Electromechanical Modelling of Trapezoidal Microstructures

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A thesis submitted for the degree of
Doctor of Philosophy
of the
University of London

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Dedicated to
To my wife Kelly
Mum and Dad, and
my sisters Ling, Ching, Yee and Wah.
Abstract

This thesis is concerned with the assessment of the effects of non conformity of shape when using low cost fabrication methods. The use of deep reactive ion etching and synchrotron sources (LIGA process) allows for almost vertical side walls in an etched or exposed material. However a standard micro-electromechanical systems (MEMS) fabrication process, which uses basic UV lithography and anisotropic wet etching, creates sloping sidewalls thus leaving us with structures that are no longer square or rectangular in cross section, but trapezoidal. This affects the electrostatic actuation of comb electrodes, since they are now asymmetrical and also support beams that may experience twisting or bending. In this case the assumption that structures have simple square or rectangular cross-section is no longer valid and the sloping side wall has to be taken into account.

Standard scientific techniques have been employed into the investigation of trapezoidal microstructures namely an analytical process, followed by computer simulation and then comparisons with the fabricated structures. A variational method has been used to calculate the torsional behaviour of trapezoidal beams, while a conformal transformation approach has been used to model the electrostatic behaviour of the comb electrodes. On the computer simulation side ANSYS was used for the finite element modelling, while MATLAB was used for symbolic mathematical calculations of the variational method. As a comparison to the variational method and the FEM results, electrostatic and mechanical trapezoidal structures were made using nickel electroforming and silicon bulk etching.
Acknowledgements

I would like to acknowledge my supervisor, Andrew Holmes. Andrew has been my supervisor from the days when I was a final year undergraduate at Imperial College, and indeed it was he who first proposed the idea of doing a PhD to me. Over the years he has given me constant support and encouragement, and I have enjoyed working with him and learning from him. Half my time has been spent in the chemistry lab and thanks go to Munir Ahmad who has guided me through the do’s and don’ts with chemicals. James Hampson - I owe all my PC knowledge to him. Mino Green - the ‘Prof’ who lives down the road, thanks for making my morning journey into university more enlightening. Eric Yeatman - for employing me to work on the International Journal of Electronics.

To my friends who started their PhD’s at the same time Ryan, Emmanuel, Sonia, A, Ty, Regina, Gerald I have really enjoyed working with all of you and getting to know you, it is a shame that we will be parting company soon. To those who are in the middle of their research, Miguel, Shubo, Anke, Nick and Athanasios keep on going the end is in sight! To those who are just starting Yuk, Michael, Trevor and Valerio, the experience of the next few years will have their ups and downs but it will be all worthwhile. Guodong and Helin - best of luck. To the rest of the staff in our group, Richard Syms, Kristel Fobelets, Tom Tate and Dave Morris I have seen this group expand over the years and all your efforts in making this research group a great place to study has not gone unnoticed. Special thanks go to my old university friends Arash, Tay and Ganesh, when I am with you guys the fun is endless, thank god you all live in London, and Ganesh the PhD is not that bad at the end of it you can be proud of you achievements; to my old school chums Adrian, James, Kai, Johnny, Al and Jules and the girls Dom, Louise, Cat, Drew and Horace - after 170 pages is anyone up for a pint; more uni friends Andrew DS, Ivan, Maria - this is begining to look like a Felix end of year acknowledgment; Debs - thanks for keeping me fit.

And finally to my parents for all the support they have given me for all of my life. You have always been there to help, and with those endless words of encouragement you have helped me through those difficult times.
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Chapter 1

Introduction

The subject area of this thesis is microelectromechanical structures and the problems incurred when using low cost fabrication techniques. Standard low cost methods often have the penalty of creating structures with sloping rather than vertical side-walls. Sloping side-walls limit the minimum clearances in a device, but more importantly the mechanical and electrostatic behaviours of trapezoidal elements differ significantly from their rectangular counterparts.

On the mechanical side an investigation has been made into the torsional behaviour of trapezoidal beams, with the aim of creating a numerical factor that will enable the torsional stiffness to be obtained for a given cross-section. When solving the torsional problem another problem naturally arises, namely finding the shear centre; this problem is also addressed.

The electrostatic problem of trapezoidal electrodes has also been investigated. With rectangular electrodes the electric field lines, in the absence of a ground-plane, are symmetrical about a line through the electrode centres, leading to a purely in-plane electrostatic force. In the presence of a ground-plane this symmetry is broken, resulting in an overall out-of-plane force that is a function of various electrode parameters, including the side-wall angle. Several methods are applied to the problem of calculating this force: a purely analytical approach using conformal transforms, a numerical method also based on conformal transforms, and finite element analysis.

The predictions of the mechanical and electrostatic analyses are compared with measurements made on real devices. To test the torsion analysis silicon mirrors were made using a low cost silicon etching process that created trapezoidal beams. Trapezoidal electrodes were fabricated by electroplating UV exposed photoresist moulds to test the electrostatic analysis. As will be seen later UV exposed photoresist can create sloping side-walls, and the side wall relief is replicated by the electroplated material.
In the remainder of this chapter a survey of MEMS actuation, and fabrication processes is covered, while the methods of mechanical and electrostatic analysis used in this research are covered in the following chapter.

1.1 Microelectromechanical Systems (MEMS)

The field of microelectromechanical systems or MEMS has a wide variety of names around the globe; micromachines, microengineering and micromechanics being a few of them. However, all these names relate to the same idea: the application of integrated circuit design and manufacturing techniques to create devices that integrate mechanical elements with sensors, actuators, and electronics on a common substrate. A common feature of MEMS devices is that their sizes may range from a few microns to a few millimetres. The same revolution that electronic circuits enjoyed in the sixties and seventies, with the move from vacuum valves to semiconductor devices, is now being applied to electromechanical systems. MEMS technology is pushing forward the idea of mass production of mechanical devices, using repeatable, and commercially viable processes.

1.2 Actuation Methods

There are many different methods for exciting a micro-mechanical structure, the most common being electrostatic [1-3], piezoelectric [4-6], thermal [7-9] and magnetic [10-12] actuation. These mechanisms are described below, together with examples of typical devices that employ them.

1.2.1 Electrostatic Actuation

Comb Electrodes

Since their first application to a lateral resonator in the late eighties [2] comb electrodes have become the ubiquitous driving force behind MEMS devices. When a potential difference is applied between two opposing comb electrodes an attractive force occurs pulling the two sides together.
To calculate the electrostatic forces the internal energy is first calculated. When the electrodes are charged to a voltage $V$ and have a capacitance $C$ the energy $U$ is:

$$U = \frac{1}{2} CV^2$$

Ignoring fringing fields the capacitance of the overlapping combs is given by:

$$C = \frac{\varepsilon_0 wt}{g}$$

Where $w$ is the overlap length, $g$ is the electrode gap and $t$ is the thickness of the comb electrodes. If the overlapping distance is increased by $dx$ the resulting capacitance will rise by:

$$dC = \varepsilon_0 t dx$$

As a result of an increase in overlap distance the internal energy will increase by:

$$dU = \frac{1}{2} V^2 dC$$

The charge stored on the capacitor is defined as $Q = CV$, so if the voltage is constant a change in capacitance must be accompanied by a change in stored charge. The extra charge is supplied by the external energy source, and the energy supplied is $dE = dCV^2$. Half of this energy is used to increase the internal energy stored in the capacitor, while the remainder is used to perform mechanical work. Assuming that the work is done by a force $F$ acting in the x-direction, the energy is $dU = F dx$.

$$F = \frac{\partial U}{\partial x} = \frac{1}{2} V^2 \frac{\partial C}{\partial x}$$

This reduces to:

$$F = \frac{1}{2} \frac{\varepsilon_0 t}{g} V^2$$
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With respect to a series of comb electrodes this force applies to each gap. Assuming that there are \( n \) electrodes, and neglecting fringing fields, the overall force is given by

\[
F = (n - 1) \varepsilon_0 V^2 / 2g,
\]

where \( \varepsilon_0 \) is the dielectric constant of free space, \( V \) is the voltage and \( g \) is the electrode gap [2].

Parallel Plate Electrodes

Simple parallel plate actuators are extremely easy to make but suffer from the problem of snap-down. A pair of variable gap parallel plate electrodes with a spring suspension becomes unstable at a certain critical gap. This non-linear behaviour has been modelled both by analytical methods and by FEM using ABAQUS [13] and ANSYS [14]. Non-linearity and instability are undesirable for actuators in general, but may be advantageous for bistable devices e.g. switches [15].

Device Examples

Comb electrodes have been used almost exclusively in laterally driven devices, such as linear resonators [3, 16]. One important feature of typical comb electrodes is that the capacitance is linear with displacement, and therefore the electrostatic force is constant over the length of the combs except at the ends of the range of travel [17]. In addition to standard in-plane resonators more novel devices have utilised comb electrodes. For example, electrostatic combs have been used in micro-grippers [18], laser microscanners [19, 20], accelerometers [21] and gyroscopes [22].

Parallel plate actuators have been used previously to actuate torsional mirrors. For example in [23] a torsional device consisting of an aluminium mirror was driven using two electrodes directly beneath the mirror. In this kind of device a rocking motion is created when the two sets of electrodes are driven in anti-phase to one another.

Micromechanical switches have also been fabricated [24]. When compared with conventional relays these offer the obvious advantages of being smaller and consuming less power. However, as with most MEMS devices the ability to integrate them with other electronic devices on a single wafer is their most attractive property.

Other devices come in the form of electrostatic motors [25, 26] [27–30], and capacitive transducers [31]. With electrostatic motors the parallel plates are on the stator and the rotor, and the electrostatic forces tend to align the two plates in a horizontal motion rather than a vertical motion. Most of the electrostatic micromotors are similar to the one described in fig(1.1). Each segment of the stator is driven at the same frequency, but with a certain phase difference, much in the same way the commutator of a conventional dc motor magnetises the coils at the right position to maximise the torque on the axle.
Each pole on the rotor is attracted to a charged stator segment, which creates a torque on the rotor. Before the attractive force reaches a minimum and the rotor is overcome by friction, the next segment is charged up, which pulls the rotor round further, resulting in a spinning motion. To reduce friction a bushing is designed into the rotor; the bushing reduces the contact surface area between the rotor and the substrate allowing for a smoother motion.

These types of electrostatic motors are prone to wear, and so a next generation of micromotor, called the wobble motor, was developed in which the rotor moves by rolling rather than sliding on the stator as describe in fig(1.2). A wobble motor’s motion is much like that of a spinning coin, which has been spinning for a while so that the angle it makes with respect to the ground is quite low. In contrast to a conventional electrostatic motor, the stator poles are located lower than the rotor so that the rotor is pulled down to the substrate during rotation. With the reduction in friction a higher torque can be generated, which is important if these types of motors are to be used for coupling mechanical energy out for driving purposes.

1.2.2 Piezoelectric Actuation

Piezoelectric films have the characteristic that when a voltage is applied to the material a stress is induced. The most popular type of piezoelectric material used in MEMS structures is polycrystalline zinc oxide (ZnO). Piezoelectric materials experience a change in volume when a voltage is applied but the changes in volume are small. To obtain larger movements a bilayer must be formed, for example ZnO on a silicon substrate. When a voltage is applied to the ZnO it contracts causing a stress gradient normal to the whole bilayer, which causes it to bend. The ZnO can be deposited, for example, by RF magnetron sputtering using a manganese doped target with the substrate held at around
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Another type of piezoelectric film that offers an attractive route is lead zirconate titanate or more simply PZT. Apart from the traditional deposition methods of sputtering, e-beam evaporation, and chemical vapour deposition, some films can be deposited using solution deposition. Since the materials are in a solution this allows for good intermixing between the precursor chemicals. They can also be deposited on to samples using simple spin coating. The solution methods can be categorised into sol-gel methods and non-polymeric routes. The aim of PZT films is to make the precursors create hetero linkages (Pb O Ti and Pb O Zr). However in sol-gels homogeneous bonding can occur more readily (Pb O Pb, Ti O Ti and Zr O Zr) degrading the quality of the film.

A non-polymeric method developed here in the Optical and Semiconductor Devices Group of Imperial College has the advantage of using commercially available materials that have a high stability against humidity and a simple application method [32]. This basically allows the material to be mixed at room temperature and in open air. The method requires the use of a metallo-organic compound such as lead 2-ethyl hexanoate and zirconium n-butoxide mixed in xylene to prevent polymerisation together with Ti (diisopropoxoide) bis (2,4-pentanedioniate). After mixing the precursor materials and spin coating, the final films are baked at high temperature 600-700°C for up to 30min which starts the crystallisation process.

Device Example

Valves and micropumps [33–35] are typical fluidic MEMS devices fabricated using bulk-micromachining technology and using piezoelectric actuation. Fig(1.3) shows a micropump made from two bonded silicon substrates and actuated using a piezoelectric
actuator. The cavities sloped relief is endemic to silicon that has been anisotropically etched. This type of bulk micromachining is explained later in this chapter. The larger arrows in fig(1.3) indicate a high flow rate associated with the two principle actuating positions, while the non-return valves maintain the flow direction.

![Piezoelectric Actuator](image)

Figure 1.3: A simple micropump design with piezoelectrically actuated membrane

### 1.2.3 Thermal Actuation

The concept behind thermal actuation is simple: when a material like silicon is heated it expands. Now if a structure is designed with some asymmetry, one side may expand more than the other resulting in bending and hence thermal actuation.

![Thermal Actuation](image)

Figure 1.4: Example of thermal actuation

When a potential difference is applied to the anchors of fig(1.4) a current flows through the beams heating them up. The shape's asymmetry will lead to some differential heating, and the resulting difference in expansion will cause the beams to bend. A
thermally actuated device using this so-called bi-morph effect might at first seem difficult to control, since cooling and heating have to be strictly controlled. However it has been shown that useful devices such as micromirrors can be designed and fabricated using industrial 0.8μm [9] and 1.2μm [36] double metal CMOS processes. Other designs using thermal actuation have included microgrippers [8].

Other thermal actuators use two different types of material with different thermal expansion coefficients, sandwiched together to create a bi-material fig(1.5), as opposed to fig(1.4) which employs one type of material. As the temperature is increased from $T_1$ to $T_2$ the bottom layer expands more than the top layer, in fig(1.5). This expansion differential causes the whole material to bend.

\[ \frac{1}{r} = \frac{6 (\alpha_2 - \alpha_1) (T_2 - T_1) (1 + m)^2}{t \left[ 3 (1 + m)^2 + (1 + mn) (m^2 + 1/mn) \right]} \]  

(1.1)

where

- $\alpha_1$ and $\alpha_2$ = temperature coefficients of expansion
- $E_1$ and $E_2$ = Young’s modulus
- $t_1$ and $t_2$ = thickness of the strips
- $t$ = total thickness
- $r$ = radius of curvature of strip
- $T_1$ and $T_2$ = temperatures
- $m = \frac{t_1}{t_2}$
- $n = \frac{E_1}{E_2}$
1.2.4 Magnetic Devices

Electrostatic forces are most effective over short distances, whereas magnetic forces can act over longer distances and can achieve larger displacements. Magnetic forces also have the ability to be either attractive or repulsive, whereas electrostatic forces can only supply attraction.

Magnetic actuation has been demonstrated by several research groups [10, 11, 39]. There are some drawbacks to using magnetic actuation, one of them being the amount of area required for creating a coil. For example, in the case of [39] a coil of 17 turns occupying a 5mm by 5mm area was used. There are also some concerns over power consumption, with [11] and [39] quoting power consumptions of 1.6W and 1.3W respectively for their devices. However if the devices are operated at resonance the driving currents can be reduced, while increasing the deflection. Despite these problems there are commercially available products based on magnetic actuation, such as a magnetic relay [40].

1.2.5 Polymer Actuators

These types of actuators are a departure from the previous devices, which utilise IC design and manufacturing techniques, but they give an insight into what can be achieved with wet-soft-polymers.

Electroactive polymers (EAP) is the term given to polymers that can be deformed when given an electrical stimulus. These materials can be thought as wet piezoelectrics. Ionic EAP's involve the migration or diffusion of ions and are classed as Ionic Polymer Gels. DuPont Nafion is one such ionic polymer with a composition of tetrafluoroethylene combined with a sulfonyl fluoride vinyl ether. The bulk of the material is negatively charged and this is usually countered by mobile cations. When this polymer is soaked in a cation solution, containing Li⁺ or Na⁺ ions, and then an electric field is applied, the material will bend. Under the electric field the cations migrate towards the cathode dragging with them a certain number of water molecules. As the number of water and Na⁺ molecules build up at the cathode there is obviously a depletion of ions at the anode side, therefore a stress gradient occurs which deforms the material. More simply one side of the polymer becomes swollen due to an increase in the number of water molecules, which creates a stress gradient, which in turn causes bending, see fig(1.6).

Fig(1.7) shows the movement of a Nafion, membrane 5mm by 10mm, soaked in Li⁺ ions and applied with a voltage of approximately 1.5V.

Electrodes need to be created on the polymer in order to be able to apply a potential difference. The main method of depositing metal onto the polymer is by cation reduction on the polymer surface. To form the electrodes the Nafion is first immersed in a solution
of Pt\((\text{NH}_3)_2^+\) ions, long enough to ensure that the polymer is saturated with platinum ions. The platinum ion solution is then chemically reduced to platinum metal using a suitable borohydride such as sodium borohydride NaBH\(_4\). The borohydride is able to diffuse some distance into the polymer matrix reducing the platinum ions to platinum metal. This allows good adhesion between the polymer and metal as the two materials are effectively meshed together. Thick electrodes lead to good conductivity, however prolonged exposure to the reducing chemical will cause the metal to bridge through the polymer leading to a short circuit. Thicker layers also lead to stiffer membranes, so other materials such as gold have also been used to form the electrodes.

Over time, as the material is being actuated, the number of bending cycles and magnitude of deflection slowly diminish. This is due to two main factors. Firstly a potential difference of a few volts is enough electrolyse the solution forming H\(^+\) and OH\(^-\) group, which reduces the hydrostatic pressure. The second effect is evaporation; the membrane eventually dries out requiring it to be ‘recharged’ with fresh metal ion solution. Determination of the dynamic behaviour of these membranes is fairly complicated but a few attempts have been described in [41, 42].

These types of polymers have a slow response time when compared to ‘hard’ actuators like comb electrodes and piezoelectrics. However unlike hard actuators their wet nature offers a great compatibility with biomechanical applications, where the main goal of EAPs is to create artificial muscles [43].
1.3 Fabrication Technology

1.3.1 Basic Processes

The main subject of this thesis is the mechanical and electrostatic behaviour of trapezoidal beams and electrodes, and it is important to appreciate how such structures are created. The coming sections describe the following most commonly used methods of fabricating microstructures:
1. UV lithography of photoresist;
2. material deposition such as sputter coating, evaporation, electroplating and chemical vapour deposition;
3. wet and dry etching of materials;
4. release methods.

UV Lithography

Photolithography plays a pivotal role in microfabrication and is the most important technique for patterning VLSI circuits. It is ideally suited for the manufacture of microsystems because of its high resolution, good alignment accuracy and fast throughput. Lithography is used to transfer geometric shapes from a mask to a photoresist layer, which coats an underlying substrate. If the photoresist is then UV exposed and developed the underlying layer is accessible. It is then possible to selectively remove undesired areas of the underlying film or modify them by ion implantation. Most lithography for MEMS is performed by contact or proximity printing, where the UV light simply passes through the mask directly onto the wafer. In contrast lithography for microelectronics is nowadays performed by projection lithography. With this technique the mask pattern is imaged onto the wafer by a reduction lens allowing a higher density of devices on the wafer, an advantage for VLSI circuits. Projecting the image also means that the resolution requirements on the mask fabrication process are relaxed.

One of the aims of a MEMS device is to combine the electronics and mechanical parts onto the same substrate. The mechanical parts benefit from lithography by having the same accurate definition of dimensions. With photolithography the manufacture of small, well defined mechanical features becomes possible. There are a few factors that define the profile shape of exposed photoresist:

1. the penetration depth of the UV rays into the photoresist;
2. diffraction at the boundaries between opaque and transparent regions of the mask;
3. development time after exposure;
4. the material used for the underlying layer - if the photoresist is spin coated on a reflective material the incident energy can be reflected back through the material.

The smallest feature that can be defined in photoresist using a mask is defined by eq(1.2) where \( \lambda \) is the wavelength of light, \( s \) is the mask wafer separation, \( d \) is the thickness of
the photoresist and \(2b\) is the grating period. Eq(1.2) only applies to contact or proximity printing and is not applicable to projection lithography, which uses optical reduction to scale down the mask features.

\[
2b_{\text{min}} = 3\sqrt{\lambda\left(s + \frac{1}{2}d\right)}
\] (1.2)

Fig(1.8) shows a set of electroplated nickel comb electrodes designed and manufactured as part of the work reported in this thesis. Photoresist was used as an electroplating mould, and hence the electroplated nickel takes on the same shape as the voids created after patterning and developing. As can be seen in fig(1.8a) the side walls are not perfectly vertical. Measurements of the wall angle on this device and others used in this thesis indicate a nominal value of 6° with respect to the vertical.

Fig(1.9) shows a 80\(\mu\)m thick T-shape nickel structure [44] electroformed from AZ4562 mould, the same resist used in this thesis. Here again the nickel clearly shows a sloped profile.

Fig(1.9b) shows a positive photoresist mould of a 17\(\mu\)m thick STR110 Shipley photoresist [45]. Devices are never plated right to the top of the photoresist as over-plating will create a T-shape structures fig(1.9a).

Positive photoresists are typically made up of three component materials consisting of a base resin, a photoactive compound and a solvent. Once the resist has been baked the base resin and photoactive compound make up the bulk of the material. The photoactive compound serves to inhibit the dissolution of the photoresist in an aqueous developer after exposure and so can be described as an inhibitor. The exposed and developed
photoresist in fig(1.9b) has what can only be described as a vase shaped profile, with a short widening neck and then a taper towards the base. The profile near the top surface initially widens due to diffraction (see fig(1.10)), because the energy of the diffracted rays is above the critical dose of the photoresist, thus initially creating a widening channel. If the diffraction of a rectangular hole is considered the diffracted energy takes the shape of \( y = \frac{\sin \theta}{\theta} \), using the example of Fraunhofer diffraction [46], so this explains the initial increase in line width of the channel. As one goes deeper into the photoresist towards the substrate the channel width starts to decrease. The diffracted photons have to pass through more photoresist than those in the middle of the channel, since the diffracted photons move laterally too. As a result the diffracted rays can no longer maintain the channel width widening at increasing depths. After the incident light passes through the aperture the incident energy is no longer collimated, as such the smoother extremes are reflected in a sloped side-wall profile after the initial widening. However there are several other factors that also affect the overall profile these include: the baking temperature and time; the type of photoactive compound; the overall thickness of the photoresist; the exposure time; and the developer concentration.

This vase shape is a common feature of such resist structures described in the literature - a concave profile with the narrowest part at approximately 2/3 of the resist height. Devices made as part of this thesis do not exhibit this ‘vase-like’ shape as they are never electroplated to the very top of the photoresist. Furthermore plating to the same level of the photoresist is difficult to achieve as there is always a slight variation of plated material height over the whole wafer. The risk of over-plating would also result in
CHAPTER 1. INTRODUCTION

Figure 1.10: Explanation of the side wall profile

the aforementioned T-shape structures rendering the structures useless, therefore as a general rule only 2/3 of the resist height is plated.

By considering the diffraction of light at the edge of the mask, and assuming an exponentially decreasing light intensity, it has been possible to calculate the structure profiles for thick photoresist layers up to 100μm [47]. While fig(1.11) shows a structure profile simulation of 16μm thick AZ4562 resist at different dose values, after 5 minutes of prebake and 1000 seconds of development time with the mask edge at x=5μm [23].

Figure 1.11: AZ4562 profile simulation [48]

Various research groups have developed sophisticated simulators including SAMPLE (Simulation And Modelling of Profile) developed at the University of California at Berkeley [49], PROLITH (the Positive Resist Optical LITHography model), a commercial
simulator [50], and SOLID-C which can model the lithography process steps [51]. Most of the background mathematics was developed by Dill in the 1970s [52–54]; these papers gave birth to the field of lithography modelling, and gave a simple model for image formation with incoherent illumination and an empirical model for photoresist profile calculation.

AZ4562 Photoresist Characteristics

Photoresist is typically applied by spin coating onto wafers. A realistic coating is less than 20μm thick, since large edge beads are formed with thicker layers and the surface suffers from poor uniformity. These factors limit the contact between mask and wafer during lithography. The methods used to remove the edge bead as part of this thesis have been: (1) augmented spin cycles, whereby a very short and fast acceleration is applied at the end of the spin cycle to remove the excess photoresist, and (2) edge bead removal by exposing and developing the unwanted photoresist.

A range of photoresists have been used as UV moulds, the most important being the AZ4000 series developed by Hoechst. In this thesis AZ4562 has been used as the main photoresist for UV electroforming. Typical values of the thickness with spin speed is given in fig(1.12).

Figure 1.12: Measured spin-coating characteristic for AZ4562 photoresist on a 3 inch wafer

Chemical Vapour Deposition

Chemical vapour deposition (CVD), also known as vapour phase epitaxy (VPE), is a method of depositing thin layers, semiconductors or metals, onto a substrate. Deposition
CHAPTER 1. INTRODUCTION

is achieved by passing reactive gases over a hot substrate. The gases are chosen so that they decompose depositing the required material onto the substrate. In VLSI circuits silicon, in addition to being used for the substrate, is used for transistors and for wiring conductors. Fig(1.13) shows a CVD reactor for depositing silicon. This is based on a common process using silicon tetrachloride mixed with hydrogen. At 1200°C the following reaction will deposit silicon onto the substrate:

\[
\text{SiCl}_4(g) + 2\text{H}_2(g) \rightarrow \text{Si}(s) + 4\text{HCl}(g)
\]

Figure 1.13: A CVD reactor for depositing epitaxial silicon using SiCl4 and H2

CVD is not limited to depositing silicon. Table 1.1 shows a list of gases that can be used for depositing other materials, and their functions in VLSI.

<table>
<thead>
<tr>
<th>material</th>
<th>reaction</th>
<th>conditions</th>
<th>material function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Epitaxial Silicon</td>
<td>(\text{SiH}_4 \rightarrow \text{Si} + 2\text{H}_2)</td>
<td>950-1050°C</td>
<td>transistors, or wiring</td>
</tr>
<tr>
<td></td>
<td>(\text{SiCl}_4 + 2\text{H}_2 \rightarrow \text{Si} + 4\text{HCl})</td>
<td>1100-1200°C</td>
<td>transistors, or wiring</td>
</tr>
<tr>
<td>Polysilicon</td>
<td>(\text{SiH}_4 \rightarrow \text{Si} + 2\text{H}_2)</td>
<td>580-650°C</td>
<td>transistors, or wiring</td>
</tr>
<tr>
<td>Low-Temp Oxide</td>
<td>optional dopants (\text{SiH}_4 \rightarrow \text{Si} + 2\text{H}_2)</td>
<td>300-450°C</td>
<td>insulation or passivation</td>
</tr>
<tr>
<td></td>
<td>(4\text{PH}_3 + 5\text{O}_2 \rightarrow 2\text{P}_2\text{O}_5 + 6\text{H}_2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2\text{B}_2\text{H}_6 + 3\text{O}_2 \rightarrow 2\text{B}_2\text{O}_3 + 6\text{H}_2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Med-Temp Oxide</td>
<td>((\text{OC}_2\text{H}_5)_4 \rightarrow \text{SiO}_2 + 2\text{CO} + \text{H}_2 + 3\text{C}_2\text{H}_6)</td>
<td>350-750°C</td>
<td></td>
</tr>
<tr>
<td>High-Temp Oxide</td>
<td>(\text{SiCl}_4 + 2\text{N}_2\text{O} \rightarrow \text{SiO}_2 + 2\text{N}_2 + 2\text{HCl})</td>
<td>900</td>
<td></td>
</tr>
<tr>
<td>Silicon Nitride</td>
<td>(3\text{SiH}_4 + 4\text{NH}_3 \rightarrow 3\text{SiN}_4 + 12\text{H}_2)</td>
<td>700-900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3\text{SiCl}_2\text{H}_2 + 4\text{NH}_3 \rightarrow 3\text{SiN}_4 + 6\text{HCl} + 6\text{H}_2)</td>
<td>650-750</td>
<td></td>
</tr>
<tr>
<td>Tungsten</td>
<td>(\text{WF}_6 + 3\text{H}_2 \rightarrow \text{W} + 6\text{HF})</td>
<td>&gt;300</td>
<td>wiring conductors</td>
</tr>
<tr>
<td></td>
<td>(2\text{WF}_6 + 3\text{SiH}_4 \rightarrow 2\text{W} + 3\text{SiF}_4 + 6\text{H}_2)</td>
<td>400-500</td>
<td></td>
</tr>
<tr>
<td>Copper</td>
<td>(\text{H}_3 + \text{organo copper} \rightarrow \text{Cu} + \text{volatile product})</td>
<td>200-450</td>
<td>wiring conductors</td>
</tr>
<tr>
<td>Nickel</td>
<td>(\text{Ni} + (\text{CO})_4 \rightarrow \text{Ni} + 4\text{CO})</td>
<td>175-300</td>
<td>commercial Nickel CVD</td>
</tr>
</tbody>
</table>

Table 1.1: CVD deposition of various materials
Metal Deposition

There are various different methods for depositing metals, and in particular nickel, onto a substrate. In a research environment one could use evaporation, sputter coating, electrolysis or electroless plating.

Evaporation and Sputter Coating  Evaporation and sputter coating are generally used for thin coatings, of the order of a few 1000Å, as thicker layers tend to take too long to deposit and tend to fail as a result of internal stress. In the case of thermal evaporation where a metal is placed in a tungsten boat, increasing the filament current (the current through the boat) will increase the deposition rate. However if the current is increased too much the metal boils too quickly ejecting large quantities from the evaporation boat resulting in insufficient base material to evaporate or uneven coatings.

Electroless Plating and Electroplating of Nickel  Electroless plating is generally used for depositing thin layers onto dielectric surfaces where no conducting layer is available. An example of this is creating silver mirrors by the reduction of silver nitrate. Electroplating is more suitable for thicker layers and where a conducting layer is available. When depositing good quality films electroplating can achieve much higher deposition rates than vacuum deposition methods. Nickel is typically electroplated at rates of 12μm/hr or 2000Å/min, while when sputter coating chrome and copper values of 100Å/min to 200Å/min are more often used. Electroplating has another advantage of being a low temperature process (<60°C typically) that can be done at atmospheric pressure. The electroplating process is applicable to a wide range of metals. However this thesis will focus on nickel because it is most relevant to the fabrication of micromechanical parts.

In electroless plating the metal ions in solution are reduced chemically rather than by an electric current. Several reducing agents are mentioned in literature that can reduce nickel salts to nickel metal.

<table>
<thead>
<tr>
<th>Reducing Agent</th>
<th>Nickel Salt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sodium Hypophosphite</td>
<td>Nickel Sulphate</td>
</tr>
<tr>
<td>Sodium Boron Hydride</td>
<td>Nickel Chloride</td>
</tr>
<tr>
<td>Hydrazine</td>
<td>Nickel Acetate</td>
</tr>
</tbody>
</table>

Hypophosphite solutions have been used to create nickel microstructures by electroless plating both for MEMS [55] and for flip chip technology [56]. However electroless nickel deposits reduced with hypophosphites tend to have a large phosphorus impurity, which affects the overall density. Gawrilov [57] reported in 1979 that a commercially available
electroless nickel plating solution sold by Kanigen contained about 7.5% phosphorus, which dropped the density of the plated nickel to 7.92g/cm³. As long as the density of the plated nickel is known this should not pose a problem if the nickel is to be used as a mechanical structure.

Boron hydrides require high pH levels (pH11-pH14) [58], and sodium hydroxide is often used as a pH stabilising solution. This poses a problem with photoresists, especially AZ4562, as they are dissolved in alkali solutions.

L Pessel in December 1944 applied for a patent (US Pat 2430581 1947) on an electroless bath, which contained hydrazine as the reducing agent. Hydrazine reduction can produce nickel coatings with a high purity (97%-99% [59]), however hydrazine is normally avoided since it is extremely hazardous, toxic, flammable and explosive.

There are two popular methods of electroplating nickel. Most commercial plating solutions are based on the one name after Watts, with the main ingredient being nickel sulphate, which is a relatively cheap and widely available source of nickel ions. The other type, which is used in the fabrication of devices used in this thesis, is base on a sulphamate bath. The advantages of electroplating with nickel sulphamate [Ni(NH₂SO₃)₂] are the high rate of deposition and more importantly, the low stress in the deposit. One important application of nickel sulphamate plating is in the hot embossing of vinyl records in the music industry. The low internal stress is essential to prevent distortion of the grooves that are to be transferred onto the vinyl.

<table>
<thead>
<tr>
<th>type of bath</th>
<th>chemical composition</th>
<th>nickel sulphate NiSO₄·7H₂O</th>
<th>nickel sulphamate Ni(NH₂SO₃)₂·4H₂O</th>
<th>nickel chloride NiCl₂·6H₂O</th>
<th>boric acid H₃BO₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional</td>
<td></td>
<td>300-450</td>
<td>0-15</td>
<td>30-45</td>
<td></td>
</tr>
<tr>
<td>Sulphamate Bath</td>
<td></td>
<td>-</td>
<td>37-53</td>
<td>30-45</td>
<td></td>
</tr>
<tr>
<td>Watts Bath</td>
<td></td>
<td>240-330</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.2: Typical composition of baths used for nickel deposition [60]

<table>
<thead>
<tr>
<th>type of bath</th>
<th>ultimate tensile strength (N/mm²)</th>
<th>yield strength (N/mm²)</th>
<th>elongation (%)</th>
<th>internal stress (N/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Watts</td>
<td>380-450</td>
<td>220-280</td>
<td>20-30</td>
<td>140-170</td>
</tr>
<tr>
<td>Conventional</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sulphamate Bath</td>
<td>500-800</td>
<td>500</td>
<td>8-13</td>
<td>7-70</td>
</tr>
</tbody>
</table>

Table 1.3: Range of physical properties of nickel electrodeposited from table(1.2) [60]

Low stress is an important quality of any deposited material, as mechanical deformation under internal stress is often a factor limiting the success of fabricated devices. Our electroplating bath uses a nickel sulphamate solution designed by Schlötter Ltd. It has been designed to produce very low-stressed deposits when plated at 10mA/cm² at a pH4.0 at 50°C, producing a deposition rate of 12μm/hr.
CHAPTER 1. INTRODUCTION

Wet and Dry Etching

The main process of lithography transfers a mask pattern to a photoresist layer. Areas of the underlying material can then be selectively removed using an etchant. Wet etchants are ideal for this task as they are cheap and readily available. Silicon can be wet etched with a number of different aqueous alkali solution, some of which are described later in this chapter. Silicon oxide is also often used in the fabrication of MEMS devices, and the simplest method of removal involves hydrofluoric acid (HF).

Not all etching processes involve the use of photoresist as an etch mask. For example in the LIGA process, described later in this chapter, metal seed layers are deposited to act as conducting layers for the electroplating processes, and to form electrical isolation in the final device these seed layers need to be removed. Suitable commercial metal etchants can usually be found, but it is important that the etchant is highly selective so that other materials present are not attacked.

<table>
<thead>
<tr>
<th>material</th>
<th>etchant</th>
<th>process</th>
</tr>
</thead>
<tbody>
<tr>
<td>silicon</td>
<td>EDP; KOH; TMAH</td>
<td>bulk micromachining</td>
</tr>
<tr>
<td>silicon dioxide</td>
<td>NH;HF (6:1)</td>
<td>general oxide removal</td>
</tr>
<tr>
<td>chrome</td>
<td>Cerium ammonium nitrate; CH₂COOH (22%;8%)</td>
<td>chrome mask etching</td>
</tr>
<tr>
<td>chrome</td>
<td>K₂Fe(CN)₆; NaOH</td>
<td>chrome etching in the presence of copper</td>
</tr>
<tr>
<td>copper</td>
<td>H₃PO₄; HNO₃; CH₂COOH</td>
<td>general copper etching</td>
</tr>
<tr>
<td>gold</td>
<td>HCl;HNO₃ (3:1)</td>
<td>general gold etching</td>
</tr>
<tr>
<td>aluminium</td>
<td>H₂PO₄; HNO₃; CH₂COOH</td>
<td>same composition as copper etch</td>
</tr>
<tr>
<td>metal oxides</td>
<td>10% HCl</td>
<td>removal of native metal oxide</td>
</tr>
</tbody>
</table>

Table 1.4: Common etch chemistries

An alternative method to wet etching uses reactive ion etching (RIE). In this process etching takes place in a low pressure plasma containing reactive gaseous species. Most RIE process are geared towards etching silicon and its compounds (oxides and nitrides), requiring highly reactive fluorine and chlorine based compounds. By adjusting the chamber pressure, etchant gas composition, RF power and substrate temperature it is possible to control the etching process to the users requirements, whether it be high anisotropy for creating comb electrodes or high isotropy for undercutting a suspended device. One key advantage of undercutting by dry etching is that the problem of stick down can be avoided, as explained in the next section. However with the large number of parameters it can be time consuming to obtain the desired conditions.

Device Release with Freeze Drying

During the fabrication cycle liquids are often used and they pose a problem with suspended structures. Fig(1.14) shows a cross section of a suspended beam with a small amount of water evaporating beneath it. As the volume of liquid reduces with
Table 1.5: Common dry etch chemistries

evaporation surface tension pulls the structure to the substrate. If the restoring force of the suspended beam is less than the surface tension the structure will eventually touch the surface, at which point intermolecular forces (Van der Waals) may prevent the structure from detaching.

There have been several different methods suggested in the literature to overcome this stick down problem. The main methods can be summarised into evaporation, sublimation and super critical drying. A comparison of these technologies can be found in [61]. Evaporation and sublimation are the easiest methods to implement. For this thesis sublimation was used, using an Edwards freeze dryer connected to a rotary pump.
Evaporation at Elevated Temperatures

Evaporating at an elevated temperature only succeeds in a slight reduction in the surface tension. At 20°C the surface tension of water is about 75mN/m and just below boiling point this drops to about 60mN/m. Better results can be achieved with methanol, at RTP the surface tension is about 25mN/m and just below its boiling point at ~65°C this reduces to about 20mN/m.

Sublimation Drying

Sublimation methods require the liquid to be frozen and then pumped at low pressure. In fig(1.15) the liquid is first frozen, moving from a liquid state to a solid one. From the phase diagram it can be seen that if the pressure is now reduced there is a direct transition from solid to gas. This bypasses the liquid phase and removes the problem of surface tension.

Supercritical Drying with CO$_2$

This method relies on high pressures, 800psi to 1350psi, and elevated temperatures. When carbon dioxide is above 800psi it is in a liquid form rather than a gas. When the temperature and pressure are raised above carbon dioxide's critical point, it forms a gas and because of the molecular state of the liquid, a meniscus does not form during drying. Without a meniscus there is no surface tension.

Other Techniques

A newer technique, HF vapour-phase etching (VPE) [62, 63] has also been developed which avoids the use of liquids. However this technique can only be applied to oxide sacrificial layers and requires a complicated setup.

1.3.2 Silicon MEMS

There are two basic types of silicon process for fabrication MEMS devices. The first is bulk silicon micromachining, which involves removing selected areas of the silicon substrate. The second type is surface micromachining involves building up layers on the silicon wafer and then selectively removing unwanted areas. The silicon lattice is arranged in a tetrahedral pattern with each silicon atom connected with four other atoms. Fig(1.16) shows two of the important crystal planes mentioned frequently in this thesis and they
are the \(<111>\) plane and the \(<100>\) plane. If four silicon atoms are considered, that lie within the corners of a cube, the \(<100>\) plane corresponds to any of the faces of the cube, while the \(<111>\) plane atoms that lie on a diagonal.

![Figure 1.16: Crystallographic planes of silicon](image)

**Bulk Silicon Micromachining**

Bulk silicon micromachining involves the modification of the silicon substrate and is based upon anisotropic and isotropic etching. Anisotropic etching relies on the fact that certain etchants attack silicon at differing speeds in different crystallographic directions. For example EDP etches more slowly along the \(<111>\) direction of silicon when compared with the \(<100>\) direction, by almost 30-40 times. Therefore a square that is defined on the etch mask of a \(<100>\) orientated wafer will result in a pyramidal pit, see fig(1.17). As the etchant removes silicon it exposes the \(<111>\) planes and, since these planes are etched very slowly, a V-shaped pit is created with a wall angle of 35.26° with respect to the vertical or 54.74° with the horizontal fig(1.18a). This geometry is defined by the angle
between \(<111>\) and \(<100>\) directions [64, 65]. Isotropic etching, on the other hand, is not orientation dependent so an opening defined on the etch mask of a \(<100>\) orientated wafer will result in a shape as in fig(1.18b).

![Diagram](a) anisotropic etching

(b) isotropic etching

Figure 1.18: Anisotropic and isotropic etching of silicon

The precise mechanism of anisotropic etching is not well understood, although there are known to be several key factors which are involved in the etching process. One factor is that the silicon surface on the \(<111>\) plane exposes a higher surface density of atoms. Another is that the silicon atoms on the \(<100>\) plane have two spare electrons while the ones on the \(<111>\) only have one spare electron thus requiring more energy to remove them from the bulk.

The ethylenediamine in EDP reacts with water to produce \(\text{OH}^+\) groups (see fig(1.19)). These ions react with the silicon creating \(\text{Si(OH)}_4\), an unstable species that would normally be expected to decompose into silicon dioxide, creating an oxide layer on the silicon. The result would lead to a cessation in the silicon etching. However, instead of forming an oxide a complex is formed with the catechol which is soluble in water. Etch rates up to \(1.5\mu\text{m/min}\) are possible.

Tetra-methyl ammonium hydroxide (TMAH) is another popular anisotropic silicon etchant. It is less toxic than EDP, but requires a thicker mask layer as it attacks \(\text{SiO}_2\). TMAH's most desirable property is that it is a CMOS compatible process with the etching characteristics of EDP.

Potassium Hydroxide etches very slowly in the \(<111>\) plane, some 10 times slower than EDP. With such a high anisotropy little undercutting occurs, and this can be a disadvantage in some applications. For example, in the creation of a suspended oxide cantilever, fig(1.20), where the silicon must be removed from beneath the beam, \(<111>\)
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etching is desirable. <111> etching in essence results in some lateral etching, thus freeing the SiO₂ beam.

Hydrazine and water is a purely isotropic etch, where there is no preference between the etching of the <111> and <100> planes. The resulting etch shape is ellipsoidal or spherical as in fig(1.18b).

Bulk Micromachined Structures

Silicon anisotropic bulk etching is a popular method for creating, for example, cantilever beams [66, 67] and micromirrors [36]. The method is very simple and involves only one mask. To create a simple suspended SiO₂ beam, a thick layer of oxide is first grown on a silicon wafer (step 1 of fig(1.20)). The oxide is then simply patterned with a U shape (2). After removing the exposed U shaped oxide with HF, the wafer is left in EDP or TMAH, until all the silicon is removed from beneath the beam (3). Since the <111> and <110> planes are etched under cutting occurs which removes the silicon directly below the beam (4).

Rather than using an oxide a version of this method uses a boron doped layer of silicon as an etch stop layer [68]. A layer of boron is ion implanted into a silicon wafer, then using an RIE the boron layer is patterned with the device features (e.g. beams and comb electrodes). An anisotropic etch is then used to under etch the features leaving a suspended boron doped device. The advantage of using a doped layer is that it is now a fairly good conductor so devices like comb electrodes can be made. If an n-type substrate is used electrical isolation, with respect to the substrate, is also achieved between individual features without the need of an oxide insulator. Isolation occurs since
the boron ion implantation creates a pn junction [3]; each feature is then isolated by back-to-back diodes.

<table>
<thead>
<tr>
<th>etchants</th>
<th>typical composition</th>
<th>temp</th>
<th>mask</th>
<th>etch rate</th>
<th>Anisotropic (100):(111) etch ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>HF HNO$_3$ water+CH$_3$COOH</td>
<td>10ml 30ml 80ml</td>
<td>22</td>
<td>SiO$_2$ (30nm/min)</td>
<td>0.7-3.0</td>
<td>1:1</td>
</tr>
<tr>
<td></td>
<td>25ml 50ml 25ml</td>
<td>22</td>
<td>Si$_3$N$_4$</td>
<td>40</td>
<td>1:1</td>
</tr>
<tr>
<td></td>
<td>9ml 75ml 30ml</td>
<td>22</td>
<td>SiO$_2$ (70nm/min)</td>
<td>7.0</td>
<td>1:1</td>
</tr>
<tr>
<td>Ethylene diamine Pyrocatechol water</td>
<td>750ml 120g 100ml</td>
<td>115</td>
<td>SiO$_2$ (.2nm/min)</td>
<td>0.75</td>
<td>35:1</td>
</tr>
<tr>
<td></td>
<td>750ml 120g 240ml</td>
<td>115</td>
<td>SiO$_2$ (.2nm/min)</td>
<td>1.25</td>
<td>35:1</td>
</tr>
<tr>
<td>TMAH</td>
<td>80</td>
<td>85</td>
<td>Si$_3$N$_4$</td>
<td>1.4</td>
<td>400:1</td>
</tr>
<tr>
<td>KOH Water and IPA</td>
<td>44g 100ml</td>
<td>50g 100ml</td>
<td>Si$_3$N$_4$</td>
<td>1.0</td>
<td>400:1</td>
</tr>
</tbody>
</table>

Table 1.6: Different types of silicon etches

Surface Micromachining

In surface micromachining devices are fabricated on the surface of the silicon wafer through a series of lithography, depositions and etch steps. In a sacrificial layer process two materials are used, one for the mechanical part and the other as a sacrificial material that is eventually etched away to release the moving parts of the structure. The mechanical structure is generally polysilicon or silicon nitride, while the sacrificial material is often silicon dioxide. CVD (page 31) is commonly used in the surface
micromachining process to deposit the required materials, while thermal oxidation is the preferred method to form the first oxide layer.

In section 1.2.1 an electrostatic motor was described and surface micromachining is one method of fabricating this device. A simplified version of this device is a gear wheel on a fixed axle and Fig.1.21 outlines its fabrication process. (1) An oxide is first thermally grown or deposited by CVD and then a polysilicon layer is deposited and patterned with the gear wheel. (2) The whole device is coated with oxide, and since the axle must be attached to the substrate a hole is patterned and etched down to the substrate. (3) A second layer of polysilicon is deposited and then patterned and etched to form the axle. (4) The final step involves removing the oxide using HF.

The thickness of surface micromachined structures are limited by the amount of material that can be deposited using CVD and the resulting stress in the deposited layer. Bonded silicon on insulator (BSOI) allows for thick silicon structures to be formed, and rather than depositing the sacrificial oxide and mechanical silicon layers, the separate layers are bonded together. A BSOI wafer consists of a thick oxide layer sandwiched between two silicon layers. Such a structure is created by sandwiching an oxidised wafer with a plane silicon wafer and then annealing at high temperatures. At high temperatures the oxide bonds with the secondary silicon layer fusing the two wafers together. To create a device a deep dry etch process is first used to pattern the top silicon surface and etching stops when the oxide dielectric is exposed. The oxide beneath the structures is then removed using HF and the devices are freed using a freeze drying process. Using this technique various devices have been demonstrated, such as the 100μm high gyroscope shown below.

1.3.3 The LIGA Process

The LIGA technique uses X-rays from a synchrotron source to expose photoresist. The typical peak wavelength used is 3Å, and at these short wavelengths the X-rays can
penetrate deep into the resist with little diffraction. This allows for high aspect ratio structures to be formed in the resist, and structures up to several millimetres are possible. If a metallic base is used as a substrate to the resist the defined cavities can then be electroplated with metal and thus a replication process is possible.

This process only allows for only one copy to be made because the surrounding resist must be removed before the metallic structure is freed. However if the electroplating is allowed to continue above the height of the resist a hot embossing or an injection moulding die can be made, allowing for mass reproduction.

The LIGA process was originally developed at the Nuclear Research Centre in Karlsruhe, Germany as a technique for manufacturing separation nozzles for the enrichment of uranium [70]. However it was later realised that the technique could be used for microstructure fabrication [71].

**LIGA Fabrication Steps**

Fig(1.23) shows the LIGA process with the following steps:

1. metallise substrate if silicon is used or prepare solid metal substrate;
2. apply resist;
3. expose resist to short wavelength radiation (X-rays from a synchrotron source) using a mask;
4. develop pattern;
5. electroform with metal (usually nickel);
6. overplate and then remove metal plug to create an embossing tool;
7. apply suitable base and then injection mould;
8. release;
9. replate with metal.

Figure 1.23: The LIGA process

The LIGA process seems an ideal method for creating microstructures as tall structures with almost vertical side-walls can be achieved. However there are several drawbacks with LIGA that have prevented it from being widely adopted:

1. The cost. A typical wafer that undergoes all the steps in the LIGA process costs up to £10,000, including the mask fabrication and the use of a synchrotron source. The LIGA mask needs to be made of an x-ray absorbing material such as gold or tungsten with a thickness of 12µm to 15µm [72] and often cost around £4,000. The x-ray source is usually obtained from a synchrotron and obtaining time on a synchrotron is expensive. In the UK as of March 2002 there is only
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one available synchrotron lab Synchrotron Radiation Sources (SRS), Daresbury. A second synchrotron ring, Diamond at the Rutherford Appleton Laboratory, Oxford, is due for completion by September 2006.

2. Incompatibility with other integrated processes. Integrating electronics and mechanical devices onto the same wafer is one of the primary aims of MEMS, and the high energy radiation sources used in LIGA make this difficult.

Low Cost UV LIGA

LIGA processes based on UV lithography [5, 12, 71-78] have the advantage of being much simpler to implement than the original x-ray process. They can be carried out using commercially available materials and standard clean room equipment, ideal conditions for an academic environment. This low-cost LIGA approach forms the main fabrication technique used in this thesis. However there are some drawbacks of using optical lithography. In particular the longer wavelength and lower power of the source mean that the pattern resolution and the exposure depth limited. For example, optical absorption limits the maximum depth to around 50μm in conventional positive photoresists, while line widths are difficult to control to better than ~5μm at this depth due to diffraction. Using optical lithography rather than x-rays to define a mould in a polyimide Frazier and Allen [79, 80] were the first to transfer the x-ray process to a UV one, though their structures were limited to thickness of ~40μm. Despite these problems the consumables are widely available and make up the standard materials found in semiconductor technology.

The low cost LIGA approach also allows for multi-level structures. By repeated steps of lithography and electroforming using a spin-on method of positive photoresist, micro-coils have been formed [74]. A multilevel structure was necessary in their design of a planar coil, which required a contact from the centre of the spiral coil to be brought to outside of the device. The process is not limited to wet positive resists as negative dry film resists have also been used [77, 81].

Moving Structures Using Photoresist Sacrificial Layers

As early as 1967 photoresist sacrificial layer processing was used to fabricate resonant gate transistor (RGT) devices [82] (fig(1.24)). In these initial experiments gold was used as the structural material to create a cantilever beam as a resonant gate for a field-effect transistor. During the fabrication steps photoresist was used as the sacrificial material, onto which the gold beam was plated. To release the structure the photoresist was removed using an organic solvent, leaving a suspended device.
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A more recent example of a movable structure, which uses a two layer sacrificial photoresist, is described in [83, 84]. The first layer consisted of a thin layer of photoresist (~3μm of Shipley S1828) and was used as the sacrificial layer while the exposed areas were electroplated to form anchors. On top of this sacrificial layer was the device layer created by electroforming a thicker layer of photoresist (12μm of AZ4562). Once the sacrificial layer was removed a suspended structure was formed which could then be actuated by driving its comb electrodes.

Other Methods for Structuring Moulds

There are several other methods of structuring polymer moulds. These include reactive ion etching [85] see fig(1.26), excimer [86] and YAG laser micromachining (fig(1.27)). Excimer lasers have also been used in conjunction with halftone masks to create resist moulds with continuous relief fig(1.28).
Figure 1.26: High aspect ratio polyimide etching. 32μm thick polyimide with 1μm gaps using O₂ plasma using an electron cyclotron resonance source

Figure 1.27: Silicon laser machining with Nd:YAG at 255nm
Figure 1.28: 193nm UV ablation of LM5000 photoresist using a halftone mask [87]
Bibliography


Chapter 2

Background to the Mathematical Approaches

The following chapter outlines some of the theory and methods that are used to solve mechanical and electrostatic problems. The mechanical problem is broken down into two problems a torsional and a bending problem. The approach to solving the electrostatic problem uses conformal transforms and a brief introduction is given to the use of Schwarz-Christoffel transforms.

2.1 Mechanical Structures

In most MEMS devices with moving parts, the moving elements are attached to the substrate by built-in flexural or torsional support beams. Hinges and pinned joints allowing rotation are possible, but require complex processing and are subject to friction and wear. For parts attached by built-in support beams movements can occur only by bending and/or twisting of the beams as shown in fig(2.1).

We have already seen that both anisotropic etching and electroplating of UV exposed photoresist moulds lead to sloping side-walls. The question that will be addressed in the next chapter is: how does the behaviour of a trapezoidal beam, subjected to bending and/or twisting, differ from that of a rectangular one?

2.1.1 The Torsion Problem of a Cantilever Beam

The mechanical behaviour of beams and cantilevers is obviously not a new subject. The torsion problem or a circular section beam was first approached analytically by Coulomb
in 1784 [1], who deduced the torsional moment’s relationship with the angular twist and the fourth power of the beam’s diameter. Coulomb’s relationship was applied to beams for all cross sections until Cauchy’s attempt in 1829 to develop a relationship for square and rectangular cross-sections [2].

Ten years later Saint-Venant, in a private letter to Cauchy, found Cauchy’s analysis to be faulty, since he had assumed that a beams cross-section always remains in plain when twisted and so did not take into account the warping of the beam’s cross-section. In 1843 Saint-Venant [3] corrected Cauchy’s expression for the rectangle, and later in 1847 [4] he correctly formulated the general torsion problem and solved it for the rectangle and ellipse cases. The torsion problem is now commonly called the Saint-Venant’s torsion problem after this eminent French elastician.

Many of the structures that have been analysed over the years are either simple ones such as beams with a circular or rectangular cross-section, or those that have been used in heavy construction and railway lines such as I, H, L and U beams. The simple cross-sections have closed form solutions while the I, H, L and U beams are assumed to have very slender parts, which allows for assumptions to be made on how the stresses are distributed through the structure.

2.1.2 Methods for Solving the Torsion Problem

Saint-Venant’s method for solving simple cross-sections (circular, square, triangular, etc.) all employed his semi-inverse method. This involved making certain assumptions as to the deformation of the twisted beam and then showing they satisfied the equilibrium conditions (2.1) and the boundary conditions (2.2)\(^1\). (\(\tau\) - shear stress; \(\sigma\) - normal stress; \(X\), \(Y\) and \(Z\) - body forces; and \(\overline{X}, \overline{Y}\) and \(\overline{Z}\) - surface forces)

\(^1\)For a detailed discussion on this topic please refer to chapter 3
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\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \]
\[ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + Y = 0 \]
\[ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0 \]  
(2.1)

However with more complex cross-sections it becomes difficult to guess the initial starting conditions.

Higgins [5] gives an excellent overview of almost all of the most interesting papers written on the torsion problem, covering a period from the mid 1800s to the 1940s. The methods mentioned include: the inverse method by Saint-Venant; the method of images invented by Kelvin and applied by Hay [6]; and Lamé's method applied by Clebesch.

After the 1930s we start to see the emergence of more common methods that are in use today. Conformal mapping was used by Seth to solve the torsion problems of T and L shaped cross-sections in 1934 [7]. The same author then published a method to calculate the torsion of regular polygons by using conformal mapping onto a circle [8]. By the 1940s it was becoming apparent that finding solutions for every possible type of cross-section would be difficult to do. In the torsion problems of T and L shaped cross-sections the mathematics is extremely complicated and not very applicable to other shapes.

Finite Element Analysis (FEA) was first developed in 1943 by R. Courant [9], who utilised the Ritz method [10, 11] of numerical analysis and variational calculus to obtain approximate solutions to vibration systems. Shortly thereafter, a paper published in 1956 by Turner et al. [12] established the modern day form of the finite element concept through the derivation of the stiffness matrix for the triangular element.

2.1.3 Bending of a Cantilever Beam

The problem of pure bending is a much simpler one to analyse. For a cantilever beam with a transverse point load \( P \) applied at one, the end displacement \( u \) is simply given by: eq(2.3) [13].

\[ u = \frac{PL^3}{3EI} \]  
(2.3)

Where \( E \) is the Young's modulus and \( I \) is the moment of inertia. In this case of pure bending only one non-zero component of stress is considered, \( \sigma_z \), and this is proportional to the distance from the neutral axis.
2.1.4 Shear Centre

With the problem of torsion and bending comes the problem of the shear centre. Later in this thesis there are comparisons of different ways of calculating the torsional constant of a beam (FEM and Ritz method). These calculations assume that pure torsion is occurring, meaning that the beam twists without any bending. The same methods are then used to determine the shear centres of trapezoidal beams. The shear centre is of practical importance because it identifies the point through which a transverse load must act to produce pure bending of a beam.

2.2 Electrostatic Modelling Survey

As far as the author is aware the electrostatic behaviour of trapezoids has not been investigated before from a MEMS perspective. Comb electrodes formed by a process involving reactive ion etching or the LIGA process have near vertical side walls and for most cases it is undesirable to have sloping walls as it limits the maximum thickness of the device.

An approach for simplifying complicated structures to achieve an approximate solution to the forces of comb electrodes was mentioned by Warne [14]. The method used by Warne to simplify the electrostatic force problem involved splitting a complicated electrostatic problem into simpler regions. In fig(2.2), a top view of a set of comb electrodes, region 2 shows an area where the electric field may be fairly constant and so it can be modelled like two parallel plates, region 1 is a bit more complicated but it is far easier than modelling the whole problem.

![Figure 2.2: Simplification of the electrostatic forces in a comb actuator](image)
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When comparing the electric fields of trapezoidal electrodes and rectangular ones, it is readily seen that in the rectangular case the field distribution is symmetrical about a horizontal line through the electrode centres see fig(2.3), while in the trapezoidal case it is not. Out-of-plane forces on comb electrodes have been investigated previously by Tang [15] but only for the case of rectangular electrodes in the vicinity of a ground plane.

Figure 2.3: Electric field comparison between trapezoidal and rectangular comb electrodes

2.2.1 Methods for Solving the Electrostatic Problem

There are several different ways of solving electrostatic problems, FEM analysis, and conformal mapping being the most popular. The following paragraphs will describe the use of conformal transforms, and in particular the Schwarz-Christoffel transformation, that may simplify the trapezoidal electrode. It will then be shown how to use the transformation to calculate the force.

Conformal Mapping

Suppose it is required to determine the field distribution between two equipotential boundaries represented by fig(2.4) in the (complex) z-plane. Suppose further that the two boundaries are of awkward shape as shown in the diagram.

Such boundaries could be those of the rotor and stator problem of an electrostatic micro-motor. In such a problem it may be possible to find and apply a transformation or mapping from this z-plane (the domain of the actual object) to a new plane called the χ-plane, in which the shapes of the boundaries become something recognisable. For example a transformation may be found that gives the boundaries and equipotential as parallel straight lines.
A mapping in the plane is said to be angle preserving or conformal if it preserves angles between oriented curves in both magnitude and sense. The transform or conformal map, when applied to the known solution in the $\chi$-plane, leads to a solution in the $z$-plane thus solving the actual problem.

If the mapping $\chi(z)$ is an analytic function, then it is referred to as a conformal transformation. A mapping in the plane is said to be angle preserving or conformal if it preserves angles between oriented curves in both magnitude and sense. If the solution of the field problem in the $\chi$-plane is known and is represented by the analytic function $g(\chi)$, where either the real or imaginary part of $g$ is the potential function, then the required solution in the $z$-plane is represented by the function $f(z) = g(\chi(z))$.

The Schwarz-Christoffel Transformation

The Schwarz-Christoffel transformation is a conformal transformation that connects the complex upper half-plane to the interior of a polygon, as in fig(2.5), with the real axis mapped to the polygon boundary. The theory of a Schwarz-Christoffel transformation allows an infinite number of polygon points to be mapped to the upper half plane.

The discoverers behind this type of transform were Schwarz [16] and Christoffel [17] who formulated this independently; hence the co-joined name. Further information can be found in chapter six of [18], which has an in depth analysis of the theory and applications of the Schwarz-Christoffel transformation.

The mapping between the upper-half of the $\chi$-plane and the polygon boundary in the $z$-plane is found by integrating eq(2.4). It must be noted that we desire to go from the polygon to the upper-half plane and that integrating eq(2.4) gives $z = f(\chi)$ whereas the inverse mapping $\chi = g(z)$ is more desirable. The main difficulty when dealing with
Schwarz-Christoffel transforms arises when it is necessary to invert the transform \( z = f(\chi) \) analytically. Often it is only possible to do this for simple equations; otherwise numerical techniques have to be used.

\[
\frac{dz}{d\chi} = S(\chi - a)^{\alpha - 1}(\chi - b)^{\beta - 1}(\chi - c)^{\gamma - 1}(\chi - d)^{\delta - 1} \ldots \tag{2.4}
\]

In eq(2.4) \( S \) is a scaling constant and \( a, b, c \) and \( d \) are points on the real axis of the transformed plane, which correspond to the vertices of the polygon and \( \alpha, \beta, \lambda \) and \( \gamma \) are the interior angles of the polygon. The scaling factor \( S \) can be complex therefore the orientation of the polygon can be fixed arbitrarily.

![Diagram](image.png)

Figure 2.5: The Schwarz Christoffel-transform, showing mapping of points \( a-d \) on the real axis in the \( \chi \)-plane to point \( A-D \) on a polygon boundary in the \( z \)-plane.

To describe a polygon we need a minimum of three points in the \( z \)-plane. In the \( \chi \)-plane the polygon vertices lie on the real axis. Furthermore the nature of the transform is that it transforms the whole upper half plane into the interior of a polygon. The whole upper half plane extends to \( \pm \infty \) along the real axis so one vertex of the polygon in the \( z \)-plane is the join between the two ends of the real axis in the \( \chi \)-plane. This means that one vertex of the polygon is always mapped to \( -\infty \) and \( +\infty \) in the \( \chi \)-plane, in fig(2.6) this vertex is chosen to be \( C \). The implication of having a point at infinity in the \( \chi \)-plane is that it does not appear in the transform, so for the example described in fig(2.6) eq(2.4) leads to:

\[
\frac{dz}{d\chi} = S(\chi - a)^{\alpha - 1}(\chi - b)^{\beta - 1}
\]

Closed polygons are not the only shape that can be transformed. By placing one vertex at infinity in the \( z \)-plane another type of problem can be modelled. In fig(2.7) point \( C \) in the \( z \)-plane is moved to infinity making the parallel lines and again since the transformed point \( c \) is infinite it doesn't appear in the transformation. From fig(2.7) \( \alpha = \frac{\pi}{2} \) and \( \beta = \frac{\pi}{2} \).
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Figure 2.6: Transform to a closed polygon

which transforms the upper half $\chi$-plane to the inside of the polygon.

Figure 2.7: Transform to an open polygon

\[
\frac{dz}{d\chi} = S (\chi - a)^{\frac{1}{2} - 1} (\chi - b)^{\frac{1}{2} - 1}
\]
\[
\frac{dz}{d\chi} = S (\chi - a)^{\frac{1}{2} - 1} (\chi - b)^{\frac{1}{2} - 1}
\]
\[
\frac{dz}{d\chi} = \frac{S}{\sqrt{(\chi - a) (\chi - b)}}
\]

When applying the transform to fig(2.7) it is possible to use the external angles of $A$ and $B$. If $\alpha = \beta = \frac{3\pi}{2}$ is used the result is a transform from the upper half $\chi$-plane to the outside of the polygon.

In engineering terms the previous two examples are not very useful as the $z$-plane polygons are not very common. The final example is the transformation of two parallel plates extending an infinite distance in both directions parallel to $x$ axis, as in fig(2.8).
Fig(2.8) is not a polygon but the Schwarz-Christoffel transform can still be applied to it. Point $B$ is not really necessary since the angle it makes is $\pi$ and so will vanishes in the Schwarz-Christoffel transform. From fig(2.8) $a = 0$, $\alpha = 0$ and $\beta = \pi$.

$$\frac{dz}{d\chi} = S (\chi - a)^{\frac{1}{s} - 1} (\chi - b)^{\frac{1}{s} - 1}$$

Integrating the Schwarz-Christoffel transformation can also cause problems, as there are often singularities in the integrand. Simpson’s rule can be used with limits displaced from the vertices, or alternatively integration by Gaussian procedures and by Gaussian procedures with limits displaced from singularities can be applied. The Gauss-Chebyshev and Gauss-Jacobi quadrature formulae seem to be suitable methods of handling singularities in integrations and these have been used in the numerical computation of the Schwarz-Christoffel transformation [19].

Trapezoidal electrodes have been analysed before but only in the field of waveguides, in particular monolithic microwave integrated circuits and electroptic devices requiring thick-electrode waveguides made from LiNbO$_3$. A paper by Goano et al. [20] describes the use of a Schwarz-Christoffel Toolbox for Matlab, which allows the capacitance, inductance and characteristic impedance to be calculated. While the cross-sections they model do not correspond exactly to those of comb electrodes, their approach is applicable to the comb electrode problem. It involves using the Schwarz-Christoffel transformation to get to the upper-half plane followed by elliptic integrals to form a rectangle, converting their problem from a complicated polygon to a simple ideal parallel plate capacitor.
Calculating the Force from a Schwarz-Christoffel Transform

As noted previously the Schwarz-Christoffel transformation gives the result in the form $\frac{dz}{d\chi}$. The relationship between electric field strengths in the $\chi$ and $z$-planes can be expressed simply in terms of this derivative as $E_z = E_\chi \left| \frac{dz}{d\chi} \right|$. The force exerted on any (straight) section of a bounding electrode in the $z$-plane is calculated using $F_z = \frac{1}{2\varepsilon_0} \int |D_z|^2 \, dz$. This is a line integral in the $z$-plane between specified points on the boundary. Unfortunately flux density in the $z$-plane is initially unknown, and it is for this reason that a transformation is applied. Rearranging the force calculation to include the transformation leaves us with $F_z = \frac{\varepsilon_0}{2} \int \left( E_\chi \left| \frac{dz}{d\chi} \right| \right)^2 \, dz$. Notice that the derivative has been inverted but the transform is still in terms of $\chi$, and here lies the main problem with conformal transformations. The transform $z = f(\chi)$ is often a complicated equation that cannot be inverted readily. For simple problems that involve right angles, bends with multiples of $\pi$, and a few vertices exact solution exist. However for trapezoidal electrodes having an arbitrary wall angle a closed form solution may be impossible to find.

Coming back to the force equation it can be simplified by a change of variable leading to:

\begin{align*}
F_z &= \frac{\varepsilon_0}{2} \int \left( E_\chi \left| \frac{dx}{dz} \right| \right)^2 \, dz \\
F_z &= \frac{\varepsilon_0}{2} E_\chi^2 \int \left| \frac{dx}{dz} \right| \left| \frac{d\chi}{dz} \right| \, dz \\
F_z &= \frac{\varepsilon_0}{2} E_\chi^2 \int \frac{d\chi}{dz} \frac{d\chi}{dz} \, (2.5)
\end{align*}

In the above equation $E_\chi$ has been taken outside the integral; this is only true if $E_\chi$ is a constant. It will be shown in chapter 4 that by choosing the right transformation $E_\chi$ can be made to be a constant.

A change of variable simplifies the problem as the transform stays in terms of $\chi$. However the discrete integration limits for eq(2.5) need to be transformed from the $z$-plane to the $\chi$-plane too. At first it would be thought that $\chi = g(z)$ is again needed and if we new $\chi = g(z)$ we would not need to change the integration variable. Fortunately this is often not necessary because an integration will most likely be done between two vertices for which the transformed and original locations will be known.

Some quantities are invariant under conformal transformation, namely energy, charge and voltage. For example the charge invariance means that the charge between any two points on the boundary in the $z$-plane is the same as the charge between the corresponding points on the boundary in the $\chi$-plane.
Bibliography


Chapter 3

Torsional Analysis of Trapezoidal Beams

It was described in the previous chapter how structures formed by anisotropic etching of silicon and electroplating of UV exposed photoresist moulds are trapezoidal in cross-section. In this chapter a semi-analytical model is used to determine the torsional constants and shear centres of trapezoidal beams as a function of aspect ratio and sidewall angle. The results are compared with FEM calculations in ANSYS.

An introduction to relevant parts of elasticity theory is first given to provide a framework for subsequent analysis. The Saint-Venant formulation of the torsion problem is then outlined. The mathematics at first seems quite complex but the main aim is to find the moment due to twisting or bending. The problem is formulated in terms of a stress function that satisfies Poisson’s equation over the beam’s cross-section, and is zero on the boundary. An approximate solution to this problem is derived using the Ritz Method. The moment is obtained by integrating the stress function over the cross-section and once the moment is known the torsional and shear centre problems are easily solved.

3.1 Formulation of the Torsion Problem

3.1.1 Conditions of Equilibrium and Compatibility

To analyse the torsion problem it is helpful to develop the mathematics from first principles. Consider a small cube as in fig(3.1) that has dimensions $\delta x$, $\delta y$ and $\delta z$, and is experiencing a force per unit volume $X$, $Y$ and $Z$ in the direction $x$, $y$ and $z$. In equilibrium these forces are balanced by internal stresses on the surfaces of the cube. These stresses are of two types: shearing stresses, which act in the plane, and are denoted by $\tau$; and normal stresses $\sigma$, which act perpendicular to the plane.
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Figure 3.1: Stress components of a small cube

Fig(3.1) shows the stresses on the visible faces of a cube (face 1, 3 and 5). The stresses on the hidden surface can be described in the same way (face 1 is opposite to 2, 3 is opposite to 4 and 5 is opposite to 6). If the element is in equilibrium then the forces in each direction must sum to zero.

\[
\begin{align*}
[(\sigma_x)_1 - (\sigma_x)_2]\delta y\delta z + [(\tau_{xy})_3 - (\tau_{xy})_4]\delta x\delta z + [(\tau_{xz})_5 - (\tau_{xz})_6]\delta x\delta y + X\delta x\delta y\delta z &= 0 \\
[(\sigma_y)_3 - (\sigma_y)_4]\delta x\delta z + [(\tau_{xy})_1 - (\tau_{xy})_2]\delta y\delta z + [(\tau_{yz})_5 - (\tau_{yz})_6]\delta x\delta y + Y\delta x\delta y\delta z &= 0 \\
[(\sigma_z)_5 - (\sigma_z)_6]\delta x\delta y + [(\tau_{yz})_3 - (\tau_{yz})_4]\delta x\delta z + [(\tau_{xz})_1 - (\tau_{xz})_2]\delta y\delta z + Z\delta x\delta y\delta z &= 0
\end{align*}
\]

(3.1)

Eq(3.1) applies to any small cube-like element and it is necessary for this model to be applicable to every point inside an object. Taking the limit as \(\delta x \to 0, \delta y \to 0\) and \(\delta z \to 0\) eq(3.1) becomes:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0
\end{align*}
\]

(3.2)

Eq(3.2) applies to the interior of a body. However as we approach the surface the elastic forces must also be in equilibrium with any external forces. Taking a tetrahedron OBCD so that the sides BCD coincide with the surface of the body, we define \(N\) as the direction of the normal to BCD and \(l, m\) and \(n\) as the cosines of the angles between \(N\) and the coordinate axes:

\[
l = \cos (Nx) \quad m = \cos (Ny) \quad n = \cos (Nz)
\]

(3.3)
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The components of force per unit volume \( X \), \( Y \), and \( Z \) are now replaced by forces per unit area \( \bar{X} \), \( \bar{Y} \), and \( \bar{Z} \) at the surface, giving:

\[
\begin{align*}
\bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{xz} n \\
\bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\
\bar{Z} &= \sigma_z n + \tau_{zx} l + \tau_{yz} m
\end{align*}
\]

(3.4)

Conditions of Compatibility

The elastic strains in a material are subject to a set of constraints known as the conditions of compatibility. There are two types of strain: elongation \( \varepsilon \) and shearing strain \( \gamma \). In three dimensions these are described by six strain components, with associated displacements \( u \), \( v \) and \( w \) along \( x \), \( y \) and \( z \).

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} & \gamma_{zz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} & \gamma_{yz} &= \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y}
\end{align*}
\]

(3.5)

The conditions of compatibility are readily obtained by differentiation of the above relations and are given by:

\[
\begin{align*}
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial z \partial x}
\end{align*}
\]

\[
\begin{align*}
2 \frac{\partial^2 \varepsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
2 \frac{\partial^2 \varepsilon_y}{\partial x \partial z} &= \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
2 \frac{\partial^2 \varepsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right)
\end{align*}
\]

(3.6)

Eq(3.6) is formed using strains, but it can be recast in terms of stresses (\( \sigma \) and \( \tau \)) by using Hooke's law (Timoshenko [1] page 238-239). Using the notation \( \Theta = \sigma_x + \sigma_y + \sigma_z \) and
\( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) eq(3.6) is transformed into the following form:

\[
(1 + v) \nabla^2 \sigma_x + \frac{\partial^2 \Theta}{\partial x^2} = 0 \quad (1 + v) \nabla^2 \sigma_y + \frac{\partial^2 \Theta}{\partial y^2} = 0 \\
(1 + v) \nabla^2 \sigma_z + \frac{\partial^2 \Theta}{\partial z^2} = 0 \quad (1 + v) \nabla^2 \tau_{yz} + \frac{\partial^2 \Theta}{\partial x \partial y} = 0 \quad (3.7)
\]

The solution of any elasticity problem involves finding stress components satisfying equilibrium conditions eq(3.2) and eq(3.4), together with the compatibility conditions eq(3.7).

### 3.1.2 Torsion of a Bar

Consider a uniform bar, as described in fig(3.2), which is rigidly constrained at one end and is experiencing a twist. The corresponding displacements of a cross-section of the beam in the \( x \) and \( y \) directions are \( u = -\theta z y \) and \( v = \theta z x \). Here \( \theta z \) is the angle of rotation of the cross-section at a distance \( z \) from the origin, ie \( \theta \) is the rotation per unit length, and \( u \) and \( v \) are the displacement in the \( x \) and \( y \) directions. Assuming points on the cross-section also undergo some axial displacement due to warping we define a warping function, \( \psi \), such that \( w = \theta \psi (x, y) \).

![Figure 3.2: A uniform bar](image)

For a beam experiencing torsion the components of strain, eq(3.5), simplify to the following:

\[
\epsilon_z = \epsilon_y = \epsilon_x = \gamma_{xy} = 0 \\
\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \left( \frac{\partial \psi}{\partial x} - y \right) \\
\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left( \frac{\partial \psi}{\partial y} + x \right) \quad (3.8)
\]
The only displacements that exist are \( u = -\theta z y \) and \( v = \theta z x \) [2], while the relationships of displacement and strains are \( \varepsilon_z = \frac{\partial u}{\partial z}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z} \) etc. (eq(3.5), hence \( \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{zy} = 0 \).

The first line of eq(3.8), \( \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{zy} = 0 \), states that there is no elongation (tensile strain) in the twisted beam, and no shear strain in the plane of the beam’s cross-section. The second and third lines of eq(3.8) indicate that there is some shear strain in the other two axes.

Eq(3.8) is described using strains. However by using the relationship between strain, stress and the modulus of rigidity, \( G \), (\( \gamma = \frac{\sigma}{E} \)), eq(3.8) can be re-cast in terms of stresses.

\[
\begin{align*}
\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\
\tau_{xz} &= G\theta \left( \frac{\partial \psi}{\partial x} - y \right) \\
\tau_{yz} &= G\theta \left( \frac{\partial \psi}{\partial y} + x \right)
\end{align*}
\]  

(3.9)

When the beam is under torsion the equilibrium condition must be maintained. Considering the third relation in eq(3.2), \( \frac{\partial \sigma_x}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0 \), and substituting the previous expressions for the stresses, eq(3.9), leads to:

\[
\frac{\partial}{\partial x} \left[ G\theta \left( \frac{\partial \psi}{\partial x} - y \right) \right] + \frac{\partial}{\partial y} \left[ G\theta \left( \frac{\partial \psi}{\partial y} + x \right) \right] = 0
\]

which simplifies to:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
\]  

(3.10)

Thus, for a beam under torsion about the z-axis, and free from any surface forces, the warping function \( \psi \) satisfies Laplace’s equation.

**Boundary Condition**

The boundary condition on \( \psi \) must now be considered. For a beam under torsion there are no surface forces, so that \( \bar{X}, \bar{Y}, \bar{Z}=0 \) in eq(3.4). furthermore, on the boundary of a cross-section the direction cosine \( n \) between the normal and the \( z \)-axis is zero. Thus, with the stresses as in eq(3.9), eq(3.4) is reduced to \( \tau_{xz} l + \tau_{yz} m = 0 \). The direction cosines \( l \) and \( m \) between the normal \( N \) and the \( x \) and \( y \) axes can be expressed as \( \frac{ds}{dx} \) and \( -\frac{ds}{dy} \) where \( ds \) is a small path element on the boundary of the cross-section. Substituting these expressions together with \( \tau_{xz} \) and \( \tau_{yz} \) from eq(3.9) into \( \tau_{xz} l + \tau_{yz} m = 0 \), the full boundary condition is formed:

\[
\left( \frac{\partial \psi}{\partial x} - y \right) \frac{dy}{ds} - \left( \frac{\partial \psi}{\partial y} + x \right) \frac{dx}{ds} = 0
\]  

(3.11)
Eq(3.11) is fairly complex looking and finding a function \( \psi \) that satisfies eq(3.10) and the boundary condition looks daunting. Fortunately Prandtl [3] introduced a stress function, \( \phi \), that greatly simplifies the boundary condition. The simplification involves introducing a new function \( \phi \) such that \( \frac{\partial \phi}{\partial y} = \tau_{xz} \) and \( -\frac{\partial \phi}{\partial x} = \tau_{yz} \), so that eq(3.9) becomes to:

\[
\frac{\partial \phi}{\partial y} = G\theta \left( \frac{\partial \phi}{\partial x} - y \right) \quad \text{and} \quad -\frac{\partial \phi}{\partial x} = G\theta \left( \frac{\partial \phi}{\partial x} + x \right)
\]

(3.12)

Substituting eq(3.12) into eq(3.11) the boundary condition becomes:

\[
\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{d\phi}{ds} = 0
\]

(3.13)

Eq(3.13) is easier to deal with than eq(3.11), since the boundary condition on \( \phi \) is simply \( \phi = \text{constant} \) on the boundary of the cross-section.

By introducing \( \phi \) the idea was to eliminate the need for \( \psi \) with its complicated boundary condition. By differentiating the first term of eq(3.12) with respect to \( y \) and the second term with respect to \( x \) and then summing the terms together, it is found that the stress function must satisfy Poisson's equation, with the boundary condition \( \frac{d\phi}{ds} = 0 \).

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta
\]

(3.14)

**Calculating the Torsion Constant from the Stress Function \( \phi \)**

To calculate the torsional constant the moment at the end of the beam is required. This can be determined by integrating the moments of the stresses \( \tau_{xz} \) and \( \tau_{yz} \) over the cross-section of the beam:

\[
M_z = \int \int (\tau_{yz}x - \tau_{xz}y) \, dx \, dy
\]

In terms of the stress function this may be written as:

\[
M_z = -\int \int \frac{\partial \phi}{\partial x} \, x \, dx \, dy - \int \int \frac{\partial \phi}{\partial y} \, y \, dx \, dy
\]

By integrating this by parts and observing that \( \phi = 0 \) at the boundary the moment becomes:

\[
M_z = -\int x \phi \, dy + \int \int \phi \, dxdy - \int y \phi \, dx + \int \int \phi \, dxdy
\]

\[
M_z = 2 \int \int \phi(x, y) \, dx \, dy
\]

(3.15)
Once the moment is known, torsional constant, \( k \), can be formed by simply dividing the moment by the rotation \( \theta L \):

\[
k = \frac{M_t}{\theta L}
\]  

(3.16)

Summary

Before proceeding to finding a solution for \( \phi \) a brief summary of what has been covered will be given. With the help of a stress function and with certain assumptions about the stresses and strains, the torsion problem can be reduced to solving Poisson’s equation. Suitable methods for finding a stress function, \( \phi \), that satisfies Poisson’s equation \( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \) subject to the boundary condition \( \frac{d\phi}{dz} = 0 \) will now be discussed.

3.1.3 Solving for the Stress Function \( \phi \) Using the Energy Method

The stress function is the key to the solution; find an expression for this and we can find the stresses and then the torsional constants. Rather than dealing with the differential equation itself Ritz [4] approached this problem by considering the strain energy of the twisted beam. The Ritz method is covered extensively in books by Timoshenko [1] and Sokolnikoff [5]. A modern day implementation of the Ritz Method using computers [6], has approached this type of problem, but only for simple cross-sections (elliptic, triangular and rectangular cross sections).

Strain Energy Formulation

Before delving into calculating the strain energy of the whole problem, let us investigate the work done on a small element which is subject to a stress in one direction only. The application of this stress causes an elongation (a strain) on this element. The force is simply the pressure integrated over the area, \( \sigma_x dydz \), and the elongation is simply \( \varepsilon_x dx \). Therefore the work done is:

\[
dV = \frac{1}{2} \sigma_x \varepsilon_x dx dydz
\]

More generally if all the stress components (3 normal and 3 shear) are included, the work done is given by:

\[
dV = \frac{1}{2} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yx} \gamma_{yx} + \tau_{xz} \gamma_{xz} \right) dx dydz
\]

By using Hooke’s Law this can be expressed in terms of stress components only.

\[
dV = \left\{ \frac{1}{2E} \left( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right) - \frac{v}{E} \left( \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x \right) + \frac{1}{2G} \left( \tau_{xy}^2 + \tau_{yx}^2 + \tau_{xz}^2 \right) \right\} dx dydz
\]
By applying the assumptions stated in eq(3.9), the above expression reduces to:

\[
dV = \frac{1}{2G} (\tau_{xx}^2 + \tau_{yz}^2) \, dx \, dy \, dz
\]

In terms of the stress function \( \phi \), the total strain energy per unit length may now be written as:

\[
V = \frac{1}{2G} \int \int \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \, dx \, dy
\]  \hspace{1cm} (3.17)

The Ritz Method

The Ritz method is a variational method based on the principle of virtual work. If we consider a small variation \( \delta M_z \) of the applied moment, then the corresponding variation in the strain energy (per unit length) should be \( \delta V = \theta \delta M_z \) where \( \theta \) is the rotation per unit length as before. Using eq(3.15) and eq(3.17) we can express this in terms of the stress function as:

\[
\frac{1}{2G} \delta \int \int \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \, dx \, dy = 2 \delta \int \int \phi \, dx \, dy \, * \theta
\]

or \( \delta S = 0 \) where \( S \) is given by:

\[
S = \int \int \left\{ \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] - 2G \theta \phi \right\} \, dx \, dy \]  \hspace{1cm} (3.18)

To apply the Ritz method, a trial stress function is constructed that satisfies the boundary conditions and contains one or more unknown parameters. The parameters are then adjusted to minimise eq(3.18), yielding an approximate solution to the true stress function. The accuracy of the result depends on the particular form of trial function chosen.

The trial stress function can take the form of a series:

\[
\phi = u_0 \phi_0 + u_1 \phi_1 + u_2 \phi_2 \ldots
\]

where the expansion coefficients \( u_n \) are the unknown parameters that need to be determined, and \( \phi_n \) are basis functions that satisfy the boundary conditions. However, rather than having \( n \) different functions that satisfy the boundary condition (but not necessarily the equilibrium conditions) it is more convenient to express \( \phi \) in the form:

\[
\phi = \text{(boundary only solving term)} \times \text{(equilibrium solving term)}
\]
The boundary solving term is simply a geometrical representation of the cross-section of the beam. In the simple case of a square cross-section the boundary solving term can be in the form of \((x + \frac{1}{2}a) (x - \frac{1}{2}a) (y + \frac{1}{2}a) (y - \frac{1}{2}a)\) see fig(3.3a).

The side length of the square is defined by \(a\) and as \(x\) or \(y\) approaches \(a\) the term tends to zero, hence satisfying the boundary condition \(\phi = 0\). The equilibrium part may be expressed as a polynomial series in the form:

\[
\sum_n (u_n x^n + v_n y^n)
\]

It is more convenient to express this using one variable set of parameters, \(u_n\), in the following manner:

\[
u_0 x^0 + u_1 y^0 + u_2 x + u_3 y + \ldots + u_n x^{\frac{n}{2}} + u_{n+1} y^{\frac{n-1}{2}}
\]

Minimising eq(3.18) is mathematically achieved by finding suitable values for the factors \(u_n\). Firstly eq(3.18) is differentiated with respect to each of the unknown factors, which forms \(\frac{dS_a}{du_n}\). This leads to \(n\) linear equations with \(n\) unknowns. A mathematical problem with \(n\) linear equations with \(n\) unknowns is simply solved using Gaussian elimination.
3.2 Calculation of Torsional Constants for Trapezoidal Beams

Torsional Constant Formulation

Torsional constants for beams with simple cross-sections such as rectangles and triangles are well known. In the case of the rectangular beam, for example, there is an exact solution for the moment in the form of an infinite trigonometric series (page 312 of [1]).

\[ M_z = \frac{1}{3} G \theta a^3 b \left\{ 1 - \frac{192}{\pi^5} \frac{a}{b} \sum_{n=odd}^{\infty} \frac{1}{n^5} \tanh \frac{n \pi b}{2a} \right\} \] (3.19)

where \( a \) and \( b \) are the side lengths of the cross-section. This expression can be quite tedious to evaluate, especially when one requires high accuracy. For this reason tables of values have been published for the bracketed part of eq(3.19). This factor, normally referred to as the \( \beta \) value, depends only on the ratio of the width to height \( \left( \frac{b}{a} \right) \) of the rectangular cross section. Having the \( \beta \) values in a tabulated form makes it easy to calculate \( M_z \) using the following relation:

\[ M_z = \beta G \theta a^3 b \] (3.20)

Combining eq(3.16) and eq(3.20) leads to the following expression for the torsional constant.

\[ k = \frac{\beta a^3 b G}{L} \] (3.21)

In the remainder of this section, the Ritz method is used to determine \( \beta \) values for trapezoidal beams, allowing eq(3.21) to be extended to trapezoids. In this case \( \beta \) depends both on the aspect ratio of the beam and on the degree of taper. The results are presented as a table showing the variation of \( \beta \) with \( \left( \frac{b}{a} \right) \) and wall angle, these parameters being defined as shown in fig(3.4)

3.2.1 Formulation of the Stress Function

By combining the boundary solving term and the equilibrium solving term eq(3.22) is formed, thus creating our stress function. The first four bracketed terms of eq(3.22) define the boundary solving term of the trapezoidal cross-section, and the coefficients \( u_n \) define the polynomial series.

\[ \phi(x, y) = y (y - a) \left( x + my - \frac{1}{2} b \right) \left( x - my + \frac{1}{2} b \right) \left( u_0 x^0 + u_1 y^1 + u_2 x + u_3 y + u_4 x^2 + u_5 y^2 \ldots \right) \] (3.22)
Due to symmetry the final form of the stress function for trapezoids, eq(3.23), does not require the odd $x$ terms, when the cross section is defined as in fig(3.3b).

$$\phi(x,y) = y(y + a) \left(x + my - \frac{1}{2}b\right) \left(x - my + \frac{1}{2}b\right) (u_0 + u_1y + u_2x^2 + u_3y^2 + u_4y^3 + u_5x^4 \ldots) \quad (3.23)$$

Matlab’s Symbolic Toolbox was used to implement the energy minimisation, while numerical values were given for $b$, $a$ and $m$. Eq(3.23) was formed in Matlab as a symbolic equation with the following notation. (NB to simplify the equation $u_n$ were replaced by $A$ to $Z$.)

$$f = y(y-a) \times (x+n*y-b) \times (x-n*y+b) \times (Z+A*y+B*x^2+C*y^2+\ldots+D*y^3+E*x^4+F*y^4+G*y^5)$$

The integral part of eq(3.18) was then formed as a temporary equation

$$\text{temp} = \left(\frac{\text{diff}(f, 'x')^2 + \text{diff}(f, 'y')^2}{2} - 2f\right)$$

as this allowed $\phi$ to be differentiated with respect to $x$ and $y$.

Together with the temporary equation the whole part of eq(3.18) was then formed:

$$S = \int \left(\int \text{temp}, 'x', n*y-b, -n*y+b\right), 'y', 0, a$$

Fortunately in MATLAB there is a function that automatically solves $n$ linear functions with $n$ unknowns. So in one step $S$ is differentiated by each of the unknown factors ($A$ to $G$) and solved.

$$[A B C D E F G Z] = \text{solve(diff(S,'Z'), diff(S,'A'), diff(S,'B'), \ldots, diff(S,'G'))}$$

The moment was then solved by integrating $f$ over the boundary.
The Required Number of Coefficients

It is important to establish how many unknown parameters should be included in the trial function. Table 3.1 shows the $\beta$ values for a series of trapezoidal beams with aspect ratios of 1 and 5 and a range of wall angles, calculated with an increasing number of coefficients. Over the range of aspect ratios under investigation 7 coefficients are sufficient for the results to stabilise to 4 decimal places.

<table>
<thead>
<tr>
<th>coefficients</th>
<th>$b/a=1$ angle=0°</th>
<th>$b/a=1$ angle=6°</th>
<th>$b/a=5$ angle=36°</th>
<th>$b/a=5$ angle=0°</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0.05555555555556</td>
<td>0.04014435630943</td>
<td>0.88219153472543</td>
<td>1.38888888888889</td>
</tr>
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<td>1</td>
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<td>0.10867686736602</td>
<td>0.20453023413141</td>
<td>0.26709401709402</td>
</tr>
<tr>
<td>2</td>
<td>0.13888888888889</td>
<td>0.10984451464289</td>
<td>0.22089364248491</td>
<td>0.26709401709402</td>
</tr>
<tr>
<td>3</td>
<td>0.1363694394574</td>
<td>0.11022920877651</td>
<td>0.22921889576430</td>
<td>0.28898505534428</td>
</tr>
<tr>
<td>4</td>
<td>0.1403910068191</td>
<td>0.11120069368562</td>
<td>0.23000494851518</td>
<td>0.28898505534428</td>
</tr>
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</tbody>
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Table 3.1: $\beta$ values for 0 to 12 coefficients

Correction factors

Table 3.2 shows $\beta$ correction factors for trapezoidal beams, solved using the aforementioned method, where the numerical factor depends not only on the ratio of the base width to the height, but also on the wall angle. For a square beam the numerical factor is $\beta=0.1404$, while for a trapezoidal beam with $\frac{b}{a} = 1$ and a 6° wall angle $\beta=0.1112$. In this case tapering the beam leads to a reduction in the torsional constant of 20.8%, corresponding to a 11% shift in the resonant frequency of a torsional mirror (for example). For beams formed by anisotropic etching the effects are even more pronounced.

The approximate wall angle for the electroplated structures fabricated by the author was 6° so it was natural for a 6° step size to be used in defining the wall angles. Secondly after converting to radians a step size of 6° creates fractions that are easier to deal with when compared with a step size for example 5° (36°=\frac{\pi}{5} and 35°=\frac{7\pi}{36})

3.2.2 Previous Analyses of Trapezoidal Beams

The main problem that has been tackled so far is the torsional modelling of trapezoidal beams. In Roark's formula book for mechanical structures [7], there is already a formula
CHAPTER 3. TORSIONAL ANALYSIS OF TRAPEZOIDAL BEAMS

<table>
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Table 3.2: $\beta$ values for trapezoidal beams calculated by the Ritz method

for the torsion of a trapezoidal beam. The analysis is taken from [8] in their paper titled *Structural Beams in Torsion*. Their analysis covers a standard I-beam where the flanges are long and tapered i.e. trapezoidal. The problem of the I-beam was broken down into parts with each flange being modelled as a trapezoidal that was further approximated as a curvilinear rectangle bounded by two concentric circular arcs and two radii.

![I-beam](image)

The torsional stiffness for these trapezoids was calculated using the following formula:

$$K = \frac{1}{12} (b + n) (b^2 + n^2) - V_L b^4 - V_S n^4$$  \hspace{1cm} (3.24)

where

$$V_L = 0.10504 - 0.10000s + 0.08480s^2 - 0.06746s^3 + 0.05153s^4$$
$$V_S = 0.10504 + 0.10000s + 0.08480s^2 + 0.06746s^3 + 0.05153s^4$$
$$s = \frac{b - n}{a}$$

Lyse’s formula is only valid for long tapered beams with the parallel sides being the shorter lengths of the quadrilateral. When Lyse’s formula is applied to cross-sections that are wider than they are high negative torsional constants are obtained.
Fig(3.6) shows the discrepancy between torsional constants of a trapezoidal beam calculated using the Ritz method and using the formula developed by Lyse. The latter does not have sufficient accuracy for a beam with $\frac{b}{a} = 1$.

![Graph showing comparison between Lyse formula and Ritz method](image)

Figure 3.6: Lyse and Ritz comparison with $\frac{b}{a} = 1$

If length $a$ in fig(3.5) is now allowed to increase the accuracy starts to improve. Fig(3.7) shows another torsional constant comparison, for the case of $\frac{b}{a} = 0.5$. In this case the difference in values is 2-3% at most.

Eq(3.24) is accurate to a certain degree for long tapered beams but fails for low aspect ratio cross-sections ($b > a$). The reason for this is that Lyse’s method is based on a geometrical approximation - approximating a trapezoid to a curvilinear rectangle bounded by two concentric circular arcs - and for low aspect ratios this approximation is not sufficiently accurate.

### 3.3 Shear Centre Calculations

In the previous section an energy formulation of the Saint-Venant method was used to calculate the torsional constants of trapezoidal beams. In this section a similar approach is used to determine how the position of the shear centre depends on the aspect ratio and side-wall angle. The shear centre is defined as the point, on the end of a cantilever beam,
through which a transverse force must act in order to cause pure bending i.e. bending without any twisting. For a simple rectangular beam, this point is the geometrical centre of the cross-section, so that a transverse force applied half way up the beam will cause pure bending as shown on the left of fig(3.8). For a trapezoidal beam, the shear centre lies somewhere on the line of symmetry, but not generally at its mid-point. Consequently a transverse load applied halfway up the beam will, in general, cause both bending and twisting as shown on the right of fig(3.8). This has implications for the operation of MEMS devices based on flexural suspensions. In particular it means that in-plane forces acting on an inertial element supported by trapezoidal flexure beams will, in general, produce spurious out-of-plane deflections.

To outline the approach used here a simple trapezoidal beam is constrained at one end and a transverse load is applied to the other end. A stress function is then derived by
assuming that the load produces pure bending i.e. that the load is acting through the (currently unknown) shear centre. Once this stress function is known, the moment due to the associated shear stresses can be calculated at any point on the line of symmetry. The point where this moment goes to zero is the shear centre.

3.3.1 Background Equations for a Cantilever Beam

To calculate the shear centre it is first necessary to make certain textbook assumptions regarding the stresses in a cantilever beam bent by a point load.

1. The normal stresses over any cross-section of the beam are distributed in the same manner as for pure bending. For a transverse force $P_x$ applied at one end of a beam of length $L$, the stress $\sigma_z$ normal to the $z$-plane, at a distance $z$ along the beam is given by:

$$\sigma_z = -\frac{P_x (L - z) z}{I_y}$$  \hspace{1cm} (3.25)

where $I_y$ is the second moment of area about the $y$-axis

2. Over any cross-section there are also shear stresses, which may be resolved into components $\tau_{xz}$ and $\tau_{yz}$.

3. The normal stress components, $\sigma_x$ and $\sigma_y$ and the shear component, $\tau_{xy}$ are zero.

With these assumptions, and the conditions of equilibrium and compatibility eq(3.7), we can establish the differential equations satisfied by the stress function. By applying the assumptions to eq(3.7) we obtain:

$$\nabla^2 \tau_{yz} = 0$$

$$\nabla^2 \tau_{xz} = -\frac{P_x}{I_y (1 + v)}$$  \hspace{1cm} (3.26)

Similarly, by applying the above assumptions to the differential equations of equilibrium we obtain:

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \quad \frac{\partial \tau_{yz}}{\partial z} = 0 \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{P_x y}{I_y}$$  \hspace{1cm} (3.27)
These equilibrium equations are satisfied if the stress components $\tau_{xz}$ and $\tau_{yz}$ take the form:

$$
\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{P_x y^2}{2I_y} + f(y)
$$

$$
\tau_{yz} = -\frac{\partial \phi}{\partial x}
$$

(3.28)

where $\phi(x, y)$ is the stress function to be determined and $f(y)$ is a function of $y$ to be determined from the boundary conditions.

If the shear stresses from eq(3.28) are substituted back into the reduced conditions of compatibility, eq(3.26), we obtain:

$$
\frac{\partial}{\partial x} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0
$$

$$
\frac{\partial}{\partial y} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \frac{v}{1 + v} \frac{P_x}{I_y} - \frac{d^2 f}{dy^2}
$$

These equations may be integrated to obtain.

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{v}{1 + v} \frac{P_x y}{I_y} - \frac{df}{dy} + c
$$

(3.29)

where $c$ is a constant of integration which can be taken as zero in the case of pure bending (Timoshenko [1] page 356).

Turning now to the boundary conditions, as in the earlier torsional analysis eq(3.4) reduces to $\tau_{xz} l + \tau_{yz} m = 0$ on the boundary of any cross-section. Using the relation $l = \frac{dy}{ds}$ and $m = -\frac{dz}{ds}$, and the expressions for $\tau_{xz}$ and $\tau_{yz}$ in eq(3.28) this may be written as:

$$
\frac{\partial \phi}{\partial s} = \left[ \frac{P_x x^2}{2I_y} - f(y) \right] \frac{dy}{ds}
$$

(3.30)

The function $f(y)$ is chosen so that the right-hand side of eq(3.30) is zero everywhere on the perimeter of the cross-section. In this case $\phi$ is constant on the boundary, and we can choose this constant to be zero.

Thus the problem can be reduced to solving Poisson's equation for $\phi$, subject to the boundary condition $\phi = 0$. Eq(3.29) can be solved using the Ritz method, the same approach that was used to solve the torsion problem in the previous section.

The transverse force applied to the end of the beam as described in eq(3.25) is parallel to the x-axis has a line of action which passes through the shear centre. The moment $M_z$ of this force about any point on the line of symmetry must be exactly balanced by
the moment due to the shear stresses \( \tau_{xz} \) and \( \tau_{yz} \) and can be calculated in terms of these stresses using the relation:

\[
M_z = \int \int (\tau_{xz} y - \tau_{yz} x) \, dx \, dy
\]

By calculating the moment at several points along the line of symmetry, a point can be identified by interpolation where the moment will equal zero. This point is the shear centre.

### 3.3.2 Formation of the Stress Function

A stress function for the shear centre problem of a trapezoidal beam can be formed in the same way as for the torsion problem. Firstly the boundary function is formed to ensure that \( \phi = 0 \) on the boundary of the trapezoid fig(3.3b):

\[
y (y - a) \left( x + my - \frac{1}{2} b \right) \left( x - my + \frac{1}{2} b \right)
\]

The function \( f(y) \) is then chosen to ensure that the right-hand-side of eq(3.30) is zero on the boundary of the trapezoid. The upper and lower boundaries are catered for by the \( \frac{dy}{dx} \) term, and so it is necessary only to ensure that the term in square brackets goes to zero on the sides. A suitable choice for \( f(y) \) is:

\[
f(y) = \frac{P_z}{2I_y} \left(-my + \frac{1}{2} b\right)^2
\]

The differential equation is more complicated than the torsion problem and with this form of \( f(y) \), the Poisson's equation for \( \phi \), eq(3.29), becomes:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{v}{1 + v} \frac{P_z y}{I_y} + \frac{P_z}{I_y} \left(-my + \frac{1}{2} b\right) m
\]

This can be solved by constructing a trial stress function formed as a series:

\[
\phi(x, y) = (y + \frac{1}{2} a) \left( y - \frac{1}{2} a \right) \left( x + my - \frac{1}{2} b \right) \left( x - my + \frac{1}{2} b \right) \sum_n (u_n x^n + v_n y^n)
\]

and then minimising eq(3.31).

\[
S = \int \int \left\{ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right\} + \phi \left[ \frac{v}{1 + v} \frac{P_z y}{I_y} - \frac{df(y)}{dy} \right] \, dx \, dy \quad (3.31)
\]
Eq(3.31) was derived by Timoshenko in [9]. Timoshenko’s paper uses a membrane analogy whereby a membrane is stretched over a frame with the same shape as the beam’s cross-section. By deforming the membrane with a uniform pressure an analogy between shear stresses in a real beam and the deformed membrane’s slope can be created. Comparing eq(3.31) with the Ritz integral obtained in the case of torsion (eq(3.18)), it can be seen that the two are of similar form, but that the term $-2G\theta$ in eq(3.18) has been replaced by the expression on the right-hand-side of eq(3.29). This reflects the difference in Poisson’s equation satisfied by the stress functions for the two problems.

3.3.3 Implementation in MATLAB

The shear centres were calculated using MATLAB with its symbolic toolbox. For the trial stress function $x^m$ and $y^m$ where taken to the fifth power and, as in the case of the torsion problem, the odd powers of $x$ yielded no results.

After solving for the stress function $\phi$, the moments at two different positions on the line of symmetry were calculated. From these two points (0.4 and 0.6 of the height of the beam) it was possible to determine the point where there was zero moment, hence finding the shear centre.

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Table 3.3: Normalised shear centre positions for trapezoidal beams as a function of width/height ratio and wall angle

Two points are sufficient to extrapolate the result as the relationship between the moment and vertical distance is linear. Fig(3.9) shows the moment calculations at a few points between 0.4 and 0.6, for a beam with an aspect ratio of 1 and a wall angle of 12°.
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3.3.4 Results

Table 3.3 and fig(3.10) show the calculated shear centre positions normalised with respect to the height of the beam, for different wall angles and width to height ratios. For example at a wall angle of 24° and with a width to height ratio of 3, the normalised shear centre is at 0.536147. This means the shear centre point is displaced away from the wide base of the trapezoid in this case.

Comment on Results

The analytical results for the shear centre seem to be counter-intuitive. With a rectangular beam the shear centre is at the geometrical centre and as the sides are tapered it might be thought that the shear centre would simply move towards the wide base. This is indeed the case for large wall angles. However the results show that the shear centre point initially moves away from the wide base, above the centre line. Then as the wall angle increases further, it starts moving back towards the wide base. This behaviour has been confirmed using ANSYS, as described in the next section.
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3.4 ANSYS Torsion and Shear Centre Simulations

ANSYS was employed to test the aforementioned methods for calculating the torsional constants and shear centres of trapezoidal beams. To find the shear centre of a given beam, ANSYS simulations were run with point loads applied at different positions on the line of symmetry and the resulting angular rotations were recorded. The position where no rotation would occur was then found by interpolation. To find the $\beta$ correction factor for the torsional constant a torque was applied to the end of the beam, and the angular rotation was recorded. Knowing the angular rotation, the applied torque and the length of the beam, the torsional constant could be found and hence the $\beta$ correction factor could be calculated.

3.4.1 ANSYS Torsion Solution

When constructing the model for a given beam in ANSYS it was necessary to decide on a suitable length. In the earlier analysis it was assumed that each cross-section of the beam warps in the same way i.e. that there are no end effects. However for a simple cantilever beam under torsion, this will not be true; the whole surface of the constrained end will be forced to have zero displacement (and hence no warp), while at the other end the load will not, in general, be distributed over the cross-section in the manner required.

Figure 3.10: Normalised shear centre position for trapezoidal beams as a function of width/height ratio and wall angle using Ritz method.
CHAPTER 3. TORSIONAL ANALYSIS OF TRAPEZOIDAL BEAMS

by the Saint-Venant solution. These end effects are taken into account by ANSYS, and are expected to lead to discrepancies between the two different methods. However as discussed in Timoshenko [1] page 239, such discrepancies can be made arbitrarily small by modelling a sufficiently long beam.

Fig(3.11) shows the $\beta$ values for beams with lengths from 50 units to 2000 units for a trapezoidal beam with $\frac{b}{a} = \frac{24}{12} = 2$ and a wall angle of $36^\circ$. For beam lengths over 600 units the $\beta$ values start to converge to about 0.0958 so it can be assumed, in this case, that a beam length of 1000 units is sufficiently long enough that the end effects can be ignored. In this case a beam must be about 42 times longer than widest part of the cross-section. 42 may seem like a large number for a real device, however in a real device the ends will not be constrained in the same manner as the ANSYS simulations. The end face of the beam will not be held completely rigid as the join with the substrate will allow some flexure.

![Graph showing $\beta$ values for beam with lengths from 50 to 2000 units long](image)

Figure 3.11: $\beta$ values for beam with lengths from 50 to 2000 units long

**Method** Each beam was constructed in ANSYS with the required same cross-sectional shape and with the following basic essential parameters:
Solid 95 is a type of finite element used in ANSYS used to mesh a volume. Solid95 is a brick element with 20 defined by 20 nodes (at the corners and on the mid sides), and is a quadratic element rather than a linear one, which offers more accurate results.

A torque was then applied to the free end of the beam and the model was solved. The torque was applied in the following way: two nodes were defined on one end of the beam, centred around the mid point and on the line of symmetry. Then two equal but opposite forces applied perpendicular to the line of symmetry were attached to the nodes.

After the simulation the angular rotation was calculated using the nodal displacements. To avoid the end effects the nodes were chosen away from the ends of the beam and their positions before and after were recorded and the angle solved. The torsional constant was calculated using $k = \frac{M}{\theta^2}$ and from this the $\beta$ values $\beta = \frac{KL}{G\alpha^2 b}$, where $a$ is the beam height and $b$ is the width of the wide base.

The mesh density will have an impact on the results obtained in ANSYS so a simulation of element size with $\beta$ values will also be done.
Results Table 3.4 shows the equivalent $\beta$ values extracted from the ANSYS FEM simulations. These cover the same range of wall angle and $\frac{b}{a}$ as the semi-analytical results. Figure(3.14) compares the two sets of data.

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Table 3.4: $\beta$ values using ANSYS
Figure 3.14: Comparison of $\beta$ values obtained from ANSYS and the Ritz method

Figure 3.15: $\beta$ percentage error between Ritz method and ANSYS with increasing mesh density
CHAPTER 3. TORSIONAL ANALYSIS OF TRAPEZOIDAL BEAMS

3.4.2 ANSYS Shear Centre Simulation

ANSYS has no in-built function for calculating the shear centre of a beam. A suitable method for finding it is to construct the beam, constrain it at one end, and then load the other end with forces applied at different points, searching for the place where no rotation occurs. This method is computationally wasteful. However, assuming that the analytical values are roughly correct, the search area can be reduced to between 0.34 to 0.56, the minimum and maximum values from table 3.3.
CHAPTER 3. TORSIONAL ANALYSIS OF TRAPEZOIDAL BEAMS

Method  A beam was modelled with a length of 1000 units and a height of 12 units. The cross sections were meshed with a 3 unit spacing and a length of 40 units. A force was applied and the angle of rotation was calculated. To reduce computation time the angular rotation was found for only two points. The point with zero rotation was then obtained by interpolation, giving the results shown in table 3.5.

<table>
<thead>
<tr>
<th>wall angle</th>
<th>b/a 1</th>
<th>b/a 2</th>
<th>b/a 3</th>
<th>b/a 4</th>
<th>b/a 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.50000</td>
<td>0.50000</td>
<td>0.50000</td>
<td>0.50000</td>
<td>0.50000</td>
</tr>
<tr>
<td>6.00</td>
<td>0.48282</td>
<td>0.50919</td>
<td>0.51029</td>
<td>0.50918</td>
<td>0.50799</td>
</tr>
<tr>
<td>12.00</td>
<td>0.44841</td>
<td>0.51687</td>
<td>0.52072</td>
<td>0.51871</td>
<td>0.51632</td>
</tr>
<tr>
<td>18.00</td>
<td>0.39384</td>
<td>0.52150</td>
<td>0.53105</td>
<td>0.52861</td>
<td>0.52510</td>
</tr>
<tr>
<td>24.00</td>
<td>0.33252</td>
<td>0.52038</td>
<td>0.54080</td>
<td>0.53887</td>
<td>0.53442</td>
</tr>
<tr>
<td>30.00</td>
<td>n/a</td>
<td>0.50899</td>
<td>0.54889</td>
<td>0.54931</td>
<td>0.54436</td>
</tr>
<tr>
<td>35.26</td>
<td>n/a</td>
<td>0.48545</td>
<td>0.55292</td>
<td>0.55821</td>
<td>0.55355</td>
</tr>
<tr>
<td>36.00</td>
<td>n/a</td>
<td>0.48084</td>
<td>0.55310</td>
<td>0.55940</td>
<td>0.55487</td>
</tr>
</tbody>
</table>

Table 3.5: Shear centre positions calculated using ANSYS

Discussion  Fig(3.17) is a graphical comparison between the results obtained using the Ritz method and those obtained using ANSYS.

Figure 3.17: Comparison between shear centre results obtained using the Ritz method and ANSYS
The results show a good correlation between the two methods. As discussed earlier the shear centres of beams with large aspect ratios initially move above the centre line away from the wide base, and then as the angle increases further they move back towards the wide base. From these results it seems feasible to make an anisotropically etched silicon device in which the shear centre of each support beam is placed exactly halfway up the line of symmetry. This would minimise any out-of-plane rotations if the device were driven in-plane. For an anisotropically etched silicon beam the \( \frac{b}{a} \) ratio that will place the shear centre at the mid point was calculated to be \( \approx 2.17 \).

3.5 Hexagonal Beams

It is possible to create beams with hexagonal cross-sections by anisotropic etching of silicon. For example, if etching is allowed to occur on the front and back sides of a silicon wafer at the same time, through similar masks, eventually the etch paths will meet at the centre of the wafer. This type of etching will create beams with hexagonal cross-sections having two axes of symmetry. If etching is delayed on one side then hexagons with only one axis of symmetry are created instead. The former case is of particular interest because the shear centre is halfway up the beam regardless of aspect ratio.

3.5.1 Modelling of Regular Hexagonal Beams using the Ritz Method

The cross-sections remain convex and can be modelled using straight line equations to define the boundary of the hexagon (see fig(3.18)).

\[
\phi(x, y) = (y - \frac{1}{2}a) (y + \frac{1}{2}a) (x + my - \frac{1}{2}b) (x - my + \frac{1}{2}b) \cdots \\
\cdots (x - my - \frac{1}{2}b) (x + my + \frac{1}{2}b) \sum_n (u_n x^n + v_n y^n) \tag{3.32}
\]

Using the trial stress function in eq(3.32) \( \beta \) values for a range of hexagonal beams were calculated by the Ritz method. The results are shown in Table 3.7.

Results using the Ritz method

Since the cross sections were symmetrical in both the \( x \) and \( y \) axes only the even terms of the coefficients were used in the series and for the table of results terms up to \( x^6 \) and \( y^6 \) were included.

The results show a good correlation between the two methods. The largest difference occurs at \( 0^\circ \) but this can be overcome by modelling it as a rectangle rather than a hexagon.
Figure 3.18: Hexagon boundary lines and equations

Figure 3.19: $\beta$ percentage error between Ritz method and ANSYS for a hexagonal beam
### 3.6 Summary

A method has been developed for calculating the torsional stiffness of trapezoidal beams. To make the results more general a numerical factor $\beta$ has been introduced that depends only on the aspect ratio of the beam and the wall angle. From the $\beta$ value and the dimensions of the beam the torsional stiffness can be calculated. Having the $\beta$ values in a tabulated form allows a MEMS designer to quickly design, for example, a torsion mirror, without requiring too much computational time in the early stages of the design process.
The values have been shown to be very accurate, closely matching those extracted from ANSYS.

The variational method used for calculating the torsional stiffness, has also been applied to finding the shear centres of trapezoidal beams. The significance of the shear centre is that a transverse force applied at this point produces pure pending without any twisting. It has been shown that the shear centre can be engineered to lie half way up the beam by adjusting the aspect ratio.
Bibliography


Chapter 4

Electrostatic Analysis of Comb Drives with Trapezoidal Electrodes

A simple method for calculating the electrostatic force for a pair of trapezoidal electrodes within a comb drive is first attempted. This involves breaking the structure down into simpler geometries that can be solved by conformal mapping. An approximate analytical solution with closed form expressions for the force is developed and is used to investigate the relative contributions from the top, bottom and side-wall of the trapezoidal electrode. The problem is then solved using the Schwarz-Christoffel (SC) Toolbox within MATLAB. The numerical solution using the SC Toolbox can handle non-coplanar electrodes with or without a ground-plane. Both methods are compared with FEM calculation in ANSYS.

All the calculations, analytical and FEM, typically assume 6° wall angle on the comb electrodes. This value was chosen because it is the nominal wall angle of the low cost LIGA process used by the author to fabricate test devices. However it is important to emphasise that the methods developed are general and are applicable to electrodes with other wall angles.

4.1 Approximate Analytical Model

From elementary field theory the local electrostatic force on a conductor surface lies parallel to the electric field, and hence normal to the surface. In the situation of two adjacent trapezoidal beams, held at different potentials, out-of-plane forces arise from the asymmetry in the fringing fields on the top and bottom surfaces, and also from the vertical component of the electric field in the gap between the sloping surfaces.

Previous analysis of trapezoidal comb electrodes considered only the force contribution from the sloping surfaces [1]. This field was approximated to that inside a wedge formed
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by two infinitely long conductors separated by an angle $\alpha$, as in fig(4.1).

![Diagram of trapezoidal electrodes](image)

Figure 4.1: Simple wedge model used in [1] to approximate the field in the gap between trapezoidal electrodes

Neglecting fringing fields (i.e. assuming that $R_1 \to 0$ and $R_2 \to \infty$ in fig(4.1)), the potential and the electric field are given by:

$$V(r, \phi) = V_0 \left(1 - \frac{\phi}{\alpha}\right) \quad \mathbf{E}(r, \phi) = -\left(\frac{1}{r}\right) \frac{\partial V}{\partial \phi} = \frac{V_0}{r\alpha} \quad (4.1)$$

The force normal to the surface of either electrode, when $\phi = 0$ or $\phi = \alpha$, is given by:

$$F_n = \frac{\varepsilon_0}{2} \int_{R_1}^{R_2} \left\{ |\mathbf{E}(r, 0)|^2 \, dr \right\} \quad (4.2)$$

By taking the vertical component of this force, $F_n \sin \left(\frac{\alpha}{2}\right)$, and neglecting the force contributions from the top and bottom surfaces, the overall vertical force on one half of each trapezoidal beam can be estimated.

The method used by Tay et al. in [1] ignores the top and bottom surfaces of the electrode, and also neglects hence the concentration of charge (and field) at the top and bottom of the side-wall. As a result the force values the force values obtained are too inaccurate to be of any real use.

4.1.1 Coplanar Electrode Model

An early attempt by the author to find an improved approximate solution involved breaking the structure down into three separate parts, as shown in fig(4.2). Johnson and Warne [2] used this approach to determine the approximate charge for rectangular comb electrodes. The upper and lower surfaces, ignored by Tay [1], were treated as pairs of co-planar electrodes, for which the electric field may be found by using conformal
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transforms. The problem of the resistance between two co-planar strips was described in the book by Gibbs [3], and the approach was adapted to the electrostatic problem of a coplanar array. Details of this analysis are contained in appendix B.

The advantage of using the coplanar array was that an analytical solution existed for the electric field. However it was found that the electric field of the coplanar electrodes could not accurately model the top and bottom sides of the trapezoidal electrodes. Furthermore the charge concentration on the side-walls was still neglected, resulting in a poor model for the sloping side-wall. Overall the results were rather inaccurate, although useful experience was gained in using and manipulating SC transforms.

4.1.2 Stepped Channel Model

Based on the experience gained from the coplanar electrode model, a new model was proposed as shown in fig(4.3). Here the sloping surfaces are approximated as stepped side-walls, allowing the electrodes to be treated as a pair of channels i.e. back-to-back L-shaped electrodes. Using a channel as the basic primitive employed to build up the model.

The top and bottom force contributions are given by $F_t$ and $F_b$ in fig(4.3). The force contribution for the sloping side-wall is taken as a contribution from each of the channel legs $F_0$ and $F_1$. These are assumed to be good approximations for the actual force contributions from the sloping sidewall, allowing the total vertical force to be calculated using:

$$F_{total} = F_t - F_b - (F_0 + F_1) \sin \left( \frac{\alpha}{2} \right)$$  \hspace{1cm} (4.3)

where $\frac{\alpha}{2}$ is the wall angle. A method for setting the relative lengths of the upper and
lower sections of the stepped side-wall is given later in this chapter.

There are two main advantages of the stepped side-wall approach, as shown in fig(4.3), over the simplification used by Tay et al. [1]. Firstly it includes the top and bottom surfaces of the electrode and secondly it includes the charge concentration at the corners, therefore creating a more accurate model.

### 4.2 Formulation of Stepped Channel Model

#### 4.2.1 Channel Description

The arms and legs of the two channels in fig(4.3) have finite lengths, but in order to produce analytical approximations for the fields they are modelled initially as shown in fig(4.4). Each part is represented as an infinitely long open channel of width $g$, bounded either side by two L-shaped electrodes, both extending to infinity. With the electrodes at potentials 0 and $V$, the line of symmetry is an equipotential $\frac{V}{2}$ and this symmetry helps simplify the conformal transformation.

Figure 4.4: Channel model, with equipotential line of symmetry
4.2.2 Conformal Transformation of the Channel

As noted above, the channel problem is mirror-symmetrical with respect to the line of symmetry. This simplifies the problem as it reduces it from a two right-angled problem to a single right-angled problem. The potential on the line of symmetry between the two electrodes is taken to be $Y/2$, while the remaining electrode is taken to be at $0V$. 

![Figure 4.5: Desired mapping of half the channel to two infinitely long parallel plates. Red lines represent equipotentials.](image)

The field due to a channel in which the potential alternates between two values can be obtained using conformal mapping. The aim is to transform half of the channel into an infinitely long pair of parallel plates one as illustrated in fig.4.5. Knowing the solution of the electric field in the parallel plate configuration, and the transform that links the two domains, the electric field and hence the electrostatic force for the channel problem can be calculated. The basic principles of conformal mapping were outlined in chapter 2.

Unfortunately it is not easy to go straight from the channel domain to the parallel plate domain in one step. Therefore an intermediate domain is employed so that standard transforms can be used. In this case the modelled region (bounded by the electrodes in fig.4.5) is mapped to the upper half plane with all points lying on the x-axis. In the following discussion the initial, intermediate, and final domains will be referred to as the $z$, $w$, and $z$-domains respectively.

**Transformation from the $w$-plane to the $z$-plane**

Referring to fig(4.6) there is one corner in the $z$-plane at $j\gamma$ with an angle of $\frac{3\pi}{2}$ and a zero angle at $-j\infty$. This leads to a Schwarz-Christoffel transformation of the form:

$$\frac{dz}{dw} = S\sqrt{w + \frac{1}{w}}$$
Integrating this equation gives an expression for \( z \) in terms of \( w \):

\[
z = S \left\{ 2\sqrt{w+1} + \ln \frac{\sqrt{w+1} - 1}{\sqrt{w+1} + 1} \right\}
\]  

(4.4)

Figure 4.6: Transformation from z-plane to w-plane

Determination of Parameter \( S \)

As seen in fig(4.6) the channel width in the z-plane remains constant at \( \frac{g}{2} \), so we may write:

\[
\frac{g}{2} = \int_P dz
\]

where \( P \) is a straight integration path from \( z = jy \) to \( z = \frac{g}{2} + jy \) for any \( y \). In the limit as \( y \) tends to minus infinity the corresponding path in the w-plane is a vanishingly small semicircle centred at the origin. This allows us to express the above integral in term of \( \theta \):

\[
\frac{g}{2} = S \int_0^\pi \frac{1}{r} e^{-j\theta} \sqrt{r^2 + 1} e^{j\theta} d\theta
\]

As \( r \to 0 \) the above equation reduces to:

\[
S = \frac{g}{2\pi}
\]

This seems a complicated method, since solving eq(4.4) with \( z = \frac{1}{2}g \) and \( w = -1 \) yields the same result. However, it is common to use contour integration to verify the \( S \) value, even when the integrated transform is known. More generally path integration is the only possible approach in situations where the original transform cannot be integrated.
Transformation from the $\chi$-plane to the $w$-plane

The second transformation is simply:

$$\chi = \frac{1}{\pi} \ln w$$

which maps the problem of two coplanar, semi-infinite electrodes in the $w$-plane to a parallel plate configuration in the $\chi$-plane.

Figure 4.7: transformation from $w$-plane to $\chi$-plane

<table>
<thead>
<tr>
<th>$w$-plane</th>
<th>$z$-plane</th>
<th>$\chi$-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>$+\infty + j$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$2g$</td>
<td>$-\infty + j$</td>
</tr>
<tr>
<td>$0$</td>
<td>$-j\infty$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$+j\infty$</td>
<td>$+\infty$</td>
</tr>
</tbody>
</table>

Table 4.1: Mappings of key points under transformations shown in fig(4.6) and fig(4.7)

4.2.3 Force Calculation

The conformal transform is in the form of a derivative, so one obvious question that may not be clear is, how is it applied? The quantities we are interested in are the force and to calculate the force we require the flux density or electric field. So to use the transform we simply multiply it by a known quantity in one plane to find the unknown in the other plane, e.g. the electric field in the $\chi$-plane multiplied by the transform $\frac{d\chi}{dw}$ will give the electric field in the $z$-plane.

$$D_z = D_\chi \left| \frac{d\chi}{dz} \right|$$ (4.5)
Using electric fields instead of flux densities yields a similar result.

\[ E_z = E_x \left| \frac{dx}{dz} \right| \]

The electric flux density in the z-plane, \( D_z \), can be obtained as the product of two transforms, one from z-plane to w-plane and then w-plane to the \( \chi \)-plane. Ideally we would like to have one transform from the w-plane to the \( \chi \)-plane, but to simplify the problem two transforms were used. In this case eq(4.5) can be expanded, accommodating the two transforms \( \frac{dx}{dw} \) and \( \frac{dz}{dw} \).

\[ D_z = D_x \left| \frac{dx}{dz} \right| = D_x \left| \frac{dx}{dw} \frac{dw}{dz} \right| \quad (4.6) \]

The flux density in \( \chi \)-plane is simply:

\[ D_\chi = \frac{V \varepsilon_0}{2} \]

Using eq(4.6) the flux density in the z-plane is given by:

\[ D_z = D_x \times \left| \frac{1}{\pi w} \times \frac{2\pi w}{g \sqrt{w^2 + 1}} \right| \]

\[ D_z = \left| \frac{V \varepsilon_0}{g} \frac{2}{\sqrt{w^2 + 1}} \right| \]

the force acting on any part of the channel in the z-plane can be found using the integral in eq(4.7).

\[ F_z = \frac{1}{2 \varepsilon_0} \int |D_z|^2 \, dz \]

\[ F_z = \frac{1}{2 \varepsilon_0} \int |D_\chi|^2 \left| \frac{dx}{dw} \frac{dw}{dz} \right|^2 \, dz \quad (4.7) \]

By a change of variable eq(4.7) can be expressed in terms of \( w \) as:

\[ F_z = \frac{1}{2 \varepsilon_0} \int |D_\chi|^2 \left| \frac{dx}{dw} \right|^2 \left| \frac{dw}{dz} \right| \, dw \]

\[ = \frac{V^2 \varepsilon_0}{4 \pi g} \int \left| \frac{1}{w \sqrt{w^2 + 1}} \right| \, dw \]

\[ = \frac{V^2 \varepsilon_0}{2 \pi g} \text{atanh} (\sqrt{w + 1}) \quad (4.9) \]

Thus the force calculation yields a close formed answer in the case where the corner
electrodes extend to infinity, and by changing the variable we avoid inverting \( z(w) \), eq(4.10).

\[
z = j \frac{g}{2\pi} \left\{ 2\sqrt{w+1} + \ln \frac{\sqrt{w+1} - 1}{\sqrt{w+1} + 1} \right\}
\]

**(4.10)**

**Force Contributions from Top and Bottom**

\( F_t \) and \( F_b \) in fig(4.3) are calculated using eq(4.10) with two limits. The first limit is known and it is the corner in the \( z \)-plane; this point maps onto \( w = -1 \). The second point in the \( z \)-plane is the sum of the half the gap and half the comb width. The related point in the \( w \)-plane is found by using the mapping in eq(4.10). Rather than trying to invert this equation, the \( w \) values may be found numerically by iteration and this is done in MATLAB. After finding \( w \) this value is then substituted into eq(4.9) to find the force.

**Contributions from Side-wall**

The force contribution \( F_0 \) and \( F_1 \) from the legs of the channels can also be solved using eq(4.9) and again only a finite length of each channel leg is required. This is given by \( kh \) in the case of the upper channel, and \((1 - k) h \) for the lower channel, where \( h \) is the electrode height. The parameter \( k \) is chose such that, in the absence of fringing effects, the sum of the two forces \( F_0 \) and \( F_1 \) would be equal to the force \( F_n \) predicted by the simple wedge model.

The total vertical force can now be calculated using \( F_{total} = F_t - F_b - (F_0 + F_1) \sin \left( \frac{\theta}{2} \right) \), where \( \frac{\theta}{2} \) is the wall angle.
Calculating Parameter $k$

The force between two semi-infinite plates at an angle $2\theta$ (in eq(4.1) this was previously defined as $\alpha$) between $R_2$ and $R_1$ can be calculated using eq(4.11), which is derived from eq(4.1) and eq(4.2).

$$F_n = \frac{\varepsilon_0}{2} \left( \frac{V}{2\theta} \right)^2 \left( \frac{\Delta R}{R_2 R_1} \right)$$

(4.11)

With

$$R_1 = \frac{g_0}{2} \csc \theta \quad R_2 = \frac{g_1}{2} \csc \theta \quad \Delta R = h \sec \theta$$

The electric field in section of the channel is assumed to be constant $E_0 = \frac{V}{g_0}$ and $E_1 = \frac{V}{g_1}$, providing horizontal forces $F_0 = \frac{\varepsilon_0}{2} \frac{V^2}{g_0} kh$ and $F_1 = \frac{\varepsilon_0}{2} \frac{V^2}{g_1} (1 - k) h$ respectively. Setting $F_n = F_0 + F_1$ we obtain:

$$\frac{\varepsilon_0}{2} \left( \frac{V}{2\theta} \right)^2 \left( \frac{\Delta R}{R_2 R_1} \right) = \frac{\varepsilon_0}{2} \frac{V^2}{g_0^2} kh + \frac{\varepsilon_0}{2} \frac{V^2}{g_1^2} (1 - k) h$$

$$\frac{1}{4\theta^2} \left( \frac{h \sec \theta}{\frac{g_0}{2} \csc \theta \frac{g_1}{2} \csc \theta} \right) = \frac{kh}{g_0^2} + \frac{(1 - k) h}{g_1^2}$$

$$\frac{h}{g_0 g_1} \frac{\sin^2 \theta}{\theta^2 \cos \theta} = \frac{kh}{g_0^2} + \frac{(1 - k) h}{g_1^2}$$

when $\theta < \frac{\pi}{15} \frac{\sin^2 \theta}{\theta^2 \cos \theta} \approx 1$

$$\frac{1}{g_0 g_1} = \frac{k}{g_0^2} + \frac{(1 - k)}{g_1^2}$$

$$\frac{g_0}{g_0 g_1} = \frac{g_0^2 k + g_0^2 (1 - k)}{g_0^2}$$

$$k \approx \frac{g_0}{g_0 + g_1}$$

Therefore as a first approximation the amount of channel to include in each part of the stepped side-wall force calculation depends only on the ratio of the gap widths.

4.3 Electrostatic Simulations Using ANSYS

Finite element analysis in ANSYS was used to test the validity of the results obtained by conformal mapping. As described in the ANSYS documentation this type of analysis uses the standard h-method. The h-method uses h-elements, which have a fixed shape function polynomial, and therefore convergence is achieved by refining the mesh in regions where higher accuracy is needed.
A cross-section consisting of two half-electrodes within a comb array was modelled. The ANSYS procedure of implementing an electrostatic problem requires the use of electrostatic elements and far-field elements. The far-field or infinite elements allow the model to be more realistic, by allowing for far-field decay in the electrostatic analysis without having to assume the air modelled around the electrodes extends to infinity. The infinite elements then need to be flagged with the infinite surface option and the surface that requires a force to be calculated needs to be flagged with Maxwell surface option. In ANSYS the potential field lines are always perpendicular to any unconstrained boundary. The ANSYS and conformal mapping models only deal with a pair of comb electrodes and due to their repeating pattern only the adjacent halves need to be calculated. So the term comb half width will be used which means half the distance of the widest width of the trapezoidal comb electrode, though the real comb will be double this value.

\[ \frac{d\phi}{dn} = 0 \]

\[ \frac{d\phi}{dn} = 0 \]

\[ \text{surface tagged, e.g., 60V} \]

\[ \text{infinite elements} \]

\[ \text{electrostatic air elements} \]

\[ \text{infinite elements} \]

\[ \text{infinite elements} \]

\[ \text{gap, half width, } g \]

\[ \text{comb height, } a \]

\[ \text{air height} \]

\[ \text{wall angle} \]

\[ \text{air height} \]

\[ \text{infinite elements} \]

\[ \text{infinite elements} \]

\[ \text{infinite elements} \]

Figure 4.9: Comb electrode modelling in ANSYS

4.3.1 ANSYS Electrostatic Simulation of Comb Electrodes

ANSYS electrostatic models require a certain amount of care at the design stage. When doing any electrostatic simulation it is the air surrounding the electrodes that is modelled rather than the electrodes themselves. If too little air is modelled the answers will be inaccurate; too much and the simulation time will be prohibitive. The element size will also have an impact. There are two possible methods for determining the validity of the results.

1. Increase the amount of air surrounding the electrodes until the calculated force or
energy reaches a constant value. This method will test for convergence.

2. Use an energy calculation from the conformal mapping of coplanar electrodes and then increase the amount of air in the simulation until the value converges to the analytic answer. The same amount of air required for the simulation can then be applied to the trapezoidal electrodes. This method could be used to test accuracy but only for a simple problem where the conformal mapping is known to give an accurate energy value.

The ANSYS guides do not specify how much air to model, and so tests were carried out to answer this question for the case of trapezoidal electrodes. The graph in fig(4.10) shows how the calculated electrostatic force on each half-electrode varies with increasing amounts of air and at different mesh sizes. This simulation was for a comb half width of 2μm (\frac{1}{2}b in fig(4.9)), comb electrode height of 12μm (a in fig(4.9)) with a 6° wall angle and a 4μm gap. With more than 12μm of air there is little change in the force. From these results 12μm or more of modelled air will produce the best results. Therefore at least 12μm of air will be modelled in future simulations together with a mesh element size of 0.6μm.

![Effect of varying mesh size and air on force calculation](image)

Figure 4.10: Effect of varying mesh size between 0.2μm and 1.8μm and amount of modelled air on force calculation in ANSYS
4.4 Results for Approximate Analytical Model

Fig(4.11) shows a comparison of the force contributions calculated using the stepped side-wall model and ANSYS for a fixed electrode half width of 6μm while varying the wall angle between 0° and 12°. The contribution from the top side using the analytic model is constant as expect, while the ANSYS result matches the analytic results very closely. The bottom side trends are similar for the analytic and ANSYS result, as the wall angle increases the bottom electrode width decreases with an increasing gap and this is represented as a decreased force with wall angle. As for the sloping side-wall the method using the ratio \( \frac{d_0}{g_0+g} \) shows a good match with ANSYS however as the wall angle increases it can be seen that these two values start to diverge.

![Graph showing force contributions and total force with wall angle at constant electrode width](vf-sep-sides01.ps)

Figure 4.11: Force contributions and total force with wall angle at constant electrode width

The total force results for the stepped side-wall simplification, as shown in fig(4.12), has a similar trend to those obtained using ANSYS. At a comb half width of about 4μm the total force results start to diverge reaching a difference of about 20% at 6μm comb half width. The results from the sloping side-wall show a similar trend to one another and as expect the overall force does not change a great deal as the comb width increases.

**Discussion** By analysing the individual force components it can be seen that the analytic method, using a channel to represent the top and bottom side electrodes and a stepped side-wall channel for the sloping side wall, correlates well with the ANSYS
Figure 4.12: Force contributions and total force with electrode width at a constant wall angle results. However when all three forces are combined the results look less impressive since the sum of the errors become more significant.

4.5 The Schwarz-Christoffel Toolbox

The Schwarz-Christoffel (SC) Toolbox for MATLAB is a set of MATLAB functions that calculates the conformal maps from the disk, half-plane, strip, and rectangle domains to polygon interiors [4]. To deal with trapezoidal electrodes the toolbox calculates the mapping points for the corners of the polygon into the rectangular domain.

A compound form of Gauss-Jacobi quadrature is used to evaluate the Schwarz-Christoffel integral numerically. A predecessor to the Matlab Schwarz-Christoffel Toolbox describes the numerical computation of the Schwarz-Christoffel transformation using this technique [5].

A Gauss-Jacobi quadrature formula can convert an integral such as $\int_{-1}^{1} f(x)(1-x)^a(1+x)^b dx$, which is a common form of the Schwarz-Christoffel transform, into a summation $\sum_{i=1}^{N} w_i f(x_i)$. In a traditional Gauss quadrature formula the values $w_i$ are often obtained from formulae books [6]. The advantage of Gauss quadratures is their ability to cope with singularities. Further information of Gauss quadratures can be found in [6, 7].
4.5.1 Notes on Usage

Fig(4.13) illustrates what can be achieved with the MATLAB Toolbox. Since the toolbox deals with the whole problem it is expect to be more accurate than the previous methods. More importantly it allows for the case of trapezoidal electrodes, that are no longer coplanar, to be modelled.

![Diagram](image)

Figure 4.13: Mapping between trapezoidal and rectangular domains using SC Toolbox

In this section the toolbox usage will be briefly explained. Starting in the $z$-plane the vertices and angles of the electrodes are first described. For example, table 4.2 shows the entries for a set of electrodes with a comb half width of 5$\mu$m, a gap of 4$\mu$m, a height of 12$\mu$m and a wall angle of 6° (or $\frac{\pi}{30}$). The points are described as complex numbers, while the angles are calculated as multiples of $\pi$.

<table>
<thead>
<tr>
<th>z-point</th>
<th>angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+j\infty$</td>
<td>0</td>
</tr>
<tr>
<td>$-7 + 6j$</td>
<td>+0.5</td>
</tr>
<tr>
<td>$-2 + 6j$</td>
<td>+1.5 + $\frac{1}{30}$</td>
</tr>
<tr>
<td>$-3.254 - 6j$</td>
<td>+1.5 + $\frac{1}{30}$</td>
</tr>
<tr>
<td>$-7 - 6j$</td>
<td>+0.5</td>
</tr>
<tr>
<td>$-j\infty$</td>
<td>0</td>
</tr>
<tr>
<td>$+7 - 6j$</td>
<td>+0.5</td>
</tr>
<tr>
<td>$+3.254 - 6j$</td>
<td>+1.5 + $\frac{1}{30}$</td>
</tr>
<tr>
<td>$+7 + 6j$</td>
<td>+1.5 + $\frac{1}{30}$</td>
</tr>
<tr>
<td>$+7 + 6j$</td>
<td>+0.5</td>
</tr>
</tbody>
</table>

Table 4.2: Description of points from SC toolbox

After the points and angles have been defined the problem can be solved using the `rectmap` command. The whole problem can be solved with the following commands:
The resulting answer ($f$) is the map from the rectangular domain to the trapezoidal domain.

If the transformed points in the rectangular domain ($w$) are required from known points in the $z$-plane, the command evalinv is used (NB evalinv can be used to find any point in one domain to from the other one). The following code calculates 2 points in the $w$-plane from points in the $z$-plane.

$$ww = \text{evalinv}(f, [-7+6i; 7+6i])$$

To obtain the corresponding points in the $z$-plane from the points in the $w$-plane, the eval command is used:

$$zz = \text{eval}(f, [-\pi/2; \pi/2])$$

### 4.5.2 Calculating the Force from Schwarz-Christoffel Toolbox

Unfortunately one function the SC toolbox is unable to offer is force calculation, and since the calculations used to solve the SC transforms are numerical, explicit algebraic equations are unavailable.

For a given point in one plane the toolbox gives the transformed point in the other domain. In order to calculate the electric field, and hence the force, discrete points in the $z$-plane and the transformed points in the $w$-plane are required. $\frac{\Delta w}{\Delta z}$ can be calculated from these discrete points, this suggests a method for solving the force. Eq(4.12) shows the progression from the force integration to a summation, it must be remembered that $E_z = E_w \left| \frac{dw}{dz} \right|$ and that $E_w$ is a constant value found from the parallel plate configuration.

$$F_z = \frac{\varepsilon_0}{2} \int |E_z|^2 dz$$

$$F_z = \frac{\varepsilon_0}{2} \int \left| E_w \frac{dw}{dz} \right|^2 dz$$

$$F_z = \frac{\varepsilon_0}{2} \sum_i \left\{ \left| E_{w_i} \right|^2 \left| \frac{\Delta w_i}{\Delta z_i} \right|^2 \right\} \quad (4.12)$$
CHAPTER 4. ELECTROSTATIC ANALYSIS OF COMB DRIVES WITH ...

From eq(4.12) the terms $\Delta w_i$ and $\Delta z_i$ need to be found. Suppose the force is required on a top side electrode which exists between $2\mu m$ and $7\mu m$ in the $z$-plane. In this case we might use a constant $z$-domain step size, say, of $\Delta z = 0.1\mu m$. Applying the inverse transform to the range of values in the $z$-plane (e.g. $2\mu m$ to $7\mu m$ with a $0.1\mu m$ step size) will give $w_i$, the $z$-plane points transformed to the $w$-plane. The $\Delta w_i$ are then calculated from $w_i - w_{i+1}$. Together with $E_w$, there is enough information to find the force.

Since a numerical approximation is being obtained from another numerical approximation a degree of error may be evident in the answers.

4.6 Comparison of Analytic Results to the SC Toolbox

Previously in section 4.2.2 the problem of an open channel was solved and was found to have a closed form analytic solution. Force calculations using the SC toolbox will now be compared with this solution of a channel in order to validate the SC Toolbox method. Since the channel’s L-shaped electrodes extend to infinity, the SC Toolbox will map this onto two infinitely long parallel lines. It will be shown later that the trapezoidal electrodes will be mapped onto a rectangle.

**Method** A channel with a gap of $4\mu m$ was set up with at a potential of $30V$. ANSYS is not able to simulate the whole channel, which extends horizontally to infinity, but a large amount, $40\mu m$, was modelled in both directions. Analytically the force was calculated over a finite distance from the corner and with ANSYS the force was calculated over the same region. Comb half width of $2\mu m$ to $6\mu m$ were used, to simulate comb widths of $4\mu m$ and $12\mu m$ respectively.

**Results** As shown in fig(4.14) the forces calculated using the SC Toolbox compare favourably with the exact results. The ANSYS match less well and the reason for this is attributed to the fact that ANSYS is unable to simulate the whole channel in it entirety.

4.7 Improving the Schwarz-Christoffel Model

It was hoped that the SC toolbox would provide accurate results, but after viewing the unmodified SC results in fig(4.16), it is apparent that the model of the trapezoidal electrodes does not produce accurate result as in the case of the channel. The main reason for this is due to a different type of mapping. The trapezoids are now mapped onto a rectangle while the channel was mapped onto two infinitely long parallel lines. A difference could then arise since the two transforms are mathematically different.
When inspecting the force contribution from the top side of the trapezoidal electrode, it was found to be greater than when compared with the equivalent topside force contribution of the channel simplification. A reason for this could be that there is a singularity at the corner of the electrode, and there are two methods that could be used to avoid this problem: (a) change the summation range to avoid the singularity, or (b) change the model to include smooth corners.

The above methods both have their advantages and disadvantages. For example, changing the summation range does not require any modification of the transform, but it is necessary to choose an appropriate avoidance distance (distance away from the singularity). Ideally there should be a golden value applicable to all cases, but this may not be true in practice. With the addition of smooth corners this problem is avoided, but the Schwarz-Christoffel transformation is based on polygons rather than smooth curves, so a curve has to be defined with many points.

The simplest case of smoothing is to introduce a bevel at the corner, and this was investigated with a view to improving the SC Toolbox results. The analytic model was used as the benchmark, the justification for this being the close agreement between the analytical closed form results and ANSYS in this case of the open channel (with fig(4.14).
Method The model was redrawn with the corners clipped as in fig(4.15). The amount of bevel used depended on a fraction of the top side electrode length. More of the corner was removed until the force calculations for the top side of the electrode came within 10% of the channel force calculation. The bottom corner was then clipped by the same fraction.

Results Fortunately a bevel of $10^{-4} \%$ of the length of the top side of the electrode was enough to bring the results within the 10% limit of the channel force calculation. The overall results for a corner with $10^{-4} \%$ show about a 20% deviation over the ANSYS results. Reducing the bevel to $5 \times 10^{-5} \%$ of the top side length provides a better match with ANSYS. The modified values are displayed in fig(4.16).

4.8 P-method Electrostatic Analysis

A second way of simulating electrostatic problems in ANSYS is possible, namely the p-method. To calculate electrostatic problems, the p-method employs higher order polynomial levels (p-levels) of the finite element shape functions to approximate the real solution. The finite elements used in the previous models only used quadratic elements (though linear ones were available).

The p-method works by taking a finite element mesh, solving it at a given p-level, increasing the p-level selectively, and then solving the mesh again. After each iteration the results are compared for convergence against a set of convergence criteria. In this particular case these criteria are applied to the electrostatic force in the y-direction. The higher the p-level, the better the finite element approximation is to the real solution.
Figure 4.16: Comparison between SC Toolbox and ANSYS for trapezoidal electrodes

However, when using the p-method one feature is unavailable - the use of infinite elements. As previously stated if enough air surrounding the electrodes is modelled the overall simulation can be quite accurate. However it was found that ANSYS would not converge or it would crash when more than 3 times the electrode height was included, corresponding to about 36µm of air. Despite not using infinite elements the results show a correlation between the p-method and the previous ANSYS simulations, see fig(4.16).

4.9 Comparison of Methods

Several different methods have been use to calculate the electrostatic forces on trapezoidal electrodes. ANSYS (using the h-method) has always achieved the most reliable results. In tests carried out on an open channel geometry, which has a simple closed form formula for the force, eq(4.9), the analytical and FEM simulation results showed very close agreement as shown in fig(4.14). FEM simulations using the p-method showed similar trend to those of the h-method. However without the use of infinite elements to model the field to infinity, the p-method simulation model was somewhat less accurate.

The approximate analytical methods using conformal transformations have had partial success. The general trend of the force as the comb width increased followed a similar trend to the ANSYS results, though the results showed a 20% deviation at a comb half width of 6µm.
The SC Toolbox gave mixed results, as shown in fig(4.16). The initial model without any modifications produced a total force contribution some four times less than the ANSYS ones. However by bevelling the corners by a small amount, the results could be brought within 10% of the ANSYS results. The amount of bevel was found by comparing the top side force with the calculated analytical method of the stepped side-wall channel. The bevel was increased until the analytic and SC Toolbox value for the top side of the electrode were within 10% of one another. A $10^{-4}$% bevel gave a result that compared well with the analytic result but by reducing this by half gave an overall force result that compared better with ANSYS.

4.10 Electrostatic Forces on Non-Coplanar Electrodes

Up to this point it has been assumed that all the trapezoidal comb electrodes are coplanar. In this section the problem is taken a step further by considering the case where there is some out-of-plane deflection. Typically the electrodes are attached alternately to the substrate and to a mass on a flexural suspension. In this configuration any out-of-plane forces arising from the applied voltage will cause vertical displacement of the mass. To calculate the new equilibrium position the variation of the electrostatic force with displacement must be known.

The approximate analytical method developed in section 4.1 cannot deal with the non-coplanar case. However, both the SC Toolbox and ANSYS can be applied, and the results are compared below. Both methods can also deal with the important case where there is a ground-plane beneath the electrode array. This applies to most devices fabricated by sacrificial layer processing.

4.10.1 Without a Ground-Plane

Out-of-plane forces were calculated using ANSYS and the SC toolbox over a range of displacements, and the results were compared. An electrode height of 12μm was assumed, and displacement in the rate of -12 to +12μm were considered.

Method The ANSYS simulation was straightforward: all that was required was to redraw one electrode with the required displacement and then re-run the simulation. The SC calculation was also easy to implement, only requiring the model to include a relative displacement between the electrodes. Fig(4.17) shows a typical field plot obtained from the SC Toolbox.
CHAPTER 4. ELECTROSTATIC ANALYSIS OF COMB DRIVES WITH...

Figure 4.17: A typical field plot obtained from the SC Toolbox

Results Fig(4.18) shows the results obtained for two cases having the same gap and wall angle, but different electrode half-widths. In both cases the ANSYS and SC Toolbox calculations show very close agreement for relative displacements below 4μm. However, when the right-hand electrode (on which the force is calculated) is displaced in the negative direction beyond about -4μm the results start to diverge. This effect is less pronounced for positive displacements, with good agreement even up to the point where the electrodes no longer overlap.

Figure 4.18: Vertical force with displacement for trapezoidal electrodes without a ground-plane, calculated using ANSYS and SC Toolbox
For small displacements between -4µm and +4µm the vertical force looks like a linear restoring force, tending to bring the electrodes back into alignment. At larger displacements <±10µm the restoring force falls off because the electrodes are no longer overlap by a significant degree.

The ANSYS results are not symmetrical like the SC Toolbox results i.e. forces for the positive displacement are large in magnitude than for negative displacements when a point at +8µm and -8µm are compared. This discrepancy is due to the asymmetry in the ANSYS simulation rather than any other cause.

4.10.2 With a Ground-Plane

The presence of a ground-plane can have a dramatic effect on the vertical force. The reason for this is easily explained with the aid of fig(4.19), which compares field plots for coplanar electrodes with and without ground-planes. In both diagrams the right-hand electrode is held at zero potential, while a positive voltage is applied to the left-hand electrode. The ground-plane if fig(4.19b) is held at zero volts.

![Equipotentials without ground plane](image1) ![Equipotentials with a ground plane](image2)

Figure 4.19: Equipotential field plots for comb electrodes with and without a ground-plane (electrode half width 6µm, gap 4µm, 30V applied to left-hand electrode)

We see that in the presence of a ground-plane the region around the bottom of the grounded electrode is almost equipotential, indicating that the electric field in this region is negligible. In physical terms this is because the electric field lines emanating from the bottom of the high potential electrode flows directly to the ground-plane, and never reach the grounded electrode. Consequently the vertical force on the grounded electrode has contributions only from the sloping side-wall and the topside [8].

**Method** In order to make use of the SC transformation from a polygon to a rectangle, (the rectangle having a simple electric field), the modelled problem has to be slightly
changed. The issue here is that the rectangle domain has only two electrodes, while adding the ground-plane introduces a third, making the problem intractable by the SC method. Fortunately a simple solution is suggested by fig(4.19): the region beneath the grounded electrode is almost equipotential, so shorting this electrode to the ground-plane should have negligible effect on the solution, while at the same time reducing the number of electrodes to two. This idea is illustrated in fig(4.20a), where an additional section of conductor has been inserted between point $d$ and $e$.

Figure 4.20: SC model including ground-plane

An alternative method could be to extend points $d$ and $e$ to infinity, as shown in fig(4.20b), but still have points $c$ and $h$ mapping to two corners of the rectangle.

Results Fig(4.21) shows the results of ANSYS and SC Toolbox force calculations including a ground-plane. The electrode parameters were as in fig(4.18), the ground-plane was positioned 3$\mu$m below the electrodes, and for the SC calculations the model in fig(4.20a) was used. Note that in both cases there is a significant upward force on the grounded electrode at zero displacement. This arises because the downward force on the bottom of the electrode is suppressed by the ground-plane, and so cannot balance the upward force on the top surface. The general trend of force versus displacement is similar to that seen without a ground-plane. However, the agreement between the ANSYS and SC Toolbox results is less good, particular for large positive displacements.

The results for the case where the grounded electrode and the ground-plane are extended to infinity in the SC Toolbox model are given in fig(4.22). This model clearly gives closer agreement with ANSYS in the positive displacement region of the graph.
Figure 4.21: Vertical force with displacement in the presence of a ground-plane; SC calculation based on model of fig(4.20a)

Figure 4.22: Results obtained using SC model of fig(4.20b), with right-hand electrode and ground-plane extended to infinity
From fig(4.23) it can be seen why the SC ground-plane model breaks down at large displacements. In fig(4.20) a connection was made from the electrode to the ground-plane to maintain the transformation from polygon to rectangle. This is expected to give reasonably accurate results while the region beneath the grounded electrode remains equipotential as in fig(4.23a). However, this assumption breaks down as the displacement is increased, as shown in fig(4.23b) and fig(4.23c). This effect is more pronounced for narrow electrodes.

Extending the grounded electrode to infinity improved the model to a certain degree. However with small electrode widths (eg 2μm) the results are only really accurate to about 2μm displacement.

![Figure 4.23: Equipotential field plots for different displacements with ground-plane](image)

4.10.3 Conclusion

From the previous results it is clear the SC Toolbox can be applied successfully to the electrostatic force problem when the electrodes are out-of-plane. The results from ANSYS and the SC Toolbox show close agreement for the case where there is no ground-plane.
At 30V the static deflection is around -40nm, a very small amount to measure. The simulations were re-run however the voltage was doubled to 60V. This time the deflections are more reasonable -0.12μm without a ground-plane and 0.5μm with a ground-plane.

### 4.12 Summary

The main aim of this chapter was to calculate the electrostatic force of trapezoidal electrodes. The first method used a stepped side-wall employing the close formed solution for the force of a channel. The individual force contributions matched well with the results obtained with ANSYS. The total force contribution followed a similar trend with ANSYS but had a roughly 20% difference.

The Schwarz-Christoffel toolbox for MATLAB was then used, using the methodology gained in the previous attempt and a few extra calculations the electrostatic force could be obtained from the SC toolbox. At first the results were worse than the ANSYS model and the stepped side-wall model, but this was soon attributed to the corners of the electrodes. With the help of the close formed channel force a small amount was clipped from the corners of the electrodes until the top side force matched the analytical ones. After this slight adjustment the result compared favourably with ANSYS.

The next step was to model the electrostatic forces when the comb electrodes move in and out-of-plane. This problem was modelled using ANSYS and the SC-toolbox, in one
situation having a ground-plane and another without a ground plane. The final analysis involved estimating the out-of-plane displacement of the comb electrodes when there is a restoring force supplied by support beams.
Bibliography


Chapter 5

Electrostatic Results

Chapter 4 described methods for calculating the electrostatic forces for trapezoidal electrodes. In this chapter the results obtained using ANSYS and the Schwarz-Christoffel Toolbox are compared with a real world device.

5.1 Description of Electrostatic Devices

To test the simulation models two devices were fabricated by electroplating into UV-exposed photoresist moulds, with a sacrificial photoresist layer. The sacrificial layer allows the movable part of the device to be suspended above the substrate surface. Fig(5.1a) shows the CAD layout of a device supported by two beams while fig(5.1b) shows a device supported by four beams.

![CAD layouts of nickel devices](image)

(a) device supported by two beams, nickel device 1

(b) device supported by four beams, nickel device 2

Figure 5.1: CAD layouts of nickel devices
Nickel device 1 employs two support beams as the intension was to drive this device torsionally as well as out-of-plane. However the torsional behaviour of this device was not investigated. Nickel device 2 is supported by four beams and the intension here was to drive this device only out-of-plane. Doubling the number of support beams leads to a stiffer device but helps it endure the rigours of the fabrication process. Fig(5.2) shows the main mass, support beams, lattice area, and support pads of nickel device 1 without the extra drive electrodes.

![Figure 5.2: The main components of nickel device 1 without the drive electrodes](image)

To measure the small displacements a ZYGO interferometer was used. This piece of equipment can measure surface topography using interferometry. White light interferometry provides measurements by recording interference patterns formed by light reflected from the sample surface and a reference surface. The optical interference between the two reflected beams is related to the surface height which is given as a function of the light intensities from the sample surface and the reference surface.

### 5.2 Initial Process Development

In this section the process development for the nickel structures will be described. A few difficulties were encountered during the fabrication process; these will be outlined and methods used to overcome them will be described.

A key step involved in the photoresist sacrificial layer process requires thermally evaporating metals onto photoresist. The main problem encountered here is due to the high temperatures involved (copper melts at 1085°C) and the tendency of photoresist to reflow and/or out gas at elevated temperatures. In the evaporation chamber the photoresist is far away from the heated source but the elevated temperatures can cause the photoresist to reflow. Depositing copper at low rates, in order to keep the temperature down, would seem to be the best way to approach this problem. However the wafer is still exposed to the heat for a long period allowing for the photoresist to out gas or reflow.
It was observed many times that bubbles were formed on the wafer when evaporating at low rates for extended periods (see fig(5.3)), suggesting that out-gassing was the cause of the problem. It was found that evaporating at a high deposition rate $10-12\text{Å/s}$, rather than a slow $3\text{Å/s}$, alleviated this problem. Most probably the short time prevented the resist from melting too much or out-gassing before the surface was sealed with copper.

![Blistering of photoresist after copper evaporation](image)

Figure 5.3: Blistering of photoresist after copper evaporation

Another limitation was found with the design of the chrome mask. After exposing and developing the photoresist it was found that the comb gaps of the initial trial nickel device designs were too narrow. The comb electrodes were designed with a $4\mu\text{m}$ gap. However, after exposing and developing the electrodes the gap became narrower. This widening of the openings in the resist mould is attributed to a mixture of diffraction, over exposure and partial dissolution of unexposed photoresist. The $4\mu\text{m}$ gap became on average $2.5\mu\text{m}$, and with these small widths it became difficult to electroplate the devices without any bridged electrodes (see fig(5.4b)). Once the successful devices were released the small gaps made it easy for the comb electrodes to touch one another and then become welded when a driving voltage was applied.

A second limitation of the first designs was the design of the supporting structure. An oversight meant that the designs did not have a solid mass joining directly to the support beams. The movable structure was instead joined to the beams via a frame, as shown in fig(5.5). At first this would not seem to be a problem but with the internal stress it was found that the device would distort. As the whole structure shrunk it would pull on the frame causing it to change shape. The only solution to the above two problems was unfortunately to redesign the mask.

The initial idea for the first generation of devices was to remove the substrate from beneath the comb electrodes. This was tackled by etching windows from the back side
of the wafer. The main problem encountered here was that the electroplated nickel was starting to be etched after prolonged exposure to the EDP etch. One of the components of EDP is catechol and it was found that after long etches catechol would deposit itself on the nickel, forming pink flakes. These flakes were difficult to remove and more problematically it appeared that they attacked the nickel. Fig(5.6) shows a nickel pad after 4 hours in EDP and after the catechol flakes have been removed. The nickel has been etched leaving what can only be described as crows feet impressions. To reduce the exposure time of the nickel to EDP the wafers were first etched half-way through from the back-side, so that later etching cycles would be reduced. However since the back side of the wafer was no longer flat it was not easy to hold the wafer on vacuum chucks. When these wafers were spun with photoresist the wafer would often detach from the chuck with disastrous consequences. To avoid these problems a sacrificial layer process was developed as described in the next section.
5.3 Nickel Device 1

Fig(5.7) shows a interferometric scan of a small section of comb electrodes for the first nickel device. At rest with no driving voltage the comb electrodes are not coplanar, with the suspended structure sagging below the fixed electrodes by a few microns. This unwanted deflection in the structure has been attributed to stresses in the electroplated nickel.

5.3.1 Out-of-plane Measurements

The static out-of-plane displacements were measured using the ZYGO interferometer for a range of voltages between 0-63V, fig(5.9a). To compare the recorded displacements with both the FEM simulations and the Schwarz-Christoffel results the following steps were taken. A 2D force analysis of a pair of combs was performed over a range of distances, -3.5μm to 0μm, the same range as the measured displacements and over a range of
voltages. The force per unit length per gap was then multiplied by the overlap length of the comb electrodes and then by the total number of electrode gaps, as described in chapter 4. For each voltage the force-distance relationship was then plotted on a graph. ANSYS was then used to mechanically model the whole device with the aim of finding the force versus distance relationship (stiffness) at the same point where the ZYGO measurements were taken, fig(5.9b). This was required since the lattice structure is flexible and simply taking the support beam stiffness is incorrect. This force-distance relationship was then plotted on the same graph as the electrostatic force curves. The position where the stiffness line intersected the force curves gave the simulated vertical displacement for a given voltage. These results can be compared with the ones obtained using the ZYGO.

5.3.2 Mechanical and Electrostatic Modelling in ANSYS

A device model was designed in ANSYS with the critical dimensions, such as the beam dimensions and device thickness, obtained using the ZYGO rather than from the original mask designs. The alignment feature, fig(5.10a), shows the individually electroplated nickel layers and from fig(5.10b) the height of the devices is approximately 10.4μm while the sacrificial layer is approximately 3.6μm. From the interferometer several important beam and comb electrode dimensions were gathered and are shown in table 5.1. The mask dimensions could not be used since during the lithographic process the line widths increased due to diffraction and dissolution of unexposed photoresist.
(a) height profile measurement using the ZYGO

(b) ANSYS stiffness simulation

(c) device with electrical contacts

Figure 5.9: Nickel device 1 at different stages of analysis

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</tr>
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<tr>
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<td>320</td>
</tr>
</tbody>
</table>

Table 5.1: Important dimensions of nickel device 1

The device features a lattice area, as shown in fig(5.11b), and rather than modelling it as a solid entity it was redesigned in ANSYS as a series of beams. Beams modelled in ANSYS only require two points and a line, while the thicknesses and moments of inertia are specified as meshing constants. This simplification reduces the number of nodes which in-turns reduces the computation time without loss of accuracy.

In ANSYS forces were then placed at a series of points at the same location as the comb
5.3.3 Results

Fig(5.9b) shows the modelled device in ANSYS with all the dimensions accurately measured from the optical interferometer. A total force of 50μN force was applied to the perimeter of the mass and the overall displacement at a corner of the mass was 1.98μm (the same place where the ZYGO measurements were performed): this gave a stiffness of 25.25Nm$^{-1}$. 
(a) ANSYS results: 6° wall angle

(b) ANSYS results: vertical wall angle

(c) Schwarz-Christoffel results: 6° wall angle

Figure 5.12: Vertical force with displacement at different voltages for nickel device 1
CHAPTER 5. ELECTROSTATIC RESULTS

Fig(5.12) shows a series of curves for the vertical electrostatic force for a series of displacements at different voltages, obtained using ANSYS and the SC Toolbox. Overlayed onto the graphs is the apparent stiffness of nickel device 1, obtained by mechanically modelling the whole device in ANSYS. The intersections give the expected vertical displacements corresponding to the different applied voltages. A total of three plots are shown in fig(5.12): two are models of trapezoidal electrodes fig(5.12a and 5.12c) and the other is for rectangular electrodes (fig(5.12b)).

The resulting displacement versus voltage curves for nickel device 1 are plotted in fig(5.13), together with experimental measurements. The experimental measurements have a certain error. The main contributor is the surface roughness of the electroplated nickel and to a lesser extent the limitation of the ZYGO interferometer. The electroplated nickel has a roughness of approximately ±0.2μm, while the ZYGO has a quoted resolution of ±10nm. The error in the measured voltages is too small to be considered as a 3 decimal point voltmeter was employed. The measured out-of-plane displacements tie in very well with the simulated results obtained using ANSYS. At voltages below 30V very little movement occurs, while above 30V the displacement has an almost linear relationship with the applied voltage. The results using the SC Toolbox are also in close agreement. As explained in chapter 4, the SC Toolbox model of the electrodes with a ground plane is only valid for a short distance above the ground-plane. With a comb half-width of 2μm, a gap of 4μm and an electrode height of 12μm, the limit of validity was about 5μm above the ground plane when the ground plane was at -3μm. For the same parameters, except with a half comb width electrode of 6μm, the limit was about 11μm above the ground-plane. Fortunately the displacements achieved from nickel device 1 were within the correct limits to achieve good accuracy.

5.4 Nickel Device 2

Nickel device 2 has a suspended nickel lattice supported by four, 500μm long beams. Each side of the lattice has a total of 116 comb electrodes. Using the ZYGO the following data was extracted in a similar fashion to nickel device 1.
This device unfortunately had a 10.4μm initial displacement and so the moving electrodes sat above the fixed electrodes. ANSYS was used as before to find the stiffness of the device at the mid point along one side of the suspended structure; this was found to be 85Nm⁻¹. The vertical displacement measurements with respect to applied voltage were taken at the same mid point along one side of the lattice.

The simulated displacement versus voltage curves for nickel device 2 are shown in fig(5.16), along with the experimental measurements. The simulated curves were derived from the graphs in fig(5.15).
CHAPTER 5. ELECTROSTATIC RESULTS

Figure 5.14: Nickel device 2 at various stages of analysis

Figure 5.15: Vertical force with displacement at different voltages for nickel device 2
Figure 5.16: FEM, SC Toolbox and experimental results for displacement versus voltage for nickel device 2

5.5 Process Development: Sacrificial Layer Process

Process Overview A 3 inch diameter silicon wafer is thermally oxidised and coated with a chrome adhesion layer and a copper seed layer (step 1 in fig(5.17)). A thin resist layer is spun onto the wafer (2). The features for the nickel anchors are exposed into the resist (3), and the wafer is then developed (4) and then electroplated (5). A second copper seed layer is deposited (6), and a thicker layer of photoresist is deposited. This layer is exposed with the features of the moving parts (7), developed (8) and finally electroplated (9). The resist layers and copper seed layers are then removed by wet processing and the sample is freeze dried to release the structure (10).

5.5.1 A Detailed Process Development of Nickel Electroplated Structures

It was found that oxidising the wafer at 1100°C for 12 hours in oxygen was sufficient to produce an oxide thickness of 0.4µm. This layer would electrically isolate the drive combs from the movable structure when driving it with a voltage. The oxide could be grown at 1200°C in half the time, but the quality of the oxide obtained at this temperature was very poor and lead to pinholes when silicon wet etching was used.

The first chrome and copper metal layers were used as a seed layer for electroplating nickel. A double seed layer is required since chrome adheres well to oxide but not to
nickel while copper adheres well to both chrome and nickel but not to oxide. Thermal evaporation and sputter coating was used for this process but it was found that sputter coating yielded better results. Cleanliness of the machine was probably the biggest factor with the adhesion. The Nordiko sputter coater used for this work has a turbo-molecular pump, whereas the evaporator has an oil based diffusion pump which can introduce oil into the chamber if back streaming occurs contaminating the wafer.

A standard Shipley photoresist was then used to define the anchor layer. Shipley photoresist S1828 produces approximately 3μm of resist when spun at 4000rpm for 35-40s. Either oven backing or hotplate baking was suitable for curing the photoresist, but it must be noted that oven backing takes much longer 30 minutes at 90°C, while only 90-120s is required for hotplate baking at the same temperature. Before spinning the photoresist the wafer was left on the hot plate for about 15-30 minutes at 100°C to drive off any moisture which could hinder adhesion.

After baking the photoresist the anchor features were exposed using a Quintel mask aligner, each exposure being calibrated on a test wafer to minimise over-exposure. The photoresist was developed using undiluted M319 developer, a standard alkali developer, for about 2 minutes.
Before electroplating the anchor layer the wafer was first baked again to drive off any moisture absorbed while developing and rinsing. Again 15-30 minutes on a hotplate at 100°C was used. This bake also helps harden the photoresist and creates a sturdier structure that can resist the rigours of electroplating. Just before electroplating the wafer was dipped in a 10% solution of sulphuric acid for 30s to remove any oxide formed on the exposed copper. Deposited layers of copper oxidise quite readily in the open air and to ensure a good bond between the electroplated nickel and copper this must be removed. After the acid dip the wafer was kept in water to prevent further oxidation.

The wafer must be connected to the cathode (negative terminal) and this was done using a simple metal contact onto an exposed pad of copper. It is important for this contact not be overly exposed to the electrolyte, as this area will receive preferential plating. Before placing the wafer into the plating solution the contact resistance was measured to ensure there was a good contact.

The amount of nickel deposited depends on the current density and the area of exposed copper. The nickel sulphamate solution has been calibrated to deposit 12μm per hour at 10mA/cm², so an accurate area measurement is required. For this a computer program was written to read in the mask data file and then to calculate the total area of the features. To encourage the electroplating process the current density was increased by 50% for the first 15 seconds and then brought back down to the correct value.

After the electroplating the sample was rinsed in copious amounts of water because nickel sulphamate dehydrates to form crystals that are difficult to remove. The wafer was then baked again for another 30 minutes at 90°C before evaporating copper for the next level of features. (Depending on whether some time has passed before the evaporation is done, an acid dip may be required to remove any native nickel oxide). Evaporating copper at a fast rate, 10-12Å/s, produced around 1500-2000Å thickness, and was preferable to evaporating at a lower rate for periods greater than 10 minutes (see earlier discussion in section 5.2).

Creating the second layer of features proceeded in a similar way to the first. Instead of using Shipley S1828 photoresist, however, Hoechst AZ4562 was used. As explained in chapter 1 this resist is able to form relatively thick films. Spinning at 1200rpm for 30s was found to produce layer of almost 15μm thickness. With a safety margin of 3μm to avoid overplating this produced devices that were approximate 12μm thick.

Exposure of AZ4562 is a critical point. Thick photoresists tend not to be very flat and, together with the underlying layer of photoresist, the electroplated features and the edge bead, the top surface tends to be fairly uneven. When the mask aligner is set to vacuum mode the pressure behind the wafer is increased, so that there is a good contact between the wafer and the mask. If this is not done fine features such as comb electrodes with gaps of less than 4μm are not defined very well and after electroplating bridging can
occur. After exposure the photoresist was developed using dilute AZ400K (1 developer: 4 water) for about 4 minutes. The wafer was then electroplated forgoing the extra baking phase as was done with S1828 resist. Re-baking AZ4562 after developing can cause the photoresist to reflow.

After the second electroplating step the wafer was usually diced into smaller portions. To remove the photoresist the samples were refluxed in acetone at 65°C for one hour. The copper seed layer was then removed using an aluminium wet etchant supplied by Laporte Electronics. This etchant works well on copper and the main ingredients include 4 parts H$_3$PO$_4$, 4 parts acetic acid, 1 part HNO$_3$ and 1 part water. The S1828 sacrificial layer of photoresist was removed in the same way as the Hoechst resist, but left refluxing for longer (2hrs) to ensure that all the photoresist under the suspended structures was removed.

The first copper layer was then removed using the same aluminium etchant. Previously a standard chrome etch was used to remove the chrome seed layer. However, it was found that it also attacked the copper seed layers causing delamination. Another method used to remove the chrome was sputter etching, but this had the detrimental effect of contaminating the sputter coater. Finally an etch based on potassium ferro-cyanide was found that had the required selectivity.

To release the structures freeze drying was used. This process involves using a liquid that can displace the liquid trapped beneath any suspended structures, will not contract or expand too much while freezing, and can sublime. A solution of 1 part methanol to 4 parts water was used for this process. De-ionised water was found not to be pure enough for this purpose: when the solution sublimed under low pressure and temperature, impurities where left on the sample. As the solid sublimes the impurities are left behind eventually becoming deposited, as a white flake, uniformly over the sample. Distilled water reduced this problem but the simplest solution to this problem was to freeze dry the samples upside down.

5.6 Conclusion

Two devices have been fabricated, using electroplated nickel with a photoresist mould, to test the FEM and SC models developed in the previous chapter. The FEM model results for nickel device 1 show a very good correlation with the results obtained from the actual device. With nickel device 2 at low voltages (<50V) the correlation seems good (<4% difference) but as the voltages reaches 63V the deviation is almost 15%. The results obtained from the Schwarz-Christoffel Toolbox also match well with the results from nickel device 1, even though parts of the model had to be simplified to maintain
conformal transformation to a rectangle (see chapter 4). The main reason for the good correlation with the SC Toolbox is that the comb electrode movement is small and close to the ground-plane. The results from nickel device 2 do not correspond as well to the measured results since the movable comb electrode is far above the ground-plane and above the fixed electrodes. In this region, as expected, the SC Toolbox result is not very accurate, resulting in a deviation of about 7%.

During the process development for the test devices, a number of key issues had to be addressed. One factor was the effect of processing on the comb electrode gaps: during UV exposure and development it was found that the line widths increased reducing the comb electrode gaps, and this had to be taken into account at the mask design stage. Another factor is the type of wet chemical etchants used: it is important to have selective etches, for example it was shown that EDP would attack nickel after prolonged exposure. The final problem that was overcome was the difficulty in depositing metals onto photoresist: faster deposition rates require higher temperatures but the underlying photoresist did not bubble or reflow when compared with low temperatures and slow deposition rates.
Chapter 6

Torsional Measurements

6.1 Introduction

In chapter 3 a method for calculating the torsional constant of a trapezoidal beam, by means of a $\beta$ correction factor, was developed. To test the validity of the results silicon mirrors were created using EDP etching, which produces trapezoidal support beams. The method used to drive the silicon torsional mirrors meant that it was not possible to use the ZYGO interferometer as in the previous chapter. Therefore rather than measuring static deflections of the beam, resonant frequencies were used as a comparison. Driving a device at its resonant frequency has the advantage that the amplitudes can be much greater than when driven statically, therefore requiring a lower level of sophistication for deflection and detection.

Without knowing exact values of Young's modulus and without taking into account the slight variations of thickness of the EDP etched silicon, it is expected there will be some variation in the results.

6.2 Device Descriptions

Several different types of mirror were designed and fabricated using EDP etching, four of which were measured. Fig(6.1a,b) shows the actual devices etched into a silicon wafer and the process steps are explained later in this chapter.

Table 6.1 is a description of the dimensions of all the mirrors together with the calculated torsional frequencies. $a$ is the thickness of the mirror and hence the trapezoidal beam height; $b$ is the width of the wide base of the trapezoidal beams; $L$ is the beam length; $\beta$ is the calculated torsional factor as described in chapter 3; $I$ is the moment of inertia, $I_y$.
for mirrors 1 and 2 and $I_x$ for mirror 3; and $k$ is the calculated torsional constant using $k = \frac{Eb^3G}{L}$. Finally the resonant frequency, $f$, is then calculated using $f = \frac{1}{2\pi} \sqrt{\frac{2k}{I}}$. The torsional stiffness, $k$, and hence the frequency require the shear modulus $G$. The shear modulus can be calculated from $G = \frac{E}{2(1+v)}$, where $E$ is the Young’s modulus for silicon, 162.6GPa and $v$ is Poisson’s ratio, 0.21. This leads to a value for the shear modulus of $G = 67.2$GPa;

The torsional beams join to the supporting structures and the main mass at a wall angle of 57.4°. To take account of this the beam lengths were taken as the average length over the beam height.

Dimensional measurements were taken using an optical microscope with a CCTV attachment together with a digital measurement unit attached to an external monitor. The error margin of these measurements was estimated to be around ±2μm. The moments of inertia were calculated using ANSYS because the geometries were not simple enough to do by hand.

<table>
<thead>
<tr>
<th>mirror</th>
<th>a (μm)</th>
<th>b (μm)</th>
<th>L (mm)</th>
<th>$\beta$</th>
<th>$I$ (kg.m$^2$)</th>
<th>$k$ (N.m$^{-1}$)</th>
<th>calculated f (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>171</td>
<td>340</td>
<td>1.88</td>
<td>0.0982</td>
<td>3.7554E-11</td>
<td>0.0060</td>
<td>2827.7</td>
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<tr>
<td>2</td>
<td>175</td>
<td>433</td>
<td>2.38</td>
<td>0.1396</td>
<td>3.2180E-11</td>
<td>0.0090</td>
<td>3774.4</td>
</tr>
<tr>
<td>3</td>
<td>189</td>
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<td>0.88</td>
<td>0.1386</td>
<td>3.6496E-12</td>
<td>0.0329</td>
<td>21240.1</td>
</tr>
<tr>
<td>4</td>
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<td>390</td>
<td>1.88</td>
<td>0.0877</td>
<td>6.3522E-11</td>
<td>0.0108</td>
<td>2879.6</td>
</tr>
</tbody>
</table>

Table 6.1: Silicon mirror descriptions

### 6.3 Process Development for Bulk Micromachining

Using silicon as the bulk material, torsional mirrors were created by etching right through the wafer. The process for making these silicon mirrors requires very few fabrication
steps, and fortunately very few problems were encountered.

A 3 inch diameter silicon wafer was thermally oxidised at 1100°C in oxygen for about 12 hours (step 1 in fig(6.2)), and then coated with photoresist on both sides (2). The front was patterned with the structures and the reverse with the windows (3). The windows are required so that the wafer is locally thinned during etching. An average wafer is 400μm and this is too thick for creating usable movable structures. If the whole wafer thickness is used, then the support beams have to be made narrower and longer to achieve a similar torsional constant, and long narrow beams lead to devices that are too delicate. The exposed oxide was then removed in HF (4). After removing the photoresist the wafer was placed in the EDP etch. Etching occurred simultaneously from the front and back and the wafer was removed from the EDP when the wafer was etched right through (5).

A chrome layer of a few 100nm was then sputtered on the reverse side of the wafer to create a conducting layer. This was required for electrostatically driving the mirrors.

![Figure 6.2: Silicon process](image)

### 6.4 Experimental Setup for Testing

A specially designed split electrode was placed behind the mirror which allowed it to be excited torsionally rather than in and out of plane, fig(6.4). Both electrodes were driven driven with square wave signals. These were in anti-phase to one another to achieve a
torsional driving force. The mirror’s movement was recorded by reflecting a laser beam off the surface and measuring the deflection amplitude using the setup in fig(6.3).

![Diagram of torsional measurement setup]

Figure 6.3: The setup used for measuring the mirror’s deflection by deflecting a laser beam

To align the whole system the following procedure was used. First the mirror and the electrode had to be aligned so that their surfaces were parallel. This was done by reflecting the laser beam off the surface of the electrode only and then recording the position on the measurement surface. Then the silicon mirror was returned to the setup and again the laser beam was reflected off its surface. The mirror was tilted until the reflected beam was aligned to the first recorded spot on the measurement surface. When these two points were aligned it was safe to assume that the electrode and mirror were squarely aligned.

![Diagram of torsional actuation]

Figure 6.4: Torsional actuation

The electrode was placed 20μm away from the mirror. To achieve this distance the electrode was slowly moved forward using a micrometer screw gauge until it just...
touched the mirror. A small deflection of the laser beam was observed when the electrode just touched the mirror. From this point the electrode was then moved back by 20µm. A gap of 20µm was chosen as it was the smallest distance that could be reliably achieved using the micronometer screw gauge trolley.

6.5 FEM Simulations

Each torsional mirror was designed with the above parameters and including the 57.4° wall angle on the all the edges, leading to ANSYS design in fig(6.5). The resonant frequencies were obtained from ANSYS using a harmonic analysis, and fig(6.6) shows the desired harmonic for each of the mirrors.

![Figure 6.5: ANSYS modelling of the torsional mirrors](image)

(a) silicon mirror 1  
(b) silicon mirror 2

(c) silicon mirror 3  
(d) silicon mirror 4

6.6 Results

For mirrors 1 and 2 the calculated resonant frequencies, obtained using the β values in Table 6.1, showed excellent agreement both with the ANSYS simulations and the
(a) silicon mirror 1 2982Hz
(b) silicon mirror 2 3806Hz
(c) silicon mirror 3 19426Hz
(d) silicon mirror 4 3050Hz

Figure 6.6: Harmonic analysis of silicon mirrors
measurements on the real devices. However, the results obtained for the higher frequency device (mirror 3) and the device with the 'bow-tie' mirror (mirror 4) were less consistent.

<table>
<thead>
<tr>
<th>mirror</th>
<th>calculated f (Hz),</th>
<th>ANSYS (Hz)</th>
<th>measured (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2827.7</td>
<td>2982</td>
<td>3087</td>
</tr>
<tr>
<td>2</td>
<td>3774.4</td>
<td>3806</td>
<td>3768</td>
</tr>
<tr>
<td>3</td>
<td>21240.1</td>
<td>19426</td>
<td>23344</td>
</tr>
<tr>
<td>4</td>
<td>2879.6</td>
<td>3050</td>
<td>2540</td>
</tr>
</tbody>
</table>

Table 6.2: Comparison of measured resonant frequencies with values obtained by analytical approximation and ANSYS

![Graphs](a) silicon mirror 1  (b) silicon mirror 2
(c) silicon mirror 3  (d) silicon mirror 4

Figure 6.7: Frequency response of silicon mirrors

6.7 Discussion and Conclusion

At low frequencies the calculated values of the resonant frequencies for the rectangular mirrors were found to lie close to those values obtained through the torsional mirror
experiments. This agreement was particularly good for mirror 2, where all three methods were with 1% of each other. However for the higher frequency device it was found that the values did not match so well. Given that both the analytical and FEM calculations underestimated the frequency in this case, a likely explanation is that errors in the dimensional measurements were at least partly responsible; such error are expected to have a more pronounced effect in a smaller device.

Dimensional errors are also the most likely cause of the relatively poor results obtained for mirror 4. The mirror on the device has a bowtie shape, which was designed with angles other than 90°, and because of this it is difficult to accurately measure its dimensions. For example the wall angle on the sloped feature is no longer predictable at 35.26°. For this reason, vertical side-walls were assumed in the ANSYS model of the mirror. Etching also takes place on convex corners and as can be see in fig(6.8b) the corner has been slightly clipped.

![Image](image.png)

(a) reverse view of mirror 4  
(b) topside view of mirror 4

Figure 6.8: Mirror 4

In conclusion the use of the Ritz method to calculate the torsional constant of silicon beams has been demonstrated and comparisons made with FEM and a real device have shown consistently close results. Given more accurate data on the effects of processing on the device dimensions, it should be possible to create trapezoidal beams and to be assured of their torsional stiffness without having to resort to FEM modelling.
Chapter 7

Discussion and Conclusions

7.1 Summary of Achievements

7.1.1 Torsion

The influence of the side-wall angle on the torsional behaviour of micromachined beams was first investigated. The main implication of having a sloping side wall is its impact on the torsional stiffness of a trapezoidal beam. For example with only a 6° slope on a beam with an aspect ratio of one the reduction in stiffness is 20.8% (from $\beta=0.1404$ to $\beta=0.1112$). This is a significant amount and cannot be ignored. When the ratio of base width to height increases this effect reduces as expected, since the cross-section will appear more rectangular. With ratios of 4 to 5 the reduction in stiffness is 2 to 3%, though this percentage increases 10 fold when anisotropically etched silicon is taken into consideration since the wall angle is much greater.

A relatively straightforward method was developed to calculate the torsional stiffness of a trapezoid. The problem was solved from first principles using the Saint-Venant method, which leads to the formation of a stress function satisfying Poisson's equation. It was found that, by using the principle of virtual work, the Ritz method could be used to solve this problem without having to deal directly with Poisson’s equations. Solutions for the stress function for a trapezoidal beam were derived and once the stress function was known it was a simple matter to calculate the moment of the twisted beam and hence find the torsional constant.

The results were presented in terms of a correction factor for the torsional constant that depends only on the beam’s width to height ratio and the wall angle. Values for this correction factor were tabulated over a range of beam shapes. The results obtained using the Ritz method matched very favourably with those derived using ANSYS. The method
was then applied to hexagonal beams and again the results matched well with those from ANSYS.

7.1.2 Shear Centre

The methodology used to solve the torsion problem was then applied to the problem of the shear centre of a trapezoidal beam. Again the problem was developed from first principles, and the Ritz method was used to solve for the stress function.

To find the shear centre the moment due to the shear stresses in the trapezoidal beam was calculated at two points along the line of symmetry. From these values the point of zero moment, corresponding to the shear centre, could be found by interpolation. Again the results were compared with ANSYS and again the two results matched each other very well.

In a MEMS device the position of the shear centre is most important when a suspended mass is supported by two beams. Using the example of an accelerometer with capacitive sensing, as shown in fig(7.1), the following problem could arise: when the device experiences some in-plane acceleration, the inertial force may lead to some unwanted twisting of the support beams in addition to the desired in-plane deflection. In this case any change in capacitance due to an increase in longitudinal comb electrode overlap might be reduced by the out-of-plane movement of the electrodes.

Methods to over come this type of problem are relatively simple. If predominantly in-plane movement is desired an extra set of beams can be included to increase the torsional stiffness of the overall structure and hence reduce the out-of-plane movement. Alternatively, if a significant increase in torsional stiffness is undesirable, the shear centre can be designed so that it lies within the geometrical centre of the beam, as described in chapter 2.
Reducing Out-of-Plane Movement with Extra Beams

Adding an extra set of beams to resist torsion is a method to prevent out-of-plane movement due to trapezoidal beams. Two devices were designed in ANSYS and are shown in fig(7.2). These model two situations one with the beams close together and the other with the beams are spaced further apart. A harmonic analysis was performed, which sweeps through a range of frequencies between 10kHz to 20kHz with a 80μN in-plane driving force.

The force in the x-direction was applied to the four corners on the right-hand side of the mass. A simple damping coefficient of 0.02 was applied to the device to make the harmonic response more realistic. ANSYS has three different types of damping coefficient, the simplest one was used in this model which applies the same amount of damping over the whole frequency range. The out-of-plane measurements were then taken on the upper left most corner of the mass. A wall angle of 35.26° was used, the same angle as anisotropically etched silicon.

Fig(7.3a) and fig(7.3b) show the amplitude response of the device and an out-of-plane movement can clearly be seen. When the beams are close together, fig(7.3a), it is obvious that the torsional behaviour cannot be adequately constrained, resulting in a large out-of-plane movement at ~14kHz. Even when the beams are spaced further apart, fig(7.3b), the mass experiences an out-of-plane movement and this is wholly attributable to the trapezoidal support beams. At ~14kHz the in-plane mode is excited but the since the force is applied off the shear centre there is an associated twist therefore an out-of-plane movement occurs. At ~17kHz a torsional out-of-plane mode is excited as some of the in-plane energy is coupled directly into this mode, again this is due to the shear centre. As a result stiffening the structure by adding extra beam helps but it cannot completely remove any out-of-plane movement.
Another method to reduce the out-of-plane deflection is to design the beams so that the shear centre lies at midpoint of the line of symmetry. Using the shear centre calculations it was found that a beam with a base width of approximately 2.17 times the height would place the shear centre half way up the beam for an anisotropically etched silicon beam. Fig(7.4) shows a harmonic analysis of such a device. To keep the resonant frequencies similar to those found in fig(7.3) the beams have been lengthened to 7mm. The maximum displacement is slightly less than with the previous device, but the out-of-plane movement is about 5 times less - an obvious improvement.

7.1.3 Electrostatics

In chapter 3 it was shown that comb electrodes generally produce an out-of-plane force, and that this force varies with the aspect ratio and sidewall angle of the electrodes. In the case of an in-plane resonator, the existence of an out-of-plane force could cause an out-of-plane mode to be excited. The in-plane problem was first simplified by breaking it down into sections, using a channel as the basic primitive shape. The top and bottom sides were modelled using the arms of the channel while the sloping side-wall was modelled as a stepped side-wall. The individual force contributions compared well with the results calculated using ANSYS, but when the individual sections were combined the errors were cumulative resulting in a poorer than expect match.

The next method used to calculate the electrostatic forces involved using the Schwarz-Christoffel Toolbox within MATLAB. The SC Toolbox was not the magical solution,
CHAPTER 7. DISCUSSION AND CONCLUSIONS

Figure 7.4: Harmonic analysis for a device in which the trapezoidal beams are designed to have their shear centres half way up the line of symmetry

providing answers to all the problems. However, it did offer the advantage over the analytical models of being able to deal with the trapezoidal problem without simplification. The results at first did not match well with ANSYS and it was suspected that electric field singularities at the corners of the electrode were responsible for the discrepancies. To overcome this problem a very small proportion of the corners of the electrodes were removed. This crude smoothing gave encouraging results that matched well with those from ANSYS.

Having shown that comb electrodes produce out-of-plane forces, the next question of how would these forces would affect a real device was addressed. To answer this the electrostatic forces had to be calculated for the case of non-coplanar electrodes with and without a ground-plane. In the absence of a ground-plane the SC Toolbox and ANSYS results matched very well except at displacements comparable to the electrode height. However, with a ground-plane present the SC Toolbox results were found to be valid only for displacements of up to a few microns. This was because the SC Toolbox model had to be simplified by linking the movable electrode to the ground-plane.

7.1.4 Results from Real Devices

Silicon micromirrors were created using anisotropic wet etching to test the torsion results for a trapezoidal beam. By actuating these mirrors torsionally their natural frequencies were found experimentally. The natural frequencies were then compared with results
obtained from ANSYS and those using the Ritz method for calculating the torsional stiffness of the beams. The semi-analytical results produced natural frequencies close to those obtain in a real device, again reinforcing the validity of the Ritz method.

Electrostatic devices were formed by electroplating photoresist moulds with the photore sist as a sacrificial layer. The electrostatic devices showed that the sloping side walls had a significant effect on the electrostatic forces. Two devices were compared and due to the process limitations one had its suspended comb electrodes close to the ground plane and the other had them above the fixed electrodes. Out-of-plane displacements were measured using a white light interferometer, and for the first device the results compared very well with those obtained using the SC toolbox and ANSYS. The results from the second devices compared less well, although the observed trends were as expected.

7.2 Future Work

7.2.1 Exploitation of Out-of-plane Forces in Torsional Devices

Comb electrode arrays have been used previously to drive micro-optical scanner devices in which a mirror on a torsional suspension is excited into resonant torsional oscillations. In such devices the driving force tends to be impulsive because the comb electrodes only generate a significant force when there is overlap between the moving and fixed electrodes. An interesting area of work would be to find the optimal driving waveform to achieve the maximum ratio of oscillation amplitude to peak drive voltage. The force versus displacement curves in chapter 3 represent a first step in solving this problem.

7.2.2 Design of Electrodes to Reduce Out-of-plane Forces

The work with ANSYS and the SC-toolbox has demonstrated shown simple methods for calculating the electrostatic forces. Using these tools different electrode shapes can be designed and modelled with the aim of reducing the out-of-plane force when the electrodes are in plane.

7.2.3 Capacitive Sensing

Capacitive sensing allows the position of a moving element to be determined, and traditionally this has been used mainly for sensing in-plane movement. It has been shown that the out-of-plane situation can be modelled using conformal transformation, and the SC-toolbox is the ideal tool for doing the calculations. The capacitance can be calculated once the comb electrode cross-section is transformed into the rectangular
domain, and this is relatively simple since, in the transformed domain, the capacitance only depends on the rectangle's geometry.

7.2.4 Torsion and Shear Centre

The torsion and shear centre problems could be solved using the SC toolbox. By transforming the trapezoids into a circular domain it should be possible to find the stresses for a cylinder and then map it back to the trapezoidal problem. This will provide further solutions that can be used for comparisons.

7.3 Conclusions

Microfabrication processes generally produce structures that are trapezoidal rather than rectangular in cross-section. This effect is most pronounced with low cost fabrication methods such as anisotropic silicon etching and UV LIGA, but also occurs with more advanced technologies such as deep reactive ion etching.

The thesis contains a detailed study of the mechanical and electrostatic implications of non-vertical side-walls. Methods have been developed for calculating the torsional constants and shear centres of trapezoidal support beams, and for determining the electrostatic forces on trapezoidal comb electrodes. These methods provide the MEMS designer with useful alternatives to full FEM simulation. The results have been validated by tests on silicon and nickel devices fabricated by the author.

To date the work has resulted in two publications relating to the original electrostatic model [2] and development of the UV LIGA process [1]. Two further papers on the torsional behaviour of the trapezoidal beams and electrostatic excitation of torsional mirrors are in preparation.
Bibliography


Appendix A

Program Listings

A.1 ANSYS Command Line Scripts

A.1.1 Electrostatic Trapezoidal Electrodes

The following scripts and ANSYS command listings were used to model the electrostatic forces. Using scripts saves a great deal of time, as dimensions can be changed simply by changing a model's parameter. The electrostatic problem consists of one core program which sets up some parameters and then two other scripts that describe the electrostatic problem for cases with and without a ground-plane. The first program is:

```plaintext
!c:\ansys56\bin\intel\ansys56.exe -p ansysrf -i t.txt -o test.out#
"C:\Program Files\Ansys Inc\ANSYS60\bin\intel\ansys60.exe"
/begin
/nopr
/clear
/prep7
/pnum,line,1
/pnum,kp,1
/pnum,area,1
/units,si
*cfopen,ansys-displacement-nogp,txt,append
*dim,tt,9,2
*vread,tt(1,1),zygo,txt,jik,2,9
(2e10.0)
!*do,iv,4e-6,7e-6,1e-6
!*do,iii,4e-6,12e-6,8e-6
!fine range -1 to 1um at 0.05um
!corse range -4 to 4um at 0.5um
!*do,count,110,140,10
!*do,ii,-12e-6,-3e-6,1e-6
/prep7
/voltage=30
/displace=12e-6
```
APPENDIX A. PROGRAM LISTINGS

\[
\begin{align*}
\text{pi} &= 3.1415 \\
\text{ang} &= 6 \\
\text{gap} &= 4e-6 \\
\text{comb} &= 12e-6 \\
\text{bb} &= 0 \\
\text{extreme} &= (\text{gap} + \text{comb})/2 \\
\text{height} &= 12.0e-6/2 \\
\text{elsize} &= .2e-6 \\
\text{multiply} &= 6 \\
\text{infmult} &= 1 \\
\text{esize}, \text{elsize} \\
\text{et}, 2, \text{plane121} \\
\text{et}, 1, \text{infin110} \\
\text{mp}, \text{perx}, 1, 1 \\
!/\text{input}, \text{tftest}, \text{txt} \\
!/\text{input}, \text{tftestground}, \text{txt} \\
\text{finish} \\
!/\text{input}, \text{trapforce\_solve}, \text{txt} \\
!/\text{input}, \text{reset}, \text{txt} \\
*cfclos \\
/gopr
\end{align*}
\]

The second script, which models the case of no ground-plane.

\[
\begin{align*}
\text{a} &= \text{gap}/2 \\
\text{b} &= \text{a} + \text{height} \times 2 \times \tan(\text{ang} \times (\text{pi})/180) \\
\text{k, a} &= \text{height} + \text{displace} \\
\text{k, 1, b, -height} &= \text{displace} \\
\text{k, 2, b, - (height*multiply)} &= \text{a, 1, 4, 16, 13} \\
\text{k, 3, b, - (height* (multiply+infmult))} &= \text{a, 1, 2, 14, 13} \\
\text{k, a, (height* (multiply+infmult))} &= \text{a, 5, 6, 18, 17} \\
\text{k, a, (height*multiply)} &= \text{a, 2, 3, 15, 14} \\
\text{k, a, (height* (multiply+infmult))} &= \text{a, 4, 5, 11, 10} \\
\text{k, extreme+bb, -height} &= \text{displace} \\
\text{k, extreme+bb, - (height*multiply)} &= \text{a, 1, 2, 8, 7} \\
\text{k, extreme+bb, - (height* (multiply+infmult))} &= \text{a, 5, 6, 12, 11} \\
\text{k, extreme+bb, - (height* (multiply+infmult))} &= \text{a, 2, 3, 9, 8} \\
\text{(height* (multiply+infmult))} &= \text{a, 16, 17, 23, 22} \\
\text{k, extreme+bb, height} &= \text{displace} \\
\text{k, extreme+bb, (height*multiply)} &= \text{a, 13, 14, 20, 19} \\
\text{k, extreme+bb, (height* (multiply+infmult))} &= \text{a, 17, 18, 24, 23} \\
\text{k, extreme+bb, (height* (multiply+infmult))} &= \text{14, 15, 21, 20} \\
\text{k, a, -height} &= \text{lesize, 11a, 1} \\
\text{k, a, - (height*multiply)} &= \text{lesize, 13a, 1} \\
\text{k, a, - (height* (multiply+infmult))} &= \text{lesize, 16a, 1} \\
\text{k, a, height} &= \text{lesize, 24a, 1} \\
\text{k, a, (height*multiply)} &= \text{lesize, 26a, 1} \\
\text{k, a, (height* (multiply+infmult))} &= \text{lesize, 34a, 1} \\
\text{k, extreme, -height} &= \text{lesize, 36a, 1} \\
\text{k, extreme, - (height*multiply)} &= \text{div=gap/elsize} \\
\text{k, extreme, - (height* (multiply+infmult))} &= \text{lesize, 2a, div!elsize!} \\
\text{k, extreme, height} &= \text{lesize, 4a, div!gap/elsize} \\
\text{k, extreme, (height*multiply)} &= \text{lesize, 6a, div} \\
\text{k, extreme, (height* (multiply+infmult))} &= \text{lesize, 9a, div!gap/elsize} \\
\text{k, extreme, (height* (multiply+infmult))} &= \text{lesize, 12a, div}
\end{align*}
\]
This program script allows for a groundplane to be modelled:

```
 ************** k_w-extreme,-height
 !I=-6e-6 k_w-extreme,-ground
 a=+gap/2+height*2*tan(6*(pi)/180) k_w-extreme,-ground-le-6
 b=a+height*2*tan(ang*(pi)/180) k_w-extreme,height
 ground=3.0e-6+height k_w-extreme,(height*multiply)
 !right hand side
 k,1,b,-height+displace extreme,(height*(multiply+infmult))
 k,2,b,-ground
 k,3,b,-ground-le-6
 k,a, height+displace a,1,4,16,13
 k,a, (height*multiply) a,4,5,17,16
 k,a, (height*(multiply+infmult)) a,1,2,14,13
 k,extreme+bb,-height+displace a,5,6,18,17
 k,extreme+bb,-(ground) a,2,3,15,14
 k,extreme+bb,-(ground)-le-6 a,4,5,11,10
 k,extreme+bb,height+displace a,5,6,12,11
 k,extreme+bb,(height*multiply) a,2,3,9,8
 k,extreme+bb,(height*(multiply+infmult))a,16,17,23,22
 !left hand side
 k_w-b,-height a,13,14,20,19
 k_w-b,-ground a,17,18,24,23
 k_w-b,-ground-le-6 a,14,15,21,20
 k_w-a,height a,16,17,23,22
 k_w-a, (height*multiply) a,4,5,17,16
 k_w-a, (height*(multiply+infmult)) a,1,2,14,13
```

APPENDIX A. PROGRAM LISTINGS

```
A.2 MATLAB Programs

A.2.1 Torsion Code

The $\beta$ correction factor for the torsion of a trapezoidal beam is calculated with the following code. This version of the code is a shortened version which only includes the parameters $A$ to $G$ and $Z$.

```matlab
function [beta]=torsion01(a,b,angle)
% beta value from trapezoidal beam with a height b width angle degrees
syms x y Z A B C D E F G H I J K L M
b=b/2
i=angle
n=tan(i*pi/180)
f=y*(y-a)*(x+n*y-b)*(x-n*y+b)*(x+y+b)+x*y+x*y+y^2+2*y^2+3*y^3+4*y^4+G*y^5);
temp = (diff(f,'x')^2)+(diff(f,'y'))^2/2/2*f;
S = int(int(temp,'x',n*y-b,-n*y+b),'y',0,a/2)+int(int(temp,'x',n*y-b,-n*y+b),'y',a/2,a);
```

APPENDIX A. PROGRAM LISTINGS

lesize,24",1
lesize,26",1
lesize,34",1
lesize,36",1
div=gap/elsize
lesize,2",div/elsize
lesize,4",div/gap/elsize
lesize,6",div
lesize,9",div/gap/elsize
lesize,12",div
lesize,15",div/gap/elsize
lesize,17,19,2
lesize,20,22,2
lesize,23,29,2
lesize,30,32,2
lesize,33,35,2
lesize,all,elsize
ndivs=20
stretch=.1
lesize,7",ndivs,stretch
lesize,5",ndivs,1/stretch
lesize,28",ndivs,stretch
lesize,18",ndivs,stretch
lesize,8",div/s,1/stretch
lesize,10",div/s,stretch
lesize,31",div/s,stretch
lesize,21",div/s,stretch
dl,19",volt,0
dl,1",volt,0
dl,22",volt,0
sfl,1,1,mxwf
```
APPENDIX A. PROGRAM LISTINGS

[\[ A B C D E F G Z \] = solve(diff(S,'Z'),diff(S,'A'),diff(S,'B'),
   diff(S,'C'),diff(S,'D'),diff(S,'E'),diff(S,'F'),diff(S,'G'))

beta=2*eval(int(int(f,'x'),n*y-b,-n*y+b)),'y',0,a/2)/a^3*2*b)
   +2*eval(int(int(f,'x'),n*y-b,-n*y+b)),'y',a/2,a)/(a^3*2*b)

beta=double(beta);

A.2.2 Shear Centre Code

The shear centre is extracted from two moments calculations. Therefore one function first
calculates the moment at a point, cc, for a beam with a defined height, width and wall
angle. The second calculates the intersections point. This version of the code is a shorted
version which only includes the parameters A to G and Z.

function Mz=moment01(height,width,angle,cc)
%Mz=moment01(height,width,angle,cc)
%use inconjuction with runshearcentre
syms xyZABCDEFG
a=height
b=width/2;
v=0.3;
P=1;
i=angle;
n=tan(i*pi/180);
Iy=double(int(int(x''2,'x',n* (y+cc)-b,-n* (y+cc)+b), 'y', -cc,a-cc))
ff=P/2/Iy*(-n*(y+cc)+b)^2
f=(y+cc)* (y+a+cc)* (x+n*(y+cc)-b)* (x-
n*(y+cc)+b) *Z+A*y+B*x''2+C*y^2+D*y^3+E*x''4+F*y''4+G*y''5);
temp=((diff(f,'X')^2) + (diff(f, 'y')''2))/2+f*(0.3/1.3*P*y/Iy-diff(ff, 'y'));
S=int(int(temp,'x',n*(y+cc)-b,-n*(y+cc)+b), 'y', -cc,a-cc);
[A B C D E F G Z]=solve(diff(S,'Z'),diff(S,'A'),diff(S,'B'),
   diff(S,'C'),diff(S,'D'),diff(S,'E'),diff(S,'F'),diff(S,'G'));
A=double(A), B=double(B), C=double(C), D=double(D),
E=double(E), F=double(F), G=double(G), Z=double(Z);
Mz=eval(int(ttz*y-tyz*x,'x',n*(y+cc)-b,-n*(y+cc)+b), 'y', -cc,a-cc))

Second program to extrapolate the shear centre from two moment calculations at 0.4 and
0.6 of the height of the beam.

for comb=1.5:1:2.5
for ang=0:6:36
ml=double(moment01(1,comb,ang,0.4))
m2=double(moment01(1,comb,ang,0.6))
shearcentre=0.4-ml*(0.6-0.4)/(m2-ml)
end
end
A.2.3 Hexagonal Code

Matlab function to find the $\beta$ values for a hexagonal beam.

```matlab
function [beta,x,y,f]=hexagon(a,b,angle)
% [beta,x,y,f]=hexagon(a,b,angle) a height b width angle eg 18 degrees
syms x y Z A B C D E F G H I J K L M N
a=a/2
b=b/2
i=angle
n=tan(i*pi/180)
f=(y+a)*(y-a)*(x+n*y+b)*(x-n*y-b)*(x-n*y+b)*(x+n*y+b)*2+C*x^4+D*y^4+E*x^6+F*y^6)
temp=((diff(f,'x')^2)+diff(f,'y')^2)/2-2*f
S=int(int(temp,'x'),-n*y-b, +n*y+b,'y',-a,0)+int(int(temp,'x',+n*y-b, -n*y+b),'y',0,a)
[A,B,C,D,E,F,Z]=solve(diff(S,'Z'),diff(S,'A'),diff(S,'B'),diff(S,'C'),diff(S,'D'),diff(S,'E'))
Z=double(Z), A=double(A), B=double(B)
C=double(C), D=double(D), E=double(E), F=double(F)
beta=2*(eval(int(int(f,'x'),-n*y-b, +n*y+b),'y',-a,0))+eval(int(int(f,'x',+n*y-b,-n*y+b),'y',0,a))/((a*2)^3*(b*2))
```

A.2.4 Electrostatic Actuation No Ground-plane

Matlab function to calculate the electrostatic force for trapezoidal electrodes without a groundplane, using the clipped corners method. Note that the displacement of the comb electrode (disp) must be entered in complex number

```matlab
function [fl,ftot]=eforce05(comb,gap,height,angl,disp,V,clip)
% calculates the energy using SC toolbox with comb,gap,height,disp and then using
% clipped corners using efield
sccmapopt('tol',le-30,'trace','off')
e0=8.854e-12;
ang2=angl/pi
s=height*tan(angl)
gg=0
h=height/2*1i;
g=3e-61
a=gap/2;
b=a+comb/2;
cc=clip*comb/2
dd=cc*1
ccl=clip*(comb/2-a)
ddi=ccl*1
w=[0+inf*1i,-b+h,-a+h-cc,-a+h-dd,-a-s-h-ddl,-a-s-h-ccl,-b-h,0-inf*1i,
b-h-disp,a+s-h-ccl+disp,
ang2,1.25+ang2,1.25-ang2,1.25..5,0..5,1.25-
ang2,1.25+ang2,1.25..5];
p= polygon(w,alf);
fl= rectmap(p,[2 7 9 14]);
```
APPENDIX A. PROGRAM LISTINGS

step=(real(b)-real(a))/1000;
zz=real(a)+gg:step:real(b);
zz=zz+h+disp;
xx=evalinv(f1,zz);
ftop=sum(imag(diff(xx)).^2/step)*(V/pi)^2*e0/2
zz=real(c)+gg:step:real(b);
zz=zz-h+disp;
xx=evalinv(f1,zz);
ftbot=sum(imag(diff(xx)).^2/step)*(V/pi)^2*e0/2
a=a+s-h
zz1=real(a):((real(c)-real(a))/sstep):real(c);
size(zz1)
zz2=imag(a):((imag(c)-imag(a))/sstep):imag(c);
size(zz2)
if ang2==0
zz1=1
end
zz=zz1+zz2.*i+disp;
xx=evalinv(f1,zz);
xx1=xx;

A.2.5 Electrostatic Actuation With Ground-plane

Matlab function for calculating electrostatic force for trapezoidal electrodes with a groundplane, using the clipped corners method.

function [f1]=eforce06(comb,gap,height,angl,V,disp,clip)
%calculates the energy using S-
%C toolbox with comb,gap,height,disp and then using
%virtual work to calculate force
%clipped corners using groundplain
scmapopt('tol',le-30,'trace','off')
o=8.854e-12;
ang2=angl/pi
s=height*tan(angl)
h=height/2*li;
g=3.6e-6i
a=gap/2;
b=a+comb/2;
cc=clip*comb/2
dd=cc*i
cc1=clip*(comb/2-s)
dd1=cc1*i
cc=1e-7
d=1e-71
w=[0+inf*li,-b+h,-a+h-cc,-a+h-dd,-a-s-h-dd,-a-s-h-cc,-b-h,-b-h-g,
\texttt{inf-h-g,a+s-h+cc+disp,a+s-h+dd+disp,a+h-dd+disp,a+h+cc+disp,b+h+disp)}

\texttt{alf=[0,.5,1.25,1.25+ang2,1.25-ang2,1.25,.5,0,1.25,1.25-}

\texttt{ang2,1.25+ang2,1.25,.5,0,1.25,1.25-}

\texttt{ang2,1.25+ang2,1.25,.5];}

\texttt{p=polygon(w,alf);}  
\texttt{f1=rectmap(p,[2 7 8 14]);}  
\texttt{step=(real(b)-real(a))/1000;}  
\texttt{zz=real(a):step:real(b);}  
\texttt{zz=zz+h+disp;}  
\texttt{xx=evalinv(f1,zz);}  
\texttt{ftop=sum(imag(diff(xx)).^2/step)*(V/pi)^2*e0/2}  
\texttt{c=a+s-h;}  
\texttt{zz=real(c):step:real(b);}  
\texttt{zz=zz-h+disp;}  
\texttt{xx=evalinv(f1,zz);}  
\texttt{fbot=sum(imag(diff(xx)).^2/step)*(V/pi)^2*e0/2}  
\texttt{c=a+s-h;}  
\texttt{a=a+h;}  
\texttt{sstep=100}  
\texttt{zzl=real(a):(real(c)-real(a))/sstep:real(c);}  
\texttt{size(zzl)}  
\texttt{zz2=imag(a):(imag(c)-imag(a))/sstep:imag(c);}  
\texttt{size(zz2)}  
\texttt{if ang2==0}  
\texttt{zzl=1}  
\texttt{end}  
\texttt{zz=zzl+zz2.*i+disp;}  
\texttt{xx=evalinv(f1,zz);}  
\texttt{xxl=xx;}  
\texttt{fmid=sum((diff(xx)).^2/(c-a)/sstep)*(V/pi)^2*e0/2;}  
\texttt{fmid=abs(fmid)*sin(angl)}  
\texttt{ftot=ftop-fbot-fmid};
Appendix B

Coplanar Electrode Model

This analysis forms part of an early attempt by the author to model the electrostatic forces of trapezoidal electrodes. The results obtained using this method were horrendously incorrect. However the methodology included in this appendix is nonetheless useful.

As noted in chapter 3, the field due to an array of coplanar electrodes in which the potential alternates between two values can be obtained using conformal mapping, the details of which will be given here.

The aim is to transform the coplanar problem into a parallel plate one as illustrated in fig(B.1). Here only half-period of the coplanar array (the region between the mid-points of adjacent electrodes) is shown. Knowing the solution of the electric field in the parallel plate configuration, and the transform links the two domains, will allow the electric field and hence the electrostatic force for the coplanar problem to be calculated. The basic principles of conformal mapping were outline in chapter 1.

Figure B.1: Desired mapping of coplanar electrode array to parallel plate configuration. Red lines represent equipotentials.
APPENDIX B. COPLANAR ELECTRODE MODEL

As with the case before it is not easy to go straight from the coplanar array domain to the rectangular domain in one step. Therefore an intermediate domain is employed so that standard transforms can be used. In this case the modelled region (bounded by the blue dashed lines and the electrodes in fig(B.1)) is mapped to the upper half plane with all points lying on the x-axis. In the following discussion the initial, intermediate, and final domains will be referred to as the z, w, and \( \chi \)-domains respectively.

The coplanar electrodes that represent the top and bottom sides of the electrodes are modelled using two electrodes at the edge of a strip. From fig(B.2) the electrodes are placed apart by a distance \( g \) (distance \( BC \)), with the extremes at \( A \) and \( D \) at \( \pm rg/2 \). The factor \( r \) becomes a scaling factor which gives the ratio of \( OD \) to \( OC \).

**First Transformation**

Referring to (B.2), the boundary of the modelled region in the \( z \)-plane has two right angles where the electrodes meet the lines of symmetry of the field pattern. The Schwarz-Christoffel transformation, shown in it's general form eq(B.1), can be used on the \( z \)-plane boundary to 'straighten out' these two right angles, mapping the modelled region to the \( w \)-plane - the upper half plane. In physical terms the \( w \)-plane solution corresponds to the field of a single pair of coplanar electrodes [1, 2].

\[
\frac{dz}{d\chi} = S (\chi - a)^{\frac{a}{e}-1} (\chi - b)^{\frac{b}{e}-1} (\chi - c)^{\frac{c}{e}-1} (\chi - d)^{\frac{d}{e}-1} \ldots \quad (B.1)
\]

<table>
<thead>
<tr>
<th></th>
<th>( w )-plane</th>
<th>( z )-plane</th>
<th>( z )-plane angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(-\frac{1}{r} )</td>
<td>(-\frac{1}{2}rg )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>B</td>
<td>(-1 )</td>
<td>(-\frac{1}{2}g )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>C</td>
<td>( 1 )</td>
<td>( \frac{1}{2}g )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{1}{r} )</td>
<td>( \frac{1}{2}rg )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>E</td>
<td>(-\infty )</td>
<td>(-\frac{1}{2}rg + j\infty )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>F</td>
<td>(+\infty )</td>
<td>(+\frac{1}{2}rg + j\infty )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>O</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>(- )</td>
</tr>
</tbody>
</table>

Table B.1: Mappings of key points under transformation shown in fig(B.2)

It is usual to define the gap \( BC \) between the electrodes in the \( z \)-plane as \( g \), and the width \( AD \) of the modelled region (representing the pitch of the actual electrodes) as \( rg \). The factor \( r \) becomes a scaling factor which gives the ratio of \( OD \) to \( OC \). The two electrodes in the \( w \)-plane are spaced apart by 2 units, with their extremes at \( \pm \frac{1}{r} \). Table B.1 shows the required mapping of key points on the boundary and the \( z \)-plane angles are the internal angles made by each of the vertices.
Applying the Schwarz-Christoffel transformation to this problem yields eq(B.2). As explained in chapter 1 vertices placed at $\pm \infty$ do not appear in the transformation.

\[
\frac{dz}{dw} = \frac{S}{\sqrt{w-k^{-1}} \sqrt{w+k^{-1}}} 
\]  
(B.2)

Figure B.2: First transformation of electrodes from the $z$-plane into the upper half $w$-plane

Finding $S$

In eq(B.2) there is a scaling variable, $S$, that needs to be calculated. This can be done by manipulating eq(B.2) at very large limits. In the $z$-plane the distance between $A$ to $D$ is constant at $rg$, so that:

\[
\int_P dz = -rg 
\]  
(B.3)

where $P$ is a straight path from $z = +\frac{1}{2}rg + jy$ to $z = -\frac{1}{2}rg + jy$ for any $y$.

In the limit of large $y$, the equivalent path $P'$ in the $w$-plane is a semicircle with a large radius, $R$, centred at the origin. This semicircle is expressed mathematically in the $w$-plane by $w = Re^{i\theta}$. Differentiating with respect to $\theta$ gives:

\[
dw = jRe^{i\theta} d\theta 
\]  
(B.4)

With the aid of eq(B.4) and eq(B.2) we can change variables in eq(B.3) to express the integral in terms of $\theta$:

\[
\int dz = \int_0^\pi \frac{jRe^{i\theta} S}{\sqrt{R^2e^{2i\theta} - 1/k^2}} d\theta 
\]
APPENDIX B. COPLANAR ELECTRODE MODEL

As \( R \to \infty \) the above integral simplifies to:

\[
-g_{r} = \int_{0}^{\pi} jSd\theta
\]

\[
A = \frac{-rg}{j\pi}
\]

Substituting this value of \( S \) into eq(B.2) and integrating with respect to \( w \) one obtains:

\[
z = \frac{-rg}{\pi} \int \frac{dw}{j\sqrt{w^2 - k^2}}
\]

\[
= \frac{-rg}{\pi} \arcsin kw
\]

(B.5)

Second Transformation

The second transformation (\( w \)-plane to \( \chi \)-plane), illustrated in fig(B.3) is a standard transform that maps the upper half of the \( w \)-plane into the interior of a rectangle in the \( \chi \)-plane using elliptic integrals. Table B.2 shows the mappings of the key points on the boundary.

<table>
<thead>
<tr>
<th>( \chi )-plane</th>
<th>( w )-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>A ( -K + jK' )</td>
<td>( \frac{-1}{k} )</td>
</tr>
<tr>
<td>B ( -K )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>C ( +K )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>D ( +K + jK' )</td>
<td>( \frac{1}{k} )</td>
</tr>
<tr>
<td>O 0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table B.2: Mappings for transformation show in fig(B.3)

\( K \) is the complete elliptic integral with modulus \( k \), while \( K' \) is the complete elliptical integral with modulus \( k' = \sqrt{1 - k^2} \). \( k' \) is called the complimentary modulus of \( k \). The elliptic function which transforms the upper half-plane to a rectangle is of the form \( \chi = sn^{-1}(w) \). It is normally unnecessary to mention the modulus but if the modulus has to be brought in explicitly the equation can be written as \( \chi = sn^{-1}(w, k) \) (see Gibbs [2] page 140).

Applying the Transforms to Calculate the Force in the \( z \)-plane

The transform is in the form of a derivative, so one obvious question that may not be clear is, how is it applied? The quantities we are interested in are the force and to calculate the
force we require the flux density or electric field. So to use the transform we simply multiply it by a known quantity in one plane to find the unknown in the other plane, e.g. the electric field in the $\chi$-plane multiplied by the transform $\frac{dx}{dw}$ will give the electric field in the $z$-plane.

$$D_z = D_\chi \left| \frac{d\chi}{dz} \right|$$

Using electric fields instead of flux densities yields a similar result.

$$E_z = E_\chi \left| \frac{d\chi}{dz} \right|$$

Now the reason for using the parallel plate configuration becomes apparent. In the $\chi$-plane the field is that of a parallel plate capacitor with a gap of $2K$, so the expression for $D_\chi$ or $E_\chi$ are particularly simple.

$$E_\chi = \frac{V}{2K} \text{ or } \frac{\varepsilon_0 V}{2K}$$

Now returning to the $z$-plane, the electric flux density, $D_z$, can be obtained as the product of two transforms, one from $z$-plane to $w$-plane and then $w$-plane to the $\chi$-plane. Ideally we would like to have one transform from the $w$-plane to the $\chi$-plane, such as:

$$D_z = D_\chi \left| \frac{d\chi}{dz} \right|$$

(B.6)
However to simplify the problem two transforms are used, which allows eq(B.6) to be expanded, accommodating the two transforms $\frac{dx}{dw}$ and $\frac{dz}{dw}$.

\[
D_z = D_x \left| \frac{dx}{dz} \right| = D_x \left| \frac{dx}{dw} \frac{dw}{dz} \right|
\]

The derivative $\frac{dw}{dz}$ is obtained by substituting $S = -\frac{r}{j\mu}$ into eq(B.2), giving:

\[
\frac{dw}{dz} = \frac{\pi}{jrgk} \sqrt{1 - k^2 w^2}
\]

while differentiation of the second transform gives:

\[
\chi = \frac{\sin^{-1}(w)}{\sqrt{(1 - w^2)(1 - k^2 w^2)}}
\]

\[
\frac{d\chi}{dw} = \frac{1}{\sqrt{(1 - w^2)(1 - k^2 w^2)}}
\]

Combining the two transforms leads to the flux density calculation:

\[
D_z = D_x \left| \frac{dx}{dz} \right| = D_x \left| \frac{dx}{dw} \frac{dw}{dz} \right|
\]

\[
D_z = \frac{1}{\sqrt{(1 - w^2)(1 - k^2 w^2)}} \cdot \frac{\pi}{r g k} \sqrt{1 - k^2 w^2}
\]

\[
D_z = \frac{D_x \pi}{r g k \sqrt{1 - w^2}}
\]

The force acting on each half-electrode in the $z$-plane can be found using the integral in eq(B.7) with limits $C$ and $D$ as shown in fig(B.2). The path is integration along the electrode surface, which corresponds to the real axis, so in $z = x + jy$, $y$ can be taken as zero.

\[
F_z = \frac{1}{2 \varepsilon_0} \int_C^D |D_z|^2 dz
\]
\[ F_x = \frac{1}{2\varepsilon_0} \left( \frac{D_X \pi}{rgk} \right)^2 \int_{C}^{D} \left| \frac{1}{\sqrt{1 - w(z)^2}} \right|^2 dz \]

\[ F_z = \frac{1}{2\varepsilon_0} \left( \frac{D_X \pi}{rgk} \right)^2 \int_{\frac{t}{2}}^{\frac{t_2}{2}} \left| \frac{1}{\sqrt{1 - \frac{k^2}{k^2} \sin \left( \frac{\pi}{r_2} \right)^2}} \right|^2 dx \] (B.8)

A similar method was used in a paper by O'Connor [3], which was derived from a book by Binns [1], however both methods were intended for magnetic problems.
Bibliography

