Three Essays on the Co-Evolution of the Real and Financial Sector During the Growth Process

by

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Abstract

In a seminal work by Gurley and Shaw, they established that economic development is typically accompanied by a transition from self-financed to intermediated investment. This work seeks to analyse the co-evolution of the real and financial sectors during this growth process.

We study the biases generally observed in the AR(1) OLS estimation, and apply the result of these to causality tests between finance and growth. As originally identified by Sawa these biases are present in regressions using small samples, while OLS will normally be consistent if the error term is i.i.d. We assess and find analytical expressions for these biases.

The phase-average methodology has been often used to study the growth effect of financial deepening; notably by Barro and Sala-i-Martín. We study the properties of phase averages of the variables when estimating parameters in growth regressions. We find that phase averages artificially increase the size of estimated parameters, introduce inefficiencies and bias the estimation by OLS. We conclude that phase averaging is more likely to produce problems than benefits, as aggregation from annual data entails loss of information and does not reduce other data problems such as serial-correlation.
Chilean financial markets have experienced extensive development, increased participation, deregulation and new legislation over the last two decades. We analyse a financial database of equity, traded on the Chilean stock market between 1989 and 2000. We develop a model for the relationship between risk and return, under market frictions. We find that market frictions like compulsory holdings and short sale restrictions have a negative effect over asset prices for given levels of risk. We study the evolution of these frictions using the Chilean data.
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Chapter 1

Preliminaries

1.- The co-evolution of real and financial wealth.

Based on cross-country comparisons Gurley and Shaw (1955 and 1960) established in their seminal work, that economic development is typically accompanied by a transition from self-financed to intermediated investment. In their account of the evolution of real and financial wealth they claimed: "... financial assets accumulate as income and wealth grow. Our suggestion is that accumulation reflects division of labor and productions, saving and investment, and intermediation. Specialization in the use of productive factor generates a rising stream of income and a rising stock of both real and financial wealth." (Gurley and Shaw 1967, p. 260)

Availability of funds plays an essential role when a firm wants to implement a profitable project; through making these funds available financial development will exert beneficial effects on growth. For example, consider two economies expecting large amounts of new technological development that will arise only after some investment is made. If the two economies have opposite investment possibilities, the economy that has the funds is likely to experience high growth, while the second one will not.
There are many attributes of the financial system that can promote economic growth. We identify two main forms of financial institutions, namely: financial intermediaries, like banks or insurance companies and stock markets. On the one hand, financial intermediaries play an important role assessing risk and profitability of different projects. They are seen to provide information on which, among the projects available in the economy, are the best investments to be made. Therefore, these intermediaries should direct the available funds to businesses with the largest prospects for growth. Stock markets are believed to perform a similar role, as the information about investment flows through investors in the market place. Furthermore, the existence of a market where a large number of firms are traded, allows for the possibility of hedging risk among all the participants in the economy. As the risk of overall investments decreases with the existence of a stock market, financing a project occurs more often in an economy with this kind of market than in one without.

The poor performance, in the early 1970s, of investment and growth in developing countries was ascribed to interest rate ceilings, direct credit programmes and other measures considered repressive on the financial system. This theory, where innovation in the financial sector increases the interest elasticity of money demand, by Shaw (1973) and Mckinnon (1973) advocated for liberalisation in countries with repressed financial systems. With low-yielding investment projects eliminated and real interest rates adjusting to their equilibrium level, savings and the total real supply of credit would expand. Economic growth would be stimulated through a higher volume of

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1 The emergence of capital markets has been generally modelled as a more efficient response to hedge liquidity risk. For example: Bencivenga and Smith (1991), Levine (1993) and Greenwood and Smith
investment and through increases in the average productivity of capital. However, the experience from the liberalisation experiments in the 1970s was catastrophic for the banking sector that collapsed in the 1980s in many Latin American countries. In particular, the empirical evidence from these experiments was not clear.

Generally, financial intermediaries provide finance through debt while financial markets allow the issue of equity by firms. They are part of the same system but they perform different roles in it. The attempt to liberalise Latin-American economies at the end of the 1970s was partly a failure because of the neo-liberal argument that debt and equity were perfect substitutes (see Fry, 1995). So the attempt to remove barriers in the financial system failed to assess the ability of the existent financial structure to deal with agency problems, likewise there were no attempts made to deal with these problems or to improve the level of information in the economy.

In a second wave of the theory, Townsend (1983) established a distinction between fiat money and credit with regard to their capacity for lowering trade costs. Money allows consumers to exchange goods at a given moment in time; in a similar way financial institutions allow them to trade over time, eliminating the costs of having to wait long periods for consumption.

More recently endogenous growth theorists have devoted their attention to the growth enhancing benefits of the financial system, see for example Barro and Sala-I-Marti (1995). Levine (1996) identified a number of mechanisms through which finance helps

(1997).
economic growth. In his framework, financial markets and financial institutions are created when information and transaction frictions exist. Hence, financial institutions perform in order to resolve these frictions and expedite the exchange and allocation of economic resources across space and time.

Merton and Bodie (1995) established six basic functions performed by the financial sector. First, a financial system provides ways of settling and clearing payments to facilitate the exchange of goods, services and assets. Second, a financial system provides a mechanism for the pooling of funds, to undertake large-scale indivisible enterprises or for the subdividing of shares in enterprises to facilitate diversification. Third, it provides ways of transferring economic resources through time, across geographic regions and among industries. Fourth, it provides ways to manage uncertainty and control risk. Fifth, a financial system provides price information that helps co-ordinate decentralised decision-making in various sectors of the economy. Finally, a financial system provides ways to deal with the incentive problems when one party to a financial transaction has information that the other party does not, or when one party is an agent for another.

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2 For related schemes on the financial functions and the economic role of the financial sector see also: Cole and Slade (1991), Levine (1996) and, more recently, Santomero (1997).
3 This role has been emphasised particularly by Greenwood and Smith (1997)
4 Sirri and Tufano (1995) provide a detailed theoretical and empirical analysis of this function.
5 Levine (1996) divides the financial forces into five basic functions. First, financial intermediaries and institutions facilitate the trading, hedging, diversifying and pooling of risk. Second, they command resource allocation. Third, they help to monitor managers and exert corporate control. Fourth, they ease the mobilisation of savings. And last, financial institutions facilitate the exchange of goods and services.
6 In a similar scheme Cole and Slade (1991) identify seven functions where, if well performed by the
These six basic functions impact upon both capital accumulation and technological innovation, as more or less resources are available for investment in working capital and research. The impact that more finance has on capital and innovation, finally translates in higher growth rate for the economy. As a result of this evidence, financial liberalisation still has its advocates; see for example Fry (1997).

From the empirical point of view we are less able to precisely measure the extent of this alleged beneficial effect. Some evidence suggests that the development of the financial system is associated with higher rates of growth. Nonetheless, the size of the financial sector is endogenous to the process of growth, as is the size of any other sector of the economy; see for example Greenwood and Smith (1998). It is not clear if the connection found between financial deepening and growth rates isolates the effect of the exogenous improvements in the financial system on growth. In many of the financial sector, the latter could produce positive effects on growth. Through reducing the cost of financial resources to investors, improve the allocation of resources, enhance stability of the economy and reduce risk, the financial sector can contribute to development. The financial function can be divided into seven elements. First, the financial system provides a safe, convenient and beneficial form in which savers can hold their resources. Second, it transfers the control of resources from savers to investors (or producers) at low transaction costs, compared to the costs of direct transfers, inflation or government intermediation (like taxes or foreign savings). Third, it channels purchasing power to a variety of investors with varying needs, degrees of risk and return. This permits a more diversified (and more efficient) investment. Fourth, financial institutions exert pressure on investors to use resources more efficiently and more productively, in order to repay existing debt and to qualify for new financing. Fifth, the financial system rewards more efficient and capable financial managers who make wise, lucky choices or develop innovative financial instruments; in this way financial structures develop themselves. Sixth, it maintains stable indexes to compare values across countries and time. Seventh, it reduces the risks and costs of financial transactions through the financial system of payments.
empirical works we will examine, it is possible that the, apparently significant, parameters reflect the impact of good growth prospects on the incentive to develop the financial sector, rather than a significant effect in the opposite direction. Therefore, it is difficult to judge the direction of causality between these two variables. Using data from Germany, South Korea and the US, Arestis and Demetriades (1997), find mixed evidence on the causal direction of the dependence between finance and growth. Demetriades and Hussein (1996) find causal evidence in both directions in an empirical analysis of 16 developing countries.

Opposing voices, like Lucas (1988), criticise the excess attention devoted by economists to a rather unlikely causal connection between growth and financial development. Singh (1997) points out that financial liberalisation is more likely to hinder rather than to favour growth in developing countries. This kind of criticism, however, focuses mainly on the negative effects, experienced in some developing countries, after the application of the financial liberalisation-repression analysis, suggested by Shaw and McKinnon.

2. **Finance and growth: a simultaneous system.**

The co-evolution of financial institutions and economic growth has been much studied recently, see for example Bencivenga and Smith (1998), Santomero (1997) and Greenwood and Smith (1997). It is normally claimed that markets enhance growth to
the extent that they serve to allocate resources to the places in the economic system where their economic return is greatest. It is also argued that market formation permits increased specialisation, allowing faster advance and creation of new technology, see for example Merton and Bodie (1995) and Sussman (1996). In addition, market structures affect agents’ incentives to accumulate various types of capital and assets.

The degree of development of financial markets, and the particular kind of institutions forming it, play an important role in economic development. The value of financial institutions to the economy rests primarily on their ability to screen and finance wealth-enhancing projects in the economy. In addition, the existence of monitoring institutions often leads to higher returns, as investors respond to on-going supervision by increasing effort. Monitoring induces managers to make operating decisions closer to the original purpose, when the financial resource was granted.

Bencivenga and Smith (1998) and Greenwood and Smith (1997) have pointed out that the development of financial markets is an endogenous process. Setting up a trade agreement or arranging the delivery of goods on a certain date, time and place requires resource expenditures. So, poorer economies will devote fewer resources to this process than wealthier ones. Therefore, economic development should potentially lead to increasing financial-sector activity and this increased activity will further stimulate economic growth.
a.- Financial channels to enhance growth.

There are many ways in which the financial system may affect the rate of growth of an economy. First, well-performing financial institutions should have the ability of detecting wealth-enhancing projects in the economy. Financial resources would be directed to those projects with the highest social return, increasing in this way the speed of economic growth. Second, in general, more productive investments tend to be riskier. The existence of a financial system that insures investors against idiosyncratic risk investors will enable them to shift portfolios towards higher-return investment. Third, the increase in specialisation required by new technologies is facilitated by financial intermediaries. The financial system helps agents to trade in those specialised services and goods required by new technologies.

A fourth growth enhancing channel is performed by the financial system through monitoring and exertion of corporate control. This function can easily be seen as performed by banks as they routinely monitor debt in companies. Financial institutions induce borrowers to make productive decisions in the run of investment financed with loans. This role can also be seen as performed by capital markets as a more-developed capital market allows for better corporate control and easier take-over.

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7 This analysis follows Levine (1996) closely. Levine also includes the role markets play on mobilising savings. This function however, does not enhance growth without some explicit externality in the production function.

8 Take-overs however involve several agency problems that make it less clear for us to conclude that they are effective ways of control. For example, Stiglitz (1985) identifies the free-rider problem in
b. Growth of the financial sector.

Gurley and Shaw (1960) stressed the role of transaction costs in the evolution of the financial system. The existence of fixed costs of asset evaluation means that intermediaries have an advantage over individuals because they spread these costs among small savers.

The endogenous growth literature considers the development of financial markets and economic growth as a bi-directional process. The explanation that more resources are devoted to financial activities when an economy becomes larger is linked to costs of intermediation. The two main forms of financial institution, namely banks and financial markets present different kinds of cost to the users of the respective services. Greenwood and Smith (1997) identify financial market activities as more costly, compared to the costs incurred by agents using banks or other financial intermediaries. However, given that either type of institution will produce increasing profitability if savings are channelled through them, investors will find it convenient at a high level of income to engage in such activities.

Regarding less developed economies, Greenwood and Smith (1997) point out that:

"Arranging and effecting trade requires resource expenditures [...] But poor economies are less well placed to devote substantial resources to the trading process[...]" (op. cit. p.146) However, the amount of resources in a particular economy will affect the degree of involvement of its financial institutions in the development process in many take-overs.
different ways. Financial institutions perform a number of tasks related to the transference of resources among different investors and across time. Difference in the size of the economies will be reflected in all the aspects of financial intermediation.

Greenwood and Jovanovic (1990) and Greenwood and Smith (1997) point out that a poorer economy is composed of individuals that cannot afford the cost of starting a market for finance. In other words, it is profitable, in terms of the increase on savings return, to either establish a bank or set up a stock market. However, this is not the case for poor agents, whose overall investment is too small to account for the increased return, after discounting the fixed costs incurred. Levine (1993) extends this idea by introducing fixed-entry fees or transaction costs, of increasing magnitude, for more sophisticated financial services. In his analysis, the simpler, initial financial structures become more complex as the individual wealth grows.


a. Panels of countries.

Many recent empirical studies have indicated the significant relationship between financial development and economic performance. Arestis, Demetriades, Luintel (2001); Arestis and Demetriades (1997); Harris (1997); Atje and Jovanovic (1993) and King and Levine (1993) find some econometric evidence suggesting that the degree of stock-market development has positive consequences for GDP growth and the general
level of GDP. While Luintel and Khan (1999); Arestis and Demetriades (1997); Levine (1996) Demetriades and Hussein (1996); Barro and Sala-i-Martin (1995) and King and Levine (1993) find empirical evidence that financial-depth measurements, for example bank lending and liquid liabilities, which exert a positive influence on economic growth.

King and Levine (1993) attempt to study the effects of financial development over growth, finding some strong correlation between the two. They find in particular some 'predictive' content in these relations, that is those countries that displayed fast grow over 1970 and 1989 had, between 1960 to 1969, larger financial systems as measured by the proportion of the income represented by liquid assets. On the other hand, faster growing economies, over 1970-1989, displayed a greater share of lending undertaken by banks than by the central bank, and had a higher share of lending to the private sector than to the public sector over 1960-1969 (op.cit. Table 6.4, p. 174).

Barro and Sala-I-Marti (1995) find a significant positive coefficient of 0.16 between per-capita growth and liquid-liabilities-to-output-ratio. This coefficient means that one-standard-deviation increase in the liquid-liabilities ratio (an increase of 0.26 over the period 1964 to 1975) would rise the per capita growth by 0.4 percentage points. King and Levine (1993) apply a similar model for the average growth over the period 1960 to 1989, finding a coefficient for liquid-liabilities ratio of 0.032. Their coefficient means that a country that increased its liquid-liabilities-output ratio from the lowest

\[ \text{\textsuperscript{9}} \] In a model that controls by human capital, terms of trade and other government variables, they estimate parameters for a panel with a big number of countries.
growing countries (0.2) to the level of the fastest-growing countries (0.6) would raise its growth by 1.2 per cent.

Figure 1.1. Liquid liabilities output ratio against previous per-capita growth.¹⁰

King and Levine's (1993) cross-country evidence illustrates how resources are allocated to financial activities among economies with different development characteristics. For example, they find a correlation of 0.72 between real per capita GDP in 1970 and the proportion of domestic credit granted by deposit banks. They also find a 0.61 correlation coefficient in 1985 (op. cit. Table 6.1, p. 167 and Table 6.2, Source data from R. G. King and R. Levine, World Bank (1994), Regressions estimated by OLS using between 79 and 93 countries in 7 time periods; 1960 to 1990 in five-year periods. Liquid liabilities are defined as M3, money; coins and notes in circulation plus all other private-sector bank deposits and certificates of deposit (Buttler and Isaacs (1996)); these are measured at the beginning of each period. Liquid liabilities corresponds to lines 34+35 in the IMF's International Finance Statistics. Liquid liabilities output ratio is defined as liquid liabilities divided by real GDP in the same period. Real GDP is taken from Penn World Table mark 5.

¹⁰ Source data from R. G. King and R. Levine, World Bank (1994). Regressions estimated by OLS using between 79 and 93 countries in 7 time periods; 1960 to 1990 in five-year periods. Liquid liabilities are defined as M3, money; coins and notes in circulation plus all other private-sector bank deposits and certificates of deposit (Buttler and Isaacs (1996)); these are measured at the beginning of each period. Liquid liabilities corresponds to lines 34+35 in the IMF's International Finance Statistics. Liquid liabilities output ratio is defined as liquid liabilities divided by real GDP in the same period. Real GDP is taken from Penn World Table mark 5.
These figures indicate an increasing level of activity performed by deposit banks for increasing levels of per capita GDP. In addition, they also find a correlation 0.53 and 0.70 in 1970 and 1985 respectively between GDP per capita and the ratio of gross claims on private sector to GDP. When measuring financial size by liquid liabilities and comparing the richest top 25 per cent against the poorest 25 per cent, they find that wealthier countries held 31 per cent of a year's income in liquid assets. At the other extreme in 1970, citizens of the poorest countries held only 5 per cent of a year's income. In 1985, the proportions where 60 per cent against 11 per cent. They confirm the presence of increasing liquidity in wealthier economies.

Figure 1.1 shows the relation between the five previous years growth in per-capita output and the five-year average of liquid liabilities output ratio. We use data from the World Bank's financial growth data set (R. G. King and R. Levine, 1994) covering 119 countries and seven time observations, equally spaced five-year periods, covering from 1960 to 1990. Each line and group of observations is a cross-country comparison of per-capita GDP growth and their initial level of liquid liabilities. The figure illustrates a very strong positive relation between these two variables. However, as shown in figure 1.2, the correlation between the five previous years' average of liquid-liabilities level is highly correlated with the current five-year average of liquid-liabilities-output-ratio. This suggests that finding the right effect of growth over financial size will be an arduous exercise.

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This was in addition to their monetary liabilities; King and Levine (1993) use for these comparisons the measurement:
b.- Measuring financial sophistication and the financial functions.

One difficulty with cross-country comparisons is the difference in structure and operation of the different financial systems, and therefore to find the relevant measure of financial activity. A first consideration is whether to focus on financial intermediaries like banks or rather examine measurements on stock market activity.

\[
(\frac{LL-M1}{GDP})
\]

12 Source data from R. G. King and R. Levine, World Bank (1994). Regressions estimated by OLS using between 79 and 93 countries in 7 time periods; 1960 to 1990 in five-year periods. Liquid liabilities are defined as M3, money: coins and notes in circulation plus all other private-sector bank deposits and certificates of deposit (Buttler and Isaacs (1996)); these are measured at the beginning of each period. Liquid liabilities correspond to lines 34-35 in the IMF's International Finance Statistics. Liquid liabilities output ratio is defined as liquid liabilities divided by real GDP in the same period. Real GDP is taken from Penn World Table mark 5.
Although it is generally agreed that financial intermediaries and stock markets perform different functions, there is some intersection in their roles.

The liquid-liabilities-to-output ratio has been used often and successfully, by for example: Atje and Jovanovic (1993), King and Levine (1993) and Barro and Sala-I-Marti (1995). Turnover, market capitalisation or bank lending has also been tested against growth. However, these measures of the size of the financial institutions might not be an appropriate approach when examining the operation of the financial sector or, at least, it might not provide a complete picture. The amount of debt and equity borne by firms may be unrelated to the easiness with which these are available; for example differential tax treatments will normally bias against equity.

Stiglitz and Weiss (1981) argued that when the issue of new security, by companies, is seen as a signal of the quality of the borrowing, this makes equity finance more expensive, as debt investments are more secure and therefore lower the cost of debt relative to equity. Therefore, the choice between debt and equity might depend on elements outside of the financial system.

For example, Arestis et al. (2001) use data from industrialised countries to analyse the relationship between stock-market development and economic growth. They find, both banks and stock markets promote economic growth, but identify stronger effects from financial intermediaries than from stock-market activity.

In addition, the size or the organisation of the financial sector is not necessarily what matters, if they do not reflect how well the financial system is able to perform those
growth-enhancing functions. A useful measure of financial activity should reflect the way the financial structure in place performs the functions defined by authors like Merton and Bodie (1995) or Levine (1996).

Measurements of financial repression, or degree of liberalisation, may act in opposite directions with regards to their link to economic growth. The neo-liberal approach to financial liberalisation was based on the assumption that financial growth was beneficial for economic growth. By extension, this approach argued that any impediments to financial deepening, specifically nominal interest rate ceilings, exchange rate controls, and strict import-tariff and quota restrictions, should be eliminated. However, a measurement based on the degree of liberalisation will not account for agency problems in the financial system. Summing up, a measurement of financial easiness will not register quality shortcomings and can be negatively correlated with growth in economies where these shortcomings exist.

4.- Overview of the thesis.

In Chapter 2 we study the causality tests between financial development and growth. We concentrate on the small sample bias arising in such tests when using individual country time-series data. Small sample bias problems can be particularly relevant for developing countries where data is sparse. We find that statistical inferences derived from dynamic models with short time series, present significant biases. We find expressions for the biases in the estimation of the relationship between economic growth and various measures of financial sophistication. We apply our results to a
database from Latin America and the Caribbean from 1970 to 1995. This period covers increasing financial liberalisation in these markets. We study the causality relationship between a variety of bank credit and monetary measurements and economic growth. These financial development variables are: liquid liabilities, quasi-liquid liabilities, private non-guaranteed debt, credit to the private sector and domestic credit provided by the banking sector. Our conclusions indicate a non-significant relationship between these variables and growth.

In Chapter 3 we analyse the evolution of stock market activity as measured by asset prices and the coverage of the markets within the whole economy. We study a model of the relationship between risk and return, under market frictions prevalent in developing economies. We find that market frictions like compulsory holdings and short sale restrictions have a negative effect on asset prices for given levels of risk. In particular, we show that growth in stock-market activity, wherein an increasing number of firms are traded, brings about increasing asset prices.

In Chapter 4 we study the evolution of these frictions using Chilean stock market data. Chilean financial markets have experienced extensive development, increased participation, deregulation and new legislation over the last two decades. We analyse a financial database of equities, traded on the Chilean stock market between 1989 and 2000. We find some evidence that, over time, Chilean markets have increased their coverage, pushing asset prices up, but that this growth is not necessarily a permanent feature in this economy. To study this process we also develop in Chapter 4 a non-linear estimation method based on seemingly unrelated regression. We demonstrate that these estimates enjoy such properties as efficiency, unbiasedness and asymptotic
normality.

In Chapter 5 we analyse financial development and growth in the context of growth models. In particular, we study the empirical effects of time aggregation in terms of the biases, and efficiency of the estimates. We show that the most appropriate procedure to test the effect of weakly exogenous variables entails using the data in the most disaggregated form possible. We do not find reasons to average the explanatory variables over periods of time. Likewise we find no reasons to drop the observations in the interior of the time sample, and work with the rate of growth between both ends of the period, as is usually done in growth regression using countries' cross sections. In addition, we present Monte-Carlo evidence on the biases and inefficiencies incurred when using phase averages for different levels of aggregation.

Following these principles, we disaggregate over time and reassess the empirical evidence from the finance-led growth literature. We find some positive but weak effects of financial depth over growth. Finally, we study this model over different levels of time aggregation for a database of South American southern countries, Argentina, Bolivia, Brasil, Chile, Paraguay and Uruguay. In particular, we study the benefits of private credit over growth and find no significant evidence that this variable produces positive effects on growth.

The main objective of this study is to assess the wealth-enhancing effects of the financial system. Our overall finding is that such evidence that exists in the literature of effects flowing from financial liberalisation to growth is contaminated by faulty statistical procedures. Our assessment does not find strong evidence of such effects.
Chapter 2

Short Sample Bias and Causality Tests

1. - Introduction and Background

This chapter examines the properties of causality tests in the context of economic growth and development using small-sample data. In particular, we perform such causality tests for the relationship between economic growth and a number of different measures of financial deepening.

Performing individual country tests, using time series, allows us drawing conclusions about the significance of the causality relationship in that particular country. Given the specific policies that have been implemented in that region during the time period it would be possible to draw conclusions about the usefulness of these policies to stimulate the variables of interest. However, when using yearly data for an individual country the problem of small sample usually arises, in particular when using data from developing countries. Kendall (1954) first identified the problem where statistical inference from dynamic models, using short time series, presents significant biases due to the finite size of the sample. These biases are likely to affect the conclusions based on single time series
and yearly data. Subsequently, White (1961) provided approximations and Sawa (1978) identified an exact formula for the bias in the auto-regressive model.

The use of panels of data rather than individual country time series opens the possibility of increasing the degrees of freedom to measure the average effect across regions or countries. In the case of causality tests, when using panels that combine data from several countries, it becomes possible to draw conclusions that go beyond particular policies that have been implemented in each country. However, differences in parameter size in the cross section are likely to produce problems with the estimation method. Robertson and Symons (1992) study the properties of long and short time series and long and short panels. They find that even for small parameter variation, biases can be severe. They also find that allowing for fixed effects to control for panel heterogeneity, an Anderson and Hsiao-type estimator, does not reduce the problems in practice. The problem with the false imposition of parameter equality among panels is that it induces serial correlation of the residuals. This, in turn, increases the problem of bias in the auto-regressive model.

Country specific-fixed or random effects can produce problems with the estimation, in the presence of non-independent or correlated error terms. These estimation problems are similar to those arising in time series when a serially correlated error term is not independent from the explanatory variables; for example lagged values of the dependent variable. The use of instruments as earlier values of the dependent variable is not always possible, as for most dynamic structures of the model, they are still correlated with the error term. Therefore, individual country time-series regression has the advantage over
panel estimation, of not having this country fixed or random effect problem.

Pesaran and Smith (1995) study the properties of stationary regressors and, pooling time series and cross section data, they find that this produces inconsistent estimates. They also identify a tendency of fixed effects to underestimate short run effects and overestimate long run effects. In work related to endogenous growth estimation Lee, Pesaran and Smith (1998) identify inconsistent estimators, when there is county heterogeneity in growth effects and in speeds of convergence.

Caselli, Esquivel and Lefort (1996) study this problem in the context of endogenous-growth regressions. In their panel estimation they identify two problems: first, countries' fixed effects are correlated with the right-hand-side variables and second, some regressors are endogenous. These two problems will induce to biased estimates in most cases. Furthermore, estimating the within model or performing quasi-demeaning will not solve the problem. Taking away the mean, or a fraction of it, from the explanatory variables produces variables that are correlated with all past and futures observations of the error term. When quasi-demeaning, unless the theta 'effect' is very small, the correlation between the regressors and the error term will be important. The addition of endogenous regressors to non-independent error term increases the problems with the estimates. Caselli et. al. propose an Arellano-Bond-type estimation method to account for this problem. They identify strikingly faster convergence rates, of a country to its own steady state, than was previously believed to be the case.
Finally, Attanasio, Picci and Scouru (1998) study causality test using panel data techniques and applying the Arellano-Bond method and coefficient heterogeneity. Under these assumptions, they find significant differences in the causal relationships substantiated by each technique.

The rest of this chapter is organised as follows. In Section 2 we study the asymptotic properties of causality tests in relation to different stochastic representations of the process. We identify different conditions that allow for OLS consistency under AR(1) regressors. This is necessary as the short sample formulas rely on consistency of the estimators. In this section we also set out the main specification of the test that will be used to establish the size of bias. In Section 3 we use the specification from section 2 and identify formulas for the bias on the lagged dependent variable and other regressors. This section makes extensive use of recursive formulas for expectation of stochastic matrices. In Section 4 we study the biases over the usual Wald F-statistics, using simulation techniques, under different lengths of the dynamic structure. In Section 5 we use these findings to analyse causality tests on a Latin-American and Caribbean financial, banking oriented, database. Finally, Section 6 concludes the chapter.

2. - *Stochastic representation and asymptotic properties.*

The study follows an approach similar to those in Grubb and Symons (1987), Kiviet and Phillips (1993), Galbraith et al. (1999), Kiviet et al. (1999), in order to study the biases generally observed in the AR(1) OLS estimation and apply these result to the causality
test. In their papers, these authors study the biases affecting OLS estimated parameters from one time series. These biases are present in regressions using small samples, while OLS will normally be consistent if the error term is i.i.d.

Takamitsu Sawa originally studied this problem in 1978. The intuition behind his results is that the error term $\theta$ cannot be independent and is not necessarily uncorrelated with the term $\sum y^2_{t,k}$. At least some of the terms in this summation will be correlated with each error term and thus biases will arise as the difference between the OLS estimate and the population parameter which is$^{13}$:

$$\hat{\beta} - \beta = \frac{\sum u_t y_{t,1}}{\sum \gamma^2_{t,1}}$$

We will also examine here the biases emerging on a test statistic based on the reduction of the sum of the squared residuals, for example Wald or multiple restriction test, when relaxing a null hypothesis that a subset of the parameters are zero.

**a. - Moving average representation and the causality tests.**

In order to obtain approximate formulas for the biases under study, we are interested on

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$^{13}$ In general the results of the studies listed above apply only to regressions over a single time series. In the context of dynamic panels of data there can be extra difficulties. Standard estimation techniques as OLS or fixed effects may be inconsistent. Nickell (1981) shows the $N$ inconsistency for homogeneous dynamic panels, Pesaran and Smith (1995) show the large $T$ inconsistency for heterogeneous panels.
the stochastic process generating the time series. Here we follow closely the analysis in Geweke (1984) to set out the properties of these series. Let us define the vector of time series under study as a vector of size $m$, $q_t$. We focus on wide sense stationary and purely non-deterministic time series. Let us assume that there is a moving average representation for $q_t$, so that it can be written as:

$$q_t = \sum_{\theta} A_\theta u_t, \quad E[u_t] = 0, \quad \text{var}(u_t) = \Omega$$

Here, $A_\theta$ is a sequence of deterministic matrices such that none of the roots of the generating function $\sum_{\theta} A_\theta q^\theta$ falls below one in absolute value$^{14}$. We also require from this sequence that $\sum_{\theta} ||A_\theta||_2^2 < \omega_0$ where $||.||_2$ is the natural matrix norm induced by the L2-norm$^{15}$. Therefore, we are associating our time series with a system where the observations are generated by the innovations that constitute vector $u_t$. The set of variables in vector $q$ are affected by different proportions of $m$ different innovations in every time period$^{16}$. Given the square-summability condition above, the effect of a previous innovation tends to disappear after a long time. As in Geweke (1984) we assume that the innovations $u_t$ are not serially correlated; so the only way the error term $u_t$ affects the current value of $q_t$ is

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$^{14}$ As in Geweke (1984), for complex roots, they should be no less than one in modulus.

$^{15}$ Also known as the spectral norm and associated with the maximum eigenvalue of the product between the matrix and its adjoint.

$^{16}$ We can think of the factors $u_t$ as only a few and less than $m$. However, it is more likely that these
through the coefficients in the matrix $A_t$.

We are interested in an $AR(1)$ expression of this model. This moving average representation can be inverted if $q_t$ can be expressed in terms of $u_t$ and $Q_{t-1}$, the set of past observations of $q$:

$$u_t = \sum_{s=0}^{\infty} B_s q_{t-s}, \quad \text{or} \quad q_t = -\sum_{s=1}^{\infty} B_s q_{t-s} + u_t,$$

Here, $B_s$ is a sequence of deterministic matrices\(^{17}\) that produce the linear projection of $q_t$ on $Q_{t-1}$.

A condition required for us to obtain an inverted system as above is that there exists a real constant $c$, larger than one, such that when subtracted from the spectral density matrix the outcomes are positive definite matrices as follows: $f(\lambda)-c^{-1}I_m$ and $c I_m - f(\lambda)$ are positive definite. This property is much more than just asking for the coefficients $A_t$ to go to zero; we need that these coefficients to decrease at such a rate that the spectral density $f(\lambda)$ can be bound away from zero\(^ {18}\).

The conditions above allow us to analyse finite or infinite order moving averages and the random factors are many (or infinite) and so they can be grouped into $m$ non-perfectly correlated factors.

\(^{17}\) Without any loss of generality $B_0=I$.

\(^{18}\) In a simple example with real coefficients if $q_t= u_t + A_1 u_{t-1}$, the modulus $|I+A_1|$ has to be strictly larger than zero. If $q_t= u_t + A_1 u_{t-1} + A_2 u_{t-2}$, then the modulus $|I+A_1+A_2|/ |I-A_1+A_2|$ have to be strictly larger than zero.
causal relationships they imply. If we now take a vector \( q_t \) with two components \( y_t \) and \( z_t \) and if the conditions above are met then \( q_t \) is invertible and it can be written as the projection onto the set of past observations \( Q_{t-1} \) and \( u_t \).

\[
q_t = - \sum_{s=1}^{t-1} B_s q_{t-s} + u_t,
\]

This can be written in terms of the components \( y_t \) and \( z_t \) as follows:

\[
y_t = - \sum_{s=1}^{t-1} (\beta y_{t-s} + \gamma z_{t-s}) + u_{yt}, \quad \text{and} \quad z_t = - \sum_{s=1}^{t-1} (\delta y_{t-s} + \lambda z_{t-s}) + u_{zt},
\]

Under the assumption that the invertibility conditions hold for \( q_t \), analogous conditions will be met for each of the components\(^\text{10} \), and so each of them can be inverted individually; in particular \( y_t \) will have an auto-regressive expression:

\[
y_t = - \sum_{s=1}^{t-1} b_s y_{t-s} + v_{yt},
\]

\( Z \) causes \( Y \) if and only if the \( \chi \) are different than zero or equivalently if \( \text{var}(\chi_{yt}) \) is strictly larger than \( \text{var}(\chi_{yt}) \).

The formula above is the projection of \( y_t \) on the set of past observations \( Y_{t-1} \). Comparing this formula with the projection of \( y_t \) on the set of past observations \( Y_{t-1} \) and \( Z_{t-1} \), we can conclude that \( Z \) causes \( Y \) if and only if some of the \( \chi \)'s are different than zero or

\(^{10}\text{See Geweke (1984, page 1110) for a proof of this. The idea is if the matrices } f(\lambda) - c' I_m \text{ and } c I_m f(\lambda) \text{ based on the spectral density of the system are positive definite then similar bounds can be obtained for each variable } y \text{ and } z.\)
equivalently if \( \text{var}(v_{yr}) \) is strictly larger than \( \text{var}(u_{yr}) \).

In order to study causality between \( Y \) and \( Z \) we can perform multiple-restriction tests on the sum of the residuals using the restricted and unrestricted versions of the model considered earlier. In addition, we can perform individual tests on each of the coefficients \( \gamma_s \), \( (s=1,2,3,\ldots) \). In subsequent sections we will analyse the properties of these tests in a small sample.

**b.- Auto-regressive representation and the causality test**

The causality test specification outlined in the last section can also be retrieve from a vector of variables \( q, \) that is represented as an auto-regression. Let us assume that the variables in \( q, \) respond to their own past values and to some random not serially-correlated shocks. Following Anderson (1971) an auto-regressive representation for \( q, \) is:

\[
q_t = - \sum_{s=1} A_s q_{t-s} + u_t, \quad E[u_t] = 0, \quad \text{var}(u_t) = \Omega
\]

According to the analysis in the last section, if there are two components, \( y, \) and \( z, \) in a vector of variables the auto-regressive representation for \( y, \) can be written as:

\[
y_t = \sum_{s=1} \left( \beta y_{t-s} + \psi z_{t-s} \right) + u_{yr},
\]

Applying again the invertibility conditions found in Geweke (1984), \( q, \) can be written as a moving average:
Finally, using the lag operator from the auto-regressive expression for $y$, we can write:

$$y_t = \sum_{i=1}^p \beta_i y_{t-i} + \sum_{i=1}^q \gamma_i t^{0} B_{s} u_{t-i} + \epsilon_{t}.$$

In a similar fashion as in the previous section, the formula above is the projection of $y_t$ over the set of, exclusively, its own past observations $Y_{t-i}$. Given that the terms arising from $Z = \sum_{i=1}^p \gamma_i t^{0} B_{s} u_{t-i}$ depend on all $u_i$ and it has been assumed that these are not serially-correlated, $Z$ causes $Y$ if and only if some of the $\gamma$ are different than zero. Again, for $Z$ to cause $Y$, the variance of the model including $Z$ will have to be strictly smaller than the one excluding it.

**c.- Conditions for OLS consistency under AR(1) regressors.**

Often biases in OLS estimation of AR(1) processes are asymptotically zero; for example, in Grubb and Symons (1987), the estimated parameters are consistent. Their OLS estimations are consistent given the auto-regressive nature of the variables involved and given that the error term is independent, equally-distributed and normal. Under these conditions lagged values of the dependent and explanatory variables, involved in the estimation, are uncorrelated with the error term.
In the case of causality tests, variable $z_{i,t}$ is, by assumption, correlated with $y_{i,t}$ and with past values of the error term. Both of the explanatory variables are $AR(1)$; depending on the parameters, they can be seen as linear combinations of two correlated univariate $AR(1)$'s. This makes establishing consistency more difficult than in the case of independent and identically-distributed explanatory variables.

A different approach to study consistency in auto-regressive models can be found in Anderson (1971, p. 208, Lemma 5.5.10), using more restrictive assumptions. Using the theory developed by Mann-Wald (1943) it is shown that $\text{plim} \hat{\beta} = \beta$, where $\hat{\beta}$ is the maximum likelihood estimator and in Anderson’s case, assuming normal disturbance, $\hat{\beta}$ is the same as the OLS estimator. Anderson presents a simple proof for a Gaussian multivariate $AR(1)$ estimated by multiple-equation maximum likelihood. In that study, the conditions over the model and variables are: firstly, that the error term has to be independently distributed. Secondly, the error term has to be either identically distributed or verify $E[|u_t|^4] < M$ for some $\delta$ and $M$. Two more restrictive assumptions are a fixed (not random) explanatory variable $z_t$ and that this variable has to verify a series of square summability conditions. Finally, Anderson also requires that the eigenvalues of the

\[ \frac{1}{T} \sum_{t=1}^{T} (z_t)^2, \]

These are: $z_t^2$ uniformly bound for all $t$ by some constant $M$ and the existence of the following limits when $T \to \infty$:
transition matrix are all below one in modulus; in our case this translates to $\beta < 1$. Under these conditions the OLS estimators, that is $\beta$, $\gamma$, and $\sigma^2$, will converge in probability to their population counterparts. Under normality assumptions these estimators will also be the maximum-likelihood estimators. Anderson's analysis provides asymptotic properties for maximum-likelihood estimation, however, under these regularity conditions OLS converges to the true parameters, regardless of the distribution of the error term.

Finally, when $z$ is not random, Anderson requires that this series does not grow too fast so the sums $\sum_{i=1}^{T} (\sum_{j=0}^{T-2} (-\beta)^j z_{i-j-1})^2 / T$ converge. As $\beta$ in our model, is below one, this formula means that $x_t$ can be allowed to grow but not so fast that this growth overcomes the damping effect of $\beta^t$. For example, if $z_t$ and $z_{t-1}$ were related by the non-random equation: $z_t = \rho z_{t-1}$, the square-summability conditions imply $|\rho| < 1$.

Although, in our case, we want to use OLS for a univariate system, the formula for the estimator will be the same as Anderson's multivariate likelihood-maximisation method outlined above. If our problem was not a causality test, and the variable $z$ was not stochastic, we could directly apply the result in the last paragraph. However, the necessary conditions when the variable $z$ is random are not too different to the ones required for non-stochastic $z$. In the former case, we require $E/z_t^2 / \gamma^8$ for some $\gamma > 0$ and

$$\frac{1}{T} \sum_{i=1}^{T} (\sum_{j=0}^{T-2} (-\beta)^j z_{i-j-1}) z_t$$

and

$$\frac{1}{T} \sum_{i=1}^{T} (\sum_{j=0}^{T-2} (-\beta)^j z_{i-j-1})^2$$
$r > 1$ to be uniformly bound. The more memory the random variable $z$ has, the larger $r$ will have to be and thus we require more of the moments of it to be uniformly bound. It is also required that the following matrix, defined for the random variable $z$, is uniformly positive-definite:

$$\frac{1}{T} E \left[ \sum_{t=1}^{T} (z_t)^2 \right].$$

This is more than the simple existence of some limits over $z_t$ which is required when the explanatory variable is non-random.

Mann and Wald (1943) established the consistency and asymptotic normality of OLS when the regressors include lagged values of the dependent variable and other i.i.d. regressors. In that case it is necessary that all the roots of the auto-regressive part of the equation fall outside the unit circle$^{21}$. In addition, the error term in the current period should consist of only current innovations and must not be correlated with previous realisations of the regressors or previous realisations of the dependent variable$^{22}$. Finally, the regressor $z_t$ must be stationary and ergodic. Stationary means that correlations between values of $z_t$ at different points in time can vary only with how far apart they are and not with which particular moment in time, they are calculated. Ergodicity means that any two events are, at the limit, independent when one of them is shifted to the future. In

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$^{21}$ Corresponding to the invertibility conditions studied here in sections a.- and b.-.

$^{22}$ More Specifically $E[u_t | x_{t-1}, y_{t-1}] = 0; E[u_t^k | x_{t-1}, y_{t-1}] = M < \infty$, for $k \geq 2; E[y_t | x_t] = 0$; and $E[u_t y_t] = 0$. 

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this context, a moving-average process, with constant parameters and regularity conditions on the distribution of the disturbances, will follow the ergodic property. As the correlation disappears for two events, sufficiently far away, the average probability of two events, when one of them is repeatedly shifted to the future, will equal the product of the probabilities.

In our treatment of the problem when a non-infinitely lived auto-regressive process, that includes lagged values of variable \( y \), generates the right-hand side variable \( z \), the observations will also be heterogeneously distributed. In order to establish limits in the way \( z \) can evolve, so that consistency holds, we will use the mixing condition. This condition prevents \( z \) from drifting away, by accumulating all the past innovations, without disremembering a substantial part of these shocks over time. Therefore, on testing whether \( Z \) causes \( Y \), the problem is establishing whether one can reasonably believe that the variable being tested, \( z \), verifies one of the mixing conditions.

The result we wish to apply, to prove consistency, is theorem 3.51 in White (1984 pp. 47-48). In this, if the explanatory variables in a model, display either the uniform or the strong mixing conditions, then the OLS estimated parameters would converge almost-surely to their population counterparts. In particular, exercise 3.53, establishes sufficient conditions for a model like the present one, these condition that are \( \beta < 1 \) and \( \gamma < \infty \) and that \( z_{t-1} \) and \( y_t \) have to be uniformly or strongly mixing. As we will see below, the uniform and mixing conditions are not so easily satisfied by a general \( AR(1) \) process. A very popular case where this condition holds, is when the observations of \( z_t \) are generated from an
AR(1) process with normally-distributed disturbances.

The strong mixing condition, $\alpha(m)$, reflects the degree of dependence between events separated by at least $m$ time periods. If $\alpha(m)$ goes to zero, as $m$ grows, then the events are asymptotically independent. Formally, $\alpha(m)$ is defined as an upper bound for the dependency between Borel sets, separated by $m$ periods. For a real valued random variable $z$, the Borel field $B$ is the smallest collection of sets including: (a) all sets $\{z|z \in \mathcal{G}\}$; (b) all the complements of such sets, and; (c) all unions of such sets. For the observations $n$ through $n+m$ of a time series, $\{z_k(w), k=n,...,n+m\}$, the Borel field $B_{n+m}^n$ is extended as the cartesian product of the Borel sets for each individual observation, all the complements and all the infinite unions. The Borel sets represent the information contained between observations $n$ and $n+m$. Similarly, the Borel sets $B_n^m$ and $B_{n+m}^m$ are the smallest collection of unions of $B_n^m$ and $B_{n+m}^m$ respectively when $m$ goes to $\infty$. We can interpret $B_{n+m}^m$ and $B_n^m$ as the collection of events able to contain all the information of the sequence $z_i$ from the past and future respectively.

To study the mixing condition we are interested in the dependency of events that become more distant over time. We define the strong and uniform mixing conditions respectively as:

$$\alpha(m) = \sup_s \sup_{\{G,B_n^m,H\in B_n^m\}} |P[H \cap G] - P[H]P[G]|, \quad \text{and}$$

$$\phi(m) = \sup_s \sup_{\{G,B_n^m,H\in B_n^m\}} |P[H | G] - P[H]|$$
For example, a Gaussian AR(1) process is a mixing sequence with $\sigma(m) = O(m^{r})$ with $r > r/(r - 1)$ for any $r > 1$ (White, 1984; Example 3.46, p. 46); since $\sigma(m)$ goes to zero when $m$ goes to $\infty$, this process is strongly-mixing. However, these processes are not uniformly mixing, as $\sigma(m)$ does not go to zero when $m$ becomes large. A $\gamma$-independent sequence (one where $z_{t}$ is independent from $z_{t+\tau}$ for any $\tau$ greater than $\gamma$) verify the property $\sigma(m) = \sigma(m) = 0$ for all $m$ greater than $\gamma$ and therefore they are strongly and uniformly mixing. It will be difficult for an AR(1) to be $\gamma$-independent unless the distribution of the error term is time-dependent, so that the correlation becomes exactly zero between $z$'s for two points distant in time.

We want to relax the Gaussian assumption over the innovations which intervene in the

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23 Defined as those auto-regressive processes where the error term is independent, identically-distributed and normal with mean zero and variance 1.

24 This is based on Ibragimov and Linnik's Theorem 17.3.2 (1971, p.312), where a Gaussian sequence satisfies the uniform mixing condition only if the Borel fields $B_{n}$ and $B_{n+m}$ are independent for all $n$ for some sufficiently large $m$. We can see from this that this property holds for moving averages but not for auto-regressive processes.

25 Ibragimov and Linnik (1971) introduce an example of uniformly-mixing sequences; this is the homogeneous Markov chain with a finite number of states and for which the probability of moving from one state to another is not zero. They also present a stationary sequence based on a normal moving average that does not verify any of the mixing condition and whose average does not converge to a normal.
processes of generating variables $y$ and $z$, but keep the independent equally-distributed property of these sequences of innovations. For a more general case, when $z_t$ is not necessarily Gaussian or $\gamma$-independent, let us define simple events as intervals in the following manner:

$$G = \ldots \bigcap_{i=0}^\infty \mathbb{R} \bigcap_{j=0}^\infty \mathbb{R} \bigcap_{k=0}^\infty \mathbb{R} \ldots$$

and

$$H = \ldots \bigcap_{i=0}^\infty \mathbb{R} \bigcap_{j=0}^\infty \mathbb{R} \bigcap_{k=0}^\infty \mathbb{R} \ldots$$

Here, the interval in event $G$ is in the position corresponding to index number $n$ and the interval in event $H$ is in the position corresponding to $n+m$. In this set-up, and using the strong mixing condition, we can calculate the degree of dependence between the two events $H$ and $G$ by computing the probability:

$$\frac{P[H \cap G] - P[H]P[G]}{P[H]P[G]}$$

$$= P\left[\bigcap_{i=0}^\infty \mathbb{R} \bigcap_{j=0}^\infty \mathbb{R} \bigcap_{k=0}^\infty \mathbb{R} \ldots \right] - P\left[\bigcap_{i=0}^\infty \mathbb{R} \bigcap_{j=0}^\infty \mathbb{R} \bigcap_{k=0}^\infty \mathbb{R} \ldots \right]$$

The $AR(1)$ structure of our random variable $z_t$ allow us to separate $z_{n+m}$ into two components $z_n$ and $w_{n,m}$:

$$z_{n+m} = \sum_{i=0}^{m-1} \rho^i u_{n+i} + \sum_{i=0}^{m-1} \rho^i u_{n+m-i}$$

$$x_{n+m} = \rho^n z_n + \sum_{i=0}^{m-1} \rho^i u_{n+m-i} = \rho^n z_n + w_{n,m}$$

$$w_{n,m} = \sum_{i=0}^{m-1} \rho^i u_{n+m-i}$$

The components $z_n$ and $w_{n,m}$ will be independent only under the independence over time
assumption about the innovation term $u_n$. It follows that:

$$|P[H \cap H] - P[H]P[G]| = |P[z_0 \in A] - P[z_0 \in A \cap z_1 \in B]|$$

$$= |P[z_0 \in A] - P[z_0 \in A \cap z_1 \in B]| - P[z_0 \in A \cap z_1 \in B]$$

$$+ P[z_0 \in A \cap z_1 \in B] - P[z_0 \in A \cap z_1 \in B]$$

Figure 2.1, below, depicts the events in the vector space of $z_n$ (or $p^m z_n$) and $w_{n,m}$. The diagonal line represents the event $z_{n,m} = z_n + p^m w_{n,m} = a_{n,m}$, the vertical line represents $z_n = a_n$.

The measure of dependence between events corresponding to the strong mixing conditions is:

$$\alpha(m) = \sup_n \sup_{[a, \infty)} |P[H \cap G] - P[H]P[G]|$$

$$= \sup_n \sup_{[a, \infty)} |P[z_n \leq a_n] \cap [z_{n,m} \leq a_{n,m}]| - P[z_n \leq a_n]P[z_{n,m} \leq a_{n,m}]$$

$$= \sup_n \sup_{[a, \infty)} |P[A] - P[A \cap C]P[A \cup B]|$$

Figure 2.1. Plot of $z_n$ and $w_{n,m}$ space.

Sets $A$, $B$ and $C$ in figure 1 are disjoint and they are respectively: the area below the
diagonal and to the left of \( z_n = a_n \) (the shaded area), the area below the diagonal and to the right of \( z_n = a_n \) and the area above the diagonal to the left of \( z_n = a_n \). In order to calculate \( \alpha(m) \) we hold the distance \( m \) fixed and find the supreme over the sets \( C \) and \( B \). If \( z_n \) is non-identically distributed then also the supreme over \( n \), holding \( m \) fixed, should be taken; this would be the case when the initial value, \( z_0 \), is a fixed value or a random value drawn from a independent process.

When \( m \) goes to \( \infty \) the diagonal line in Figure 2.1 \((z_n + p^{m}w_{k,m} = a_{n,m})\) becomes more horizontal. The difference in value between \( P[A] \) and \( P[A \cup C]P[A \cup B] \) will be exactly zero when the line dividing \( A \cup B \) and \((A \cup B)^c\) is horizontal because \( z_n \) and \( w_{n,m} \) are independent. Therefore, if the distributions of \( z_n \) and \( w_{n,m} \) were regular enough, so that it was possible to interchange limits and supreme in the strong mixing condition, then the following will be true:

\[
\lim_{m \to \infty} \alpha(m) \leq \sup_n \sup_{P[A \cup C]P[A \cup B]} \lim_{m \to \infty} |P[A] - P[A \cup C]P[A \cup B]| = 0
\]

More generally, the strong mixing conditions holds, as long as the supreme over \( n \) and \( B \) and \( C \) of the expression above, is a decreasing function of \( m \). A simple case that illustrates how the mixing condition operates is when the variables in the axis of Figure 2.1 are

---

26 The set \( A \) is simply the intersection between \( B \) and \( C \).

27 If the process generating the data were a finite moving average then that line would become exactly horizontal for all values of \( m \) larger than the length of the moving average.
distributed uniformly\textsuperscript{28}. Without loss of generality, we can analyse the case when $z_n$ and $w_{n,m}$ are distributed uniformly between zero and one. The support of the joint distribution of these two variables is the bounded and closed subset of $\mathbb{R}^2 \times [0,1] \times [0,1]$. In this case the probabilities we need to calculate are simply the areas of the portions of $A$, $B$ and $C$ confined within $[0,1] \times [0,1]$. Given the dimensions of the distributions' support, in this example, we only need to consider the mixing coefficient for sets $B$ and $C$ defined by values of $a_n$ and $a_{n+m}$ below one. The following proposition examines the strong mixing condition for this process.

**Proposition 1:** Under the assumption that both $a_n$ and $a_{n+m}$ are below one and if $z_n$ and $w_{n,m}$ are distributed uniformly between zero and one:

\[
\alpha(m) = \rho^m / 8
\]

\[
\lim_{m \to \infty} \alpha(m) = 0
\]

**Proof:** The area of $A$ is simply the trapezoid with one horizontal length equal to $a_n$ and two vertical lengths equal to $a_{n+m}$ and $a_{n+m} - \rho^m a_n$, therefore:

\[
P[A] = a_n a_{n+m} - \rho^m a_n^2 / 2
\]

\textsuperscript{28} A note of caution must be made here, if $Z$ is generated as an auto-regressive process, variables $z_n$ and $w_{n,m}$ cannot be distributed as uniform, except as an approximation. However, the support of these distributions can be a compact set. This example then illustrates how the mixing coefficient, $\alpha(m)$, behaves for large values of $m$. 

50
The area of $A \cup C$ is simply the value $a_n$.

$$P[A \cup C] = a_n$$

The area $A \cup B$ is the zone defined between the two axis $z_n=0$ and $w_{n,m}=0$ and the diagonal line $z_n = \rho_m w_{n,m} = a_{n+m}$. As $m$ gets larger the intersection of this diagonal line and the axis $w_{n,m}=0$ moves to the right. As we are interested in the limit values of $\mathcal{A}(m)$, the case of smaller values of $m$, when this intersection is still below $a_n$ or 1, is not relevant to the argument here. We will only calculate the probability of $A \cup B$, as in Figure 2.2, when the diagonal intersects the horizontal axis in a value larger than 1; i.e. $m$ is large enough such that:

$$\rho_m < a_{n+m}.$$ 

**Figure 2.2. Plot of $z_n$ and $w_{n,m}$ distributed uniform [0,1].**

![Figure 2.2](image.png)

Therefore, the area of $A \cup B$ is the trapezoid with one horizontal length equal to 1 and two
vertical lengths equal to \( a_{n+m} \) and \( a_{n+m-\rho} \), therefore:

\[
P[A \cup B] = l(a_{n+m} + a_{n+m-\rho})/2 = a_{n+m-\rho}/2
\]

\[
P[A] - P[A \cup C] P[A \cup B] = a_n a_{n+m} - \rho^m a_n^2/2 - a_n (a_{n+m-\rho}/2)
\]

\[
= \rho^m a_n (1-a_n)/2
\]

This expression achieves its uniform maximum over \( n \) when \( a_n \) is equal to 0.5 and this proves the result.

The example above can be generalised to other distributions of \( z_n \) with a bound support that is independent of \( n \).

Finally, and considering again Gaussian sequences, Ibragimov and Linnik (1971, pp. 313-314) bound the strong mixing coefficient with the auto-correlation function. Then, they established a bound for the auto-correlation function, which goes to zero with \( m \), for any Gaussian sequence having a spectral density that is bounded above zero. For example, this is the case for a AR(1) Gaussian sequence, with auto-regressive coefficient \( \rho \) and variance of the error term \( \sigma^2 \), where the spectral density is:

\[
f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{1 + \rho^2 - 2\rho \cos(\lambda)}
\]

This density is bounded from below by \( 1/2\pi(1+\rho^2) \), which is strictly larger than zero when \( \rho \) is strictly below one.
Greene (2000) provides the spectral density for other general ARMA processes. In general, the spectral density can be bound above zero for $AR(p)$ processes with coefficients below one. The conditions on the moving average part are the same conditions needed for invertibility we found in Section 2.a. above.

d. - **Biases in a simple Causality test.**

From the analysis done in sections a.- and b.-, the general Causality test will be performed based on a regression of variable $y_t$ on its lagged values and lagged values of variables from the set $z_t$. At the same time, the same test can be performed over $z_t$, regressed against lags of $y_t$ and $z_t$, to finally establish the direction of causation.

The relationship used to test $Z$ causing $Y$ is related to an auto-regressive process dependent on two inputs, one uncorrelated term, $u_t$, and one $AR(1)$. Indeed, let us take a pair of equations used to test causality, which include a single lag of each variable on the right hand side:

\[
y_t = \beta y_{t-1} + \gamma z_{t-1} + u_t
\]

\[
z_t = \phi y_{t-1} + \lambda z_{t-1} + \nu
\]

We are interested in the case $\gamma=0$ so $Z$ does not cause $Y$ but $Y$ possibly causes $Z$. We can
change the input variable \( z_t \) in the \( y_t \) equation with a purely auto-regressive process. To see this more clearly, we can define the simple auto-regressive process state variable \( x_t \), a linear combination of \( z_t \) and \( y_t \), as:

\[
x_t = \rho x_{t-1} + \eta \quad \text{and} \quad x_t = y_t + \delta z_t
\]

Replacing \( y_t \) and \( z_t \) in the previous system:

\[
x_t = \beta y_{t-1} + \gamma z_{t-1} + u_t + \delta \phi y_{t-1} + \delta \lambda z_{t-1} + \delta v
\]

\[
= (\beta + \delta \phi) y_{t-1} + (\gamma + \delta \lambda) z_{t-1} + u_t + \delta v
\]

\[
= (\beta + \delta \phi) (y_{t-1} + (\gamma + \delta \lambda)(\beta + \delta \phi) z_{t-1}) + u_t + \delta v
\]

Thus, \( \delta \) and \( \rho \) should equal the expressions \((\gamma + \delta \lambda)/(\beta + \delta \phi)\) and \((\delta + \delta \phi)\) respectively.

Finding a solution for \( \delta \) is possible only under the condition that:

\[
4\phi \gamma < (\beta - \lambda)^2.
\]

Thus, when this condition holds, we will be able to assemble our auto-regressive variable \( x_t \). By solving the following quadratic expression:

\[
\phi \delta^2 + (\beta - \lambda) \delta - \gamma = 0
\]

we obtain:

\[
\delta = \frac{(\lambda - \beta)}{2\phi} \left(1 \pm \sqrt{1 + \frac{4\phi \gamma}{(\beta - \lambda)^2}}\right), \quad \text{and}
\]
\[ \rho = \frac{(\lambda + \beta)}{2} \pm \frac{(\lambda - \beta)}{2} \sqrt{1 + \frac{4\phi\gamma}{(\beta - \lambda)^2}} \]

Using these new parameters the system can be written now as:

\[ y_t = (\beta - \gamma) y_{t-1} + \gamma x_{t-1} + u_t \text{ where } u_t \sim \text{IID } (0, \sigma_u^2) \]

\[ x_{t-1} = \beta x_{t-2} + \eta_{t-1} \text{ where } \eta \sim \text{IID } (0, \sigma_\eta^2) \]

\[ \eta_{t-1} = u_{t-1} + \delta \nu_{t-1} \]

These equations correspond to the specification in Grubb and Symons (1987, p. 374) with the addition of correlated error terms. Furthermore, if variable \( x_t \) was observable we could estimate either the equation above or the original causality relationships involving variable \( z_t \). In our model, variable \( z_t \) is only weakly exogenous, as we would expect it to be the case in causality relationships. This setting is different than the one in Grubb and Symons identified, inasmuch as they want to estimate only with the first equation and their variable \( z_t \) is strongly exogenous.

Given that \( \gamma \) is zero under the null hypothesis, the estimates using variable \( x \) should

---

29 In Grubb and Symons (1987) it is not explicit whether the process \( u_t \) and \( \eta_{t-1} \) can be correlated or not. However a condition for consistency is that \( u_t \) and \( \eta_{t-1} \) must be uncorrelated. In our case the latter is true but \( u_{t-1} \) is correlated with \( \eta_{t-1} \) and the correlation is:

\[ \text{corr}(u_{t-1}, \eta_{t-1}) = \frac{\delta \sigma_u}{\sqrt{\delta^2 \sigma_u^2 + \sigma_\eta^2}} \]
approximate $\beta$ and $\theta$ respectively$^{30}$. Because the true parameters are unknown it is not
possible to construct the state-variable $x_t$ even if we observe $y_t$ and $z_t$ we would still need
to know $\gamma$ and $\delta$.

The population and estimated parameters are different when working with a model that
uses variable $z_{t-1}$ instead of variable $x_{t-1}$. Comparing these two regressions:

\[
y_t = \beta y_{t-1} + \gamma x_{t-1} + u_t;
\]
\[
y_t = \beta y_{t-1} + \gamma z_{t-1} + u_t;
\]

And given that:

\[
\begin{bmatrix}
y_{t-1} \\
z_{t-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
-1/\delta & 1/\delta
\end{bmatrix}
\begin{bmatrix}
y_{t-1} \\
x_{t-1}
\end{bmatrix}
\]

The population parameters and the OLS estimated parameters that we obtain when using
$y_{t-1}$ and $z_{t-1}$ are related to the ones obtained when $y_t$ is regressed against $y_{t-1}$ and

\[30\] If a system like this was specified in Grubb and Symons (1987) we hold that the amount bias in the
OLS estimation of $(\beta - \gamma\delta)$ using their formula (18) would be:

\[
B = E [\hat{\beta} - (\beta - \gamma\delta)]
\]
\[
= -TB = \frac{\gamma\delta\beta(1-(\beta-\gamma\delta))\rho}{1+(\gamma\delta)(1-(1-(\beta-\gamma\delta)\rho))/1+(\gamma\delta)(1-(1-(\beta-\gamma\delta)\rho)\sigma_q^2/\sigma_p^2)} + O(1)
\]

For $\gamma=0$ as in the Granger causality test, the bias $B$ is just the bias in the OLS estimation of parameter $\beta$.
If that is the case: $\rho = \lambda$, $\delta = (\lambda-\beta)/\phi$, the condition $4\phi\gamma=0 < (\beta - \lambda)^2$ is true as long as $\beta$ is not exactly
equal to $\lambda$ and the bias formula is simpler:

\[
-T E [\hat{\beta} - \beta] = \frac{\lambda\beta(1-(\beta-\lambda))\rho}{1+(\gamma\delta)(1-(1-(\beta-\gamma\delta)\rho)\sigma_q^2/\sigma_p^2)} + O(1)
\]
x_{t-1} by the same linear relationship:
\[
\hat{\beta} = \beta - \gamma \\
\hat{\gamma} = \delta \gamma
\]

Therefore the biases when regressing \(y_i\) against \(y_{t-1}\) and \(z_{t-1}\) will be:
\[
E[\hat{\beta} - \beta] = E[\hat{\beta} - \beta] - E[\hat{\gamma} - \gamma] \\
E[\hat{\gamma} - \gamma] = \delta E[\hat{\gamma} - \gamma]
\]

The formulas above imply that, when you use \(z\) instead of \(x\), the bias on \(\gamma\) will be damped by \(\delta\) and the bias on \(\beta\) will depend on how much \(\gamma\) is biased.

In addition, if OLS estimates consistent parameters when using the model in \(y\) and \(x\), then the model in \(y\) and \(z\) does. Therefore, studying the biases in one model or another generates results that are linearly related and that are adequate in the sense that they converge to the same value in long samples. In addition, the error in approximate consistent formulas, for the bias in each model, must be linearly related and the difference between these errors must go to zero.

Therefore, this analysis indicates that we can simply search for expressions of the biases in the OLS estimation of the system:
\[
y_i = \beta y_{t-1} + \gamma x_{t-1} + u_i, \quad \text{where } \beta \sim \text{NID}(0,\sigma_\beta^2)
\]
\[
x_i = \rho x_{t-1} + \eta, \quad \text{where } \eta \sim \text{NID}(0,\sigma_\eta^2)
\]
We add to this model the distinctive feature that the error terms $u$, and $\eta$ are neither independent nor uncorrelated. We assume that the error terms $u$, and $\eta$ are no serially correlated, and that they are independent from $y_{t-1}$ and $x_{t-1}$ and identically distributed with zero mean and constant variance.

3.- Small sample and causality tests.

a.- The regression model.

Let $\beta$ be a parameter smaller than one and $\gamma$ a constant parameter so that the model is formulated as:

$$y_t = \beta y_{t-1} + \gamma x_{t-1} + u_t$$

Let us call $u$ the vector containing all past observations of the error term at time $t$ and $Y_1$ the vector containing all observations of the lagged dependent variable. Define $X_1$ as the vector with all the observations of the variables in $x_{t-1}$. The variable $x$ is defined as previously in section 2. as an auto-regressive process whose observations $x_t$ are correlated with $u_t$. We define $M$ and $W$ as:
\[
M = (I - X_i (X_i'X_i)^{-1}X_i')
\]
\[
W = (I - Y_i (Y_i'Y_i)^{-1}Y_i')
\]

**Proposition 2.** When estimating this equation by OLS the difference between the true parameters and those estimated by OLS can be written as:

\[
\hat{\beta} - \beta = Y_i'Mu / Y_i'M Y_i
\]
\[
y - y = X_i' W u / xy W X_i = (X_i'X_i)^{-1}X_i'u - (\hat{\beta} - \beta)(X_i'X_i)^{-1}X_i'y_i
\]
\[
\hat{u}'u - u'u = -u'(I + WX_i(X_i'X_i)^{-1}X_i'W)u
\]

**Proof:** See appendix 2A.

At this point it is convenient to write \(Y_i'Mu, Y_i'M Y_i\) and other expressions in terms of the basic components of the \(AR(1)\) processes, namely: the equation parameters and the i.i.d. random innovations. For a given sample size \(T\), we define matrices \(A\) and \(B\) as:

\[\]
The variables and expressions can thus be written in terms of these matrices as\:^{32}:

\[ Y_i = Bu, \quad \text{and} \quad X_i = A\eta = A(u + \delta \varphi) \]

\[ M = I - \Lambda \eta \Lambda' \Lambda \eta'^{-1} \eta' \Lambda' \]

Therefore, \( \hat{\beta} - \beta \) can be written as:

\[
\hat{\beta} - \beta = \frac{u'(I - \Lambda \eta \Lambda' \Lambda \eta'^{-1} \eta' \Lambda')Bu}{u'B'(I - \Lambda \eta \Lambda' \Lambda \eta'^{-1} \eta' \Lambda')Bu}\]
\[= \frac{u'B'u \eta' \Lambda' \Lambda \eta - u' \Lambda \eta \eta' \Lambda' Bu}{u'B'Bu \eta' \Lambda' \Lambda \eta - u'B' \Lambda \eta \eta' \Lambda' Bu}\]

Defining:

\[\eta' \Lambda' \Lambda \eta = \Lambda \eta \Lambda' \eta' \Lambda' \]

\(^{32}\) Under regularity conditions on the distribution of the initial values, \( y_0 \) and \( x_0 \), they are dominated in the bias expression by terms of larger order and go to quickly zero. For example, in Symons and Grubb these terms are assumed independently distributed from \( u \) and \( \varepsilon \) and their influence on the bias can be ignored.

For simplicity of the exposition, we have made the assumption that these initial values are zero.
We obtain:

\[ E[\hat{\beta} - \beta'] = E[N / D] \]

As \( E[\hat{\beta} - \beta'] \) is equal to \( E[N/D] \), the former can be approximated using Taylor expansion around \( E[N] \) and \( E[D] \) by:

\[
E[N/D] = E[N]/E[D] - E[N]/E[D] \cdot (D - E[D])/E[D]^2 + \cdots
\]

We demonstrate, under certain conditions on the parameters, that it is possible to use only the first few terms from this expansion to obtain an accurate approximation of the bias; this approach follows closely the one in Grubb and Symons (1987). The crucial result that allowing to use this approximation is that \( N/E[D] \) and \( (D - E[D])/E[D] \) are \( O(T^{-1}) \). When these two conditions are true, \( (D - E[D])/E[D] \) and higher powers on this term go to zero more rapidly than \( O(T^{-1}) \). We provide an approximation based on the expressions derived from \( N/E[D] \) and \( (D - E[D])/E[D] \).

The conditions for the goodness of the approximation are alternatively a large \( T \), or \( \lambda \) and \( \beta \) well below one. In addition, in order to apply a formula for the evaluation of expectations of stochastic matrices below we need an error term that is normally, independent and identically distributed. From the analysis in section 2.c.- we determined that the distribution of the error term in an auto-regressive process needs to be regular.
enough in order to produce consistent OLS estimates, so that the bias be zero asymptotically. One of the distributions for which the OLS estimates have this property is the normal.

\textbf{b. - Approximation for }E[N], E[D] \text{ and } E[ND].

Taking into account the analysis in section a.- and if we enjoy the rapid convergence to zero of \((D-E[D])/E[D]\) and higher powers, when looking at the Taylor expansion of \(E[N/D]\) only the first two terms in the sum will be relevant. Defining \(\Delta D\) as the difference between \(D\) and \(E[D]\) and using Taylor expansion around \(E[N]\) and \(E[D]\), the ratio between \(N\) and \(D\) can be written as:

\[
\frac{N}{D} = \frac{N}{E[D]}(1 - \frac{\Delta D}{E[D]} + (\frac{\Delta D}{E[D]})^2 - (\frac{\Delta D}{E[D]})^3 + \ldots)
\]

To study the order of probability of this approximation we need to study the order of probability of both \(N\) and \(\Delta D\), and the order of magnitude of \(E[D]\) and \(E[N]\). The latter are simple expectations of products of stochastic matrices. In particular, for \(E[D]\) we have:

\[
E[D] = E[u'B\eta'] - E[u'B\eta'] - E[u'B\eta'] - E[u'B\eta']
\]

Calling \(\epsilon = \delta\), we have that \(\eta = u + \epsilon\). Therefore:
\[ E[D] = E[u'B'Bu(u + \varepsilon)\Lambda'(u + \varepsilon)\Lambda'Bu] - E[u'B'\Lambda(u + \varepsilon)(u + \varepsilon)\Lambda'Bu] \\
= E[u'B'Bu\Lambda'\Lambda u] - E[u'B'\Lambda uu'\Lambda'Bu] \\
+ E[u'B'Bu\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda u] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
= E[u'B'Bu\Lambda'\Lambda u] - E[u'B'\Lambda uu'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
+ E[u'B'Bu\varepsilon'\Lambda'\Lambda \varepsilon] - E[u'B'\Lambda \varepsilon u'\Lambda'Bu] \\
\]

**Proposition 3.** When the error term vectors are distributed as normal independent, the following relationships for expectations of stochastic matrix products hold:

a) \[ E[u'Pu'\varepsilon\varepsilon'Qu] = \alpha^2 \delta^2 Tr(P)Tr(Q) \]

b) \[ E[u'P\varepsilon\varepsilon'Qu] = \alpha^2 \delta^2 Tr(PQ) \]

c) \[ E[u'Pu'\varepsilon'Qu] = \alpha^2 (Tr(P)Tr(Q) + Tr(PQ) + Tr(P'Q)) \]

By calling \( P = H_B = \Lambda' \) and \( C = H_A \) a consequence of the formulas above is:

\[ E[D] = \sigma^2 \delta^2 Tr(P)Tr(Q) + Tr(PQ) + Tr(P'Q) - Tr(C'C) - Tr(C^2) \]

**Proof:** a) is a consequence of \( E[u'Pu'\varepsilon\varepsilon'Qu] = E[u'Pu]E[\varepsilon\varepsilon'Qu] \) because of independence of \( u \) and \( \varepsilon \). Again by independence of \( u \) and \( \varepsilon \) using the law of iterated expectations over \( \varepsilon \) b) can be derived from:

\[ E[u'P\varepsilon\varepsilon'Qu] = E[u'P\varepsilon\varepsilon'Qu] = \alpha^2 E[u'PQu] \]

By applying the general formula for the evaluation of expectations of stochastic matrix products in Srivastava and Triwari (1976) and Misra (1972) we obtain the result. Obtaining c) is direct from formula (3.9) page 137 in Srivastava and Triwari (1976), applying it to two general matrices \( P \) and \( Q \) and under the assumption uncorrelated and
independent $u$ and $\varepsilon$

Therefore, we can write the formula for $E[D]$ in terms of the matrices $B$ and $\Lambda$:

$$E[D] = \sigma^4_n (\text{Tr}(B'B)\text{Tr}(\Lambda'\Lambda) + 2\text{Tr}(B'B\Lambda'\Lambda))$$

$$- \text{Tr}(B'B) - \text{Tr}(B'B\Lambda'\Lambda)$$

$$+ \sigma^2_n \sigma^2_m (\text{Tr}(B'B)\text{Tr}(\Lambda'\Lambda) - 2\text{Tr}(B'B\Lambda'\Lambda))$$

For the matrix $B$ defined earlier the trace of $BB'$ is the sum of terms in $\beta$:

$$\text{Tr}(B'B) = \sum_{i=2}^{I} \sum_{j=1}^{I} \beta^{i-j}$$

In general, traces of any number of products between $B, \Lambda$ and their respective transposes can be written as more complex sums of $\beta$ and $\lambda$. In section d.- below we study the order of these traces and provide explicit formulas for some of these products. For practical purposes, however, it is enough to use the expressions in the matrix trace and evaluate them numerically.

Turning to the problem of finding an expression for the expected value of $N$:

$$E[N] = E[u' Bu \eta' \Lambda' \Lambda \eta] - E[u' \Lambda \eta' \Lambda' Bu]$$

Using again the decomposition $\eta = u + \varepsilon$ we have that:
\[ E[N] = E[u'Bu(u + \varepsilon)' \Lambda' \Lambda(u + \varepsilon)] - E[u' \Lambda(u + \varepsilon)(u + \varepsilon)' \Lambda' Bu] \\
= E[u'Buu' \Lambda' \Lambda u] - E[u' \Lambda uu' \Lambda' Bu] \\
+ E[u'Bu\varepsilon' \Lambda' \Lambda \varepsilon] - E[u' \Lambda \varepsilon u' \Lambda' Bu] \\
+ E[u'Bue' \Lambda' \Lambda \varepsilon] - E[u' \Lambda \varepsilon \varepsilon' \Lambda' Bu] \\
+ E[u'Bu\varepsilon' \Lambda' \Lambda u] - E[u' \Lambda uu' \Lambda' Bu] \\
+ E[u'Bue' \Lambda' \Lambda \varepsilon] - E[u' \Lambda \varepsilon \varepsilon' \Lambda' Bu] \]

**Proposition 4.** The following relationships for expectations of stochastic matrix products hold:

a) \[ E[u'Bu\varepsilon^2Q] = 0 \]

b) \[ E[u'\Lambda \varepsilon^2 C'u] = \sigma^2 \sigma^2 \text{Tr}(\Lambda C') \]

c) \[ E[u'Bu\varepsilon^2 Q u] = \sigma^4 \left( \text{Tr}(BQ) + \text{Tr}(B'Q) \right) \]

By calling \( P = B' Q = \Lambda \Lambda' \) and \( C = B' A \), a consequence of the formulas above is:

\[ E[N] = \sigma^4 \left( 2 \text{Tr}(B\Lambda' \Lambda) - \text{Tr}(\Lambda \Lambda' B) - \text{Tr}(\Lambda' \Lambda' B) \right) - \sigma^2 \sigma^2 \text{Tr}(\Lambda \Lambda' B) \]

**Proof:** a) is a consequence of \( E[u'Bu\varepsilon^2Q] = E[u'Bu]E[\varepsilon^2Q] \) because of independence of \( u \) and \( \varepsilon \). As the diagonal of both \( B \) and \( \Lambda \) are zero, then \( E[u'Bu] = 0 \). The same argument can be applied to \( E[u'\Lambda \varepsilon^2 P Q \varepsilon] = 0 \).

The formula in b) can be derived from applying the law of iterated expectations over \( \varepsilon \) and using its independence from \( u \):

\[ E[u'\Lambda \varepsilon^2 C'u] = E[u'\Lambda E[\varepsilon^2]C'u] = \sigma^2 E[u'\Lambda C'u] \]

Applying the formula c) from proposition 3, replacing matrix \( P \) by \( B' \) and using the independence and lack of correlation between \( u \) and \( \varepsilon \) we obtain:
The result comes from the observation that $\text{Tr}(B) = 0$.

Finally, $E[N]$ derives from applying all three formulas to the expression:

$$E[N] = E[u'Buu'\Lambda\Lambda'u] - E[u'\Lambda uu'\Lambda'Bu] + E[u'Bue'\Lambda\Lambda'e'] - E[u'\Lambda e e'\Lambda'Bu]$$

Applying the formulas for the expectations of stochastic matrices:

$$E[N] = (\text{Tr}(B)Q + \text{Tr}(B'Q) - \text{Tr}(AC) - \text{Tr}(A'C')) - \text{Tr}(A'A)$$

In the following sections we will make extensive use of the following result.

**Proposition 5.** If $F$, $G$ and $H$ are general squared matrices of size $T$ and $w$ is a vector of random independent observations from a Normal $(0, \sigma^2)$ distribution, then:

$$E[w'Fww'Gww'Hw] = \sigma^2 \{\text{Tr}(F)\text{Tr}(G)\text{Tr}(H) + \text{Tr}(G)\text{Tr}(FH) + \text{Tr}(G)\text{Tr}(FH')$$

$$+ \text{Tr}(GH)\text{Tr}(F) + \text{Tr}(FG) + \text{Tr}(FG'H')$$

$$+ \text{Tr}(G'H)\text{Tr}(F) + \text{Tr}(FG'H) + \text{Tr}(FH')$$

$$+ \text{Tr}(H)\text{Tr}(FG) + \text{Tr}(FGH) + \text{Tr}(FG'H')$$

$$+ \text{Tr}(H)\text{Tr}(FG') + \text{Tr}(FG'H') + \text{Tr}(FH'G')\}$$

**Proof:** See appendix 2A

Finally, we can turn to find an expression for the expectation of $ND$, expanding this expected value around $u$ and $\eta$. 

66
\[ E[ND] = E[(u'B_u'Q_u' \eta - u'A_u'C_u') (u'P_u'Q_u' \eta - u'C_u'P_u')]
\]
\[ = E[u'B_u'Q_u'P_u'Q_u'] - E[u'B_u'Q_u'C_u'P_u'] - E[u'A_u'Q_u'C_u'Q_u'] + E[u'A_u'C_u'Q_u'P_u']
\]
\[ = E[u'B_u'Q_u'P_u'Q_u'] - E[u'B_u'Q_u'C_u'P_u'] - E[u'A_u'C_u'Q_u'P_u'] + E[u'A_u'C_u'Q_u'C_u']
\]

The most sensible approach is to analyse each of these four terms separately.

**Proposition 6.** Defining \( \eta = u + \varepsilon \), \( Q = A'B ' \), \( P = B'B ' \) and \( C = B'A ' \) the following relationships hold:

a) \[ E[u'B_u'P_u'Q_u'Q_u'] = E[u'B_u'u'u'Q_u'u'u'] + 2\sigma^4_i \sigma^2_z \text{Tr}(BP)\text{Tr}(Q)^2 + 4\sigma^6_i \sigma^3_z \text{Tr}(Q)(\text{Tr}(BP)\text{Tr}(Q) + \text{Tr}(BQ)\text{Tr}(P)) + O(T^2)
\]

b) \[ E[u'B_u'C_u'Q_u'C_u'] = E[u'B_u'Q_u'Q_u'] + \sigma^2_i \sigma^2_z \text{Tr}(Q)(\text{Tr}(B(C + C'))\text{Tr}(C + C') + O(T^2)
\]

c) \[ E[u'A_u'C_u'C_u'] = E[u'A_u'Q_u'C_u'] + \sigma^4_i \sigma^2_z \text{Tr}(Q)(\text{Tr}(A(C + C'))\text{Tr}(P) + \text{Tr}(PA)\text{Tr}(Q))\text{Tr}(P)\text{Tr}(Q) + \text{Tr}(C + C')A\text{Tr}(Q)\text{Tr}(P)\text{Tr}(Q)) + O(T^2)
\]

d) \[ E[u'A_u'C_u'Q_u'C_u'] = E[u'A_u'Q_u'C_u'Q_u'] + 3\sigma^6_i \sigma^3_z \text{Tr}(A'C)\text{Tr}(Q)^2 + O(T^2)
\]

**Proof:**

a) The proof attempts to expanding the expressions around \( \varepsilon \) and \( u \) subsequently to use
independence to analyse the terms involving each of them separately. In addition, we apply recursively the general formula in Srivastava and Triwari (1976). First expand \( \eta \):

\[
E[u'Bu'u'\eta'\eta'\eta]' = E[u'Bu'u'(\eta'\eta)^3]
= E[u'Bu'u'(u + \varepsilon)'Q(u + \varepsilon))^3]
= E[u'Bu'u'((u'Qu)^3 + 4u'Qee'Qu + \varepsilon'Qee'Qe + 4u'Que'Qu + 2\varepsilon'Que'Qe + 4u'Qee'Qe)]
\]

Applying the law of iterated expectations it is possible to eliminate expectations involving first moments of \( \varepsilon \). Furthermore, making the assumption that the distribution of \( \varepsilon \) is symmetric, all odd powers of this random term will vanish. Consequently:

\[
E[u'Bu'u'\eta'\eta'\eta]' = E[u'Bu'u'(\eta'Qu)^3 + \varepsilon'Qee'Qe + 4u'Que'Qu + 2\varepsilon'Que'Qe]
\]

Now, applying the general formula for the evaluation of expectations of stochastic matrix to \( uu'Puu' \), and keeping in mind that \( Tr(B) = 0 \):

\[
E[u'Bu'u'] = Tr(BE[uu'Puu']) = 2\sigma_i^2 Tr(BP)
E[\varepsilon'Qee'Qe] = Tr(QE[\varepsilon'Qee']) = \sigma_i^4 (Tr(Q)^2 + 2Tr(Q^2))
\]

For the third term we can apply law of iterated expectations, keeping in mind that \( u \) and \( \varepsilon \) are independent:

\[
E[u'Bu'u'uu'Qee'Qu] = E[u'Bu'u'Puu'QE[\varepsilon']Qu] = \sigma_i^2 E[u'Bu'u'Puu'Q^2u]
\]

Now, applying recursively over the last expression, the formula for the evaluation of
expectations of stochastic matrix:

\[ E[u'Buu'Puu'Q^u] = \sigma^2_{\epsilon^2} \{2\text{Tr}(BQ^)\text{Tr}(P) + 2\text{Tr}(Q^)\text{Tr}(BP) + 4\text{Tr}(BPQ^) + 4\text{Tr}(BQ^P)\} \]

Finally, for the last term we have:

\[ E[u'Buu'Puu'Que'Q^e'] = E[u'Buu'Puu'Qu]E[\epsilon'Q\epsilon] = \sigma^2_{\epsilon^2}\text{Tr}(Q)E[u'Buu'Puu'Qu] \]

Applying again, recursively, over the last expression the formula for the evaluation of expectations of stochastic matrix we obtain:

\[ E[u'Buu'Puu'Qu] = \sigma^2_{\epsilon^2} \{2\text{Tr}(BQ)\text{Tr}(P) + 2\text{Tr}(Q)\text{Tr}(BP) + 4\text{Tr}(BPQ) + 4\text{Tr}(BQP)\} \]

Replacing these formulas and keeping only the \( O(T^j) \), namely: \( \text{Tr}(BQ)\text{Tr}(Q)\text{Tr}(P) \) and \( \text{Tr}(BP)\text{Tr}(Q)^2 \), and after some algebra, we obtain the result.

b) Following the lines of part a) we expand the expressions around \( \epsilon \) and \( u \) and use independence to analyse the terms involving each of them separately. We finish by applying recursively the general formula for products of stochastic matrices:

\[ E[u'Buu'C\eta'\eta'C'u] = E[u'Buu'C(u + \epsilon)(u + \epsilon)'Q(u + \epsilon)(u + \epsilon)'C'u] \]

\[ = E[u'Buu'C(uu'Quu' + uu'Que' + uu'Qee + uu'Qee') \]

\[ + uu'Quu' + uu'Que' + uu'Qee + uu'Qee'] \]

\[ = E[u'Buu'C(uu'Quu' + uu'Qee + uu'Que' + uu'Qee')C'u] \]

\[ = E[u'Buu'C(\epsilon'Qee + \epsilon'Que' + \epsilon'Qee + \epsilon'Qee')C'u] \]

We have again eliminated expectations involving first and odd moments of \( \epsilon \). Terms like
\(\varepsilon'Qe\) or \(u'Qe\) are scalars therefore they can be seen as commutative and symmetric one by one matrices. Hence:

\[
E[u'Buu'C\eta'?Q\eta'?C'u] = E[u'Buu'C(uu'Quu' + 2uu'Qee + uu'e'Qee')C'u] + ee'u'Qu + 2ee'Quu' + ee'e'Qee'C'u]
\]

Applying iterated expectations:

\[
E[u'Buu'C\eta'?Q\eta'?C'u] = E[u'Buu'Cuu'C'u] + \sigma^2 (2E[u'Buu'Cuu'QC'u] + Tr(Q)E[u'Buu'Cuu'C'u] + E[u'Buu'Cuu'Qu] + 2E[u'Buu'CQuu'C'u] + E[u'Buu'C-Tr(Q)I + 2Q)C'u])
\]

Only the terms \(E[u'Buu'Cuu'Quu'C'u]\) and \(Tr(Q)E[u'Buu'Cuu'C'u]\) have \(O(T^4)\) components, the rest of the terms are all of orders equal or below to \(O(T^3)\). The second of these expressions:

\[
E[u'Buu'Cuu'C'u] = 1/4 E[u'Buu'(C + C')uu'(C' + C')u] = \sigma^2 (4Tr(B)Tr(C + C')^2 + 2Tr(B(C + C'))Tr(C + C') + 8Tr(B(C + C')^2))
\]

Now, applying recursively the formula for the evaluation of expectations of stochastic matrix over the last expression:

\[
E[u'Buu'Puu'Q^2u] = \sigma^2 (2Tr(BQ^2)Tr(P) + 2Tr(Q^2)Tr(BP) + 4Tr(BPQ^2) + 4Tr(BQ^2P))
\]

After algebraic manipulation we obtain the result.
c) In the same lines as part a) and b) of the demonstration we expand $\eta$ and eliminate all odd moments of $\varepsilon$ and $u$. We obtain:

$$
E[u' \Lambda \eta' Q \eta' C' u' u' P u] = E[u' \Lambda u' Q u' C' u' u' P u] + 2 \sigma^2 E[u' \Lambda u' Q C' u' u' P u] + \sigma^2 E[u' Q u' C' \Lambda u' P u] + \sigma^2 E[u' C' \Lambda u' P u] + 2 \sigma^2 E[u' C' Q \Lambda u' P u]
$$

In this case the $O(T^3)$ terms are the first, third, fourth and fifth:

$$
E[u' Q \Lambda u' C' u' u' P u] = \frac{1}{2} E[u' Q \Lambda u'(C + C') u' u' P u] = \sigma_6^6 / 2 \{ Tr(QA)Tr(C + C')Tr(P) + 2 Tr(QA)Tr(C + C') \}
$$

$$
E[u' \Lambda u' C' u' u' P u] = \frac{1}{2} E[u' \Lambda u'(C + C') u' u' P u] = \sigma_6^6 / 2 \{ Tr(QA)Tr(C + C')Tr(P) + 2 Tr(QA)Tr(C + C') \}
$$

$$
E[u' Q u' C' \Lambda u' P u] = \frac{1}{2} E[u' Q u'(C + C') \Lambda u' P u] = \sigma_6^6 / 2 \{ Tr(QA)Tr((C + C') \Lambda P) + 2 Tr(QA)Tr((C + C') \Lambda) \}
$$

$$
E[u' \Lambda \eta' Q \eta' C' u' u' P u] = E[u' \Lambda u' Q u' C' u' u' P u] + O(T^3)
$$

Replacing these formulas and keeping only the $O(T^3)$, namely: $Tr(QA)Tr(C+C')Tr(P)$,
After algebraic manipulation of the relevant terms we obtain the result.

d) Finally and applying a similar procedure as in a) b) and c) we obtain:

\[ E[u'yCuu'Cee'C'u] = E[u'yCuCuu'Cee'C'u] + 3\sigma_v^2E[u'yCuCuu'Cee'C'u] \]

In this case the \(O(T^3)\) terms are the first, third and fourth:

\[ E[u'C'\Lambda uu'Cuu'C'u'] = \sigma_u^2/4E[u'C'\Lambda uu'(C'C')uu'(C+C')u] \]

\[ = \sigma_u^2/4\{Tr(C'\Lambda)Tr((C+C')^2)+8Tr(C'\Lambda(C+C')+(C+C'))Tr(C+C') \] + 2Tr(C'\Lambda)Tr((C+C')^2) + 8Tr(C'\Lambda(C+C')^2) \}

Finally to solve the term \(E[u'\Lambda ee'Cuu'Cee'C'u]\) we apply recursively the formula in Srivastava and Triwari (1976):

\[ E[u'\Lambda ee'Cuu'Cee'C'u] = \sigma_u^2\sigma_v^2\{3Tr(C'\Lambda)Tr(C'C)+6Tr(\Lambda C'C'C') \}

c. **The expected value of ND when \(\varepsilon=0\).**

To obtain the results in section b.- we relied on knowing the component \(E[ND]\) when this depends only on the error term \(u_i\). This can be seen as a particular case of our model when the variance of the error term \(\varepsilon\) is zero; in terms of the state variable \(z\), the process generating it will not have an independent innovation term. This component also corresponds to the value of \(E[ND]\) that emerges in a regression where the variable \(x\) is an
auto-regressive process \((x_t = x_{t-1})\) whose random component depends entirely on the random component that enters in the process that generates \(y_t\). The expectation we need to calculate is:

\[
E[ND]_{\varepsilon=0} = E[(u'Buu'Qu - u'Auu'C'uu'C'u)(u'Puu'Qu - u'Cuu'C'u)]
\]

\[
= E[u'Buu'Quu'Puu'Qu] - E[u'Buu'Quu'Cuu'C'u]
\]

\[
- E[u'Auu'C'uu'Puu'Qu] + E[u'Auu'C'uu'Cuu'C'u]
\]

The following result is essential in finding the expected values in the last formula:

**Proposition 7.** If matrix \(B\) is defined as previously, \(w\) is a random vector as defined in proposition 5 and \(F, G\) and \(H\) are general matrices:

\[
E[w'Bww'Fww'Gww'hw] = \sigma^2 \{Tr(F)Tr(G)Tr(B(H + H'))
\]

\[
+ Tr(F)Tr(H)Tr(B(G + G')) + Tr(G)Tr(H)Tr(B(F + F'))
\]

\[
+ Tr(B(G + G'))Tr(G(H + H'))
\]

\[
+ Tr((B + B')(F + F')(G + G')(H + H'))
\]

\[
+ Tr((B + B')(G + G')(F + F')(H + H'))
\]

\[
+ Tr((B + B')(G + G')(F + F')(H + H'))
\]

\[
+ Tr((B + B')(G + G')(F + F')(H + H'))
\]

With the aid of proposition 7 it is direct to calculate the expected value of \(ND\) when \(\varepsilon\) is fixed in zero. The best strategy is to analyse four terms separately:

\[
E[ND]_{\varepsilon=0} = E[u'Buu'Quu'Puu'Qu] - E[u'Buu'Quu'Cuu'C'u]
\]

\[
- E[u'Auu'C'uu'Puu'Qu] + E[u'Auu'C'uu'Cuu'C'u]
\]
**Proposition 8.** Defining $\eta = u + \varepsilon, \quad Q = \Lambda \Lambda, \quad P = B'B$ and $C = B'C$ the following relationships hold:

\[E[u'Bu'u'Quu'Quu'Qu] = \sigma^8 \{ 4 \text{Tr}(Q) \text{Tr}(P) \text{Tr}(BP) + 2 \text{Tr}(Q) \text{Tr}(Q) \text{Tr}(BP) + 8 \text{Tr}(Q) \text{Tr}(BPQ) + 4 \text{Tr}(P) \text{Tr}(BPQ^2) + 16 \text{Tr}(BPQP) + 16 \text{Tr}(BPQ^2) \{ \}
\]

\[b) \quad E[u'Bu'u'Quu'Cuu'C'u] = \sigma^8 \{ 2 \text{Tr}(Q) \text{Tr}(C) \text{Tr}(BP) + 2 \text{Tr}(C) \text{Tr}(C) \text{Tr}(BP) + 4 \text{Tr}(C) \text{Tr}(C) \text{Tr}(B(C + C')) + 4 \text{Tr}(B(C + C')^2) + 4 \text{Tr}(B(C + C')Q) + 4 \text{Tr}(B(C + C')Q(C + C')) \} \]

\[c) \quad E[u'Auu'Cuu'Puu'Quu'Qu] = \sigma^8 \{ 2 \text{Tr}(C) \text{Tr}(P) \text{Tr}(AP) + 4 \text{Tr}(C) \text{Tr}(AP) \text{Tr}(AP) + 2 \text{Tr}(P) \text{Tr}(AP) \text{Tr}(AP) + 2 \text{Tr}(P) \text{Tr}(AP) \text{Tr}(AP) + 4 \text{Tr}(AP) \text{Tr}(AP) \text{Tr}(AP) + 4 \text{Tr}(AP) \text{Tr}(AP) \text{Tr}(AP) + 4 \text{Tr}(AP) \text{Tr}(AP) \text{Tr}(AP) \} \]

\[d) \quad E[u'Auu'Cuu'Cuu'C'u] = \sigma^8 \{ 3 \text{Tr}(C) \text{Tr}(C) \text{Tr}(AP) + 3 \text{Tr}(C) \text{Tr}(AP) \text{Tr}(AP) + 3 \text{Tr}(C) \text{Tr}(AP) \text{Tr}(AP) + 3 \text{Tr}(C) \text{Tr}(AP) \text{Tr}(AP) \} \]


**Proof:** This expressions are a direct consequence of applying the formula in proposition 7. See appendix 2A for more details.

One important caveat, regarding the formulas identified here, is that for \( \lambda \) and \( \beta \) close to one the terms in the traces that are related to higher powers of \( P, Q \) and \( C \), will consist of large coefficients. This will have the consequence that terms of lower order, like \( O(T^x) \) or smaller, may be overtaken by \( O(T^y) \) terms very slowly. In cases like this, the formula will only be accurate for large values of \( T \). However, for large values of \( T \), as the formulas assume normally distributed error terms and in the light of the results in section 2., these biases will be approaching zero. The formulas presented here will be accurate for smaller values of \( \lambda \) and \( \beta \). In section e.- we present the final version of the approximations we use and a number of evaluations for different values of the parameters.

**d.- Order of probability of the approximation.**

The accuracy loss when using our simple approximation for the bias will depend on the order of probability of terms involving higher powers of \( N(D-E[D])^{1/E[D]} \); \( i=2,3,4,... \). For every \( i \) the expression can be divided in one term depending on \( N \) and one depending on \( D \):

\[
\frac{N(D-E[D])^i}{E[D]^{i+1}} = \frac{N}{E[D]} \left( \frac{D-E[D]}{E[D]} \right)^i
\]

By knowing the probability order of \( N/E[D] \) and \( (D-E[D])/E[D] \) we can infer the order of
probability for any products of these terms.

**Proposition 9.**

a) If $E[(D-E[D])^2]$ is $O(T^4)$ the order of probability of $(D-E[D])/E[D]$ is $O(T^{4^4})$.

b) If $E[N^2]$ is $O(T^4)$ the order of probability of $N/E[D]$ is $O(T^{4^4})$.

**Proof:**

a) In the light of the results in proposition 3 we know $E[D]^2$ is of order $O(T^4)$. If we assume that $E[(D-E[D])^2]$ is $O(T^4)$ and by using the Chebyshev inequality, we can find the order of probability of $((D-E[D])/E[D])$. Indeed, if $d_\varepsilon$ is a constant Chebyshev inequality states that:

$$P\left[\left|\frac{D-E[D]}{E[D]}\right| \geq d_\varepsilon\right] \leq \frac{E\left(\frac{(D-E[D])^2}{E[D]}\right)}{d_\varepsilon^2} = O(T^{4^4}) / d_\varepsilon^2$$

b) Again we use the fact that $E[D]^2$ is of order $O(T^4)$ and take a constant $n_\varepsilon$. The Chebyshev inequality implies:

$$P\left[\left|\frac{N}{E[D]}\right| \geq n_\varepsilon\right] \leq E\left(\frac{N}{E[D]}\right)^2 / n_\varepsilon^2 = O(T^{4^4}) / n_\varepsilon^2$$

Therefore to evaluate the goodness or quality of the approximation we need to find the order in $T$ of the expressions $E[N^2]$ and $E[\{(D - E[D])^2\}]$, i.e. the values of $l$ and $k$ in proposition 9.
**Proposition 10.** \( E[(D - E[D])^2] \) is of order \( O(T^*) \).

**Proof:**

\[
E[(D - E[D])^2] = E[D^2] - E[D]^2
\]

\[
E[D^2] = E[(u' P \eta' Q \eta - u' C \eta' C' u)(u' P \eta' Q \eta - u' C \eta' C' u)]
\]

\[
= E[u' P \eta' Q \eta u' P \eta' Q \eta] - E[u' P \eta' Q \eta u' C \eta' C' u]
\]

\[
- E[u' C \eta' C' u u' P \eta' Q \eta] + E[u' C \eta' C' u u' C \eta' C' u]
\]

Expanding this term around \( \eta = (u + \varphi) \), applying the general formula for the evaluation of expectations of stochastic matrix products in Srivastava and Triwari (1976) and from proposition 3 we see that \( E[D^2] \) and \( E[D]^2 \) have the same \( O(T^*) \) terms. These terms are:

\[
\sigma^8 \{ \text{Tr}(P)^3 \text{Tr}(Q)^2 - 2 \text{Tr}(P) \text{Tr}(Q) \text{Tr}(C)^2 + \text{Tr}(C)^4 \},
\]

\[
\sigma\xi^2 \sigma_z \{ \text{Tr}(P)^3 \text{Tr}(Q)^2 - \text{Tr}(P) \text{Tr}(Q) \text{Tr}(C)^2 \}, \text{ and}
\]

\[
\sigma^4 \sigma^2 \{ \text{Tr}(P)^2 \text{Tr}(Q)^2 \}
\]

Therefore, these terms cancel each other and the remaining are \( O(T^3) \) or less:

\[
E[(D - E[D])^2] = O(T^3)
\]

**Proposition 11.** \( E[N^2] \) is of an order equal or less than \( O(T^3) \).

**Proof:** Expanding the expected value:
\[
E[N^2] = E[(u'Bu' \eta'Q\eta - u' \Lambda \eta' C' u)(u'Bu' \eta'Q\eta - u' \Lambda \eta' C' u)]
\]
\[
= E[u'Bu' \eta'Q\eta'u'Bu' \eta] - E[u'Bu' \eta'Q\eta'u' \Lambda \eta' C' u] + E[u' \Lambda \eta' C' uu' \Lambda \eta' C' u]
\]

Now expanding around \( r = (u + \vartheta) \) and in the light of the results in proposition 7 we see that the only possibly \( O(T^2) \) terms are:

\[
\sigma_x^2(Tr(B)^2 Tr(Q)^2 - 2Tr(B)Tr(Q)Tr(A)Tr(C) + Tr(A)^2 Tr(C)^2) = 0,
\]
\[
\sigma_x^4 \sigma_x^{-4} \{Tr(B)^2 Tr(Q)^2 - Tr(B)Tr(Q)Tr(A)Tr(C)\} = 0,
\]
\[
\sigma_x^4 \{Tr(B)^2 Tr(Q)^2\} = 0
\]

These terms are all zero because they involve traces of either \( B \) or \( A \) which are zero. The remaining term can only be \( O(T^3) \) or smaller.

The only remaining issue is the order of magnitude of the approximation identified. Applying the results in proposition 10 and 11 to proposition 9 we conclude that:

\[
P \left[ \left| \frac{D - E[D]}{E[D]} \right| \geq d_x \right] \leq O(T^{-1}) / d_x^2
\]
\[
P \left[ \left| \frac{N}{E[D]} \right| \geq n_x \right] \leq O(T^{-1}) / n_x^2
\]

Finally, we can confirm that all the remaining terms in the Taylor expansion that were not considered for our formula are \( O(T^2) \) or smaller.
Formula for the bias in $\gamma$

We found in section 3.a., the difference between the true parameter on $x_{i}$ and the one estimated by OLS can be written as:

$$\gamma - \gamma = X_{i}' W u / X_{i}' W X_{i}$$

Under the assumptions made earlier over the initial values and holding parameter $\gamma$ equal to zero, the variables and expressions can thus be written in terms of the matrices $B$ and $A$ as:

$$Y_{i} = Bu, \quad \text{and} \quad X_{i} = A \eta = A(\mu + \delta \psi)$$

$$W = I - Bu(u' B' Bu)^{-1} u' B'$$

Therefore, $\gamma - \gamma$ can be written as:

$$\gamma - \gamma = \eta' \Lambda (I - B \eta (\eta' B' B \eta)^{-1} \eta' B') u$$

$$= \eta' \Lambda' uu' B' B u - \eta' \Lambda' B uu' B \eta$$

Defining $D$ as before and $R$ as:

$$R = \eta' \Lambda' uu' B' B u - uu' B uu' \Lambda' B u$$

We obtain:

$$E[\gamma - \gamma] = E[R / D]$$

As $E[\gamma - \gamma]$ is equal to $E[R / D]$, the former can be approximated using a similar Taylor
expansion as done with the bias in $\beta$. This time we expand around $E[R]$ and $E[D]$ by:

$$R/D = (R/E[D])(1 - \Delta D/E[D] + (\Delta D/E[D])^2 + (\Delta D/E[D])^3 + \ldots)$$

$$E[R/D] = 2E[R]/E[D] - E[R(D-E[D])/E[D]]^2 + E[R(D-E[D])/E[D]]^3 + \ldots$$

As before, we can find an expression for the expected value of $R$ by analysing two separate expectations:

$$E[R] = E[\eta' \Lambda uu'B'Bu] - E[\eta' \Lambda' Bu]$$

Applying again the decomposition of the error term in the second equation: $\eta = u + \varepsilon$ and keeping in mind that $u$ and $\varepsilon$ are independent identically-distributed and normal we have:

$$E[N] = E[u' \Lambda (u + \varepsilon)'u'B'Bu] - E[u' Bu(\varepsilon' \Lambda' Bu)]$$
$$= E[u' \Lambda uu'B'Bu] - E[u' Buu' \Lambda' Bu]$$
$$+ E[u' \Lambda uu'B'Bu] - E[u' Bu \varepsilon' \Lambda' Bu]$$
$$= E[u' \Lambda uu'B'Bu] - E[u' Buu' \Lambda' Bu]$$

By calling $P = BB' C = BB'$ we can obtain the following result:

$$E[u' \Lambda uu' Pu] = \sigma_0^2 (Tr(\Lambda P) + Tr(\Lambda' P))$$

Using also properties of the traces of matrices and symmetry of $P$, we obtain the expectation of $R$ as:

$$E[R] = \sigma_0^2 (2Tr(\Lambda P) - Tr(\Lambda C) - Tr(\Lambda' C))$$

We also have to find an expression for the expectation of $RD$. Expanding this expected
value around $u$ and $\eta$

$$E[RD] = E[(u' \Lambda \eta u' Pu - u' Bu \eta' C' u)(u' Pu \eta' Q \eta - u' C \eta \eta' C' u)]$$

$$= E[u' \Lambda \eta u' Pu \eta' Q \eta] - E[u' \Lambda \eta u' C \eta u' Pu \eta' C' u]$$

$$- E[u' Bu \eta' C' \eta' Q \eta u' Pu] + E[u' Bu \eta' C' \eta u' C' \eta' C' u]$$

The best approach is to study each of these terms separately. In sub-sections i.- to iv.- following we solve these four expressions up to $O(T^3)$. We use $\eta = u + \varepsilon$, $Q = \Lambda A P = B'B$ and $C = B' A$ the result in propositions 3 to 7.

i.- Expanding first $\eta$

$$E[u' Pu u' Pu' \Lambda \eta \eta' Q \eta] = E[u' Pu u' Pu' \Lambda (u + \varepsilon)(u + \varepsilon)' Q (u + \varepsilon)]$$

$$= E[u' Pu u' Pu' \Lambda u u' Qu] + E[u' Pu u' Pu' \Lambda u \varepsilon' Q \varepsilon]$$

$$+ 2E[u' Pu u' Pu' \Lambda \varepsilon \varepsilon' Qu]$$

$$= E[u' Pu u' Pu' \Lambda u u' Qu] + \sigma^2 u^2 E[u' Pu u' Pu' \Lambda u] Tr(Q)$$

$$+ 2\sigma^2 \varepsilon u E[u' Pu u' Pu' \Lambda Qu]$$

Here we have applied the law of iterated expectations, independence of $u$ and $\varepsilon$ and eliminated expectations involving odd moments of $\varepsilon$

Now, applying the result in proposition 5 and keeping in mind that $Tr(\Lambda) = 0$, we obtain:

$$E[u' \Lambda u u' Pu u' Pu] = \sigma^4 u^4 \{4Tr(P)Tr(\Lambda P) + 8Tr(\Lambda P^3)\}$$

$$E[u' \Lambda Q u u' Pu u' Pu] = \sigma^4 u^4 \{4Tr(P)Tr(\Lambda Q P) + 8Tr(\Lambda Q P^3)$$

$$+ 2Tr(P^2)Tr(\Lambda Q) + Tr(\Lambda Q)Tr(P)\}$$

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ii.- The second term is:

\[
E[u' \Lambda u' P uu' C(u + \epsilon)'(u + \epsilon)'C'u] = E[u' \Lambda(u + \epsilon) u' P uu' C(u + \epsilon)'(u + \epsilon)'C'u] \\
= E[u' \Lambda uu' P uu' Cu' u'C'u] + E[u' \Lambda uu' P uu' Ce' C'u] \\
+ 2E[u' \Lambda ee' C'u uuP uu' Cu'] \\
= E[u' \Lambda uu' P uu' Cu' u'C'u] + \sigma^2 E[u' \Lambda uu' P uu' CC'u] \\
+ 2\sigma^2 E[u' \Lambda C' uuP uu' Cu']
\]

From proposition 5, the first term \(E[u' \Lambda uu' P uu' CC'u]\) is of order equal to \(O(\tau^4)\) while the second expression is:

\[
E[u' \Lambda C' uuP uu' Cu'] = \sigma^6 \{Tr(\Lambda C')Tr(P)Tr(C) + Tr(P)Tr(\Lambda C)C' + Tr(C)Tr(P)Tr(\Lambda C') + Tr(\Lambda C' P) + Tr(C'P) + 2Tr(\Lambda C' C') + 2Tr(\Lambda C' PC) + 2Tr(\Lambda C' C'P)\}
\]

iii.- In the same lines as part i.- and ii.- of this analysis we expand \(\eta\) and eliminate all odd moments of \(\epsilon\) and \(u\). We obtain:

\[
E[u' Buu' C\eta' Pu\eta' Q] = E[u' Buu' C(u + \epsilon) u' Pu(u + \epsilon)' Q(u + \epsilon)] \\
= E[u' Buu' C\eta' PuuQ] + E[u' Buu' C\eta' Pu' Q\epsilon] \\
+ 2E[u' Buu' C\eta' Puu' Q' Pu] \\
= E[u' Buu' C\eta' PuuQ] + \sigma^2 E[u' Buu' C\eta' Pu]Tr(Q) \\
+ \sigma^2 E[u' Buu' C\eta' Pu]
\]

In this case the \(O(\tau^4)\) terms are only the first and the second. The expectation in the second term can be written as:

\[
E[u' Buu' C\eta' Pu] = \sigma^6 \{2Tr(C)Tr(BP) + 2Tr(BP(C + C')) \\
+ 2Tr(B(C + C')P) + Tr(P)Tr(B(C + C'))\}
\]
iv. Finally and applying a similar procedure as in i.- ii.- and iii.- we obtain:

\[ E[u'Buu'Cuu'C'u'] = E[u'Buu'C(u + \epsilon)u'C(u + \epsilon)(u + \epsilon)'C'u'] \]
\[ = E[u'Buu'Cuu'C'u'] + E[u'Buu'Cuu'Cuu'C'u'] \]
\[ + 2E[u'Buu'Cuu'Cuu'C'u'] \]
\[ = E[u'Buu'Cuu'Cuu'C'u'] + 2\sigma^{2}E[u'Buu'Cuu'C'u'] \]

In this case there are no \( O(T^{5}) \) terms except the first.

We have avoided solving the expectations involving eight products of \( u \). The reason for this is that this problem has been already solved in proposition 8. If we interchange the roles of \( \beta \) and \( \lambda \) in the formula for \( E[ND]_{\epsilon=0} \) in proposition 8, \( A \) becomes \( B \) and vice-versa, \( P \) becomes \( Q \) and vice-versa and \( C \) remains the same. In consequence, \( E[RD]_{\epsilon=0} \) is equal to \( E[ND]_{\epsilon=0} \) when we swap \( \beta \) with \( \lambda \). For the same reason we find that \( E[R^2] \) is of an order equal or less than \( O(T^{5}) \), and finally, that the \( E[RD] \) can be approximated to the \( O(T^{5}) \) by the following formula:

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\[ E[RD] = E[u'Puu'Puu'Auu'Qu] + \sigma^2 \sigma^6 (4 \text{Tr}(P)\text{Tr}(\Lambda P) + 8 \text{Tr}(\Lambda P^2))\text{Tr}(Q) + 2 \sigma^2 \sigma^6 (4 \text{Tr}(P)\text{Tr}(\Lambda QP) + 8 \text{Tr}(\Lambda QP^2)) + 2 \text{Tr}(P)\text{Tr}(\Lambda Q) + 2 \text{Tr}(\Lambda Q)\text{Tr}(P) \} \\
- E[u'\Lambda uu'Puu'Cu'u'C'u'] - 2 \sigma^2 \sigma^6 \{ \text{Tr}(\Lambda C')\text{Tr}(P)\text{Tr}(C) + \text{Tr}(P)\text{Tr}(\Lambda C'C) + \text{Tr}(P)\text{Tr}(\Lambda C'C) + \text{Tr}(P)\text{Tr}(\Lambda C'C) + 2 \text{Tr}(\Lambda C'PC) + 2 \text{Tr}(\Lambda C'C') \} \\
- E[u'Buu'Cuu'PuuQu] - \sigma^2 \sigma^6 (2 \text{Tr}(C)\text{Tr}(BP) + 2 \text{Tr}(BP(C + C')) + 2 \text{Tr}(B(C + C')P) + 2 \text{Tr}(P)\text{Tr}(B(C + C'))\text{Tr}(Q) + E[u'Buu'Cuu'Cuu'C'u'] \\

\text{f.- Evaluation of the formula.}

Tables 2.1 and 2.2 below present the results of the evaluation of the \(O(T^4)\) approximation we formulated in sections 2.d.- and 2.e.-. Appendix 2C provides formulas for the matrix traces needed to evaluate the approximation of the bias. The tables below have been evaluated from the analytical expressions using Mathematica version 2.2.

These values approximate the biases arising from OLS estimation when the equation to be estimated, \(y_i = \beta x_{i1}\), wrongly includes a variable \(x_{i2}\) and this is correlated with \(y_{i1}\) as specified in section 2.a.-. We evaluate the approximation for three different values of each \(\beta\) and \(\sigma^2\), four values of \(\sigma^6\) and six values of \(\rho\). Because of the diagonal form of the \(y/x\) system described in section 2.a.- is it not necessary to evaluate the negative values of \(\beta\).
For any given value of $\beta$ and $\rho$ these biases will be equal to minus the corresponding bias for $-\beta$ and $-\rho$.

Table 2.1 below presents the biases on the parameter of the lagged dependent variable evaluated for different parameter values. As the system is diagonal, both $y_t$ and $x_t$ depend exclusively on their own past values and the mutually correlated normal error terms: $u_t$ and $u_{t+s}$. The variances of $u_t$ and $\theta$ take the values 0.05, 0.1, 0.5, 1.0 and 0.1, 0.5 and 1 respectively. The structure of the system implies that for a small variance of $u_t$ and large one on $\theta$, the variables $y_t$ and $x_t$ can be better distinguished from each other than in the opposite case.

The biases are also evaluated for six values of $\rho$, the autorregressive parameter on $z_t$ and three values for the corresponding parameter on $y_t$, $\beta$. These values are -0.9, -0.5, -0.1, 0.1, 0.5 and 0.9 for $\rho$ and 0.1, 0.5 and 0.9 for $\beta$. Sawa (1978) determined that the bias on the autorregressive parameter, $\beta$, decreases with its size. In the present case this should also be true in the absence of a variable $z_t$ on the right hand side. Given that $z_t$ is correlated with $y_t$, the bias will persist even for small values of $\beta$. However, and as predicted by Sawa, when the variance of $u_t$ is small while the variance of $\theta$ is large the bias tends to zero as $z_t$ is less correlated with $y_t$. 


Table 2.1 Biases in $\beta$ for the $y/x$ system.$^{33}$

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<th>0.1</th>
<th>0.9</th>
<th>0.5</th>
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From Table 2.1 it becomes apparent that the bias will be significant when the variance $\sigma^2$ is small. The structure of the model confirms this, as the variance component $\sigma^2$ is the only characteristic that can drive $x_i$ away from $y_i$. Accordingly, the bias is linked to the degree of independence between $x_i$ and $y_i$.

---

$^{33}$ Values calculated in Mathematica version 2.2. Biases for OLS estimation using time series with 100 observations. Model:

\[
y_i = \beta y_{i-1} + u_i, \quad u_i \sim IID (0, \sigma^2_u)
\]

\[
x_i = \rho x_{i-1} + \eta_i
\]

\[
\eta_i = u_i + \epsilon_i, \quad \epsilon_i \sim IID (0, \sigma^2_{\epsilon})
\]
Table 2.1 (continued) Biases in $\beta$ for the $y/x_1$ system

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<th>$\sigma^2$</th>
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Table 2.2 below describes the behaviour of the estimated parameter for $x_1$ for different values of the autoregressive parameters $\beta$ and $\rho$ and the system variances. Our primary interest in this chapter is the bias arising on the causality relationship between the original variables $z_1$ and $y_2$. However, we established in section 2.d.- that this bias can be retrieved from the $y/x_1$ system by using the linear relationship:

$$E[\gamma - \gamma] = \delta E[\gamma - \gamma]$$

Here $\delta$ is the parameter of $z_1$ in the equation for $x_i$. As the values of interest are linearly related to those in table 2.2 we can draw correct inferences about the biases in causality test.

The biases are substantially higher for small variances of $\sigma$, with no regard to the size of $u_i$.

For smaller variances of $u_i$ the bias takes smaller values but this is still highly inflated by a
small disturbance $\theta$.

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<tr>
<td></td>
<td></td>
<td>0.0150</td>
</tr>
</tbody>
</table>

Table 2.2 Biases in $\gamma$ for the $y/x$ system. 34

Another important effect on the size of the bias is the increased magnitude arising from larger values of $\beta$. The result is somehow natural, as for high values of the autorregressive parameter, this parameter itself will be biased. This result coupled with a high degree of correlation between the two disturbances $u_i$ and $\theta$ should translate into a large bias on $\gamma$.

34 See footnote 33.
Table 2.2 (continued) Biases in \( \gamma \) for the \( y/x \) system

<table>
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<tr>
<th>( \sigma^2 )</th>
<th>( \sigma^2 )</th>
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<th>0.5</th>
<th>0.1</th>
<th>0.9</th>
<th>0.5</th>
<th>0.1</th>
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<td>-0.0038</td>
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<tr>
<td>0.1</td>
<td>0.1</td>
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<td>-0.0112</td>
<td>-0.0837</td>
<td>-0.0569</td>
<td>0.0242</td>
<td>-0.1200</td>
<td>-0.0083</td>
<td>-0.0275</td>
</tr>
<tr>
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<td>-0.0030</td>
<td>-0.0011</td>
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<td>1</td>
<td>-0.0053</td>
<td>-0.0202</td>
<td>-0.0112</td>
<td>-0.0249</td>
<td>-0.0569</td>
<td>-0.0236</td>
<td>-0.1200</td>
<td>-0.0227</td>
<td>-0.0136</td>
</tr>
</tbody>
</table>

From Table 2.2 also emerges that for positive values of \( \beta \) and \( \rho \) the value of the bias is negative and large values of \( \rho \) are coupled with large biases. However, these two results should be considered with care when translated to biases in causality test. Given that, under the null hypothesis, \( \rho \) and \( \delta \) are related to \( \lambda \) through:

\[
\delta = \frac{\lambda - \beta}{\phi}, \text{ and } \rho = \lambda
\]

These expressions are obtained by replacing \( \gamma \) with zero in the formulas for \( \delta \) and \( \rho \) we identified in section 2.d. Therefore, larger values of \( \rho \) are equivalent to larger values of \( \lambda \). However, a negative bias can be due to either a positive or a negative bias on the corresponding parameter of variable \( x \), and this will depend on the size of \( \beta \) and \( \lambda \) and the sign of \( \phi \).
4.- Causality test simulations

*a.- Results for simultaneous hypothesis tests.*

Tables A2.1 through A2.3 (in Appendix 2B) present the values of the usual Wald F-statistic that computes the joint significance of the corresponding explanatory variable.

The simulation model is:

\[
gnp_i = \alpha_0 + \beta_1 gnp_{i-1} + \ldots + \beta_k gnp_{i-k} + \gamma_1 z_{i-1} + \ldots + \gamma_k z_{i-k} + \epsilon_i
\]

\[
z_i = \delta_0 + \phi_1 gnp_{i-1} + \ldots + \phi_k gnp_{i-k} + \lambda_1 z_{i-1} + \ldots + \lambda_k z_{i-k} + \nu_i
\]

The names \(gnp_i\) and \(z_i\) we used above arise from the fact that we are interested in the causality relationship between GNP and a measurement of financial sophistication. Therefore, the test is checking whether including measurements of the financial structure improve the estimation prediction of GNP. In contrast, and given that the null hypothesis is that there is no causality from finance to GNP, the data generation uses lagged value GNP to enter in the first equation; i.e. all \(\gamma\)s are zero. We use simple normal disturbance \((\epsilon_i, \nu_i)\) with each component independent from the other and independent over time.

Finally, we test the model with different lag lengths, but we generate the data with only one lag of each variable. Therefore, the true relationship is:

\[
gnp_i = \beta gnp_{i-1} + \epsilon_i
\]

\[
z_i = \phi gnp_{i-1} + \lambda z_{i-1} + \nu_i
\]

We create simulations with a range of values of the parameters: parameter \(\beta\) ranges
between the values 0.85 and 0.9844222; parameter $\phi$ takes values in the interval 0.00416246 and 0.06461862; and parameter $\lambda$ varies between 0.8 and 0.95. The intervals were chosen based on preliminary estimations using a Latin American and Caribbean database (see the data section 5.b for details).

The variance of the error terms has also an effect on the resulting causality-test statistic. We then produce simulations for a range of these values. However, only one of the variances can be moved independently, while the other one will have an effect of simply re-scaling the variables. In other words, only one of the equations’ variances is relevant. For example, we can generate values for $z$, using the $gnp$, series and a variance of $\sigma_v$. If we subsequently increase this variance and parameter $\phi$ by 10, then we will generate the same variable $z_n$, but ten times larger. In a case like this, only the scale of $z_n$ will be affected, but the causality test will produce the same result.

With the set-up above there will be 91 different parameter choices, producing 91 different distributions of a test statistic to test the hypothesis:

35 When diagonalising the system using the eigenvalues, one of the transformed variables is the GNP itself, while the other is a linear function of GNP and the financial size measurement. This diagonal system has only four parameters two in the diagonal and two for each variance. As a consequence, five parameters cannot be independent in the original model. To see this more clearly, in variable $z$'s equation, after setting $\sigma_v$, we can always adjust parameters $\lambda$ and $\sigma_v$ to produce the same $z_n$ data. The variance of the transformed eigenvalue is:

$$\phi (1 + \beta/(\lambda - \beta)) \sigma_v + \sigma_v$$
\[ \text{H}_0: \gamma_1 = 0 \text{ and } \gamma_2 = 0 \ldots \text{and } \gamma_k = 0 \]

\[ \text{H}_A: \gamma_1 \neq 0 \text{ or } \gamma_2 \neq 0 \ldots \text{or } \gamma_k \neq 0, \text{ and} \]

\[ \text{H}_0: \phi_1 = 0 \text{ and } \phi_2 = 0 \ldots \text{and } \phi_k = 0 \]

\[ \text{H}_A: \phi_1 \neq 0 \text{ or } \phi_2 \neq 0 \ldots \text{or } \phi_k \neq 0 \]

The statistic is a linear-Wald test (Wald statistic divided by the number of hypotheses) based on the sum of the squared differences when imposing and relaxing \( \text{H}_0 \). As usual, if the null hypothesis is true, and the error terms are normal, this statistic will follow an \( F \)-distribution with degrees of freedom equal to the number of restriction in \( \text{H}_0 \) and the number of data points. The short-sample distribution of the test will be an \( F \) only under these rather restrictive conditions, on the right-hand side. Given that \( z_n \), or any of its lags, does not intervene in the \( GNP \) equation and given that \( \beta \) is smaller than one, the null hypothesis is true and the statistic will follow asymptotically an \( F \)-distribution with the appropriate degrees of freedom. Under more general conditions the limiting distribution of the Wald statistic is a chi-square. In our case, as the bias in the estimation disappears for large samples, this statistic will also have the limiting chi-square distribution.

In the next section, we wish to apply the values of the Wald test's short-sample distribution to regressions using a data set from Latin-America. This data has 23 observation for most countries, therefore, we simulated 23 data points and we use one, two and three lags for the simulated causality tests. When the maximum lag is 1 then there are only 22 observations left for the estimation. When the maximum lag is 2, then only 21 observations will be usable for the estimation and when the lags are 3,
only 20 observations will be available. Therefore, we compare our values with an $F$-distribution with 1 and 19 (equal to 22-2-1) degrees of freedom, 2 and 16 (equal to 21-4-1) degrees of freedom and 3 and 13 (20-6-1) degrees of freedom respectively.

From Tables A2.1 to A2.3 it can be seen that the effect on the $F$-statistic is important for any of the parameters used. For example, for a parameter choice $\beta=0.9844222$, $\sigma_e=0.05$, $\phi=0.01754792$, $\lambda=0.9367262$, $\sigma_f=0.02723165$ the $F$-statistic is 4.9854851 much larger than the 5 per cent level from an $F$-distribution which is 3.410534077. The corresponding probability to the right, from an $F$-distribution with 3 degrees of freedom in the numerator and 13 in the denominator, for a value like 4.9854851, is 1.62 per cent rejecting non-causality when in fact there is no causal relation.

Another way of seeing this problem, is that the usual 5 per cent $F$-distribution critical value of 3.410534077 corresponds, for this data-generating process, a remaining 13.355 per cent of probability to the right. Therefore, if one obtains an $F$-statistic of 3.411 and uses the usual $F$-distribution one would be accepting there is causality at a true level of approximately 13 per cent.

Some patterns can be drawn from the tables in terms of how the parameters affect the size of the bias. For example, an increase in the variance in each of the equations decreases the size of the bias. This is true for any of the three types of test simulated. Furthermore, this is an expected effect as the equation variances represent the size of the term that has been assumed to be independent; this is an exogenous innovation. This effect is also very intuitive, if the independent innovation is very large then the series GNP and financial measurement will follow patterns easy to differentiate.
In addition, patterns depending on the values of \( \beta \), \( \phi \) and \( \lambda \) can be deduced from the simulations. The number of observations left for estimation is 19 when using three lagged values of the explanatory variables, the 95 per cent critical value for an \( F \)-distribution with 3 and 19 degrees of freedom, is 3.41. For the range of parameters used in the simulation, every 0.01 increase in \( \beta \) will lead to an approximate increase of 0.06 in the critical value. For a simulation with \( \beta=0.9844 \) we found that the 5 per cent critical values could be as high as 4.99 and they were 4.32 on average. In the case of 20 observations, and two lagged dependent variables, the critical value for the usual \( F \)-distribution is 3.63. For every 0.01 increase in \( \beta \) the critical values, in the actual distribution, moves to the right by 0.12. The average \( F \)-value obtained in simulations with \( \beta=0.9844 \) was 4.86 but some of these values were as high as 6.08. Finally, using 21 observations and only one lag the critical value from an \( F \)-distribution is 4.38. In our simulations the actual 5 per cent critical value goes up by 0.28 when \( \beta \) grows in 0.01 and the average critical value, when is \( \beta=0.9844 \), reaches 6.69, while the highest 5 per cent critical value obtained was 8.23.

The parameter \( \lambda \) shows a less clearly defined, negative, effect over the critical values. This small negative effect shows up only when using one lagged value and 21 observations. Therefore, increases in \( \lambda \) makes the bias smaller. This is also an expected effect, as a larger \( \lambda \) will push variable \( z \) to follow its own pattern and in consequence be less correlated with \( GNP \).

Finally, the feedback from \( GNP \) on variable \( z \) has a negative effect on the size of the
bias. The smaller the parameter $\phi$ is, the larger is the departure from the values of the usual $F$-statistic. A decrease of 0.01 in $\phi$ will make the critical values grow by 0.08. For the values simulated from $\phi=0.004$ to $\phi=0.065$ we found that the 5 per cent critical values could be as high as 4.31 and they were 4.04 on average; the value taken by the usual $F$-distribution should be 3.41. In the case of 20 observations, and two lagged dependent variables, the critical values according to an $F$-distribution should be 3.63. However, for every 0.01 decrease in $\phi$ the critical value moves to the right by 0.16. The average value of the actual distribution in the simulations, for values of $\phi$ ranging from 0.004 to 0.065, is 4.30. In this range of values of $\phi$ we obtained values for the actual distribution as high as 4.79. Finally, using 21 observations and one lagged value the critical value for an $F$ distribution is 4.38. In our simulations the 5 per cent critical value goes up by 0.38 when $\phi$ falls by 0.01; the average critical value is 5.37 and the maximum 5 per cent critical value obtained was 6.65.

Given the characteristics of the short-sample distribution of the Wald test it would be better to use higher critical levels. For example, in the case of 3 and 13 degrees of freedom, it would be more advisable to use the values from an $F$-distribution but for a critical level of 2 to 2.5 per cent. For the 2 and 16 degrees of freedom, we should use 1 to 1.5 per cent critical levels of an $F$-distribution. Finally, for 1 and 19 degrees of freedom test, it would be better to use an $F$-distribution critical level from 0.5 to 1 per cent. In summary, drawing conclusions for causality tests based on the usual 5 per cent critical levels from an $F$-distribution are not satisfactory; a 1 per cent critical value would be more adequate.
5.- Estimation

a.- Variable choice and methodology.

It is generally argued that financial deepening affects growth through increased productivity. For example, Greenwood and Jovanovic (1998) or Bencivenga and Smith (1991) argue that funds intermediated by financial institutions are available for investment in productive capital. This, in turn, will bring economies with different financial structures to different income levels. For example, Levine et al. (1999) perform causality tests on a panel of countries and include in their analysis a set of control variables to take care of regional specific effects. They study the extent of the improvement in growth produced by the introduction of financial measures like liquid liabilities and private credit. Barro and Sala-i-Martin (1995) support an hypothesis that liquid liabilities positively affect growth rate in a cross section of countries.

Arestis and Demetriades (1997) find mixed empirical evidence on the causal direction between finance and growth using data from Germany, South Korea and the US. Also Demetriades and Hussein (1997) find causal evidence in both directions in an empirical analysis of 16 developing countries. Specifically, they study the causal direction from a number of financial depth ratios to per-capita GDP. In particular, they use, as measures of financial sophistication, liquid liabilities to GDP and the ratio domestic bank credit to GDP.

Other studies including financial-depth effects over both growth rate and output level
In a newer line of research, based on the ideas of Sussman (1995), Harrison et al. (1999) study the feedback between the real and financial sector on a data panel from US states. They perform a test of level of Gross State Output on costs of financial intermediation as the latter are affected by "... both the wage and the specialization [of financial intermediation services] effect." In their work set-up costs of intermediation reflect how the financial sector affects the real sector development.

There has been a large degree of uncertainty towards the treatment of stochastic and deterministic trends in the empirical specification of causality tests. For example, Demetriades et al (1998) using data from South East Asia find evidence that the order of integration of their financial savings variable is one in some cases and zero in others. Demetriades and Hussein (1996) identify different degrees of integration for bank deposit liabilities to GDP ratio and claims on private sector to GDP ratio using a sample of developing countries. On the other hand, many studies that consider cross-country and time series panels of data assume deterministic trends and treat time series as stationary with roots close to but below one.

b.- The data

In this section we study the causal relationship between finance and growth over a comprehensive database from Latin-America and the Caribbean. The data is taken from the World Bank's (1997) World Development Report. The data include
information on per capita GNP growth, money, private credit, bank credit and debt. Generally, the data is available for 31 countries and 26 years from 1970 to 1995 and there are incomplete data for some periods and regions. The variables we use for the tests are:

- the logarithm of GNP per capita
- liquid liabilities\(^{36}\) as per cent of GDP
- quasi-liquid liabilities as % of GDP\(^{37}\)
- private non-guaranteed debt as per cent of external debt
- credit to private sector as % of GDP and
- domestic credit provided by banking sector as % of GDP.

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<th>Variable Code</th>
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<td>Logarithm GNP per capita, Atlas method (US$)</td>
</tr>
<tr>
<td>dldgpcap</td>
<td>Annual growth in the logarithm of GNP per capita (US$)</td>
</tr>
<tr>
<td>liqliab</td>
<td>Liquid liabilities (M3) as % of GDP</td>
</tr>
<tr>
<td>qulliab</td>
<td>Quasi-liquid liabilities (% of GDP)</td>
</tr>
<tr>
<td>nonguar</td>
<td>Private non-guaranteed debt (% of external debt)</td>
</tr>
<tr>
<td>privcred</td>
<td>Credit to private sector (% of GDP)</td>
</tr>
<tr>
<td>bankcred</td>
<td>Domestic credit provided by banking sector (% of GDP)</td>
</tr>
</tbody>
</table>

\(^{36}\) M3, money: coins and notes in circulation plus all other private-sector bank deposits plus certificates of deposit (Butler and Isaacs, 1996). It corresponds to lines 34 and 35 in International Finance Statistics from the International Monetary Fund.

\(^{37}\) Quasi-liquid liabilities correspond to the definition of liquid liabilities minus M1. M1 is normally defined as notes and coins in circulation plus private-sector current accounts and deposit accounts that can be transferred by cheque (Butler and Isaacs, 1996). It corresponds to lines 35 in International Finance Statistics by the International Monetary Fund.


c.- Empirical Results

Tables 2.4 to 2.8 below present the results of the causality test performed over the 31 countries of Latin America and The Caribbean using the level of per capita output. The estimates display very weak causality in both directions. Table 2.9 presents a summary of the results in terms of the number of significant tests, achieved by each of the financial measurements. Only in six cases, from all five financial measurements and 31 countries, result in a test significant at the 1 per cent level. Given the generally small size of the sample and according to the analysis and simulation results earlier in this chapter, the 1 per cent confidence level is more reliable.

There is some evidence of causality in some of the larger economies: Argentina, Brasil and México of finance causing per capita GNP level but only in one case, the one of non-guaranteed debt causing per capita GNP level, this is strong at the 1% level for Mexico.

The evidence on the opposite causality relationship, level of per capita output causing liquid liabilities and credit, are very mild. Only nine cases out of all 31 regions and out all five variables are significant at 1 per cent. This evidence is not observed in any of the main regional economies.
Table 2.4. Causality relation between liquid liabilities and per capita GNP level.

<table>
<thead>
<tr>
<th></th>
<th>liqliab cause Ingnpcap</th>
<th>Ingnpcap cause liqliab</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F-value</td>
<td>degrees of freedom</td>
</tr>
<tr>
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</tr>
<tr>
<td>Argentina</td>
<td>4.22 *</td>
<td>F(2,19)</td>
</tr>
<tr>
<td>The Bahamas</td>
<td>0.56</td>
<td>F(2,19)</td>
</tr>
<tr>
<td>Barbados</td>
<td>1.88</td>
<td>F(2,19)</td>
</tr>
<tr>
<td>Belize</td>
<td>7.17 *</td>
<td>F(2, 8)</td>
</tr>
<tr>
<td>Bolivia</td>
<td>0.34</td>
<td>F(2,19)</td>
</tr>
<tr>
<td>Brazil</td>
<td>3.96 *</td>
<td>F(2,19)</td>
</tr>
<tr>
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</tr>
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<td>1.42</td>
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</tr>
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</tr>
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<td>Mexico</td>
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Number significant at 5%: 4
Number significant at 1%: 2
Table 2.5. Causality relation between private credit and per capita GNP level.

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Number significant at 5% | 4 | 6 |
Number significant at 1% | 2 | 1 |
Table 2.6. Causality relation between quasi-liquid liabilities and per capita GNP level.

| Country                  | Quilliab cause Ingnpicap | F-value | Degrees of freedom | P>|F| % | Ingnpicap cause quilliab | F-value | Degrees of freedom | P>|F| % |
|--------------------------|--------------------------|---------|--------------------|----|------------------------|---------|--------------------|------|---------------|
| Antigua and Barbuda      | F(2, 6)                  | 96.3    |                    | 0.25 F(2, 6) | 78.4 |
| Argentina                | F(2, 19)                 | 33.5    |                    | 0.04 F(2, 19) | 95.6 |
| The Bahamas              | F(2, 19)                 | 24.0    |                    | 4.77 * F(2, 19) | 2.1 |
| Barbados                 | F(2, 19)                 | 17.1    |                    | 6.83 ** F(2, 19) | 0.6 |
| Belize                   | F(2, 8)                  | 2.9     |                    | 1.69 F(2, 9) | 23.9 |
| Bolivia                  | F(2, 19)                 | 57.1    |                    | 1.47 F(2, 19) | 25.6 |
| Brazil                   | F(2, 19)                 | 6.5     |                    | 1.07 F(2, 19) | 36.3 |
| Chile                    | F(2, 19)                 | 34.4    |                    | 0.83 F(2, 19) | 45.1 |
| Colombia                 | F(2, 19)                 | 29.2    |                    | 1.22 F(2, 19) | 31.7 |
| Costa Rica               | F(2, 19)                 | 38.5    |                    | 5.61 * F(2, 19) | 1.2 |
| Dominica                 | F(2, 14)                 | 73.2    |                    | 4.32 * F(2, 14) | 3.5 |
| Dominican Republic       | F(2, 19)                 | 94.6    |                    | 0.63 F(2, 19) | 54.2 |
| Ecuador                  | F(2, 19)                 | 39.5    |                    | 0.08 F(2, 19) | 91.9 |
| El Salvador              | F(2, 19)                 | 1.2     |                    | 0.57 F(2, 19) | 57.5 |
| Guatemala                | F(2, 19)                 | 36.1    |                    | 2.59 F(2, 19) | 10.1 |
| Guyana                   | F(2, 19)                 | 7.7     |                    | 2.25 F(2, 19) | 13.2 |
| Haiti                    | F(2, 19)                 | 14.8    |                    | 1.96 F(2, 19) | 16.9 |
| Honduras                 | F(2, 19)                 | 58.6    |                    | 3.23 F(2, 19) | 6.2 |
| Jamaica                  | F(2, 19)                 | 99.8    |                    | 0.94 F(2, 19) | 40.7 |
| Mexico                   | F(2, 19)                 | 4.0     |                    | 1.02 F(2, 19) | 38.1 |
| Nicaragua                | F(2, 19)                 | 23.8    |                    | 0.27 F(2, 19) | 76.9 |
| Panama                   | F(2, 19)                 | 15.5    |                    | 0.06 F(2, 19) | 94.4 |
| Paraguay                 | F(2, 19)                 | 76.3    |                    | 3.02 F(2, 19) | 7.3 |
| Peru                     | F(2, 19)                 | 70.8    |                    | 3.12 F(2, 19) | 6.7 |
| St. Kitts and Nevis      | F(2, 7)                  | 25.6    |                    | 0.38 F(2, 7) | 69.7 |
| St. Lucia                | F(2, 2)                  | 49.4    |                    | 4.72 F(2, 2) | 17.5 |
| St. Vincent-the Grenadines| F(2, 14)                | 14.5    |                    | 1.58 F(2, 14) | 24.0 |
| Suriname                 | F(2, 19)                 | 19.2    |                    | 3.30 F(2, 19) | 5.9 |
| Trinidad and Tobago      | F(2, 19)                 | 18.1    |                    | 7.92 ** F(2, 19) | 0.3 |
| Uruguay                  | F(2, 19)                 | 5.7     |                    | 0.70 F(2, 19) | 50.7 |
| Venezuela                | F(2, 19)                 | 72.3    |                    | 4.23 * F(2, 19) | 3.0 |

Number significant at 5%: 3
Number significant at 1%: 0

103
Table 2.7. Causality relation between non-guaranteed debt and per capita GNP level.

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Number significant at 5%: 4  
Number significant at 1%: 3
Table 2.8. Causality relation between banking credit and per capita GNP level.

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Number significant at 5%  5  6
Number significant at 1%  2  2

105
Table 2.9. Summary causality relation between finance and per capita GNP level.

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<td>0</td>
<td>4</td>
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<tr>
<td>Uruguay</td>
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<tr>
<td>Venezuela</td>
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<td>0</td>
<td>1</td>
<td>0</td>
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<td><strong>Total significant</strong></td>
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<td><strong>6</strong></td>
<td><strong>25</strong></td>
<td><strong>9</strong></td>
<td></td>
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</tbody>
</table>

Table 2.10 to 2.14 present the results of causality test using GNP growth and financial depth for the same countries and time period. Again, the tests are generally non-significant in both directions of causality. In this case there is no evidence that the growth of the main regional economies is driven by any of the financial measurements under study.
Table 2.15 presents a global summary of the results with the overall number of significant tests in each country. Only nine times, considering all five financial measurements and the 31 countries, the results are significant and the 1 per cent level. When looking at the 5 per cent level, only 24 experiments out of 150 are significant. The causality relationship from GNP growth to finance is significant only in nine cases at 5 per cent and in one case at 1 per cent. In general, the Caribbean shows stronger connection between finance and growth than Latin America does.
Table 2.10. Causality relation between liquid liabilities and per capita GNP growth.

| liqliab cause dlgnpcap | F-value | distribution degrees of freedom | P(>|F|) % | dlgnpcap cause liqliab | F-value | distribution degrees of freedom | P(>|F|) % |
|------------------------|---------|--------------------------------|----------|------------------------|---------|--------------------------------|----------|
| Antigua and Barbuda    | 0.49    | F(2, 5)                        | 64.1     | 0.75                   | F(2, 5) | 52.0                           |
| Argentina              | 2.13    | F(2,18)                        | 14.8     | 0.89                   | F(2,18) | 42.9                           |
| The Bahamas            | 0.39    | F(2,18)                        | 68.5     | 2.69                   | F(2,18) | 9.5                            |
| Barbados               | 0.58    | F(2,18)                        | 56.7     | 1.94                   | F(2,18) | 17.2                           |
| Belize                 | 1.29    | F(2, 8)                        | 32.6     | 1.03                   | F(2, 9) | 39.6                           |
| Bolivia                | 0.33    | F(2,18)                        | 72.4     | 0.98                   | F(2,18) | 39.4                           |
| Brazil                 | 0.49    | F(2,18)                        | 61.8     | 0.50                   | F(2,18) | 61.7                           |
| Chile                  | 1.39    | F(2,18)                        | 27.5     | 0.08                   | F(2,18) | 92.0                           |
| Colombia               | 0.21    | F(2,18)                        | 81.4     | 2.29                   | F(2,18) | 13.0                           |
| Costa Rica             | 2.75    | F(2,18)                        | 9.0      | 1.95                   | F(2,18) | 17.1                           |
| Dominica               | 0.47    | F(2,14)                        | 63.4     | 0.17                   | F(2,14) | 84.9                           |
| Dominican Republic     | 0.02    | F(2,18)                        | 98.3     | 1.97                   | F(2,18) | 16.8                           |
| Ecuador                | 1.00    | F(2,18)                        | 38.7     | 0.38                   | F(2,18) | 69.1                           |
| El Salvador            | 0.14    | F(2,18)                        | 87.2     | 0.02                   | F(2,18) | 98.4                           |
| Guatemala              | 1.14    | F(2,18)                        | 34.3     | 1.40                   | F(2,18) | 27.1                           |
| Guyana                 | 5.21 *  | F(2,18)                        | 1.6      | 3.02                   | F(2,18) | 7.4                            |
| Haiti                  | 1.68    | F(2,18)                        | 21.4     | 0.94                   | F(2,18) | 41.1                           |
| Honduras               | 4.18 *  | F(2,18)                        | 3.2      | 0.73                   | F(2,18) | 49.4                           |
| Jamaica                | 0.01    | F(2,18)                        | 99.4     | 1.51                   | F(2,18) | 24.8                           |
| Mexico                 | 1.23    | F(2,18)                        | 31.6     | 1.74                   | F(2,18) | 20.4                           |
| Nicaragua              | 1.82    | F(2,18)                        | 19.1     | 2.65                   | F(2,18) | 9.8                            |
| Panama                 | 3.47    | F(2,18)                        | 5.3      | 4.62 *                 | F(2,18) | 2.4                            |
| Paraguay               | 1.58    | F(2,18)                        | 23.3     | 0.93                   | F(2,18) | 41.4                           |
| Peru                   | 0.50    | F(2,18)                        | 61.4     | 0.17                   | F(2,18) | 84.4                           |
| St. Kitts and Nevis    | 5.63 *  | F(2, 6)                        | 4.2      | 0.29                   | F(2, 6) | 75.8                           |
| St. Lucia              | 0.78    | F(2, 1)                        | 62.6     | 10.88                  | F(2, 1) | 21.0                           |
| St. Vincent-the Grenadines | 3.12   | F(2,14)                        | 7.6      | 0.97                   | F(2,14) | 40.4                           |
| Suriname               | 6.48 ** | F(2,18)                        | 0.8      | 0.61                   | F(2,18) | 55.5                           |
| Trinidad and Tobago    | 7.95 ** | F(2,18)                        | 0.3      | 1.65                   | F(2,18) | 21.9                           |
| Uruguay                | 1.20    | F(2,18)                        | 32.4     | 0.06                   | F(2,18) | 94.0                           |
| Venezuela              | 3.00    | F(2,18)                        | 7.5      | 0.32                   | F(2,18) | 73.2                           |

Number significant at 5%: 4
Number significant at 1%: 1
Table 2.11. Causality relation between private credit and per capita GNP growth.

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<td>degrees of freedom</td>
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<td>4.37 * F(2,18)</td>
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<td>0.50 F(2, 8)</td>
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<tr>
<td>Bolivia</td>
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</tr>
<tr>
<td>Brazil</td>
<td>0.14 F(2,18)</td>
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</tr>
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<td>St. Vincent-the Grenadines</td>
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</tbody>
</table>

Number significant at 5%: 7, 2
Number significant at 1%: 2, 0
| Country                        | F-value  | degrees of freedom | P>|F| | F-value  | degrees of freedom | P>|F| |
|-------------------------------|----------|--------------------|------|----------|--------------------|------|
| Antigua and Barbuda           | 0.02     | F(2, 5)            | 97.7 | 0.58     | F(2, 5)            | 59.1 |
| Argentina                     | 1.78     | F(2,18)            | 19.7 | 0.47     | F(2,18)            | 63.5 |
| The Bahamas                   | 1.15     | F(2,18)            | 33.8 | 2.67     | F(2,18)            | 9.6  |
| Barbados                      | 0.76     | F(2,18)            | 48.4 | 1.00     | F(2,18)            | 38.8 |
| Belize                        | 0.67     | F(2, 8)            | 53.9 | 1.10     | F(2, 9)            | 37.4 |
| Bolivia                       | 0.61     | F(2,18)            | 55.3 | 0.71     | F(2,18)            | 50.6 |
| Brazil                        | 0.90     | F(2,18)            | 42.4 | 0.52     | F(2,18)            | 60.4 |
| Chile                         | 0.08     | F(2,18)            | 91.9 | 0.20     | F(2,18)            | 82.4 |
| Colombia                      | 0.32     | F(2,18)            | 72.7 | 2.12     | F(2,18)            | 14.9 |
| Costa Rica                    | 1.39     | F(2,18)            | 27.5 | 2.76     | F(2,18)            | 9.0  |
| Dominica                      | 0.08     | F(2,14)            | 92.8 | 0.06     | F(2,14)            | 93.9 |
| Dominican Republic            | 0.33     | F(2,18)            | 72.7 | 0.53     | F(2,18)            | 60.0 |
| Ecuador                       | 0.65     | F(2,18)            | 53.2 | 0.86     | F(2,18)            | 43.8 |
| El Salvador                   | 0.49     | F(2,18)            | 62.2 | 0.28     | F(2,18)            | 76.1 |
| Guatemala                     | 0.52     | F(2,18)            | 60.3 | 1.35     | F(2,18)            | 28.5 |
| Guyana                        | 3.27     | F(2,18)            | 6.1  | 2.86     | F(2,18)            | 8.3  |
| Haiti                         | 3.54     | F(2,18)            | 5.0  | 0.60     | F(2,18)            | 56.2 |
| Honduras                      | 3.61     | F(2,18)            | 4.8  | 0.83     | F(2,18)            | 45.0 |
| Jamaica                       | 0.05     | F(2,18)            | 95.5 | 1.33     | F(2,18)            | 28.9 |
| Mexico                        | 2.60     | F(2,18)            | 10.2 | 0.70     | F(2,18)            | 50.8 |
| Nicaragua                     | 1.25     | F(2,18)            | 31.1 | 2.73     | F(2,18)            | 9.2  |
| Panama                        | 2.88     | F(2,18)            | 8.2  | 3.27     | F(2,18)            | 6.1  |
| Paraguay                      | 0.91     | F(2,18)            | 42.1 | 0.95     | F(2,18)            | 40.6 |
| Peru                          | 0.07     | F(2,18)            | 93.1 | 1.36     | F(2,18)            | 28.1 |
| St. Kitts and Nevis           | 7.31     | F(2, 6)            | 2.5  | 0.57     | F(2, 6)            | 59.4 |
| St. Lucia                     | 0.51     | F(2, 1)            | 70.2 | 5.54     | F(2, 1)            | 28.8 |
| St. Vincent-the Grenadines    | 4.09     | F(2,14)            | 4.0  | 1.20     | F(2,14)            | 32.9 |
| Suriname                      | 7.10     | F(2,18)            | 0.5  | 0.04     | F(2,18)            | 95.9 |
| Trinidad and Tobago           | 7.08     | F(2,18)            | 0.5  | 1.73     | F(2,18)            | 20.6 |
| Uruguay                       | 1.33     | F(2,18)            | 29.0 | 0.20     | F(2,18)            | 81.8 |
| Venezuela                     | 4.02     | F(2,18)            | 3.6  | 0.31     | F(2,18)            | 74.0 |

Number significant at 5%: 5
Number significant at 1%: 0
Table 2.13. Causality relation: non-guaranteed debt and per capita GNP growth.

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<th>dlgnpcap cause nonguar</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F-value</td>
<td>distribution</td>
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<tr>
<td>Antigua and Barbuda</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<tr>
<td>The Bahamas</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Barbados</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Belize</td>
<td>0.11 F(2,13)</td>
<td>89.8</td>
</tr>
<tr>
<td>Bolivia</td>
<td>0.15 F(2,18)</td>
<td>86.2</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.35 F(2,18)</td>
<td>28.3</td>
</tr>
<tr>
<td>Chile</td>
<td>0.82 F(2,18)</td>
<td>45.7</td>
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<td>Colombia</td>
<td>0.83 F(2,18)</td>
<td>45.2</td>
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<td>Costa Rica</td>
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<tr>
<td>Dominican Republic</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Dominican Republic</td>
<td>0.33 F(2,18)</td>
<td>72.5</td>
</tr>
<tr>
<td>Ecuador</td>
<td>0.10 F(2,18)</td>
<td>90.2</td>
</tr>
<tr>
<td>El Salvador</td>
<td>0.27 F(2,18)</td>
<td>76.5</td>
</tr>
<tr>
<td>Guatemala</td>
<td>2.31 F(2,18)</td>
<td>12.8</td>
</tr>
<tr>
<td>Guyana</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Haiti</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Honduras</td>
<td>2.29 F(2,18)</td>
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<td>Paraguay</td>
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<td>Peru</td>
<td>0.14 F(2,18)</td>
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<td>St. Kitts and Nevis</td>
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<td>St. Lucia</td>
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<tr>
<td>St. Vincent-the Grenadines</td>
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<td>-</td>
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<tr>
<td>Suriname</td>
<td>-</td>
<td>-</td>
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<tr>
<td>Trinidad and Tobago</td>
<td>0.10 F(2,18)</td>
<td>90.3</td>
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<tr>
<td>Uruguay</td>
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Number significant at 5% 0                                          3
Number significant at 1% 0                                          1
Table 2.14. Causality relation between banking credit and per capita GNP growth.

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<td>Argentina</td>
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<td>F(2,18)</td>
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<td>F(2,18)</td>
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<tr>
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<td>Peru</td>
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<td>6.06 **</td>
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<td>F(2,18)</td>
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<td>Uruguay</td>
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<td>Venezuela</td>
<td>5.85 *</td>
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Number significant at 5%: 5
Number significant at 1%: 3
Table 2.15. Summary causality relation between finance and per capita GNP growth.

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<th>Country</th>
<th>Finance cause 5% level</th>
<th>Finance cause 1% level</th>
<th>Growth cause 5% level</th>
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6.- Concluding remarks.

This chapter has considered the short sample characteristic of causality test between financial development and economic growth. From the theoretical point of view we
have concentrated on time series and autorregresive processes and the biases arising on the estimation of these. From the empirical point of view we have studied the causality relationship for measurements involving financial intermediation and mainly banking activity.

A recent study Demetriades and Hussein (1996) perform an extensive analysis of the causal relationship between financial deepening and growth\(^{38}\) finding similar evidence to ours. They modify causality tests to allow for integrated variables. After they find that GNP and financial sophistication co-integrate, they compute the error correction model and perform a multiple test of restricting either or both the financial variable and the error correction term to zero. A significant test will imply that a causality relationship exist from either GNP to financial development or vice-versa.

They test two dimensions of financial development, namely: the ratio of bank deposit liabilities to GNP and the ratio of claims on private sector to GNP. For the first one of these measures, they find some evidence of bi-directional causality for the 10% and 5% confidence levels they present. They reject non-causality from GNP to financial development in 62% of the countries using the 10% critical value form an F distribution. In addition, they find in 38% of the cases non-causality is significant at

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\(^{38}\) Demetriades and Hussein (1996) perform the test using data from the IMF publication *International Financial Statistics* (1993). Their criteria to include countries is they have to be a less developed country in 1960 and they must have at least 27 continuos observations on the variables of interest and its population must have exceeded 1 million people in 1990. They consider sixteen countries: Costa Rica, El Salvador, Greece, Guatemala, Honduras, India, Korea, Mauritius, Pakistan, Portugal, South
5% while the inverse causality from financial development to GNP is significant in 38% of the cases at the 10% critical value and in 31% of the cases when the 5% critical level is used.

According to our model a 1% significance test will be safer, given the possibility of short sample biases. Using the 1% significance levels only in 38% of the cases GNP causes finance development while just 15% of the countries show evidence of causality from finance to GNP\textsuperscript{39}.

They draw a similar picture when using the second financial measurement, claims on private sector to GNP ratio. At the 10% level two thirds of the countries present causality from GNP to financial development and one-third from finance to GNP\textsuperscript{40}. At the 1% level only one third of the countries present GNP to finance causality an none of them display the opposite causation. Overall the procedure does not seem to be conclusive, there is little evidence of causality unless the critical level are relaxed greatly. From a different perspective the financial measurements they use, related to banks lending, may not reflect properly the evolution of the financial sector. As the authors illustrate with the case of Pakistan, the Pakistani banking system plays a minor role in the financing of agricultural activities which constitute a large part of this country's output.

\textsuperscript{39} Only for the two countries India and Mauritius

\textsuperscript{40} At the 5% level GNP to finance causality occurs only for 44% of the countries while in one third of the countries the opposite causality is still observed.
In a more recent work Levine, Loayza and Beck (1999) find evidence of legal and accounting reform that strengthens creditor rights, contract enforcement and accounting practices accelerates economic growth by helping financial development. Their causality tests however, involve estimation of cross-sections and dynamic panels of data. The short sample properties of their results are uncertain. As we have seen here, average results using the time series data do not show strong effects. Future research ought to study and explain these differences.
Chapter 3
Compulsory Holdings and
the Risk Return Relationship

1.- Introduction.

Sharpe (1964) first set out the capital asset pricing model (CAPM) by realising that through diversification some of the risk inherent in an asset can be avoided and that for this reason the total risk of the asset is not the relevant influence on its price. Until Sharpe's formulation little had been said concerning which particular risk component was the relevant factor to determine price. Based on portfolio-selection theory, developed by Markowitz (1959), Sharpe aimed to construct a market equilibrium theory of asset prices under conditions of risk. Later contributions by Lintner (1965) and Mossin (1966) created the resulting CAPM as a model of investors' return expectations, which dominated as a research paradigm for a few decades.

In the usual solution of the CAPM many investors select efficient portfolios by maximising the expected return for a given level of expected risk (or variance). There are \( N \) assets available with expected variance covariance matrix \( \Omega \) and vector of expected excess returns \( \bar{r} \) per unit invested. The excess return is measured with
respect to a risk free asset available to everyone who wants to borrow or invest in it. If we take an investor identified by its index number \( i \), who is willing to bear a maximum risk equivalent to a variance \( \sigma_i^2 \), his maximisation problem is:

\[
\text{Max } \omega_i^T \bar{r} \quad \text{subject to } \sigma_i^2 = \omega_i^T \Omega \omega_i
\]

Under the assumptions that every investor can purchase any of the assets in any divisible amounts, that there is enough supply to satisfy the demand and that these two are in equilibrium, all investors will arrive at the same distribution of investment among the assets. In equilibrium, two investors will only differ in the total amount of investment. The lagrangean first conditions are:

\[
\frac{\partial L}{\partial \omega_i} = \bar{r} - \lambda_i \Omega \omega_i = 0
\]
\[
\frac{\partial L}{\partial \lambda_i} = -\sigma_i^2 + \omega_i^T \Omega \omega_i = 0
\]

The solution for investor \( i \) will verify:

\[
\omega_i = \lambda_i \Omega^{-1} \bar{r}
\]
\[
\lambda_i^2 = \frac{\sigma_i^2}{\bar{r}^T \Omega^{-1} \bar{r}}
\]

Therefore, all investors use the same distribution of wealth among assets investing different total amounts depending on \( \lambda_i \) that depends itself on the level of risk \( \sigma_i \). Since the market demand and supply are in equilibrium the sum of all individual demands will be equal to the market portfolio.

\[
\omega_M = \sum_{i=1}^{K} \omega_i = (\sum_{i=1}^{K} \lambda_i) \Omega^{-1} \bar{r} = \lambda_M \Omega^{-1} \bar{r}
\]
The usual CAPM expression comes from multiplying the relationship above by a vector \( \tilde{e}_j \) whose component are all zero except for the \( j \)-th component that records the supply asset \( j \)-th:

\[
\begin{align*}
\omega^T_M \Omega \omega_M &= \lambda_M \omega^T_M \tilde{r} \\
\tilde{e}_j^T \Omega \omega_M &= \lambda_M \tilde{e}_j^T \tilde{r}
\end{align*}
\]

Dividing the two expressions above we obtain:

\[
\begin{align*}
r_j &= \lambda_M \sigma^2_{M} r_M \\
r_j &= \beta_j r_M
\end{align*}
\]

The CAPM provides theory of asset prices for a market that is in equilibrium, it can be regarded as an equilibrium version of mean-variance analysis. The only restriction that the CAPM actually imposes on the way the equilibrium is achieved is that the market portfolio is mean-variance efficient i.e. that given the total market variance \( \sigma^2_M \), no other hypothetical combination of assets will produce a higher expected return.

Empirical tests of the CAPM and the mean-variance efficiency hypothesis have frequently identified the presence other factors as a constant term or the amount of leverage. This has been observed in the past by for example: Fama and MacBeth (1973); in Black, Jensen and Scholes (1972); in Gibbons, Ross and Shanken (1989) and Fama and French (1992 and 1993). On the other hand, as indicated by Roll (1977) given the impossibility of measuring the market portfolio a realistic empirical implementation of the CAPM is problematic. The exclusion all non-marketable assets
from the market portfolio and from the expression of the market equilibrium are crucial and give scope for the inclusion of other variables in the simple risk-return relationship.

The additional components in the CAPM have been generally associated with the existence of restrictions in borrowing from the riskless asset and to short-sale restrictions, for example Black (1972) and Ross (1977). We develop a model that incorporates such restrictions and study how they are related to the information and the development of the market.

The rest of chapter 3 is organised as follows. In Section 2 we identify the scope for our consideration of compulsory holdings in the chapter. We also study the role played by compulsory holdings and other restrictions in terms of the information they convey and the development of stock markets. In Section 3 we develop a model of asset pricing with compulsory holdings and short-sale restrictions. Section 4 concludes.
2. Compulsory Holdings.

The usual assumptions of the CAPM can be summarised in the following five groups. Firstly, all investors maximise expected utility of nominal wealth in one period, they choose among alternative portfolios on the basis of mean and variance of the assets' returns. All investors agree on their expectations and planning horizons. Second, all investors can borrow or lend unlimited amounts of money at an exogenously-given risk-free rate of interest and there are no restrictions in short sales of any asset. Next, all assets are perfectly divisible, perfectly liquid and all assets are marketable. Fourth, there are no transaction costs and no taxes. Finally, all investors are price takers and the quantities of all assets are given.

Our analysis incorporates any of three distinctive features with respect to the theory outlined above. In our model a) some agents cannot short-sell some assets in unlimited amounts, b) some agents have to maintain strictly positive claims in some assets or c) these restrictedly available assets may or may not be measurable so that they can be part of a market index.

a. Mandatory holdings and short sale restrictions.

The three differences in our analysis relate somehow to the marketability of these investments, their divisibility or their liquidity. Due to limited market coverage some investors have to hold non-negative claims without any possibility of trading them. These positive holdings do not maintain any connection with the risk return trade-off of the corresponding assets. There are two types of assets in the theory presented
here: on the one hand, the assets freely available in the market, that any investor can
buy, sell, take upon short positions or lend. On the other hand, there are those assets
that some investors cannot trade freely. For the latter, there are rigid lower bounds
that apply to the demand of some agents. At the same time, there are no upper limits
on the agents' demand. Finally, it is also likely that these constrained assets are
mainly traded outside the established stock markets and thus, their value does not
contribute to any stock market aggregate.

The problem of mandatory holdings has been analysed in the past by Chen (1995) and
Cvitanic and Karatzas (1993). Chen allows for such restrictions to study the
incentives for innovation in financial markets. Chen's "economy [...] differs from the
standard economies only in that there are realistic trading constraints" (p. 118). Given
that "investors can no longer achieve perfect risk sharing even when there are as many
traded securities as states of nature; investors will have different marginal valuations
of both issued and not-yet-issued securities. The existence of such valuation
 clienteles provides the necessary pretext for financial innovations (p. 118)." In Chen's
 proposal, "by opening markets for new securities [...], innovators help enhance the
effectiveness of the arbitrage valuation approach, because innovation brings the
arbitrage-free price closer to the equilibrium price (p. 119)." A key finding in this
work is the deterioration of equilibrium asset prices under mandatory holdings: "In
economies where [all securities are subject to non-negative constrains] the replication
based arbitrage approach tend to be above the equilibrium (p. 129)."

Cvitanic and Karatzas study various geometries for investors facing a number of
restrictions. Investors facing mandatory holdings do not consider the securities risk-
return trade off to allocate their wealth among the assets. In such a case they will not be able to maximise their expected return for an acceptable level of risk.

During the early development stages of the CAPM theory, short-sale constraints were a predominant concern. These trade restrictions, where an investor cannot trade in negative amounts of an asset, can be seen as a particular case of mandatory holdings. The linearity of CAPM under such restrictions, and more generally its efficacy, were first studied in Black (1972), Fama (1976) and Ross (1977). Black (1972) found that restrictions on borrowing from a riskless asset retain the properties of risk-return trade off. "The expected return of any risky asset is a linear function of its β, just as it is without any restrictions or borrowing" (page 455). In Ross's (1977) geometric approach the CAPM linearity holds only for those assets that face no restrictions to short positions. Ross's work eminently corroborates the validity of CAPM under a fairly extensive collection of constraints, including restrictions in borrowing and lending from the riskless asset.

In a more recent work Markowitz (1992) creates a set of geometries to represent different sets of binding constraints on short sales. For consecutive levels of risk Markowitz generates a path that at each level achieves maximum return. This path is characterised by assets that disappear from the efficient portfolio as risk increases.

He and Modest (1995) model a consumption-CAPM including short-sale restrictions in an attempt to account for the rejection of the intertemporal maximisation first order conditions implied by this model. They include other forms of market frictions such as no borrowing constraints, solvency constraints and transaction costs. Under large
frictions in the market they accept the intertemporal CAPM. Their model identifies a structure of market equilibria under diverse frictions, "[...]the results suggest that a combination of these market frictions can drive a large enough wedge between IMRS [Intertemporal Marginal Rates of Substitution] so that the apparent violations may not be inconsistent with market equilibrium." (p. 96)

Mathematically, compulsory holdings and short-sale restrictions can be seen as the same mechanism where a different set of boundaries has been applied. In practice, however, these very different. On the one hand, performing a short sale involves performing a special procedure within the market place; by borrowing assets from a broker (or gain the broker’s permission or agreement). On the other hand, compulsory long positions can be applied outside a stock market and to a section of the public (for example small business or students) by simply restricting them to market some assets.

b.- Restrictions to free trade and information.

Short positions, and restrictions imposed on the use of them in trade, play a fundamental role over the information communicated to the market. The theory identifies that allowing short positions can make the market more efficient by quickly conveying negative information about a firm's future prospects. However, the conclusions and empirical evidence from this theory are ambiguous.

Diamond and Verrecchia (1986) formulated a model where short sales constraints are predictors of bad news and future under-performance. Short-sale constraints will drive the informed investor out from trading these assets. In their model driving away
some investors does not bias prices but it slows down the adjustment of price process. Therefore, short-sale restrictions dilute bad news over time. No trading on short-sale constrained stock will signal the market that the stocks will under-perform in the future.

In an empirical approach Asquith and Meulbroek (1996) study short selling and negative interest in an attempt to test Diamond and Verrecchia's predictions. They find that the greater the short interest, the more the firm will underperform in the future. Large short selling is a good predictor of future underperformance. These and other studies have instigated claims, in relation to short selling U.S. regulation, like the following: "... based on the latest evidence, the existing restrictions faced by short-sellers interfere with useful market signals." (The Economist, 1995)

L'Her and Suret (1995) study Canadian stocks under a model of normal asset prices and short sales. They find a short-sell opportunity cost related to dispersion of the agents' beliefs and of the liquidity level of the securities. In their model asset prices are equal to the rate that would prevail in the absence of short-sale constraints. Increasing amount of dispersion affects returns positively while decreasing liquidity pushes prices down. The relationship, however, is no sanctioned by their empirical tests.

In a different approach, Wang et al. (1999) find that short-sale constraints do not affect, in practice, the possibility of diversification for American investors. "Although the diversification benefit decreases when short-sales constraints are imposed on all countries, they are not affected by those constraint on emerging markets." (p. 19) In
this case, international diversification enables American investors investing in emerging markets to compensate for the increased risk that arise from short-sale constraints in those markets.

Hornst et al. (1998) study restrictions faced by individual investors to take short positions in Dutch mutual funds or their underlying assets. They provide a modified efficient frontier based on the short-sales constraint work by Markowitz (1992). They identify groups of equity managed by Dutch mutual funds whose performance is unaffected by short-sale restrictions and that provide investors with an extension of the efficient set.

Excessive freedom to short sell has also been regarded as dangerous as it leaves markets at the mercy of speculators' attacks. For example, this newspaper article about metal prices in the London Metal Exchange (LME): "A key factor in the drop in aluminium and other metal prices in recent months to their lowest levels in several years has been an onslaught of funds engaged in short selling... " (Financial Times, 1998a). Short sellers were signalling bad news by short selling massively the metal the bad news perception was increased proportional to the short positions. "With Dollars 120bn under management compared with the combined value of LME warehouse stocks of less than Dollars 2bn, it is clear that these funds have more than enough muscle to move metals' prices substantially." (Ibid.)

In a different market, the Japanese government regarded speculative short positions in banks as damaging for bank stocks. Accordingly, short-sale legislation was tightened. "Faced with the imminent implementation of the new rules, hedge funds and other
investors have scrambled to unwind some of their positions this week... This has sharply pushed up shares in Japanese banks, since many hedge funds have been "shorting" bank stocks." (Financial Times, 1998b) In this case, according to the article, investors, being forced to close their short positions, pushed prices up

Generally, when short-sales restrictions are imposed short sellers are immediately bound to close their positions. It has been alleged about Malaysia that this led to a price fall due to a combination of excess supply and investor's lost confidence. "On August 28 [1997] the Malaysian government announced that it would ban so-called short-selling of stocks on the Kuala Lumpur exchange, in effect locking a number of big foreign investors into the market, at least temporarily, and discouraging others from entering it. The aim was to halt the market's decline but instead Dr. Mahathir scored a spectacular own-goal: on the day the news came out, the stockmarket dropped 4%." (The Economist, 1997)

In this case the sudden imposition of short-sale restrictions has driven markets and analysts immediately to leave Malaysia. The Diamond Verrecchia story of information delivery slowing down does not seem to be sustained when constraints appear abruptly. Generally, sudden imposition of short-sales restrictions is regarded as damaging to investors' confidence. Following the argument of Diamond and Verrechia the dissolution of such frictions as short-sale restrictions or compulsory holdings leads to increasing market participation from both investors and securities. Empirically, however, it is unclear what the short and medium-term effects are of such trade conditions easiness.
c.- *Market frictions and financial development.*

Markets where some stocks can only be traded above certain boundaries, and where these bounds are unrelated to the assets risk-return relationship, need not to be associated with an idea that the agents trading in those markets are unhappy with this arrangement. Although it is true that the amount of investment in these assets is not the result of an expected utility maximisation, investors may be prevented from trading these stocks for a variety of reasons. Where, for example, there is a desire to maintain some power position or keep control of a firm, as in the case of medium-size family business or large family conglomerates. The Catholic Church keeps control of many assets, land and others, for historical, cultural or political reasons. In these two examples, a change in the economic conditions is not likely to change the way these assets are traded. Moreover, only changes in the preferences of the investors holding these assets have the power to eliminate these constraints.

In contrast, a market to trade some assets might not exist, given that, the assets are not divisible and none of the investors being wealthy enough to trade the existing large units of them. High costs of finance may also lead to the existence of such frictions. Furthermore, the lack of a trade over some assets may be due to an extreme dispersion of the investors and their inability to pool their resources. Finally, there is no market for some non-tangible assets as, for example: claims on own human capital, future salary income or health.

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41 Efficient portfolios are also the result of expected utility maximisation for either agents having quadratic expected utilities or when asset returns normally distributed; see Markowitz (1952).
From the observation that relatively underdeveloped economies tend to have less developed financial systems, Gurley and Shaw (1967) sketched a theory of co-evolution between financial and economic development. In the early stages of economic development, most investment tends to be self-financed, and financial market activity is severely limited or even non-existent. As the economy develops, borrowing and lending flourish. Further development is associated with the appearance of banking and the intermediation of investment. And, ultimately, even more sophisticated financial institutions—such as equity markets—come to be observed.

A plausible account is offered in Bencivenga and Smith (1998) where the costs of financial intermediation are prohibitive for poor economies. "When utilization of the financial system involves real resource costs [...] economies with sufficient initial wealth may experience enough income growth that they take advantage of the benefits of financial market development, despite the associated costs. Such economies may, therefore, attain a steady state where the financial system is fully developed. [...] At the same time, initially poor economies may never grow enough that they can afford to incur the costs of a developed financial system. Such economies may then attain a steady state in which they are permanently poor, and in which their financial markets are permanently underdeveloped." (p. 364) In their model, financial development supports the high long-run level of real activity. According to Bencivenga and Smith (1998), financial systems can reach a state of underdevelopment because real activity is low, and real activity is low because the financial system is underdeveloped.

Sussman (1993) explains how, as capital stock grows, the financial system develops
and the cost of intermediation decreases. Due to competition and free entry the mark-up will decrease when the capital stock grows and pushes up the market for financial intermediation. "Each bank serves a smaller market share and gets closer to its competitors. As a result, the mark-up falls via the specialization effect and the decrease of market power." (p. 30) Sussman provides with a rational for financial intermediation as the optimal way of monitoring loans on behalf of the lenders: "Efficiency considerations require that only one lender will perform the monitoring, on behalf of other lenders." (p. 32)

In another attempt to consider the fundamental monitoring role performed by the financial system, Levine (1997) suggests that a variety of skills are required to evaluate production technologies and monitor management. If such skills become more common, as an economy becomes more developed, this would provide still another mechanism by which real development promotes financial development.

Townsend (1983) pioneered the work of establishing the co-evolution of financial activity and the level of development in the economy. He explains how real and monetary phenomena are consequences of the same underlying process of trade. In a similar way as fiat money allows consumers to exchange goods at a given moment of time, financial institutions allow them to trade over time. This work establishes a clear distinction between money and credit with regard to their capacity of lowering trade costs; credit can eliminate high costs of waiting one period for consumption. "The key idea is that the degree of interconnectedness of traders determines both the amount of production and trade as well as the types of assets which are used to facilitate exchange." (p. 896) In Townsend's model financial markets eliminate the
cost of waiting one period to consume goods that are not produced by the consumer. Without financial markets the consumer can only wait until his production period is complete to trade with other type of goods. This gives consumers an incentive to use financial institutions. Therefore, market fragmentation plays a role in economic activity and in the theory of finance.

Given the way financial markets evolve over time, we will study in the next section what effect this evolution has over asset prices. When individuals perceive themselves as poor they are not willing to pay for the fixed costs involved in trading assets. As their businesses grow they will find it worthwhile selling shares of their assets to be able to exchange these by other securities allowing them to hedge the risks they face. During this process, asset prices will affect the perception agents have about their wealth. This in turn will affect the rate of grow of the financial market activities.

The model we develop in the next section studies a segmented economy where a significant fraction of the investors have control over assets that are too small to trade in the stock market. In this economy, there is an established market with a reasonable number of participants and trade, but there are also an important number of dispersed assets that are only available to a number of dispersed investors. In addition, since these assets are not traded in an established market they are not accounted for in any of the established market indices. The price of some of these assets will be linked to land or property prices, price of industrial goods or consumer durable goods. In those cases it maybe possible to account for their price movements. However, it will be difficult to measure price variation in assets related to claims in human capital, like
level of education, health, life span or working life span.
3. The model.

a. Establishing individual demands.

There are two types of investor: investor type $A$ has to hold an asset, called “asset $\infty$”, in a fixed amount $\bar{\omega}_A$ and investor type $B$ who has no restrictions to participate in the established market. In this case the market can trade asset $\infty$ but investor type $A$ prefers, or has, to hold an amount without any regard for the risk and return of this asset. Investor $A$ will use the rest of the securities to hedge or diversify the risk induced by $A$. On the other hand, type $B$ investor will allocate wealth among assets according to Lagrange maximisation as described above.

Depending on the amount of asset $\infty$ there are three zones of maximisation in investor $A$’s problem. In the first zone the amount of compulsory holding induces an amount of risk larger than what investor $A$ wants to bear; in this zone it is not possible to add any security with positive expected return without increasing the risk. In the second, investor $A$ can increase the risk but he is already holding asset $\infty$ in an amount that will exceed the desired risk level unless part of this risk can be perfectly diversified using the rest of the assets. Third, the amount of asset $\infty$ is very small compared with the risk that the investor is willing to hold and so by simple Lagrange maximisation, which yield optimal hedging, the final amount of asset $\infty$ will be larger than the compulsory holding.

In the following sub-sections we will study the optimal investment allocation in each of these zones. Following the original portfolio allocation theory, Markowitz (1952),
we study the problem of portfolio allocation, where for a given risk tolerance an investor will select the allocation among risky assets that provides the highest expected return. Although there is a duality between expected return maximisation and expected risk minimisation, we believe that in our context the latter is a more appropriate way of studying the behaviour of an agent that faces investment constraints. A risk-averse investor that is forced to rely on random prospects from one resource, a lottery that he has not freely chosen to enter, cannot increase the risk beyond a certain level.

On the other hand, agents without facing any mandatory investments can freely solve an intertemporal problem where they will maximise utility at a point where expected income can be achieved with minimum variance. For them the usual duality between expected return maximisation and expected risk minimisation is valid. For these investors it does not make any difference which one of these dual problems is solved to establish the distribution of wealth among securities.

\textit{i. Compulsory holding exceeds desired risk.}

From the assumption made in section 2 that all individuals maximise expected utility of nominal wealth and using utility functions for risk-averse investors, it is clear that agents will prefer either to increase their wealth or to minimise the risk associated with the expected gains. Markowitz (1952) rejects the idea that there might be a portfolio that combines both maximum expected return and the minimum variance. He explains: "the portfolio with maximum expected return is not necessarily the one with minimum variance. There is a rate at which the investor can gain expected
return by taking on variance, or reduce variance by giving up expected return."(p. 79)

In the present case the compulsory holding induces an amount of risk larger than the agent's tolerance risk and even by hedging it is not possible to bring this variance below this benchmark. From the above it would be concluded that a non-satiable investor will try to achieve the highest possible return. In our set up when this investor is already above the desired level of risk, there is no theoretical limit to the amount of risk he would now take. Therefore, he would just go to infinite return. We see that the only possibility is that our investor will respond like a pure hedger. In other words, he will behave like an agent that takes positions in assets only to reduce his exposure to risk. Therefore, he will minimise the risk for the given level of compulsory holding⁴².

The investor minimises:

\[ \text{Min } \omega^T \Omega \omega \quad \text{subject to } \omega_\omega \geq \bar{\omega}_\omega \]

Where \( \omega^T \) is a vector of amounts held in each security \( \omega^T = [\omega_1, \ldots, \omega_n, \omega_\omega] \)

and \( \Omega \) is a matrix of variance covariance of all assets including asset \( \omega_\omega \). We partition matrix \( \Omega \) though the last column and row:

⁴² Notice that in our analysis agents have chosen a level of risk or variance to select afterwards a portfolio that preserves that level of risk. An alternative view is that agents decide first level of consumption (alternatively the wealth necessary to achieve that consumption) and subsequently they select a portfolio that minimises the risk. In both cases, for an agent that is forced to keep a too high amount of asset \( \omega_\omega \), there will be a one to one relationship between expected return and variance. This agent will be willing to swap some return in order to lower the variance, and at the same time, achieving the lower desired level of consumption.
\[ \Omega = \begin{bmatrix} \Omega_{1N} & \sigma_{1N,\omega} \\ \sigma_{1N,\omega}^T & \sigma_{\omega}^2 \end{bmatrix} . \]

We call \( \sigma_{1N,\omega}^T \) the vector of covariance between each security and asset \( \omega \) 

\( ( \sigma_{1N,\omega}^T = [\sigma_{1\omega}, \sigma_{2\omega}, \ldots, \sigma_{N\omega}]^T ) \).

The compulsory holding induces a level of risk:

\[ \bar{\omega}_\omega^2 \sigma_{\omega}^2 > \sigma_\omega^2 \]

**Proposition 1.** Under the condition that no combination of assets \( I \) to \( N \) is perfectly correlated with asset \( \omega \) and that:

\[ \sigma_\omega^2 < \bar{\omega}_\omega^2 (\sigma_{\omega}^2 - \sigma_{\omega,1N}^2 \Omega_{1N}^{-1} \sigma_{1N,\omega} ) \]

The minimisation above is equivalent to finding the best fit between asset \( \omega \) and the rest of the securities. Therefore, type \( A \) investor will buy a portfolio equal to:

\[ \bar{\omega}_\omega = \left[ -\Omega_{1N}^{-1} \sigma_{1N,\omega} \right] \bar{\omega}_\omega \]

**Proof:** see appendix 3A.

The condition in proposition 1 intends to rule out cases where the risk induced by compulsory holding can be brought below the desired level of risk. In a case like that, or when there is perfect correlation between asset \( \omega \) and other assets, investors would instead eliminate all or part of the compulsory risk and then maximise return using all assets again. This case is analysed in the next section.
ii.-  *Compulsory holding exceeds optimal hedging but not the whole desired risk.*

In this case the investor is forced to hold some asset $\infty$ but the compulsory amount does not induce a risk higher than his desired level. Here the residual variance between asset $\infty$ and assets 1 to $N$, is not larger than the desired level of risk:

$$\sigma_d^2 \geq \bar{\omega}_\infty \left( \sigma_\infty^2 - \bar{\sigma}_m \Omega_{1N} \Omega_{N1}^{-1} \bar{\sigma}_m \right)$$

This means that after using all other assets to approximate the compulsory risk and insure against it, the investor has not exceeded the desired level of risk. Notice that, it is not important if the risk induced by the compulsory holding is greater or smaller than the desired level. The relationship above establishes an upper bound for the compulsory holding; however, it is still possible that $\sigma_d^2 < \bar{\omega}_\infty \sigma_\infty^2$. In addition to this, the compulsory holding has to be more than the amount that would be allocated to asset $\infty$ by simple Lagrange maximisation. If this were not the case, the investor would not be facing any restrictions.

In the present case the investor will maximise the return using the rest of the assets; however, the proportion allocated between asset 1 to $N$ and asset $\infty$ will be given by the compulsory holding. Now the problem is how to allocate the rest of the risk among the first $N$ assets.

*Proposition 2.* Adding any amount of asset $\infty$ does not achieve optimal return maximisation.
Proof: see appendix 3A.

Proposition 3. Suppose an investor has to hold a compulsory amount of asset \( \infty \) larger than the maximised value according to the expected return and variance of all the assets. Assume further that the compulsory holding is smaller than the saturation level, this is:

\[
\sigma_A^2 \geq \bar{\omega}_m \left( \sigma_m^2 - \sigma_{m,1,N}^T \Omega_{1,N}^{-1} \sigma_{m,1,N} \right) \quad \text{and} \quad \bar{\omega}_m > \omega_m^*
\]

Then, the investor will allocate the rest of his wealth only in the assets \( I \) to \( N \) and according to the formula:

\[
\bar{\omega}_{1,N} = \sqrt{\frac{\sigma_A^2 - \bar{\omega}_m^2 \left( \sigma_m^2 - \sigma_{m,1,N}^T \Omega_{1,N}^{-1} \sigma_{m,1,N} \right)}{\sigma_m^2 - \bar{\omega}_m^2 \Omega_{1,N}^{-1} \sigma_{m,1,N}}} \Omega_{1,N}^{-1} \frac{\bar{\omega}_m^T \sigma_m}{\sigma_m^2 - \bar{\omega}_m^2 \Omega_{1,N}^{-1} \sigma_{m,1,N}}
\]

Proof: see appendix 3A.

iii. Compulsory holding is less than the optimal choice for the desired risk.

In this case the investor does not face any restriction and he maximises

\[
\text{Max } \bar{\omega}_A \bar{\sigma} \quad \text{subject to } \sigma_A^2 \geq \bar{\omega}_A \Sigma \bar{\omega}_A
\]

Proposition 4. The solution to the problem above is:

\[
\bar{\omega}_A = \frac{r_A \Omega^{-1} \bar{\sigma}}{\bar{\sigma}^T \Omega^{-1} \bar{\sigma}} \quad r_A^2 = \sigma_A^2 \bar{\sigma}^T \Omega^{-1} \bar{\sigma}
\]
**Proof:** see appendix 1.

**b.- Aggregation.**

Assume that the three types of investors coexist in the market, namely: the investor constrained to hold asset $\approx$ beyond his saturation level (we called this investor $A$), the investor constrained below his saturation level (investor $B$) and the unconstrained investor (investor $C$). The total market demand will be equal to the sum of the demands of each of these investors.

Without loss of generality we will assume that there is one representative agent of each type. In other words: the demand of agent type $C$ will represent the aggregate demand of all investors of this type. Each investor makes different choices regarding the total amount of risk he is willing to bear. However, as they all maximise expected return for the given level of risk, all of them choose portfolios that have an equal proportion of assets. The representative type $C$ agent will therefore hold a risk equal to the sum of all individual risks measured by the standard deviation. Furthermore, for type $A$ agents, the sum of their portfolios is equal to the portfolio held by a representative investor with a compulsory holding equal to the sum of all compulsory holdings of this type. Given the result in proposition 1 all type $A$ investors hedge the risk using equal proportions of assets $I$ to $N$. Finally and using the results in proposition 2 for type $B$ agents, the representative investor is one who holds the sum of all the type $B$'s compulsory holdings, $\bar{\omega}_{a,b}$, plus a portfolio of assets $I$ to $N$ equal
to:

\[
\sqrt{\frac{\sigma_B^2 - \bar{\omega}_{-A}^2 (\sigma_{-A}^2 - \sigma_{-A,1N}^2 \Omega_{-A,1N}^{-1} \sigma_{-A,1N})}{r_{1N}^T \Omega_{1N}^{-1} r_{1N}}} \Omega_{1N}^{-1} r_{1N}
\]

where \( \sigma_B^2 \) is such that:

\[
\sqrt{\sigma_B^2 - \sum_{k \in \text{TypeB}} \bar{\omega}_{A,B}^2 (\sigma_{A,B}^2 - \sigma_{A,B,1N}^2 \Omega_{A,B,1N}^{-1} \sigma_{A,B,1N})} = \sqrt{\sum_{k \in \text{TypeB}} \sigma_k^2 - \bar{\omega}_{-A}^2 (\sigma_{-A}^2 - \sigma_{-A,1N}^2 \Omega_{-A,1N}^{-1} \sigma_{-A,1N})}
\]

\[
\sigma_B^2 = \sum_{k \in \text{TypeB}} \bar{\omega}_{A,B}^2 (\sigma_{A,B}^2 - \sigma_{A,B,1N}^2 \Omega_{A,B,1N}^{-1} \sigma_{A,B,1N}) + \left( \sum_{k \in \text{TypeB}} \sigma_k^2 - \bar{\omega}_{-A}^2 (\sigma_{-A}^2 - \sigma_{-A,1N}^2 \Omega_{-A,1N}^{-1} \sigma_{-A,1N}) \right)^2
\]

All of these terms are well defined, as each type \( B \) investor does not hold asset \( \infty \) beyond his saturation level.

In a similar fashion as in the usual CAPM, if this market is in equilibrium demand and supply are equal. As a result of the trade, asset prices will adjust until these two quantities match each other.

**Proposition 5.** The total market return, \( r_M \), is equal to:

\[
r_M = r_{m,1N} I_m + \lambda r_{1N}^T \Omega_{1N}^{-1} r_{1N}
\]

where,

\[
I_m = \bar{\omega}_{-A} + \bar{\omega}_{-B} + \frac{r_{m,1N}}{RV_m} \sqrt{\frac{\sigma_C^2}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{m,1N}^2 / RV_m}}
\]

\[
\lambda = \sqrt{\frac{\sigma_B^2 - \bar{\omega}_{A,B}^2 RV_m}{r_{1N}^T \Omega_{1N}^{-1} r_{1N}}} + \sqrt{\frac{\sigma_C^2}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{m,1N}^2 / RV_m}}
\]

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\textbf{Proof:} see appendix 3A.

$L_\omega$ corresponds to the sum of investment in asset $\omega$ by the three types of investors. The role of $\lambda$ resembles the Lagrange multiplier's role in a problem where there is only unconstrained investors and therefore its name. These two variables play a role in the market variance and the covariance between the market and individual assets in a way that resembles the usual free market equilibrium.

\textbf{Proposition 6.} The total market variance, $\sigma^2_M$, is equal to:

\[
\sigma^2_M = RV_m((\bar{\omega}_{\omega, A} + \bar{\omega}_{\omega, B})^2 - \bar{\omega}_{\omega, B}^2) + \sigma^2_B + \sigma^2_C + 2 \sqrt{\frac{\sigma^2_C}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega, 1N}^2} / RV_m} (\bar{\omega}_{\omega, A} + \bar{\omega}_{\omega, B}) r_{\omega, 1N} + \sqrt{\frac{\sigma^2_B - \bar{\omega}_{\omega, B}^2 RV_m}{r_{1N}^T \Omega_{1N}^{-1} r_{1N}}} 2 r_{1N}^T \Omega_{1N}^{-1} r_{1N} + \frac{r^2_{\omega, 1N}}{RV_m})
\]

\textbf{Proof:} see appendix 3A.

Although the formula above can be derived directly from applying the three types of investment to the variance covariance matrix, a more convenient rearrangement of terms will allow us to relate total market risk to total market return:

\[
\sigma^2_M = RV_m I_m^2 + \frac{\sigma^2_C}{RV_m} \sqrt{\frac{\sigma^2_C}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega, 1N}^2} / RV_m} \lambda = \sigma^2_B + \sigma^2_C - RV_m \bar{\omega}_{\omega, B}^2 + \sqrt{\frac{\sigma^2_B - \bar{\omega}_{\omega, B}^2 RV_m}{r_{1N}^T \Omega_{1N}^{-1} r_{1N}}} (2 r_{1N}^T \Omega_{1N}^{-1} r_{1N} + \frac{r^2_{\omega, 1N}}{RV_m})
\]

Also,
\[(r_M - r_{w,1N}(\bar{\omega}_{w,A} + \bar{\omega}_{w,B}))\lambda = \sigma_B^2 + \sigma_C^2 - RV_{w} \bar{\omega}_{w,B} \]
\[+ \sqrt{\frac{\sigma_C^2}{\Omega^{-1}_{1N} r_{1N}} + \frac{\sigma_B^2 - \bar{\omega}_{w,B}^2 RV_{w}}{\Omega^{-1}_{1N} r_{1N}^2}} \left( \frac{2r_{1N}^2 \Omega^{-1}_{1N} r_{1N} + r_{w,1N}^2}{RV_{w}} \right) \]

Therefore,

\[(r_M - r_{w,1N} I_{w})\lambda = \sigma_M^2 - RV_{w} I_{w}^2 \]

**Proposition 7.** The covariance between one of the first \(N\) assets and the market portfolio, \(\sigma_{i,M}\), is equal to:

\[\sigma_{i,M} = r_i \lambda \]

**Proof:** see appendix 3A

**Proposition 8.** The return of one of the first \(N\) assets and the market return are related via:

\[r_i = \frac{\sigma_{i,M}}{\sigma_M^2 - RV_{w} I_{w}^2} (r_M - r_{w,1N} I_{w}) \]

**Proof:** By dividing the formula in proposition 7 by the relationship found before:

\[(r_M - r_{w,1N} I_{w})\lambda = \sigma_M^2 - RV_{w} I_{w}^2 \]

and eliminating \(\lambda\) we obtain the result.

The relationship in proposition 8 resembles the usual CAPM risk return trade-off. In particular, the expected return of any of the free traded assets is positively related to
its diversifiable risk, namely $\sigma_{\alpha,M}$. By being able to measure the right indexes: 

\[ r_M - r_{m,IN}I_\alpha \quad \text{and} \quad \sigma^2_M - RV_\alpha I^2_\alpha, \]

all the usual conclusions about CAPM can be obtained in this generalisation. Some additional properties are:

- The proportions of constrained investors (measured by their holdings of asset $\alpha$: $\bar{\omega}_{\alpha,\alpha}$ and $\bar{\omega}_{\alpha,\beta}$) are irrelevant as determinant of the risk-return relationship. Only the total investment in asset $\alpha$, $I_\alpha$, matters.

- The risk tolerance of each representative investor ($\sigma^2_\alpha$, $\sigma^2_\beta$, and $\sigma^2_\gamma$) or the distribution of them is irrelevant, as in the usual CAPM only the total market risk matters.

- The expected return and the variance of asset $\alpha$ do not play a direct role in the risk return relationship. Only measurements of the residual from regressing $r_\alpha$ on the returns of asset $I$ to $N$ are relevant, namely: $RV_\alpha$ and $r_{m,IN}$.

- From proposition 6 we can see that the terms $r_M - r_{m,IN}I_\alpha$ and $\sigma^2_M - RV_\alpha I^2_\alpha$ are always positive\(^{43}\).

- The effect on price of an increasing amount of compulsory holding will depend entirely on the realisation of $r_{m,IN}$. This can be positive or negative. However, the

\[ \lambda = \frac{\sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2}{\bar{\omega}_{\alpha,\beta}^2 + 2r_{IN}^T \Omega_{IN}^{-1} r_{IN}^T / RV_\alpha} \]

and

\[ \lambda = \frac{\sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2}{\bar{\omega}_{\alpha,\beta}^2 + 2r_{IN}^T \Omega_{IN}^{-1} r_{IN}^T / RV_\alpha} \]

and given that $\Omega_{IN}$ is positive definite and $\sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2$ is non-negative, they are both positive.

\(^{43}\) They are equal respectively to:

\[ \sigma^2_c^2 + \sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2 + 2r_{IN}^T \Omega_{IN}^{-1} r_{IN}^T / RV_\alpha \]

and

\[ \lambda = \frac{\sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2}{\bar{\omega}_{\alpha,\beta}^2 + 2r_{IN}^T \Omega_{IN}^{-1} r_{IN}^T / RV_\alpha} \]

and given that $\Omega_{IN}$ is positive definite and $\sigma^2_\alpha - RV_\alpha \bar{\omega}_{\alpha,\beta} \bar{\omega}_{\alpha,\beta}^2$ is non-negative, they are both positive.
total market return is positively affected by an increasing $I_\infty$ and this effect is larger than the one depending on $r_{\infty,1:N}$. This will become evident in the next section after using a feasible measurement of the market return.

- The measurement $\beta_i = \sigma_{i,M} / \sigma_M^2$ will overestimate the true market beta (defined as the market premium) when there is more mandatory investment in asset $\infty$ and when the other $N$ assets cannot effectively replicate the return of asset $\infty$ so the residual variance $RV_\infty$ is large.

- In a well developed market, where there is enough diversification with assets $1$ to $N$, market frictions produced by mandatory holdings will be mild or even completely ineffective as the two terms $RV_\infty$ and $r_{\infty,1:N}$ will go to zero.

- For too large amounts of mandatory holdings, even a well diversified market will be affected. If the asset subject to mandatory holdings cannot be well replicated by the free asset $1$ to $N$, the effect over the risk-return relationship will be important.

c.- Asset $\infty$ is not traded in the market.

A more realistic assumption in terms of the relevant market aggregation is that this market index will not include any information on asset $\infty$, its variance, its return or the amount of trade in it. A market index will include stock market trade in equity that is normally not subject to any frictions, like short-sales restriction or mandatory
holdings. Our asset corresponds to claims in assets not traded in the market\textsuperscript{14}. Indeed, non-measurable assets like contingent claims on human capital are left out from the market index. Therefore, in the same way as it happens with any feasible test of the CAPM, the relevant market return is not measurable. The market index will therefore consist of the observable assets; those freely traded in the stock market: asset 1 to \( N \).

This limitation of the market index will introduce problems when trying to estimate or test of the CAPM. Roll (1977) argues that since the market portfolio must include all claims in the economy, any test that uses an aggregation of only marketable assets has a low power. We identify below an explicit expression for the loss described by Roll\textsuperscript{45}.

Consider \( r_M \) and \( \sigma_M^2 \), the return and variance of a market consisting only of the investments in assets 1 to \( N \). We can define the covariance between this exclusive market aggregate and one individual asset, \( i \), as \( \sigma_{M,i} \).

**Proposition 9.** \( r_M \) and \( r_M^* \) are related via:

\[
    r_M = r_M^* + r_m I_m
\]

\textsuperscript{14} We identified a few exceptions to this in section 2 in this chapter. When restrictions are imposed on assets that are traded in the established market these are normally not permanent.

\textsuperscript{45} There is some evidence that this effect may not be, in practical terms, as important as Roll indicated. Stambaugh (1982) studied the properties of the CAPM using various proxies of the market portfolio. His identical inferences under different market proxies suggested that the loss in power is not too significant.
**Proof:** see appendix 3A.

**Proposition 10.** $\sigma_{M_\ast}^2$ and $\sigma_{M_\ast}^2$, are related via:

$$\sigma_{M_\ast}^2 = \sigma_M^2 - 2I_m\sigma_{M_\ast} + I_m\Omega_{\omega,N}^{-1}\sigma_{\omega,N} - I_m^2 R\sigma_{\omega,N}$$

and,

$$\sigma_M^2 = \sigma_{M_\ast}^2 + 2I_m\sigma_{M_\ast} + I_m\Omega_{\omega,N}^{-1}\sigma_{\omega,N} - I_m^2 R\sigma_{\omega,N}$$

Where we have called $\sigma_{M_\ast}$ and $\sigma_{M_\ast}$ each per unit correlation of asset $\omega$ with the respective market index $M$ and $M_\ast$.

**Proof:** see appendix 3A.

**Proposition 11.** $\sigma_{M,N}$ and $\sigma_{M_\ast,N}$ are related via:

$$\sigma_{M,N} = \sigma_{M_\ast,N} + \sigma_{\omega,N}$$

**Proof:** see appendix 3A.

**Proposition 12.** The risk return relationship is now:

$$r_i = \frac{\sigma_{i,M_\ast} + \sigma_{i,N}I_m}{\sigma_{M_\ast}^2 + 2I_m\sigma_{M_\ast} + I_m\Omega_{\omega,N}^{-1}\sigma_{\omega,N} - I_m^2 R\sigma_{\omega,N}}(r_{M_\ast} + \sigma_{\omega,N}^T\Omega_{\omega,N}^{-1}\sigma_{\omega,N} - I_m^2 R\sigma_{\omega,N})$$

**Proof:** see appendix 3A.
If we call \( \hat{\sigma}_i = \hat{\sigma}_{i,1/N}^T \Omega_{1/N}^{-1} \hat{\sigma}_{1/N} \) given that this is the best predictor of \( r_m \) and \( \beta_i \) the covariance-variance ratio above, we have that:

\[
    r_i = \beta_i (r_M + \hat{\sigma}_i I_{1/N})
\]

From this expression it is clearer what the effect of the disappearing market frictions like compulsory holdings or short sale restrictions will be, over the risk-return relationship. Assuming that the return of the mandatory holding is positive, positively correlated with the market and asset \( i \) and that \( \beta_i \) is not greatly affected by changes in \( I_{1/N} \), a lower amount of mandatory investment will decrease the return for an equal level of risk. Consequently, asset prices will rise.
Chapter 4

Empirical Analysis of the
Chilean Stock Exchange Data.

1.- Empirical tests of the CAPM

If we can measure return of an individual asset, $r_{it}$, the return of the market portfolio, $r_{Mt}$ or the covariance between the market and the returns of asset $i$, $\beta_i$, the CAPM theory leads to the following specification:

$$ r_{it} = \alpha_i + \beta_i r_{Mt} + \epsilon_{it} $$

The first empirical test of the CAPM developed by Black, Jensen and Scholes (1972) used the approach outlined below. Taking the equation above and, if we know the value $\beta_i$ for each asset, the regression of $r_i$, the expected return of asset $i$, on $\beta_i$ is:

$$ r_i = \gamma_1 + \gamma_2 \beta_i + \epsilon_i $$

Under the assumptions of the CAPM, this regression must result in $\gamma_1 = r_f$ and $\gamma_2 = r_m - r_f$. Black, Jensen and Scholes found that they could not reject the CAPM and they were able to explain almost all the variability of the cross-sectional data. The problem with their approach is that as long as variances and covariance of assets are
unknown, there are no data available to test directly the CAPM. To overcome this problem they estimated the variances, from the volatility of prices themselves, using the Centre for Research in Security Prices (CRSP) panel of cross-section and time-series data on asset returns. Their procedure induced the problem of error in variables, which is particularly important when the number of estimated variances is high. The problem is that given a large number of assets, and a median length for the time series, the degrees of freedom go down drastically, making the error in the variables still more important. Black et al. (1972) used beta-ranked portfolios, instead of individual securities, to avoid this problem. This technique involves grouping assets with equal beta value, requiring a much lower number of variances to be estimated. However, they induced problems of error term correlated with regressors.

The empirical test of the CAPM by Fama and MacBeth (1973) used three different periods of data to aggregate portfolios and to test the CAPM. They grouped portfolios by beta in one period, recalculated them in a different non-overlapping period and tested the equation on the remaining portion of the series. With this methodology, they reduced the problem of under-and over-estimating the betas. Nevertheless, they found in their estimations, a constant term, positive in average, that led them to reject the CAPM.

However, the test carried out by Black, Jensen and Scholes, and later by Fama and MacBeth, had the additional restriction that the market-portfolio return was a particular weighted index of asset prices. This means the rejecting a value of zero for \( \gamma_i \) or finding evidence that \( \gamma_i \neq r_m - r_f \), can either be evidence against the CAPM or it can be evidence against the proxy used for the market portfolio. Richard Roll
(1977) first identified this and it became one of the main criticisms about such empirical tests. Roll argued, in his 1977 article, that since the market portfolio must include other non-measurable assets (for example contingent claims on human capital) any test that uses an aggregation of marketable assets has a low power⁴⁶.

Taking into account the problem described above Gibbons, Ross and Shanken (1989) applied a different procedure to test the CAPM. What they really did was to test the ex-ante mean-variance of the CRSP equally weighted index. Using approximately the same grouping of assets as Black, Jensen and Scholes' study, they estimated β, and α in the equation and compared the estimated ˆα with zero. Under the assumption of ε_i being jointly normally distributed they used a multivariate ‘Hotelling’s $T^2$ test’ to test the joint hypothesis of $H_0$: $\alpha = 0$. In order to reduce the problem of error in variables they aggregated portfolios, applying the same procedure as Black, Jensen and Scholes. Since $\alpha$ was never significantly different from zero they could not reject the hypothesis of ex-ante mean-variance efficiency of CRSP. They argue, consequently, that if that index is taken as the true-market portfolio, then the Sharpe-Litner version of the CAPM cannot be rejected⁴⁷.

Evidence on CAPM rejection can been found in a more recent work by Fama and

⁴⁶ Subsequent work by Stambaugh (1982) showed that this effect may not be as important as Roll indicated.

⁴⁷ From the evidence by Fama (1976), the normality assumption is only a good working approximation to the true distribution. Therefore, it is not clear how reliable the results can be when the exact small sample F-distribution is adopted. If instead, an asymptotic test is used to prove the hypothesis they would reject H_0 more often. From Table I in Gibbons et al.'s work, it can be seen that using an asymptotic Wald test the p-values are higher and therefore one is inclined to reject more often zero as the value of alpha. De-Ramon (1996) studies the short sample and asymptotic properties of this tests.
French (1992). They detected that the strong relation between $\beta$ and the return for the pre-1969 period disappear for the more recent 1969 to 1990 period. They also found that such financial measures as leverage, size, book-to-market equity and earning-price ratio come to be more important in explaining the return. Using data from the CRSP they made an extensive study considering a number of different groupings by periods and portfolios. They found a strong association between return and the ratios book asset-to-book equity and book asset-to-market equity (Table II, pp. 436-437).

Fama and French (1992) describe as a contradiction the empirical evidence, they and other authors find, indicating that leverage is an important variable in the returns' regression. As they consistently argue, "It is plausible that leverage is associated with risk and expected return but in the Sharpe-Lintner-Black model this should be captured by market $\beta$." (p. 427)

The rest of the chapter is organised as follows. In Section 2 we study some of the statistical problems associated with the CAPM regressions and devise a convenient non-linear OLS estimator. In Section 3 we study the development of the Chilean stock market through the twentieth century and in particular the last 20 years. In addition, we perform some preliminary CAPM estimations for the average beta. In Section 4 we create a number of proxies for the market coverage and apply the non-linear OLS from section 2. Section 5 concludes and summarises the findings of the chapter.
2.- Methodology

a.- The panel of asset returns.

We concentrate in this section on some of the statistical problems associated with estimating and testing the parameters in the CAPM. One major assumption is that the parameters are constant over periods of time. Black, Jensen and Scholes (1972) assume constant parameters over sixty months. Fama and MacBeth (1973) use as much as eight years. In the present study we assume that parameters are constant over a period of 24 months. Recursive residual tests for parameter-constancy supports this assumption.

For a given cross section of assets:

\[ r_i = \gamma_1 + \gamma_2 \beta_i + \epsilon_i \]

Here \( r_i \) is the return on asset \( i \), \( \beta_i \) is its covariance with the marker return divided by the market variance and \( \gamma_1 \) and \( \gamma_2 \) are constants. When there are no frictions in the market, the CAPM theory implies specific hypotheses over the model above, these can be summarised as follows:

a) Linearity; \( E[r_i] = \gamma_1 + \gamma_2 \beta_i \)

b) The constant matches the risk free rate of interest; \( \gamma_1 = r_F \)

c) The risk premium is positive; \( \gamma_2 > 0 \)

d) Mean-variance efficiency; \( \gamma_2 = E[r_{M}] - r_F \)
However, testing these hypotheses is complicated by the fact that $\beta_i$ is not directly observable. To overcome this problem the tests can be conditioned on prior estimates of the equation parameters. For example, Fama and MacBeth (1973) condition their tests on estimates of $\beta_i$. Stambaugh (1982, 1983) conditions on a prior estimate of the market average return $\bar{r}_m$.

Here we closely follow Stambaugh (1982) richer approach to develop a framework for testing the model. Let $\bar{r}_i^T = (r_{i1}, r_{i2}, \ldots, r_{iT})$ be the vector of returns for an asset indexed by $i$, with $i=1,\ldots,k$. A panel of $k$ equations, one for each of the $k$ securities, can be written in a compact way:

$$\bar{r} = I \otimes \bar{\alpha} + I \otimes \bar{r}_m \bar{\beta} + \bar{\varepsilon}$$

Where $\bar{r}$ is the vector of size $kT$ resulting after stacking all $\bar{r}_i$, $\bar{\alpha}$ and $\bar{\beta}$ are size $k$ vectors, of all betas and alphas; $\bar{1}_T$ is a vector of ones of size $T$; $\bar{r}_m$ is the vector of $T$ market returns; $\bar{\varepsilon}$ is a vector of residuals for all assets and all time periods and $\otimes$ is the Kronecker product.

From the earlier cross section, describing the relationship between expected returns and betas, and from the hypothesis on linearity we can write $k$ relationships:

$$r_i = \gamma_1 + \gamma_2 \beta_i + \varepsilon_i \quad \text{or in expected value}$$

$$E[r_i] = \gamma_1 + \gamma_2 \beta_i$$

Matching this with the panel of returns above will impose $k-2$ restrictions over the vector of constants $\bar{\alpha}$.
\[ \alpha = \gamma_1 \bar{1}_k + (\gamma_2 - E[r_m]) \bar{\beta} \]

The \( k \) parameters in vector \( \alpha \) can be written in terms of the vector \( \bar{\beta} \) and \( \gamma_1 \) and \( \gamma_2 \).

Hypothesis b) and d) impose one additional restriction:
\[ \bar{\alpha} = \gamma_1 (\bar{1}_k - \bar{\beta}) \]

However, if the market aggregate \( r_M \) used in the test differs from the 'true' market, \( r_M^* \), index by a constant \( c \), hypothesis b) and d) will imply:
\[ E[r_M] + c - \gamma_2 = E[r_M^*] - \gamma_2 = \gamma_1 \]
\[ \bar{\alpha} = \gamma_1 \bar{1}_k + (c - \gamma_1) \bar{\beta} \]

We call \( \theta = (c - \gamma_1) \) and the condition above imposes \( k-2 \) restrictions on our system.

The number of parameters to estimate is \( k+2 \), \( k \) betas, \( \gamma_1 \) and \( \theta \). Therefore, our panel will be:
\[ \bar{r} = I \otimes \bar{1}_r (\gamma_1 \bar{1}_k + \theta \bar{\beta}) + I \otimes \bar{r}_M \bar{\beta} + \bar{\varepsilon} \]
\[ = \gamma_1 I \otimes \bar{1}_r \bar{1}_k + I \otimes (\bar{r}_M + \theta \bar{1}_r) \bar{\beta} + \bar{\varepsilon} \]
\[ = \gamma_1 \bar{1}_n + I \otimes (\bar{r}_M + \theta \bar{1}_r) \bar{\beta} + \bar{\varepsilon} \]

In summary, estimating the parameters with the mean-variance efficiency restriction involves dealing with a non-linear system. Stambaugh (1982) incorporates the additional assumption that assets are normally-distributed, and thus is the error term. He then estimates the model using the maximum-likelihood method, allowing for common shocks, i.e. correlated error term across the assets.

We develop in the next section the formulas for OLS estimation for this system.
Finding simple expressions for the non-linear OLS is possible because the right-hand side variable, the market return, is the same for all the panels. This estimator provides us with a way to deal with a more general error term. Given the special form of the CAPM equation, the normality assumption used in Stambaugh (1982, 1983) is an unnecessary and often unrealistic requirement for the data. Fama (1976) finds extensive evidence that the assets return distribution deviates from the normal.

b.- Non-linear SURE

For the sake of clarity we will temporarily replace $r_{jt}$, the return on asset $j$ at time $t$, with $y_{jt}$ and $r_{Mt}$, the market return, by $x$. From the analysis in the previous section, the model to estimate using data from $k$ securities and $T$ time periods is:

$$y_{jt} = \gamma_1 + \beta_j (x_t + \theta) + \varepsilon_{jt}$$

A distinctive property of this model is that the variation over time is due to the changes in the explanatory variable while the differences between securities is due to the changes in parameter $\beta_j$. Of course, the error term, $\varepsilon_{jt}$, influences both dimensions time and assets, and it is reasonable to believe that a shock at a given point in time will be common to all assets. Subsequently we will use the following notation: $y_\bullet$, is the average of $y$ across the panels, $y_\bullet t$ is the average of $y$ over time, $\beta_\bullet$ is the average of $\beta$ across the panels and $x_\bullet$ is the average of $x$ over time. Averaging the equation for the panel of assets, across the securities and over time, we obtain two alternative models:
When taking the average across the assets, all cross-equation correlation will disappear. Therefore, a natural estimate for the average $\beta_*$ parameter is the OLS estimate from the equation from this averaged model:

$$\hat{\beta}_* = \frac{\sum_i (y_{i*} - \bar{y}_*) (x_i - \bar{x}_*)}{\sum_i (x_i - \bar{x}_*)^2}$$

As long as the average error term, $\varepsilon_{*,i}$, is well behaved, the estimated average beta will have all the desirable properties, namely: unbiasedness, efficiency (for the average data) and consistency. The constant term, $a$, estimated from this equation will be related to parameters $\gamma_i$ and $\theta$ by the equation:

$$a = \gamma_i + \beta_i \theta$$

The following proposition identifies an exact formula for the non-linear model under study.

**Proposition 1.** Let the model:

$$y_{jt} = \gamma_i + \beta_j (x_i + \theta) + \varepsilon_{jt}$$

The OLS estimates of this model are:

a) $\hat{\beta}_j = \frac{\sum_i (x_i + \hat{\theta})(y_{jt} - \hat{\gamma}_i)}{\sum_i (x_i + \hat{\theta})^2}$

b) $\hat{\gamma}_i = y_{i*} - \hat{\beta}_i (x_i + \hat{\theta})$
c) \( \hat{\theta} = -x_{\cdot}^2 + \frac{V_x V_y - V_{xy} + \sqrt{(V_x V_y - V_{xy})^2 - 4V_x V^2_y}}{2V_{xy}} \)

Where:

\( V_x = (x^2)_{\cdot} - x^2_{\cdot} \)

\((x^2)_{\cdot}\) is the average over \( t \) of the square of all \( x_t \).

\( V_y = \sum_j (y_{j\cdot} - y_{\cdot \cdot})^2 \)

\( V_{xy} = \sum_j ((xy)_{j\cdot} - x_{\cdot \cdot} y_{\cdot \cdot})^2 - (\sum_j (xy)_{j\cdot} - x_{\cdot \cdot} y_{\cdot \cdot})^2 / T \)

\((xy)_{j\cdot}\) is the average over \( t \) of all \( x_t \) times \( y_{jt} \).

\( V_{yxy} = \sum_j (y_{j\cdot} - y_{\cdot \cdot})(xy)_{j\cdot} - x_{\cdot \cdot} y_{\cdot \cdot} \)

**Proof:** see Appendix 4A.

From the demonstration (in Appendix 4A) it becomes clear that this estimator will be consistent and asymptotically normal. In general, if the disturbances are normally distributed, the parameter estimates via non-linear OLS are efficient as they are in the case of linear OLS. In our case, for any given values of \( \theta \) and \( \beta_j \) the pseudo-regressors are:

\( x_{0,\cdot} = 1 \)

\( x_{i,\cdot} = x_i + \theta \), and

\( x_{3,\cdot} = \beta_j \)

The asymptotic properties of the estimator are those for when \( T \) goes to \( \infty \), when the
size of the cross-section of assets is fixed. We are interested here in the properties of
the estimator of a CAPM for a given set of assets in a market (not even necessarily all
the trade assets) when the number of observations of each asset (the time series) is
large. Therefore, as $\theta$ and $\beta_j$ are constants (the true parameter values) by just
imposing conditions on $x_i$ as if it were for linear OLS, we have the necessary
conditions for the non-linear system. The following proposition states conditions for
the consistency of the method.

**Proposition 2.** Under the conditions:

a) $p \lim \sum_{t=1}^{T} x_i^T x_j / T = Q$  
   a positive-definite matrix
b) $p \lim \sum_{t=1}^{T} x_i^T \varepsilon_j / T = 0$  
   $\forall j = 1, ..., k$

c) $\sum_{t=1}^{T} x_i^T \varepsilon_j / \sqrt{T} \rightarrow N(0, \sigma^2 Q)$  
   $\forall j = 1, ..., k$

The non-linear OLS estimates are consistent and asymptotically normal\(^49\).

**Proof:** This is direct by applying the conditions to the pseudo-regressors.

The next step, from this, is to apply SURE to the panel equation. For an estimate of
the variance-covariance matrix, we use the estimated matrix from the non-linear
method outlined above. Under the conditions of proposition 2, the weighting of the
objective function for non-linear GLS minimisation by different constant weights will

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\(^{48}\) For a general treatment see Green, 2000 p. 419.

\(^{49}\) We believe these conditions to be reasonable for our vector of variables $x_i$; among these one of the
variables is the market return. Under the alternative hypothesis over the market return, namely: the
market return is a trend following $t$, or the market return is $I(1)$ variable, the necessary condition is:
produce consistent estimates of the variance-covariance matrix. We can therefore use an identity matrix as weight, which is equivalent to the non-linear OLS method, and obtain a consistent estimate for the variance-covariance matrix. The second step, minimising the non-linear objective function with weights from the estimated variance-covariance matrix, will produce fully efficient and consistent estimates.

\textit{c.- Simulation results.}

We present here simulation results for the estimated parameters by non-linear SURE from a system like the one outlined in sections \textit{b.-} and \textit{c.-}. We generate random returns for three variables $y$ (three securities), using parameter values: $\gamma_1=1$, $\theta=-1$, $\beta_1=2$, $\beta_2=0.5$ and $\beta_3=1$. The average beta is approximately 1.1667 and corresponds to the simple average between $\beta_1$, $\beta_2$ and $\beta_3$. The error term is correlated between equations, according to a variance-covariance matrix, that leads to a structure between equations:

$$
\Sigma^{1/2} = \begin{bmatrix} 0.1 & 0.05 & 0.025 \\ 0.05 & 0.1 & 0.1 \\ 0.025 & 0.1 & 0.02 \end{bmatrix}
$$

We generate 10 observations for a common variable $x_t$ (the market return) uniformly distributed between zero and one, and 30 observations of a random variable $u_t$ distributed as a normal with zero mean and variance, taking one of three values 0.1, 0.15, 0.2. In addition, we have the following condition:

$$
p \lim (x'x/T)^{-1}(x'e_j) = 0 \quad \forall \ j = 1,...,k
$$
0.01 and 0.001. The actual error term entering the equations is \( e_\mu \) and it will be the transformed random term \( u_\mu \) though the matrix \( \Sigma^{1/2} \). \( y_\mu \) is generated as:

\[
y_\mu = 1 + \beta_j (x_j - 1) + e_\mu
\]

\[
\varepsilon = (\Sigma^{1/2} \otimes I_j) \tilde{u}
\]

Table 4.1 below presents a summary of the results for three different values of the variance of \( u_\mu \): \( \sigma^2 = \{0.001, 0.01, 0.1\} \). We replicate the model 5000 times for each value of \( \sigma^2 \). The estimation method leads to precise estimation of the parameter, on average, over a long series of experiments. For example, the mean of the 5000 estimates and the variance of the mean for \( \sigma^2 = 0.001 \) are accurate to the third decimal place. Simultaneously, the asymmetry of the distribution of \( \gamma_i \) and \( \theta \) emerges when the system is affected by larger variances. For example, compare the mean of the 5000 estimates with the variance and the maximum and minimum estimate; the mean or \( \theta \) is below the median and closer to the maximum. The mean of the estimated values of \( \gamma_i \) is closer to the left-hand side of the distribution, showing an asymmetry of this estimate too.

While the distribution of \( \bar{\beta} \) remains fairly normal, even for larger variances, the maximum estimate for \( \theta \) is seven standard deviations away from the mean while the minimum is only four standard deviations. This asymmetry is in turn transferred to the individual betas.
Table 4.1. Simulation results for non-linear SURE\textsuperscript{50}.

<table>
<thead>
<tr>
<th></th>
<th>(\bar{\beta})</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(\beta_3)</th>
<th>(\theta)</th>
<th>(\gamma_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_0^2=0.001)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>1.1667</td>
<td>2.0000</td>
<td>0.5000</td>
<td>1.0000</td>
<td>-1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Median</td>
<td>1.1667</td>
<td>2.0000</td>
<td>0.5000</td>
<td>1.0000</td>
<td>-1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.00138</td>
<td>0.00132</td>
<td>0.00170</td>
<td>0.00140</td>
<td>0.00048</td>
<td>0.00112</td>
</tr>
<tr>
<td>Minimum</td>
<td>1.1599</td>
<td>1.9932</td>
<td>0.4925</td>
<td>0.9939</td>
<td>-1.0031</td>
<td>0.9950</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.1728</td>
<td>2.0062</td>
<td>0.5073</td>
<td>1.0063</td>
<td>-0.9981</td>
<td>1.0061</td>
</tr>
<tr>
<td>(\sigma_0^2=0.01)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>1.1665</td>
<td>1.9999</td>
<td>0.4999</td>
<td>0.9999</td>
<td>-1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Median</td>
<td>1.1668</td>
<td>1.9997</td>
<td>0.5001</td>
<td>1.0002</td>
<td>-1.0000</td>
<td>1.0001</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0137</td>
<td>0.0134</td>
<td>0.0167</td>
<td>0.0138</td>
<td>0.0048</td>
<td>0.0110</td>
</tr>
<tr>
<td>Minimum</td>
<td>1.1013</td>
<td>1.9412</td>
<td>0.4053</td>
<td>0.9358</td>
<td>-1.0349</td>
<td>0.9419</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.2130</td>
<td>2.0587</td>
<td>0.5601</td>
<td>1.0456</td>
<td>-0.9661</td>
<td>1.0504</td>
</tr>
<tr>
<td>(\sigma_0^2=0.1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>1.1625</td>
<td>1.9956</td>
<td>0.4954</td>
<td>0.9964</td>
<td>-1.0035</td>
<td>1.0044</td>
</tr>
<tr>
<td>Median</td>
<td>1.1597</td>
<td>1.9948</td>
<td>0.4940</td>
<td>0.9950</td>
<td>-0.9996</td>
<td>0.9982</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.1397</td>
<td>0.1343</td>
<td>0.1717</td>
<td>0.1423</td>
<td>0.0490</td>
<td>0.1131</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.6241</td>
<td>1.4665</td>
<td>-0.3768</td>
<td>0.3749</td>
<td>-1.4535</td>
<td>0.5475</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.7233</td>
<td>2.5426</td>
<td>1.1917</td>
<td>1.5783</td>
<td>-0.8051</td>
<td>1.8264</td>
</tr>
</tbody>
</table>

Another feature of the simulations in table 4.1 above is that the estimates are apparently unbiased for small values of the variance. For example, the mean of the 5000 estimates for \(\gamma_1\), \(\theta\), \(\beta_1\), \(\beta_2\), \(\beta_3\) or the average beta; all these values are well within two standard deviations of the mean from the true parameter value for \(\sigma_0^2=0.001\) and \(\sigma_0^2=0.01\). This is also true for the median. However, as it would be expected, for \(\sigma_0^2=0.1\) the average estimates from 5000 experiments are not so close to their true expected value and this is due to use of an the estimated variance covariance matrix.

In theory, the estimates will only converge to their mean for a larger sample. In the present case the simulation generates a very small sample of only 10 observations.

To look at the distribution of the estimates from another point of view, we present

\textsuperscript{50} Simulations implemented using Excel 97 and Visual Basic 97. Results are averages over 5000 simulation results for each variance level.
probability plots of these in Figures 4.1, 4.2 and 4.3. These plots show the distributions of the simulated values of $\bar{\beta}$, $\gamma_i$ and $\theta$ respectively. From these figures we notice that independent of the variance size, normality is broadly true for the average beta$^{51}$. As expected, the distribution of $\theta$ is asymmetric to the right while the distribution of $\gamma_i$ is asymmetric to the left$^{52}$.

$^{51}$ Simulation results indicate that this property on the average beta is maintained with even larger variances.

$^{52}$ The horizontal axis plots the probability, under normal assumption, of the estimated parameters, while the vertical plots the actual distribution of the points.
Figure 4.1.

Probability Plot for average $\beta$. 
Figure 4.2

Probability Plot for $\theta$. 

![Probability Plot](image-url)
Figure 4.3

Probability Plot for $\gamma_t$. 

\begin{center}
\includegraphics[width=\textwidth]{figure4.3.png}
\end{center}
2.- Data Considerations

Here we study the characteristics of the data and the Chilean stock market in an attempt to identify how the development of the Chilean exchanges influences the risk-return relationship we studied in Chapter 3. In general, at the time of the initial emergence of stock exchanges only a few companies participate and issue equity. As the stock market develops more companies will be allowed to participate. According to the theoretical relationship explored in Chapter 3, equilibrium asset prices are positively affected by increasing amounts of participation in a free stock market and by decreasing amount of compulsory holdings. As argued by Gallego and Loayza (1999), stock markets were established in Chile in 1893 but they only made any significant progress during the last twenty years.

Government legislation to deal with agency problems or facilitating the allocation of financial resources to the market is likely to promote the growth of the financial markets. In the case of the Chilean stock market, major policy changes have been experienced in the last 15 years. Chilean financial markets were also affected by extreme liberalisation experiments during the late 1970s and the early 1980s. The results of these experiences were generally damaging for investment and forced subsequent governments to tackle existing problems with financial legislation.

In section a.- we put the markets for securities in Chile in their historical context. Section b.- studies these markets in more detail the period covered by the data, the last fifteen years.

The Chilean 'Bolsa de Santiago' (BS) was founded on 27\textsuperscript{th} November 1893. Chilean stock markets were very active during this early period. For example, by the end of the 19\textsuperscript{th} century there were 329 publicly-listed companies, mainly in mining sector activities. Until 1929, the stock market lived through a period of prosperity, as did its counterparts in New York and Europe. September 1929 witnessed a general fall in mining equity prices, and in particular nitrate mining which was the main economic activity in Chile at the time.

Couyoumdjian, Millar and Tocornal (1992) explain that the following 30-year period, 1930 to 1960 was characterised by adverse conditions for stock-market operations. The Great Depression affected the Chilean economy until 1932 and it was followed by a subsequent rise in economic activity. However, from 1938 onwards, another period of deterioration was experienced as a consequence of the Second World War. This period was also a time of increasing government intervention and nationalisation. For these reasons stock-market activities, and in general privately-owned business operations, were greatly jeopardised. There were persistent asset price loses until 1973. Inflation and hyperinflation (up to 400 per cent) between 1960 and 1978 increased these problems; see table 4.2 below.
From 1974 onwards, the Chilean economy experienced a 25-year period of economic reform. For example, in 1976, a stock register was created, and the public disclosure of financial information was made mandatory. In 1980 a fully-funded pension system began to operate, and private institutions started to manage the pension funds by investing them in various financial instruments. The old pay-as-you-go system was replaced by an individual capitalisation system, administered by private companies called 'Asociación de Fondos de Pensiones' (AFP, or pension fund administrators). In 1981, a series of laws destined to protect minority shareholders and prevent the misuse of privileged information were enacted. In addition, other institutional investors started participating in the market: insurance companies, mutual funds and investment funds were also allowed to become members of BS. In 1980, insurance market rates were liberalised, while prudential regulations on insurance companies' portfolios were implemented. From 1983 onwards and after economic reform, the

---

Table 4.2 Chilean stock markets development 1930-1980

<table>
<thead>
<tr>
<th>Year</th>
<th>Market Capitalisation (millions of 1975 pesos)</th>
<th>Value Traded</th>
</tr>
</thead>
<tbody>
<tr>
<td>1930</td>
<td>3555.1</td>
<td>479</td>
</tr>
<tr>
<td>1935</td>
<td>4716.5</td>
<td>1094</td>
</tr>
<tr>
<td>1940</td>
<td>3562.4</td>
<td>668</td>
</tr>
<tr>
<td>1945</td>
<td>2786.3</td>
<td>385</td>
</tr>
<tr>
<td>1950</td>
<td>1287.1</td>
<td>323</td>
</tr>
<tr>
<td>1955</td>
<td>1829.4</td>
<td>1162</td>
</tr>
<tr>
<td>1960</td>
<td>1128.2</td>
<td>179</td>
</tr>
<tr>
<td>1965</td>
<td>1082.1</td>
<td>208</td>
</tr>
<tr>
<td>1970</td>
<td>561.6</td>
<td>77</td>
</tr>
<tr>
<td>1975</td>
<td>735.7</td>
<td>99</td>
</tr>
<tr>
<td>1978</td>
<td>4091.4</td>
<td>317</td>
</tr>
</tbody>
</table>

---


54 In 1981 as well, a new law came into place to facilitate long-term bonds issuance by companies. See
Chilean stock market witnessed major expansion, characterised by a substantial increase of the market operations, new equity issues, diversification of available instruments and development of new markets.

However, until the late 1980s there were important adjustments in Chilean stock markets, with new institutions emerging and old ones disappearing. For example, in 1983 'Bolsa de Valores de Valparaíso' (BVV) was unable to adapt their operation to these new markets and equity laws and was dissolved by the government regulatory body. Valparaíso is the second city in Chile and the activities of the old BVV had been of historical importance. For example, as identified in Couyoumdjian et al. (1992), during the early 20th century BVV had a matching volume of transactions with Santiago's BS. However, between the 1930s and the 1960s Santiago gradually won four fifths of the market. After the closure of BVV, Valparaíso resumed activities some years later under a new name and revised operations. In 1987 the new 'Bolsa de Valores de Corredores - Bolsa de Valores' (BV) came to life, inheriting the activities of 'Bolsa de Valparaíso'. It began its activities, mainly in brokerage, in 1988.

Brokerage firms in Chile are required to own a seat on a securities exchange. Although BS handled most of the country's stock-market transactions, it was difficult for new market makers to join, as BS chose to maintain itself as a small-sized company. In 1989 a new competitor entered the market: 'Bolsa Electrónica de Chile' (or ESEC, Electronic Stock Exchange of Chile), created by a group of securities dealers including bankers interested in expanding their brokerage and in stock-market

activities. ESEC became the first electronic trading system in Latin America and it opened its doors to traders on November 1989. Glen (1994) examined, in detail, the ESEC trading system and documents the rapid growth in market share that it has captured from its competitor BS.

Another important development in the 1980s was the opening of Chilean companies to international markets by the issue of ADR (American Depository Receipts) held by banks in the USA. In addition, many Chilean companies started to participate in the ownership of foreign, mainly Latin American, businesses.


It is claimed in Gallego and Loayza (1999) that Chile has experienced a remarkable development in its financial system in the last 15 years. In their view, this started with market-oriented policies, in the mid-1970s, and the implementation of a regulatory framework in the 1980s. According to their analysis, the Chilean stock markets experienced moderate improvement in the 1980s in terms of size, activity, and efficiency and remarkable growth in the 1990s. However, according to their assessment and drawing comparison with the world average capitalisation, and particularly traded value, the Chilean stock market is still underdeveloped.

Market capitalisation in Chile, experienced fast growth since the mid-1980s. As shown in Table 4.3 below, capitalisation has grown an average of 33 per cent per
annum since 1985\textsuperscript{55}. Capitalisation/GDP ratio has grown at a 12 per cent per annum reaching over a 100 per cent in 1993.

Table 4.3 Market Capitalisation in Chilean Stock Markets 1985 - 2000\textsuperscript{56}

<table>
<thead>
<tr>
<th>Year</th>
<th>Market Capitalisation (Millions of pesos)</th>
<th>Capitalisation /GNP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>370,567</td>
<td>13.97</td>
</tr>
<tr>
<td>1986</td>
<td>831,552</td>
<td>24.32</td>
</tr>
<tr>
<td>1987</td>
<td>1,241,498</td>
<td>27.34</td>
</tr>
<tr>
<td>1988</td>
<td>1,740,762</td>
<td>29.42</td>
</tr>
<tr>
<td>1989</td>
<td>2,819,642</td>
<td>38.34</td>
</tr>
<tr>
<td>1990</td>
<td>4,596,608</td>
<td>49.72</td>
</tr>
<tr>
<td>1991</td>
<td>10,555,566</td>
<td>87.23</td>
</tr>
<tr>
<td>1992</td>
<td>11,333,749</td>
<td>74.64</td>
</tr>
<tr>
<td>1993</td>
<td>19,234,004</td>
<td>107.00</td>
</tr>
<tr>
<td>1994</td>
<td>27,348,505</td>
<td>127.83</td>
</tr>
<tr>
<td>1995</td>
<td>29,739,998</td>
<td>114.93</td>
</tr>
<tr>
<td>1996</td>
<td>27,981,726</td>
<td>98.99</td>
</tr>
<tr>
<td>1997</td>
<td>31,592,248</td>
<td>100.08</td>
</tr>
<tr>
<td>1998</td>
<td>24,545,666</td>
<td>72.99</td>
</tr>
<tr>
<td>1999\textsuperscript{(a)}</td>
<td>36,146,943</td>
<td>105.01</td>
</tr>
<tr>
<td>2000\textsuperscript{(a)}</td>
<td>35,934,650</td>
<td>96.43</td>
</tr>
</tbody>
</table>

Average annual growth

|                | 33.10% | 12.83% |

Additionally, as shown in Table 4.4 below, value traded has grown at a faster rate of around 40 per cent per year\textsuperscript{57}. However, according to Gallego and Loayza (1999), the ratio value traded to GDP remains low, compared with the world average of 17 per cent. This ratio has grown at a fast pace of 0.33 per cent per annum.

\textsuperscript{55} In US dollar terms the growth is only around 19 per cent.

\textsuperscript{56} Source: Compiled from Bolsa de Santiago Reseña Annual 1985 to 2000 issues. Values in millions of current national currency. (a) Provisional data. Annual growth for capitalisation calculated as the geometric average over the period. Annual growth for capitalisation over GNP-ratio is the simple arithmetic average over the period.

\textsuperscript{57} Only 18\% in dollar terms.
Table 4.4. Value Traded in Chilean Stock Markets 1985 - 2000.\textsuperscript{58}

<table>
<thead>
<tr>
<th>Year</th>
<th>Value Traded (Millions of pesos)</th>
<th>Value Traded /GNP %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>8,603</td>
<td>0.32</td>
</tr>
<tr>
<td>1986</td>
<td>57,235</td>
<td>1.67</td>
</tr>
<tr>
<td>1987</td>
<td>110,353</td>
<td>2.43</td>
</tr>
<tr>
<td>1988</td>
<td>150,200</td>
<td>2.54</td>
</tr>
<tr>
<td>1989</td>
<td>224,687</td>
<td>3.06</td>
</tr>
<tr>
<td>1990</td>
<td>234,362</td>
<td>2.53</td>
</tr>
<tr>
<td>1991</td>
<td>663,514</td>
<td>5.48</td>
</tr>
<tr>
<td>1992</td>
<td>735,798</td>
<td>4.85</td>
</tr>
<tr>
<td>1993</td>
<td>1,131,801</td>
<td>6.30</td>
</tr>
<tr>
<td>1994</td>
<td>2,204,108</td>
<td>10.30</td>
</tr>
<tr>
<td>1995</td>
<td>4,393,080</td>
<td>16.98</td>
</tr>
<tr>
<td>1996</td>
<td>3,487,296</td>
<td>12.34</td>
</tr>
<tr>
<td>1997</td>
<td>3,120,943</td>
<td>9.89</td>
</tr>
<tr>
<td>1998</td>
<td>2,034,540</td>
<td>6.05</td>
</tr>
<tr>
<td>1999\textsuperscript{(a)}</td>
<td>3,455,321</td>
<td>10.04</td>
</tr>
<tr>
<td>2000\textsuperscript{(a)}</td>
<td>2,093,830</td>
<td>5.62</td>
</tr>
</tbody>
</table>

Average annual growth: 40.97% 0.33%

Table 4.5 below shows the number of listed companies and the 'Indice General de Precios de Acciones' (IGPA) general price index between 1985 to 2000. Over the period the number of companies listing in the stock market has grown at a rate of 2.39 per cent per annum on average; this is over five new listing per annum. The general price index has grown at an average 40 per cent per annum.

\textsuperscript{58} See footnote 11 above.
Table 4.5. Listed companies and price index 1985 - 2000.59

<table>
<thead>
<tr>
<th>Year</th>
<th>Listed companies</th>
<th>IGPA index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>228</td>
<td>-</td>
</tr>
<tr>
<td>1986</td>
<td>216</td>
<td>-</td>
</tr>
<tr>
<td>1987</td>
<td>209</td>
<td>343.64</td>
</tr>
<tr>
<td>1988</td>
<td>203</td>
<td>459.81</td>
</tr>
<tr>
<td>1989</td>
<td>213</td>
<td>757.59</td>
</tr>
<tr>
<td>1990</td>
<td>215</td>
<td>1166.69</td>
</tr>
<tr>
<td>1991</td>
<td>223</td>
<td>2483.69</td>
</tr>
<tr>
<td>1992</td>
<td>244</td>
<td>2733.46</td>
</tr>
<tr>
<td>1993</td>
<td>263</td>
<td>3915.49</td>
</tr>
<tr>
<td>1994</td>
<td>279</td>
<td>5425.17</td>
</tr>
<tr>
<td>1995</td>
<td>282</td>
<td>5739.97</td>
</tr>
<tr>
<td>1996</td>
<td>290</td>
<td>4902.59</td>
</tr>
<tr>
<td>1997</td>
<td>294</td>
<td>4794.41</td>
</tr>
<tr>
<td>1998</td>
<td>287</td>
<td>3594.75</td>
</tr>
<tr>
<td>1999</td>
<td>282</td>
<td>5167.72</td>
</tr>
<tr>
<td>2000(0)</td>
<td>291</td>
<td>4752.81</td>
</tr>
</tbody>
</table>

Average annual growth 2.39% 20.64%

C.- The panel of assets prices.

We study a database from Datastream60 (supplied by Primark), including daily prices of securities traded in Chilean stock markets. We examine around 80 per cent of the shares traded in Chilean markets, leaving out those with a low level of trade, for example 'Comercio' the name of the share for BS, or assets for which too little data is available during the period considered. The database covers a period from July 1989 to November 2000 following daily prices for 233 securities. We also study data from eight aggregate market indexes and seven daily lending rates.

60 Datastream is financial online database run by Primark. They provide daily price quotations and analysis of securities from around the world.
IGPA is the general price index measuring the capitalisation of all quoted shares in BS; its base is 100 on December 1980. IPSA (Indice de Precios Selectivos de Acciones) considers the 40 most-traded assets in BS and these are updated quarterly to reflect the variation of the most active securities in the market. The market index International 10 (Intl10), is available from 1994 and reflects price movements on the 10 most-traded Chilean ADR quoted in American markets. The Intl10 index would only reflect the Chilean market-portfolio if markets were fully integrated. Finally, the JCLAC market index is maintained by the International Financial Corporation (IFC) and includes measures of domestic trade, ADR's and Chilean bank rates.

Tables 4.6 to 4.9 show a linear CAPM test performed over the averaged model. As explained in Section 1.c.- the average model provides unbiased, consistent estimates of the averaged beta. We present the estimates of this model over six two-year periods and using four different market indexes.

Table 4.6 shows the result using IGPA as the market index. In general, the residuals from the estimation are not normal except for the more recent data. Heteroscedasticity seems to be a problem only in the periods 1989-1991, 1991-1993 and 1997-1999. Serial correlation does not seem to be a major problem as we are looking at a critical value of around 1.69. The residuals from the period 1995 to 1997 seem to be the best-behaved ones. In this case we cannot reject linearity of the CAPM but we reject the constant being equal to the Chilean risk free 30-day-average-rate. The value of the constant indicates a risk free interest rate of -10.05%.
Over the whole time period, 1989 to 2000, both the average beta and the level of risk-free interest rate tend to decrease. This has been driven partly by decreasing inflation and Central Bank interest rates. Inflation has gone down from annual 21% in 1980 to 4.3% in 2000. This also indicates that for the first three time periods, where risk free is not rejected, the 30-day-average-rate might not be a good proxy of risk free rate as there is too much volatility from inflation.

Table 4.6 Average beta using IGPA as market index.  

<table>
<thead>
<tr>
<th>Period</th>
<th>7.79</th>
<th>2.791</th>
<th>5.793</th>
<th>7.795</th>
<th>4.797</th>
<th>2.799</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Average Beta</td>
<td>0.7227</td>
<td>0.6783</td>
<td>0.5481</td>
<td>0.4807</td>
<td>0.4460</td>
<td>0.3308</td>
</tr>
<tr>
<td>Average Alfa</td>
<td>(0.0381)</td>
<td>(0.0366)</td>
<td>(0.0261)</td>
<td>(0.0325)</td>
<td>(0.0194)</td>
<td>(0.0537)</td>
</tr>
<tr>
<td>Annual risk free rate(^{(a)}) (%)</td>
<td>42.31</td>
<td>32.04</td>
<td>18.16</td>
<td>-10.05</td>
<td>-17.29</td>
<td>-1.53</td>
</tr>
<tr>
<td>Risk free=ad30d(^{(b)}) F(1,102)</td>
<td>0.0491</td>
<td>0.3080</td>
<td>0.0688</td>
<td>21.6596</td>
<td>34.6164</td>
<td>1.6722</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>82.51</td>
<td>58.01</td>
<td>79.36</td>
<td>0.00</td>
<td>0.00</td>
<td>20.83</td>
</tr>
<tr>
<td>Number of assets</td>
<td>70</td>
<td>138</td>
<td>168</td>
<td>189</td>
<td>205</td>
<td>177</td>
</tr>
<tr>
<td>Number of observations</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>104</td>
<td>104</td>
<td>26</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.7787</td>
<td>0.7706</td>
<td>0.8111</td>
<td>0.6816</td>
<td>0.8388</td>
<td>0.6122</td>
</tr>
<tr>
<td>Durbin-Watson</td>
<td>2.1317</td>
<td>1.4168</td>
<td>1.7375</td>
<td>1.5098</td>
<td>1.5526</td>
<td>1.8504</td>
</tr>
<tr>
<td>RESET F(3,99)</td>
<td>0.7961</td>
<td>0.9191</td>
<td>0.3348</td>
<td>2.0423</td>
<td>0.7483</td>
<td>0.3496</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>49.89</td>
<td>43.46</td>
<td>80.02</td>
<td>11.29</td>
<td>52.58</td>
<td>78.99</td>
</tr>
<tr>
<td>Heteroscedasticity chi2 (1)</td>
<td>9.3312</td>
<td>7.7840</td>
<td>0.0614</td>
<td>2.7751</td>
<td>10.3432</td>
<td>0.3180</td>
</tr>
<tr>
<td>Probability &gt; chi2 (%)</td>
<td>0.23</td>
<td>0.53</td>
<td>80.43</td>
<td>9.57</td>
<td>0.13</td>
<td>57.28</td>
</tr>
<tr>
<td>Probability Skewness (%)</td>
<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
<td>4.68</td>
<td>17.37</td>
<td>0.69</td>
</tr>
<tr>
<td>Probability Kurtosis (%)</td>
<td>0.04</td>
<td>0.08</td>
<td>0.53</td>
<td>10.56</td>
<td>2.98</td>
<td>0.29</td>
</tr>
</tbody>
</table>

\(^{(a)}\) Annual risk-free rate is calculated by converting the weekly rate (the regression constant) to an annual basis.

\(^{(b)}\) ad30d is the Chilean 30-day-average-rate; the test is a Wald statistic on the difference between period average of ad30d and the estimated annual risk free rate.

ADR, American Deposit Receipt, these are Chilean issues held by banks in the USA.

The dependent variable is the average weekly logarithm of the return of all assets and the explanatory variable is the weekly logarithm return on IGPA. The parameters Average Alfa and Average Beta are calculated by ordinary least squares.
The results in table 4.6 do not reject linearity in any period; this is reflected in the low values of the RESET test.

Table 4.7 estimates the average beta and the constant using IPSA, the index of most actively traded assets. The results are similar to the ones using IGPA. As IPSA includes more volatile assets the average of securities has a smaller correlation with this market index. However, the downward tendency of the average beta over the period 1989 to 2000 is reflected by these estimates too.

<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>2.7.93</th>
<th>5.7.91</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IPSA</td>
<td>28.6.91</td>
<td>30.6.95</td>
<td>25.6.94</td>
<td>27.6.97</td>
<td>25.6.99</td>
<td>20.10.00</td>
</tr>
<tr>
<td>Average Beta</td>
<td>0.5292</td>
<td>0.5059</td>
<td>0.3893</td>
<td>0.3183</td>
<td>0.2582</td>
<td>0.2080</td>
</tr>
<tr>
<td></td>
<td>(0.0344)</td>
<td>(0.0398)</td>
<td>(0.0287)</td>
<td>(0.0244)</td>
<td>(0.0219)</td>
<td>(0.0246)</td>
</tr>
<tr>
<td>Average Alfa</td>
<td>0.0033</td>
<td>0.0014</td>
<td>0.0023</td>
<td>-0.0017</td>
<td>-0.0024</td>
<td>-0.0010</td>
</tr>
<tr>
<td></td>
<td>(0.0014)</td>
<td>(0.0013)</td>
<td>(0.0009)</td>
<td>(0.0005)</td>
<td>(0.0009)</td>
<td>(0.0007)</td>
</tr>
<tr>
<td>Annual risk free rate (%)</td>
<td>44.20</td>
<td>15.51</td>
<td>21.43</td>
<td>-12.15</td>
<td>-15.37</td>
<td>-6.39</td>
</tr>
<tr>
<td>Risk free = ad300^th F (1, 102)</td>
<td>0.1729</td>
<td>0.1388</td>
<td>0.3719</td>
<td>37.7490</td>
<td>20.3450</td>
<td>9.6135</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>67.84</td>
<td>71.02</td>
<td>54.33</td>
<td>0.00</td>
<td>0.00</td>
<td>0.28</td>
</tr>
<tr>
<td>Number of assets</td>
<td>70</td>
<td>138</td>
<td>168</td>
<td>189</td>
<td>205</td>
<td>177</td>
</tr>
<tr>
<td>Number of observations</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>104</td>
<td>104</td>
<td>69</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.6994</td>
<td>0.6126</td>
<td>0.6415</td>
<td>0.6245</td>
<td>0.5762</td>
<td>0.5171</td>
</tr>
<tr>
<td>Durbin-Watson</td>
<td>2.0024</td>
<td>1.3291</td>
<td>1.4572</td>
<td>1.2659</td>
<td>1.3303</td>
<td>1.6689</td>
</tr>
<tr>
<td>RESET F (3, 99)</td>
<td>0.7065</td>
<td>0.2164</td>
<td>0.1841</td>
<td>1.8913</td>
<td>0.1558</td>
<td>2.1220</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>55.04</td>
<td>88.48</td>
<td>90.70</td>
<td>13.60</td>
<td>92.57</td>
<td>10.61</td>
</tr>
<tr>
<td>Heteroscedasticity chi2(1)</td>
<td>4.4259</td>
<td>10.2800</td>
<td>0.0629</td>
<td>4.5563</td>
<td>11.4283</td>
<td>3.5492</td>
</tr>
<tr>
<td>Probability &gt; chi2 (%)</td>
<td>3.54</td>
<td>0.13</td>
<td>80.20</td>
<td>3.28</td>
<td>0.07</td>
<td>5.96</td>
</tr>
<tr>
<td>Probability Skewness (%)</td>
<td>0.09</td>
<td>0.03</td>
<td>0.01</td>
<td>38.79</td>
<td>0.05</td>
<td>76.48</td>
</tr>
<tr>
<td>Probability Kurtosis (%)</td>
<td>1.26</td>
<td>0.73</td>
<td>1.18</td>
<td>17.83</td>
<td>0.12</td>
<td>5.32</td>
</tr>
</tbody>
</table>

In the same way, as with IGPA, linearity of CAPM is generally accepted, in table 4.7, while the hypothesis of the constant being equal to the risk-free rate is not rejected for

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63 See footnote 61 above. The dependent variable is the average weekly logarithm of the return of all assets and the explanatory variable is the weekly logarithm return on IPSA.
the period between 1989 to 1994 and it is rejected afterwards. Serial correlation and non-normality of the residuals seems to be a more significant problem in this case.

Table 4.8 Average beta using Intl10 as market index.\(^{64}\)

<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>2.7.93</th>
<th>5.7.91</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>int10</td>
<td>int10</td>
<td>int10</td>
<td>int10</td>
<td>int10</td>
<td></td>
</tr>
<tr>
<td>Average Beta</td>
<td>0.2371</td>
<td>0.2315</td>
<td>0.1524</td>
<td>0.1689</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.1777)</td>
<td>(0.0249)</td>
<td>(0.0224)</td>
<td>(0.0240)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Alfa</td>
<td>0.0099</td>
<td>-0.0021</td>
<td>-0.0026</td>
<td>-0.0005</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.0045)</td>
<td>(0.0007)</td>
<td>(0.0012)</td>
<td>(0.0008)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annual risk free rate(^{64}) (%)</td>
<td>95.67</td>
<td>-13.35</td>
<td>-14.78</td>
<td>-3.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk free=ad30dF( 1 , 10 )</td>
<td>3.2183</td>
<td>36.0297</td>
<td>15.6081</td>
<td>5.2363</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>10.31</td>
<td>0.00</td>
<td>0.01</td>
<td>2.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of assets</td>
<td>168</td>
<td>189</td>
<td>205</td>
<td>177</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of observations</td>
<td>12</td>
<td>104</td>
<td>104</td>
<td>69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R-squared</td>
<td>0.1512</td>
<td>0.4592</td>
<td>0.3124</td>
<td>0.4250</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Durbin-Watson</td>
<td>0.7831</td>
<td>1.2086</td>
<td>1.3631</td>
<td>1.4820</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RESET F(3, 7)</td>
<td>0.6432</td>
<td>3.1666</td>
<td>5.1141</td>
<td>0.8346</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>61.13</td>
<td>2.78</td>
<td>0.25</td>
<td>47.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heteroscedasticity chi2(1)</td>
<td>0.5286</td>
<td>1.8935</td>
<td>0.4062</td>
<td>1.2944</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability &gt; chi2 (%)</td>
<td>46.72</td>
<td>16.88</td>
<td>52.39</td>
<td>25.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability Skewness (%)</td>
<td>8.93</td>
<td>52.48</td>
<td>0.00</td>
<td>11.61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability Kurtosis (%)</td>
<td>89.76</td>
<td>12.21</td>
<td>0.11</td>
<td>5.19</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8 above estimates the CAPM using the index of internationally traded assets Intl10; this is only available after 1993. The results are not very satisfactory in terms of the serial correlation that seems to be a problem in the four periods under study. However, homoscedasticity, normality and linearity are not rejected in most of the cases. In this case too, the estimates of the constant and the slope show a downward trend over the period. As the Intl10 index records the trade of Chilean securities in U.S. markets, the covariance of the average Chilean stock with this index is smaller than the one obtained when using IGPA or IPSA.

\(^{64}\) See footnote 61 above. The dependent variable is the average weekly logarithm of the return of all assets and the explanatory variable is the weekly logarithm return on Intl10.

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Table 4.9 uses the IFC index for Chilean trade JCLAC to test the CAPM. We obtain similar results from those using IGPA. JCLAC includes most Chilean trade, foreign trade and other market returns like bank interest rate. These characteristics are reflected on the smaller average beta, implying reduced correlation with the market index. However, the properties over time are similar to those from using other market indexes, as they are the rejection and non-rejection for linearity and risk free interest rate. The statistical properties of the residuals vary in each of the six periods considered but they are in general good.

<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>2.7.93</th>
<th>5.7.91</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>28.6.91</td>
<td>30.6.95</td>
<td>25.6.94</td>
<td>27.6.97</td>
<td>25.6.99</td>
<td>20.10.00</td>
</tr>
<tr>
<td>Average Beta</td>
<td>0.5277</td>
<td>0.5342</td>
<td>0.3611</td>
<td>0.2744</td>
<td>0.2499</td>
<td>0.1849</td>
</tr>
<tr>
<td>(0.0427)</td>
<td>(0.0484)</td>
<td>(0.0313)</td>
<td>(0.0258)</td>
<td>(0.0197)</td>
<td>(0.0486)</td>
<td></td>
</tr>
<tr>
<td>Average Alfa</td>
<td>0.0037</td>
<td>0.0021</td>
<td>0.0031</td>
<td>-0.0016</td>
<td>-0.0023</td>
<td>-0.0001</td>
</tr>
<tr>
<td>(0.0016)</td>
<td>(0.0014)</td>
<td>(0.0010)</td>
<td>(0.0006)</td>
<td>(0.0009)</td>
<td>(0.0015)</td>
<td></td>
</tr>
<tr>
<td>Annual risk free rate (%))</td>
<td>50.78</td>
<td>26.37</td>
<td>28.97</td>
<td>-10.98</td>
<td>-14.77</td>
<td>-0.41</td>
</tr>
<tr>
<td>Risk free=ad30d F(1,102)</td>
<td>0.4078</td>
<td>0.0782</td>
<td>1.7595</td>
<td>30.7448</td>
<td>21.6834</td>
<td>0.8202</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>52.45</td>
<td>78.04</td>
<td>18.76</td>
<td>0.00</td>
<td>0.00</td>
<td>37.94</td>
</tr>
<tr>
<td>Number of assets</td>
<td>70</td>
<td>138</td>
<td>168</td>
<td>189</td>
<td>205</td>
<td>177</td>
</tr>
<tr>
<td>Number of observations</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>104</td>
<td>104</td>
<td>17</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.5998</td>
<td>0.5348</td>
<td>0.5639</td>
<td>0.5266</td>
<td>0.6126</td>
<td>0.4911</td>
</tr>
<tr>
<td>Durbin-Watson</td>
<td>1.8221</td>
<td>1.2966</td>
<td>1.3291</td>
<td>1.1771</td>
<td>1.2530</td>
<td>1.6331</td>
</tr>
<tr>
<td>RESET F(3,99)</td>
<td>1.5917</td>
<td>1.5270</td>
<td>0.4731</td>
<td>2.0718</td>
<td>0.5974</td>
<td>0.1340</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>19.62</td>
<td>21.22</td>
<td>70.17</td>
<td>10.88</td>
<td>61.82</td>
<td>93.79</td>
</tr>
<tr>
<td>Heteroscedasticity chi2(1)</td>
<td>2.3037</td>
<td>10.1161</td>
<td>2.1013</td>
<td>1.0804</td>
<td>8.3965</td>
<td>1.4451</td>
</tr>
<tr>
<td>Probability &gt; chi2 (%)</td>
<td>12.91</td>
<td>0.15</td>
<td>14.72</td>
<td>29.86</td>
<td>0.38</td>
<td>22.93</td>
</tr>
<tr>
<td>Probability Skewness (%)</td>
<td>0.00</td>
<td>0.13</td>
<td>0.01</td>
<td>20.88</td>
<td>0.24</td>
<td>6.39</td>
</tr>
<tr>
<td>Probability Kurtosis (%)</td>
<td>0.13</td>
<td>8.23</td>
<td>0.17</td>
<td>56.27</td>
<td>0.55</td>
<td>2.66</td>
</tr>
</tbody>
</table>

See footnote 61 above. The dependent variable is the average weekly logarithm of the return of all assets and the explanatory variable is the weekly logarithm return on JCLAC.
3.- Non-linear CAPM tests.

In this section we use the non-linear SURE method outlined in section 2 to study the database of securities traded in Chilean stock markets from Datastream. With this method we are able to estimate parameters for individual asset or portfolios as well as the average beta, $\gamma$ and $\theta$ as defined in section 2.

a.- Portfolio choice.

The non-linear method we apply requires the estimation of the variance-covariance matrix in order to perform the second stage of the estimation. If we are to calculate these parameters for each of the assets in the sample the number of variances to estimate can become large. For example, for the period 1991 to 1993 we can use up to 138 assets (see table 4.6) which means calculating 9453 variances. The residuals available for this evaluation in that period are only two years of weekly data; i.e. 104 observations. For the period 1997 to 1999 with 205 assets this problem turns out to be much more prominent. Our attempts estimating parameters for all assets in the first and second period 1989 to 1991 and 1991 to 1993 confirmed this problem existed for these data. Although in those two periods the estimation was possible, the variance-covariance matrix was very badly conditioned for inversion or for applying the Cholesky decomposition. This, in turn, introduced excessive noise in the second stage estimates of $\beta$, $\gamma$ and $\theta$.

To avoid this problem with the degrees of freedom, we group the Chilean shares in a
number of portfolios. Empirical tests of the CAPM by means of portfolio returns have been used in the past by, for example: Gibbons, Ross and Shanken (1989) and Black, Jensen and Scholes (1972). We run auxiliary individual OLS regressions for each security against IGPA to measure the corresponding individual betas. We obtain estimates ranging between approximately -1.6 and 2.5. With these estimates we classified the assets in approximately 20 groups according to the values of beta. The clusters are formed so that the largest distance between the maximum and the minimum beta is smaller than a fixed value for all groups. After some experimentation we select 0.7 as this largest value so that we can ensure having approximately 20 clusters in each time period. Therefore, the number of groups varies over time: in the period 1989 to 1991 we form 19 clusters. In 1991 to 1993, 1993 to 1995 and 1997 to 1999 we can create 20 groups. In the period 1995 to 1997 we have 24 groups and finally in 1999 to 2000 we form 23 groups.

b.- Market coverage variable.

In this section we are interested on testing the model studied in chapter 3. In that model we identified deviations from the usual CAPM line due to compulsory holdings (or short sale restrictions). We determined that these differences are related to the factor \( \beta_i \hat{r}_m \), where \( \hat{r}_m \) corresponds to a proxy of total return on the assets that are subjected to these restrictions. Under the assumption that the excess return on the mandatory holding is positive and that is positively correlated with the market, a decrease in these market frictions will decrease the return for an equal level of risk and asset prices will rise. Therefore, an increasing degree of participation in the market will be expected to have a negative effect on the CAPM constant.
One problem with testing this effect is the difficulty to measure the return on asset $\phi$ and the amount of investment in this variable, $I_\omega$. Table 4.10 below summarises the proxies used for the market coverage variable. IGPA capitalisation is an index of total value of the stock traded, when divided by quarterly GDP is becomes a proxy of the fraction of return attributable to freely traded asset as compared to the whole economy return, this, in turn, will be negatively correlated with $r_\omega I_\omega$. Therefore, an increase on variable 'capgdp' will imply a decrease on the compulsory return, raising asset prices and decreasing returns. Similar effects should happen with the variable 'capima', where 'Indice Mensual de Actividad Económica' (IMACEC) plays the role of quarterly GDP. IMACEC is a monthly index measuring mainly Chilean industrial activity and sales. Both 'capgdp' and 'capima' will be a good proxy of the market coverage we are looking for as long as GDP and IMACEC are correlated with future returns in the overall economy. Furthermore, they will only be good proxies as long and the returns in the marketable assets are not correlated with the non-traded assets.

In addition, we use two smoothed versions of these variables: 'capgdps' and 'capimas'. In these we have smoothed the value of IGPA capitalisation over a month to avoid too much short-term price variability.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Code</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>capgdp</td>
<td>IGPA capitalisation divided by quarterly GDP</td>
</tr>
<tr>
<td></td>
<td>capima</td>
<td>IGPA capitalisation divided by monthly IMACEC</td>
</tr>
<tr>
<td></td>
<td>capgdps</td>
<td>IGPA capitalisation divided by quarterly GDP smooth</td>
</tr>
<tr>
<td></td>
<td>capimas</td>
<td>IGPA capitalisation divided by monthly IMACEC smooth</td>
</tr>
<tr>
<td></td>
<td>lnassets</td>
<td>log of number of assets traded</td>
</tr>
<tr>
<td></td>
<td>trend</td>
<td>log of a time trend</td>
</tr>
</tbody>
</table>
'lnassets' is simply the logarithm of the number of assets traded in each week. This will be a good proxy of $1 - I_m$ and under the assumption of constant returns on the compulsory holdings over time, it will be a good proxy for the market coverage. We identified in section 2 above an increasing amount of market participation through growing listing in Chilean markets during the period under consideration. This is evidenced by the increasing amount of total traded assets at a rate of 2.4% additional companies per annum. We measure this effect by simply counting the assets traded each week.

In the next section we estimate the model on approximately 20 portfolios in each two-year period using the non-linear SURE method outlined in section 1.c.- and study the effects of our six market coverage proxies over the data.

c.- Results.

Tables 4.11 to 4.16 below present the estimation results for the six different proxies of market coverage. Each table presents two sets of estimates: in the section above the first stage, non-linear OLS estimates and below the non-linear SURE stage. Except for the last time period, 1999 to 2000, there are 104 weeks (two years) of data in each of the estimations. The number of parameter to estimate per model are as many as portfolios (say 20 in average), the market coverage parameter 'Delta' and the non linear parameters defined in the previous section, Theta and Gamma; a rough average of 23 parameters for 20 portfolios and 104 data points.

In table 4.11 we present the first of the coverage variables 'capgdp'. Comparing this
table with the results with the simple OLS estimation in section 2.c.- (Table 4.6 for this model) we see that the tendency towards decreasing average correlation with the market index is not so remarkable in this estimation. Instead of this effect, the market coverage variable presents a negative sign increasingly significant (see Table 4.11, non-linear SURE results) over time. Again the period 1995 to 1997 seems to be the one with best properties, and in that the market coverage variable scores a parameter of -0.0267 with standard deviation 0.0107, implying a t-test of 2.5 well above the 1 per cent value for a normal distribution. This period corresponds to the last period, in recent Chilean stock-market history, of growing market participation in terms of number of assets traded, and is the period with highest growth in capitalisation and highest turnover; see Tables 4.3 to 4.5.

We detect now in all periods a disagreement between the implied risk free interest rate and the Chilean 30-day-average-rate, except perhaps for the period 1995 to 1997. We perform additionally the linear hypothesis that the sum of theta and gamma is equal to zero. This is generally not rejected, except in the period 1991 to 1993 and 1995 to 1997. This indicates there are no significant problems with the market index being used (IGPA) as a proxy of the true market portfolio.
Table 4.11 Average results using CAPGDP as market coverage.\(^6\)

<table>
<thead>
<tr>
<th>Period</th>
<th>Market index</th>
<th>Market Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>capgdp</td>
</tr>
<tr>
<td>7.7.89</td>
<td>2.7.91</td>
<td>capgdp</td>
</tr>
<tr>
<td>5.7.93</td>
<td>7.7.95</td>
<td>capgdp</td>
</tr>
<tr>
<td>2.7.97</td>
<td>4.7.97</td>
<td>capgdp</td>
</tr>
<tr>
<td>2.7.99</td>
<td></td>
<td>capgdp</td>
</tr>
<tr>
<td>28.6.91</td>
<td>30.6.93</td>
<td>capgdp</td>
</tr>
<tr>
<td>25.6.95</td>
<td>27.6.97</td>
<td>capgdp</td>
</tr>
<tr>
<td>25.6.99</td>
<td>20.10.00</td>
<td>capgdp</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step one: Non-Linear OLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of asset</td>
</tr>
<tr>
<td>Number of weeks</td>
</tr>
<tr>
<td>Number of observations</td>
</tr>
<tr>
<td>R-squared</td>
</tr>
<tr>
<td>Portfolio Beta</td>
</tr>
<tr>
<td>Delta</td>
</tr>
<tr>
<td>Gamma</td>
</tr>
<tr>
<td>Theta</td>
</tr>
<tr>
<td>Annual risk free rate(^a) (%)</td>
</tr>
<tr>
<td>Risk free=ad30d(^b) (%)</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
</tr>
<tr>
<td>Test Gamma+Theta=0(^c)</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
</tr>
</tbody>
</table>

One important caveat on how to compare the average beta results here with those on Table 4.6, is to realise that the average beta is not the taken over the same set of assets. In table 4.6 the average is calculated over all the assets in the sample while in table 4.11 this is taken only over twenty portfolios as defined in section 3.a.-. As these portfolios include only assets with similar beta values the average over the

\(^6\) The dependent variable is the portfolio weekly return in logarithm and the explanatory variables are the weekly return on IGPA in logarithm and CAPGDP. The parameters Delta and Beta correspond to the market coverage variable and the market return respectively. Asymptotic standard deviations are in brackets underneath the parameter estimates.

\(^a\) Annual risk-free rate is calculated by converting the weekly rate (equal to \(\gamma + \beta \theta\)) to an annual basis.

\(^b\) ad30d is the Chilean 30-day-average-rate; the test is a Wald statistic on the difference between period average of ad30d and the estimated annual risk free rate.

\(^c\) Corresponds to the \(R^2\) on the original dependent variable: the return on the logarithm of the weekly portfolio return.

\(^d\) This is a Wald statistic for \(\gamma\), equal to \(-\theta\); not rejection means there is no discrepancy between the
entire set of asset might be misrepresented in just 20 portfolios.

In the last two data periods, from 1997 to 2000, the estimates for the market coverage variable are non-significant. This is coupled with the fact that in this period, Chilean stock markets have shown a significant slowdown in participation. Growth in both, capitalisation and turnover has slowed significantly and there have been no increases or even decreases on the number of traded assets; see tables 4.3 to 4.5.

Table 4.11 (continued) Average results using CAPGDP as market coverage.

<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>27.91</th>
<th>5.7.93</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>28.6.91</td>
<td>30.6.93</td>
<td>25.6.95</td>
<td>27.6.97</td>
<td>25.6.99</td>
<td>20.10.00</td>
<td></td>
</tr>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>capgdp</td>
<td>capgdp</td>
<td>capgdp</td>
<td>capgdp</td>
<td>capgdp</td>
<td>capgdp</td>
</tr>
</tbody>
</table>

Step two: Non-Linear SURE
R-squared: 0.2929 0.1834 0.3542 0.0955 0.4786 0.1501
Corrected R-squared: -1.8547 -7.4090 -0.0716 -0.0184 0.0300 0.1078
Portfolio Beta: 0.6632 2.7638 0.4496 0.7616 0.4982 0.5871
Delta: -0.0045 -0.0326 0.0058 -0.0267 -0.0027 0.0250
Gamma: 0.0127 0.0070 0.0008 -0.0007 -0.0032 -0.0040
Theta: 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
Annual risk free rate: 93.01 43.98 3.99 -3.35 -15.56 -19.06
Risk free=ad30d*: 18.3892 6.5595 18.0782 3.9118 45.8493 31.3256
Probability > F (%): 36.13 0.72 34.55 1.21 44.95 31.18

Table 4.12 below uses 'capima' as a market coverage proxy and IGPA as the market portfolio proxy. The results are similar to those in Table 4.11 and we detect an even more significant negative effect performed by the market coverage variable.
<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>2.7.91</th>
<th>5.7.93</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>capima</td>
<td>capima</td>
<td>capima</td>
<td>capima</td>
<td>capima</td>
<td>capima</td>
</tr>
<tr>
<td>28.6.91</td>
<td>30.6.93</td>
<td>25.6.95</td>
<td>27.6.97</td>
<td>25.6.99</td>
<td>20.10.00</td>
<td></td>
</tr>
</tbody>
</table>

**Step one: Non-Linear OLS**

- Number of asset: 19
- Number of weeks: 104
- Number of observations: 1976
- R-squared: 0.2267
- Portfolio Beta: 0.7994
- Delta: -0.0021 (0.0062)
- Gamma1: 0.0147 (0.0020)
- Theta: -0.0141 (0.0122)
- Annual risk free rate (\%) 
  - Risk free=ad30d^* 
  - Probability > F (%): 0.00
- Test Gamma1+Theta=0^* 
  - Probability > F (%): 54.84

**Step two: Non-Linear SURE**

- R-squared: 0.2928
- Corrected R-squared (d): -1.9619
- Portfolio Beta: 0.6259
- Delta: -0.0038 (0.0055)
- Gamma1: 0.0127 (0.0012)
- Theta: -0.0207 (0.0099)
- Annual risk free rate (\%) 
  - Risk free=ad30d^* 
  - Probability > F (%): 0.00
- Test Gamma1+Theta=0^* 
  - Probability > F (%): 42.02

The results with smoothed versions of the market coverage variables, in Tables 4.13 and 4.14, display similar characteristics with the results in Tables 4.11 and 4.12. In

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67 See footnote 65 above. The explanatory variables are the weekly return on IGPA in logarithm and CAPIMA.
the case of 'capgdps' the negative effect seems to be stronger and more consistent over time.

Table 4.13 Average results using CAPGDPS as market coverage.

<table>
<thead>
<tr>
<th>Period</th>
<th>7.7.89</th>
<th>2.7.91</th>
<th>5.7.93</th>
<th>7.7.95</th>
<th>4.7.97</th>
<th>2.7.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>capgdps</td>
<td>capgdps</td>
<td>capgdps</td>
<td>capgdps</td>
<td>capgdps</td>
<td>capgdps</td>
</tr>
</tbody>
</table>

Step one: Non-Linear OLS

| Number of asset | 19 | 20 | 20 | 24 | 20 | 23 |
| Number of weeks | 104 | 104 | 105 | 104 | 104 | 69 |
| Number of observations | 1976 | 2080 | 2100 | 2496 | 2080 | 1587 |
| R-squared | 0.2267 | 0.1371 | 0.1484 | 0.0623 | 0.4177 | 0.1959 |
| Portfolio Beta | 0.7985 | 0.6744 | 0.9376 | 0.6906 | 0.8735 | 0.6498 |
| Delta | 0.0009 | -0.0266 | -0.0146 | -0.0010 | 0.0012 | -0.0335 |
| Gamma1 | 0.0148 | 0.0080 | 0.0005 | 0.0008 | -0.0046 | -0.0038 |
| Theta | -0.0106 | -0.1995 | -0.1041 | -0.0088 | 0.0119 | -0.2495 |
| Annual risk free rate (%) | 114.66 | 51.52 | 2.57 | 4.36 | -21.30 | -17.94 |
| Risk free=ad30d (%) | 18.6039 | 4.5291 | 0.7135 | 0.8622 | 25.0154 | 16.3876 |
| Probability > F (%) | 0.00 | 3.34 | 39.84 | 35.32 | 0.00 | 0.01 |
| Test Gamma1+Theta=0 | 0.0058 | 1.6812 | 1.7391 | 0.0071 | 0.0524 | 1.2745 |
| Probability > F (%) | 93.91 | 19.49 | 18.74 | 93.26 | 81.89 | 25.91 |

Step two: Non-Linear SURE

| R-squared | 0.2948 | 0.1973 | 0.3358 | 0.0928 | 0.4799 | 0.1474 |
| Corrected R-squared | -2.8467 | -0.1953 | -0.0200 | -0.0077 | 0.0348 | 0.1127 |
| Portfolio Beta | 0.3992 | 1.0441 | 0.4855 | 0.7102 | 0.5000 | 0.5962 |
| Delta | -0.0034 | -0.0207 | -0.0036 | -0.0241 | -0.0044 | -0.0399 |
| Gamma1 | 0.0127 | 0.0068 | 0.0007 | -0.0006 | -0.0032 | -0.0040 |
| Theta | -0.0409 | -0.1557 | -0.0265 | -0.1733 | -0.0308 | -0.2956 |
| Annual risk free rate (%) | 93.10 | 42.30 | 3.78 | -3.06 | -15.57 | -18.83 |
| Risk free=ad30d (%) | 17.9252 | 5.2374 | 18.7809 | 3.7184 | 45.7631 | 30.5261 |
| Probability > F (%) | 0.00 | 2.22 | 0.00 | 5.39 | 0.00 | 0.00 |
| Test Gamma1+Theta=0 | 0.3320 | 1.5350 | 0.3131 | 4.5453 | 1.3812 | 1.1841 |
| Probability > F (%) | 56.46 | 21.55 | 57.58 | 3.31 | 24.00 | 27.67 |

68 See footnote 65 above. The explanatory variables are the weekly return on IGPA in logarithm and CAPGDPS.
### Table 4.14 Average results using CAPIMAS as market coverage.\(^6^9\)

<table>
<thead>
<tr>
<th>Period</th>
<th>7.789</th>
<th>2.791</th>
<th>5.793</th>
<th>7.795</th>
<th>4.797</th>
<th>2.799</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>capimas</td>
<td>capimas</td>
<td>capimas</td>
<td>capimas</td>
<td>capimas</td>
<td>capimas</td>
</tr>
</tbody>
</table>

**Step one: Non-Linear OLS**

<table>
<thead>
<tr>
<th>Number of asset</th>
<th>19</th>
<th>20</th>
<th>20</th>
<th>24</th>
<th>20</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of weeks</td>
<td>1976</td>
<td>2080</td>
<td>2100</td>
<td>2496</td>
<td>2080</td>
<td>1587</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.2267</td>
<td>0.1370</td>
<td>0.1478</td>
<td>0.0625</td>
<td>0.4178</td>
<td>0.1953</td>
</tr>
<tr>
<td>Portfolio Beta</td>
<td>0.7990</td>
<td>0.6781</td>
<td>0.9467</td>
<td>0.6739</td>
<td>0.8750</td>
<td>0.6578</td>
</tr>
<tr>
<td>Delta</td>
<td>0.0019</td>
<td>-0.0218</td>
<td>-0.0051</td>
<td>-0.0060</td>
<td>0.0024</td>
<td>-0.0032</td>
</tr>
<tr>
<td>Gamma1</td>
<td>0.0148</td>
<td>0.0080</td>
<td>0.0004</td>
<td>0.0008</td>
<td>-0.0046</td>
<td>-0.0038</td>
</tr>
<tr>
<td>Theta</td>
<td>-0.0141</td>
<td>-0.0340</td>
<td>-0.0055</td>
<td>-0.0090</td>
<td>0.0069</td>
<td>-0.0024</td>
</tr>
<tr>
<td>Annual risk free rate(^{(a)}) (%)</td>
<td>114.72</td>
<td>51.69</td>
<td>2.35</td>
<td>4.13</td>
<td>-21.30</td>
<td>-17.96</td>
</tr>
<tr>
<td>Risk free=(\sigma_{0}^2) (%)</td>
<td>18.5978</td>
<td>4.5919</td>
<td>0.7337</td>
<td>0.9239</td>
<td>25.0299</td>
<td>16.4143</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>0.00</td>
<td>3.22</td>
<td>39.18</td>
<td>33.65</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>Test Gamma1+Theta=0(^{(g)})</td>
<td>0.0021</td>
<td>1.2742</td>
<td>0.1684</td>
<td>0.4829</td>
<td>0.1245</td>
<td>0.0155</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>96.34</td>
<td>25.91</td>
<td>68.16</td>
<td>48.72</td>
<td>72.42</td>
<td>90.10</td>
</tr>
</tbody>
</table>

**Step two: Non-Linear SURE**

<table>
<thead>
<tr>
<th>R-squared</th>
<th>0.2953</th>
<th>0.1943</th>
<th>0.3475</th>
<th>0.0965</th>
<th>0.4782</th>
<th>0.1446</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corrected R-squared(^{(a)})</td>
<td>-3.3833</td>
<td>-0.4019</td>
<td>-0.0395</td>
<td>-0.0189</td>
<td>0.0424</td>
<td>0.1108</td>
</tr>
<tr>
<td>Portfolio Beta</td>
<td>0.3121</td>
<td>1.1994</td>
<td>0.4722</td>
<td>0.7151</td>
<td>0.5051</td>
<td>0.5930</td>
</tr>
<tr>
<td>Delta</td>
<td>-0.0020</td>
<td>-0.0211</td>
<td>-0.0022</td>
<td>-0.0261</td>
<td>-0.0025</td>
<td>-0.0318</td>
</tr>
<tr>
<td>Gamma1</td>
<td>0.0127</td>
<td>0.0069</td>
<td>0.0007</td>
<td>-0.0060</td>
<td>-0.0032</td>
<td>-0.0041</td>
</tr>
<tr>
<td>Theta</td>
<td>-0.0176</td>
<td>-0.0326</td>
<td>-0.0033</td>
<td>-0.0034</td>
<td>-0.0016</td>
<td>-0.0473</td>
</tr>
<tr>
<td>Annual risk free rate(^{(a)}) (%)</td>
<td>93.14</td>
<td>24.80</td>
<td>3.90</td>
<td>3.19</td>
<td>-15.38</td>
<td>-19.08</td>
</tr>
<tr>
<td>Risk free=(\sigma_{0}^2) (%)</td>
<td>17.8349</td>
<td>5.5562</td>
<td>18.3978</td>
<td>3.8032</td>
<td>45.6576</td>
<td>30.9509</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>0.00</td>
<td>1.85</td>
<td>0.00</td>
<td>5.13</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Test Gamma1+Theta=0(^{(g)})</td>
<td>0.1592</td>
<td>1.8152</td>
<td>0.1268</td>
<td>10.4885</td>
<td>0.6379</td>
<td>0.6591</td>
</tr>
<tr>
<td>Probability &gt; F (%)</td>
<td>69.00</td>
<td>17.80</td>
<td>72.18</td>
<td>0.12</td>
<td>42.46</td>
<td>41.70</td>
</tr>
</tbody>
</table>

Table 4.15 below presents the results when the logarithm of traded assets is the

\(^6^9\) See footnote 65 above. The explanatory variables are the weekly return on IGPA in logarithm and CAPIMAS.
market coverage variable. In this case, a negative effect is obtained only over the period from 1993 to 1995 but this is in general non-significant.

Table 4.15 Average results using traded assets as market coverage.\(^70\)

<table>
<thead>
<tr>
<th>Period</th>
<th>7.789</th>
<th>2.791</th>
<th>5.793</th>
<th>7.795</th>
<th>4.797</th>
<th>2.799</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>28.691</td>
<td>30.693</td>
<td>25.695</td>
<td>27.697</td>
<td>25.699</td>
<td>20.100</td>
</tr>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>lnasset</td>
<td>lnasset</td>
<td>lnasset</td>
<td>lnasset</td>
<td>lnasset</td>
<td>lnasset</td>
</tr>
</tbody>
</table>

Step one: Non-Linear OLS

<table>
<thead>
<tr>
<th>Number of asset</th>
<th>19</th>
<th>20</th>
<th>20</th>
<th>24</th>
<th>20</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of weeks</td>
<td>104</td>
<td>104</td>
<td>105</td>
<td>104</td>
<td>104</td>
<td>69</td>
</tr>
<tr>
<td>Number of observations</td>
<td>1976</td>
<td>2080</td>
<td>2100</td>
<td>2496</td>
<td>2080</td>
<td>1587</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.2268</td>
<td>0.1364</td>
<td>0.1480</td>
<td>0.0626</td>
<td>0.4179</td>
<td>0.1957</td>
</tr>
<tr>
<td>Portfolio Beta</td>
<td>0.7959</td>
<td>0.6962</td>
<td>0.9453</td>
<td>0.7006</td>
<td>0.8741</td>
<td>0.6523</td>
</tr>
<tr>
<td>Delta</td>
<td>0.0051</td>
<td>0.0084</td>
<td>-0.0479</td>
<td>-0.0273</td>
<td>-0.0216</td>
<td>0.0111</td>
</tr>
<tr>
<td></td>
<td>(0.0105)</td>
<td>(0.0218)</td>
<td>(0.0533)</td>
<td>(0.0328)</td>
<td>(0.0224)</td>
<td>(0.0118)</td>
</tr>
<tr>
<td>Gamma</td>
<td>0.0148</td>
<td>0.0079</td>
<td>0.0005</td>
<td>0.0008</td>
<td>-0.0046</td>
<td>-0.0038</td>
</tr>
<tr>
<td></td>
<td>(0.0021)</td>
<td>(0.0020)</td>
<td>(0.0028)</td>
<td>(0.0018)</td>
<td>(0.0014)</td>
<td>(0.0013)</td>
</tr>
<tr>
<td>Theta</td>
<td>-0.0423</td>
<td>-0.0485</td>
<td>0.2505</td>
<td>0.1426</td>
<td>0.1197</td>
<td>-0.0566</td>
</tr>
<tr>
<td></td>
<td>(0.0507)</td>
<td>(0.1095)</td>
<td>(0.2778)</td>
<td>(0.1732)</td>
<td>(0.1212)</td>
<td>(0.0630)</td>
</tr>
</tbody>
</table>

Step two: Non-Linear SURE

| R-squared       | 0.2949 | 0.1943 | 0.3434 | 0.0844 | 0.4745 | 0.1501 |
| Corrected R-squared \(^{(4)}\) | -2.1873 | -0.1288 | -0.0318 | -0.0719 | 0.0404 | 0.1127 |
| Portfolio Beta  | 0.4679 | 1.0421 | 0.4778 | 0.8662 | 0.5091 | 0.5795 |
| Delta           | 0.0028 | 0.0118 | -0.0129 | 0.0061 | 0.0120 | 0.0153 |
|                 | (0.0093) | (0.0180) | (0.0311) | (0.0284) | (0.0202) | (0.0143) |
| Gamma           | 0.0127 | 0.0068 | 0.0007 | -0.0006 | -0.0032 | -0.0039 |
|                 | (0.0016) | (0.0013) | (0.0005) | (0.0006) | (0.0008) | (0.0010) |
| Theta           | -0.0272 | -0.0646 | 0.0665 | -0.0328 | -0.0625 | -0.0771 |
|                 | (0.0450) | (0.0908) | (0.1621) | (0.1501) | (0.1095) | (0.0765) |

---

\(^{70}\) See footnote 65 above. The explanatory variables are the weekly return on IGPA in logarithm and LN\text{ASSETS}.
Table 4.16 summarises the results for time trend as a market coverage variable. These results seem to have bad properties in terms of the residuals.

<table>
<thead>
<tr>
<th>Step one: Non-Linear OLS</th>
<th>7.789</th>
<th>2.791</th>
<th>5.793</th>
<th>7.795</th>
<th>4.797</th>
<th>2.799</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>28.691</td>
<td>30.693</td>
<td>25.695</td>
<td>27.697</td>
<td>25.699</td>
<td>20.100</td>
</tr>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Step two: Non-Linear SURE</th>
<th>7.789</th>
<th>2.791</th>
<th>5.793</th>
<th>7.795</th>
<th>4.797</th>
<th>2.799</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>28.691</td>
<td>30.693</td>
<td>25.695</td>
<td>27.697</td>
<td>25.699</td>
<td>20.100</td>
</tr>
<tr>
<td>Market index</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
<td>igpag</td>
</tr>
<tr>
<td>Market Coverage</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
<td>trend</td>
</tr>
</tbody>
</table>

See footnote 65 above. The explanatory variables are the weekly return on IGPA in logarithm and T.
4.- Concluding remarks

The results from tables 4.11 to 4.16 prove that it is very difficult to determine the effect of compulsory holdings in an empirical test. This difficulty is related to the problem identified by Roll (1977) in terms of the ability to measure the market return. In our approach we have been able to divide Roll’s uncertainty in tow: the known market index return and the unknown return of the compulsory holdings.
1.- Introduction

Empirical evidence suggests that the expansion of the financial sector is associated with higher rates of growth. Most notably the work by Barro and Sala-i-Martin (1995) and King and Levine (1993) find evidence in this direction. It is not clear, however, whether these empirical measurements of the relationship have succeeded in isolating the effect, on growth, of the exogenous improvements in the financial system. The natural feedback from economic growth to the activity of the financial sector is likely to overshadow the purely financial innovations. For this reason it is difficult to assess empirically the extent of this effect.

Much of the empirical work on endogenous growth has used an approach where the data is averaged over time to estimate subsequently cross-section regressions. Variations of this approach consider two, three or four periods of time; it averages the data over these periods and it estimates the parameters from this shrinking panel of data. We call this process phase-averaging as these periods define particular data phases. Campos, Ericsson...
and Hendry (1990) have shown that phase averaging is more likely to produce more problems than benefits, as aggregation from annual data entails loss of information and does not reduce other data problems such as serial-correlation.

In our case, when the process generating the financial activity variable has some dynamic components, the use of phase averaging can introduce additional bias in the estimates of the model. Depending on the dynamic structure of the process generating the error term, problems related to serial-correlation, in the error term can be increased by phase averaging. In summary, when using phase averages, the degree of exogeneity on the explanatory variable has to be much higher in order to obtain reasonable estimates compared to using annual frequencies in the estimation.

We find that the length of the phase average produces increased distortions on the value of the estimated parameter in a regression, where the regressor is not strictly exogenous.

Creating the phase-averaged series entails losing information from the original data. First, it is necessary to decide where to start with the phase-averaged series and forfeit the previous observations; it is also probable that observations will be lost at the end of the time period. Second, the data in interior of the averaged periods are dropped. It is clear from our examination that these observations should be considered and that we do not need to evaluate only the period averages or to consider only the rate of growth between the two end-periods.

Likewise, we study the use of instrumental-variables to overcome the simultaneity and counteract biases as in Atje and Jovanovic (1993) and Harris (1997). We find that this is
inferior to some more direct specification of the model. Instrumental variables estimation is also increasingly inefficient with the size of the phase average.

We do not find reasons to average the explanatory variables over periods of time in the growth model under study. We show that the most convenient procedure to test the effect of financial size on growth in a time series entails using the data in the most disaggregate form possible.

Pesaran and Smith (1995) study the properties of time aggregation when the regressors are stationary and when they are integrated. In the stationary case, and using a slightly different averaged model than ours, they study the between-regression and find this produces inconsistent estimators. They also study a long-run version of the model and find that the corresponding OLS estimates converge to their population counterparts for long time series, but they are biased in a short time series even when the cross-section is large. In another work Lee, Pesaran and Smith (1998) also identify inconsistent estimators, in dynamic panels, when there is country heterogeneity in growth effects and in speeds of convergence.

Arestis et al. (2001) find that the growth-enhancing effect provided by stock markets may have been exaggerated by empirical studies using cross-country growth regressions.

Bergstrom (1984) studies continuous-time stochastic models and the properties of aggregation over time. In his work he is concerned about the use of quarterly or yearly measurements when more time disaggregate data is not available. He advocates for the study of continuous-time structural models rather than discrete versions of the models.
one is considering the discrete versions of the models this should be based on the continuous specification. From his observation that aggregation over time is an inconvenience, created by the frequency in which the data can be collected, he considers it unrealistic "to assume that the economy moves in discrete jumps between successive positions of temporary equilibrium at intervals whose interval coincides with the observation period." (op.cit. p. 1147). He demonstrates that the actual form taken by the discrete model is not invariant with the level of time aggregation: "The simultaneity in the unlagged endogenous variables [...] is necessary in order to avoid the unrealistic assumption that the minimum lag in any causal dependency is nor less than the observation period." (ibid.). He shows that a discrete model can be drawn for the observation period and that this "is [...] of no importance except for the fact that the shorter the observation period the more observations there will be and the more efficient will be the estimates of the structural parameters." (op.cit. p. 1149) We acknowledge two of Bergstrom's findings: first, long term relationships do not necessarily have the same form as short term ones and second, estimating parameters from a more time-disaggregated model is more efficient if this model is the known structural relationship.

The rest of this chapter is organised as follows. We study in Section 2 the effects of phase averaging in the OLS and IV estimation of a simple dynamic system; we look at the effect in the size and bias over the parameters and efficiency of the estimation method. In section 3 we disaggregate over time and estimate parameters for the models in Barro and Sala-i-Martin (1995) and King and Levine (1993). We identify some positive but weak effects of finance on growth. In section 4 we study the effect of different phase average length applied to a Latin-American database on finance and growth. A frequently used measurement of financial activity in empirical analysis is the liquid-liabilities-to-output
ratio. It has been successfully used in Atje and Jovanovic (1993), King and Levine (1993) and Barro and Sala-i-Martin (1995). We analyse extensively the effect of this financial characteristic on growth. Section 5 contains some concluding remarks and identifies new areas for future research.

2.- Phase Average

The model presented in this section is similar to the cobweb models used by Engle, Hendry and Richard (1994) to analyse exogeneity issues. The phase average procedure is based on the mechanism detailed in Campos, Ericsson and Hendry (1990). The structure is a customary control rule for an output variable \( y_t \) being manipulated through the input variable \( z_t \). In our case \( y_t \) would be the logarithm of the output a given economy at time \( t \), while \( z_t \) will be the size or activity in its financial system\(^{72} \). From the analysis in chapter 1, if we consider a short enough time interval, the financial sector will not change until after the realisation \( y_{t,1} \) is known.

\[
\begin{align*}
    y_t &= \beta_1 y_{t-1} + \beta_2 z_t + \epsilon_t \\
    z_t &= \lambda_1 y_{t-1} + \lambda_2 z_{t-1} + \nu_t \\
    \epsilon_t &= \rho \epsilon_{t-1} + \omega \\
    \text{with} & \quad |\rho| < 1 \quad \text{and} \quad \{\omega, \nu\} \sim iid(0, \Omega)
\end{align*}
\]

\(^{72} \) If we believed that the political system is also weakly endogenous to output, we could think of \( z_t \) being any measurement reflecting political, social or legal conditions. For example, democracy as measured in a positive scale related to participation in elections or political instability in a negative scale measured by
Our objective is to make correct inferences about the parameters in the first equation based on \( T \) time series observations. In particular, to make inferences about \( \beta_2 \) that measures the feedback from financial activity and possibly other weakly exogenous variables.

In order to deal with data problems as missing observations, and in order to smooth measurement errors in the variables, the observations of \( y_t \) and \( z_t \) are often averaged and/or differentiated over time in empirical work. After transforming the data the relation to be estimated is:

\[
y_t = b_1 y_{t-k} + b_2 z_t + u_t
\]

The phase-averaging transformation is often applied to the data, see for example Barro et al. (1995) or King and Levine (1993). In their procedure, the original five-year frequency data is averaged over a ten-year period\(^7\). The parameter estimation method is then applied using all the non-overlapping ten-year separated observations. The span of the phase-average also implies that after averaging and smoothing the time observations, some of the transformed data are dropped from the sample. As an example, in Barro et al. (1995), the original panel data of countries is available almost always in any five years period from 1960 to 1965, 1970, 1975, 1980, 1985 and 1990\(^4\). However, the parameter estimation stage is performed only over two elements comprising information from ten

---

\(^7\) King and Levine (1993) apply also a twenty and thirty years phase-averaging, while Atje and Jovanovic (1993), Harris (1997), Barro (1991) and Barro (1997) use more often a ten-year as well as other time intervals.

\(^4\) Some of the variables are at the end of the period as GDP, investment to product ratio, government expenditures and so on. Other observations are averages over the five years period; for example political stability or liquid-liabilities-to-GDP ratio.
years: 1965 to 1975 and 1975 to 1985. This transformation of the data involved both procedures of averaging the values over ten-year phases, and dropping the overlapping observations 1960 to 1970, 1970 to 1980 and 1980 to 1990.

The convenience of averaging data before performing an estimation procedure is often related to alleged mitigation of measurement errors. However, as has been summarised by Ericsson and Hendry (1994) phase average raises three questions: (i) what are the theoretical effects of phase averaging? (ii) what are the observed effects of phase averaging? and, (iii) what are the effect of selecting the interval over which phase average?

A first negative effect we can identify, about phase averaging, is that it loses information. This can be seen in the frequency domain, where averaging the time series is equivalent to dropping some of the frequencies of the series. Therefore, unless there are good reasons to weight by zero these frequencies, phase averaging will be an inefficient method. It will not be advisable averaging the observations without knowing if it helps to obliterate these undesired measurement errors. A second problem with phase-averaged is that the tests over the parameters based in averaged data may be of low power relative to those based in annual data. This is due to the decreased degrees of freedom. Even if the corresponding annual parameters can be recovered from the phase average estimations so they can be

---

Barro (1996) claims that “[T]he precise timing between growth and its determinants is not well specified at the high frequencies characteristics of business cycles.” (p.13) He explains that the underlying theory relates to long-term growth and that relationships at the annual frequencies would be likely to be dominated by measurement error due to this mistiming. However, he does not present an explicit model for the kind of data problems present at the annual level and why these would disappear when taking the averages. In our model since there is no relationship in the long-term without the relationship in the short term, averaging and dropping observations increases measurement error.
tested indirectly, the test procedures will be less sensitive to alternative hypothesis.

If all the parameters of interest, including mean values and standard deviations, can be recovered after the estimating from the phase-averaged data, the effects of transformation would be unimportant. However, it is not likely that parameters related to granger non-causality, short run variability or dynamic mechanisms whereby the economy adjust to shocks can be recovered from estimation on phase-averaged data.

More importantly, weakly exogenous variables at annual observations level, might not longer be weakly exogenous at phase averaged level. This can be seen intuitively in the following way. We pictured a system were growth $y_i$ depends on a predetermined variable $z_i$ that measures for example: the activity of the financial system or political instability. In an annual base, there can be some feedback from growth to variables like the size of markets or to some social measurements. An economy that is growing faster this year may be able to delay or stop the detonation of particularly explosive social issues. At the same time, a faster-growing economy at a given year will push markets to operate or to get more involved in the economy. This possibly small annual feedback after phase averaging becomes a large simultaneity between the two variables. At the ten-year average level financial markets are not predetermined anymore.

The basic properties of phase averaging is illustrated in a simple time series model, taken from Campos, Ericsson and Hendry (1990), let:

\[ y_t = \beta_0 + \beta_2 z_t + \varepsilon_t \]

\[ \varepsilon_t = \rho \varepsilon_{t-1} + \omega \]

\[ z_t = \lambda_0 y_{t-1} + \lambda_2 z_{t-1} + \nu_t \]

where, $|\rho|<1$ and $(\omega, \nu_t) \sim N(0, Diag(\sigma_\omega^2, \sigma_\nu^2))$
To construct a $k$-years phase average we take the mean value of every contiguous $k$ observation and, drop anything that lays overlapping the $k$-periods phase. The serial correlation of the error term $\varepsilon_t$ is then decreased by the phase average. The covariance between two contiguous observations at the disaggregated level is:

$$E[\varepsilon_t, \varepsilon_{t+1}] = \frac{\rho \sigma^2}{1 - \rho^2},$$

The length of the average damps the covariance between two contiguous observations after phase average\(^7\) as follows:

$$E[\tilde{\varepsilon}_t, \tilde{\varepsilon}_{t+1}] = \frac{\rho \sigma^2}{1 - \rho^2} \frac{(1 - \rho^4)}{k^2(1 - \rho)^2},$$

However, after phase averaging, the OLS estimates of $\beta_0$ and $\beta_2$ will be always inconsistent, except in the case of strict exogeneity of $z_t$ (when $\lambda_t=0$); even when the error term $\varepsilon_t$ is not auto-correlated ($\rho=0$) the estimation is inconsistent. For example, for an average of two observations ($k=2$) the covariance between the new phase-average variable $\tilde{z}$ and the corresponding error term is:

$$E[\tilde{z}, \tilde{\varepsilon}_t] = \frac{\lambda_0 \sigma^2 (1 + \rho)}{4(1 - \rho)(1 - \kappa \rho)},$$

In the expression above $\kappa=\lambda_2 + \beta_2 \lambda_t$. On the other hand, this correlation at the disaggregated level is smaller\(^7\):

\(^7\) Campos, Ericsson and Hendry (1990) indicate the reduction in auto-correlation is marginal only

\(^7\) This is due to $(1+\rho)>\rho/(1+\rho)$ for all $|\rho|<1$. 

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More importantly, under phase averaging the only case when there is no bias is when \( z_i \) is strictly exogenous.

**a.- Different parameter size for different phase length**

The value of the parameter in the cobweb model of growth is not invariant with the size of the time aggregation. In the simple model of growth presented earlier, where the explanatory factor is the weakly exogenous variable \( z_i \), we see that phase averaging will affect the size of the parameters in the model as following. Take the system:

\[
\begin{align*}
    y_i &= \beta_1 y_{i-1} + \beta_2 z_i + \epsilon_i \\
    \epsilon_i &= \rho \epsilon_{i-1} + \omega \\
    z_i &= \lambda_1 y_{i-1} + \lambda_2 z_{i-1} + v_i
\end{align*}
\]

where, \(|\rho|<1\) and \([\omega, v, \epsilon] \sim N(0,Diag(\sigma_\omega^2, \sigma_v^2))\)

From this basic structure we can replace backwards using lagged values of \( z_i \) \( k \) times and obtain a model for \( y_i \) on the weakly exogenous variables \( y_{i-k} \) and \( z_{i-k} \); we call this version of the model the \( k \)-lagged model:

\[
\begin{align*}
    y_i &= b_1 y_{i-k} + b_2 z_{i-k} + e_i \\
    z_i &= l_1 y_{i-k} + l_2 z_{i-k} + v_i
\end{align*}
\]

---

78 This solution was obtained after orthogonalising the original system. The variables \( x_1 \) and \( x_2 \) we use here are the eigenvalues of the transition matrix.
\[ b_1 = x_1^2 + x_1^r r^2 \left( (1 + r)^2 - 1 \right)^r \]
\[ b_2 = (x_1^r - x_2^r)(x_1 - x_2) r \left( (1 + r)^{1/2} - 2 \lambda_1 \right)^r \]
\[ l_1 = x_1^r + x_1^r r^2 \left( (1 + r)^2 - 1 \right)^r \]
\[ l_2 = x_1^r - x_2^r r^2 \left( (1 + r)^2 - 1 \right)^r \]
\[ r = \sqrt{1 + 4 \beta_2 \lambda_2 \lambda_3 \left( \lambda_3 - (\beta_1 + \lambda_1 \beta_2) \right)^{-r}} - 1 \]
\[ x_1 = \lambda_2 + 0.5 * r \left( \lambda_2 - (\beta_1 + \lambda_1 \beta_2) \right) \]
\[ x_2 = \beta_1 + \lambda_1 \beta_2 + 0.5 * r \left( \lambda_2 - (\beta_1 + \lambda_1 \beta_2) \right) \]

When the parameter \( r \) is small, or in other words when the original transition matrix is approximately diagonal, the corresponding (approximate) values for \( b_1, b_2, l_1 \) and \( l_2 \) can be more easily understood as follows:

\[ b_1 = (\beta_1 + \lambda_1 \beta_2)^r \]
\[ b_2 = \beta_2 \lambda_2 \left( \lambda_2^r - (\beta_1 + \lambda_1 \beta_2)^r \right) \left( \lambda_2 - (\beta_1 + \lambda_1 \beta_2) \right)^r \]
\[ l_1 = \lambda_1 \left( \lambda_2^r - (\beta_1 + \lambda_1 \beta_2)^r \right) \left( \lambda_2 - (\beta_1 + \lambda_1 \beta_2) \right)^r \]
\[ l_2 = \lambda_1^r \]

Because of the weak exogeneity of \( z_t \), making inferences about the parameters \( b_1 \) and \( b_2 \) only based on the model conditional to \( y_{t, k} \) and \( z_{t, k} \) (the equation \( y_t = b_1 y_{t, k} + b_2 z_{t, k} + e_t \)) would not be problematic. However, inferences for the original values \( \beta_1 \) and \( \beta_2 \) are not possible only from the conditional model for \( y_t \). For example, in a simple case, where the variable \( y_t \) does not affect \( z_t \) at the annual level so \( \lambda_1 = 0 \); \( r \) becomes equal to zero and the parameter \( b_2 \) is:

\[ b_2 = \beta_2 \lambda_2 \left( \lambda_2^r - \beta_1 \right) \left( \lambda_2 - \beta_1 \right)^{-r} \]
Except when the restriction that \( \lambda_2 \) is equal to \( \beta_i \) holds, the parameter \( \beta_2 \) cannot be recovered without studying the complete system and computing estimators for the parameters \( \lambda_2^2 \) and \( \beta_i^2 \). This problem becomes more difficult in cases where \( \lambda_i \neq 0 \).

Table 5.1 below determines the corresponding values of \( \beta_i \) and \( \beta_2 \) as functions of the values estimated in the \( k \)-lagged model for \( k=30 \) years. The values of \( \lambda_2 \) have been adjusted in such a way that the resulting \( b_1 \) and \( b_2 \) in the 30-lagged model are the same as the estimated values in the growth equation from King and Levine\(^79\) (1993).

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( \lambda_2/\beta_i )</th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90.00</td>
<td>0.9001</td>
<td>0.42</td>
</tr>
<tr>
<td>94.32</td>
<td>0.9433</td>
<td>0.26</td>
</tr>
<tr>
<td>95.00</td>
<td>0.9501</td>
<td>0.24</td>
</tr>
<tr>
<td>99.99</td>
<td>1.0000</td>
<td>0.12</td>
</tr>
<tr>
<td>105.00</td>
<td>1.0501</td>
<td>0.05</td>
</tr>
<tr>
<td>110.00</td>
<td>1.1001</td>
<td>0.02</td>
</tr>
<tr>
<td>115.00</td>
<td>1.1501</td>
<td>0.01</td>
</tr>
<tr>
<td>120.00</td>
<td>1.2001</td>
<td>0.00</td>
</tr>
</tbody>
</table>

We see that given different assumptions about the dynamic structure of the variable \( z_t \) (through the parameter \( \lambda_2 \)) we will produce very different 'guesses' for the values of parameter \( \beta_2 \), ranging from zero to 0.42 per cent. The simple back of the envelope calculation \( \beta_2 = b_2 / 30 \) can be seriously misleading. A strong effect found at the 30-year

\(^79\) We have adjusted these to by a factor of 100, so that they can be read as percentages.
aggregation can be translated into an either very important effect at the yearly level or a
negligible one.

b.- The phase average model.

The phase-average model uses the average value of $z_t$ over $t-k+1$ and $t$, instead of the $k$-th
lag, to explain the rate of growth of $y_t$ between $t-k+1$ and $t$. In particular, for our initial
one-year model, we will call the phase-average model the following structure:

\[ y_t = P_1 y_{t-k} + P_2 \bar{z}_t + u_t \]
\[ \bar{z}_t = \frac{\sum_{i=0}^{k-1} z_{t+i}}{k} \]

The idea behind using the average is to capture all the shocks and innovations occurring to
the variable $z_t$ over the period of interest and use them to explain the changes in variable $y_t$.
For example, Harris (1997) argues in favour of using the current average value of an
interaction between investment and level of financial intermediation. "The use of lagged
investment is inadequate as a solution to the endogeneity issue since it is not highly
correlated with current investment and hence not a good proxy for this variable." (p. 140)
He recognises that it is likely that current investment and in per-capita growth are jointly
determined and so OLS estimates of the stock market effect may be biased. In our model
the financial level is also jointly determined and so, not all the innovation in $z_t$ is

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80 The model in Harris (1997) considers the effect directly through investment. As in Atje and Jovanovic
(1993) and Greenwood and Jovanovic (1990) his model establishes that the level of financial
intermediation increases the return on investment.
independent of the error term \( u \), and that will produce different biases depending on the estimation method used.

In order to be more explicit about the properties of the average we apply \( k \) times backwards the original system relating \( z \) and \( y \) to write the average of \( z \), in terms of the past values and the shocks \( e \), and \( v \):

\[
\bar{z}_t = [0 \ 1] V (\Sigma_{-\phi}^{k-1} D^{k+j}) V' \left[ \begin{array}{c} y_{t-k} \\ z_{t-k} \end{array} \right] + [0 \ 1] V (\Sigma_{-\phi}^{k+j} (I-D)^j(I-D^{k+j}) V' \left[ \begin{array}{c} e_{t-k} \\ \beta \ V \end{array} \right]
\]

\[
V = 2 \times 2 \text{ matrix of eigenvectors of the original system.}
\]
\[
D = 2 \times 2 \text{ diagonal matrix of eigenvalues.}
\]

We define a convenient function of \( y_{t-k} \) and \( z_{t-k} \) as:

\[
f ( y_{t-k} , z_{t-k} ) = (1/k) [0 \ 1] V (\Sigma_{-\phi}^{k+j} D^{k+j}) V' \left[ \begin{array}{c} y_{t-k} \\ z_{t-k} \end{array} \right], \quad \text{and}
\]

\[
v_t = (1/k) [0 \ 1] V (\Sigma_{-\phi}^{k+j} (I-D)^j(I-D^{k+j}) V' \left[ \begin{array}{c} e_{t-k} \\ \beta \ V \end{array} \right]
\]

Here the term involving \( e_{t-j} \) and \( v_{t-j} \) carries all the innovations and shocks between \( t-k+1 \) and \( t \), but according to our model only those shocks related to \( v_{t-j} \) are independent from past shocks on GDP.

We established earlier the \( k \)-lagged model, the one in terms of \( y_{t-k} \) and \( z_{t-k} \), as follows:

\[
y_{t-k} = b_{1} y_{t-k+i} + \beta_{z} z_{t-k+i} + e_{t-i}, \quad \text{where}
\]

\[
e_{t-i} = [1/0] V (\Sigma_{-\phi}^{k+j} D V' \left[ \begin{array}{c} e_{t-k-j} \\ \beta \ V \end{array} \right]
\]

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By comparing the expression above with the phase-average model we obtain the result summarised in the following proposition.

**Proposition 1.** Under the assumption that the eigenvalues of the $y/z$ system are all below one in absolute value and that the error term is regular; there is a unique linear mapping between $[b_1,b_j]$ and $[P_1,P_j]$, if the following matrix $A$ is non-singular:

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \end{bmatrix} + \frac{(I\cdot k)}{D^r(I-D')^r(I-D')^r} D V \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \end{bmatrix}
$$

**Proof:** See appendix 5A.

More importantly, when proposition 1 holds we can identify the error terms in the phase-average model and the $k$-lagged version of the model. Because the phase-average model has to be correct, then all the innovations between $t-k+1$ and $t$ in the average of $z_t$ should be contained in the error term in the $k$-lagged model. We find an expression for $u_t$ in terms of the yearly error terms $e_{ij}$ and $v_{ij}$:

$$
e_t = [I/0] V(\Sigma_{j=1}^{k} \cdot D' V^r [e_{ij}^r + \beta_j v_{ij}^r]) = u_t + (P_2/k) [0/1] V(\Sigma_{j=1}^{k} \cdot (I-D')^r(I-D')^r) V^r [e_{ij}^r + \beta_j v_{ij}^r])
$$

$$
u_t = [I/0] V(\Sigma_{j=1}^{k} \cdot D' V^r [e_{ij}^r + \beta_j v_{ij}^r]) - (P_2/k) [0/1] V(\Sigma_{j=1}^{k} \cdot (I-D')^r(I-D')^r) V^r [e_{ij}^r + \beta_j v_{ij}^r])
$$

Thus, the error term $u_t$ comprises information for the period between $t-k+1$ and $t$ on two

---

81 See Appendix 5A for more details on how these formulas were derived.
types of shocks: the ones on \( y \), and the shocks on \( z \). This information cannot be separated so that the exogenous disturbances on \( z \) can be isolated, as it cannot be separated the information on the same shocks contained in \( \bar{z} \). That would be possible only in the case where the matrices \( D, V \) and the value of the parameter \( \beta_z \) were known constants.

**Proposition 2.** Under the conditions of proposition 1, the OLS estimators of the phase average model are inconsistent and biased. The size of the bias is:

\[
\begin{align*}
\text{plim}(\hat{p}_1 - p_1) &= E[w_{i,k} u_{i,k}]/(\text{Var}(y_{i,k}) - \text{Cov}(y_{i,k}, f(y_{i,k}, z_{i,k})) (\text{Var}(w_{i,k}) + \text{Var}(f(y_{i,k}, z_{i,k})))^{1/2}) \\
\text{plim}(\hat{p}_2 - p_2) &= E[w_{i,k} u_{i,k}]/(\text{Var}(w_{i,k}) + \text{Var}(f(y_{i,k}, z_{i,k}))^{2} (1-\psi^2)) \\
\psi &= \text{Correl}(y_{i,k}, f(y_{i,k}, z_{i,k}))
\end{align*}
\]

**Proof:** The regularity conditions on the error term required here are as in White (1984, p. 69), those needed to obtain asymptotic normality and consistency of the OLS and instrumental variable estimators. Although these were not necessary for proposition 1 they are needed here for consistency of the \( k \)-lagged model and to separate the probability limits above into the product of two expected values \( E[z_{i,k}] / E[z_{i,k} u_i] \). See Appendix 5A for more details on how to obtain the formulas.

We apply now the formula in proposition 2 above to the special case of \( \beta_z = 0 \). When variable \( z \) has no influence on \( y \), the population parameters \( p_2 \) and \( b_2 \) should both equal to zero as the parameters \( p_1 \) and \( b_1 \) should both be equal to \( \beta_1^k \). However, in this case, the average estimator of \( p_2 \) for a large sample will be centred around a value different from zero:
\[ p \lim (\hat{p}_2 - \hat{p}_1) = \frac{\sigma_e^2 \sum_{j=0}^{k-1} g(j)}{\sigma_e^2 \sum_{j=0}^{k-1} g(j)^2 + \sigma_i^2 \sum_{j=0}^{k-1} \left( \frac{1 - \lambda_j^i}{1 - \lambda_2} \right)^2 + (1 - \psi^2) \text{Var}(f(y_{-k}, z_{-k}))} \]

\[ g(j) = \frac{\lambda_1}{\beta_1 - \lambda_2} \left( \frac{1 - \beta_1^i - 1 - \lambda_2^i}{1 - \beta_1 - 1 - \lambda_2} \right) \]

Which is a positive quantity that only decreases when \( k \), the size of the phase average, is very large. Clearly, only when \( \lambda_j = 0 \), or in other words, only when financial sophistication is strongly exogenous, the estimated parameter is unbiased.

Figure 5.1 below plots the bias for a set of hypothetical parameters, compatible with those estimated by Barro and Sala-i-Martin (1995). The size of the bias is measured in a per cent scale and increases until a value of approximately \( k=18 \) (Barro and Sala-i-Martin use \( k=10 \)). From there, it slowly decreases towards zero, as \( k \) grows, reaching for example 47.17 per cent at \( k=100 \), 37.77 per cent at \( k=200 \) and 27.01 per cent at \( k=500 \).
Similar biases, but with the opposite sign, can be found for parameter $P_t$. The time averaging of the explanatory variable $z$, has an effect of producing smaller estimates for the parameter on $y_{r,t}$. This means that phase averaging will suggest faster convergence among countries than the actual speed, if the estimates are obtained using OLS. This is because the estimate of $(P_t - 1)$ in the following expression will be larger in absolute value than the true parameter:

$$
\Delta y_t = (P_t - 1) y_{r,t} + P_2 \bar{z}_t + u_t
$$

\textit{c.- Using instrumental variables for the average}

A possible solution to the simultaneity is the use of instrumental variables for the average
of the variable \( z \). When doing this, the desired instrument will be a variable that is as much correlated, as possible, with the regressor and as little correlated as possible with the error term in the \( y \), equation of the phase-average model.

For example, Harris (1997) uses OLS and 2SLS (instrumental variables) techniques for a model of growth that includes investment and financial traded value averaged over the period 1980-1991. The instrumental variables are lagged values of investment finance, population and so on.

King and Levine (1993) estimate models where the explanatory variables: schooling, inflation rate, total trade to GDP, government consumption and liquid liabilities, are considered in terms of their average value over the sample. The sample is 1960 to 1989 and the average is taken over these 30 years. To correct for endogeneity problems they undertake an additional exercise where the explanatory variables are the average over the previous ten years. In that case the samples are the growth over 1970 to 1989 and the explanatory variables are the average over 1960 to 1969 of inflation rate, total trade to GDP ratio, government consumption and liquid liabilities. Atje and Jovanovic (1993) use only lagged values as explanatory variables.

Barro and Sala-i-Martin (1995) include averages of public expenditure on education, government consumption, black market premium, political instability, terms of trade, liquid liabilities to GDP\(^{82}\) as explanatory variable. The values at the beginning of the period are instruments for each of these.

\(^{82}\) For liquid liabilities to GDP ratio, the initial value at each decade is the instrumental variable.
The phase average model we described in section 5.b. can be regarded as a model that presents error in the explanatory variable problem and as such, instrumental variables can overcome this obstacle. This is because the basic model, where averages of \( z_{t-k} \) over the period \( t-k \) and \( t \) are used, is simply a version of the \( k \)-lagged model, from section 5.a, with measurement error. We can see this more clearly by splitting the average value into two terms:

\[
\tilde{z}_t = f(y_{t-k}, z_{t-k}) + w_t
\]

We therefore have:

\[
f(y_{t-k}, z_{t-k}) = (1/k) [0/1] VD(I-D)^t(I-D')V' \left[ \frac{y_{t-k}}{z_{t-k}} \right]
\]

\[
w_t = (1/k) [0/1] V(\Sigma_{i,j} (I-D)^t(I-D')V' \left[ \frac{\varepsilon_{t-j}}{\nu_{t-j}} \right] + \beta z_{t-j})
\]

Thus, the first term, \( f(y_{t-k}, z_{t-k}) \), measures everything up to time \( t-k \) while the second, \( w_t \), comprises, in a one-dimensional summary, all the innovation between \( t \) and \( t-k \), that comes from the original system's error terms: \( \varepsilon_{t-j} \) and \( \nu_{t-j} \).

The first thing to notice is that not all the information included in \( w_t \) corresponds to independent innovations entering only in \( \tilde{z}_t \). A great deal of it corresponds to shocks also entering to variable \( y_t \), the terms \( \varepsilon_{t-j} \) that we assimilate shocks to GDP. Secondly, it is not possible to separate the effects of every \( \varepsilon_{t-j} \) and \( \nu_{t-j} \) over the period \( t-k+l \) and \( t \). The longer the period is, the scalar summary \( w_t \) will comprise larger amounts of different kinds of shocks. Therefore, to justify the use of phase-averages because these should include all
the information arriving between $t-k+1$ and $t$ is not too clear, as these effects will become more and more diffuse as $k$ grows.

In the third place, as $k$ grows the term $f(\ y_{i,k}, \ z_{i,k})$ becomes a less accurate approximation for $z_{i,k}$. Let us write the formula for this function in terms of the eigenvalues of the transition matrix is:

$$f(\ y_{i,k}, z_{i,k}) = \frac{r+2}{2r+2} \left[ y_{i,k} \cdot z_{i,k} \right] \cdot \left[ \frac{r(\lambda_2 - \lambda_1 \beta_2 - \beta_1)}{2\lambda_2 \beta_2} \cdot \left( \frac{x_1(1-x_i^l)}{1-x_1} - \frac{x_2(1-x_i^l)}{1-x_2} \right) \right]$$

If in a case where the changes in $y$, are much more correlated with its own past values than $z$, is with its past values\(^{83}\), we see that as $k$ grows the variable $f(\ y_{i,k}, z_{i,k})$ becomes increasingly correlated with $y_{i,k}$ and less correlated with $z_{i,k}^{84}$. If the growth of $y$, dominates the system and given that $f(\ y_{i,k}, z_{i,k})$ reflects the average between very old and new values of $z$, the longer is the interval $k$, the less important the initial value, $z_{i,k}$, will be.

Figure 5.2 below illustrates this effect using the values from a simulation of variable $y$, and variable $f(\ y_{i,k}, z_{i,k})$ and while applying different average sizes: 5, 10, 20 and 50. It is very apparent that for larger sizes of $k$, $f(\ y_{i,k}, z_{i,k})$ has a correlation of almost one with $y$.

---

\(^{83}\) This will be reflected by a ratio $\chi_i/\chi_2$ smaller than 1. We use this assumption for the simulation implemented in the next section and figure 5.2.

\(^{84}\) Corr($f(\ y_{i,k}, z_{i,k}), y_{i,k}$) = 1 and Corr($f(\ y_{i,k}, z_{i,k}), z_{i,k}$) = Corr($y_{i,k}, z_{i,k}$)

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Therefore, the information conveyed by \( f(y_{t-k}, z_{t-k}) \) is not neutral to the size of the phase average. In our case, different values of \( k \) will produce different information about the initial state of the GDP-financial size system. If the growth of \( z_t \) dominates the transition matrix, so that the relation between the eigenvalues \( (\lambda_1/\lambda_2) \) is larger than one, or if the two eigenvalues are equal; then the variable \( f(y_{t-k}, z_{t-k}) \) becomes increasingly correlated with \( z_{t-k} \) and less correlated with \( y_{t-k} \). In such a case, a model with a large phase average will become equal to the \( k \)-lagged model, times a scale factor for \( \mu_2 \), except for the additional error component \( w_t \), included in \( \tilde{e}_n \), that in this case will be pure measurement error.

These results indicate that for a system where \( y_t \) dominates the transition matrix, the phase-

---

85 Based on a simulation of the system \( y_t = \beta_1 y_{t-1} + \beta_2 z_{t-1}; z_t = \lambda_1 y_{t-1} + \lambda_2 z_{t-1} \), with parameters: \( \beta_1 = .9951; \beta_2 = .0023; \lambda_1 = 1.861 \) and \( \lambda_2 = 0.9432 \). Starting values \( y_0 \) and \( z_0 \) are set to zero. The corresponding values of \( r_1, x_1 \) and \( x_2 \) are respectively: 0.2620, 0.9363 and 1.0024. The error terms \( e_t \) and \( v_t \) are generated as random, independent and distributed as Normal(0,1).
average model will be increasingly inefficient in estimating the effect of average $z$, with the size of the average. In general, we can find expressions for this inefficiency in terms of the matrix $A$ identified in proposition 1. Proposition 3 compares the performance of the $k$-lagged and phase-average models in a short sample and asymptotically.

**Proposition 3.** Under the regularity conditions of proposition 1 and 2, and defining

$$\tilde{b} = A'\tilde{p}$$

where $\tilde{p}$ is the instrumental variables estimate for the phase-average model and $A$ is the matrix identified in proposition 1 we have the results:

a) If the following matrix is positive definite:

$$E[(Z'Z)^{-1} - E[(Z'Z + Z'\xi A^{-1})^{-1}Z'Z(Z'Z + A^{-1}\xi'Z)^{-1}]]$$

and $\sigma_z^2 < \sigma_e^2$ then the instrumental-variable-phase-average estimates are more efficient.

b) Asymptotically the instrumental-variable-phase-average estimate is more efficient if and only if $\sigma_z^2 < \sigma_e^2$.

c) When $\beta_z = 0$, both methods have the same asymptotic variance and therefore are asymptotically equally efficient.

**Proof:** See appendix 5A.

The last proposition provides conditions for when the phase-average model will be a better estimation method than the $k$-lagged model. If, as claimed for example in Harris (1997), the relevant variable explaining the effect of the financial system in growth is the average
and not the initial values and this translates into the phase-average model having a smaller error term, phase-average will be more efficient. In such a case, $\sigma^2_\epsilon < \sigma^2_\epsilon$ and therefore, asymptotically, there will be reasons to use the phase-average model. However, under short samples comparing the performance of the two methods is more involved. The analysis in terms of the function $f(y_{t-k}, z_{t-k})$ and its correlation with the variables $y_{t-k}$ and $z_{t-k}$, suggests that the $k$-lagged model will be more efficient in a general case.

Proposition 3 is useful to compare efficiency of the phase-average as compared with a $k$-lagged model for a given size $k$. In terms of the optimal choice of $k$, the evidence presented above suggests a loss of efficiency with longer averages and lags.

3.- Econometric Results.

We reassess in this section the empirical evidence from endogenous growth: firstly, we apply the finding related to phase-average to the empirical work by Barro and Sala-i-Martin (1995) and the work by King and Levine (1993). Secondly, the application of our phase-average model and different estimation alternatives for a data-set of South-American countries from the southern cone: Argentina, Brasil, Bolivia, Chile, Paraguay and Uruguay. These countries are the current subscribers and associate members to the trade agreement Mercosur. These countries have experience similar economic development over the period under consideration and have common historical background.
In this section we closely reproduce the estimation results for the specification in both Barro and Sala-i-Martin (1995) and King and Levine (1993).

Table 5.1 below corresponds to Barro's Table 12.3 column (1) in which two ten-year phase averages have been applied to the data. The dependent variable is the growth rate of per capita GDP and the sample corresponds to 79 countries, for the period 1965 to 1975 and 94 for the period 1975 to 1985. Although the sample sizes differ slightly we obtain results very close to those of Barro and Sala-i-Martin (op.cit.). For simplicity, we have not included yet the financial measurement. However, the results in this table are not robust to time disaggregation, particularly for the variables that could be considered endogenous or weakly exogenous to growth, namely political instability, government consumption, investment ratio and education expenditure. This can be seen in Table 5.2.

---

86 Barro and Sala-i-Martín (1995) work with 87 observations in the first period and 97 for the second.
Table 5.1 Barro’s regression for growth rate of real per capita GDP.\textsuperscript{87}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.0266</td>
<td>0.0032</td>
<td>-8.20</td>
</tr>
<tr>
<td>Male secondary education</td>
<td>0.0131</td>
<td>0.0051</td>
<td>2.55</td>
</tr>
<tr>
<td>Female secondary education</td>
<td>-0.0053</td>
<td>0.0057</td>
<td>-0.94</td>
</tr>
<tr>
<td>Male higher education</td>
<td>0.0606</td>
<td>0.0230</td>
<td>2.63</td>
</tr>
<tr>
<td>Female higher education</td>
<td>-0.0827</td>
<td>0.0300</td>
<td>-2.74</td>
</tr>
<tr>
<td>Log(life expectancy)</td>
<td>0.0638</td>
<td>0.0170</td>
<td>3.86</td>
</tr>
<tr>
<td>Log(GDP)*human capital</td>
<td>-0.2940</td>
<td>0.0760</td>
<td>-3.87</td>
</tr>
<tr>
<td>Education expenditure ratio</td>
<td>0.0604</td>
<td>0.1300</td>
<td>0.48</td>
</tr>
<tr>
<td>Investment ratio</td>
<td>0.0756</td>
<td>0.0220</td>
<td>3.50</td>
</tr>
<tr>
<td>Government consumption</td>
<td>-0.0523</td>
<td>0.0260</td>
<td>-2.01</td>
</tr>
<tr>
<td>Black market premium</td>
<td>-0.0531</td>
<td>0.0078</td>
<td>-6.79</td>
</tr>
<tr>
<td>Political instability index</td>
<td>-0.0275</td>
<td>0.0100</td>
<td>-2.73</td>
</tr>
<tr>
<td>Growth rate terms of trade</td>
<td>0.1430</td>
<td>0.0370</td>
<td>3.85</td>
</tr>
</tbody>
</table>

R\textsuperscript{2} 0.5751 0.5902
Number of countries 79 94

Correlation Matrix (%) 100.00 2.04 2.04 100.00

Table 5.2 Growth of real per capita GDP; five years phase average.\(^8\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.0176</td>
<td>4.0E-04</td>
</tr>
<tr>
<td>Male secondary education</td>
<td>0.0115</td>
<td>8.0E-04</td>
</tr>
<tr>
<td>Female secondary education</td>
<td>-0.0038</td>
<td>1.1E-03</td>
</tr>
<tr>
<td>Male higher education</td>
<td>0.0210</td>
<td>1.3E-02</td>
</tr>
<tr>
<td>Female higher education</td>
<td>-0.0301</td>
<td>1.9E-02</td>
</tr>
<tr>
<td>Log(life expectancy)</td>
<td>0.0371</td>
<td>9.9E-03</td>
</tr>
<tr>
<td>Log(GDP)*human capital</td>
<td>-0.4010</td>
<td>6.6E-01</td>
</tr>
<tr>
<td>Education expenditure ratio</td>
<td>-0.0860</td>
<td>5.2E-01</td>
</tr>
<tr>
<td>Investment ratio</td>
<td>0.0861</td>
<td>1.6E-02</td>
</tr>
<tr>
<td>Government consumption</td>
<td>-0.0324</td>
<td>2.4E-02</td>
</tr>
<tr>
<td>Black market premium</td>
<td>-0.0406</td>
<td>1.4E-03</td>
</tr>
<tr>
<td>Political instability index</td>
<td>-0.0116</td>
<td>2.8E-03</td>
</tr>
<tr>
<td>Growth rate terms of trade</td>
<td>0.0740</td>
<td>2.0E-02</td>
</tr>
</tbody>
</table>

| GLS (transformed variables)              |                     |             |
| Residual sum of squared deviation        | R\(^2\) N           |
| Period                                   | 0.2430              | 0.0240      | 0.3127      | 422 |
| 60-65                                    | 0.0214              | 0.0207      | 0.2341      | 50  |
| 65-70                                    | 0.0346              | 0.0227      | 0.2255      | 67  |
| 70-75                                    | 0.0415              | 0.0240      | 0.3360      | 72  |
| 75-80                                    | 0.0745              | 0.0294      | 0.3079      | 86  |
| 80-85                                    | 0.0487              | 0.0238      | 0.3407      | 86  |
| 85-90                                    | 0.0223              | 0.0191      | 0.3965      | 61  |
| R\(^2\) original dependent variable = 0.2914 |

| OLS                                      |                     |             |
| Residual sum of squared deviation        | R\(^2\) N           |
| Period                                   | 0.2420              | 0.0239      | 0.2949      |
| 60-65                                    | 0.0204              | 0.0202      | 0.2335      |
| 65-70                                    | 0.0359              | 0.0231      | 0.2105      |
| 70-75                                    | 0.0414              | 0.0240      | 0.3198      |
| 75-80                                    | 0.0732              | 0.0292      | 0.2895      |
| 80-85                                    | 0.0489              | 0.0239      | 0.3123      |
| 85-90                                    | 0.0221              | 0.0190      | 0.3861      |

Given the analysis before the best choice is to maintain the maximum number of observations, specifying the dynamic structure of the error. In our case this becomes a GLS procedure with auto-correlation in the error term. The periods considered are six, of five years of growth each: 1960 to 1965, 1965 to 1970 and so on up to 1990. GDP, education and life expectancy are measured at the beginning of the period\(^9\). Public

---

\(^8\) See Footnote for table 5.1. GLS applied over six five-year periods from 1960 to 1990 allowing for correlation over time. OLS results correspond to the first stage.

\(^9\) For the life-expectancy variable, Barro and Sala-i-Martin (1995) use the variable averaged over the five years prior to the start of each decade. They indicate that the result are essentially the same if the initial
spending on education, investment ratio, government consumption, and political instability are used as the average over the corresponding five-year period. The same applies for the variable black-market premium; a proxy of government distortions of markets. Finally, the variable terms-of-trade is also the average over the five-year period; this variable is often considered exogenous to the growth process. It is over the investment and government decision variables as well as the political instability where we think there could be some bias in the estimation with phase average. For example, Barro and Sala-i-Martin (op. cit.) make the following inference: "[...] one-standard-deviation increase in [government expenditure] (by 6.5 percentage points for 1965-75 period) is associated with a fall in the growth rate by 0.7 percentage points per year" (p. 434). After disaggregating over time this effect becomes non-significant and amount to only 0.2 per cent per year. We pointed out earlier that, given our dynamic specification, the parameter at the yearly level cannot be inferred from the phase-average estimation. Therefore, an increase in government expenditure cannot be proved to produce a detrimental effect of neither 0.7 nor 0.2 to growth every year. A second important change in the estimation is the size of the parameter on political instability. According to the results of Barro and Sala-i-Martin an increase in 0.12 in the 1965-75 period (a one-standard-deviation increase in political instability) in the index of political instability lowers the growth rate by 0.4 percentage points. However, according to our estimation, this would not amount to more than 0.14

Using the model described in section 2.b., the previous result is not incompatible with either a system of structural yearly parameters: \( \beta_1=99.97 \) per cent, \( \beta_2=-3.38 \) per cent, \( \lambda_1=0.122 \), \( \lambda_2=-0.83 \) or with a system of structural parameters: \( \beta_1=99.48 \) per cent, \( \beta_2=0.11 \) value at each decade is used.
per cent, $\lambda_1=0.414$, $\lambda_2=1.02^{10}$. In the first case, the convergence parameter is almost nil and the weakly exogenous factor $z_t$ has a negative effect in growth. In the second case, countries converge slowly and financial size has a positive almost zero effect in growth. In any case, the size of the effect of financial sector is smaller than originally suggested by a regression that uses data aggregated over time.

Now including liquid-liabilities to GDP ratio, the measurement of financial size used in Barro and Sala-i-Martin. (op. cit.), we obtain the estimates in Table 5.3 below. These estimates are equally compatible with the two examples introduced in the above paragraph. Using these examples, at an annual level the financial size could have a positive or negative effect on growth. Moreover, the rate of convergence between countries could be almost zero.

---

90 For the first case, the system is: $y_t=0.9997y_{t-1}-0.0338z_t$ and $z_t=0.122y_{t-1}-0.83z_{t-1}$. Therefore, the growth is: $\Delta y_t=-0.0003 y_{t-1}+0.0338 z_t$. In the second case, the system becomes: $y_t=0.9948y_{t-1}+0.0011z_t$ and $z_t=0.414y_{t-1}+1.02z_{t-1}$. Therefore, the growth is: $\Delta y_t=-0.0052 y_{t-1}+0.0011z_t$. 
Table 5.3 Growth of real per capita GDP; five years phase average.\footnote{1}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.0184</td>
<td>4.0E-04</td>
</tr>
<tr>
<td>Male secondary education</td>
<td>0.0111</td>
<td>8.0E-04</td>
</tr>
<tr>
<td>Female secondary education</td>
<td>-0.0032</td>
<td>1.1E-03</td>
</tr>
<tr>
<td>Male higher education</td>
<td>0.0149</td>
<td>1.3E-02</td>
</tr>
<tr>
<td>Female higher education</td>
<td>-0.0196</td>
<td>1.9E-02</td>
</tr>
<tr>
<td>Log(life expectancy)</td>
<td>0.0351</td>
<td>1.0E-02</td>
</tr>
<tr>
<td>Log(GDP)*human capital</td>
<td>-0.4240</td>
<td>7.1E-01</td>
</tr>
<tr>
<td>Education expenditure ratio</td>
<td>-0.0769</td>
<td>5.2E-01</td>
</tr>
<tr>
<td>Investment ratio</td>
<td>0.0771</td>
<td>1.7E-02</td>
</tr>
<tr>
<td>Government consumption</td>
<td>-0.0357</td>
<td>2.4E-02</td>
</tr>
<tr>
<td>Black market premium</td>
<td>-0.0404</td>
<td>1.4E-03</td>
</tr>
<tr>
<td>Political instability index</td>
<td>-0.0114</td>
<td>2.8E-03</td>
</tr>
<tr>
<td>Growth rate terms of trade</td>
<td>0.0752</td>
<td>2.0E-02</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>0.0097</td>
<td>1.7E-03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GLS (transformed variables)</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>( R^2 )</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>0.2420</td>
<td>0.0240</td>
<td>0.3152</td>
<td>422</td>
</tr>
<tr>
<td>60-65</td>
<td>0.0214</td>
<td>0.0207</td>
<td>0.2327</td>
<td>50</td>
</tr>
<tr>
<td>65-70</td>
<td>0.0349</td>
<td>0.0228</td>
<td>0.2184</td>
<td>67</td>
</tr>
<tr>
<td>70-75</td>
<td>0.0417</td>
<td>0.0241</td>
<td>0.3337</td>
<td>72</td>
</tr>
<tr>
<td>75-80</td>
<td>0.0734</td>
<td>0.0292</td>
<td>0.3183</td>
<td>86</td>
</tr>
<tr>
<td>80-85</td>
<td>0.0489</td>
<td>0.0239</td>
<td>0.3364</td>
<td>86</td>
</tr>
<tr>
<td>85-90</td>
<td>0.0217</td>
<td>0.0189</td>
<td>0.4118</td>
<td>61</td>
</tr>
<tr>
<td>( R^2 ) original dependent variable =0.2941</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>OLS</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>0.2410</td>
<td>0.0239</td>
<td>0.2977</td>
</tr>
<tr>
<td>60-65</td>
<td>0.0203</td>
<td>0.0202</td>
<td>0.2345</td>
</tr>
<tr>
<td>65-70</td>
<td>0.0361</td>
<td>0.0232</td>
<td>0.2046</td>
</tr>
<tr>
<td>70-75</td>
<td>0.0416</td>
<td>0.0240</td>
<td>0.3173</td>
</tr>
<tr>
<td>75-80</td>
<td>0.0721</td>
<td>0.0290</td>
<td>0.3002</td>
</tr>
<tr>
<td>80-85</td>
<td>0.0492</td>
<td>0.0239</td>
<td>0.3087</td>
</tr>
<tr>
<td>85-90</td>
<td>0.0216</td>
<td>0.0188</td>
<td>0.4001</td>
</tr>
</tbody>
</table>

A correct technique to estimate the model above would be to use instrumental variables. We do this in Table 5.4, using instruments for the variables: public spending in education, investment ratio, government consumption, political instability and liquid liabilities ratio. In each case the instrument is the value of the variable over the corresponding five previous years.

\footnote{1}{See Footnote for tables 5.1 and 5.2. Liquid liabilities correspond to the ratio of liquid liabilities to GDP. Source: Barro and Lee (1994).}
Table 5.4 Instrumental variables for growth of real per capita GDP.  

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.0193</td>
<td>4.0E-04</td>
</tr>
<tr>
<td>Male secondary education</td>
<td>0.0136</td>
<td>9.0E-04</td>
</tr>
<tr>
<td>Female secondary education</td>
<td>-0.0065</td>
<td>1.2E-03</td>
</tr>
<tr>
<td>Male higher education</td>
<td>0.0031</td>
<td>1.4E-02</td>
</tr>
<tr>
<td>Female higher education</td>
<td>-0.0031</td>
<td>2.1E-02</td>
</tr>
<tr>
<td>Log(life expectancy)</td>
<td>0.0316</td>
<td>6.7E-03</td>
</tr>
<tr>
<td>Log(GDP)*human capital</td>
<td>-0.4670</td>
<td>8.9E-01</td>
</tr>
<tr>
<td>Education expenditure ratio</td>
<td>0.1082</td>
<td>9.3E-01</td>
</tr>
<tr>
<td>Investment ratio</td>
<td>0.0599</td>
<td>3.4E-02</td>
</tr>
<tr>
<td>Government consumption</td>
<td>-0.0611</td>
<td>5.3E-02</td>
</tr>
<tr>
<td>Black market premium</td>
<td>-0.0433</td>
<td>2.8E-03</td>
</tr>
<tr>
<td>Political instability index</td>
<td>-0.0031</td>
<td>1.3E-02</td>
</tr>
<tr>
<td>Growth rate terms of trade</td>
<td>0.0667</td>
<td>2.2E-02</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>0.0180</td>
<td>2.5E-03</td>
</tr>
</tbody>
</table>

GLS (transformed variables)  

<table>
<thead>
<tr>
<th>Period</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>R²</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>0.2390</td>
<td>0.0293</td>
<td>0.2666</td>
<td>372</td>
</tr>
<tr>
<td>60-65</td>
<td>0.0357</td>
<td>0.0231</td>
<td>0.1907</td>
<td>67</td>
</tr>
<tr>
<td>65-70</td>
<td>0.0457</td>
<td>0.0252</td>
<td>0.2770</td>
<td>72</td>
</tr>
<tr>
<td>75-80</td>
<td>0.0829</td>
<td>0.0310</td>
<td>0.2271</td>
<td>86</td>
</tr>
<tr>
<td>80-85</td>
<td>0.0511</td>
<td>0.0244</td>
<td>0.3080</td>
<td>86</td>
</tr>
<tr>
<td>85-90</td>
<td>0.0236</td>
<td>0.0197</td>
<td>0.3701</td>
<td>61</td>
</tr>
</tbody>
</table>

R² original dependent variable = 0.2443

OLS  

<table>
<thead>
<tr>
<th>Period</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>R²</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
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<td>0.0253</td>
<td>0.2456</td>
<td></td>
</tr>
<tr>
<td>60-65</td>
<td>0.0369</td>
<td>0.0235</td>
<td>0.1882</td>
<td></td>
</tr>
<tr>
<td>65-70</td>
<td>0.0450</td>
<td>0.0250</td>
<td>0.2615</td>
<td></td>
</tr>
<tr>
<td>75-80</td>
<td>0.0825</td>
<td>0.0310</td>
<td>0.1991</td>
<td></td>
</tr>
<tr>
<td>80-85</td>
<td>0.0513</td>
<td>0.0244</td>
<td>0.2767</td>
<td></td>
</tr>
<tr>
<td>85-90</td>
<td>0.0229</td>
<td>0.0194</td>
<td>0.3633</td>
<td></td>
</tr>
</tbody>
</table>

The main differences between this estimation and Barro's are the size of the political instability parameter, which is not significantly detrimental to growth anymore, and that government consumption is not negatively related to growth. In addition, convergence is slower than the one reported by Barro. Using the parameters from Table 5.4 we conclude

---

92 See Footnote for tables 5.1, 5.2 and 5.3. Instruments are value at the beginning of the period for: public spending on education, investment ratio, government consumption, political instability, black market premium, terms of trade and ratio of liquid liabilities to GDP.
that convergence occurs at the rate of 2.03 per cent per year which is 1 per cent slower than the rate found in previous work\textsuperscript{93}.

\textbf{Table 5.5 Growth of real per capita GDP, using instrumental variables and twenty years phase average.\textsuperscript{94}}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.3385</td>
<td>1.7E-01</td>
<td>-1.99</td>
</tr>
<tr>
<td>Male secondary education</td>
<td>-0.0310</td>
<td>1.1E-02</td>
<td>-2.71</td>
</tr>
<tr>
<td>Female secondary education</td>
<td>0.0110</td>
<td>7.0E-03</td>
<td>1.54</td>
</tr>
<tr>
<td>Male higher education</td>
<td>-0.0088</td>
<td>7.5E-03</td>
<td>-1.16</td>
</tr>
<tr>
<td>Female higher education</td>
<td>0.0004</td>
<td>2.5E-02</td>
<td>0.02</td>
</tr>
<tr>
<td>Log(life expectancy)</td>
<td>0.0138</td>
<td>3.1E-02</td>
<td>0.44</td>
</tr>
<tr>
<td>Log(GDP)*human capital</td>
<td>0.1340</td>
<td>4.8E-02</td>
<td>2.82</td>
</tr>
<tr>
<td>Education expenditure ratio</td>
<td>0.5895</td>
<td>9.4E-01</td>
<td>0.63</td>
</tr>
<tr>
<td>Investment ratio</td>
<td>0.1078</td>
<td>1.1E-01</td>
<td>1.02</td>
</tr>
<tr>
<td>Government consumption</td>
<td>-0.1009</td>
<td>6.3E-02</td>
<td>-1.61</td>
</tr>
<tr>
<td>Black market premium</td>
<td>0.0262</td>
<td>6.7E-02</td>
<td>0.39</td>
</tr>
<tr>
<td>Political instability index</td>
<td>0.0540</td>
<td>1.8E-01</td>
<td>0.30</td>
</tr>
<tr>
<td>Growth rate terms of trade</td>
<td>0.0610</td>
<td>8.5E-02</td>
<td>0.72</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>0.0148</td>
<td>1.5E-02</td>
<td>1.02</td>
</tr>
</tbody>
</table>

The effect of averaging through the twenty year period is as expected. In the first place, convergence seems to be much faster at 4.3 per cent a year. In the second place, the estimated parameters for the weakly exogenous variables tend to have large variances and in consequence smaller t values. Finally, the effect of liquid liabilities decreases as well as the effect of political instability, becoming a positive effect on growth, while the negative effect of government consumption increase, however is still non-significant.

\textsuperscript{93} Barro estimates 3 per cent convergence per year. Our results are calculated using a formula from Barro and Sala-i-Martin (1995) Chapter 12, footnote 4, page 422.

\textsuperscript{94} See Footnote for tables 5.1, 5.2, 5.3 and 5.4. Cross-section of 47 countries where the only observation over time is the phase average from 1965 to 1985.
The second empirical study we reassess is the extensive work to measure the effect of financial sector on growth by King and Levine (1993). The authors summarise the data studying only one cross section of the growth through 30 years. The growth is considered over the period 1960 to 1989 for four different measurements, namely: real per capita GDP growth rate, growth in capital stock, investment share of GDP and, finally, measure of efficiency growth equal to real GDP growth discounted from 0.3 of the growth in capital stock. Their results support the hypothesis that the size of the financial sector is positively correlated with growth. The result is also robust to several specifications they apply. In general the specifications consist of a convergence-type model, including measures of education, 'openness', inflation and government activities as explanatory variables. We present the results in table 5.6 below.

We closely reproduce the regression where growth over 1960 to 1989 is explained by the per capita GDP at the beginning of the period, secondary school enrolment in 1960 and averages over the 30-year period of: government consumption, annual inflation rate, total trade and liquid liabilities to GDP ratio. The only significant parameters are the liquid-liabilities ratio and secondary school enrolment, while the implied rate of convergence is rather slow: 0.28 per cent per annum.
Table 5.6 King-Levine regression for growth rate of real per capita GDP.\textsuperscript{95}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>0.670</td>
<td>0.550</td>
<td>1.22</td>
</tr>
<tr>
<td>Secondary school enrolment in 1960</td>
<td>-0.270</td>
<td>0.250</td>
<td>-1.08</td>
</tr>
<tr>
<td>Government consumption ratio</td>
<td>2.590</td>
<td>1.080</td>
<td>2.40</td>
</tr>
<tr>
<td>Average annual inflation</td>
<td>-3.690</td>
<td>3.540</td>
<td>-1.04</td>
</tr>
<tr>
<td>Imports plus exports ratio</td>
<td>-0.001</td>
<td>0.002</td>
<td>-0.57</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>3.020</td>
<td>0.720</td>
<td>4.18</td>
</tr>
<tr>
<td>R\textsuperscript{2}</td>
<td>0.365</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>88</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Global adjustment F test</td>
<td>8.412</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residual standard deviation</td>
<td>1.330</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residual sum of squares</td>
<td>155.570</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

However using the disaggregated data and estimating over a panel of six time periods (1960 to 1965, 1965 to 1979, etc. until 1985 to 1990), and the cross section of countries, the results appear to be very different as presented in table 5.7. In the first place, the lagged value of per capita GDP estimates a positive value, implying a result of divergence across countries (-0.14 per cent per year). The variable, secondary education enrolment appears much less important than in King and Levine's regression and non-significant, while government and trade seem to have a stronger effect. The size of the financial sector is almost significant at the 5% level but decreases in size, with respect to the single cross

\textsuperscript{95} The data is taken from King and Levine (1994) Data Set "Finance, Entrepreneurship, and Economic Growth Data", World Bank. The sample size in our case is larger than the one reported on King and Levine (1993), seven extra observations are included in our sample. In addition, the initial per capita GDP in our estimations differs from the total initial GDP used by King and Levine (1993). These two differences account for the small discrepancies between our results and theirs. Notice that the dependent variable has been scaled by 100 and thus all the parameters are 100 time those in King and Levine (1993). Regressions estimated by OLS using one time period corresponding to the growth 1960 to 1989. Liquid liabilities are defined as M3, money: coins and notes in circulation plus all other private-sector bank deposits and certificates of deposit (Buttler and Isaacs (1996)); these are measured at the beginning of each period. Liquid liabilities corresponds to lines 34+35 in the IMF's International Finance Statistics. Liquid liabilities output ratio is defined as liquid liabilities divided by real GDP in the same period. Real GDP is taken from Penn World Table mark 5.

226
Table 5.7 Growth of real per capita GDP; five years phase average.\(^6\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>0.145</td>
<td>0.226</td>
</tr>
<tr>
<td>Secondary school enrolment</td>
<td>1.309</td>
<td>0.879</td>
</tr>
<tr>
<td>Government consumption ratio</td>
<td>-9.484</td>
<td>2.636</td>
</tr>
<tr>
<td>Average annual inflation</td>
<td>-0.005</td>
<td>0.002</td>
</tr>
<tr>
<td>Imports plus exports ratio</td>
<td>1.206</td>
<td>0.439</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>1.238</td>
<td>0.646</td>
</tr>
</tbody>
</table>

GLS (transformed variables)  

<table>
<thead>
<tr>
<th>Period</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>(R^2)</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>3510.20</td>
<td>2.5982</td>
<td>0.0800</td>
<td>520</td>
</tr>
<tr>
<td>60-65</td>
<td>226.59</td>
<td>1.9117</td>
<td>0.0337</td>
<td>62</td>
</tr>
<tr>
<td>65-70</td>
<td>366.66</td>
<td>2.0892</td>
<td>0.1133</td>
<td>84</td>
</tr>
<tr>
<td>70-75</td>
<td>772.69</td>
<td>2.8370</td>
<td>0.0904</td>
<td>96</td>
</tr>
<tr>
<td>75-80</td>
<td>790.83</td>
<td>2.8852</td>
<td>0.0481</td>
<td>95</td>
</tr>
<tr>
<td>80-85</td>
<td>923.90</td>
<td>3.0396</td>
<td>0.0439</td>
<td>100</td>
</tr>
<tr>
<td>85-90</td>
<td>429.59</td>
<td>2.2750</td>
<td>0.1750</td>
<td>83</td>
</tr>
</tbody>
</table>

OLS

<table>
<thead>
<tr>
<th>Period</th>
<th>Residual sum of squared</th>
<th>Residual standard deviation</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>3454.5</td>
<td>2.5775</td>
<td>0.1155</td>
</tr>
<tr>
<td>60-65</td>
<td>252.09</td>
<td>2.0164</td>
<td>0.0723</td>
</tr>
<tr>
<td>65-70</td>
<td>383.71</td>
<td>2.1373</td>
<td>0.1466</td>
</tr>
<tr>
<td>70-75</td>
<td>724.83</td>
<td>2.7478</td>
<td>0.1253</td>
</tr>
<tr>
<td>75-80</td>
<td>766.08</td>
<td>2.8397</td>
<td>0.0854</td>
</tr>
<tr>
<td>80-85</td>
<td>896.49</td>
<td>2.9941</td>
<td>0.0786</td>
</tr>
<tr>
<td>85-90</td>
<td>431.29</td>
<td>2.2795</td>
<td>0.2085</td>
</tr>
</tbody>
</table>

The overall conclusion of this exercise is that King and Levine (1993) results are not robust to changes on the size of the phase average.

Table 5.8 presents the same model when correctly estimated using instrumental variables. The estimates in this case should not be biased, however the size of the parameters do not imply properties over the annual values, and the estimates are less efficient than those based in a lagged model or the estimates with the annual frequencies.

\(^{6}\) See Footnote for table 5.6. GLS applied over six five-year periods from 1960 to 1990 allowing for correlation over time. OLS results correspond to the first stage.
Table 5.8 Instrumental variables for growth of real per capita GDP.\textsuperscript{97}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter</th>
<th>Standard deviation</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(initial GDP)</td>
<td>-0.097</td>
<td>0.281</td>
<td>-0.35</td>
</tr>
<tr>
<td>Secondary school enrolment</td>
<td>0.856</td>
<td>1.056</td>
<td>0.81</td>
</tr>
<tr>
<td>Government consumption ratio</td>
<td>-3.838</td>
<td>3.651</td>
<td>-1.05</td>
</tr>
<tr>
<td>Average annual inflation</td>
<td>0.008</td>
<td>0.006</td>
<td>1.31</td>
</tr>
<tr>
<td>Imports plus exports ratio</td>
<td>0.533</td>
<td>0.587</td>
<td>0.91</td>
</tr>
<tr>
<td>Liquid liabilities ratio</td>
<td>3.290</td>
<td>0.905</td>
<td>3.64</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GLS (transformed variables)</th>
<th>Residual sum of squared deviation</th>
<th>Residual standard deviation</th>
<th>R\textsuperscript{2}</th>
<th>N</th>
<th>OLS Residual sum of squared deviation</th>
<th>Residual standard deviation</th>
<th>R\textsuperscript{2}</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>Total 2862.1</td>
<td>2.6884</td>
<td>0.0712</td>
<td>396</td>
<td>2836.6</td>
<td>2.6764</td>
<td>0.0955</td>
<td>---</td>
</tr>
<tr>
<td>60-65</td>
<td>249.02</td>
<td>2.1087</td>
<td>0.1107</td>
<td>56</td>
<td>252.82</td>
<td>2.1248</td>
<td>0.1342</td>
<td>---</td>
</tr>
<tr>
<td>70-75</td>
<td>712.32</td>
<td>2.9295</td>
<td>0.0471</td>
<td>83</td>
<td>688.18</td>
<td>2.8795</td>
<td>0.0589</td>
<td>---</td>
</tr>
<tr>
<td>75-80</td>
<td>652.46</td>
<td>2.7706</td>
<td>0.0842</td>
<td>85</td>
<td>633.07</td>
<td>2.7291</td>
<td>0.1113</td>
<td>---</td>
</tr>
<tr>
<td>80-85</td>
<td>829.29</td>
<td>3.0023</td>
<td>0.0431</td>
<td>92</td>
<td>827.59</td>
<td>2.9993</td>
<td>0.0705</td>
<td>---</td>
</tr>
<tr>
<td>85-90</td>
<td>419.04</td>
<td>2.2887</td>
<td>0.1180</td>
<td>80</td>
<td>434.93</td>
<td>2.3317</td>
<td>0.1477</td>
<td>---</td>
</tr>
</tbody>
</table>

We used as instruments, in Table 5.8, for the variables: government consumption ratio, average annual inflation, imports plus exports ratio, liquid liabilities ratio, the average in the previous five years. The implied convergence estimate is 0.10 per cent per annum.

Again, government consumption does not appear to be important throughout different levels of aggregation over time. However, financial size appears to be the only factor guiding growth from the original Levine-King model. Surprisingly, the average level of inflation has now a high positive effect on growth. Although the estimate is non-significant, if South-American countries had had an inflation rate of 3 per cent on average, instead of the actual 141 per cent, over the period, the parameter obtained implies for that hypothetical situation that the growth rate of the economy had been 1.16 per cent slower

\textsuperscript{97} See Footnote for tables 5.6 and 5.3. Instruments are the value at the beginning of the period for: government consumption ratio, average annual inflation, imports plus exports ratio and liquid liabilities ratio.
in average.

**b.- Results for the South-American southern-cone data.**

The second part of the empirical work in this chapter studies a data set from six South-American countries from the southernmost region: Argentina, Brasil, Bolivia, Chile, Paraguay and Uruguay. The data includes observations from 1960 to 1994 and the variables considered are per-capita GDP growth, domestic private credit and initial GDP level in international dollars. Table 5.9 below presents the averages of the variable over the sample period by country.

<table>
<thead>
<tr>
<th></th>
<th>Private credit</th>
<th>per capita GDP growth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average (%)</td>
<td>Standard deviation (%)</td>
</tr>
<tr>
<td>Argentina</td>
<td>11.09</td>
<td>15.80</td>
</tr>
<tr>
<td>Bolivia</td>
<td>13.54</td>
<td>11.77</td>
</tr>
<tr>
<td>Brazil</td>
<td>27.44</td>
<td>14.98</td>
</tr>
<tr>
<td>Chile</td>
<td>32.09</td>
<td>24.90</td>
</tr>
<tr>
<td>Paraguay</td>
<td>12.75</td>
<td>2.86</td>
</tr>
<tr>
<td>Uruguay</td>
<td>34.82</td>
<td>17.27</td>
</tr>
</tbody>
</table>

Given the number of observations available the phase-average procedure can be performed with 34 the possible different sizes. There are also many ways of choosing the sample after phase averaging. For example, if the phase average is two observations, we can select the

---

98 The data is taken from World Development Indicators, World Bank (1996). The data for GDP per capita growth is measured in constant local currency and the initial level of GDP is considered as the natural logarithm of per capita GDP measured in constant 1987 US$. Domestic private credit corresponds
observations made out of one odd year and the following, or we can select the averages made of one even year and the following. These different choices are illustrated in figure 5.3 below.

Figure 5.3. Generation and selection of sample in phase average.

\[
\begin{array}{ccccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

It is not always clear what is the right sample to choose. For example, for the Latin-American database under study and in the case of a phase average of size 4, we can form four possible sets of observations: two of them allow for 48 observations, one has a sample size of 45 and one of them includes 42 observations. If we were choosing the maximum coverage of the sample we still would not be able to decide for either of the two sets of 48 observations.

The following estimations include all the possible subsets of observations for each phase-average size. When no phase-average is done we have only one possible group of observations, the whole set. For a phase-average of two we can do two experiments for each country. For an average of three observations we experiment with the three possible sets. The maximum amount of subgroups occurs when the phase average is 17; in that case the number of possible subsets is 17 per country. Thereafter, for a phase average of size 18 we have only 16 candidates. The possible combinations decrease one by one until
to domestic credit minus claims on central and local government as a percentage of GDP.
there is again a unique subset; this is when the phase average is done with the 34 observations.

We produce below estimates for three estimation methods where the explanatory variable is private credit: phase average model by simple OLS; instrumental variable for the phase average model; and $k$-lagged model where the explanatory variable is the previous level of private credit.

Figures 5.4 to 5.8 show the estimation results for the three different estimation methods. Figure 5.4 below shows the result of estimating the parameter $\beta_1$ in the simple phase-average model estimated by OLS. This model connects GDP growth with the previous level of output and current level of private credit. As discussed in section 2.a., the estimation by simple OLS of this model is biased and if we have positive feedback from finance to growth and vice-versa the observed pattern would be as illustrated in this figure. Initially, the parameter is positive but non-significant, becoming larger and more significant as we increase the size of the phase average. For our data, when $k$ takes values larger than 30, the bias tends to decrease and the estimates are not significantly different from zero.

Figure 5.5 and 5.6 below illustrates the results in proposition 3, where we established that estimating the phase average model via instrumental variables is less efficient than using the lagged model, and that this inefficiency grows with $k$. From the data the size of the variance appears to decrease again for very large values of $k$. This appears to be true for both the lagged and the phase average model. From figure 5.6 we can see that the size of the parameter does not change as much in the lagged model as it does for the phase average model using instrumental variables. Instrumental variable estimation produces
parameters that are non-significant but very large in size and variance.

Figures 5.7 and 5.8 present all the possible lower bounds of the 95% confidence intervals built around the instrumental variable phase-average estimation and the $k$-lagged model estimation. These confidence intervals are much wider when $k$ is around ten to twenty observations. In addition, the confidence intervals for the phase average model using instrumental variables are much wider than the intervals for the lagged model. Instrumental variables can give in some cases and interval 80 times wider than the lagged model.

An additional problem common to all of these methods is that when using averages of about 17 observations, there are around 17 sub-samples to choose. Each of these sub-samples produce very different results. We observe this effect in all the figures from 5.4 to 5.8.

In summary, these data confirms that phase-average tend to overestimate the size of financial measurement and convergence. The lagged model produce more efficient estimates and it is less affected by the size of the aggregation over time. The conclusions based on averages over the whole sample are affected by the procedure and the most appropriate method would be using yearly data.
Figure 5.4 Simple OLS parameter for private credit for different phase-average size.

Data from World Development Indicators, World Bank (1996). See footnote for table 5.9. Private credit averaged over the $k$ observations, average GDP growth over the
Figure 5.5 Instrumental variables parameter for private credit for different phase-average size.

Parameter on Private Credit

period. Initial level of GDP measure at the beginning of the period.
Figure 5.6 Parameter for private credit for different lags, k-lagged model by OLS.\textsuperscript{101}

\textsuperscript{100} See footnote for figure 5.4.
\textsuperscript{101} See footnote for figure 5.4.
Figure 5.7 Instrumental variables confidence interval for private credit parameter for different phase-average size. \(^{102}\)

\[ k, \text{ Size of the average} \]

Lower bound of a 95% confidence interval

\(^{102}\) See footnote for figure 5.4.
Figure 5.8 Confidence interval for private credit parameter for different lags, $k$-lagged model by OLS.\textsuperscript{103}

\textsuperscript{103} See footnote for figure 5.4.
4.- Concluding Remarks

We have show that, in the absence of any known desirable properties from time aggregation, the most adequate procedure to test the effect of a weakly exogenous variable entails using the data in the most disaggregated form possible. Any data averaging over time implies perverse effects over the estimated parameters. Firstly, OLS is increasingly biased when the size of the phase-average increases. Correcting phase-average biases using instrumental variable estimation is increasingly inefficient with the size $k$. Without an explicit average procedure, that clearly reduces measurement errors, there is no reason to average the explanatory and dependent variables over periods of time. In addition, there are no reasons to drop the observations in the interior of the time sample and work with the rate of growth between both ends of the period.

Most international panels of data allow for a maximum disaggregation down to five-year averages, particularly because some educational attainment measurements and life expectancy are mainly available at that frequency. Future directions in this research, and following the time disaggregation recommendation, are the study other approaches to maximise the use of these five-year periods data sets. Dropping observations is only one of the possible solutions to the problem of missing observations. At least two different solutions can be implemented in the context of OLS estimation; on the one hand, as shown in Little (1992) or Little and Rubin (1987), OLS can be fully efficient with missing observations. On the other hand, five-year average data could be used as proxies to the corresponding annual frequencies (see Afifi and Clark (1996)).
The present study sought to establish how the financial system develops and what are its links with overall economic growth. We gave a particular regard to this process in Latin America. Working in a framework in which real and financial wealth evolve in a simultaneous system we reassessed the existence of some links as it has been recently suggested.

Gurley and Shaw (1955 and 1960) first observed that finance and growth was a simultaneous system. More recently, this hypothesis has been also sustained by for example: Townsend (1983); Greenwood and Jovanovic (1990); Sussman (1996); Santomero (1997); Greenwood and Smith (1997 and 1998) and Bencivenga and Smith (1998). Consistent with this hypothesis is the alleged beneficial effects that a larger and more developed financial system has over economic growth. Greenwood and Jovanovic (1990) introduced an explicit model for such relationship and Merton and Bodie (1995) among others have hypothesised a number of mechanisms though which these beneficial effects could be conveyed.

Empirical research has attempted to determine the extent of the beneficial effect of the financial system on economic growth. The availability of time series for an increasing
number of countries has allowed empirical studies to analyse this relationship around the world and use this hypothesis as one of the conditioning variables in growth regressions. Most notably the empirical work by King and Levine (1993) and Barro and Sala-i-Martin (1995) that find strong finance growth links and which has been confirmed in more recent empirical work by: Arestis and Demetriades (1997), Harris (1997) and Arestis, Demetriades, Luintel (2001).

In terms of reassessing the empirical evidence the present study identified two ways in which such assessments can be improved. Keeping in mind that the hypothesis that deeper finance is beneficial for growth should be fully integrated with a simultaneous system of financial and economic growth, we first study the properties of causality tests under such scheme. We find a substantial amount short sample biases and identify an $O(T^{-1})$ formula to measure such biases. Among other effects the formula shows that the extent of the bias is higher when the evolution of the financial depth variable depends heavily on growth and not on other independent shocks. A financial measurement will tend to show a significant estimate when in fact there is not such causality relationship. The formula also indicates that bias is increased if economic growth variable has roots too close to the unity. In addition, and using simulation techniques we show that the sizes of the test for Granger causality do not match the nominal sizes. We conclude that roughly for 5% confidence in the test, a 1% rather than 5% critical value would produce a more reliable analysis.

Using a Latin American and Caribbean database and the results above, we study causality links between six financial measurement related to bank lending and growth. We find little evidence of any causality neither on the level of GDP or GDP growth.
In a second approach we study growth regressions and in particular the properties of time aggregation under the simultaneous model of financial and real wealth growth. In particular, we study the effects of phase average, as defined by Campos et al. (1990) in growth regressions. The study concentrates on the consequences of time aggregation on the size of the estimated parameters and the extent of bias on them. We find that depending on the dynamic structure of the process generating growth and the financial variable regression problems related to serial-correlation can be increased by phase averaging. In general, we find that the use of longer phase averages produce increased distortions on the value of the estimated parameter in regressions where the right hand side variables are not strictly exogenous. We also study the consequences of time aggregation when instrumental variable least squares is used as a remedial procedure for endogeneity. We find evidence indicating that data should be used in the most disaggregate way as possible.

We reassess the empirical evidence from the work of Barro and Sala-i-Martin (1995) and King and Levine (1996) using our phase average model. We time-disaggregate the data from both these studies and find large changes in the estimates under different size of the aggregation, in particular with financial size and political instability variables. Under different phase average size the financial size under study is significant in some cases and non-significant in others. The evidence from this study also suggests that more disaggregated data tend to estimate stronger parameters.

We finally, study all possible levels of time aggregation with data set from six South-American countries from the southernmost region. The data use observations from
1960 to 1994 for per-capita GDP growth and banking related financial measurements. The estimations illustrate all the possible models and results that can arise from different phase averages.

We finally study the effects on asset prices as a stock market develops. Drawing from Townsend's (1983) hypothesis that stock markets and more complex financial institutions emerge from a richer economy, we study the development of a market that allows investors to undo undesired compulsory holdings of assets and other restrictions. Using a CAPM based theory we find that an increased coverage of stock markets, coupled with a decrease in restrictions to investors, will have a positive effect on asset prices for a given level of risk. At the same time this decrease in restrictions will produce decreasing return for given levels of risk.

We study the Chilean stock market over the last 12 years to attempt to identify these effects. We examine price data for around 80 per cent of the shares traded in Chilean markets. The database covers a period from 1989 to 2000 following daily prices for 233 securities. We also study data from eight aggregate market indexes and seven daily lending rates. In a similar way as the Roll critique we find that an exact measurement of market coverage is difficult to obtain. We attempt to measure market coverage using a number of proxies and analyse the connections with asset prices over a number of periods. We do not find strong evidence of such connection from the data.
Appendix 2A

Proof of proposition 2.

The OLS estimation of the model:

\[ Y = \beta Y_i + \gamma X_i + u \]

Lead to the equation:

\[
\begin{bmatrix}
\hat{\beta} - \beta \\
\hat{\gamma} - \gamma
\end{bmatrix} = \begin{bmatrix}
Y_i Y_i & Y_i X_i \\
X_i Y_i & X_i X_i
\end{bmatrix}^{-1} \begin{bmatrix}
Y_i u \\
X_i u
\end{bmatrix}
\]

Using the partitioned matrix inverse formula.

\[
\begin{bmatrix}
Y_i Y_i & Y_i X_i \\
X_i Y_i & X_i X_i
\end{bmatrix}^{-1} = \begin{bmatrix}
(Y_i Y_i)^{-1} (I + Y_i X_i (X_i^2 W X_i)^{-1} Y_i Y_i)^{-1} (Y_i Y_i)^{-1} & -(Y_i Y_i)^{-1} Y_i X_i (X_i^2 W X_i)^{-1} \\
-(X_i^2 W X_i)^{-1} X_i Y_i (Y_i Y_i)^{-1} & (X_i^2 W X_i)^{-1}
\end{bmatrix}
\]

And multiplying the second row:

\[ \hat{\gamma} - \gamma = (X_i^2 W X_i)^{-1} X_i^2 W u \]

Exchanging the roles of \( X_i \) and \( Y_i \) in the partitioned matrix inverse formula.

\[
\begin{bmatrix}
Y_i Y_i & Y_i X_i \\
X_i Y_i & X_i X_i
\end{bmatrix}^{-1} = \begin{bmatrix}
(X_i^2 X_i)^{-1} (I + X_i^2 Y_i (Y_i^2 M Y_i)^{-1} X_i^2 Y_i)^{-1} (X_i^2 X_i)^{-1} & -(X_i^2 X_i)^{-1} X_i^2 Y_i (Y_i^2 M Y_i)^{-1} \\
-(Y_i^2 M Y_i)^{-1} X_i^2 Y_i (X_i^2 X_i)^{-1} & (Y_i^2 M Y_i)^{-1}
\end{bmatrix}
\]

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Now multiplying the first row:

$$\hat{\beta} - \beta = (Y_{\cdot,1}^t MY_{\cdot,1})^{-1} Y_{\cdot,1}^t Mu$$

The formula for the sum of square of residuals:

$$\hat{u}^t \hat{u} = u^t \left( I - [Y_{\cdot,1} X_{\cdot,1}] \begin{bmatrix} Y_{\cdot,1} Y_{\cdot,1} & Y_{\cdot,1} X_{\cdot,1} \\ X_{\cdot,1} Y_{\cdot,1} & X_{\cdot,1} X_{\cdot,1} \end{bmatrix}^t \begin{bmatrix} Y_{\cdot,1} \\ X_{\cdot,1} \end{bmatrix} \right) u$$

Applying the inverse of a partitioned matrix again:

$$\hat{u}^t \hat{u} - u^t u = u^t (I - W + WX(X'WX)^{-1} X'WX) u$$

**Proof of proposition 5.** Let $F$, $G$ and $H$ are general squared matrices of size $T$ and $w$ is a vector of random independent observations from a Normal $(0, \sigma_w^2)$ distribution. Using the fact that $E[u'Fuu'Guu'Hu']$ is scalar:

$$E[u'Fuu'Guu'Hu'] = Tr(E[u'Fuu'Guu'Hu'])$$

$$= Tr(FE[uu'Guu'Hu'])$$

Now we apply the formula for evaluation of expectations of products of two stochastic matrices found in Srivastava and Tiwari (1976) formula 2.10 (p. 137):

$$E[u'Fuu'Guu'Hu'] = \sigma_w^6 Tr(F(Tr(G)(Tr(H)I + H + H')$$

$$+ Tr(GH)I + HG + G'H + Tr(G'H)I + G'H + H'G$$

$$+ Tr(H)G + GH + GH' + Tr(H)G' + HG' + H'G'))$$

After some manipulation we get to the desired result:
Proof of Proposition 7. Let matrix $B$ be defined as above and $F$, $G$ and $H$ general squared matrices of size $T$ and $w$ is a vector of random independent observations from a Normal $(0, \sigma^2_w)$ distribution. Using the fact that $E[u'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u'Fu'u'Gu'u']$ is scalar:

\[
E[u'u'Fu'u'Gu'u'Hu'] = \sigma_w^4 \{ \text{Tr}(F)\text{Tr}(G)\text{Tr}(H) + \text{Tr}(G)\text{Tr}(FH) + \text{Tr}(G)\text{Tr}(FH') \\
+ \text{Tr}(GH)\text{Tr}(F) + \text{Tr}(FG) + \text{Tr}(FH') \\
+ \text{Tr}(G'H)\text{Tr}(F) + \text{Tr}(FG'H) + \text{Tr}(FH'G) \\
+ \text{Tr}(H)\text{Tr}(FG) + \text{Tr}(FG'H) + \text{Tr}(FH'G') \}
\]

Now keeping in mind that $\text{Tr}(B)$ is zero because $B$ has only zeros in its main diagonal and using formula 2.10 from Srivastava and Tiwari (p. 137) we get that:
\[ E[\mu' Bu' Fu' Gu' Hu] = \sigma^8 \{ \text{terms involving } G, H, F \} \]

After collecting terms with common factors from the last expression we obtain a somehow simpler formula.
\[ E[u'Bu'Fuu'Guu'Hu] = \sigma^8 \frac{1}{3} \left\{ \text{Tr}(F)\text{Tr}(G)\text{Tr}(B(H + H')) + \text{Tr}(G)\text{Tr}(H)\text{Tr}(B(F + F')) + \text{Tr}(B)\text{Tr}(F + G')(H + H') + \text{Tr}(B)\text{Tr}(F + G')(H + H') + \text{Tr}(B)\text{Tr}(F + G')(H + H') \right\} \]

This expected value involves three general matrices \((F, G\text{ and } H)\) and one triangular matrix with zero diagonal \((B)\). The final expression has made use of a symmetric transformation of the matrices, \(F+F'\) and so on; this makes explicit the symmetric role played by each expression, \(u'Fu\) and so on, in the expected value. In addition, the last formula reflects the commutative property that each of the components, \(u'Fu\) and so on, enjoy within the expression. In general, expressions for the expected values of stochastic matrices are simpler if they involve symmetric matrices on repeated matrices.

**Proof of Proposition 8.** Each expression arises from the direct application of the formula in proposition 7. Indeed:
\[ E[u \; Buu' Quu' Puu' Qu] = \sigma^B_u \{ Tr(Q) Tr(P) Tr(B(Q + Q')) \\
+ Tr(Q) Tr(Q) Tr(B(P + P')) + Tr(P) Tr(Q) Tr(B(Q + Q')) \\
+ Tr(Q) Tr(B(P + P')(Q + Q')) \\
+ Tr(P) Tr(B(Q + Q')(Q + Q')) + Tr(Q) Tr(B(Q + Q')(P + P')) \\
+ Tr(B(Q + Q')) Tr(P(Q + Q')) \\
+ Tr((B + B')(Q + Q')(P + P')(Q + Q')) \\
+ Tr((B + B')(Q + Q')^2 (P + P')) \\
+ Tr((B + B')(P + P')(Q + Q')^2 )\} \]

\[ E[u \; Buu' Quu' Cuu' Cu] = \sigma^B_u \{ Tr(Q) Tr(C) Tr(B(Q + Q')) \\
+ Tr(Q) Tr(C) Tr(B(C + C')) + Tr(C) Tr(Q) Tr(B(Q + Q')) \\
+ Tr(Q) Tr(B(C + C')(C + C')) \\
+ Tr(C) Tr(B(Q + Q')(C + C')) + Tr(Q) Tr(B(Q + Q')(C + C')) \\
+ Tr(B(Q + Q')) Tr(C(C + C')) \\
+ Tr((B + B')(Q + Q')(C + C')(C + C')) \\
+ Tr((B + B')(C + C')(Q + Q')(C + C')) \}

\[ E[u \; Auu' Cuu' Puu' Qu] = \sigma^B_u \{ Tr(C) Tr(P) Tr(A(Q + Q')) \\
+ Tr(C) Tr(Q) Tr(A(P + P')) + Tr(P) Tr(Q) Tr(A(C + C')) \\
+ Tr(C) Tr(A(P + P')(Q + Q')) \\
+ Tr(P) Tr(A(C + C')(Q + Q')) + Tr(Q) Tr(A(C + C')(P + P')) \\
+ Tr(A(C + C')) Tr(P(Q + Q')) \\
+ Tr(A(P + P')) Tr(C(P + P')(Q + Q')) + Tr(A(Q + Q')) Tr(C(P + P')) \\
+ Tr((A + A')(C + C')(P + P')(Q + Q')) \\
+ Tr((A + A')(C + C')(Q + Q')(P + P')) \\
+ Tr((A + A')(P + P')(C + C')(Q + Q')) \} \]
\[ E[w' A w' C w' C w] = \sigma^8 \{ Tr(C) Tr(C) Tr(\Lambda(C + C')) + \lambda Tr(C) Tr(C) Tr(\Lambda(C + C')) + \lambda Tr(C) Tr(\Lambda(C + C')(C + C')) + \lambda Tr(C) Tr(\Lambda(C + C') C + C')) + \lambda Tr(\Lambda(C + C')(C + C')) + \lambda Tr((\Lambda + \Lambda')(C + C')(C + C')(C + C')) \}

The expressions in proposition 8 can be obtained by collecting similar terms from above and by applying the property \( Tr(ABCD = Tr(DABC) \).
Appendix 2B
Table A2.1 Small sample distribution of the test. N of lags 1

| β  | σ | y  | z  | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) |
|----|---|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
Table A2.1 (continued) Small sample distribution of the test. N of lags 1

| $\beta$ | $\sigma_{\epsilon}$ | $\gamma$ | $z$ | $z$ | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |

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Table A2.1 (continued) Small sample distribution of the test. N of lags 1

\[
\begin{array}{cccccccccccccccc}
\beta & \sigma_y & y & x & x & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) & ( ) \\
\end{array}
\]
Table A2.1 (continued) Small sample distribution of the test. N of lags 1

| β  | σ_2  | y | x | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) |
|----|------|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|

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Table A2.1 (continued) Small sample distribution of the test. N of lags 1

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Table A2.1 (continued) Small sample distribution of the test. N of lags 1

| $\beta$ | $\sigma_0$ | $y$ | $z$ | $x$ | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) | ($$) |
|--------|-----------|-----|-----|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
Table A2.1 (continued) Small sample distribution of the test. N of lags 1

| β | σ | y | x | z | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
|---|---|---|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
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Table A2.2 Small sample distribution of the test. N of lags 2

| $\beta$ | $\sigma_y$ | y | z | z | (c) | (d) | (e) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) | (c) | (d) |
|---------|-----------|---|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
Table A2.2 (continued) Small sample distribution of the test. N of lags 2

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Table A2.2 (continued) Small sample distribution of the test. N of lags 2

| $\beta$ | $\sigma_x$ | y  | x  | z  | (a) | (b) | (c) | (d) | (e) | (f) | (g) | (h) | (i) | (j) | (k) | (l) | (m) | (n) | (o) | (p) | (q) | (r) | (s) | (t) | (u) | (v) | (w) | (x) | (y) | (z) |
|---------|------------|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
Table A2.2 (continued) Small sample distribution of the test. N of lags 2

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Table A2.2 (continued) Small sample distribution of the test. N of lags 2

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Table A2.2 (continued) Small sample distribution of the test. N of lags 2

| \( \beta \) | \( \sigma_{y} \) | \( y \) | \( x \) | \( x \) | \( (c) \) | \( (d) \) | \( (e) \) | \( (f) \) | \( (g) \) | \( (h) \) | \( (i) \) | \( (j) \) | \( (k) \) | \( (l) \) | \( (m) \) | \( (n) \) | \( (o) \) | \( (p) \) | \( (q) \) | \( (r) \) |
| \( \beta \) | \( \sigma_{y} \) | \( y \) | \( x \) | \( x \) | \( (c) \) | \( (d) \) | \( (e) \) | \( (f) \) | \( (g) \) | \( (h) \) | \( (i) \) | \( (j) \) | \( (k) \) | \( (l) \) | \( (m) \) | \( (n) \) | \( (o) \) | \( (p) \) | \( (q) \) | \( (r) \) |
Table A2.2 (continued) Small sample distribution of the test. N of lags 2

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

| $\beta$ | $\sigma_\beta$ | y | x | x | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
|---------|----------------|---|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Table A2.3 (continued) Small sample distribution of the test. N of lags 3

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Appendix 2C

This appendix provides formulas for the matrix traces needed to evaluate the approximation of the bias on parameters $\beta$ and $\gamma$ in Chapter 2. From the expressions identified here it also emerges which terms from $E[N]$, $E[D]$, $E[R]$, $E[RD]$ and $E[ND]$ converge more quickly to zero and can therefore be ignored with respect to those that go to zero more slowly. We obtained these expressions analytically and sometimes with the help of the software Mathematica version 2.2. As defined in Chapter 2, $P$ is a symmetric matrix equal to $B'B$; where $B$ (defined in section 3.a.- of chapter 2) is the triangular matrix with zeros in the main diagonal and the powers of $\beta$ in each diagonal of the lower triangle. $Q$ is a symmetric matrix, analogous to $P$, and equal to $A'A$. $C$ is the product of matrices $B'$ and $A$. We generally require, for the following results to be true, that $\beta$ and $\lambda$ are smaller than one. In the rest of this appendix, we offer a more detailed analysis only on those less known and obvious results.

The diagonal of both $P$ and $Q$ contains powers of $\beta^2$ and $\lambda^2$ respectively. Therefore the following formulas for the trace of products of $P$ and $Q$ hold:

1) $\text{Tr}(P) = (1 - \beta^2)^{-1} (T - 1 - \beta^2 (1 - \beta^2)^{-1} (1 - \beta^2)^{-1}) = O(T)$

2) $\text{Tr}(PQ) = (1 - \beta^2)^{-1} (1 - \lambda^2)^{-1} (T(1 - (\beta \lambda))^{-1} - T(1 - (\beta \lambda)^{-1})) + O(0)$

3) $\text{Tr}(P^2) = (1 - \beta^2)^{-2} (T(1 - \beta^2)^{-1} - T(1 - (\beta)^{-1} \lambda)^{-1}) + O(0)$

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The diagonal of $C$ contains powers of $\beta\lambda$; other off diagonals cells contain product of powers of $\beta$ and $\lambda$. The following formulas for the trace of products of $C$ hold:

4) $\text{Tr}(C) = (1 - \beta\lambda)^{-1}(T - 1 - \beta\lambda(1 - (\beta\lambda)^{T-1})(1 - \beta\lambda)^{-1}) = O(T)$

5) $\text{Tr}(C^2) = (1 - \beta\lambda)^{-4}(T(1 - \beta\lambda)(1 + \beta\lambda + 4(\beta\lambda)^{T-1}) - (\beta\lambda)(1 - (\beta\lambda)^{T})(\beta\lambda + (\beta\lambda)^{T+1} + 4))$

6) $\text{Tr}(CC') = \text{Tr}(PP')$

The following formulas for the trace of products of $C$, $P$ and $Q$ times $B$ hold:

7) $\text{Tr}(BC) = (1 - \beta\lambda)^{-1}(1 - \beta^2)^{-1}\beta(T - 2) + O(0)$

8) $\text{Tr}(BC') = (1 - \beta\lambda)^{-2}\lambda(T - 2)(1 + (\beta\lambda)^{T-1}) + O(0)$

9) $\text{Tr}(BQ) = (1 - \beta\lambda)^{-2}(1 - \beta^2)^{-1}\lambda(T - 2) + O(0)$

10) $\text{Tr}(BP) = (1 - \beta^2)^{-3}\beta((T - 2)(1 + \beta^{2(T-1)})(1 - \beta^2) - 2(\beta^2 - \beta^{2(T-1)}))$

By using the property that the trace of a matrix is equal to the trace of its transposed and formulas 7 to 10 above, we can construct other traces of matrix products; for example: $\text{Tr}(CB')$ or $\text{Tr}(QB')$. In order to compute the expectations of stochastic matrices in Chapter 2 sections 3.b.- and 3.c.- we also use the following property that holds for any two general matrices $G$ and $H$:

$$\text{Tr}(GH) = \text{Tr}(HG)$$

This property allows us to compute: $\text{Tr}(B'C)$, $\text{Tr}(C'C)$ or $\text{Tr}(QB)$. Finally, by exchanging the roles of $\beta$ and $\lambda$ we can compute traces of other products like: $\text{Tr}(AC)$, $\text{Tr}(Q)$, $\text{Tr}(Q^2)$.
or $Tr(AQ)$. Combining all these properties it is possible to calculate other traces like $Tr(A'Q)$ or $Tr(PA')$.

Larger products of matrices like $Tr(P^3)$ or $Tr(P^4)$ are also orders of $T$. Therefore, they are also dominated, in the approximation to the bias, by higher order terms like: $Tr(P)^2$ or $Tr(P)^2 Tr(Q)$.

The following formulas for the trace of powers of $P$ hold:

1) $Tr(P^3) = T(1 - \beta^2)^2 (1 - \beta^4 + 2\beta^2 (2 + \beta^2)) + O(0)$

12) $Tr(P^4) = O(T)$

The trace of $P^3$ consists of two series on powers of $\beta^i$ one with coefficients increasing with $i$, an another with coefficients $f(i)$ decreasing towards zero:

\[ a) \quad \sum_{j=1}^{T-1} \sum_{i=1}^{j} \frac{i^3(i+1)^2}{4} \beta^{2(i-1)} = O(T), \text{ and} \]

\[ b) \quad \sum_{j=T}^{\delta(T-1)} \sum_{i=1}^{j} f(i) \beta^{2(i-1)} \xrightarrow{T \to \infty} 0 \]

The trace of $P^4$ consists of two series on powers of $\beta^i$ one with coefficients increasing with $i$, an another with coefficients $f(i)$ decreasing towards zero:

\[ a) \quad \sum_{j=1}^{T-1} \sum_{i=1}^{j} \frac{i^3(i+1)^2(i+2)^2}{36} \beta^{2(i-1)} = O(T), \text{ and} \]

\[ b) \quad \sum_{j=T}^{\delta(T-2)} \sum_{i=1}^{j} f(i) \beta^{2(i-1)} \xrightarrow{T \to \infty} 0 \]
All these formulas will depend on the inverse of \((1 - \beta^2)\) therefore, when ignoring terms that go to zero faster than \(T\) and when \(\beta\) is near 1, these will go to zero very slowly. In cases like this, the accuracy of the approximation will not be as good as when the parameter \(\beta\) is well below one. In addition, the effect of the system parameters on larger products of matrices depends on \((1 - \beta^2)^{-2}\), \((1 - \beta^2)^{-3}\) or \((1 - \beta \lambda)^{-4}\) and so on; these take large values when \(\beta\) or \(\lambda\) are close to 1.
Appendix 3A.

Proof of proposition 1

The condition $\sigma_A^2 < \tilde{\omega}_w^2 (\sigma_w^2 - \sigma_{w,1N} \Omega_{1N}^{-1} \sigma_{w,1N})$ and non-perfect correlation between assets aims to rule out situations where the risk induced by compulsory holding can be brought below the desired level of risk. Therefore, the problem for the investor will be how to minimise the total risk using all assets.

The investor type A maximisation problem in chapter 3. section c.-i.-:

$$\text{Min} \; \omega_A^T \Omega \omega_A \quad \text{subject to} \; \omega_w \geq \tilde{\omega}_w$$

Has first order conditions which are:

$$\omega_w = \tilde{\omega}_w$$

$$\frac{\partial L}{\partial \omega_{1N}} = 2 \Omega_{1N} \omega_{1N} + 2 \tilde{\omega}_w \sigma_{1N,w} = 0$$

Where $\tilde{\omega}_A^T = [\tilde{\omega}_{1N} \; \omega_w]$ and $\tilde{\omega}_{1N}' = [\omega_1 \; \omega_2 \; \ldots \; \omega_N]$. Therefore:

$$\tilde{\omega}_A = - \Omega_{1N}^{-1} \sigma_{1N,w} \tilde{\omega}_w$$

The expected return and variance for this solution are:

$$\begin{bmatrix} E[r_{1N}^T] : E[r_w] \end{bmatrix} \begin{bmatrix} - \Omega_{1N}^{-1} \sigma_{1N,w} \\ 1 \end{bmatrix} \tilde{\omega}_w = \tilde{\omega}_w (E[r_w] - \tilde{\sigma}_{1N,w} \Omega_{1N}^{-1} E[r_{1N}])$$
In addition, suppose $r_\omega$ is the random return of an individual asset and $\bar{r}_N$ is a vector of random returns of $N$ other assets correlated with $r_\omega$. Call $\sigma^2_\omega$, $\sigma_{1N,\omega}$ and $\Omega_{1N}$ the variance of $r_\omega$, the covariance of $r_\omega$ and the other assets and the variance-covariance matrix of the $N$ assets respectively.

The conditional distribution of $r_\omega$ for given values of $\bar{r}_N$ follows a normal distribution with mean and variance equal to:

$$E[r_\omega] + \sigma^T_{1N,\omega} \Omega^{-1}_{1N} (\bar{r}_N - E[\bar{r}_N])$$

and

$$\sigma^2_\omega - \sigma^T_{1N,\omega} \Omega^{-1}_{1N} \sigma_{1N,\omega}$$

If $\bar{\alpha}$ is any vector of constants, then the difference below follows a normal distribution:

$$\epsilon = (r_\omega - E[r_\omega]) - \bar{\alpha}^T (\bar{r}_N - E[\bar{r}_N])$$

The following regression represents the best fit of $r_\omega$ in terms of the given values $\bar{r}_N$:

$$r_\omega - E[r_\omega] = \bar{\alpha}^T (\bar{r}_N - E[\bar{r}_N]) + \epsilon$$

This best fit will have a parameter $\bar{\alpha}^T = \sigma^T_{1N,\omega} \Omega^{-1}_{1N}$. So, the conditional expectation of $r_\omega$ given $\bar{r}_N$ is just the regression of $r_\omega$ on $\bar{r}_N$; in other words it is the best prediction of $r_\omega$ based on $\bar{r}_N$. The conditional variance of $r_\omega$ is also the residual variance of the

\[
\bar{\omega}_n \begin{bmatrix} \Omega_{1N} & \sigma_{1N,\omega} \\ \sigma^T_{1N,\omega} & \sigma^2_\omega \end{bmatrix} \bar{\omega}_n + \omega^2_\omega \begin{bmatrix} \sigma^2_\omega & \sigma^T_{1N,\omega} \\ \sigma^T_{1N,\omega} & \sigma^2_\omega \end{bmatrix} \Omega_{1N}^{-1} \sigma_{1N,\omega} = \omega^2_\omega \begin{bmatrix} \sigma^2_\omega & \sigma^T_{1N,\omega} \\ \sigma^T_{1N,\omega} & \sigma^2_\omega \end{bmatrix} \Omega_{1N}^{-1} \sigma_{1N,\omega}
\]
regression. Rearranging the regression:

\[ r_m = E[r_m] - \sigma_{1N}^T \Omega_{1N}^{-1} E[ \tilde{r}_{1N} ] + \sigma_{1N}^T \Omega_{1N}^{-1} \tilde{r}_{1N} + \epsilon \]

The term \( E[r_m] - \sigma_{1N}^T \Omega_{1N}^{-1} E[ \tilde{r}_{1N} ] \) corresponds to the regression constant.

**Proof of proposition 2**

Suppose maximising return involves investing in all the assets including an additional positive amount of asset \( \infty \). Therefore, obtaining the maximum amount of return involves holding a portfolio that includes an amount of asset \( \infty \) strictly larger than the compulsory holding. The portfolio is:

\[ \omega_A^T = [ \omega_{1N}^T, \tilde{\omega}_- ] \text{ with } \tilde{\omega}_- > \tilde{\omega}_- \]

Given that this vector maximises profit, then the amount invested in asset \( I \) to \( N \) should also be the maximum for a given investment in asset \( \infty \) equals to \( \tilde{\omega}_- \). We have ruled out cases in which maximising on the \( N+1 \) assets reach a different solution than when maximising over only the first \( N \) assets and keeping the remaining ones as a restriction. This does not hold when there are a number of unrestricted local maxima, and Lagrange maximisation does not achieve the global maximum. However, given that we consider a set of assets none of them is perfectly correlated to any other, the variance-covariance matrix \( \Omega \) and any squared sub-matrix of it has a unique inverse. Therefore, any Lagrange maximisation based on this matrix has a unique solution. The problem is therefore:

\[ \text{Max } \tilde{\omega}_A^T \tilde{\rho} \quad \text{subject to } \sigma_A \geq \tilde{\omega}_A^T \Omega \tilde{\omega}_A \quad \text{and} \quad \omega_- = \tilde{\omega}_- \]
Recall the assumptions of the problem, namely: the investor has been forced to hold more of asset $\infty$ than the global maximum obtained by unrestricted Lagrange maximisation and this holding is still below the saturation level so that:

$$\sigma_A^2 \geq \bar{\omega}_\infty^2 (\sigma_\infty^2 - \sigma_{\infty,1N}^T \Omega_{1N}^{-1} \sigma_{\infty,1N})$$

As shown in proposition 4 the solution by unrestricted Lagrange maximisation will produce a vector of investment:

$$\bar{\omega}_A = \frac{r_A}{\bar{\Omega}^T \Omega^{-1} \bar{r}}$$

$$r_A^2 = \sigma_A^2 \bar{\Omega}^T \Omega^{-1} \bar{r}$$

From here the optimal amount invested on asset $\infty$ is:

$$\omega_* = \frac{r_* - \sigma_{\infty,1N}^T \Omega_{1N}^{-1} \bar{r}_{1N}}{RV_{\infty,1N} \sqrt{\bar{r}_{1N}^T \Omega_{1N}^{-1} \bar{r}_{1N} RV_{\infty,1N} + (r_* - \sigma_{\infty,1N}^T \Omega_{1N}^{-1} \bar{r}_{1N})^2}}$$

$$RV_{\infty,1N} = \sigma_\infty^2 - \sigma_{\infty,1N}^T \Omega_{1N}^{-1} \sigma_{\infty,1N}$$

Here we have used the formula for the inverse of a partitioned matrix:

$$\begin{bmatrix} \Omega_{1N} & \sigma_{\infty,1N} \\ \sigma_{\infty,1N}^T & \sigma_\infty^2 \end{bmatrix}^{-1} = \begin{bmatrix} RV_{\infty,1N} \Omega_{1N}^{-1} + \Omega_{1N}^{-1} \sigma_{\infty,1N} \sigma_{\infty,1N}^T \Omega_{1N}^{-1} & -\Omega_{1N}^{-1} \sigma_{\infty,1N}^T \\ -\sigma_{\infty,1N} \Omega_{1N}^{-1} & 1 \end{bmatrix} / RV_{\infty,1N}$$

By assumption the compulsory holding is larger than this optimal value:

$$\bar{\omega}_\infty > \omega_*$$

According to our assumption, the actual optimal investment $\bar{\omega}_\infty$ is larger than these
two amounts:
\[ \tilde{\omega}_n > \bar{\omega}_n > \omega^* \]

We can obtain the solution for the rest of the asset by solving the maximisation above or by taking \( \omega_n = \tilde{\omega}_n \) and redefining the problem as follows:

\[
\tilde{\omega}_n^T = \begin{bmatrix} \omega_{1N}^T & \tilde{\omega}_n \end{bmatrix}
\]

\[
\sigma_A^2 \geq \tilde{\omega}_n^T \Omega_{1N} \tilde{\omega}_n + 2 \tilde{\omega}_n \tilde{\omega}_n^T \sigma_{wN} \tilde{\omega}_n + \sigma_w^2 \tilde{\omega}_n^2
\]

\[
= (\tilde{\omega}_n^T + \tilde{\omega}_n \tilde{\sigma}^T_{wN} \Omega_{1N}^{-1} \Omega_{1N} \tilde{\omega}_n + \tilde{\omega}_n \tilde{\omega}_n^T \sigma_{wN} \tilde{\omega}_n)
\]

\[
\tilde{\omega}_n^T \Omega_{1N} \tilde{\omega}_n + \tilde{\omega}_n \tilde{\sigma}^T_{wN} \Omega_{1N}^{-1} \Omega_{1N} \tilde{\omega}_n + \sigma_w^2 \tilde{\omega}_n^2
\]

Defining:
\[ z = \tilde{\omega}_n^T + \tilde{\omega}_n \tilde{\sigma}^T_{wN} \Omega_{1N}^{-1} \tilde{\omega}_n \]

\[ z^T \Omega_{1N} z \leq \sigma_z^2 = \sigma_A^2 - \tilde{\omega}_n^T (\sigma_w^2 - \sigma_{wN}^T \Omega_{1N}^{-1} \sigma_{wN}) \]

The following problem in variable \( z \) is equivalent:

\[ \text{Max } z^T r_{1N} \text{ subject to } \sigma_z^2 \geq z^T \Omega_{1N} z \]

Its solution is:

\[ z = \frac{r_z}{\tilde{r}_{1N} \Omega_{1N}^{-1} \tilde{r}_{1N}} \]

\[ r_z^2 = \sigma_z^2 \tilde{r}_{1N} \Omega_{1N}^{-1} \tilde{r}_{1N} \]

\[ \tilde{\omega}_{1N} = \frac{r_z}{\tilde{r}_{1N} \Omega_{1N}^{-1} \tilde{r}_{1N}} \Omega_{1N}^{-1} \tilde{r}_{1N} - \tilde{\omega}_n \Omega_{1N}^{-1} \sigma_{wN} \]

The expected return for this investment is:
\[ \tilde{r} = r_+^T \tilde{\omega}_{1N} + r_- \tilde{\omega}_- \]

\[ = r_+ - \tilde{\omega}_m r_+^T \Omega_{1N}^{1} \tilde{\sigma}_{m1N} + r_- \tilde{\omega}_- \]

\[ = \sqrt{\sigma_d^2 + \sigma_1^2} + \tilde{\omega}_m (r_+ - \tilde{r}_+^T \Omega_{1N}^{1} \tilde{\sigma}_{m1N}) \]

\[ = \sqrt{(\sigma_d^2 - \tilde{\omega}_m^2 (\sigma_1^2 - \tilde{\sigma}_m^T \Omega_{m1N}^{1} \tilde{\sigma}_{m1N})) + \tilde{\omega}_m (r_+ - \tilde{r}_+^T \Omega_{1N}^{1} \tilde{\sigma}_{m1N})} \]

The function above is well defined for the range of values we are interested in; that is, for compulsory holdings below the saturation level. Thus, the argument of the squared root is always positive for this range of values.

The derivative of the expression above is negative for the following set:

\[ \tilde{\omega}_m^2 > \frac{\sigma_d^2 (r_+ - \tilde{\sigma}_m^T \Omega_{m1N}^{1} \tilde{r}_+)^2}{\tilde{r}_+^T \Omega_{1N}^{1} \tilde{r}_+ + \tilde{\sigma}_m^T \Omega_{m1N}^{1} \tilde{r}_+^2} \]

For positive compulsory holdings this is equivalent to:

\[ \tilde{\omega}_m > \omega^*_m \]

This shows that the return is a decreasing function of \( \tilde{\omega}_m \) for values larger than the unconstrained optimal level of investment in asset \( \omega^*_m \). Therefore, if an investor is already forced to hold more of this asset than desirable, any addition to it will decrease the maximum return attainable when using the rest of the assets.

Therefore, profit maximising investors will not hold any additional amount asset \( \infty \) on top of a too high compulsory holding.

**Proof of proposition 3**
Given the analysis in proposition 3, a profit maximising investor will not hold any additional amount asset \( \infty \). We can solve the problem in variable \( z \) as defined in proposition 3 for the given mandatory holding \( \bar{\omega}_w \):

\[
\text{Max } z^T \bar{r}_N \quad \text{subject to } \sigma_z^2 \geq z^T \Omega_{1N} z
\]

\[
z = \bar{\omega}_{1N} + \bar{\omega}_w \Omega_{1N}^{-1} \bar{\sigma}_{w1N}
\]

\[
\sigma_z^2 = \sigma_A^2 - \bar{\omega}_w^2 (\sigma_w^2 - \bar{\sigma}_w^T \Omega_{1N}^{-1} \bar{\sigma}_{w1N})
\]

The solution is:

\[
z = \frac{r_z}{\bar{r}_N^T \Omega_{1N}^{-1} \bar{r}_N} \Omega_{1N}^{-1} \bar{r}_N
\]

\[
r_z^2 = \sigma_A^2 r_z \Omega_{1N}^{-1} r_N
\]

\[
\bar{\omega}_{1N} = \frac{r_z}{\bar{r}_N^T \Omega_{1N}^{-1} \bar{r}_N} \Omega_{1N}^{-1} \bar{r}_N - \bar{\omega}_w \Omega_{1N}^{-1} \bar{\sigma}_{w1N}
\]

Replacing \( r_z \) using its relationship with \( \sigma_z \) we have the result.

**Proof of proposition 4**

For the Lagrangean:

\[
L = \bar{\omega}_A^T \bar{r} - \frac{1}{2} \lambda_{A} (\sigma_A^2 - \bar{\omega}_A^T \Omega \bar{\omega}_A)
\]

The first order conditions are:

\[
\frac{\partial L}{\partial \bar{\omega}_A} = \bar{r} - \lambda_A \Omega \bar{\omega}_A = 0
\]

\[
\frac{\partial L}{\partial \lambda_A} = -\sigma_A^2 + \bar{\omega}_A^T \Omega \bar{\omega}_A = 0
\]
The solution for investor $A$ will verify:

$$
\bar{\omega}_A = \frac{r_A \Omega^{-1}_{\bar{r}}}{\bar{r}^T \Omega^{-1}_{\bar{r}}}
$$

$$
r_A^2 = \sigma_A^2 \bar{r}^T \Omega^{-1}_{\bar{r}}
$$

**Proof of Proposition 5.**

The total market return, $r_m$, is simply the product of the $N$ expected returns times the total investment in each asset; this is the sum of investment of all representative agent:

$$
\bar{\omega}_A + \bar{\omega}_B + \bar{\omega}_C = \begin{pmatrix}
\bar{\omega}_{\omega_A} + \bar{\omega}_{\omega_B} + \frac{r_{\omega,1N}}{R_{\omega}} \sqrt{\frac{\sigma_c^2}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega,1N}^2 / R_{\omega}}} & -\Omega_{1N}^{-1} \bar{\sigma}_{\omega,1N} \\
\Omega_{1N}^{-1} \bar{r}_{1N} & 0
\end{pmatrix}
$$

$$
\bar{\omega}_A + \bar{\omega}_B + \bar{\omega}_C = I_m \begin{pmatrix}
-\Omega_{1N}^{-1} \bar{\sigma}_{\omega,1N} \\
1
\end{pmatrix} + \lambda \begin{pmatrix}
\Omega_{1N}^{-1} \bar{r}_{1N} \\
0
\end{pmatrix}
$$

where,

$$
I_m = \bar{\omega}_{\omega_A} + \bar{\omega}_{\omega_B} + \frac{r_{\omega,1N}}{R_{\omega}} \sqrt{\frac{\sigma_c^2}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega,1N}^2 / R_{\omega}}}
$$

$$
\lambda = \sqrt{\frac{\sigma_c^2 - \bar{\omega}_{\omega_B}^2 R_{\omega}}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega,1N}^2 / R_{\omega}}} + \sqrt{\frac{\sigma_c^2}{r_{1N}^T \Omega_{1N}^{-1} r_{1N} + r_{\omega,1N}^2 / R_{\omega}}}
$$

If we multiply by the partitioned vector of returns:

$$
\begin{bmatrix}
[\bar{\omega}_A + \bar{\omega}_B + \bar{\omega}_C] = I_m \left( \bar{r}_{\omega} - \bar{r}_{1N}^T \Omega_{1N}^{-1} \bar{\sigma}_{\omega,1N} \right) + \lambda \bar{r}_{1N}^T \Omega_{1N}^{-1} \bar{r}_{1N}
\end{bmatrix}
$$
We obtain the result:

\[ r_M = r_{\omega,1N}I_\omega + \lambda r_{1N}^T \Omega_{1N}^{-1}r_{1N} \]

**Proof of Proposition 7.**

We obtain \( \sigma_{i,M} \) by right multiplying the variance-covariance matrix with the total market investments and left multiplying it by a unit vector \( C_j \). This vector is equal to zero everywhere except at position \( i \) where it is one. Recall the sum of investments is:

\[
\omega_A + \omega_B + \omega_C = I_\omega \left[ \begin{array}{c} -\Omega_{1N}^{-1} \sigma_{\omega,1N} \\ 1 \end{array} \right] + \lambda \left[ \begin{array}{c} \Omega_{1N}^{-1} \bar{r}_{1N} \\ 0 \end{array} \right]
\]

and the partitioned variance-covariance matrix post-multiplied by this sum:

\[
\left[ \begin{array}{cc} \Omega_{1N} & \sigma_{\omega,1N} \\ \sigma_{\omega,1N}^T & \sigma_\omega^2 \end{array} \right] (\omega_A + \omega_B + \omega_C) = I_\omega \left[ \begin{array}{c} 0 \\ RV_\omega \end{array} \right] + \lambda \left[ \begin{array}{c} \bar{r}_{1N} \\ \sigma_{\omega,1N}^T \Omega_{1N}^{-1} \bar{r}_{1N} \end{array} \right]
\]

Therefore, when pre-multiplying by vector \( e_i \) only the term \( \lambda r_i \) survives at the right hand side. Finally:

\[ \sigma_{i,M} = \lambda r_i \]

**Proof of Proposition 6.**

We use here elements from proposition 5 and proposition 7. The total market variance, \( \sigma_M^2 \), is equal to pre- and post-multiplying the variance-covariance matrix by the vector of total investments:
\[ \sigma_{M}^{2} = (\tilde{\omega}_{A} + \tilde{\omega}_{B} + \tilde{\omega}_{C})^{T} \begin{bmatrix} \Omega_{1N}^{\prime} & \sigma_{1N}^{\prime} \\ \sigma_{1N}^{\prime} & \sigma_{\omega} \end{bmatrix} (\tilde{\omega}_{A} + \tilde{\omega}_{B} + \tilde{\omega}_{C}) \]

\[ = (\tilde{\omega}_{A} + \tilde{\omega}_{B} + \tilde{\omega}_{C})^{T} \left( I_{\omega} \left[ \begin{array}{c} 0 \\ \sigma_{\omega}^{T} \Omega_{1N}^{\prime} \tilde{r}_{1N} \end{array} \right] \right) \]

\[ = \left( I_{\omega} \left[ \begin{array}{c} -\Omega_{1N}^{\prime} \sigma_{1N}^{\prime} \\ 1 \end{array} \right] + \lambda \left( I_{\omega} \left[ \begin{array}{c} 0 \\ \sigma_{\omega}^{T} \Omega_{1N}^{\prime} \tilde{r}_{1N} \end{array} \right] \right) \right) \left( I_{\omega} \left[ \begin{array}{c} 0 \\ \sigma_{\omega} \end{array} \right] \right) \]

Multiplying these two partitioned vectors:

\[ \sigma_{M}^{2} = I_{\omega}^{\prime} R_{\omega} + \lambda^{2} \tilde{r}_{1N}^{\prime} \Omega_{1N}^{\prime} \tilde{r}_{1N} \]

Replacing \( \lambda \) and \( I_{\omega} \) and after some rearranging we obtain:

\[ \sigma_{M}^{2} = R_{\omega} \left( (\tilde{\omega}_{A} + \tilde{\omega}_{B})^{2} - \tilde{\omega}_{B}^{2} \right) + \sigma_{\omega}^{2} + \sigma_{C}^{2} \]

\[ + 2 \cdot \sigma_{C}^{2} \cdot \frac{\sigma_{\omega}^{2}}{\sqrt{\tilde{r}_{1N}^{\prime} \Omega_{1N}^{\prime} \tilde{r}_{1N} + r_{1N}^{2} / R_{\omega}}} \left( \tilde{\omega}_{A} + \tilde{\omega}_{B} \right) r_{1N}^{\prime} + \frac{\sigma_{\omega}^{2} - \tilde{\omega}_{B}^{2} R_{\omega} \tilde{r}_{1N}^{\prime} \Omega_{1N}^{\prime} \tilde{r}_{1N}}{\sqrt{\tilde{r}_{1N}^{\prime} \Omega_{1N}^{\prime} \tilde{r}_{1N} + r_{1N}^{2} / R_{\omega}}} \]

**Proof of Proposition 9.**

Given that:

\[ \tilde{\omega}_{A} + \tilde{\omega}_{B} + \tilde{\omega}_{C} = I_{\omega} \left[ \begin{array}{c} -\Omega_{1N}^{\prime} \sigma_{1N}^{\prime} \\ 1 \end{array} \right] + \lambda \left( \Omega_{1N}^{\prime} \tilde{r}_{1N} \right) \]

The total investment in assets \( I \) to \( N \) can be obtained by deleting the last row in the vector above. Therefore, the difference between the returns of the full set of asset and the return of the first \( N \) is \( r_{M} I_{\omega} \).

\[ r_{M} - r_{M*} = r_{M} I_{\omega} \]

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Proof of Proposition 10.

Following the same argument as in proposition 9, deleting the last row in the vector of total investments will produce a variance equal to:

\[
\sigma_{M^*}^2 = \left( I_w \begin{bmatrix} -\Omega_{w,1,N} & \bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \Omega_{w,1,N} \bar{r}_{1,N} \end{bmatrix} \right)^T \left( I_w \begin{bmatrix} -\bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \bar{r}_{1,N} \end{bmatrix} \right)
\]

\[
\sigma_{M^*}^2 = \lambda^2 \bar{r}_{1,N}^T \Omega_{w,1,N}^{-1} \bar{r}_{1,N} + I_w^2 \bar{\sigma}_{w,1,N}^2 \Omega_{w,1,N}^{-1} - 2 \lambda I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N}
\]

\[
\sigma_{M^*}^2 = I_w^2 R_{wM} + I_w^2 \sigma_{w,M}^2 + \lambda^2 \bar{r}_{1,N}^T \Omega_{w,1,N}^{-1} \bar{r}_{1,N} - 2 \lambda I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N}
\]

\[
\Omega_{w,1,N}^{-1} \bar{\sigma}_{w,1,N}
\]

In addition, if we call \( \sigma_{M*,w} \) and \( \sigma_{M^*,w} \) the per-unit-correlation of asset \( \omega \) with the respective market index \( M \) and \( M^* \), we can calculate them as:

\[
I_w \sigma_{M,w} = \left( I_w \begin{bmatrix} -\Omega_{w,1,N} & \bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \Omega_{w,1,N} \bar{r}_{1,N} \end{bmatrix} \right)^T \left( I_w \begin{bmatrix} -\bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \bar{r}_{1,N} \end{bmatrix} \right)
\]

\[
= -I_w^2 R_{wM} + \lambda I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N}
\]

\[
I_w \sigma_{M^*,w} = \left( I_w \begin{bmatrix} -\Omega_{w,1,N} & \bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \Omega_{w,1,N} \bar{r}_{1,N} \end{bmatrix} \right)^T \left( I_w \begin{bmatrix} -\bar{\sigma}_{w,1,N} \end{bmatrix} + \lambda \begin{bmatrix} \bar{r}_{1,N} \end{bmatrix} \right)
\]

\[
= -I_w^2 \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N} + \lambda I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N}
\]

\[
\sigma_{M^*,w}^2 = \sigma_{w,M}^2 - 2I_w^2 R_{wM} + I_w \sigma_{M,w}
\]

\[
= \sigma_{w,M}^2 - 2I_w \sigma_{M,w} + I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N} - I_w^2 R_{wM}
\]

\[
\sigma_{M^*,w}^2 = \sigma_{w,M}^2 - 2I_w \sigma_{M,w} + I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N} + I_w \sigma_{M^*,w}
\]

\[
= \sigma_{w,M}^2 + 2I_w \sigma_{M,w} + I_w \bar{\sigma}_{w,1,N} \Omega_{w,1,N}^{-1} \bar{r}_{1,N} - I_w^2 R_{wM}
\]
Proof of Proposition 11.

Following the procedure in proposition 9 and 10 we can calculate the covariance between asset $i$ and the new market index as:

$$\sigma_{M,i} = -I_m \sigma_{m,i} + \lambda r_i$$

Therefore, using the result from proposition 7:

$$\sigma_{M,i} = \sigma_{M',i} + \sigma_{m,i} I_m$$

Proof of Proposition 12.

The formula is the direct result of applying propositions 9 through 11 in the risk return relationship obtained in proposition 8.
Appendix 4A.

**Proof Proposition 1.** Let the model:

\[ y_{jt} = \gamma_j + \beta_j (x_i + \theta) + \varepsilon_{jt} \]

The first order conditions and the problem of minimisation of the residuals are:

\[
\min S = \sum_{j=1}^{k} \sum_{t=1}^{T} (y_{jt} - \gamma_j - \beta_j (x_i + \theta))^2
\]

\[
\frac{\partial S}{\partial \gamma_j} = \sum_{j=1}^{k} \sum_{t=1}^{T} (y_{jt} - \gamma_j - \beta_j (x_i + \theta)) = 0
\]

\[
\frac{\partial S}{\partial \theta} = \sum_{j=1}^{k} \sum_{t=1}^{T} - \beta_j (y_{jt} - \gamma_j - \beta_j (x_i + \theta)) = 0
\]

\[
\frac{\partial S}{\partial \beta_j} = \sum_{t=1}^{T} -(x_i + \theta)(y_{jt} - \gamma_j - \beta_j (x_i + \theta)) = 0
\]

In the case of only two assets, one possible solution is that the two estimated betas are equal. In such a case the two beta estimates will be equal to the average beta estimate defined in section 4.1.c. The estimate of \( \theta \) will be:

\[ \theta = \frac{(xy)_{1*} - (xy)_{2*}}{y_{1*} - y_{2*}} \]

The estimate for \( \gamma_j \) will be residual from the cross-section-average relationship, and based on the estimates of average \( \beta \) and \( \theta \).

In the general case where there are more than two assets the betas have to be different,
as an equal beta will imply many inconsistent estimates for theta; except for a particular distribution of the data that has a zero probability to occur. In this case the estimate for \( \gamma_i \) will be based on the other estimates and the average relationship. This is direct from the first-order condition for \( \gamma_i \):

\[
\hat{\gamma}_i = y_{i*} - \hat{\beta}_*(x_{i*} + \hat{\theta})
\]

By repeatedly eliminating \( \hat{\beta}_j - \hat{\beta}_* \), from the second and third first-order conditions and after some algebraic manipulation we arrive at:

\[
(\hat{\beta}_j - \hat{\beta}_*)(x^2) + 2x_j\hat{\theta} + \hat{\theta}^2)^2 = (x_j + \hat{\theta})(y_{i*} - y_{i*}) + \hat{\beta}_*(x^2) - (x^2) + ((xy) - x_jy_{i*})
\]

\[
(x_j + \hat{\theta})^2V_{xy} + (x_j + \hat{\theta})V_{y} + (x_j + \hat{\theta})V_{xy} + V_{xy} = 0
\]

Where:

\[
V_x = (x^2) - x^2
\]

\( (x^2) \) is the average over \( t \) of the square of all \( x_t \).

\[
V_y = \sum_j (y_{i*} - y_{i*})^2
\]

\[
V_{xy} = \sum_j ((xy) - x_jy_{i*})^2 - (\sum_j (xy) - x_jy_{i*})^2 / T
\]

\( (xy) \) is the average over \( t \) of all \( x_t \) times \( y_{jt} \).

\[
V_{xy} = \sum_j (y_{i*} - y_{i*})((xy) - x_jy_{i*})
\]

Finally, we obtain the OLS estimates as:

\[
\hat{\beta}_j = \sum_i (x_i + \hat{\theta})(y_{ij} - \hat{\gamma}_i) / \sum_i (x_i + \hat{\theta})^2
\]
\[ \hat{y}_i = y_i - \hat{\beta}_i (x_i + \hat{\theta}) \]

\[ \hat{\theta} = -x_i + \frac{V_x V_y - V_{xy} + \sqrt{(V_x V_y - V_{xy})^2 - 4V_x V_{xy}^2}}{2V_{xy}} \]
Appendix 5A.

**Proof of Proposition 1.** We need to demonstrate that there is a unique linear mapping between \([b_1, b_2]\) and \([p_1, p_2]\), when the following matrix is non-singular:

\[
V'D^4V\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V'D^4(I-D^4)(I-D)'DV\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

We start by looking at the phase average model, the one written in terms of the average of \(z\).

\[
y_t = p_1y_{t-k} + p_2\bar{z}_t + e_t
\]

Here \(\bar{z}_t\) is the average of all \(z\) measured between periods \(t-k+1\) and \(t\). This average is written in terms of each of these observed \(z_{t,i}\) (\(i=0,\ldots,k-1\)) as:

\[
\bar{z}_t = \frac{\sum_{i=0}^{k-1} z_{t,i}}{k}
\]

In addition, we can write any of the \(z_{t,i}\) in terms of \(z_t\) as follows:

\[
z_{t-i} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[ V'D^{k-i}V^{-1} \begin{bmatrix} y_{t-k} \\ z_{t-k} \end{bmatrix} + V' \sum_{j=0}^{k-i-1} D_j \begin{bmatrix} e_{t-j-i} + \beta_1 v_{t-j-i} \\ v_{t-j-i} \end{bmatrix} \right]
\]

\[
\sum_{i=0}^{k-i} z_{t-i} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \left[ \sum_{i=0}^{k-i} D^{k-i}V^{-1} \begin{bmatrix} y_{t-k} \\ z_{t-k} \end{bmatrix} + \sum_{i=0}^{k-i} \sum_{j=0}^{k-i-1} D_j \begin{bmatrix} e_{t-j-i} + \beta_1 v_{t-j-i} \\ v_{t-j-i} \end{bmatrix} \right]}{k}
\]
\[
\bar{z}_t = \begin{bmatrix}
0 & 1
\end{bmatrix}
V(I-D)(I-D^k)DV^{-1} \begin{bmatrix}
y_{t,k} \\
z_{t,k}
\end{bmatrix} + \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} D^j \begin{bmatrix}
\epsilon_{t-j-i} + \beta_j V_{t-j-i}
\end{bmatrix}

f(y_{t,k}, z_{t,k}) = \bar{f}(y_{t,k}, z_{t,k}) + w_t
\]

\( f(y_{t,k}, z_{t,k}) \) represents the component of the average of \( z \) that depends on the event up to \( t-k \). \( w_t \) represents all the information arrived between \( t-k+1 \) and \( t \). We can write the phase average model in terms of the expansion above.

\[
y_t = \hat{\beta}_1 y_{t,k} + \hat{\beta}_2 z_{t,k} \left( \begin{array}{c}
y_{t,k} \\
z_{t,k}
\end{array} \right) + \xi_t
\]

As in the \( k \)-lagged model, the term \( \xi_t \) includes only innovations between \( t \) and \( t-k-1 \).

We have found an alternative expression of the \( k \)-lagged model where \( y_t \) is explained by the lagged values \( y_{t,k} \) and \( z_{t,k} \), and by an error term including all past values of \( \epsilon \) and \( \nu \) back to \( t-k-1 \). Therefore, the two sets of parameters being applied to \( y_{t,k} \) and \( z_{t,k} \), have to be equal.

\[
\begin{bmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2
\end{bmatrix} + V^{-1}(I-D)^{-1}(I-D^k)DV \begin{bmatrix}
0 \\
\hat{\beta}_2 / k
\end{bmatrix} = V^{-1}DV \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

\[
V^{-1}D^{-k} \begin{bmatrix}
V \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + (I-D)^{-1}(I-D^k)DV \begin{bmatrix}
0 & 0 \\
0 & 1/k
\end{bmatrix} \hat{\beta}_1
\end{bmatrix} = \begin{bmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

This proves the proposition and provides a formula to compute \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) in terms of the basic parameters \( \beta_1, \beta_2, \lambda_1 \) and \( \lambda_2 \). To find the linear relationship between \( [b_1, b_2] \) and \( [\hat{\beta}_1, \hat{\beta}_2] \), it is enough to multiply the matrix at the left hand side by \( V^\prime D^k V \).
Proof of Proposition 2. We will demonstrate that under the conditions of proposition 1, the OLS estimators of the phase average model are inconsistent and biased. The size of the bias is:

\[
\begin{align*}
\text{plim}(\hat{P}_1 - P_1) &= \frac{E[w_{t-k}u_{t-k}]}{(\text{Var}(y_{t-k}) - \text{Cov}(y_{t-k}, f(y_{t-k}, z_{t-k}))(\text{Var}(w_{t-k}) + \text{Var}(f(y_{t-k}, z_{t-k}))}^{-1}) \\
\text{plim}(\hat{P}_2 - P_2) &= \frac{E[w_{t-k}u_{t-k}]}{(\text{Var}(w_{t-k}) + \text{Var}(f(y_{t-k}, z_{t-k}))^2 (1-\psi^2) )}
\end{align*}
\]

\[
\psi = \text{Correl}(y_{t-k}, f(y_{t-k}, z_{t-k}))
\]

Under usual regularity conditions studied in section 2 of Chapter 2 the asymptotic properties of \( P \) are:

\[
\text{plim}(\hat{P} - P) = \left( E\left[ \begin{bmatrix}
1 & y_{t-k} & z_t
\end{bmatrix} \right] \right)^{-1} \left( E\left[ \begin{bmatrix}
1 & y_{t-k}
\end{bmatrix} u_t
\right] \right)
\]

The first expected value can be written as:

\[
E\left[ \begin{bmatrix}
1 & y_{t-k} & z_t
\end{bmatrix} \right] = \begin{bmatrix}
1 & E[y_{t-k}] & E[z_t] \\
E[y_{t-k}] & E[y_{t-k}^2] & E[y_{t-k}z_t] \\
E[z_t] & E[y_{t-k}z_t] & E[z_t^2]
\end{bmatrix}
\]

The average value of \( z \) can be written in terms of \( f(y_{t-k}, z_{t-k}) \) and \( w_t \). \( y_{t-k} \) is uncorrelated with \( w_t \), therefore:

\[
E\left[ \begin{bmatrix}
1 & y_{t-k} & z_t
\end{bmatrix} \right] = \begin{bmatrix}
1 & 0 & 0 \\
0 & E[y_{t-k}^2] & E[y_{t-k}f(y_{t-k}, z_{t-k})] \\
0 & E[y_{t-k}f(y_{t-k}, z_{t-k})] & E[w_{t-k}^2] + E[f(y_{t-k}, z_{t-k})^2]
\end{bmatrix}
\]

The inverse of the matrix above is straightforward and by using:
\[
E[\bar{z}, u_i] = E[f(y_{i-k}, z_{i-k})u_i + w_i u_i] = E[w_i u_i]
\]

We obtain the result that the average estimator of \(\beta_2\) for a large sample will be centred on the following value that is different from zero:

\[
p \lim (\hat{\beta}_2 - \beta_2) = \frac{\sigma^2 \sum_{j=0}^{k-1} g(j)}{\sigma^2 \sum_{j=0}^{k-1} g(j)^2 + \sigma^2 \sum_{j=0}^{k-1} \left( \frac{1 - \lambda' \beta}{1 - \lambda_2} \right) + (1 - \psi^2) \text{Var}(f(y_{i-k}, z_{i-k}))}
\]

\[
g(j) = \frac{\lambda - \beta \lambda_2}{\beta_1 - \lambda_2 \left( \frac{1 - \beta \lambda_2}{1 - \beta_1} - \frac{1 - \lambda_2}{1 - \lambda_2} \right)}
\]

**Proof of Proposition 3.**

We will use here the notation: \(Z = [y_{i-k}, z_{i-k}]\), these are the observations of the explanatory variables for the \(k\)-lagged model and \(X = [y_{i-k}, \bar{z}_i]\), the observations of the explanatory variables for the phase-average model. We identified in proposition 1 a linear mapping between the parameters of the phase-average model and the \(k\)-lagged model:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + V^{-1}(I - D)^{-1}(I - D^k)DV
\begin{bmatrix}
0 & 0 \\
0 & 1/k
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

We call this matrix \(A'\), and if \(\tilde{\beta}\) and \(\tilde{b}\) are the vector of parameters of each the phase-average and the \(k\)-lagged model respectively we have that:

\[
A' \tilde{\beta} = \tilde{b}, \text{ and}
\]

\[
X = ZA' + \xi_i, \text{ where}
\]
a) The conditional variance of the instrumental variables estimator, \( \hat{\beta} \), is:

\[
\sigma^2 \mathbb{E}[(Z'X)^{-1}Z'Z(X'Z)^{-1}]
\]

In order to be able to compare the parameters of the \( k \)-lagged model with the phase-average model we need to transform the estimate \( \hat{\beta} \) using the matrix \( A \) as follows:

\[
\text{var}(\hat{b}) = \text{var}(A'\hat{\beta})
\]

Therefore, \( \hat{b} \) is the transformed value of \( \hat{\beta} \) into the units of the \( k \)-lagged model. Only, through this relationship we can compare the performance of the two estimates (or with a transformation from the units of the \( k \)-lagged parameters to the units of the phase-average model). We compare the methods in the following way; in the one hand the variance of \( \hat{b} \) is:

\[
\text{var}(\hat{b}) = \mathbb{E}[(Z'Z + Z'\hat{\beta}_kA^{-1})^{-1}Z'Z(Z'Z + A^{-1}\hat{\beta}_kZ)^{-1}]\sigma^2\hat{\epsilon}
\]

Here, \( \sigma^2\hat{\epsilon} \) is the variance of the error term in the phase average model. On the other hand, the variance of \( \hat{b} \), the \( k \)-lagged model estimator, is:

\[
\text{var}(\hat{b}) = \mathbb{E}[(Z'Z)^{-1}]\sigma^2\hat{\epsilon}
\]

If the two variance terms, \( \sigma^2\hat{\epsilon} \) and \( \sigma^2\hat{\epsilon} \), are similar then the efficiency of \( \hat{b} \), the \( k \)-
lagged model estimator, with respect the phase average estimator will depend upon the following condition:

$$E[(I + Z'\xi A^{-1} (Z'Z)^{-1})(I + A^{-1}\xi' Z(Z'Z)^{-1})^{-1} | Z] > 1$$

More in general, the efficiency will depend on whether the unconditional expectation appearing in the variance $\text{var}(\tilde{b})$, minus $E[(Z'Z)^{-1}]$ is positive definite.

b) Let us look now at the asymptotic properties of the phase average estimator. Under the assumption that the error term is regular enough (for example normally distributed as in White 1984 p. 69), that the system matrix has all its eigenvalues below one in absolute value, and that the following two asymptotic distributions are valid for the residuals:

1. $Z' e / \sqrt{T} \sim N(0, \Omega_T)$ where $\tau = \text{var}(Z' / \sqrt{T})$
2. $Z' \tilde{e} / \sqrt{T} \sim N(0, \tilde{\Omega}_T)$ where $\tilde{\Omega}_T = \text{var}(Z' \tilde{e} / \sqrt{T})$

Then the estimated parameters for the instrumental-variable-phase-average model and the $k$-lagged model have the following distributions:

1. For the $k$-lagged model:

$$D^{-1/2}_T (\hat{b} - b) \sim N(0, I), \quad D_T = (E[Z'Z]/T)^{-1} E[Z'S_T Z/T] (E[Z'Z/T])^{-1}$$

$$S_T = \text{var}(e)I$$

2. For the phase-average model:

$$C^{-1/2}_T (\tilde{b} - \bar{b}) \sim N(0, I), \quad C_T = (E[Z'X]/T)^{-1} E[Z'\tilde{S}_T Z/T] (E[X'Z/T])^{-1}$$

$$\tilde{S}_T = \text{var}(\tilde{e})I$$
Therefore for the transformed estimator \( \tilde{b} \) we have:

\[
C_T^{-1/2} \sqrt{T} (\tilde{b} - b) = C_T^{-1/2} \sqrt{T} A^{-1} (\tilde{b} - b) \sim N(0, I)
\]

Thus:

\[
A' C_T A = (E[Z' X A^{-1}] / T)^{-1} E[Z' \tilde{S}_T Z / T] (E[A'^{-1} X' Z / T])^{-1}
\]

\[
E[Z' X A^{-1}] = E[Z' Z + Z' \xi] = E[Z' Z] + E[Z' \xi] = E[Z' Z]
\]

Here we have used that \( Z \) is independent from \( \xi \). We finally have:

\[
A' C_T A = (E[Z' Z] / T)^{-1} E[Z' \tilde{S}_T Z / T] (E[Z' Z / T])^{-1} = D_T \frac{\text{var}(\tilde{e})}{\text{var}(e)}
\]

Therefore asymptotically \( \tilde{b} = A' \tilde{b} \) will be more efficient than \( \hat{b} \) if and only if:

\[
\text{var}(e) > \text{var}(\tilde{e})
\]

c) If \( \beta_0 = 0 \) then the error term in both methods are the same, therefore using the result in b) where the asymptotic properties depend exclusively on the variance of the error term, both methods will perform equally.

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