Convex Tomography and the Characterisation of Ellipsoids

PhD Thesis

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Abstract

This thesis explores some aspects of convex tomography. We look in some detail at formulations of problems in tomography in which information is known about chords or sections supporting convex bodies.

Chapter 1 extends a result of Falconer on Hammer's X-Ray problem. Suppose that K and M are planar convex bodies containing, in a sense to be defined, *sufficiently distinct* convex bodies L_1, L_2 in $\text{int}K \cap \text{int}M$. Suppose further that whenever l is a line supporting L_1 or L_2 the chord lengths $K \cap l$ and $M \cap l$ agree. We prove that $K = M$.

Chapter 2 applies the result obtained in Chapter 1 to prove a new characterisation of the ellipsoid. Suppose K, L are convex bodies in $\mathbb{R}^3$ with $L \subset \text{int}K$, and that every section of K supporting L is centrally symmetric. We show that under certain conditions it is possible to prove that K is an ellipsoid.

Chapter 3 explores some aspects of a problem which we refer to as the one body problem. Here we are concerned again with chord-lengths of planar convex bodies. Suppose that K is a convex body containing the convex body L in its interior. Further suppose that the lengths of the chords of K supporting L are given. We ask how much can be deduced about K. Several results are presented.

In the final chapter, Chapter 4, we attempt to extend a result of Montejano. If every pair of sections of a convex body K through a point $\mathbf{p}$ are homothetic, is it true that K is a Euclidean ball? Montejano gave a positive result for the case $\mathbf{p} \in \text{int}K$. We provide a counterexample for the case $\mathbf{p} \in \partial K$ and a much restricted result for the case $\mathbf{p} \notin K$.

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Acknowledgments

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Definitions and Notation

The following notation is used, usually without definition. Most of it is standard.

$\mathbb{R}^n$	n-dimensional Euclidean space
Ω^n	n-dimensional unit sphere
B	Euclidean unit ball
$\langle \cdot, \cdot \rangle$	standard inner product
$H_\mu(\mathbf{u})$	$\{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{u} \rangle = \mu\}$
$\xi\ (\theta)$	$(\cos\theta, \sin\theta)^T$
ω_d	standard spherical surface measure on Ω^d
λ_d	standard volume measure on $\mathbb{R}^d$
$\lVert\cdot\rVert$	Euclidean norm
aff	affine hull
lin	linear hull
conv	convex hull
cl	closure
int	interior
relint	relative interior
∂K	the boundary of K; ie. cl$K \setminus$ int$K$.

Introduction

The ellipsoid is a convex body with remarkable properties. For example E is an ellipsoid if and only if every one-codimensional section of E is centrally symmetric. We can do much better than this however. In 1971 Aitchison, Petty and Rogers [1] published a proof of the False Centre Theorem. A point $\mathbf{x}$ is said to be a false centre of the convex body K if every one-codimensional section of K through $\mathbf{x}$ is centrally symmetric, but $\mathbf{x}$ is not a centre of symmetry for K. The False Centre Theorem states that any convex body with a false centre in its interior is an ellipsoid. D.G. Larman [13] later extended this result to include the case in which $\mathbf{p} \notin \mathrm{int}K$.

The False Centre Theorem was conjectured in a previous paper by Rogers [19]. This work concerned the conditions under which two convex bodies could be said to be directly homothetic; that is, related by scaling and translation alone. In the result relevant to the present work, Rogers showed the following. Let K, L be convex bodies with $\mathbf{p} \in \mathrm{int}K$ and $\mathbf{q} \in \mathrm{int}L$; suppose that for every two-dimensional subspace F the section $K \cap (F + \mathbf{p})$ is directly homothetic to the section $L \cap (F + \mathbf{q})$. Then K and L are directly homothetic.

This gives an immediate proof of a characterisation of the Euclidean ball. If every section of K through $\mathbf{p}$ is a ball then K is a ball. Of course, this can be proved by other means and is a relatively weak characterisation. However, Rogers' work highlights the potential for a stronger characterisation. In 1991 Montejano [16] published a paper giving the following result. If K is a convex body with interior point $\mathbf{p}$ such that every one-codimensional section through $\mathbf{p}$ is homothetic (related by scaling, translation and an orthogonal transformation) to every other, then K is a Euclidean ball. Montejano proves more than this. In order to prove the result, he first shows that if *homothetic* is replaced by *affinely equivalent* (related by an affine map and homothesis) then K is determined as an ellipsoid.

In Chapter 4 an attempt is made at extending Montejano's result to include the case $\mathbf{p} \notin \text{int}K$. In fact this cannot be done in general, for counterexamples exist when $\mathbf{p} \in \partial K$. Intuitively, these counterexamples are close to solid hemispheres with $\mathbf{p}$ at the centre of the ball; this is a promising place to look for a counterexample since all but one of the sections of the hemisphere through $\mathbf{p}$ are one-codimensional solid hemispheres. In the case in which $\mathbf{p}$ lies completely outside K, we provide a much restricted version of Montejano's theorem.

Over the last few decades, significant progress has been made on these problems and tomography in general. Almost always, however, the work has concentrated on chords or sections taken through a point. One exception to the rule is a relatively old result of Olovjanischnikoff [17]. Suppose that K is a convex body in Euclidean space of three dimensions or above. Let $\lambda > 0$ be given. In Olovjanischnikoff's notation, a λ-section of K is a one-codimensional section of K which divides the volume of K in the ratio $\lambda : 1$. He shows that if every λ-section of K is centrally symmetric, then K is an ellipsoid.

If K is centrally symmetric the λ-sections of K support a convex body in the interior of K, usually referred to as the floating body K_λ (see Meyer and Reisner [14]). Thus Olovjanischnikoff has used a property of sections supporting a convex body in the interior of K to deduce global information about K.

We conjecture that there is nothing special about the floating body in this context. More specifically, that if K, L are convex bodies (again of dimension at least three) with L contained in the interior of K, and if every one-codimensional section of K supporting L is centrally symmetric, then K is an ellipsoid. Again, a proof of the full conjecture has eluded us thus far. However Chapter 2 gives a restricted version.

Investigation of special cases of this problem soon led to another conjecture.

Consider the case in which $K, L \subset \mathbb{R}^3$ and both are centrally symmetric. Making further restrictions on L we used the methods of Chapter 2 to show that two-dimensional sections of K which intersect L have an interesting property. If F is such a section of K and L_F is the projection of L onto F, then every parallel pair of chords of F which support L_F have the same length. Since L_F is centrally symmetric, it seems natural to ask whether this implies that F is also centrally symmetric. If so, a restriction of the ellipsoid characterisation would follow (since all two-dimensional sections of K which *intersect* L are now centrally symmetric).

Stated in a more general form, this new problem, which we refer to as the one body problem, reads as follows. Suppose that K, M and L are planar convex bodies with L contained in the interior of both K and M. If whenever l is a supporting line of L the chords $K \cap l$ and $M \cap l$ have equal length, is it necessarily true that $K = M$? Several aspects of this problem are explored in Chapter 3, and we also look at a higher dimensional analogue.

During the initial investigation of the one body problem, Prof. Rogers suggested that we look at work carried out by K.J. Falconer [3] and [4] on Hammer's X-Ray problem for point sources, the problem first proposed by P.C. Hammer [9] at the 1961 AMS Symposium on Convexity. As the title of the problem suggests, Falconer's work concerns chord-lengths through points rather than supporting convex bodies. He gives the following result. Let K, L be convex bodies in the plane, and $\mathbf{p}, \mathbf{q}$ distinct points. Suppose that whenever l is a line passing through $\mathbf{p}$ or $\mathbf{q}$ the lengths of the chords $K \cap l$ and $L \cap l$ are equal. Then $K = L$. There are certain conditions on $\mathbf{p}$ and $\mathbf{q}$ but these may be neglected for the present. In Chapter 1 we extend this result, replacing $\mathbf{p}$ and $\mathbf{q}$ by convex bodies, in effect, putting two holes in the plane. We show that in fact the extension is locally no different from the original point oriented X-Ray problem. The result obtained is used to prove the ellipsoid characterisation in Chapter 2.

For the most part, the techniques used in the following work are elementary.

Some of the proofs are highly geometric; it is hoped that the reader will find the illustrations of help in following the arguments. This said, familiarity with the statement and application of the Stable Manifold Theorem would be an advantage to readers of Chapter 1, and in Chapter 3, use is made of spherical harmonics and the related Gegenbauer polynomials. A general background knowledge of spherical harmonics is assumed, but the work takes most of its inspiration from a short paper by Schneider [20].

For the reader interested in exploring further the field of geometric tomography, the following may be of interest. R.J. Gardner and A. Volcic give a systematic treatment of a number of tomographic problems in [7]. This paper includes many useful references. The book by R.J. Gardner [5] is also of interest.

An excellent introduction to spherical harmonics and their application to combinatorial geometric problems is given by J.J. Seidel in [21]. A discussion of the application of spherical harmonics, which is more relevant to the present work, may be found the the recent book by H. Groemer [8].

Dynamical systems theory is used increasingly in this sort of work. A comprehensive study is given by Katok and Hasselblatt in [12]. For a gentler introduction see Arrowsmith and Place [2].

Chapter 1

The Two Body Problem

1.1 Introduction

Certain X-Ray problems have received much attention in recent years leading to significant advances in technology. In particular Computer Aided Tomography has become an invaluable medical imaging technique used in diagnosis and in treatment. The mathematics of practical applications such as CAT scanning involves the reconstruction of a density function using X-Ray pictures taken from sufficiently many directions. A related class of problems is that of convex tomography. Here we are concerned with the reconstruction of a homogeneous convex body from X-Ray type information. These problems are probably of less practical value to medical physics. However, they are of independent interest, representing difficult problems in geometry and dynamical systems.

Suppose a set S of lines in $\mathbb{R}^n$ is given. If $K \subset \mathbb{R}^n$ is a convex body, define the functional

$$\begin{aligned} X_{K,S} &: S \to \mathbb{R} \\ & l \mapsto |K \cap l|, \end{aligned}$$

where $|\cdot|$ denotes Euclidean length, and $|\emptyset|$ is taken to be zero. That is, we measure the length of the chord cut by intersecting each line l with K. Using the obvious analogy call $X_{K,S}$ the X-Ray picture of K with respect to S. In convex tomography we fix S and attempt to decide whether or not it is possible to distinguish between convex bodies using their X-Ray pictures with respect to S. If so, it may be feasible to produce an algorithm or formula for the reconstruction of a convex body from its X-Ray picture.

A generic example of this type of problem was first proposed by P.C. Hammer [9] in 1961. Two formulations were presented. In the first a (finite) set D, of directions in $\mathbb{R}^n$, is chosen and S consists of all lines in $\mathbb{R}^n$ parallel to at least one member of D. Hammer asked how large D must be in order to guarantee that any convex body K is uniquely determined by $X_{K,S}$. In 1980 R.J. Gardner and P. McMullen [6] showed that, in general, four directions will suffice, although they must be chosen carefully. The second formulation proposed by Hammer is of more direct relevance to the present work; choose a (finite) set $P \subset \mathbb{R}^n$, and let S consist of all lines in $\mathbb{R}^n$ meeting at least one point of P. Again, Hammer asks how large P must be in order that X-Rays with respect to S distinguish between convex bodies. In [3] and [4] K.J. Falconer provided a partial solution.

For a convex body $K \subset \mathbb{R}^2$ and $\mathbf{x} \in \mathbb{R}^2$, let $S(\mathbf{x})$ denote the set of lines passing through $\mathbf{x}$, and $f_{K,\mathbf{x}} : S(\mathbf{x}) \longrightarrow \mathbb{R}$ be defined by $f_{K,\mathbf{x}}(l) = |l \cap K|$; in this context the function $f_{K,\mathbf{x}}$ is generally referred to as the chord-function of K at $\mathbf{x}$. A restriction of Falconer's result reads as follows:

Theorem 1.1 *Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ be distinct points and $K, L \subset \mathbb{R}^2$ be convex bodies with $\mathbf{p}, \mathbf{q} \in \mathrm{int}(K \cap L)$; suppose that $f_{K,\mathbf{p}} = f_{L,\mathbf{p}}$ and $f_{K,\mathbf{q}} = f_{L,\mathbf{q}}$. Then $K = L$.*

It is worth noting that in terms of Hammer's original question, this result is the best possible. Theorem 1.1 shows that we may choose P with $|P| \leq 2$. The following example demonstrates that $|P| \geq 2$ is required. Let

$$r_1(\theta) = 1 - \frac{1}{10}\cos\theta,$$

$$r_2(\theta) \quad = \quad 1 + \frac{1}{10}\cos\theta,$$

and let $K_i = \{\lambda\xi\ (\theta) \mid 0 \leq \lambda \leq r_i(\theta) \text{ and } \theta \in [0, 2\pi)\}$. Then the chord of K_i making an angle θ with the x-axis has length

$$l_i(\theta) = r(\theta) + r(\theta + \pi) = 2.$$

Thus we have $f_{K_1,\mathbf{0}} = f_{K_2,\mathbf{0}}$ but $K_1 \neq K_2$. In fact, both K_1 and K_2 also have chord-lengths at $\mathbf{0}$ matching those of the unit ball.

Falconer's work goes beyond the fundamental uniqueness result of Theorem 1.1. His method uses a more general definition of the chord-function, and allows the possibility that $\mathbf{p}$ or $\mathbf{q}$ may lie outside K or L. He also gives an indication of how one might reconstruct a convex body from its chord-functions. In this chapter we give an extension of the uniqueness result, Theorem 1.1, by replacing $\mathbf{p}$ and $\mathbf{q}$ by sufficiently distinct convex bodies L_1, L_2 and replacing chords through $\mathbf{p}$ or $\mathbf{q}$ by chords supporting L_1 or L_2.

1.2 Preliminary Definitions

In order to simplify the statement of the main result, it is necessary to start with a number of definitions. To begin with a suitable definition of the chord-function.

For $K \subset \mathbb{R}^2$ convex and compact let S_K denote the family of supporting lines of K. If $L \subset \mathbb{R}^2$ is convex and compact and $K \subset \mathbb{R}^2$ is a convex body, define $f_{K,L} : S_L \longrightarrow \mathbb{R}$ by $f_{K,L}(l) = |K \cap l|$; as above $|\emptyset|$ is taken to be 0. Henceforth the function $f_{K,L}$ will be referred to as the chord-function of K with respect to L.

In order to apply Falconer's method successfully, it is necessary to find a condition on a pair of convex bodies which ensures that they are sufficiently distinct whilst allowing the possibility of intersection. The next definition suffices for this purpose.

Let $K, L \subset \mathbb{R}^2$ be convex bodies. The pair (K, L) is defined as *proper* if whenever $l \in S_K \cap S_L$ the set $K \cap L \cap l$ is empty.

As an example of a proper pair of convex bodies, consider the unit ball B and its translate $\mathrm{B} + \mathbf{c}$ with $\mathbf{c}$ non-zero. There are exactly two lines supporting both these bodies (the pair of supporting lines of B, parallel to $\mathbf{c}$) neither of which meet $\mathrm{B} \cap (\mathrm{B} + \mathbf{c})$.

1.3 Statement of Main Result

Theorem 1.2 *Let $K, M, L_1, L_2 \subset \mathbb{R}^2$ be convex bodies with (L_1, L_2) proper and $L_i \subset \mathrm{int}K \cap \mathrm{int}M$ $(i = 1, 2)$. Suppose that $f_{K,L_i} = f_{M,L_i}$ $(i = 1, 2)$. Then $K = M$.*

It should be noted that whilst the proof of Theorem 1.2 follows Falconer's original work closely, it is not constructive. In order to simplify the proof of Theorem 1.2 it is necessary to begin by presenting several more definitions and a number of technical lemmas.

1.4 Preliminary Results

Suppose that $K \subset \mathbb{R}^2$ is a convex body and $L \subset \mathrm{int}K$ is convex and compact. Given $\mathbf{x} \in \mathbb{R}^2 \setminus L$ there are exactly two lines $l_L^+(\mathbf{x}), l_L^-(\mathbf{x}) \in S_L$ containing $\mathbf{x}$. Let $\mathbf{u}_L^+(\mathbf{x})$ be the point of $L \cap l_L^+(\mathbf{x})$ farthest from $\mathbf{x}$, and $\mathbf{u}_L^-(\mathbf{x})$ be the point of $L \cap l_L^-(\mathbf{x})$ closest to $\mathbf{x}$. It is assumed without loss that the lines l^+ and l^- are chosen so that the vectors $\mathbf{u}^-, \mathbf{x}, \mathbf{u}^+$ describe the vertices of a triangle in the positive sense. See figure 1.1 for clarification.

The key concept in the proof of Theorem 1.2 is now introduced. With K and L

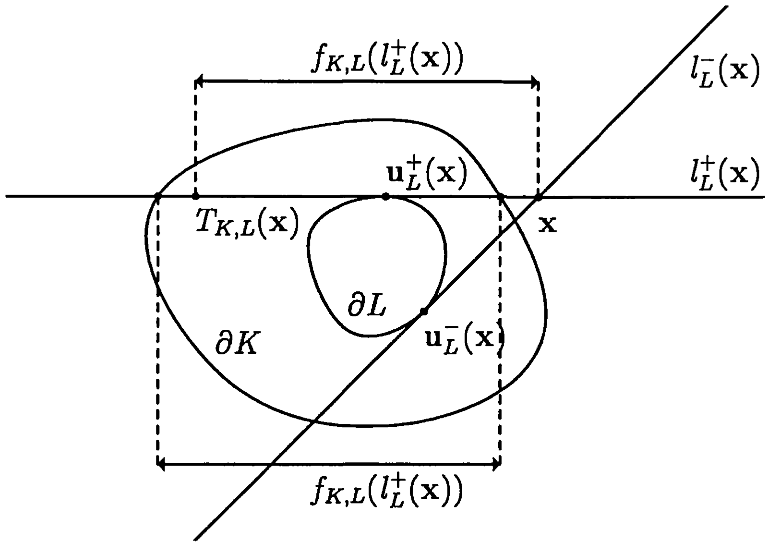

Figure 1.1: Definition of chord-function and chord-map.

as above, define the chord-map $T_{K,L} : \mathbb{R}^2 \setminus L \longrightarrow \mathbb{R}^2$ of K with respect to L by

$$T_{K,L}(\mathbf{x}) = \mathbf{x} + f_{K,L}(l_L^+(\mathbf{x})) \cdot \frac{\mathbf{u}_L^+(\mathbf{x}) - \mathbf{x}}{\left\|\mathbf{u}_L^+(\mathbf{x}) - \mathbf{x}\right\|}.$$

See figure 1.1 for an illustration. The map $T_{K,L}$ moves $\mathbf{x}$ a distance d along l_L^+ in the direction of L, where d is the length of the chord of K cut by l_L^+.

Note that, in general, the domain and image of $T_{K,L}$ do not coincide. However, it is clear that if $\mathbf{x}$ is taken to be close to the boundary ∂K of K, we may assume that $T_{K,L}(\mathbf{x}) \in \mathbb{R}^2 \setminus L$. To formalise this, set

$$D_{K,L} = \left\{\mathbf{x} \in \mathbb{R}^2 \setminus L \;\middle|\; \left\|\mathbf{x} - \mathbf{u}_L^+(\mathbf{x})\right\| < f_{K,L}(\mathbf{x})\right\},$$

and think of $T_{K,L}$ as a map from $D_{K,L}$ to $\mathbb{R}^2 \setminus L$. This definition also guarantees that $l_L^+ \cap L$ lies in the interior of the line segment joining $\mathbf{x}$ to $T_{K,L}(\mathbf{x})$.

The map $T_{K,L}$ will be required to satisfy certain Lipschitz conditions. First recall the definition of the Lipschitz constant. Let $N \subset \mathbb{R}^2$ be a (small) neighbourhood of $\mathbf{0}$, and suppose $\phi : N \longrightarrow \mathbb{R}^2$. The Lipschitz constant $L(\phi)$ of ϕ with respect to N is defined by

$$L(\phi) = \inf\left\{c > 0 \;\middle|\; \|\phi(\mathbf{x}) - \phi(\mathbf{y})\| \leq c\|\mathbf{x} - \mathbf{y}\| \;\; \forall \mathbf{x}, \mathbf{y} \in N\right\}.$$

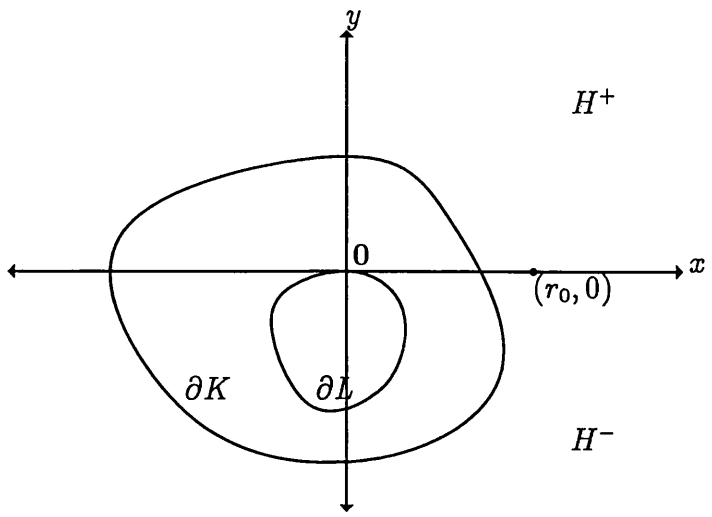

Figure 1.2: Configuration referred to in Lemma 1.4.1.

The proof of Theorem 1.2 relies on the next lemma which effectively shows that the present problem is, at least locally, indistinct from that studied by Falconer. Let $K, L \subset \mathbb{R}^2$ be convex bodies with $L \subset \text{int}K$. Using Cartesian coordinates assume that the x-axis is a support line of L, and that L is contained in the region $y \leq 0$ with $\inf\{x \in \mathbb{R} \mid (x,0)^T \in L\} = 0$. Further, let $r_0 > 0$ with $(r_0, 0)^T \in D_{K,L}$ and let H^+,H^- denote the regions $y \geq 0$ and $y \leq 0$ respectively. This situation is presented in figure 1.2.

To reduce notational clutter write $f = f_{K,L}$ and $T = T_{K,L}$. Also let $\tilde{f} = f_{K,\{\mathbf{0}\}}$ and $\tilde{T} = T_{K,\{\mathbf{0}\}}$. Here $\tilde{f}$ is the chord function of K at $\mathbf{0}$ and $\tilde{T}$ the corresponding chord map.

In what follows a neighbourhood of $(r_0, 0)^T$ will always refer to a neighbourhood of $(r_0, 0)^T$ in $\mathbb{R}^2$ restricted to H^+. We show that there exists a neighbourhood in which T and $\tilde{T}$ are Lipschitz close.

Lemma 1.4.1 *Given $\epsilon > 0$ there exists a neighbourhood of $(r_0, 0)^T$, contained in H^+, in which $L(T - \tilde{T}) < \epsilon$.*

Proof of 1.4.1 Let $\epsilon > 0$ be given. For $\theta \in [0, 2\pi)$ put $\xi\ (\theta) = (\cos\theta, \sin\theta)^T$ and

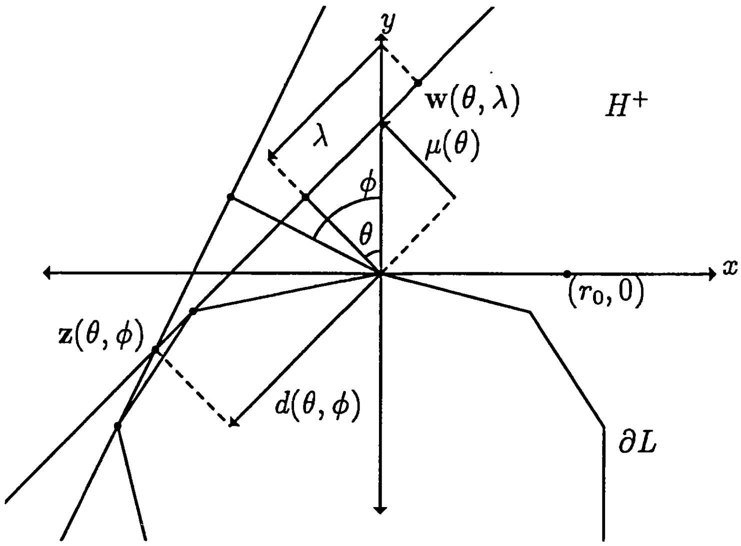

Figure 1.3: The support function μ is order θ^2 when θ is small.

let μ denote the support function of L at $\mathbf{0}$ defined by

$$\mu(\theta) = \sup\left\{\left\langle \xi\ (\theta + \frac{\pi}{2}), \mathbf{x}\right\rangle \;\middle|\; \mathbf{x} \in L\right\}.$$

That is, $\mu(\theta)$ is the distance from the origin of the supporting line of L parallel to $\xi\ (\theta)$. Convexity of L guarantees that μ is nowhere negative; that μ is continuous follows from the convexity and compactness of L.

We claim that μ is Lipschitz close to zero in a neighbourhood of $\theta = 0$; that is for $0 < \theta < \phi$ sufficiently small

$$|\mu(\phi) - \mu(\theta)| \leq \epsilon(\phi - \theta). \tag{1.1}$$

To see this put $\delta\theta = \phi - \theta$ and consider figure 1.3. Making the definitions

$$\begin{aligned} \mathbf{w}(\alpha, \lambda) &= \mu(\alpha)\xi\ (\alpha + \frac{\pi}{2}) + \lambda\xi\ (\alpha), \\ l(\alpha) &= \{\mathbf{w}(\alpha, \lambda) \mid \lambda \in \mathbb{R}\} \in S_L, \end{aligned}$$

let $\mathbf{z}(\theta, \phi)$ be the unique point of intersection between $l(\theta)$ and $l(\phi)$. Put

$$h(\lambda) = \left\langle \mathbf{w}(\theta, \lambda), \xi\ (\phi + \frac{\pi}{2})\right\rangle - \mu(\phi),$$

and let $d(\theta, \phi)$ be the $\boldsymbol{\xi}\ (\theta)$ component of $\mathbf{z}(\theta, \phi)$

$$d(\theta, \phi) = \langle \mathbf{z}(\theta, \phi), \boldsymbol{\xi}\ (\theta) \rangle.$$

Expanding the expression for h in terms of the definitions previously given we obtain

$$h(\lambda) = \mu(\theta) \cos \delta\theta - \mu(\phi) - \lambda \sin \delta\theta,$$

and by definition $h(\lambda) = 0$ only when $\mathbf{w}(\theta, \lambda)$ lies on the line $l(\phi)$. Hence $d(\theta, \phi)$ is the unique root of the linear, decreasing function h.

We claim that $d(\theta, \phi) \leq 0$. Suppose the converse. Since $l(\theta) \in S_L$, there exists $\lambda \in \mathbb{R}$ for which $\mathbf{w}(\theta, \lambda) \in \partial L$. By definition of μ,

$$\left\langle \mathbf{w}(\theta, \lambda), \boldsymbol{\xi}\ (\phi + \frac{\pi}{2}) \right\rangle \leq \mu(\phi).$$

Hence $h(\lambda) \leq 0$. But since h is a decreasing function with unique root $d(\theta, \phi)$

$$\lambda \geq d(\theta, \phi) \geq 0.$$

However if $\lambda \geq 0$ then it follows that

$$\left\langle \mathbf{w}(\theta, \lambda), \boldsymbol{\xi}\ (\frac{\pi}{2}) \right\rangle = \mu(\theta) \cos \theta + \lambda \sin \theta > 0$$

for $\theta < \frac{\pi}{2}$. That is to say, $\mathbf{w}(\theta, \lambda) \in L$ lies above the x-axis in the region H^+ contradicting the assumption $L \subset H^-$. We conclude that the inequality

$$d(\theta, \phi) \leq 0$$

holds as claimed for θ, ϕ sufficiently small.

The next step is to show that $d(\theta, \phi) > -\epsilon$ for appropriately restricted θ and ϕ. Suppose the converse; for some $\gamma < 0$, for each $k \in \mathbb{N}$ there exist $0 < \theta_k < \phi_k < \frac{1}{k}$ such that

$$d_k = d(\theta_k, \phi_k) < \gamma.$$

Since $\mathbf{z}(\theta_k, \phi_k) \in l(\phi_k) \in S_L$ there exist $\lambda_k \in \mathbb{R}$ such that

$$\mathbf{x}_k = \mathbf{z}(\theta_k, \phi_k) + \lambda_k \boldsymbol{\xi}\,(\phi_k) \in \partial L.$$

If $\lambda_k > 0$ write $\mathbf{z}(\theta_k, \phi_k) = \mathbf{w}(\theta_k, d_k)$ and $\delta\theta_k = \phi_k - \theta_k$ to deduce that

$$\left\langle \mathbf{x}_k, \boldsymbol{\xi}\,(\theta_k + \frac{\pi}{2}) \right\rangle = \mu(\theta_k) + \lambda_k \sin\delta\theta_k > \mu(\theta_k).$$

That is, $\mathbf{x}_k \notin L$. So $\lambda_k \leq 0$ for all $k \in \mathbb{N}$. Next notice that

$$\langle \mathbf{x}_k, \boldsymbol{\xi}\,(0) \rangle = -\mu(\theta_k)\sin\theta_k + d_k \cos\theta_k + \lambda_k \cos\phi_k.$$

Now $\mu(\theta_k) \geq 0$, $\lambda_k \leq 0$ and $d(\theta_k, \phi_k) < \gamma$ together imply that for sufficiently large $k \in \mathbb{N}$

$$\langle \mathbf{x}_k, \boldsymbol{\xi}\,(0) \rangle < \frac{1}{2}\gamma. \tag{1.2}$$

Assume without loss that the sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is convergent with limit $\mathbf{x} \in \partial L$. But L is bounded and $\mathbf{x}_k \in l(\phi_k)$ for all k, so $\mathbf{x}$ lies in $l(0)$, the x-axis. Hence, using inequality 1.2

$$\mathbf{x} = (t, 0)^T \in \partial L$$

for some $t < \gamma/2$. This contradicts our assumption on $L \cap l(0)$. Thus, in summary of the results so far,

$$0 > d(\theta, \phi) > -\epsilon \tag{1.3}$$

for $0 < \theta < \phi$ sufficiently small.

It is now a relatively simple matter to prove the inequality claimed in equation 1.1. In the light of inequality 1.3 it is sufficient to prove

$$\frac{|\mu(\phi) - \mu(\theta)|}{\phi - \theta} \leq \epsilon + |d(\theta, \phi)|. \tag{1.4}$$

Inspection of figure 1.3 and application of trigonometry reveals that in fact

$$\begin{aligned}
|d(\theta, \phi)| &= \mu(\theta)\tan\delta\theta + (\mu(\phi) - \mu(\theta)\sec\delta\theta)\csc\delta\theta \\
&= (\mu(\phi) - \mu(\theta))\,\delta\theta^{-1} \\
&\quad + \mu(\theta)\tan\delta\theta \\
&\quad - (\mu(\phi) - \mu(\theta))\,\delta\theta^{-1}\,(1 - \delta\theta\csc\delta\theta) \\
&\quad + \mu(\theta)\csc\delta\theta\,(1 - \sec\delta\theta).
\end{aligned}$$

A simple exercise shows that the signed terms of this last expression are all of order at most $\delta\theta$. This completes the proof of inequality 1.1.

Next consider the system of coordinates given by the definition of $\mathbf{w}$ in equations 1.2. We wish to show that it is possible to use the distance $(\delta\lambda^2 + \delta\theta^2)^{\frac{1}{2}}$ in place of the Euclidean norm for the purposes of calculating Lipschitz constants. To do this it suffices to show that the ratio of the two distances is bounded above zero in the appropriate neighbourhood.

Let $\delta\lambda = \delta\cos\eta$ and $\delta\theta = \delta\sin\eta$ with $\delta > 0$ small; for small $\theta > 0$ and $\lambda > 0$ very close to $r > 0$ consider

$$\begin{aligned}
\tilde{\delta}^2 &= \|\mathbf{w}(\theta,\lambda) - \mathbf{w}(\theta+\delta\theta, \lambda+\delta\lambda)\|^2 \\
&= \mu^2(\theta) + \mu^2(\theta+\delta\theta) + \lambda^2 + (\lambda+\delta\lambda)^2 \\
&- 2\mu(\theta)\mu(\theta+\delta\theta)\cos\delta\theta - 2\mu(\theta)(\lambda+\delta\lambda)\sin\delta\theta \\
&+ 2\lambda\mu(\theta+\delta\theta)\sin\delta\theta - 2\lambda(\lambda+\delta\lambda)\cos\delta\theta \\
&= (1-\cos\delta\theta)\left(\mu^2(\theta) + \mu^2(\theta+\delta\theta) + \lambda^2 + (\lambda+\delta\lambda)^2\right) \\
&+ 2\sin\delta\theta\delta\lambda\left((\mu(\theta+\delta\theta) - \mu(\theta)) - \mu(\theta)\right) \\
&+ \cos\delta\theta\left(((\mu(\theta) - \mu(\theta+\delta\theta))^2 + \delta\lambda^2\right).
\end{aligned}$$

Now using 1.1 and assuming that $\theta, \delta\theta, \delta\lambda$ and $|\lambda - r|$ are small enough

$$\left(\frac{\tilde{\delta}}{\delta}\right)^2 = r^2\sin^2\eta + \cos^2\eta + e, \tag{1.5}$$

where $|e| < \epsilon$. Note in particular that the right-hand side of equation 1.5 is positive for sufficiently small ϵ.

Next consider the chord-functions f and $\tilde{f}$. It is necessary to show that they are Lipschitz close in the following sense. By an abuse of notation write $f(\theta) = f(l(\theta))$ and $\tilde{f}(\theta) = \tilde{f}(l(\theta) - \mu(\theta)\xi\ (\theta + \frac{\pi}{2}))$. Thus $f(\theta)$ is the length of the chord of K cut by the supporting line of L making an angle θ with the x-axis, and $\tilde{f}(\theta)$ is the length of the parallel chord of K through $\mathbf{0}$. Set $h(\theta) = f(\theta) - \tilde{f}(\theta)$. We claim that for $0 < \theta < \phi$ sufficiently small,

$$|h(\phi) - h(\theta)| < \epsilon(\phi - \theta). \tag{1.6}$$

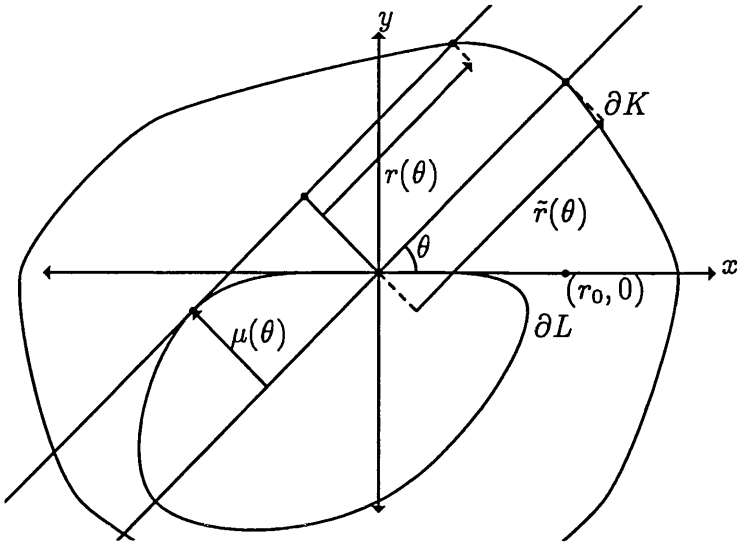

Figure 1.4: f and $\tilde{f}$ are Lipschitz close near $\theta = 0$

Referring to figure 1.4, let r and $\tilde{r}$ denote the half-chord lengths given by

$$\begin{aligned} r(\theta) &= \sup\{\lambda \in \mathbb{R} \mid \mathrm{w}(\theta, \lambda) \in K\}, \\ \tilde{r}(\theta) &= \sup\{\lambda \in \mathbb{R} \mid \lambda \xi\,(\theta) \in K\}. \end{aligned}$$

Notice that since $\tilde{r}$ is the usual radial description of the convex boundary ∂K, there exists $\gamma \in \mathbb{R}$ with

$$|\tilde{r}(\phi) - \tilde{r}(\theta) - \gamma(\phi - \theta)| \leq \epsilon(\phi - \theta). \tag{1.7}$$

Thus, using the standard properties of Lipschitz constants, we have for suitably small $0 < \theta < \phi$,

$$\left|\tilde{r}^2(\phi) - \tilde{r}^2(\theta) - 2\tilde{r}(0)\gamma(\phi - \theta)\right| < \epsilon(\phi - \theta). \tag{1.8}$$

Using the geometry it is clear that for all θ,

$$\tilde{r}^2\left(\theta + \tan^{-1}\frac{\mu(\theta)}{r(\theta)}\right) = r^2(\theta) + \mu^2(\theta),$$

so, using equations 1.8 and 1.1, for $0 < \theta < \phi$ sufficiently small,

$$r^2(\phi) - r^2(\theta) + 2\tilde{r}(0)\gamma\left(\phi - \theta + g(\theta, \phi)\right) = e_1\left(\phi - \theta + g(\theta, \phi)\right), \tag{1.9}$$

where the difference g is given by

$$g(\theta, \phi) = \tan^{-1}\left(\frac{\mu(\phi)}{r(\phi)}\right) - \tan^{-1}\left(\frac{\mu(\theta)}{r(\theta)}\right),$$

and the error $|e_1(\theta, \phi)| < \epsilon$. Now $\tan^{-1}$ is differentiable at 0 with derivative 1; we may thus assume that

$$g(\theta, \phi) = \left\{\frac{\mu(\phi)}{r(\phi)} - \frac{\mu(\theta)}{r(\theta)}\right\}(1 + e_2(\theta, \phi)),$$

with $|e_2(\theta, \phi)| < \epsilon$. Notice, also that g may be rewritten

$$\begin{aligned} g(\theta, \phi) = & \frac{1+e_2}{2r(\theta)r(\phi)}(r(\theta) + r(\phi))(\mu(\phi) - \mu(\theta)) \\ + & \frac{1+e_2}{2r(\theta)r(\phi)}(r(\theta) - r(\phi))(\mu(\phi) + \mu(\theta)). \end{aligned}$$

Now $|\mu(\phi) - \mu(\theta)| < \epsilon(\phi - \theta)$, and $\mu(\phi) + \mu(\theta)$ is small. Hence it is possible to simplify g thus:

$$g(\theta, \phi) = (\phi - \theta)e_3 + (r(\theta) - r(\phi))\, e_4, \tag{1.10}$$

where, as usual, e_3 and e_4 are small. Now using equation 1.9 and the fact that r is continuous and close to $\tilde{r}(0)$, we obtain

$$(r(\theta) - r(\phi))(2\tilde{r}(0) + e_5) - 2\tilde{r}(0)\gamma(\phi - \theta) = e_6(\phi - \theta),$$

where, for $0 < \theta < \phi$ suitably small, the error terms satisfy $|e_5|, |e_6| < \epsilon$. Finally, in a small neighbourhood, we have

$$|r(\phi) - r(\theta) - \gamma(\phi - \theta)| < \epsilon(\phi - \theta),$$

showing that r and $\tilde{r}$ are Lipschitz close near $\theta = 0$.

A similar argument holds for the pair of half-chord-lengths s and $\tilde{s}$ given by

$$\begin{aligned} s(\theta) &= \sup\{\lambda > 0 \mid \mathbf{w}(\theta, -\lambda) \in K\} \\ \tilde{s}(\theta) &= \tilde{r}(\theta + \pi), \end{aligned}$$

so the functions f and $\tilde{f}$ are Lipschitz close near $\theta = 0$ as claimed in equation 1.6.

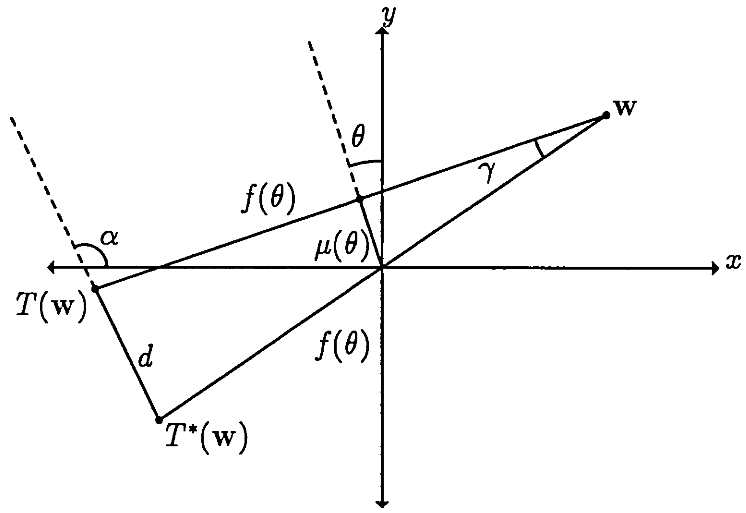

Figure 1.5: T and T^* are Lipschitz close near $(r_0, 0)^T$

Next define a new chord-map T^* by

$$T^*(\mathbf{w}(\theta,\lambda)) = \mathbf{w}(\theta,\lambda)\left(1 - \frac{f(\theta)}{\|\mathbf{w}(\theta,\lambda)\|}\right).$$

T^* is merely a convenient construction and has no particular geometric significance. The distance of $T^*(\mathbf{x})$ from $\mathbf{x}$ is the same as the distance of $T(\mathbf{x})$ from $\mathbf{x}$. However $T^*(\mathbf{x})$ lies on the line joining $\mathbf{0}$ to $\mathbf{x}$ rather than a supporting line of L. Figure 1.5 shows the relationship between T and T^* which we now explore in detail. Notice that

$$\tilde{T}(\mathbf{w}(\theta,\lambda)) - T^*(\mathbf{w}(\theta,\lambda)) = \left(f(\theta) - \tilde{f}(\theta)\right)\frac{\mathbf{w}(\theta,\lambda)}{\|\mathbf{w}(\theta,\lambda)\|},$$

so, after some calculation using equations 1.5 and 1.6, there is a neighbourhood of $(r_0, 0)^T$ in which $L(\tilde{T} - T^*) < \epsilon$. Thus to prove the lemma it suffices to show that $L(T - T^*) < \epsilon$ close enough to $(r_0, 0)^T$.

Consider figure 1.5. The definitions are as follows

$$\begin{aligned} \gamma &= \tan^{-1}\tfrac{\mu(\theta)}{\lambda}, \\ d &= \|T(\mathbf{w}) - T^*(\mathbf{w})\| = f(\theta)\sin\tfrac{\gamma}{2}, \\ \alpha &= \theta + \tfrac{\gamma}{2} + \tfrac{\pi}{2}, \end{aligned}$$

where $\mathbf{w} = \mathbf{w}(\theta, \lambda)$, α is the angle $T(\mathbf{w}) - T^*(\mathbf{w})$ makes with the x-axis in the positive sense, and γ is the angle between the line segments joining $T(\mathbf{w})$ and $T^*(\mathbf{w})$ to $\mathbf{w}$. In terms of these definitions,

$$T(\mathbf{w}) - T^*(\mathbf{w}) = f(\theta) \sin \frac{\gamma}{2} \xi\ (\alpha).$$

Suppose $\delta\theta, \delta\lambda \in \mathbb{R}$ are close to zero and set

$$\begin{aligned} \tilde{\gamma} &= \tan^{-1} \frac{\mu(\theta + \delta\theta)}{\lambda + \delta\lambda}, \\ \tilde{\mathbf{w}} &= \mathbf{w}(\theta + \delta\theta, \lambda + \delta\lambda), \\ \tilde{\alpha} &= \theta + \delta\theta + \frac{\tilde{\gamma}}{2} + \frac{\pi}{2}. \end{aligned}$$

Then if $E = 2^{\frac{1}{2}} \| T(\mathbf{w}) - T^*(\mathbf{w}) - T(\tilde{\mathbf{w}}) + T^*(\tilde{\mathbf{w}}) \|$, we have

$$E^2 = (A + B)^2 (1 - \cos \nu) + (A - B)^2 (1 + \cos \nu),$$

where A, B and ν are given by

$$\begin{aligned} A &= f(\theta) \sin \frac{\gamma}{2}, \\ B &= f(\theta + \delta\theta) \sin \frac{\tilde{\gamma}}{2}, \\ \nu &= \delta\theta + \frac{\tilde{\gamma} - \gamma}{2}. \end{aligned}$$

Using the results previously derived in the course of this proof it is now relatively easy, if somewhat tedious, to see that $E^2 < \epsilon(\delta\theta^2 + \delta\lambda^2)$ whenever $\mathbf{w}$ and $\tilde{\mathbf{w}}$ lie in a small enough neighbourhood of $(r_0, 0)^T$. Hence using equation 1.5 there is a neighbourhood of $(r_0, 0)^T$ in which $E < \epsilon \| \mathbf{w} - \tilde{\mathbf{w}} \|$; this completes the proof. □

The next result was proved by Falconer in [4]; we provide a proof here for completeness. We aim to approximate the chord-map $\tilde{T}$, the chord-map through the origin defined in Lemma 1.4.1, by a linear map. Assume that for $0 < \theta < \phi$ sufficiently small

$$\left| \tilde{f}(\phi) - \tilde{f}(\theta) - \gamma(\phi - \theta) \right| < \epsilon(\phi - \theta),$$

and put $\tilde{f}_0 = \tilde{f}(0)$. Define the linear map T_{lin} by

$$T_{\text{lin}} = \begin{pmatrix} 1 & -\gamma/r_0 \\ 0 & 1 - \tilde{f}_0/r_0 \end{pmatrix}.$$

Lemma 1.4.2 *With T_{lin} defined as above, there is a neighbourhood of $(r_0, 0)^T$, contained in H^+, in which $L(\tilde{T} - T_{\text{lin}}) < \epsilon$.*

Proof of 1.4.2 First note that using standard polar coordinates $\tilde{T}$ may be expressed by

$$T_p : (r, \theta) \mapsto (\tilde{f}(\theta) - r, \theta + \pi).$$

Let $|(x, y)| = \left\|(x, y)^T\right\|$. If $r, \delta r > 0$ and $0 < \theta, \delta\theta$ are sufficiently small

$$\begin{aligned} & |T_p(r + \delta r, \theta + \delta\theta) - T_p(r, \theta) - (\gamma\delta\theta - \delta r, \delta\theta)| \\ = & \left|(\tilde{f}(\theta + \delta\theta) - \tilde{f}(\theta) - \delta r, \delta\theta) - (\gamma\delta\theta - dr, \delta\theta)\right| \\ \leq & \epsilon\delta\theta. \end{aligned} \tag{1.11}$$

Next introduce the differentiable map

$$\Psi : (r, \theta) \mapsto (r\cos\theta, r\sin\theta)^T,$$

so that $\tilde{T} = \Psi T_p \Psi^{-1}$. Then if δx and δy are small enough,

$$\left|\Psi^{-1}(r_0 + \delta x, \delta y)^T - (r_0 + \delta x, \delta y/r_0)\right| \leq \epsilon|(\delta x, \delta y)|$$

and similarly, if δr and $\delta\theta$ are small enough,

$$\left\|\Psi(\tilde{f}_0 - r_0 + \delta r, \pi + \delta\theta) - (r_0 - \tilde{f}_0 - \delta r, (r_0 - \tilde{f}_0)\delta\theta)^T\right\| \leq \epsilon|(\delta r, \delta\theta)|.$$

Thus, in a suitably small neighbourhood of $(r_0, 0)^T$,

$$\begin{aligned} & \left\|T(x + \delta x, y + \delta y)^T - T(x, y)^T - (\delta x - \delta y\gamma/r_0, (1 - \tilde{f}_0/r_0)\delta y)^T\right\| \\ = & \left\|T(x + \delta x, y + \delta y)^T - T(x, y)^T - T_{\text{lin}}(\delta x, \delta y)^T\right\| \\ \leq & \epsilon|(\delta x, \delta y)| \end{aligned}$$

as required. □

It is possible to prove Theorem 1.2 directly, using the methods of Falconer in [3] for example. However, it is certainly much easier to follow Falconer's method in [4], invoking a substantial result from the theory of dynamical systems known as the Stable Manifold Theorem, henceforth abbreviated SMT.

The SMT is essentially a linearisation theorem, providing topological information on the behaviour of maps in the neighbourhood of a fixed point. It arises in many and various guises and is customarily stated in the form most appropriate to the context. Here we reproduce, almost verbatim, Falconer's statement of the result in [4].

A linear operator T on a Banach space $(E, |\cdot|)$ is said to be ρ-pseudo hyperbolic if the spectrum of T lies off the circle of radius ρ, centre 0, in the complex plane. Corresponding to this division of the spectrum, E may be written $E = E_1 \oplus E_2$, where E_1 and E_2 are closed and invariant under T, and, writing T_i for the restriction of T to E_i, the spectrum of T_1 lies inside the circle of radius ρ, and the spectrum of T_2 outside this circle. Under these circumstances, it is possible to define a new norm on E such that: $|T_1| < \rho$; if $x \in E_1$ then $|Tx| > \rho|x|$; finally, if $(x_1, x_2) \in E_1 \times E_2$ then

$$|(x_1, x_2)| = \max\{|x_1|, |x_2|\}.$$

In the statement of the SMT below, the Lipschitz constant is defined in terms of this norm. That is, given a neighbourhood N, $L(\psi)$ is the smallest number L such that

$$|\psi(x) - \psi(y)| \leq L|x - y|$$

for all $x, y \in N$, or ∞ if no such number exists. For our purposes $E = \mathbb{R}^2$, so norm equivalence will allow us to ignore the precise definition of the norm involved.

Theorem 1.3 (Stable Manifold Theorem) *Let T be a ρ-pseudo hyperbolic linear operator on a Banach space E, where $\rho \leq 1$. Let $E = E_1 \oplus E_2$ be the corresponding* *splitting of E. Let ψ be a continuous mapping on E defined near* 0 *such that* $\psi(0) = 0$, *and* $L(\psi - T) < \epsilon$. *Provided that ϵ is sufficiently small, there is a neighbourhood N of* 0 *such that the set*

$$W = \left\{ x \in N \;\middle|\; \lim_{j \to \infty} \left|\psi^j(x)\right| \rho^{-j} = 0 \right\}$$

is the graph of a unique continuous function from $E_2 \cap N$ to $E_1 \cap N$. Moreover, if ψ is a C^r mapping, then W is the graph of a C^r function, and the manifold W depends continuously on ψ in the C^r sense.

We will also require the following corollary.

Lemma 1.4.3 *Suppose that the hypotheses of Theorem 1.3 hold, and in addition, whenever $x \in E_1$ and $\psi(x)$ is defined we have $\psi(x) = x$. Then if ϵ is sufficiently small, W may be written*

$$W = \left\{ \mathrm{x} \in N \;\middle|\; \lim_{k \to \infty} \psi^k(x) = 0 \right\}.$$

Proofs of both these results may be found in [4], although it should be noted that the proof of Theorem 1.3 essentially consists of references to the work by Hirsch and Pugh [10] and Hirsch, Pugh and Shub [11]. The SMT and its ramifications are discussed in most introductory texts on Dynamical Systems, see Arrowsmith and Place [2] for example.

1.5 Proof of Main Result

The method used to prove Theorem 1.2 consists of three distinct stages. We first show that the boundaries of K and M meet at some fixed point of the map $T^{-1}_{K,L_2} T_{K,L_1}$; in [4] Falconer showed that such a point can be located precisely and called this step *location of the baseline*; in the present work we are content to show its existence. The second stage uses the SMT to show that ∂K and ∂M

coincide in a neighbourhood of the fixed point. Finally we use a chord-chasing argument to extend this region of coincidence to cover the whole of ∂K.

Proof of 1.2 We begin by proving that $|S(L_1) \cap S(L_2)|$ is finite. Assuming the converse, there exist distinct $l_k \in S(L_1) \cap S(L_2)$ for $k = 1, 2, \ldots$. These lines accumulate at $l \in S(L_1) \cap S(L_2)$, say. Assume without loss that l is the x-axis and after suitable choice of subsequence and transformation

$$l_k = \left\{ (x_k, 0)^T + \lambda \xi\ (\theta_k) \ \middle|\ \lambda \in \mathbb{R} \right\},$$

with θ_k monotonic decreasing to 0 and $x_k \in \mathbb{R}$. Fix $i \in \{1, 2\}$. We may further assume that

$$0 = \inf \left\{ x \in \mathbb{R} \ \middle|\ (x, 0)^T \in L_i \right\}.$$

As in Lemma 1.4.1 define the support function μ by

$$\mu(\theta) = \sup \left\{ \left\langle \xi\ (\theta + \frac{\pi}{2}), \mathbf{x} \right\rangle \ \middle|\ \mathbf{x} \in L_i \right\},$$

then using equation 1.1 and the geometry of the configuration, given $\epsilon > 0$

$$|x_k| = \mu(\theta_k) \csc(\theta_k) \leq \epsilon$$

for k sufficiently large. Thus $x_k \to 0$ as $k \to \infty$. We have shown that the points $(x_k, 0)^T \in l \cap l_k$ tend to a limit $\mathbf{x} \in l \cap \partial L_i$. However, our choice of $i \in \{1, 2\}$ was arbitrary, so the limit point $\mathbf{x}$ is contained in $l \cap L_1 \cap L_2$ contradicting the assertion that (L_1, L_2) is proper.

For reasons which will shortly emerge, we call a point $\mathbf{x}$ of ∂K or ∂M *stable* if the two following conditions hold:

$$\begin{aligned} l^+_{L_1}(\mathbf{x}) &= l^+_{L_2}(\mathbf{x}), \\ \left\| \mathbf{u}^+_{L_2}(\mathbf{x}) - \mathbf{x} \right\| &< \left\| \mathbf{u}^+_{L_1}(\mathbf{x}) - \mathbf{x} \right\|. \end{aligned} \tag{1.12}$$

If inequality 1.12 is reversed we say that $\mathbf{x}$ is *unstable*. Note that since $S(L_1) \cap S(L_2)$ is bounded, the number n of stable or unstable points on ∂K is finite. Label

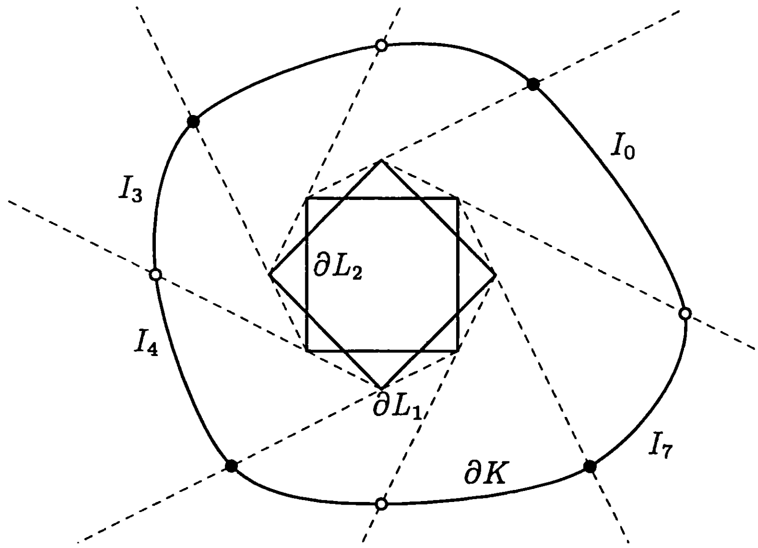

Figure 1.6: Stable and unstable points on ∂K

these points $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$ in the positive sense, and denote by I_k that section of ∂K between (in the positive sense) $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$. It is easy to see that $n \geq 2$; it will become clear that if $\mathbf{a}_k$ is stable then $\mathbf{a}_{k+1}$ is unstable, and vice versa. Figure 1.6 illustrates this notation with $n = 8$; in the diagram, filled points are stable and hollow points unstable.

Next put $T_i = T_{K,L_i}$ and introduce a new chord-map T defined by

$$T = T_2^{-1} T_1.$$

Notice that T is well defined in a neighbourhood of either ∂K or ∂M, and that by definition, T leaves both boundaries invariant. Suppose that $\mathbf{x} \in \partial K$ is a fixed point of T. Then from the definitions

$$\begin{aligned} \mathbf{x}, T_1(\mathbf{x}) &\in l^+_{L_1}(\mathbf{x}), \\ T_1(\mathbf{x}), \mathbf{x} &\in l^-_{L_2}(T_1(\mathbf{x})) = l^+_{L_2}(\mathbf{x}), \end{aligned}$$

and hence $l^+_{L_2}(\mathbf{x}) = l^+_{L_1}(\mathbf{x})$. That is, $\mathbf{x}$ is one of the points $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$, previously labeled stable or unstable.

Consider the action of T on ∂K; the maps T_1, T_2 and hence T are continuous and preserve orientation on ∂K. Hence, for each r, T leaves I_r invariant. Suppose

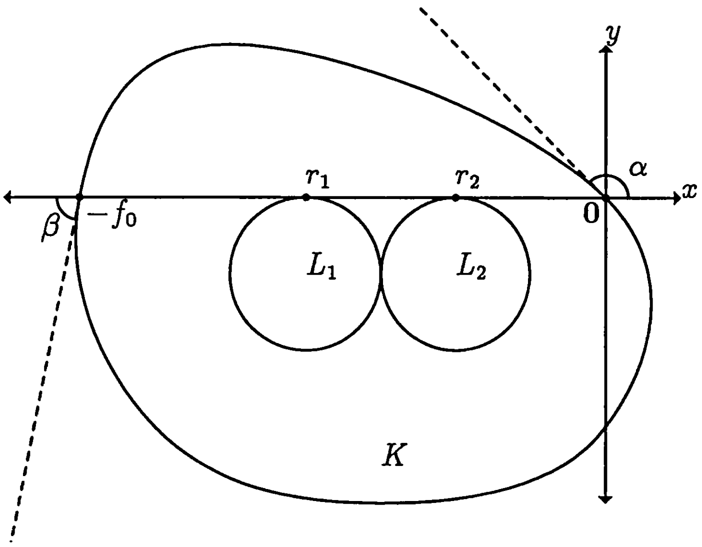

Figure 1.7: Notation used in linearisation of T near $\mathbf{0}$

that $\mathbf{x} \in I_r$, some r. The only fixed points of T on I_r are $\mathbf{a}_r$ and $\mathbf{a}_{r+1}$, so $T^m(\mathbf{x})$ tends monotonically to either $\mathbf{a}_r$ or $\mathbf{a}_{r+1}$ as m tends to infinity.

The boundaries of K and M must meet. Choose $\mathbf{x} \in \partial K \cap \partial M$, and set

$$\mathbf{x}_0 = \lim_{m \to \infty} T^m(\mathbf{x}).$$

Using what is now known about the behaviour of iterates of $\mathbf{x} \in \partial K$ under T, we may assume without loss that in fact $\mathbf{x}_0 = \mathbf{a}_0$; since $\mathbf{x} \in \partial K \cap \partial M$, which is closed and invariant under T, it follows that $\mathbf{a}_0 \in \partial K \cap \partial M$. This completes the first phase of the proof.

We now linearise T near $\mathbf{a}_0$. Assume that $\mathbf{a}_0 = \mathbf{0}$, and that the x-axis supports L_1 and L_2 which both lie in the region H^-. Referring to figure 1.7, let f_0 denote the length of the chord of K cut by the x-axis; let α denote the angle that the half-tangent, into H^+, of ∂K at $\mathbf{0}$ makes with the positive x-axis, and β the angle that the half-tangent, into H^-, of ∂K at $(-f_0, 0)^T$ makes with the negative x-axis. Clearly neither α or β are zero since $L_i \subset \text{int}K$. Finally, put

$$r_i = \sup \left\{ x \mid (-x, 0)^T \in L_i \right\}.$$

First consider the chords of K through $(-r_i, 0)^T$. Let

$$\begin{aligned} l(\theta) &= \{\lambda \xi\,(\theta) \mid \lambda \in \mathbb{R}\}, \\ \tilde{f}_i(\theta) &= \left|K \cap (l(\theta) - (r_i, 0)^T)\right|. \end{aligned}$$

Using elementary geometry it is easy to see that

$$\lim_{\theta \to 0^+} \frac{\tilde{f}_i(\theta) - f_0}{\theta} = r_i \cot \alpha + (f_0 - r_i) \cot \beta,$$

so using the fact that ∂K is convex, if $0 < \theta < \phi$ are sufficiently small,

$$\left|\tilde{f}(\phi) - \tilde{f}(\theta) - \gamma_i(\phi - \theta)\right| < \epsilon(\phi - \theta), \tag{1.13}$$

where $\gamma_i = f_0 \cot \beta + r_i(\cot \alpha - \cot \beta)$. Now write $\tilde{T}_i = T_{K,(-r_i,0)^T}$ and

$$A_i = \begin{pmatrix} 1 & -\gamma_i/r_i \\ 0 & 1 - f_0/r_i \end{pmatrix}.$$

Lemma 1.4.2 says that $L(\tilde{T}_i - A_i) < \epsilon$ restricted to neighbourhood of $\mathbf{0}$ in H^+; thus using lemma 1.4.1 there is a neighbourhood of $\mathbf{0}$ in H^+ in which $L(T_i - A_i) < \epsilon$. Furthermore, the properties of Lipschitz constants now guarantee that there is a neighbourhood of $\mathbf{0}$ in H^+ in which

$$L(T - A_2^{-1}A_1) = L(T_2^{-1}T_1 - A_2^{-1}A_1) < \epsilon.$$

Define the linearisation T_{lin} of T at $\mathbf{0}$, by

$$T_{\text{lin}} = A_2^{-1}A_1 = \begin{pmatrix} 1 & \alpha\gamma_2/r_2 - \gamma_1/r_1 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha = (r_2/r_1)(f_0 - r_1)/(f_0 - r_2)$, and immediately calculate the eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{e}_1$,$\mathbf{e}_2$ thus:

$$\begin{aligned} \lambda_2 &= 1, & \mathbf{e}_2 &= (1, 0)^T, \\ \lambda_1 &= \alpha, & \mathbf{e}_1 &= ((\alpha\gamma_2/r_2 - \gamma_1/r_1)/(\alpha - 1), 1)^T. \end{aligned}$$

Finally set

$$S(\mathbf{x}) = \begin{cases} T(\mathbf{x}) & \mathbf{x} \in H^+ \\ T_{\text{lin}}(\mathbf{x}) & \mathbf{x} \in H^- \end{cases}$$

It is easy to see that S is well defined, and furthermore, in some small region containing $\mathbf{0}$ as an interior point we have $L(S - T_{\text{lin}}) < \epsilon$.

Notice that the eigenvalues of T_{lin} depend only on the values of r_i and f_0. We may assume without loss that $r_1 > r_2$; if not we re-name L_1 and L_2 effectively inverting S and T.

The second phase of the proof proceeds. We apply lemma 1.4.3, Falconer's extension of the SMT to S at $\mathbf{0}$ and deduce that the set

$$W = \{\mathbf{x} \mid S^m(\mathbf{x}) \to \mathbf{0} \text{ as } m \to \infty\}$$

is, in the neighbourhood of $\mathbf{0}$, the graph of a unique function from the space spanned by $\mathbf{e}_2$ to that spanned by $\mathbf{e}_1$. However, previous arguments have shown that if $\mathbf{x}$ lies on I_0, iterates of $\mathbf{x}$ under T tend either to $\mathbf{a}_0$ or $\mathbf{a}_1$. Near $\mathbf{a}_0$, the linearisation of T shows that the limit must be $\mathbf{a}_0$. Analogous arguments hold for the corresponding section of ∂M close to $\mathbf{a}_0$ in H^+. Thus, the SMT shows that ∂K and ∂M coincide in a neighbourhood of $\mathbf{a}_0 = \mathbf{0}$ in H^+. This completes the second phase of the proof.

Suppose $\mathbf{x} \in \partial K \cap \partial M$ close to $\mathbf{a}_0$ in the region of coincidence. Iterates of $\mathbf{x}$ under T^{-1} tend to $\mathbf{a}_1$. Hence by continuity, ∂K and ∂M coincide in I_0; furthermore, $\mathbf{a}_1$ must be a stable point of T^{-1}. Reapplying phase two of the proof at $\mathbf{a}_1$ extends the region of intersection to $\mathbf{a}_2$, and so on. Since the number of sections $I_0, I_1, \ldots$ is finite, the proof may be completed by induction. □

1.6 Conclusions

Readers who are familiar with Falconer's work will notice that the results in [4] are considerably more general than those given here. In particular, Falconer's approach admits the characterisation of convex bodies by generalised chord-functions; if K is a convex body in $\mathbb{R}^2$ with $\mathbf{0} \in \text{int}K$, let

$$r(\theta) = \sup\{\lambda > 0 \mid \lambda\boldsymbol{\xi}\ (\theta) \in K\},$$

as usual, and for $j \in \mathbb{N}$, let

$$f_j(\theta) = r^j(\theta) + r^j(\theta + \pi).$$

Falconer showed that convex bodies are determined by chord-functions of this type at two points. It may be possible to generalise the problem in this chapter in a similar fashion when the two inner bodies L_1 and L_2 are smooth and strictly convex. However, when this is not the case, it is not possible to define a analogous generalised chord-function without introducing discontinuities.

Clearly, a convex body in the plane is not determined by its chord-lengths through a single point. Suppose, instead that we know the lengths of the chords of a convex body K cut by the supporting lines of a single convex body $L \subset \text{int}K$. We may ask whether or not K is determined. This problem, which we refer to as the one body problem, is currently open. We return to it in Chapter 3 in which several elementary observations are noted, and some partial results presented.

Chapter 2

A Characterisation of the Ellipsoid

2.1 Introduction

An ellipsoid is the image of the Euclidean ball under an affine transformation. Characterisations of the ellipsoid abound; among the better known is the False Centre Theorem [1] due to Aitchison, Petty and Rogers. A point $\mathbf{x}$ is said to be a false centre of a convex body K if all one codimensional sections of K through $\mathbf{x}$ are centrally symmetric, but $\mathbf{x}$ is not a centre of symmetry for K. The False Centre Theorem states that any convex body with a false centre in its interior is an ellipsoid. D.G. Larman [13] later extended this result to include the case in which the false centre lies outside the body. Another characterisation of the ellipsoid is due to Olovjanischnikoff [17]. Let K be a convex body, and $0 < \epsilon < 1$. Olovjanischnikoff showed that if every one codimensional section of K dividing the volume of K in the ratio $\epsilon : 1$ is centrally symmetric, then K is an ellipsoid. In [14] Meyer and Reisner showed that if K itself is centrally symmetric then the family of sections considered by Olovjanischnikoff coincide with those cut by the supporting hyperplanes of a uniquely defined convex body K_ϵ, usually referred to as the floating body. Thus, Olovjanischnikoff's result concerns sections supporting a convex body. It seems natural to generalise this by making the following conjecture.

Conjecture 2.1.1 *Let $K, L \subset \mathbb{R}^n$ be convex bodies, with $L \subset \mathrm{int}K$. If every section of K cut by a supporting hyperplane of L is centrally symmetric, then K is an ellipsoid.*

We cannot provide a proof of this conjecture. In what follows we present a partial result in $\mathbb{R}^3$ which is somewhat reminiscent both of Olovjanischnikoff's characterisation and of the False Centre Theorem. A search for a proof of Conjecture 2.1.1 led to the investigation of the one body problem investigated in Chapter 3, and later the two body problem presented in Chapter 1 upon which the main result of this chapter relies.

We will use the False Centre Theorem [1] in the final stages of the proof. We also require the well known result of F. John on maximal ellipsoids contained in convex bodies.

Theorem 2.1 *Given a centrally symmetric convex body $K \subset \mathbb{R}^n$, there is a unique ellipsoid $E \subset K$ of maximal volume, and this ellipsoid satisfies $K \subset \sqrt{n}E$.*

See Milman and Schechtman [15] for a proof. An alternative approach may be found in Pisier [18].

2.2 Main Result

Theorem 2.2 *Let $K \subset \mathbb{R}^3$ be a strictly convex body, centrally symmetric about 0. Denote by E the maximal ellipsoid contained in K, let $\alpha = \sqrt{(\sqrt{7}-1)/2}$ and* *let $L \subset \alpha E$ be a smooth, centrally symmetric, strictly convex body with centre of symmetry* $\mathbf{d} \neq \mathbf{0}$. *Suppose that whenever H is a hyperplane supporting L, the section $K \cap H$ is centrally symmetric. Then K is an ellipsoid.*

It is perhaps unfortunate that we have needed to impose restrictions on K and L. Strict convexity of K might be relatively easy to remove. That K is centrally symmetric, however, is vital. Some of the restrictions on L could possibly be circumvented whilst retaining the present method. The least aesthetic of these is the condition $L \subset \alpha E$; sadly, this is a cornerstone of the proof.

2.3 Proof of Main Result

The argument used is highly geometric. The reader might find it helpful to refer to figure 2.1 located on page 45 in which the central method of the proof is illustrated. In what follows *section* will always refer to a two-dimensional section of K.

The proof consists of the construction of many sections of K which do not support L or $-L$ but are, nevertheless, centrally symmetric. The proof can be split into three stages.

In the first stage we show that given sections $S, -S$ of K supporting L and $-L$ respectively, any section T (supporting L) which is parallel to the line joining the centres of S and $-S$ must intersect both S and $-S$.

Using this fact, the second stage of the proof demonstrates that sections R of K in between (parallel to) S and $-S$ are centrally symmetric. To do this, a number of symmetries and an application of Theorem 1.2 are used.

In the final stage, some loose ends are tied, and an application of the False Centre Theorem yields the result.

Proof of 2.2 We begin with a few definitions. For a given convex body A, let

ξ_A denote the support function of A at $\mathbf{0}$; that is, for $\mathbf{u} \in \Omega^2$,

$$\xi_A(\mathbf{u}) = \sup\{\langle \mathbf{y}, \mathbf{u}\rangle \mid \mathbf{y} \in A\}.$$

Denote by H_A the corresponding hyperplane

$$H_A(\mathbf{u}) = \{\mathbf{y} \in \mathbb{R}^3 \mid \langle \mathbf{y}, \mathbf{u}\rangle = \xi_A(\mathbf{u})\}.$$

Let $K_A(\mathbf{u})$ denote the intersection of K with $H_A(\mathbf{u})$; finally, if this section of K has a centre of symmetry, write it $\mathbf{k}_A(\mathbf{u})$.

Assume without loss that the maximal ellipsoid E contained in K is in fact the Euclidean ball B of radius one. If not, application of a suitable linear transformation rectifies the situation and does not affect the hypotheses.

Note that central symmetry of K about $\mathbf{0}$ implies that if H is a hyperplane supporting *either* L or $-L$, then $H \cap K$ is a centrally symmetric section. Let L^* denote the convex hull $L^* = \operatorname{conv}(L \cup -L)$. Then

$$\xi_{L^*}(\mathbf{u}) = \max\{\xi_L(\mathbf{u}), \xi_{-L}(\mathbf{u})\}, \tag{2.1}$$

so given $\mathbf{u} \in \Omega^2$, the section $K_{L^*}(\mathbf{u})$ supports either L or $-L$ and is thus centrally symmetric.

Let $\mathbf{u} \in \Omega^2$ be given; then the symmetry hypotheses yield

$$\begin{aligned} K_{L^*}(-\mathbf{u}) &= 2\mathbf{k}_{L^*}(-\mathbf{u}) - K_{L^*}(-\mathbf{u}) \\ &= K_{L^*}(\mathbf{u}) - 2\mathbf{k}_{L^*}(\mathbf{u}). \end{aligned}$$

That is, $K_{L^*}(-\mathbf{u})$ is a translation of $K_{L^*}(\mathbf{u})$ by $-2\mathbf{k}_{L^*}(\mathbf{u})$.

Fix $\mathbf{u}_0 \in \Omega^2$, let l be given by

$$l = \{\lambda \mathbf{d} \mid \lambda \in \mathbb{R}\},$$

the line joining the centres of symmetry of L and $-L$, and assume, for the present, that $\mathbf{k}_{L^*}(\mathbf{u}_0) \notin l$. Write $\mathbf{k}_0 = \mathbf{k}_{L^*}(\mathbf{u}_0)$. Let $\mathbf{v} \in \Omega^2$ with $\langle \mathbf{k}_0, \mathbf{v}\rangle = 0$ and consider the section $K_L(\mathbf{v})$ of K supporting L with outer normal $\mathbf{v}$.

We first show that $K_L(\mathbf{v})$ intersects both $K_{L^*}(\mathbf{u}_0)$ and $K_{L^*}(-\mathbf{u}_0)$. Notice that since $\mathbf{k}_0$ is parallel to $H_{L^*}(\mathbf{v})$ and $K_{L^*}(-\mathbf{u}_0)$ is related to $K_{L^*}(-\mathbf{u}_0)$ by a translation of $-2\mathbf{k}_0$, it is sufficient to prove that $K_L(\mathbf{v})$ intersects just one of $K_{L^*}(\pm\mathbf{u}_0)$. To see this, it is necessary to use the rather artificial condition $L \subset \alpha E$.

Let $\mathbf{x} = \xi_L(\mathbf{v})\mathbf{v} \in H_L(\mathbf{v})$, the point of $H_L(\mathbf{v})$ closest to the origin. Now $L \subset \alpha\mathrm{B}$, so $\|\mathbf{x}\| \leq \alpha$. Put $\xi = \xi_{L^*}(\mathbf{u}_0) \leq \alpha$, $H = H_L(\mathbf{v})$, and suppose that $|\langle \mathbf{x}, \mathbf{u}_0\rangle| > \xi$; since H supports L, there is a point $\mathbf{z} \in L \cap H$. However, using equation 2.1 and the central symmetry of L^* it follows that $-\xi \leq \xi_L(\mathbf{u}_0) \leq \xi$ and hence $|\langle \mathbf{z}, \mathbf{u}_0\rangle| \leq \xi$. Thus the line segment joining $\mathbf{z}$ and $\mathbf{x}$ is (by convexity) contained in $K \cap H$ and includes a point of one of $H_{L^*}(\pm\mathbf{u}_0)$ as required.

We may now assume that $|\langle \mathbf{x}, \mathbf{u}_0\rangle| \leq \xi$. By definition $\mathbf{k}_0$ is parallel to H so points $\mathbf{x} + \mu\mathbf{k}_0$ for $\mu \in \mathbb{R}$ all lie in H. Hence, if $\delta^2 = 1$, the point

$$\mathbf{z} = \mathbf{x} + \left(\delta - \frac{\langle \mathbf{x}, \mathbf{u}_0\rangle}{\xi}\right)\mathbf{k}_0$$

lies in $H_{L^*}(\delta\mathbf{u}_0)$ (recall $\langle \mathbf{k}_0, \mathbf{u}_0\rangle = \xi$). Using the facts $|\langle \mathbf{x}, \mathbf{u}_0\rangle| \leq \xi$ and $\mathbf{x} \in \alpha\mathrm{B}$, and choosing appropriate $\delta \in \{-1, 1\}$

$$\|\mathbf{z}\|^2 \leq \alpha^2 + \|\mathbf{k}_0\|^2. \tag{2.2}$$

We aim to bound this below 1 and deduce $\mathbf{z} \in \mathrm{B} \subset K$. If $\mathbf{k}_0 = \xi\mathbf{u}_0$, then $\|\mathbf{k}_0\| = \xi \leq \alpha$ and the result follows. Assume then that $r = \|\mathbf{k}_0 - \xi\mathbf{u}_0\| > 0$ and write $\mathbf{w} = (\mathbf{k}_0 - \xi\mathbf{u}_0)/r$. Clearly

$$\xi\mathbf{u}_0 - \left(1 - \xi^2\right)^{\frac{1}{2}}\mathbf{w} \in \partial\mathrm{B} \subset K,$$

and since $\mathbf{k}_0 = r\mathbf{w} + \xi\mathbf{u}_0$ is the centre of the section $K_{L^*}(\mathbf{u}_0)$ it follows immediately that

$$\mathbf{a} = \left(2r + \left(1 - \xi^2\right)^{\frac{1}{2}}\right)\mathbf{w} + \xi\mathbf{u}_0 \in K.$$

Now B is the maximal ellipsoid contained in K, so it follows from Theorem 2.1 that $\|\mathbf{a}\| \leq \sqrt{3}$; that is

$$\|\mathbf{a}\|^2 = 4r^2 + 4r\left(1 - \xi^2\right)^{\frac{1}{2}} + 1 \leq 3.$$

Solving this quadratic in r,

$$r \leq \frac{1}{2}\left(\sqrt{3-\xi^2} - \sqrt{1-\xi^2}\right).$$

It is now possible to bound $\|\mathbf{k}_0\|^2$ by

$$\|\mathbf{k}_0\|^2 = r^2 + \xi^2 \leq 1 + \frac{1}{2}\xi^2 - \frac{1}{2}\left[(3-\xi^2)(1-\xi^2)\right]^{\frac{1}{2}}.$$

Substituting this result into inequality 2.2 gives

$$\|\mathbf{z}\|^2 \leq f(\xi) = 1 + \alpha^2 + \frac{1}{2}\xi^2 - \frac{1}{2}\left[(3-\xi^2)(1-\xi^2)\right]^{\frac{1}{2}}.$$

A simple exercise now confirms that $\partial f/\partial \xi$ is positive for positive ξ. Therefore, using $\xi \leq \alpha$

$$\|\mathbf{z}\|^2 \leq f(\alpha) = 1.$$

That is $\mathbf{z} \in K \cap H_L(\mathbf{v}) \cap H_{L^*}(\mathbf{u}_0)$ as required.

Let $C_\pm = K \cap H_L(\mathbf{v}) \cap H_{L^*}(\pm\mathbf{u}_0)$. Since $\mathbf{k}_0$ is parallel to $H_L(\mathbf{v})$ and $K_{L^*}(-\mathbf{u}_0) = K_{L^*}(\mathbf{u}_0) - 2\mathbf{k}_0$, the lengths of the chords C_+ and C_- are equal. Strict convexity of K implies strict convexity of $K_L(\mathbf{v})$, so the centre of symmetry $\mathbf{k}_L(\mathbf{v})$ of $K_L(\mathbf{v})$ must lie in the plane $H_0(\mathbf{u}_0)$ half way between C_+ and C_-.

Let $\mathbf{w}$ be a unit vector such that the directions of $\mathbf{w}$ and the lines $H_L(\mathbf{v}) \cap H_{L^*}(\pm\mathbf{u}_0)$ coincide. Write $F_\mu = H_0(\mathbf{u}_0) + \mu\mathbf{k}_0$ and $K_\mu = K \cap F_\mu$. We restrict our attention to values of μ in the range $(-1, 1)$. Let m be the line cutting K in the chord $K_\mu \cap K_L(\mathbf{v})$. That is

$$m = \{\mu\mathbf{k}_0 + \mathbf{k}_L(\mathbf{v}) + \lambda\mathbf{w} \mid \mu \in \mathbb{R}\}.$$

It is important at this juncture to note that m can be written in this way only because $\mathbf{k}_L(\mathbf{v})$ lies in the plane $H_0(\mathbf{u}_0)$. Using $|\cdot|$ to denote length as usual,

$$|m \cap K_L(\mathbf{v})| = |m \cap K| = |(-m) \cap K| = |(-m) \cap K_{-L}(-\mathbf{v})|.$$

Let $\mathbf{c} = \mathbf{k}_{-L}(-\mathbf{v}) = -\mathbf{k}_L(\mathbf{v}) \in H_0(\mathbf{u}_0)$, then

$$|(-m) \cap K_{-L}(-\mathbf{v})| = |(2\mathbf{c} + m) \cap K_{-L}(-\mathbf{v})|.$$

Now, as $\mathbf{c} = -\mathbf{k}_L(\mathbf{v})$,

$$2\mathbf{c} + m = \{\mu\mathbf{k}_0 - \mathbf{k}_L(\mathbf{v}) + \lambda\mathbf{w} \mid \lambda \in \mathbb{R}\} = 2\mu\mathbf{k}_0 - m.$$

Thus $|K \cap m| = |K \cap (2\mu\mathbf{k}_0 - m)|$. Our choice of $\mathbf{v}$ perpendicular to $\mathbf{k}_0$ was entirely arbitrary, so whenever m is a line in the plane F_μ, tangent to the cylinder

$$C(L) = \bigcup_{\gamma \in \mathbb{R}} L + \gamma\mathbf{k}_0,$$

the following equalities hold.

$$|K_\mu \cap m| = |2\mu\mathbf{k}_0 - m \cap K_\mu| = |m \cap 2\mu\mathbf{k}_0 - K_\mu|.$$

Identify F_μ with $\mathbb{R}^2$ placing $\mathbf{0}$ at $\mu\mathbf{k}_0$. Let $L_1 = C(L) \cap F_\mu$ and $L_2 = -L_1$ (embedded in $\mathbb{R}^2$). Then, in $\mathbb{R}^2$, K_μ and $-K_\mu$ have the same chord-lengths supporting both L_1 and L_2; furthermore, since $\mathbf{k}_0 \notin l$, $\mathbf{0} \in \mathbb{R}^2$ is not the centre of the centrally symmetric body L_1. Hence L_2 is a non-degenerate translation of L_1 and the pair (L_1, L_2) is proper in the sense defined in Chapter 1. Application of Theorem 1.2 now shows that K_μ and $-K_\mu$ are equal in $\mathbb{R}^2$, or equivalently K_μ is centrally symmetric about $\mu\mathbf{k}_0$ in $\mathbb{R}^3$.

So far it has been assumed that $\mathbf{k}_0 = \mathbf{k}_{L^*}(\mathbf{u}_0) \notin l$. It is now necessary to eliminate the effect of the opposite case. Let

$$W = \left\{\mathbf{v} \in \Omega^2 \mid \mathbf{k}_{L^*}(\mathbf{v}) \notin l\right\}.$$

We aim to show that $\mathrm{cl}W = \Omega^2$. If not, pick $\mathbf{u}_0 \in \Omega^2$ and $0 < \beta < 1$ such that $\mathbf{v} \notin W$ whenever $\langle \mathbf{u}_0, \mathbf{v}\rangle \geq \beta$. Examining equation 2.1 it is clear that unless $\mathbf{u} \in \Omega^2$ is perpendicular to l, the support plane $H_{L^*}(\mathbf{u})$ meets only one of L and $-L$. We may therefore assume without loss that for $\langle \mathbf{u}_0, \mathbf{u}\rangle \geq \beta$ the set $K_{L^*}(\mathbf{u}) \cap -L$ is empty; smoothness and strict convexity of L now imply that there is a unique point $\mathbf{t}(\mathbf{u})$ of tangency of $K_{L^*}(\mathbf{u})$ to L^* continuously dependent on $\mathbf{u}$. Further assume without loss that $\mathbf{t}(\mathbf{u}_0) \notin l$.

Let $\mathbf{x}_0$ lie on the boundary of K, on the line joining $\mathbf{c}(\mathbf{u}_0) = \mathbf{k}_{L^*}(\mathbf{u}_0)$ and $\mathbf{t}(\mathbf{u}_0)$. Let

$$S = \left\{ \mathbf{v} \in \Omega^2 \;\middle|\; \mathbf{x}_0 \in K(L^*, \mathbf{v}) \right\}.$$

S is a topological circle on Ω^2. Next, let

$$\bar{S} = \{ \mathbf{v} \in S \mid \langle \mathbf{u}_0, \mathbf{v} \rangle \geq \alpha \}.$$

$\bar{S}$ is a non-degenerate topological line segment on Ω^2 passing through $\mathbf{u}_0$.

For all $\mathbf{v} \in \bar{S}$, the centre $\mathbf{c}(\mathbf{v})$ of $K_{L^*}(\mathbf{v})$ lies in l. Thus, either $\mathbf{c}(\mathbf{v}) = \mathbf{c}(\mathbf{u}_0)$ for all $\mathbf{v} \in \bar{S}$, in which case $\mathbf{x}_0$ and $\mathbf{c}(\mathbf{u}_0)$ occur in all the sections $K(L^*, \mathbf{v})$ contradicting smoothness of L^*; or

$$\mathbf{c}(\mathbf{v}) = \lambda(\mathbf{v})\mathbf{c}(\mathbf{u}_0),$$

for some $\lambda : \bar{S} \rightarrow \mathbb{R}$, continuous but not constant over $\bar{S}$. Then by hypothesis

$$2\lambda(\mathbf{v})\mathbf{c}(\mathbf{u}_0) - \mathbf{x}_0 \in \partial K,$$

for all $\mathbf{v} \in \bar{S}$. There is, therefore, a (small) line segment in ∂K, contradicting strict convexity of K.

We have shown that $\mathrm{cl}W = \Omega^2$. Hence, using continuity and closure of ∂K, whenever F is a section of K cutting L^*, it is centrally symmetric. In particular, any point of L^* except the origin is a false centre for K. Thus, by the False Centre Theorem [1], K is an ellipsoid as required. □

2.4 Conclusion

In order to prove Theorem 2.2 it was necessary to impose rather strong hypotheses. However, most of these restrictions were chosen to overcome the technical problems encountered. This suggests that Theorem 2.2 is a long way from being the best result possible.

The requirement that L be centrally symmetric with centre of symmetry $\mathbf{d} \neq \mathbf{0}$ can be modified. This pair of conditions is present solely to guarantee that almost all projections of the pair $(L, -L)$ are proper, allowing the application of Theorem 1.2. Hence, as indicated in the introduction, a proof of the two-dimensional one body conjecture in Chapter 3 would lead to some improvement. This is because a positive result would amount to an extension of Theorem 1.2 with the restriction (L_1, L_2) proper removed.

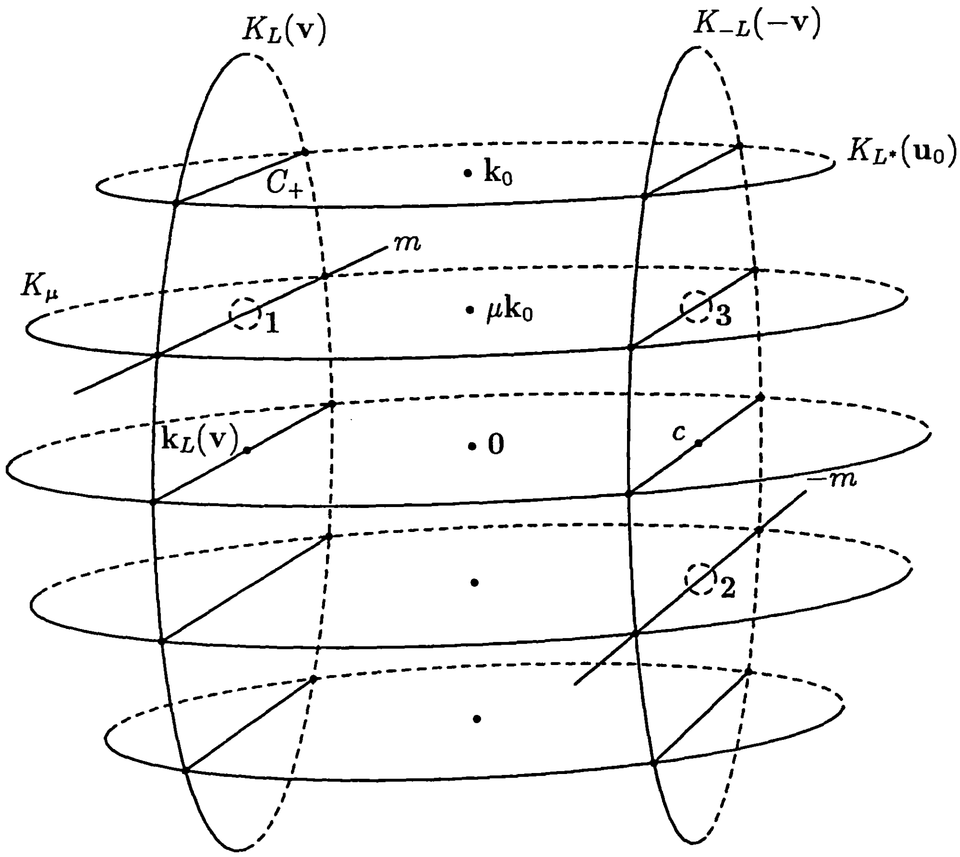

Figure 2.1: Illustration for Theorem 2.2. The proof consists of showing that the three circled chords have equal length. Various symmetries are used to map $\mathbf{1} \rightarrow \mathbf{2} \rightarrow \mathbf{3}$

Chapter 3

The One Body Problem

3.1 Introduction

In this chapter we investigate some aspects of the one body problem mentioned in Chapter 1. In essence, the aim is to decide to what extent a convex body is determined by the volumes of its sections supporting some inner body. It will be demonstrated that, at least in one respect, this problem is very much distinct from the single point X-Ray problem; there are many convex bodies with constant chord-length through some interior point, but if K is a planar convex body containing the unit ball, and all chords of K supporting the ball have the same length, then ∂K is a uniquely determined circle.

It has not been possible to apply the techniques of Chapter 1 to this situation, since the maps involved are non-hyperbolic, measure preserving, and orientation reversing. Together, these facts admit the possibility of extremely complex topology.

3.2 The Conjecture

We now proceed to formalise the problem. Suppose that $K_1, K_2, L \subset \mathbb{R}^n$ are convex bodies with $L \subset \mathrm{int}K_1 \cap \mathrm{int}K_2$; further assume that whenever $H \subset \mathbb{R}^n$ is a hyperplane supporting L, the $(n-1)$-volumes $|K_1 \cap H|$ and $|K_2 \cap H|$ are equal. We conjecture $K_1 = K_2$.

In what follows we shall discuss various partial results obtained and observations made in the course of attempting to prove this conjecture. In order to simplify the problem we have restricted L, assuming it to be the unit ball. This makes calculations easier, but does not reduce the topological complexity of the problem.

3.3 Results

We begin by considering the two-dimensional problem, and provide a positive result for the simplest case.

3.3.1 Determination of the Circle

Theorem 3.1 *Suppose that $K \subset \mathbb{R}^2$ is a convex body containing, in its interior, the unit ball* B*. Then if the chords of K cut by supporting lines of B have constant length, K is a unique multiple of* B.

Proof of 3.1 The proof is most easily obtained by considering the chord-map used in Chapter 1. Use coordinates (θ, r) defined by

$$\mathbf{x}(\theta, r) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + r \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix},$$

and suppose that K has chords of length $c > 0$ supporting B. In terms of the coordinate system just given, define a chord-map T by

$$T(\theta, r) = \left(\theta + 2\tan^{-1}(c - r), c - r\right).$$

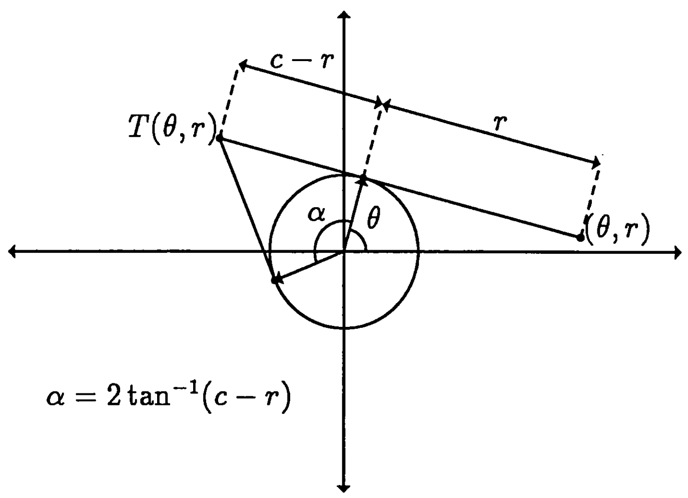

Figure 3.1: Another chord-map

Figure 3.1 illustrates the coordinates and chord-map. Clearly T maps ∂K to ∂K; hence so does T^2, which is easy to calculate:

$$T^2(\theta, r) = (\theta + 2\nu(r), r)\,,$$

where $\nu(r) = \tan^{-1} r + \tan^{-1}(c-r)$. Thus T^2 has the effect of increasing θ by a quantity dependent only on r, and keeping r constant. It is in effect a cylindrical shear.

Clearly, after $2n$ applications of T,

$$T^{2n}(\theta, r) = (\theta + 2n\nu(r), r)\,,$$

from which, if $(\theta, r) \in \partial K$, the set

$$X(\theta, r) = \{(\theta + 2n\nu(r), r) \mid n \in \mathbb{N}\}$$

is contained in ∂K. Suppose now that $\nu = \nu(r)$ is an irrational multiple of π. In this case, the set

$$\{\theta + 2n\nu(\bmod 2\pi) \mid n \in \mathbb{N}\}$$

is dense in the interval $[0, 2\pi)$ (see [12], Proposition 1.3.3 for example). We are now done, for unless ∂K is already a circle, there is surely some $(\theta, r) \in \partial K$ with

$\nu(r)/\pi$ irrational. Closure of ∂K now implies that ∂K is a circle with centre $\mathbf{0}$, and only one such circle has chords of length c supporting the unit ball. □

This result gives an indication that the problem is distinct from that of the single-point X-ray problem.

3.3.2 The General Chord-Map

We return to the general two-dimensional problem and consider the action of the chord-map. Using the coordinates (θ, r) defined in Theorem 3.1 we proceed to derive a more general form for T.

Let K be a convex body in the plane containing the unit ball. Suppose that the boundary of K is given by

$$r(\theta) = \max\{\lambda > 0 \mid \mathrm{x}(\theta, \lambda) \in K\}.$$

Let the length of the chord of K tangent to B at $\xi\ (\theta)$ be given by $l(\theta)$. Elementary geometry yields the relation

$$l(\theta) = r(\theta) + r(\theta + 2\tan^{-1}(l(\theta) - r(\theta))). \tag{3.1}$$

Given (θ, r) with r close to $r(\theta)$ set

$$T(\theta, r) = (\Phi(\theta, r), R(\theta, r)), \tag{3.2}$$

with Φ and R defined by

$$\begin{aligned} R(\theta, r) &= l(\theta) - r, \\ \Phi(\theta, r) &= \theta + 2\tan^{-1} R(\theta, r). \end{aligned}$$

It is an easy matter to show that a convex body, C, containing B in its interior has chord-lengths tangent to B given by $l(\theta)$ if and only if ∂C is invariant under T.

Investigation of the properties of T, for given K, soon leads one to ask questions concerning the existence of periodic points of ∂K under iterates of T. Specifically, for which values of $n \in \mathbb{N}$ is it possible to find $\mathbf{x} \in \partial K$ with $T^n(\mathbf{x}) = \mathbf{x}$? Of course, this situation corresponds to finding a polygon inscribed in K with edges supporting B. The next section is devoted to providing some answers in this direction.

3.3.3 Rotation Numbers and Periodic Points

A concept from the study of dynamics of homeomorphisms of the circle, known as the rotation number, will prove useful. We now introduce this idea.

Identify the circle Ω with $\mathbb{R}/\mathbb{Z}$ using the map $\Pi : x \mapsto x(\bmod 1)$. If $f : \Omega \to \Omega$ is a homeomorphism, and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a homeomorphism with the property

$$\Pi\tilde{f}(x) = f(\Pi x),$$

for all $\mathbf{x} \in \mathbb{R}$, say that $\tilde{f}$ is a lift of f. Given any lift $\tilde{f}$ of f, define the rotation number $\rho(f)$ of f by

$$\rho(f) = \lim_{n\to\infty}\left(\frac{\tilde{f}^n(x) - x}{n}\right) \bmod 1.$$

The quantity $\rho(f)$ can be shown to be independent of both the lift $\tilde{f}$ chosen, and the parameter x. Proof of this fact may be found in Proposition 11.1.1 of [12] from which the definitions of lift and rotation number are taken. It is immediately clear that the rotation number has the desirable property of identifying which maps admit periodic orbits; that is maps f for which $f^q(x) = x$ for some $x \in \Omega$, and hence $\rho(f) \in \mathbb{Q}$.

Returning to the question of polygons inscribed in K, consider the action of T restricted to ∂K by using the following map $\tilde{T}$ defined on the unit circle; for convenience make the obvious change of variable so that $\theta \in [0, 1)$.

$$\tilde{T}(\theta) = \theta + (2\pi)^{-1}\tan^{-1}(l(2\pi\theta) - r(2\pi\theta)).$$

If the rotation number $\rho(\tilde{T})$ is irrational, then clearly there can be no periodic points of T on ∂K. On the other hand, suppose that $\rho(\tilde{T}) = p/q$ with p and q coprime. Then any periodic point $\mathbf{x}$ of T on ∂K must satisfy $T^q\mathbf{x} = \mathbf{x}$ and $T^r\mathbf{x} \neq \mathbf{x}$ for $0 < r < q$. From a geometric standpoint we have shown the following to be true.

Lemma 3.3.1 *Let K be a planar convex body containing the unit ball* B *in its interior. Suppose that P and Q are polygons inscribed in K with edges supporting* B*. Then P and Q have the same number of vertices.*

Lemma 3.3.1 allows to make the following definition. Let K be a planar convex body containing the unit ball in its interior; if there is a finite polygon P inscribed in K with edges supporting B, let $N(K)$ be the number of vertices of P; if no such polygon exists, let $N(K) = -1$.

3.3.4 A Necessary Condition

The relevance of the preceding discussion to the question of whether K is determined by its chords supporting B will now be demonstrated.

Lemma 3.3.2 *Suppose that $K, M \subset \mathbb{R}^2$ are convex bodies with* $\mathrm{B} \subset \mathrm{int}K \cap \mathrm{int}M$*, and that the chords of K and M supporting* B *have equal length. Then*

$$2|N(K) = N(M).$$

Proof of 3.3.2 As usual use the chord-map T defined as in equation 3.2 using an appropriate definition for l. Using the fact that T leaves both ∂K and ∂M invariant, we deduce that $N(K) = N(M)$; that $N(K)$ is even will follow easily once we have shown that it is non-negative. To do this, we will construct an appropriate polygon. The method used was proposed by Professor C.A. Rogers.

First we show that, neglecting a reversal of orientation, T preserves a measure on $\mathbb{R}^2 \setminus \mathrm{B}$. Introduce a conjugate set of coordinates, $[\phi, s]$, given by

$$\mathrm{x}[\phi, s] = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + s \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}.$$

Let $D \subset \mathbb{R}^2 \setminus \mathrm{B}$ be a region. We claim that

$$\mu(D) := \int_{\mathrm{x}[\phi,s]\in D} \mathrm{d}\phi \mathrm{d}s = \int_{\mathrm{x}(\theta,r)\in D} \mathrm{d}\theta \mathrm{d}r. \tag{3.3}$$

This is easy to see, for $\mathrm{x}[\phi, s] = \mathrm{x}(\theta, r)$ precisely when $s = r$ and $\phi = \theta - 2\tan^{-1} r$. Hence

$$\left|\frac{\partial(\phi, s)}{\partial(\theta, r)}\right| = \begin{vmatrix} 1 & -2/(1+r^2) \\ 0 & 1 \end{vmatrix} = 1,$$

so equation 3.3 holds.

Next observe that the boundaries of K and M coincide at some point. Using the coordinates (θ, r) parameterise ∂K and ∂M by $r(\theta)$ and $\tilde{r}(\theta)$ respectively, and assume without loss that $r(0) = \tilde{r}(0)$. Further assume that for small positive excursions $\delta > \theta > 0$ we have $\tilde{r}(\theta) > r(\theta)$. Consider the region D, defined by

$$D = \{\mathrm{x}(\theta, r) \mid \theta \in [0, \delta] \text{ and } r \in [r(\theta), \tilde{r}(\theta)]\}.$$

Clearly, using the definitions of the coordinate systems (θ, r) and $[\phi, s]$, we may write

$$T(D) = \{\mathrm{x}[\phi, s] \mid \phi \in [0, \delta] \text{ and } s \in [l(\phi) - \tilde{r}(\phi), l(\phi) - r(\phi)]\}.$$

See figure 3.2 for an illustration of the method. Hence, using equation 3.3

$$\mu(T(D)) = \int_{T(D)} \mathrm{d}\theta \mathrm{d}r = \int_{T(D)} \mathrm{d}\phi \mathrm{d}s = \int_D \mathrm{d}\theta \mathrm{d}r = \mu(D),$$

showing that the regions D and $T(D)$ have the same measure with respect to μ. Now let

$$\begin{aligned} \theta_0 &= \min\{\theta > 0 \mid \tilde{r}(\theta) = r(\theta)\}, \\ D &= \{\mathrm{x}(\theta, r) \mid \theta \in [0, \theta_0] \text{ and } r \in [r(\theta), \tilde{r}(\theta)]\}. \end{aligned}$$

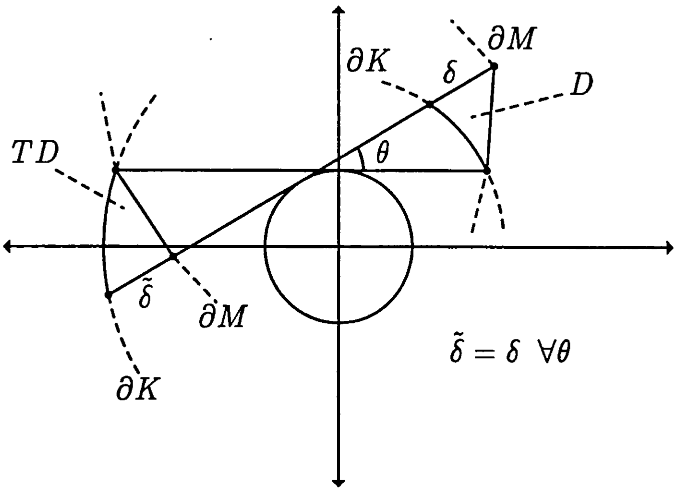

Figure 3.2: The measure preserving nature of T

By hypothesis, $\theta_0 > 0$ and hence $\mu(D) > 0$. Consider the iterates of D under T. If $T^k(D) \cap T^m(D)$ has empty interior for all k and m, then putting

$$R = \mathrm{cl}(K_1 \setminus K_2 \cup K_2 \setminus K_1)$$

it follows that $\mu(R)$ is unbounded, a clear impossibility. Therefore, without loss for some k, $I = \mathrm{int}(T^k(D) \cap D)$ is non-empty. Since T is continuous and leaves ∂K_1 invariant, at least one of $T^k(0, r(0))$ or $T^k(\theta_0, r(\theta_0))$ lies on ∂K_1 between $(0, r(0))$ and $(\theta_0, r(\theta_0))$. However, T is orientation preserving on ∂K_1 and by hypothesis, there is no solution of $r(\theta) = \tilde{r}(\theta)$ for $0 < \theta < \theta_0$. Hence $(0, r(0))$ is a fixed point of T^k as required. In fact, since $(\theta_0, r(\theta_0))$ lies in ∂K_1, it is also a fixed point of T^k and by continuity, D is invariant under T^k.

The fact that k$=N(K_1) = N(K_2)$ must be even can be seen as follows. Let x be a point of int$D$. Then $\mathbf{x} \in K_2$, but $\mathbf{x} \notin K_1$. Hence, $T(\mathbf{x}) \in K_1$, but $T(\mathbf{x}) \notin K_2$, and so on. However D is invariant under T^k, so $T^k(\mathbf{x}) \in K_2$ and k must be even.

□

This result suggests that the number of bodies for which the conjecture *fails* must be relatively small; unless one can inscribe an even polygon in K with edges supporting B, K is determined by the lengths of its chords supporting B. One might conjecture, for example, that the set of convex bodies K for which $N(K)$ is even is of first category in the set of all convex bodies (since $N(K)$ non-negative corresponds to a rational rotation number). However, the rotation number $\rho(T)$ of T does not in general depend smoothly on T, especially at rational values of $\rho(T)$! See the discussion in Chapter 11 of [12] for details of this.

The next result modifies the topology of the problem simplifying it greatly.

3.3.5 Chords Meeting an Annulus

Theorem 3.2 *Let $K_1, K_2 \subset \mathbb{R}^2$ be a convex bodies containing* B *in their interiors. Suppose that whenever l is a line supporting one of the balls* $(1-\lambda)$B *with $0 \leq \lambda \leq \epsilon$, we have $|K_1 \cap l| = |K_2 \cap l|$. Then $K_1 = K_2$.*

Of course, this result may be easily obtained as a corollary to Theorem 1.2. However, the hypothesis of Theorem 3.2 is considerably stronger. This is demonstrated by the simplicity of the following proof.

Proof of 3.2 Let $\mathrm{x} \in \partial K_1 \cap \partial K_2 \neq \emptyset$, and using the notation of Chapter 1, define the family of chord-maps T_λ by

$$T_\lambda = T_{K_i,(1-\lambda)\mathrm{B}},$$

for $\lambda \in [0, \epsilon]$. It is clear that each T_λ is continuous and invertible and that the set

$$I(\mathrm{x}) = \left\{T_\lambda^{-1}(T_{\epsilon/2}\mathrm{x}) \;\middle|\; \lambda \in [0, \epsilon]\right\}$$

is a non-degenerate topological line segment in $\partial K_1 \cap \partial K_2$; furthermore, $\mathbf{x}$ is contained in the interior of this line segment. Hence $\partial K_1 \cap \partial K_2$ is both open and closed in ∂K_i. So $K_1 = K_2$. □

Although trivial in two dimensions, the analogues in higher dimensions are more tricky. We will return to this later.

The final two-dimensional result involves another topological restriction. Assume that the ball B touches the boundary of K In this case we find that K is indeed determined by its chords tangent to B.

3.3.6 A Ball on the Boundary

Theorem 3.3 *Suppose that $K_1, K_2 \subset \mathbb{R}^2$ contain the unit ball,* B, *and that $\partial K_1 \cap$ B and $\partial K_2 \cap$ B are single points. If, further, the lengths of the chords $K_i \cap l$ are equal whenever l is a line supporting* B, *then $K_1 = K_2$.*

Proof of 3.3 To begin with we show that K_1 and K_2 meet B at the same point. Suppose without loss that B meets ∂K_1 at $(1,0)^T$. Let $l(\theta)$ denote the length of the chord cutting K_1 or K_2 supporting B at $\boldsymbol{\xi}\ (\theta)$. If $l(0) = 0$ then K_2 also meets B at $(1,0)^T$. If $l(0) \neq 0$ then using convexity, it is easy to see that l must be discontinuous at 0. Considering K_2, this can only happen where ∂K_2 meets B, so again K_2 meets B at $(1,0)^T$.

Let R be the region

$$R = \mathrm{cl}\,(K_1 \setminus K_2 \cup K_2 \setminus K_1)\,,$$

and using the notation of Theorem 3.3, let

$$\begin{aligned} l^+(\theta) &= \{\mathrm{x}(\theta, r) \mid r \geq 0\}\,, \\ l^-(\theta) &= \{\mathrm{x}(\theta, r) \mid r \leq 0\}\,. \end{aligned}$$

Since the lengths of the chords of K_1 and K_2 supporting B match, it follows that for all θ

$$r^+(\theta) = \left|R \cap l^+(\theta)\right| = \left|R \cap l^-(\theta)\right| = r^-(\theta).$$

We derive expressions for r and r^-, and show that R has empty interior.

Let Ξ be the characteristic function defined by

$$\Xi(\theta, \nu) = \begin{cases} 1 & \text{if } \sec \nu \boldsymbol{\xi}\ (\theta) \in R, \\ 0 & \text{otherwise,} \end{cases}$$

and note that if $\nu = \tan^{-1} r$

$$\mathbf{x}(\theta, r) = \sec\nu \boldsymbol{\xi}\ (\theta - \nu).$$

Thus, we may write

$$r^{+}(\theta) = \int_{\nu=0}^{\pi/2} \Xi(\theta - v, v) \sec^2 \nu \mathrm{d}\nu,$$

and if $f : \mathbb{R} \to \mathbb{R}$ has period 2π

$$\begin{aligned} \langle r^{+}, f \rangle &:= \int_0^{2\pi} f(\theta) r^{+}(\theta) \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} \Xi(\theta - \nu, \nu) f(\theta) \sec^2 \nu \mathrm{d}\theta \mathrm{d}\nu \\ &= \int_0^{2\pi} \int_0^{\pi/2} f(\theta + \nu) \Xi(\theta, \nu) \sec^2 \nu \mathrm{d}\theta \mathrm{d}\nu, \end{aligned} \tag{3.4}$$

It may similarly be shown that

$$\langle r^{-}, f \rangle = \int_0^{2\pi} \int_0^{\pi/2} f(\theta - \nu) \Xi(\theta, \nu) \sec^2 \nu \mathrm{d}\theta \mathrm{d}\nu. \tag{3.5}$$

Finally, since $r^{+} = r^{-}$ using equations 3.4 and 3.5

$$\begin{aligned} 0 &= \langle r^{+} - r^{-}, f \rangle \\ &= \int_0^{2\pi} \int_0^{\pi/2} \left(f(\theta + \nu) - f(\theta - \nu) \right) \Xi(\theta, \nu) \sec^2 \nu \mathrm{d}\theta \mathrm{d}\nu. \end{aligned} \tag{3.6}$$

We claim that whenever $\sec\nu \boldsymbol{\xi}\ (\phi) \in R$, both $\phi + \nu$ and $\phi - \nu$ lie in the range $[0, 2\pi)$. If so, we can set $f(\theta) = \theta$, for $\theta \in [0, 2\pi)$, and be sure that whenever $\Xi(\theta, \nu) \neq 0$, we have $f(\theta \pm \nu) = \theta \pm \nu$; in this case

$$2 \int_R \nu \sec^2 \nu \mathrm{d}\phi \mathrm{d}\nu = 0,$$

implying that R has empty interior and completing the proof. So it remains only prove the claim.

That $\theta \pm \nu$ lie in $[0, 2\pi)$ follows from the fact that K_1 and K_2 touch B at $(1, 0)^T$. If $\mathbf{x} = \sec\nu \boldsymbol{\xi}\ (\phi) \in K_i$

$$1 \geq \langle \sec\nu \boldsymbol{\xi}\ (\phi), \boldsymbol{\xi}\ (0) \rangle = \sec\nu \cos\phi,$$

since the line supporting B at $(1,0)^T$ is necessarily a support line of K_i for $i = 1, 2$. That is, $\cos\nu \geq \cos\phi$. The result follows. □

3.3.7 Three Dimensions and Above

So far, we have dealt exclusively with the planar problem; let us now consider the problem in higher dimensions. We begin by eliminating a number of possible formulations.

3.3.8 Sections of Codimension greater than 1

Suppose that $d > 2$ and $K \subset \mathbb{R}^d$ is a convex body containing the Euclidean unit ball B in its interior. Fix $1 \leq j \leq d-1$; let F_j denote the family of j-dimensional affine subspaces supporting B. Henceforth, if $M \subset \mathbb{R}^d$ is convex, denote by $|M|$ the standard volume of M in aff$M$. That is, if $\dim(\mathrm{aff}M) = j$, and λ_j is the usual j-dimensional volume measure, then

$$|M| = \int_M \mathrm{d}\lambda_j.$$

We aim to prove the following result.

Theorem 3.4 *Suppose that $K, M \subset \mathbb{R}^d$ are convex bodies containing* B *in their interiors. Fix $1 \leq j \leq d-2$ and further suppose that whenever $H \in F_j$ the j-volumes $|K \cap H|$ and $|M \cap H|$ agree. Then $K = M$.*

The proof is in two stages. We use a topological argument to prove a slightly different problem, then one using spherical harmonics to complete the result. The former is achieved in the following lemma.

Suppose that $d > 2$ and $K \subset \mathbb{R}^d$ is a convex body with $\mathrm{B} \subset \mathrm{int}K$. If $x, y \in \partial K$ write $\mathbf{x} \sim \mathbf{y}$ whenever there is a line l supporting B with $\mathbf{x}, \mathbf{y} \in l$.

Lemma 3.3.3 *Fix* $x_0 \in \partial K$. *There is a neighbourhood* N *of* x_0 *in* ∂K *such that whenever* $y_0 \in N$ *there exists* $\mathbf{z} \in \partial K$ *with* $x_0 \sim \mathbf{z} \sim y_0$.

This statement is relatively trivial. If the reader is convinced of its truth, the following proof, which consists mainly of calculation, may be skipped.

Proof of 3.3.3 For $\mathbf{x} \in \partial K$ set

$$C(\mathrm{x}) = \{\lambda \mathrm{x} + (1-\lambda)\mathbf{b} \mid \lambda \in \mathbb{R} \text{ and } \mathbf{b} \in \mathrm{B}\}.$$

Let $I(\mathbf{x}) = \{\mathbf{y} \in \partial K \mid \mathbf{x} \sim \mathbf{y}\}$. Then clearly

$$I(\mathbf{x}) = \partial K \cap \partial C(\mathbf{x}),$$

and we aim to show that if $\mathbf{y}_0$ is sufficiently close to $\mathbf{x}_0$ then $\mathbf{y}_0 \in I^2(\mathrm{x}_0)$. This happens if and only if there exists $\mathbf{z} \in I(\mathrm{x}_0)$ with $\mathbf{y}_0 \in I(\mathbf{z})$. However, $\mathbf{y}_0 \in I(\mathbf{z})$ if and only if $\mathbf{z} \in I(\mathbf{y}_0)$. Thus is suffices to show that

$$I(\mathrm{x}_0) \cap I(\mathbf{y}_0) = \partial K \cap \partial C(\mathrm{x}_0) \cap \partial C(\mathbf{y}_0) \neq \emptyset.$$

We establish this by a constructing continuous path $\mathbf{x}(\nu) \in \partial C(\mathrm{x}_0) \cap \partial C(\mathbf{y}_0)$ such that $\|\mathbf{x}(\nu)\| = \sec\nu$ for $\nu \in [0, \frac{\pi}{2})$. Since K is bounded, this will suffice.

First consider, for $\mathrm{x}, \mathrm{y} \in \mathbb{R}^d \setminus \mathrm{B}$, the quantity

$$d(\mathrm{x}, \mathrm{y}) = \min_{\lambda \in \mathbb{R}} \|\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\|.$$

By definition of $C(\mathbf{x})$

$$\mathbf{y} \in \begin{cases} \mathrm{int}C(\mathbf{x}) & \text{if } d(\mathrm{x}, \mathrm{y}) < 1, \\ \partial C(\mathbf{x}) & \text{if } d(\mathrm{x}, \mathrm{y}) = 1, \\ \mathbb{R}^d \setminus C(\mathbf{x}) & \text{if } d(\mathrm{x}, \mathrm{y}) > 1. \end{cases}$$

We will write

$$\begin{aligned} \mathbf{x}_0 &= \sec\nu_0 \mathbf{u}_0, \quad r_0 = \sec\nu_0, \\ \mathbf{y}_0 &= \sec\omega_0 \mathbf{v}_0, \quad s_0 = \sec\omega_0, \\ \mathbf{x} &= \sec\nu \mathbf{u}, \quad r = \sec\nu, \end{aligned}$$

with $\mathbf{u}_0, \mathbf{v}_0, \mathbf{u} \in \Omega^{d-1}$ and $\nu_0, \omega_0, \nu \in [0, \frac{\pi}{2})$. Using these variables

$$d(\mathbf{x}, \mathbf{x}_0) = \min_{\lambda \in \mathbb{R}} \left\{ \lambda^2 (r^2 + r_0^2 - 2rr_0 \langle \mathbf{u}, \mathbf{u}_0 \rangle + \lambda(2rr_0 \langle \mathbf{u}, \mathbf{u}_0 \rangle - 2r_0^2) + r_0^2 \right\}^{\frac{1}{2}}.$$

An elementary exercise shows that in fact

$$d(\mathbf{x}, \mathbf{x}_0) = \frac{r^2 r_0^2 (1 - \langle \mathbf{u}, \mathbf{u}_0 \rangle^2)}{r^2 + r_0^2 - 2rr_0 \langle \mathbf{u}, \mathbf{u}_0 \rangle},$$

from which we obtain

$$\mathbf{x} \in \begin{cases} \mathrm{int} C(\mathbf{x}_0) & \text{if } \ \langle \mathbf{u}, \mathbf{u}_0 \rangle^2 - \frac{2}{rr_0} \langle \mathbf{u}, \mathbf{u}_0 \rangle + \frac{1}{r^2} + \frac{1}{r_0^2} - 1 \ > 0, \\ \partial C(\mathbf{x}_0) & \text{if } \qquad \vdots \qquad = 0, \\ \mathbb{R}^d \setminus C(\mathbf{x}_0) & \text{if } \qquad \vdots \qquad < 0. \end{cases} \tag{3.7}$$

Substituting $\sec \mathbf{v}_0$ and $\sec \nu$ for r_0 and r yields that $\mathbf{x} \in \partial C(\mathbf{x}_0)$ if and only if

$$\langle \mathbf{u}, \mathbf{u}_0 \rangle^2 - 2 \cos \nu \cos \nu_0 \langle \mathbf{u}, \mathbf{u}_0 \rangle + \cos^2 \nu_0 + \cos^2 \nu - 1 = 0,$$

the solution to which is easily seen to be

$$\langle \mathbf{u}, \mathbf{u}_0 \rangle = \cos(\nu_0 \pm \nu). \tag{3.8}$$

By similar argument, $\mathbf{x} \in \partial C(\mathbf{y}_0)$ if and only if

$$\langle \mathbf{u}, \mathbf{v}_0 \rangle = \cos(\omega_0 \pm \nu). \tag{3.9}$$

Now fix $\mathbf{w}_0 \in \Omega^{d-1}$ with $\mathbf{w}_0$ perpendicular to both $\mathbf{u}_0$ and $\mathbf{v}_0$. Write $\gamma = \langle \mathbf{u}_0, \mathbf{v}_0 \rangle$, $\alpha = \cos(\nu_0 + \nu)$, $\beta = \cos(\omega_0 + \nu)$ and

$$\mathbf{u} = \lambda \mathbf{u}_0 + \mu \mathbf{v}_0 + h \mathbf{w}_0,$$

where

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \frac{1}{1 - \gamma^2} \begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

and, if possible,

$$h^2 = 1 - (\lambda^2 + \mu^2 + 2\lambda\mu\gamma).$$

If h, and hence $\mathbf{u}$ exist for all ν, then $\mathbf{u}$ satisfies equations 3.8 and 3.9, so that $\mathbf{x}(\nu) = \sec\nu\mathbf{u}(\nu)$ lies in the intersection of $\partial C(\mathbf{x}_0)$ and $\partial C(\mathbf{y}_0)$ for all ν. Also, $\mathbf{u}$ thus defined depends continuously on ν.

It only remains, therefore, to show the existence of h under appropriate conditions. That is

$$E^2 = \lambda^2 + \mu^2 + 2\lambda\mu\gamma < 1,$$

for all ν in the range $[0, \frac{\pi}{2})$. To see this, notice that

$$\begin{aligned} E^2 &= (\lambda, \mu)\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ &= \frac{1}{1-\gamma^2}(\alpha, \beta)^T\begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{1-\gamma^2}(\alpha^2 + \beta^2 - 2\alpha\beta\gamma). \end{aligned}$$

Thus $E^2 < 1$ if and only if

$$\alpha^2 + \beta^2 - 2\alpha\beta\gamma < 1 - \gamma^2.$$

Writing $\gamma = \cos\phi$ with $\phi > 0$ the angle between $\mathbf{x}_0$ and $\mathbf{y}_0$, and solving for $\alpha = \cos(\nu_0 + \nu)$,

$$\cos(\nu_0 + \nu) \in [\cos(\omega_0 + \nu + \phi), \cos(\omega_0 + \nu - \phi)]\,. \tag{3.10}$$

Notice that since K is bounded with B in its interior, there exists $\delta > 0$ such that whenever $\sec\theta\mathbf{a} \in \partial K$, some $\mathbf{a} \in \Omega^{d-1}$ and $\theta \in [0, \frac{\pi}{2})$ we have $\theta \in [\delta, \frac{\pi}{2} - \delta]$. If $\mathbf{x}_0$ and $\mathbf{y}_0$ are sufficiently close, we may assume that $\phi < \delta$, so that 3.10 is reduced to

$$\nu_0 + \nu \in [\omega_0 + \nu - \phi, \omega_0 + \nu + \phi]\,.$$

Therefore $E^2 < 1$ for all ν if $E^2 < 1$ for $\nu = 0$. That is

$$\cos^2\nu_0 + \cos^2\omega_0 - 2\cos\nu_0\cos\omega_0\gamma < 1 - \gamma^2. \tag{3.11}$$

However, recalling the identities

$$\begin{aligned}
\cos\nu_0 &= r_0, \\
\cos\omega_0 &= s_0, \\
\gamma &= \cos\phi = \langle \mathbf{u}_0, \mathbf{v}_0 \rangle, \\
\mathbf{y}_0 &= s_0\mathbf{v}_0, \\
\mathbf{x}_0 &= r_0\mathbf{u}_0,
\end{aligned}$$

inspection of the forms in 3.7 reveals that equation 3.11 holds if and only if

$$\mathbf{y}_0 \notin C(\mathbf{x}_0).$$

So if we can show that whenever $\mathbf{y}_0$ is sufficiently close to $\mathbf{x}_0$, we have $\mathbf{y}_0 \notin C(\mathbf{x}_0)$, we are done.

Suppose the converse; there exist $\mathbf{y}_k \in \partial K$ with $\mathbf{y}_k$ converging to $\mathbf{x}_0$ as k tends to infinity, and $\mathbf{y}_k \in C(\mathbf{x}_0)$ for $k = 0, 1, \ldots$. But then by definition of $C(\mathbf{x}_0)$, and the convexity of K, for each k, there is a point $\mathbf{b}_k \in \mathrm{B}$ lying on the line segment joining $\mathbf{x}_0$ to $\mathbf{y}_k$. Clearly, $\mathbf{b}_k$ tends to $\mathbf{x}_k$ as k tends to infinity implying that $\mathbf{x}_0 \in \mathrm{B}$ and contradicting the hypothesis $\mathrm{B} \subset \mathrm{int}K$. Hence result. □

Before proving Theorem 3.4 we mention one further lemma. Suppose that $K \subset \mathbb{R}^d$ is a convex body with $\mathbf{0} \in \mathrm{int}K$. If j is an integer, and for $\mathbf{u} \in \Omega^{d-1}$

$$r_K(\mathbf{u}) = \max\{\lambda > 0 \mid \lambda\mathbf{u} \in K\},$$

define

$$l_K^j(\mathbf{u}) = r^j(\mathbf{u}) + r^j(-\mathbf{u}).$$

Finally, let $f_K(\mathbf{u})$ denote the $d-1$-volume of the section of K through $\mathbf{0}$ perpendicular to $\mathbf{u}$. Tamvakis in [22] proved that l_K^{d-1} can be constructed using knowledge only of f_K. Subsequent investigation of spherical harmonics has yielded simpler proofs. One paper in particular provides an elegant solution. In [20], Schneider proves results which lead to the following corollary for even functions on a sphere, a proof of the invertibility of the Radon Transform for such functions; we use ω_d to denote the standard spherical surface measure.

Theorem 3.5 *Suppose that $f : \Omega^d \to \mathbb{R}$ is continuous with $f(\mathbf{u}) = f(-\mathbf{u})$ for all $\mathbf{u} \in \Omega^d$, and that given $\mathbf{v} \in \Omega^d$ the integral of f over the $(d-1)$-sphere perpendicular to $\mathbf{v}$ is zero; that is*

$$\int_{\{\mathbf{u}\in\Omega^d \ \mid\ \langle \mathbf{u},\mathbf{v}\rangle = 0\}} f(\mathbf{u}) \mathrm{d}\omega_{d-2}(\mathbf{u}) = 0.$$

Then $f \equiv 0$.

The application of Theorem 3.5 to the problem at hand is particularly simple.

Lemma 3.3.4 *Let $K \subset \mathbb{R}^d$ be a convex body with $\mathbf{0} \in \mathrm{int}K$. Then l_K^{d-1} is determined completely in terms of f_K.*

Proof of 3.3.4 If $\mathbf{u} \in \Omega^{d-1}$, write

$$\Omega^{d-2}(\mathbf{u}) = \left\{\mathbf{v} \in \Omega^{d-1} \ \middle|\ \langle \mathbf{u}, \mathbf{v}\rangle = 0\right\},$$

and assume that the standard measure, ω_{d-2}, on $\Omega^{d-2}(\mathbf{u})$ is scaled so that

$$f_K(\mathbf{u}) = \int_{\Omega^{d-2}(\mathbf{u})} l_K^{d-1}(\mathbf{v}) \mathrm{d}\omega_{d-2}(\mathbf{v}). \tag{3.12}$$

Notice that by definition $l^{d-1}(\mathbf{v}) = l^{d-1}(-\mathbf{v})$ and $f_K(\mathbf{u}) = f_K(-\mathbf{u})$.

Now if $L \subset \mathbb{R}^d$ is another convex body containing the origin in its interior and $f_K = f_L$, putting

$$h = l_K^{d-1} - l_L^{d-1},$$

h clearly satisfies the hypotheses of Theorem 3.5 and hence $h \equiv 0$ as required. □

We are now ready to give the proof of Theorem 3.4.

Proof of 3.4 First consider the case $j = 1$. In this case, whenever l is a line supporting B, the chords $K \cap l$ and $M \cap l$ have the same length. Suppose $\mathbf{x} \in$

$\partial K \cap \partial M$, and l is a line supporting B at $\mathbf{u}$ with $\mathbf{x} \in l$. If $\alpha = |K \cap l| = |M \cap l|$ then

$$K \cap l = M \cap l = \left\{ \mathbf{x} + \lambda \|\mathbf{u} - \mathbf{x}\|^{-1}(\mathbf{u} - \mathbf{x}) \;\middle|\; \lambda \in [0, \alpha] \right\}.$$

So in the notation of Lemma 3.3.3, if $\mathbf{x} \in \partial K \cap \partial M$ and $\mathbf{y} \in \partial K$ with $\mathbf{y} \sim \mathbf{x}$ we have $\mathbf{y} \in \partial K \cap \partial M$. Hence, using Lemma 3.3.3, $\partial K \cap \partial M$ is open in ∂K. Clearly, it is also true that $\partial K \cap \partial M$ is closed in ∂K. Hence $\partial K \cap \partial M$ is either empty or is equal to ∂K. However, the boundaries of K and M must meet. Hence

$$\partial K \cap \partial M = \partial K = \partial M,$$

which completes the proof.

We now turn our attention to the other cases; fix $1 < j < d - 1$. Suppose that l is a line supporting B at $\mathbf{u}$ and meeting ∂K at $\mathbf{x}$ and $\mathbf{y}$. Set

$$g_K(l) = \|\mathbf{x} - \mathbf{u}\|^j + \|\mathbf{y} - \mathbf{u}\|^j,$$

and define g_M similarly. We claim that for all l supporting B we have $g_K(l) = g_M(l)$. If so, we use an argument similar to that for the case $j = 1$ to complete the proof.

It remains, therefore, to prove the claim. This can be seen easily by using Lemma 3.3.4. Choose a $j + 1$-dimensional affine subspace H of $\mathbb{R}^d$ supporting B at $\mathbf{u}$, say. Identify H with $\mathbb{R}^{j+1}$ with the origin at $\mathbf{u}$. Clearly, $H \cap K$ and $H \cap M$ are convex bodies in $\mathbb{R}^{j+1}$ containing $\mathbf{0}$ in their interiors. By hypothesis, if $F \subset \mathbb{R}^{j+1}$ is a 1-codimensional subspace,

$$|K \cap F| = |M \cap F|.$$

Thus, using Lemma 3.3.4, if $l \subset H$ is a line supporting B at $\mathbf{u}$ we have $g_K(l) = g_M(l)$ as required. The choice of H was arbitrary so the result follows. $\square$

3.3.9 Sections of Codimension One

In this section we consider the following question. Suppose that K is a convex body containing the unit ball in its interior. To what extent is K determined by the volumes of its sections supporting the unit ball?

As in the two-dimensional case, we have not been able to provide a complete answer to this question. In fact, the best result we are able to provide at present is the following analogue of Theorem 3.2. We will use notation as follows. If $K \subset \mathbb{R}^d$ is a convex body, and $\mathbf{x} \neq \mathbf{0}$, $K(\mathbf{x})$ will denote the one-codimensional section of K cut by the hyperplane perpendicular to and containing $\mathbf{x}$. As usual $|K(\mathbf{x})|$ will mean the $d-1$-volume of $K(\mathbf{x})$

Theorem 3.6 *Suppose that $K, M \subset \mathbb{R}^d$ are convex bodies with $\mathrm{B} \subset \mathrm{int}K \cap \mathrm{int}M$ and that d is odd. Let $A \subset (0,1]$ be countably infinite and suppose that whenever $\mu \in A$ and $\mathbf{u} \in \Omega^{d-1}$ we have $|K(\mu\mathbf{u})| = |M(\mu\mathbf{u})|$. Then $K = M$.*

Proof of 3.6 In order to prove the result, we will make use of spherical harmonics and in particular results in [20]. We will show that with the given information about K it is possible to reconstruct entirely the function

$$r(\mathbf{u}) = \max\{\lambda > 0 \mid \lambda\mathbf{u} \in K\}.$$

To begin with, let $f(\mathbf{x}) = |K(\mathbf{x})|$. We derive an expression for

$$F(s_n, \mu) = \int_{\Omega^{d-1}} f(\mu\mathbf{u}) s_n(\mathbf{u}) \mathrm{d}\omega_{d-1}(\mathbf{u}),$$

where s_n is a spherical harmonic of degree n. In order to do this write

$$f(\mu\mathbf{u}) = \lim_{h\to 0} \frac{1}{2h} \int_K \delta_h(\mathbf{x}, \mathbf{u}, \mu) \mathrm{d}\lambda_d(\mathbf{x}),$$

where λ_d is the standard d-volume measure and δ_h is a functional defined by

$$\delta_h(\mathbf{x}, \mathbf{u}, \mu) = \begin{cases} 1 & \langle \mathbf{x}, \mathbf{u} \rangle \in [\mu - h, \mu + h], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that since B $\subset$ int$K$, the limit in equation 3.3.9 is uniformly convergent over Ω^{d-1} when $\mu \in [0,1]$. Thus, we begin to calculate $F(s_n, \mu)$:

$$\begin{aligned} F(s_n,\mu) &= \int_{\Omega^{d-1}} s_n(\mathbf{u}) \left(\lim_{h\to 0} \frac{1}{2h} \int_K \delta_h(\mathbf{x},\mathbf{u},\mu) \mathrm{d}\lambda_d(\mathbf{x}) \right) \mathrm{d}\omega_{d-1}(\mathbf{u}) \\ &= \int_K \left(\lim_{h\to 0} \frac{1}{2h} \int_{\Omega^{d-1}} \delta_h(\mathbf{x},\mathbf{u},\mu) s_n(\mathbf{u}) \mathrm{d}\omega_{d-1}(\mathbf{u}) \right) \mathrm{d}\lambda_d(\mathbf{x}). \quad (3.13) \end{aligned}$$

Next we will evaluate the inner integral and limit as $h \to 0$. Write

$$\mathbf{x} = t^{-1}\mathbf{u}_0,$$

with $\mathbf{u}_0 \in \Omega^{d-1}$ and $t > 0$. Note that if $t > \mu^{-1}$ then for sufficiently small $h > 0$, δ_h is zero. Assume in what follows that $t < \mu^{-1}$. For $h > 0$ define

$$I(h) = \int_{\mathbf{u}\in\Omega^{d-1}} \delta_h(t^{-1}\mathbf{u}_0, \mathbf{u}, \mu) s_n(\mathbf{u}) \mathrm{d}\omega_{d-1}(\mathbf{u}).$$

To partially evaluate $I(h)$ write the variable of integration $\mathbf{u}$ in the form

$$\mathbf{u} = \cos\theta \mathbf{u}_0 + \sin\theta \mathbf{v},$$

with $\theta \in [0, \pi]$ and $\mathbf{v} \in \Omega^{d-2}(\mathbf{u}_0)$, where

$$\Omega^{d-2}(\mathbf{w}) := \left\{ \mathbf{v} \in \Omega^{d-1} \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \right\}.$$

Notice that using this parameterisation,

$$\delta_h(\mathbf{x},\mathbf{u},\mu) = \begin{cases} 1 & \cos\theta \in [t(\mu-h), t(\mu+h)], \\ 0 & \text{otherwise}, \end{cases}$$

so that $I(h)$ can now be written in the form

$$I(h) = \int_{\theta=\cos^{-1}(t(\mu+h))}^{\cos^{-1}(t(\mu-h))} \sin^{d-2}\theta \int_{\mathbf{v}\in\Omega^{d-2}(\mathbf{u}_0)} s_n(\cos\theta\mathbf{u}_0 + \sin\theta\mathbf{v}) \mathrm{d}\omega_{d-2}(\mathbf{v}) \mathrm{d}\theta.$$

We now appeal to the results derived in [20]; the inner integral in this last expression is calculated as

$$\int_{\mathbf{v}\in\Omega^{d-2}(\mathbf{u}_0)} s_n(\cos\theta\mathbf{u}_0 + \sin\theta\mathbf{v}) \mathrm{d}\omega_{d-2}(\mathbf{v}) = |\omega_{d-2}| C_n^\nu(1)^{-1} C_n^\nu(\cos\theta) s_n(\mathbf{u}_0), \quad (3.14)$$

where $|\omega_{d-2}|$ is the surface area of the unit sphere in $\mathbb{R}^{d-1}$, $\nu = (d-2)/2$ and C_n^ν is the n^{th} Gegenbauer polynomial of order ν. Hence

$$I(h) = \kappa_n s_n(\mathbf{u}_0) \int_{\cos^{-1}(t(\mu+h))}^{\cos^{-1}(t(\mu-h))} \sin^{d-2}\theta C_n^\nu(\cos\theta)\mathrm{d}\theta,$$

with $\kappa_n = |\omega_{d-2}|C_n^\nu(1)^{-1}$. Finally we evaluate, using L'Hopital's rule, the limit

$$\lim_{h\to 0}\frac{1}{2h}I(h) = \kappa_n(1-\mu^2t^2)^{\frac{d-3}{2}}C_n^\nu(\mu t)s_n(\mathbf{u}_0).$$

Next, recalling that $\mathbf{x} = t^{-1}\mathbf{u}_0$, equation 3.13 can be written

$$F(s_n,\mu) = \kappa_n \int_{\Omega^{d-1}} s_n(\mathbf{u}) \int_{t=r(\mathbf{u})^{-1}}^{\mu^{-1}} t^{-(d+1)}(1-\mu^2t^2)^{\frac{d-3}{2}}C_n^\nu(\mu t)\mathrm{d}t\mathrm{d}\omega_{d-1}(\mathbf{u}). \quad (3.15)$$

This equation shows the complexity of the relationship between K and the spherical harmonics of the function f_K. It is apparent that spherical harmonics are, perhaps, not the most efficient tool for deducing the properties of K from f_K. However, we can, using our strong hypothesis, begin to unravel this relation.

The key observation is that when $d \geq 3$ is odd, the exponent $(d-3)/2$ is an integer. Hence we may write the integrand in equation 3.15 as the product of $t^{-(d+1)}s_n(\mathbf{u})$ and a homogeneous polynomial

$$(1-\mu^2t^2)^{\frac{d-3}{2}}C_n^\nu(\mu t) = \sum_{k=0}^{n+d-3}\lambda_n^k\mu^kt^k. \quad (3.16)$$

We do not need to know all the values of λ_n^k above; it will suffice to show that certain of them are non-zero. For the moment, we are content to see that now $F(s_n,\mu)$ can be written in the following form.

$$\begin{aligned} F(s_n,\mu) &= \kappa_n \sum_{k=0}^{n+d-3}\lambda_n^k\mu^k \int_{\mathbf{u}\in\Omega^{d-1}} s_n(\mathbf{u}) \int_{t=r(\mathbf{u})^{-1}}^{\mu^{-1}} t^{k-d-1}\mathrm{d}t\mathrm{d}\omega_{d-1}(\mathbf{u}) \\ &= \kappa_n \sum_{k=0}^{n+d-3}\lambda_n^k\mu^k(B_k(s_n,\mu) - A_k(s_n)), \end{aligned} \quad (3.17)$$

where

$$A_k(s_n) = \begin{cases} \frac{1}{k-d}\int_{\Omega^{d-1}} r^{d-k}(\mathbf{u})s_n(\mathbf{u})\mathrm{d}\omega_{d-1}(\mathbf{u}) & k \neq d, \\ \int_{\Omega^{d-1}} \log r(\mathbf{u})s_n(\mathbf{u})\mathrm{d}\omega_{d-1}(\mathbf{u}) & k = d, \end{cases}$$

and

$$B_k(s_n,\mu) = \begin{cases} \frac{1}{k-d}\int_{\Omega^{d-1}} \mu^{d-k} s_n(\mathbf{u})\mathrm{d}\omega_{d-1}(\mathbf{u}) & k \neq d, \\ \int_{\Omega^{d-1}} \log \mu s_n(\mathbf{u})\mathrm{d}\omega_{d-1}(\mathbf{u}) & k = d. \end{cases}$$

Notice that $B_k(s_n,\mu)$ is independent of K. In fact it is zero unless $n = 0$. The independence of B_k and K means that using the given information, we are able to determine the value of the functional G defined by

$$G(s_n,\mu) = \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k B_k(s_n,\mu) - F(s_n,\mu),$$

for all spherical harmonics s_n, and countably many values of $\mu \in [0,1]$. From equation 3.17, however,

$$G(s_n,\mu) = \kappa_n \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k A_k(s_n).$$

That is, G is polynomial in μ. Therefore, since $G(s_n,\mu)$ is known for countably many values of μ, it is possible to determine $G(s_n,\cdot)$ completely; or equivalently our hypotheses allow us to construct the coefficients of G given by

$$\alpha_k(s_n) = \lambda_n^k A_k(s_n).$$

We will soon see that this is enough to reconstruct K. We claim that for n even, λ_n^0 is non-zero, and that for n odd λ_n^1 is non-zero. If so given any spherical harmonics s_{2n} and s_{2n+1} of degrees $2n$ and $2n+1$ respectively, we are in a position to construct the scalar products

$$\begin{aligned} \left\langle r^d, s_{2n} \right\rangle &= \int_{\Omega^{d-1}} r^d(\mathbf{u}) s_{2n}(\mathbf{u})\mathrm{d}\omega_{d-1}(\mathbf{u}), \\ \left\langle r^{d-1}, s_{2n+1} \right\rangle &= \int_{\Omega^{d-1}} r^{d-1}(\mathbf{u}) s_{2n+1}\mathrm{d}\omega_{d-1}(\mathbf{u}). \end{aligned}$$

Using the completeness of the odd and even systems of spherical harmonics and the uniform continuity of r, we deduce that the functions

$$\begin{aligned} \phi(\mathbf{u}) &= r^d(\mathbf{u}) + r^d(-\mathbf{u}), \qquad (3.18) \\ \tilde{\phi}(\mathbf{u}) &= r^{d-1}(\mathbf{u}) - r^{d-1}(-\mathbf{u}), \end{aligned}$$

are determined for all $\mathbf{u} \in \Omega^{d-1}$ by the function f_K. It is now an easy matter to determine r, for if

$$\begin{aligned} \alpha^d + \beta^d &= a, \\ \alpha^{d-1} - \beta^{d-1} &= b, \end{aligned}$$

with $\alpha, \beta > 0$, then

$$\beta = (\alpha^{d-1} - b)^{\frac{1}{d-1}}.$$

Therefore

$$\alpha^d + (\alpha^{d-1} - b)^{\frac{d}{d-1}} = a,$$

and differentiation with respect to α of this last expression gives

$$(d-1)\alpha^{d-1} + (\alpha^{d-1} - b)^{\frac{1}{d-1}} d\alpha^{d-2} > 0.$$

Hence the solution is unique, and r is determined as claimed.

To complete the proof we still need to verify the claim that λ^0_{2n} and λ^1_{2n+1} are non-zero. In order to see this, recall the definition of λ^k_n as coefficients of the polynomial in equation 3.16. We have

$$\lambda^0_{2n} = C^\nu_{2n}(0).$$

The Gegenbauer polynomials C^ν_n may defined in terms of the expansion

$$\frac{1}{(1-2tx+x^2)^\nu} = \sum_{n=0}^{\infty} C^\nu_n(t) x^k,$$

from which it follows that $C^\nu_{2n}(0)$ is non-zero. Also

$$\begin{aligned} \lambda^1_{2n+1} &= \frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)^{\frac{d-3}{2}} C^\nu_{2n+1}(z)\right) \Bigg|_{z=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}z} C^\nu_{2n+1}(z) \Bigg|_{z=0} \\ &= \nu C^{\nu+1}_{2n}(0), \end{aligned}$$

and this is equally non-zero since $\nu = (d-2)/2$ and d is odd. This completes the result. □

Chapter 4

A Weak Characterisation of the Sphere

4.1 Introduction

We say that two sets $A, B \subset \mathbb{R}^n$ are homothetic if there exist $\lambda \in \mathbb{R}$,$\mathbf{a} \in \mathbb{R}^n$ and an orthogonal linear map ϕ such that $A = \mathbf{a} + \lambda\phi(B)$. Montejano [16] has recently proved the following result concerning sections of convex bodies:

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a convex body with $\mathbf{p} \in \mathrm{int}K$. Suppose that $2 \leq d < n$ and that every pair of d-dimensional sections of K through $\mathbf{p}$ are homothetic. Then K is a Euclidean ball.*

The aim of this chapter is to extend this result, relaxing the restriction $\mathbf{p} \in \mathrm{int}K$. We have not been entirely successful in this endeavour. However we prove a restricted result using a much stronger notion of homothesis, and also construct a counterexample for the case $\mathbf{p} \in \partial K$.

4.2 Definitions and Notation

Let O_n denote the orthogonal group of order n. That is, the group of scalar product preserving linear maps on $\mathbb{R}^n$.

As mentioned in the introduction, we use a modified definition of homothesis. Given sets $A, B \subset \mathbb{R}^n$ we say that A and B are directly homothetic, and write $A{\sim}B$, if either A or B is degenerate (by which we understand either empty or a single point), or if there exist $\mathbf{a} \in A$,$\mathbf{b} \in B$, $\lambda \in \mathbb{R}$ and $g \in O_n$ such that

$$\begin{aligned} \lambda g(B - \mathbf{b}) &= A - \mathbf{a}, \\ g|\mathrm{lin}(A - \mathbf{a}) \cap \mathrm{lin}(B - \mathbf{b}) &= 1. \end{aligned}$$

To see the motivation for this definition, consider the case in which A and B are contained in one-codimensional planes P_A and P_B in $\mathbb{R}^n$. A and B are directly homothetic if after rotation of P_B onto P_A about $P_A \cap P_B$, A and B are related by scaling and translation alone.

Throughout, $H(\mathbf{u})$ will denote the subspace perpendicular to $\mathbf{u}$. As a notational convention, given a set $A \subset \mathbb{R}^{n+1}$ write

$$A_{\mathbf{u}} = A \cap H(\mathbf{u}),$$

and in general, if F is a flat,

$$A_F = A \cap F.$$

Say that $K \subset \mathbb{R}^{n+1}$ is a body of rotation with profile $r : [0,1] \to \mathbb{R}$ if K can be written in the form

$$K = \lambda \left(\bigcup_{\mu \in [0,1]} r(\mu) \mathrm{B}_{\mathbf{u}} + \mu \mathbf{u} \right) + \mathbf{x},$$

with $\mathbf{u} \in \Omega^n$, $\lambda \in \mathbb{R}$, and $\mathbf{x} \in \mathbb{R}^{n+1}$.

If $A \subset \mathbb{R}^n$ is compact and $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$ then where they exist, make the following definitions

$$m_A(\mathbf{u}) = \min\{\lambda \in \mathbb{R} \mid \lambda \mathbf{u} \in A\},$$

$$\begin{aligned}
\mathbf{m}_A(\mathbf{u}) &= m_A(\mathbf{u})\|\mathbf{u}\|^{-1}\mathbf{u}, \\
M_A(\mathbf{u}) &= \max\{\lambda \in \mathbb{R} \mid \lambda\mathbf{u} \in A\}, \\
\mathbf{M}_A(\mathbf{u}) &= M_A(\mathbf{u})\|\mathbf{u}\|^{-1}\mathbf{u}.
\end{aligned}$$

The subscript A will almost always be omitted; the set referred to will be clear from the context.

Finally denote by C_2^n the family of n-dimensional convex bodies, whose boundaries are everywhere twice differentiable.

4.3 A Counterexample for the case $\mathbf{p} \in \partial K$

Before proving the main result of the chapter we give a construction which yields a counterexample for the case $\mathbf{p} \in \partial K$ of Theorem 4.1. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denote the standard basis for $\mathbb{R}^n$ and suppose that K is a body of rotation with profile $r : [0,1] \to \mathbb{R}$ given by

$$r^2(\lambda) = \lambda^\beta - \lambda^2,$$

with $\beta \in (0,2)$. Thus K is a convex body lying somewhere between a line segment ($\beta = 2$) and a solid hemisphere ($\beta = 0$). Write K in the form

$$K = \bigcup_{\lambda=0}^{1} r(\lambda)\mathrm{B}_{\mathbf{e}_1} + \lambda\mathbf{e}_1.$$

We show that sections through $\mathbf{0}$ inclined at an angle to the axis of rotation are homothetic to those parallel to it.

Consider the section of K cut by the hyperplane $H_{\mathbf{u}}$, where

$$\begin{aligned}
\mathbf{u} &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \\
\mathbf{v} &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,
\end{aligned}$$

and $\theta \in (0, \frac{\pi}{2})$. Clearly $K_{\mathbf{u}}$ is a body of rotation of the form

$$K_{\mathbf{u}} = \bigcup_{\lambda=0}^{M(\mathbf{v})} R(\lambda)\mathrm{B}_{H_{\mathbf{e}_1}\cap H_{\mathbf{e}_2}} + \lambda\mathbf{v}.$$

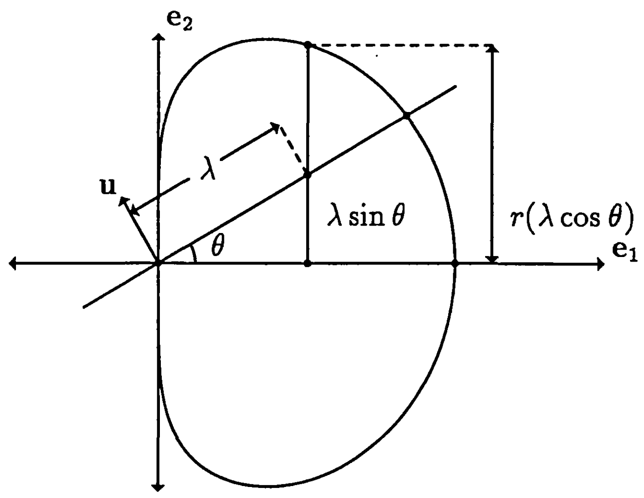

Figure 4.1: A section of K inclined with respect to $\mathbf{e}_1$

Using the geometry illustrated in figure 4.1 the profile function R is given by

$$\begin{aligned} R^2(\lambda) &= r^2(\lambda \cos \theta) - \lambda^2 \sin^2 \theta \\ &= (\lambda \cos \theta)^\beta - \lambda^2. \end{aligned}$$

This can be seen to be a scaled version of r with scale factor $\cos^{\frac{\beta}{2-\beta}} \theta$. If

$$s(\lambda) := \sec^{\frac{\beta}{2-\beta}} \theta R(\lambda \cos^{\frac{\beta}{2-\beta}} \theta),$$

then s certainly satisfies the following

$$\begin{aligned} s^2(\lambda) &= \sec^{\frac{2\beta}{2-\beta}} \theta \lambda^\beta \cos^\beta \theta \cos^{\frac{\beta^2}{2-\beta}} \theta - \lambda^2 \\ &= \lambda^\beta - \lambda^2, \end{aligned}$$

as required. Since $\mathbf{e}_2$ could have been chosen arbitrarily from $H_{\mathbf{e}_1}$ it follows that any section of K through $\mathbf{0}$ not parallel or perpendicular to $\mathbf{e}_1$ is a body of rotation with profile r up to scaling. Clearly every section of K through $\mathbf{0}$ parallel to $\mathbf{e}_1$ is a body of rotation with profile r. The only section perpendicular to $\mathbf{e}_1$ is degenerate (a single point). Hence every pair of non-degenerate sections of K through $\mathbf{0}$ are homothetic.

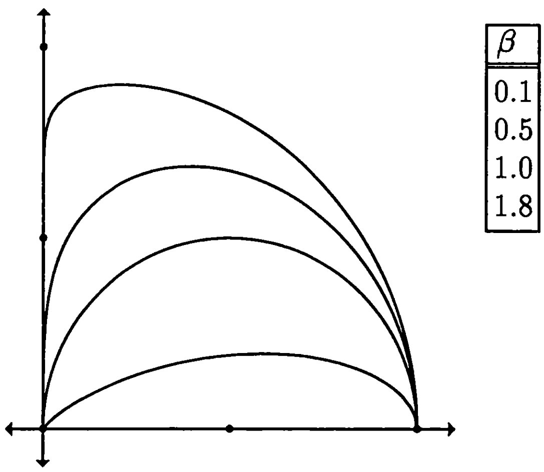

Figure 4.2: Profiles of the Counterexamples

The profiles of selected counterexamples are depicted in figure 4.2. It is worth noting that the case $\beta = 0$ corresponding to a solid hemisphere very nearly works as a counterexample. Every section of this body through $\mathbf{0}$ not perpendicular to $\mathbf{e}_1$ is a one codimensional solid hemisphere. However, the section perpendicular to $\mathbf{e}_1$ is a one-codimensional ball.

The next section introduces the results proved in the remainder of this chapter.

4.4 Results

Throughout we assume that n is an integer greater than one. The following result gives a weak version of Theorem 4.1 for the case $\mathbf{p} \notin K$.

Theorem 4.2 *Let $K \in C_2^{n+1}$ with $\mathbf{0} \notin \mathrm{int}K$. Suppose that given $\mathbf{u}, \mathbf{v} \in \Omega^n$ we have $K_{\mathbf{u}} \sim K_{\mathbf{v}}$. Then K is a Euclidean ball.*

This theorem relates to one-codimensional sections of K. It is a simple matter to extend Theorem 4.2 by induction.

Theorem 4.3 *Let $K \in C_2^{n+1}$ with $0 \notin K$. Suppose that $2 \leq d \leq n$ and that given subspaces $F, G \subset \mathbb{R}^{n+1}$ with $\dim F = \dim G = d$ we have $K_F \sim K_G$. Then K is a Euclidean ball.*

Next we provide a number of lemmas required to prove the results.

4.5 Preliminary Lemmas

Lemma 4.5.1 *Suppose that $r : [-1,1] \to \mathbb{R}$ satisfies*

$$r(x) = \gamma^{-1} r(\alpha x + \delta) + ax^2 + bx + c, \tag{4.1}$$

for all $x \in [-1,1]$, where $\alpha, \gamma, \delta, a, b \in \mathbb{R}$ satisfy $0 < \gamma \notin \{1, \alpha, \alpha^2\}$, $|\alpha| < 1$ and $\nu = \delta/(1-\alpha) \in [-1,1]$. Then r is of the form

$$r(x) = q(x) + F(x-\nu)|x-\nu|^{\beta},$$

with $\beta = \log\gamma/\log|\alpha|$, F is a function satisfying

$$F(x) = F(\alpha x) \quad \forall x \in [-1-\nu, 1-\nu],$$

and q is quadratic.

Proof of 4.5.1 For $x \in [-1,1]$ let $q(x) = Ax^2 + Bx + C$ where $A, B, C \in \mathbb{R}$ are given by

$$\begin{aligned} A &= \frac{\gamma a}{\gamma - \alpha^2}, \\ B &= \frac{\gamma b + 2\delta\alpha a}{\gamma - \alpha}, \\ C &= \frac{\gamma c + \delta^2 A + \delta B}{\gamma - 1}. \end{aligned}$$

Substitution shows that $r = q$ is a solution of equation 4.1. Writing $r(x) = q(x) + f(x)$, some $f : [-1,-1] \to \mathbb{R}$, and substituting again we find that f must satisfy

$$\gamma f(x) = f(\alpha x + \delta).$$

Now let $g(x) = f(x + \nu)$ for $x \in [-1-\nu, 1-\nu] = I$. Then

$$\gamma g(x) = g(\alpha x), \tag{4.2}$$

for all $x \in I$. Clearly, $g(x) = |x|^\beta$ is a solution to equation 4.2. We assume that g is not of this form. For $0 \neq x \in I$ there exists $F(x) \in \mathbb{R}$ such that $g(x) = F(x)|x|^\beta$. Hence

$$F(x) = F(\alpha x),$$

for $x \in I$ as claimed. This completes the proof. □

The next lemma provides a starting point for the proofs of the main results. It is well known, and can be found in [13], for example. We repeat the proof here for clarity.

Lemma 4.5.2 *Let $K \in C_2^n$ with $\mathbf{0} \notin \mathrm{int}K$. Then there exists $\mathbf{a} \in \mathrm{int}K$ such that the hyperplane $H(\mathbf{a}) + \mathbf{m}(\mathbf{a})$ supports K.*

Proof of 4.5.2 If $\mathbf{0} \in \partial K$ the result is trivial; we assume, therefore, that $\mathbf{0} \notin K$ and set

$$C = \{\mathrm{x} \in \partial K \mid \lambda \mathrm{x} \notin K \text{ whenever } \lambda < 1\}.$$

C is the set of points of ∂K visible from $\mathbf{0}$. Clearly C is a topological $(n-1)$-ball. Let $\mathbf{a} \in C$ and $\mathbf{n}$ denote the (necessarily unique) outward unit normal to K at $\mathbf{a}$. Smoothness of K ensures that the line $\mathbf{a} + \mathrm{lin}\{\mathbf{n}\}$ intersects the boundary ∂K at exactly two points, one of which is $\mathbf{a}$. The other we write $\tilde{\mathbf{a}}$. Using smoothness once more, note that $\tilde{\mathbf{a}}$ is continuously dependent on $\mathbf{a}$. Finally, let $\bar{\mathbf{a}} = \mathbf{m}(\tilde{\mathbf{a}})$. Then the map $\mathbf{a} \mapsto \bar{\mathbf{a}}$ is continuous and maps C to C, so using the Brower Fixed Point Theorem, there exists $\mathbf{a} \in C$ such that $\mathbf{a} = \bar{\mathbf{a}}$. In this case we must have $\mathbf{a}, \tilde{\mathbf{a}}$ and $\mathbf{0}$ collinear, showing that $\mathbf{a} + H(\mathbf{a})$ supports K at $\mathbf{a}$. To complete the result it is necessary only to note that the line $\mathrm{lin}\{\mathbf{a}\}$ intersects the interior of K.

□

We require one further result, due to C.A.Rogers. In [19] he proves the following.

Lemma 4.5.3 *Let $K, L \subset \mathbb{R}^n$ be convex bodies with $\mathbf{p} \in \text{int}K$ and $\mathbf{q} \in \text{int}L$. Suppose that whenever $F \subset \mathbb{R}^n$ is a two dimensional subspace the sections $(\mathbf{p} + F) \cap K$ and $(\mathbf{q} + F) \cap L$ are identical up to scaling and translation. Then K and L are identical up to scaling and translation.*

4.6 Proof of Main Results

The proof of Theorem 4.2 consists of several stages. In the first we show that under the hypotheses K must be a body of rotation. We now know that every section of K which is parallel to the axis of rotation is identical up to rotation about the axis. The second stage uses the definition of homothesis given to derive a more specific relationship between the sections of K which are not parallel to the axis of rotation and those that are. Next, the third stage uses geometric approach to predict the shape of the sections of K which are not parallel to the axis of rotation. Finally, the results from the second and third stages are combined to give an equation involving the profile of K which can then be solved using the preliminary lemmas.

Proof of 4.2 We first reduce K to a body of rotation. Using Lemma 4.5.2 let $\mathbf{a} \in \text{int}K$ be chosen so that $\mathbf{a} + H(\mathbf{a})$ supports K at $\mathbf{m}(\mathbf{a})$. Let $\mathbf{u}, \mathbf{v} \in \Omega^n_{\mathbf{a}}$. Then $K_\mathbf{u}$ and $K_\mathbf{v}$ have non-void relative interiors so by hypothesis there exist $g \in O_{n+1}$, $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned} K_\mathbf{u} &= \lambda g K_\mathbf{v} + \mathbf{x}, \\ g|H(\mathbf{u}) \cap H(\mathbf{v}) &= 1. \end{aligned} \tag{4.3}$$

We think of $K_\mathbf{u}$ and $gK_\mathbf{v}$ embedded in the one-codimensional subspace $H(\mathbf{u})$. Both $\mathbf{m}(\mathbf{a})$ and $\mathbf{a}$ belong to $H(\mathbf{u}) \cap H(\mathbf{v})$ and are thus unaffected by g. Hence for $\mathbf{x} \in K$ we have

$$\langle \mathbf{x}, \mathbf{a} \rangle = \langle g\mathbf{x}, g\mathbf{a} \rangle = \langle g\mathbf{x}, \mathbf{a} \rangle,$$

so $\mathbf{m}(\mathbf{a}) + H(\mathbf{a}) \cap H(\mathbf{u})$ supports $gK_{\mathbf{v}}$ at $\mathbf{m}(\mathbf{a})$. However, using equation 4.3 we also have

$$gK_{\mathbf{v}} = (K_{\mathbf{u}} - \mathbf{x})/\lambda,$$

so it is equally true that $(\mathbf{m}(\mathbf{a}) - \mathbf{x})/\lambda + H(\mathbf{a}) \cap H(\mathbf{u})$ supports $gK_{\mathbf{v}}$ at $(\mathbf{m}(\mathbf{a}) - \mathbf{x})/\lambda$. Thus, either $\partial K_{\mathbf{v}}$ and hence ∂K contain a line segment, or $\mathbf{x} \in \text{lin}\{\mathbf{a}\}$. But in this case, using $[\mathbf{p}, \mathbf{q}]$ to denote the line segment joining $\mathbf{p}$ and $\mathbf{q}$,

$$[\mathbf{m}(\mathbf{a}), \mathbf{M}(\mathbf{a})] = \lambda[\mathbf{m}(\mathbf{a}), \mathbf{M}(\mathbf{a})] + \mu\mathbf{a},$$

for some $\mu \in \mathbb{R}$. So $\lambda = 1$ and $\mathbf{x} = \mu\mathbf{a} = \mathbf{0}$.

We have now reduced the problem somewhat; given $\mathbf{u}, \mathbf{v} \in \Omega_{\mathbf{a}}^{n+1}$, there exists $g_{\mathbf{u},\mathbf{v}} \in O_{n+1}$ such that

$$\begin{aligned} K_{\mathbf{u}} &= g_{\mathbf{u},\mathbf{v}} K_{\mathbf{v}}, \\ g|H(\mathbf{u}) \cap H(\mathbf{v}) &= 1. \end{aligned} \tag{4.4}$$

That is, each pair of sections of K parallel to the axis $\text{lin}\{\mathbf{a}\}$ are related by an orthogonal map alone.

Let $F \subset H(\mathbf{a})$ be a two-dimensional subspace perpendicular to $\mathbf{a}$. Suppose that $\{\mathbf{u}_0, \mathbf{v}_0\}$ and $\{\mathbf{u}_1, \mathbf{v}_1\}$ are orthonormal bases for F. Writing $g = g_{\mathbf{u}_0,\mathbf{u}_1}$, we claim that

$$g\text{lin}\{\mathbf{v}_1\} = \text{lin}\{\mathbf{v}_0\}. \tag{4.5}$$

To see this, put $H = H(\mathbf{u}_0) \cap H(\mathbf{u}_1)$ and write $\mathbf{u}_0 = \lambda\mathbf{u}_1 + \mu\mathbf{v}_1$, some $\lambda, \mu \in \mathbb{R}$. Then $\mathbf{x} \in H$ if and only if $\langle \mathbf{u}_0, \mathbf{x} \rangle = \langle \mathbf{u}_1, \mathbf{x} \rangle = 0$, if and only if

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{x} \rangle &= 0, \\ \lambda\langle \mathbf{u}_1, \mathbf{x} \rangle + \mu\langle \mathbf{v}_1, \mathbf{x} \rangle &= 0. \end{aligned}$$

So either $\mu = 0$ or $\langle \mathbf{v}_1, \mathbf{x} \rangle = 0$. Without loss of generality, we assume the latter and observe that in this case $\mathbf{v}_1 \in H^\perp$ and hence $g\mathbf{v}_1 \in gH^\perp = H^\perp$ by hypothesis.

Also $\mathbf{v}_1 \in H(\mathbf{u}_1)$ so $g\mathbf{v}_1 \in H(\mathbf{u}_0)$. Finally $g\mathbf{v}_1 \in (\{\mathbf{u}_0\} \cup H)^{\perp}$; however, this is a one-dimensional subspace which by similar argument also contains $\mathbf{v}_0$.

Now fix $\mathbf{x} \in \mathrm{lin}\{\mathbf{a}\} \cap \mathrm{int}K$. For given $\mathbf{v} \in \Omega^n_F$, write

$$r(\mathbf{v}) = \max\{\lambda \in \mathbb{R} \mid \mathbf{x} + \lambda\mathbf{v} \in K\}.$$

Now $gK_{\mathbf{u}_1} = K_{\mathbf{u}_0}$ and $g\mathbf{x} = \mathbf{x}$ so using the fact that $g\mathrm{lin}\{\mathbf{v}_1\} = \mathrm{lin}\{\mathbf{v}_0\}$ we have

$$r(\mathbf{v}_1) \in \{r(\mathbf{v}_0), r(-\mathbf{v}_0)\}.$$

The choice of the bases $\{\mathbf{u}_i, \mathbf{v}_i\}$ was entirely arbitrary so by continuity of ∂K we have $r(\mathbf{u}) = r(-\mathbf{u}) = r(\mathbf{v})$ for any choice of $\mathbf{u}, \mathbf{v} \in \Omega^n_F$, and deduce that $K_{\mathbf{x}+F}$ is a Euclidean ball with centre $\mathbf{x}$. The choice of $F \subset H(\mathbf{a})$ was also arbitrary so, using Lemma 4.5.3, $K_{\mathbf{x}+H(\mathbf{a})}$ is a Euclidean ball, and K is a body of rotation.

So far, we have concerned ourselves with sections parallel or perpendicular to the axis of rotation of K. We now turn our attention to those sections of K containing $\mathbf{0}$, and inclined with respect to the axis of rotation. Let $\mathbf{v}_0 = \|\mathbf{a}\|^{-1}\mathbf{a}$. We may assume without loss of generality that K is of the form

$$K = \bigcup_{\lambda\in[-1,1]} r(\lambda)\mathrm{B}_{\mathbf{v}_0} + (\lambda + \mu_0)\mathbf{v}_0,$$

where $\mu_0 = 1 + m(\mathbf{v}_0)$ and $r : [-1,1] \to \mathbb{R}$ is concave with $r(-1) = r(1) = 0$. Fix $\mathbf{u}_0 \in \Omega^n_{\mathbf{v}_0}$ and for $\phi \in [0, \frac{\pi}{2}]$ put

$$\begin{aligned} \mathbf{v}_\phi &= \cos\phi\mathbf{v}_0 + \sin\phi\mathbf{u}_0, \\ \mathbf{u}_\phi &= \cos\phi\mathbf{u}_0 - \sin\phi\mathbf{v}_0. \end{aligned}$$

The reader might find it helpful to refer to figure 4.3 in which the present definitions are illustrated. In the figure the positive x-axis points in direction $\mathbf{v}_0$. Let

$$\Phi = \sup\left\{\phi \in [0, \frac{\pi}{2}] \;\middle|\; \mathrm{lin}\{\mathbf{v}_\phi\} \cap K \neq \emptyset\right\},$$

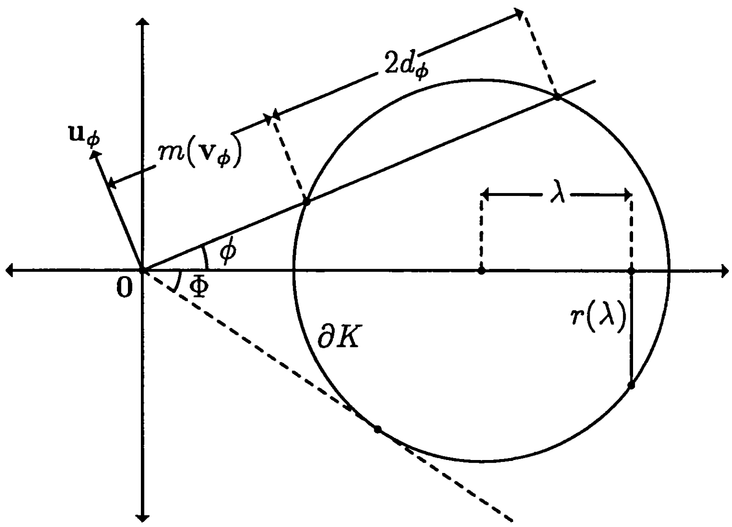

Figure 4.3: Definitions for Theorem 4.2

and for $\phi \in [0, \Phi]$ define

$$d_\phi = \frac{M(\mathbf{v}_\phi) - m(\mathbf{v}_\phi)}{2}.$$

Trivially, $\Phi \in (0, \frac{\pi}{2})$ so there exists $0 < \Phi_0 < \Phi$ such that $d_\phi < \cos\phi$ whenever $\phi \in [\Phi_0, \Phi]$.

Let $\phi \in [\Phi_0, \Phi]$ be given and consider the sections $K_{\mathbf{u}_0}$ and $K_{\mathbf{u}_\phi}$. Both have non-empty relative interior, so using the hypothesis once more, there exist $g \in O_{n+1}$, $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned} K_{\mathbf{u}_0} &= \lambda g K_{\mathbf{u}_\phi} + \mathbf{x}, \\ 1 &= g|H(\mathbf{u}_0) \cap H(\mathbf{u}_\phi). \end{aligned} \tag{4.6}$$

As in the first section, we also have

$$g\mathrm{lin}\{\mathbf{v}_\phi\} = \mathrm{lin}\{\mathbf{v}_0\}.$$

Thus, by symmetry of $K_{\mathbf{u}_0}$ and $K_{\mathbf{u}_\phi}$ about $\mathrm{lin}\{\mathbf{v}_0\}$ and $\mathrm{lin}\{\mathbf{v}_\phi\}$ respectively, $\mathbf{x}$ must lie on the line $\mathrm{lin}\{\mathbf{v}_0\}$. Finally equation 4.6, and the fact that $K_{\mathbf{u}_0}$ is a body of rotation of the form

$$K_{\mathbf{u}_0} = \bigcup_{\lambda \in [-1,1]} r(\lambda) \mathrm{B}_{H(\mathbf{v}_0) \cap H(\mathbf{u}_0)} + (\lambda + \mu_0)\mathbf{v}_0,$$

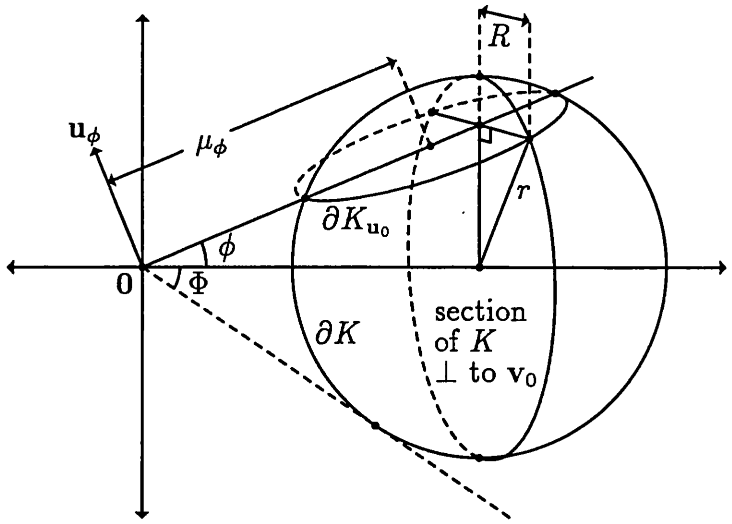

Figure 4.4: Finding the Profile R of $K_{\mathbf{u}_0}$

together imply that $K_{\mathbf{u}_\phi}$ must be a body of rotation of the form

$$K_{\mathbf{u}_\phi} = \bigcup_{\lambda \in [-d_\phi, d_\phi]} R(\lambda) \mathrm{B}_{H(\mathbf{v}_\phi) \cap H(\mathbf{u}_\phi)} + (\lambda + \mu_\phi)\mathbf{v}_\phi,$$

where $\mu_\phi = m(\mathbf{v}_\phi) + d_\phi$. The new profile function $R : [-d_\phi, d_\phi] \to \mathbb{R}$ satisfies

$$R(\lambda) = d_\phi r(\epsilon\lambda / d_\phi), \tag{4.7}$$

with $|\epsilon| = 1$, $R(-d_\phi) = R(d_\phi) = 0$ and R concave.

This is one approach to predicting the section $K_{\mathbf{u}_\phi}$. However, since K is a body of rotation, we already know the form this section must take in terms of the profile function r. Consider figure 4.4. Using Pythagoras's Theorem we obtain

$$R^2(\lambda) = r^2(\lambda \cos\phi + \mu_\phi \cos\phi - \mu_0) - (\lambda + \mu_\phi)^2 \sin^2\phi, \tag{4.8}$$

for all $\lambda \in [-d_\phi, d_\phi]$. Next, putting equations 4.7 and 4.8 together and making the change of variable $x = \epsilon\lambda / d_\phi$, the identity

$$d_\phi^2 r^2(x) = r^2\left(\epsilon d_\phi x \cos\phi + \mu_\phi \cos\phi - \mu_0\right) - \left(\epsilon d_\phi x + \mu_\phi\right)^2 \sin^2\phi \tag{4.9}$$

holds for all $x \in [-1, 1]$. Equation 4.9 is of the form required by the hypothesis of Lemma 1.4.1, which we now apply, deducing that r can be written

$$r^2(x) = q(x) + f(x - \nu).$$

Here q is quadratic and the relevant constants and conditions are

$$\begin{aligned}
\alpha &= \epsilon d_\phi \cos\phi, \\
\gamma &= d_\phi^2, \\
\delta &= \mu_\phi \cos\phi - \mu_0, \\
\nu &= \frac{\delta}{1-\alpha}, \\
f(\alpha x) &= \gamma f(x) \quad \forall x \in I = (-1-\nu, 1-\nu).
\end{aligned}$$

We consider the derivatives of f at 0. A trivial calculation shows that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(\alpha x) = \frac{\gamma}{\alpha^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x),$$

so noting that $\gamma/\alpha^2 = \sec^2\phi > 1$ it is immediately obvious that unless f is linear, it cannot possess a second derivative at 0. Hence either $\nu \in \{-1, 1\}$ contradicting the hypothesis $0 \notin K$, or f is linear and r^2 is quadratic. This completes the proof, for $r(-1) = r(1) = 0$ and inspection of the proof of Lemma 4.5.1 reveals that the leading coefficient of q is 1. □

The proof of Theorem 4.2 complete, is is a simple matter to prove the corollary, Theorem 4.3

Proof of 4.3 Let $F \subset \mathbb{R}^{n+1}$ be an $(d+1)$-dimensional subspace and suppose that relint$K_F \neq \emptyset$. Then by Theorem 4.2, K_F is a Euclidean ball. Hence, using lemma 4.5.3, K is a Euclidean ball as required. □

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