+1} - P_\alpha)(X)$. Then $X_\alpha \in \wp$ and $dens(X_\alpha) \leq |\alpha| < dens(X)$. Thus there exist sets Γ_α and one-to-one continuous linear maps

$$J_\alpha : X_\alpha \longrightarrow c_0(\Gamma_\alpha)$$

which satisfy the condition that the image of any discrete family in X_α is d. σ-d. in $c_0(\Gamma_\alpha)$.

Since $P_{\omega_0}(X)$ is separable, there is also $J_0 : P_{\omega_0}(X) \longrightarrow c_0(N)$ sharing these properties. Set

$$\Gamma = N \cup \bigcup_{\omega_0 \leq \alpha < \mu} \Gamma_\alpha$$

and define $T : X \longrightarrow \ell_\infty(\Gamma)$ by

$$T(x)(n) = J_0(P_{\omega_0}(x))(n) \text{ for } n \in N, \text{ and}$$

$$T(x)(\gamma) = \frac{1}{2} J_\alpha(P_{\alpha+1}(x) - P_\alpha(x))(\gamma) \text{ for } \gamma \in \Gamma_\alpha.$$

T is clearly a linear map and is continuous. We have

$$\|Tx\| = \sup_{\omega_0 \leq \alpha < \mu} \|\frac{1}{2} J_\alpha(P_{\alpha+1}(x) - P_\alpha(x))\| \leq$$

$$\leq \frac{1}{2} \sup_{\omega_0 \leq \alpha < \mu} \|J_\alpha\|(\|P_{\alpha+1}(x) - P_\alpha(x)\|) \leq \frac{1}{2} \sup_{\omega_0 \leq \alpha < \mu} (\|P_{\alpha+1}\|\|x\| + \|P_\alpha\|\|x\|) \leq$$

$$\leq \frac{1}{2} \sup_{\omega_0 \leq \alpha < \mu} (\|x\| + \|x\|) = \|x\|.$$

We show that $T(X) \subset c_0(\Gamma)$. By Lemma 3.2.1, ii), given $x \in X$ the set

$$\{\|T_\alpha(x)\| : \alpha \in [\omega_0, \mu)\}$$

belongs to $c_0([\omega_0, \mu))$, where $T_\alpha = P_{\alpha+1} - P_\alpha$.

So, given any $\varepsilon > 0$, there exist $\alpha_1, \ldots, \alpha_n \in [\omega_0, \mu)$ such that

$$\|T_{\alpha_i}(x)\| > \varepsilon \text{ and } \|T_\beta(x)\| \leq \varepsilon \text{ for } \beta \neq \alpha_i, i = 1, \ldots, n.$$

For each $i \in \{1, \ldots, n\}$, $J_{\alpha_i}(T_{\alpha_i}(x)) \in c_0(\Gamma_{\alpha_i})$. Hence there exist $\gamma_1, \ldots, \gamma_{m_i} \in \Gamma_{\alpha_i}$ such that $|J_{\alpha_i}(T_{\alpha_i}(x)(\gamma_k)| > \varepsilon$ for $k = 1, \ldots, m_i$. So there is only a finite collection of $\gamma \in \Gamma$ such that $|T(x)(\gamma)| > \varepsilon$, and therefore $T(x) \in c_0(\Gamma)$.

Now we prove that T is an injection. Take $x \in X$ and suppose that $T(x) = 0$. We show that $P_\xi(x) = 0$ for $\omega_0 \leq \xi \leq \mu$.

For $\xi = \omega_0$, we have $J_0(P_{\omega_0}(x)) = 0$ and, since J_0 is an injection, we conclude that $P_{\omega_0}(x) = 0$.

Fix ξ, $\omega_0 < \xi < \mu$, and suppose that $P_\alpha(x) = 0$ for $\omega_0 \leq \alpha < \xi$. If ξ is a non-limit ordinal, then $\xi = \alpha + 1$ for some $\alpha < \xi$. Now

$$J_\xi(P_\xi(x) - P_\alpha(x)) = 0 \implies P_\xi(x) = 0,$$

because $P_\alpha(x) = 0$ and J_ξ is an injection.

If ξ is a limit ordinal, we have

$$P_\xi(x) = \lim_{\alpha<\xi} P_\alpha(x) = 0.$$

For $\xi = \mu$, since μ is a limit ordinal, we have

$$P_\mu(x) = x = \lim_{\alpha<\mu} P_\alpha(x) = 0,$$

hence $x = 0$ and so T is an injection.

Now we introduce some notation. Set $Y_\alpha = P_\alpha(X)$ for $\omega_0 \leq \alpha \leq \mu$ and

$$Z_\alpha = \{y \in c_0(\Gamma) : y_t = 0 \text{ for } t \in \bigcup_{\alpha \leq \xi < \mu} \Gamma_\xi\}.$$

Define $h_\alpha : c_0(\Gamma) \longrightarrow Z_\alpha \subset c_0(\Gamma)$ by

$$h_\alpha(y) = y' \text{ with } y'_t = \begin{cases} 0 & \text{for } t \in \bigcup\{\Gamma_\xi : \alpha \leq \xi < \mu\}, \\ y_t & \text{otherwise.} \end{cases}$$

Observe that $TY_\alpha \subset Z_\alpha$ and that the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & c_0(\Gamma) \\ P_\alpha \downarrow & & \downarrow h_\alpha \\ Y_\alpha & \xrightarrow{T} & Z_\alpha \end{array}$$

has the property that $h_\alpha \circ T = T \circ P_\alpha$.

We prove now that T maps discrete families into d. σ-d. families.

Fix $\xi \leq \mu$ and suppose that for each $\alpha < \xi$ the image under T of any discrete family of subsets of Y_α is d. σ-d. in $c_0(\Gamma)$. We show that for any discrete family $\mathcal{A}= \{A_s\}_{s\in S}$ of subsets of Y_ξ the family $T\mathcal{A}$ is d. σ-d. in $c_0(\Gamma)$. We may assume that $\mathcal{A}$ is ε-discrete for some $\varepsilon > 0$.

The case $\xi = \omega_0$ is clear, since Y_ω is separable.

Now let $\xi = \alpha + 1$ be a non-limit ordinal. Take a cover

$$\mathcal{V} = \bigcup_{j=1}^{\infty} \mathcal{V}_j \text{ of } Y_\alpha$$

consisting of sets of diameter less than $\frac{\varepsilon}{4}$ and such that the families $\mathcal{V}_j$ are discrete (cover Y_α by open balls of diameter less than $\frac{\varepsilon}{4}$ and obtain a σ-discrete refinement). By the inductive assumption the families $h_\alpha^{-1}T\mathcal{V}_j = TP_\alpha^{-1}\mathcal{V}_j$ are d. σ-d. in $c_0(\Gamma)$ for all $j \in N$.

In order to prove that the family $T\mathcal{A}$ is d. σ-d. in $c_0(\Gamma)$ it is sufficient to show that for each $j \in N$ and each $U \in \mathcal{V}_j$, the family

$$T\mathcal{A}^U = \{TA_s^U\}_{s\in S}, \text{ where } A_s^U = P_\alpha^{-1}(U) \cap A_s,$$

is d. σ-d. in $c_0(\Gamma)$.

Suppose that the last assertion has already been proved. Thus, for every $j \in N$ and every $U \in \mathcal{V}_j$ there exist families $\{B_{U,j,s}^{(m)}\}_{s\in S}$, $m \in N$, such that for every $m \in N$, $\{B_{U,j,s}^{(m)}\}_{s\in S}$ is a discrete family in $c_0(\Gamma)$ and

$$TA_s^U = \bigcup_{m=1}^{\infty} B_{U,j,s}^{(m)}.$$

Now for every $j \in N$ the families $TP_\alpha^{-1}(\mathcal{V}_j)$ are d. σ-d., and therefore for every $n \in N$ there exist discrete families $\{C_{U,j}^{(n)}\}_{U\in\mathcal{V}_j}$ such that for all $j \in N$ and $U \in \mathcal{V}_j$ we have

$$TP_\alpha^{-1}(U) = \bigcup_{n=1}^{\infty} C_{U,j}^{(n)}.$$

Since $\mathcal{V}$ is a cover of Y_α, we have

$$TA_s = \bigcup_{j=1}^{\infty}\Big(\bigcup_{U\in\mathcal{V}_j} TA_s^U\Big) = \bigcup_{j=1}^{\infty}\Big(\bigcup_{U\in\mathcal{V}_j}\Big(\bigcup_{m=1}^{\infty} B_{U,j,s}^{(m)}\Big)\Big) =$$

$$= \bigcup_{j=1}^{\infty}\Big(\bigcup_{U\in\mathcal{V}_j}\Big(\bigcup_{m=1}^{\infty} B_{U,j,s}^{(m)} \bigcap TP_\alpha^{-1}(U)\Big)\Big) =$$

$$= \bigcup_{j=1}^{\infty}\Big(\bigcup_{U\in\mathcal{V}_j}\Big(\bigcup_{m=1}^{\infty} B_{U,j,s}^{(m)} \cap \bigcup_{n=1}^{\infty} C_{U,j}^{(n)}\Big)\Big) = \bigcup_{j,m,n=1}^{\infty}\Big(\bigcup_{U\in\mathcal{V}_j}(B_{U,j,s}^{(m)} \cap C_{U,j}^{(n)})\Big).$$

So we have to see that if we fix $j, m, n \in N$, then the family

$$\mathcal{F} = \{\bigcup_{U\in\mathcal{V}_j}(B_{U,j,s}^{(m)} \cap C_{U,j}^{(n)})\}_{s\in S}$$

is discrete.

So take $x_0 \in c_0(\Gamma)$, since $\{C^{(n)}_{U,j}\}_{U\in\mathcal{V}_j}$ is a discrete family, there exists an open neighbourhood of x_0, say N_0, which intersects at most one element of the family. Assume that $N_0 \cap C^{(n)}_{U_0,j} \neq \emptyset$ for some $U_0 \in \mathcal{V}_j$ and $N_0 \cap C^{(n)}_{U,j} = \emptyset$ for $U \neq U_0$, $U \in \mathcal{V}_j$. For $U_0 \in \mathcal{V}_j$ the family $\{B^{(m)}_{U_0,j,s}\}_{s\in S}$ is discrete and therefore there exists an open neighbourhood of x_0, say M_0, such that $M_0 \bigcap B^{(m)}_{U_0,j,s_0} \neq \emptyset$ for some $s_0 \in S$ and $M_0 \bigcap B^{(m)}_{U_0,j,s} = \emptyset$ for $s \neq s_0$.

Take $G_0 = N_0 \cap M_0$. G_0 is an open neighbourhood of x_0 which intersects at most the one set $C^{(n)}_{U_0,j} \cap B^{(m)}_{U_0,j,s_0}$ and therefore meets at most

$$\bigcup_{U\in\mathcal{V}_j} (B^{(m)}_{U,j,s_0} \cap C^{(n)}_{U,j}).$$

Hence we conclude that the family $\mathcal{F}$ is discrete and therefore that $T\mathcal{A}$ is d. σ-d.

So we prove that the family $T\mathcal{A}^U$ is d. σ-d. in $c_0(\Gamma)$. For $x \in A^U_s, y \in A^U_t$, with $s \neq t$ we have

$$\|P_\alpha(x) - P_\alpha(y)\| < \frac{\varepsilon}{4} \text{ and } \|x - y\| > \varepsilon.$$

(Note that $\operatorname{diam}(U) < \frac{\varepsilon}{4}$ and the family $\{A_s\}_{s\in S}$ is ε-discrete.) Therefore

$$\|(x - P_\alpha(x)) - (y - P_\alpha(y))\| > \frac{\varepsilon}{4}$$

and since

$$P_{\alpha+1}(x) = x \text{ for } x \in Y_{\alpha+1},$$

we conclude that $(P_{\alpha+1} - P_\alpha)(\mathcal{A}^U)$ is a discrete family (in fact, $\frac{\varepsilon}{4}$-discrete) in X_α. Therefore by the inductive assumption, $J_\alpha(P_{\alpha+1} - P_\alpha)(\mathcal{A}^U)$ is d. σ-d. in $c_0(\Gamma_\alpha)$. Thus the inverse image of this family under the map defined by

$$d : c_0(\Gamma) \longrightarrow c_0(\Gamma_\alpha)$$

$$y \quad \rightsquigarrow \quad y_{|\Gamma_\alpha}$$

is a d. σ-d. family. The family $T\mathcal{A}^U$ is by the definition of T a refinement of the family

$$d^{-1}(\frac{1}{2}J_\alpha(P_{\alpha+1} - P_\alpha)\mathcal{A}^U)$$

and therefore is also d. σ-d.

Now let ξ be a limit ordinal. For $x \in Y_\xi$ we have

$$\lim_{\alpha<\xi} \|Tx_{|\Gamma_\alpha}\| = 0,$$

by Lemma 3.2.1 i), and we obtain that for any $\varepsilon > 0$ and for all $x \in Y_\xi$, there exists $\alpha < \xi$ such that

$$\|P_\alpha(x) - x\| < \frac{\varepsilon}{4} \text{ and } \|Tx_{|\Gamma_\alpha}\| > \|Tx_{|\Gamma_\eta}\|, \text{ for } \eta > \alpha. \qquad (*)$$

Fix two rational numbers $r > r' > 0$ and put

$$M^\alpha = \{y \in c_0(\Gamma) : \|y_{|\Gamma_\alpha}\| > r \text{ and } \|y_{|\Gamma_\eta}\| < r', \text{ for } \eta > \alpha\}$$

We show that the family $\{M^\alpha\}_{\alpha<\eta}$ is a discrete family in $c_0(\Gamma)$. Take $x_0 \in c_0(\Gamma)$ and $\delta = \frac{r-r'}{4}$. We claim that the ball $B(x_0; \delta)$ cannot meet two distinct elements of the family. Suppose there exist α, η, say $\alpha < \eta$, such that

$$B(x_0; \delta) \cap M^\alpha \neq \emptyset \neq B(x_0; \delta) \cap M^\eta.$$

Take

$$u \in B(x_0; \delta) \cap M^\eta \text{ and } v \in B(x_0; \delta) \cap M^\alpha.$$

For $y \in c_0(\Gamma)$ write $y_\eta = y_{|\Gamma_\eta}$. We have

$$\|u_\eta - v_\eta\| \leq \|u_\eta - x_{0_\eta}\| + \|x_{0_\eta} - v_\eta\| \leq \delta + \delta = \frac{r - r'}{2}$$

and

$$\|u_\eta - v_\eta\| \geq |\|u_\eta\| - \|v_\eta\|| \geq r - r'.$$

Hence we have

$$0 < r - r' \leq \frac{r - r'}{2},$$

which is absurd.

If we consider the sets

$$A_s^\alpha = \{x \in A_s : \|P_\alpha(x) - x\| < \frac{\varepsilon}{4}\},$$

then the family $\{P_\alpha(A_s^\alpha)\}_{s \in S}$ is discrete in Y_α.

To verify this take $x_0 \in Y_\alpha$ and consider the open ball $B(x_0; \frac{\varepsilon}{8})$, and suppose that for $t \neq s$ the sets

$$B(x_0; \frac{\varepsilon}{8}) \cap P_\alpha(A_s^\alpha) \text{ and } B(x_0, \frac{\varepsilon}{8}) \cap P_\alpha(A_t^\alpha)$$

are both non-empty. Take $x_1 \in A_s^\alpha$ and $x_2 \in A_t^\alpha$ such that

$$P_\alpha(x_1) \in B(x_0; \frac{\varepsilon}{8}) \cap P_\alpha(A_s^\alpha) \text{ and } P_\alpha(x_2) \in B(x_0; \frac{\varepsilon}{8}) \cap P_\alpha(A_t^\alpha).$$

Since the family $\{A_s\}_{s\in S}$ is ε-discrete, we have

$$\|P_\alpha(x_1) - P_\alpha(x_2)\| < \frac{\varepsilon}{4} \text{ and } \|x_1 - x_2\| > \varepsilon.$$

Then

$$\varepsilon < \|x_1 - x_2\| \leq \|x_1 - P_\alpha(x_1)\| + \|P_\alpha(x_1) - P_\alpha(x_2)\| + \|P_\alpha(x_2) - x_2\| \leq \frac{3\varepsilon}{4},$$

which is absurd.

So $\{P_\alpha(A_s^\alpha)\}_{s\in S}$ is discrete in Y_α and by the inductive assumption the family $\{TP_\alpha(A_s^\alpha)\}_{s\in S}$ is d. σ-d. in $c_0(\Gamma)$. Hence

$$\{h_\alpha^{-1} TP_\alpha(A_s^\alpha)\}_{s\in S} = \{TA_s^\alpha\}_{s\in S}$$

is d. σ-d. in $c_0(\Gamma)$.

We conclude that the family $\{M^\alpha \cap TA_s^\alpha\}_{s\in S}$ is d. σ-d. (it is just a refinement of a d. σ-d. family). Since the family $\{M^\alpha\}_{\alpha<\eta}$ is discrete, the family

$$\{\bigcup_{\alpha<\xi} (M^\alpha \cap TA_s^\alpha)\}_{s\in S}$$

is d. σ-d.

We check the last assertion. Since $\{TA_s^\alpha\}_{s\in S}$ is d. σ-d., take

$$TA_s^\alpha = \bigcup_{n=1}^{\infty} B_{\alpha,s}^{(n)},$$

with $\{B_{\alpha,s}^{(n)}\}_{s\in S}$ discrete for every $n \in N$. Then

$$\{\bigcup_{\alpha<\xi}(M^\alpha \cap TA_s^\alpha)\}_{s\in S} = \{\bigcup_{\alpha<\xi}(M^\alpha \cap (\bigcup_{n=1}^{\infty} B_{\alpha,s}^{(n)}))\}_{s\in S} =$$

$$= \{\bigcup_{n=1}^{\infty}(\bigcup_{\alpha<\xi}(M^\alpha \cap B_{\alpha,s}^{(n)}))\}_{s\in S}.$$

We have to show that for every $n \in N$ the family

$$\{\bigcup_{\alpha<\xi}(M^\alpha \cap B_{\alpha,s}^{(n)})\}_{s\in S}$$

is discrete.

Take $x \in c_0(\Gamma)$. There exists an open neighbourhood N of x which meets at most one element of the family $\{M^\alpha\}_{\alpha<\xi}$, say $N \cap M^{\alpha_0} \neq \emptyset$, and $N \cap M^\beta = \emptyset$ for $\beta \neq \alpha_0$, $\beta < \xi$. For this α_0, the family $\{B_{\alpha_0,s}^{(n)}\}_{s\in S}$ is discrete. So there exists an open neighbourhood M of x which intersects at most one element of this family, say $M \cap B_{\alpha_0,s_0}^{(n)} \neq \emptyset$ and $M \cap B_{\alpha_0,s}^{(n)} = \emptyset$, $s \neq s_0$.

Set $V = N \cap M$. V is an open neighbourhood of x and meets at most $M^{\alpha_0} \cap B_{\alpha_0,s_0}^{(n)}$, and therefore V intersects at most

$$\bigcup_{\alpha<\xi}(M^\alpha \cap B_{\alpha,s_0}^{(n)}).$$

Considering all pairs of rational numbers $r > r' > 0$, by $(*)$, we cover TA_s by the sets

$$\bigcup_{\alpha<\xi}(M^\alpha \cap TA_s^\alpha)$$

associated with these pairs. Then the family $\{TA_s\}_{s\in S}$ is d. σ-d. and this ends the induction. ■

3.2.2 WCD spaces, duals of Asplund spaces and C(K) spaces

Definition 3.2.2.1

i) *A Banach space X is called an Asplund space if every convex continuous function defined on a convex open subset U of X is Fréchet differentiable on a dense G_δ subset of U.*

ii) *A Banach space X is called weakly countably determined (WCD) if there exists a countable collection $\{K_n : n \geq 1\}$ of w^*-compact subsets of X^{**} such that for every $x \in X$ and every $u \in X^{**} \backslash X$ there exists n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$.*

iii) *Let K be a compact space. K is said to be Valdivia compact if there exists a set I and a subset K_0 of $[0,1]^I$ such that K is homeomorphic to K_0 and $K_0 \cap \Sigma(I)$ is dense in K_0, where $\Sigma(I)$ is the subset of $[0,1]^I$ consisting of all functions $\{x(i) : i \in I\}$ such that $x(i)=0$ except for a countable number of i's and $[0,1]^I$ is equipped with its product topology.*

We now apply Theorem 3.2.1.1 to these three particular cases of Banach spaces. The existence of a *PRI* in these spaces can be found in [2], Chapter 6.

Corollary 3.2.2.1 *Let K be a Valdivia compact space. Then there exists a map $T : C(K) \longrightarrow c_0(\Gamma)$ with the following properties.*

i) *T is a one-to-one bounded linear map.*

ii) $T : (C(K), pointwise) \longrightarrow (c_0(\Gamma), pointwise)$ *is continuous.*

iii) *T maps discrete families of $C(K)$ into d. σ-d. families of $c_0(\Gamma)$.*

Proof. Consider the families $\{K_\alpha : \omega_0 \leq \alpha \leq \mu\}, \{r_\alpha : \omega_0 \leq \alpha \leq \mu\}$ of Valdivia compact spaces and continuous retractions that arise in the construction of a PRI on $C(K)$.

Since $w(K_\alpha) \leq |\alpha|$, we apply the inductive assumption to $C(K_\alpha)$ instead of X_α. We also can identify $C(K_\alpha)$ with Y_α. So we obtain

$$T_\alpha : C(K_\alpha) \longrightarrow c_0(\Gamma_\alpha)$$

satisfying conditions i), ii), iii), and define T as follows

$$T(f)(n) = T_{\omega_0}(P_{\omega_0}(f))(n) \text{ for } n \in N,$$

$$T(f)(\gamma) = \frac{1}{2} T_{\alpha+1}(P_{\alpha+1}(f) - P_\alpha(f))(\gamma) \text{ for } \gamma \in \Gamma_{\alpha+1}.$$

Since P_α and T_α are pointwise continuous, T is pointwise-pointwise continuous.

We have to go back to the proof of Theorem 3.2.1.1 and to modify a few things. Where we had

$$\text{“}(P_{\alpha+1} - P_\alpha)(\mathcal{A}^U) \ \textit{is discrete in} \ X_\alpha\text{”},$$

we can put

$$(P_{\alpha+1} - P_\alpha)(\mathcal{A}^U) \ \textit{is} \ \frac{\varepsilon}{4} - \textit{discrete in} \ X_\alpha,$$

and therefore discrete in $C(K_{\alpha+1}) = Y_{\alpha+1}$.

Then

$$T_{\alpha+1}(P_{\alpha+1} - P_\alpha)(\mathcal{A}^U)$$

is d. σ-d. in $c_0(\Gamma_{\alpha+1})$.

(Since μ is a limit ordinal, $\alpha + 1$ must be less than μ and therefore we can apply the inductive assumption.)

The rest of the proof remains valid. ■

Corollary 3.2.2.2 *Let X be a weakly countably determined Banach space. Then there exists a one-to-one continuous linear map $T : X \longrightarrow c_0(\Gamma)$ such that the image of any discrete family in X is d. σ-d. in $c_0(\Gamma)$.*

Proof. Let $\wp$ be the class of WCD Banach spaces. If $X \in \wp$, we obtain a *PRI* on X, and since being WCD is an hereditary property, we have that $(P_{\alpha+1} - P_\alpha)(X) = X_\alpha$ is WCD. Now we can apply Theorem 3.2.1.1 and obtain the desired result. ■

Corollary 3.2.2.3 *Let X be an Asplund space. Then there exists a one-to-one bounded linear map $T : X^* \longrightarrow c_0(\Gamma)$ such that the image under T of any discrete family in X^* is d. σ-d. in $c_0(\Gamma)$.*

Proof. Let $\wp$ be the class of duals of Asplund spaces. If $X^* \in \wp$ we can obtain a *PRI* on X^* satisfying

$$(P_{\alpha+1} - P_\alpha)(X^*) \simeq (X_{\alpha+1}/X_\alpha)^*.$$

Here $(X_{\alpha+1}/X_\alpha)^*$ is the dual of an Asplund space, i.e., another element of $\wp$ and now we only have to apply Theorem 3.2.1.1 to $\wp$. ∎

Due to the existence of equivalent *Kadec* norms for the three types of Banach spaces we have considered, our next result is known.

Theorem 3.2.2.2 *Let X be a Banach space of one of the following types: weakly countably determined, the dual of an Asplund space or a C(K) space with K being a Valdivia compact space. Then (X,weak) has a countable cover by sets of small local diameter and therefore $Borel(X, \|\cdot\|) = Borel(X, weak)$. Moreover, in the case of X=C(K), we have that (C(K),pointwise) has a countable cover by sets of small local diameter and therefore*

$$Borel(X, \|\cdot\|) = Borel(X, weak) = Borel(X, pointwise).$$

3.3 Markushevich basis.

Definition 3.3.1 Let X be a Banach space. A family $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ is called a *Markushevich basis (M-basis)* if the following conditions are satisfied:

i) $\overline{span\{x_\gamma : \gamma \in \Gamma\}} = X$;

ii) $\|f_\gamma\| = 1$, for $\gamma \in \Gamma$;

iii) $f_\gamma(x_\gamma) = 1$ for $\gamma \in \Gamma$ and $f_\gamma(x_\delta) = 0$ for $\delta \neq \gamma$, $\gamma, \delta \in \Gamma$;

iv) $\cap\{Ker\ f_\gamma : \gamma \in \Gamma\} = \{0\}$.

Let F be the norm closure of the linear span of $\{f_\gamma : \gamma \in \Gamma\}$ in X^*. The formula

$$|||x||| = \sup\{f(x) : f \in F, \|f\| \leq 1\}$$

defines a new norm on X. The M-basis is said to be *norming* if the norm $|||\cdot|||$ is equivalent to the original norm $\|\cdot\|$.

Lemma 3.3.1 *Let X be a Banach space with a norming M-basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. Then there is a continuous linear injection $T : X \to c_0(\Gamma)$.*

Proof. Consider $T : X \longrightarrow \ell^\infty(\Gamma)$ defined by $T(x) = (f_\gamma(x))$. T is clearly a continuous linear map and, by the definition of M-basis, is an injection. We now show that $T(X) \subset c_0(\Gamma)$.

Consider the set $K = \{f_\gamma : \gamma \in \Gamma\} \cup \{0\}$. K is a w^*-compact subset of X^*. Define $V_\gamma = \{x^* \in X^* : |x_\gamma(x^*)| > \frac{1}{2}\}$. V_γ is a w^*-open neighbourhood of f_γ in X^* such that $f_\delta \notin V_\gamma$ for $\delta \neq \gamma \in \Gamma$.

Now given $x \in X$ and $\varepsilon > 0$, set $U = \{x^* \in X^* : |x^*(x)| < \varepsilon\}$. U is a w^*-open neighbourhood of 0 in X^*. Now the family $\{V_\gamma : \gamma \in \Gamma\} \cup U$ is a w^*-open cover of K and therefore there exist $\gamma_1, ..., \gamma_n \in \Gamma$ such that

$$K \subset \bigcup_{j=1}^{n} V_{\gamma_j} \cup U.$$

Thus, for $\gamma \notin \{\gamma_1, ..., \gamma_n\}, \gamma \in \Gamma$ we have that $f_\gamma \in U$, i.e., $|f_\gamma(x)| < \varepsilon$ and therefore $T(x) \in c_0(\Gamma)$. ∎

The next result is part of one in [10], Theorem 1.

Proposition 3.3.1 *If X has a norming M-basis, then (X, τ_{ptwise}) is σ-fragmented using τ_{ptwise}-closed sets.*

Theorem 3.3.1 *Let X be a Banach space with a norming M-basis. Then there is a continuous linear injection $T : X \longrightarrow c_0(\Gamma)$ which is d. σ-d. and its inverse is of the first Borel class.*

Proof. Define $T : X \rightarrow c_0(\Gamma)$ by $T(x) = (f_\gamma(x))$.

It follows from Theorem 2.2.2 and Proposition 3.3.1 that T is d. σ-d. and that (X, T_{ptwise}) has a countable cover by sets of small local diameter. We show that the equivalent norm $|||\cdot|||$ is lower semicontinuous for the T_{ptwise} topology.

Take $\{x_\alpha\}_{\alpha\in I}, x \in X$ with $|||x_\alpha||| \leq 1$ and $x_\alpha \rightarrow x$ in the T_{ptwise} topology, i.e., $f_\gamma(x_\alpha) \rightarrow f_\gamma(x)$ for every $\gamma \in \Gamma$. Since $|||x||| = \sup\{f(x) : f \in F, ||f|| \leq 1\}$, given $\varepsilon > 0$ there exists $f \in F, ||f|| \leq 1$ such that

$$|||x||| \leq f(x) + \frac{\varepsilon}{3}.$$

Since $f \in F$ there exists $(\lambda_1 f_{\gamma_1} + ... + \lambda_n f_{\gamma_n})$ such that

$$f(x) \leq \lambda_1 f_{\gamma_1}(x) + ... + \lambda_n f_{\gamma_n}(x) + \frac{\varepsilon}{3}$$

and there exists α_0 such that for $\alpha >> \alpha_0$

$$\lambda_1 f_{\gamma_1}(x) + ... + \lambda_n f_{\gamma_n}(x) \leq \lambda_1 f_{\gamma_1}(x_\alpha) + ... + \lambda_n f_{\gamma_n}(x_\alpha) + \frac{\varepsilon}{3}.$$

Thus,

$$|||x||| \leq \lambda_1 f_{\gamma_1}(x_\alpha) + ... + \lambda_n f_{\gamma_n}(x_\alpha) + \varepsilon \leq |||x_\alpha||| + \varepsilon \leq 1 + \varepsilon.$$

So for any $\varepsilon > 0$ we have that $|||x||| \leq 1 + \varepsilon$ which implies that $|||x||| \leq 1$. Now since $|||\cdot|||$ is lower semicontinuous and (X, T_{ptwise}) has a countable cover by sets of small local diameter, we have by Theorem 1.3.1 that (X, T_{ptwise}) has a countable cover by differences of T_{ptwise}-closed sets of small local diameter. Since differences of T_{norm}-closed sets are T_{norm}-F_σ sets, we have, by Theorem 2.3.1, that T is d. σ-d. and its inverse is of first Borel class. ∎

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