

A FULLY DISCRETE NUMERICAL CONTROL METHOD FOR THE WAVE EQUATION*

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Abstract. We present a fully discrete finite element method for the interior null controllability problem subject to the wave equation. For the numerical scheme, piecewise affine continuous elements in space and finite differences in time are considered. We show that if the sharp geometric control condition holds, our numerical scheme yields the optimal rate of convergence with respect to the space-time mesh parameter. The approach is based on the design of stabilization terms for the discrete scheme with the goal of minimizing the computational error.

Key words. wave equation, control, numerical analysis, finite element method, stabilization

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1. Introduction. We consider the now-classical interior null controllability problem for the wave equation formulated as follows. Let $T > 0$, $\Omega \subset \mathbb{R}^n$ with $n \in \{2, 3\}$ be a connected bounded open set with smooth boundary. Finally, let $\omega \subset \Omega$ be an open set. We define $\mathcal{M} = (0, T) \times \Omega$ and $\mathcal{O} = (0, T) \times \omega$, and for each

$$(g_0, g_1, U) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\mathcal{M}),$$

consider the unique weak solution

$$u \in \mathcal{C}(0, T; H_0^1(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega))$$

of the following initial boundary value problem (IBVP):

$$(1.1) \quad \begin{cases} \square u = \partial_t^2 u - \Delta u = \chi_\omega U & \forall (t, x) \in \mathcal{M}, \\ u(t, x) = 0 & \forall (t, x) \in (0, T) \times \partial\Omega, \\ u(0, x) = g_0, \partial_t u(0, x) = g_1 & \forall x \in \Omega. \end{cases}$$

Here χ_ω is a suitable nonnegative smooth function that is localized in ω and is independent of the time parameter t .

The null controllability problem consists of determining a control function U_* , such that the solution u to (1.1) with $U = U_*$ satisfies

$$(1.2) \quad (u(T, x), \partial_t u(T, x)) = (0, 0) \quad \forall x \in \Omega.$$

This paper is concerned with a numerical scheme for solving the null controllability problem (1.1)–(1.2), based on the finite element method (FEM). In particular, we will prove optimal rate of convergence of the error in the H^1 -norm of the state variable u ,

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with respect to the space-time mesh parameter, assuming only the geometric control condition by Bardos, Lebeau, and Rauch [3, 4]. To the best of our knowledge, the present result is the first one giving optimal convergence rate in general geometries in dimensions two and three.

1.1. The geometric control condition and observability estimates. We begin by recalling the geometric control condition by Bardos, Lebeau, and Rauch.

DEFINITION 1.1 (see [4, 16]). *We say that $\tilde{\mathcal{O}} = (0, T) \times \tilde{\omega}$ satisfies the geometric control condition if every compressed generalized bicharacteristic intersects the set $\tilde{\mathcal{O}}$.*

The above definition roughly states that all geometric optic rays propagating in \mathcal{M} must intersect the region $\tilde{\mathcal{O}}$, taking into account possible reflections of the rays at the boundary. Next, we recall the following observability estimate originating from [3, 4]. The formulation here is based on [16, Proposition 1.2] and is stated as it appears in [6, Theorem 2.2].

THEOREM 1.2 (interior observability estimate). *Let $T > 0$, $\tilde{\omega} \subset \Omega$. Suppose that the set $\tilde{\mathcal{O}} = [0, T] \times \tilde{\omega}$ satisfies the geometric control condition. Let $U \in L^2(\mathcal{M})$ with $U|_{(0, T) \times \partial\Omega} \in L^2((0, T) \times \partial\Omega)$ and $\square U \in H^{-1}(\mathcal{M})$, where $H^{-1}(\mathcal{M})$ denotes the topological dual of $H_0^1(\mathcal{M})$. Then,*

$$U \in \mathcal{C}^1(0, T; H^{-1}(\Omega)) \cap \mathcal{C}(0, T; L^2(\Omega)).$$

Moreover, there exists $C_0 > 0$ such that the following estimate holds:

$$\begin{aligned} \sup_{t \in [0, T]} (\|U(t, \cdot)\|_{L^2(\Omega)} + \|\partial_t U(t, \cdot)\|_{H^{-1}(\Omega)}) \\ \leq C_0 (\|U\|_{L^2(\tilde{\mathcal{O}})} + \|\square U\|_{H^{-1}(\mathcal{M})} + \|U\|_{L^2((0, T) \times \partial\Omega)}). \end{aligned}$$

Observability estimates are one of the key tools in the study of the null controllability problem for the wave equation [22]. Although alternative geometric conditions are also available for obtaining such an estimate (see, for example, [10, 19]), it is important to note that the geometric control condition is sharp in the sense that it is both necessary and sufficient for obtaining an observability estimate.

1.2. Continuum null controllability problem. We recall the classical approach in showing the existence of a control function U that originates from [18]. Although, in general, the problem of determining a control function U solving (1.1)–(1.2) is nonunique, we may look for controls with additional constraints. The standard approach is to choose a control that additionally satisfies the (backward) wave equation as well, that is

$$(1.3) \quad \begin{cases} \square U = 0 & \forall (t, x) \in \mathcal{M}, \\ U(t, x) = 0 & \forall (t, x) \in (0, T) \times \partial\Omega, \\ U(T, x) = U_0, \partial_t U(T, x) = U_1 & \forall x \in \Omega \end{cases}$$

for some $(U_0, U_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

We recall from [17, Theorem 2.3] that (1.3) has a unique solution U in the energy space

$$\mathcal{C}^1(0, T; H^{-1}(\Omega)) \cap \mathcal{C}(0, T; L^2(\Omega)).$$

Observe that given any solution u to (1.1), and any solution V to (1.3), we have

$$(1.4) \quad \int_0^T (\chi_\omega(\cdot)U(\tau, \cdot), V(\tau, \cdot))_{L^2(\Omega)} d\tau = (\partial_t u(T, \cdot), V(T, \cdot))_{L^2(\Omega)} - (\partial_t u(0, \cdot), V(0, \cdot))_{L^2(\Omega)} - \langle u(T, \cdot), \partial_t V(T, \cdot) \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} + \langle u(0, \cdot), \partial_t V(0, \cdot) \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)}.$$

We deduce that (1.1)–(1.2) hold if and only if the following identity holds for any solution V to the wave equation (1.3):

$$(1.5) \quad \int_0^T (\chi_\omega(\cdot)U(\tau, \cdot), V(\tau, \cdot))_{L^2(\Omega)} d\tau = -(g_1, V(0, \cdot))_{L^2(\Omega)} + \langle g_0, \partial_t V(0, \cdot) \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)}.$$

Under the additional assumption that the control function U satisfies the wave equation (1.3), equation (1.5) is equivalent to the Euler–Lagrange equation for the Lagrangian functional

$$(1.6) \quad \mathcal{J}(U_0, U_1) = \frac{1}{2} \int_0^T \int_\Omega \chi_\omega |U|^2 dx dt - (g_1, U(0, \cdot))_{L^2(\Omega)} + \langle g_0, \partial_t U(0, \cdot) \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)},$$

where, for each $(U_0, U_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, U denotes the unique solution to (1.3) with this final datum.

To summarize, let $(U_{*,0}, U_{*,1})$ be a minimizer (if it exists) for the functional \mathcal{J} . Then, the solution U_* to (1.3) with this final datum yields a control function that drives the solution u_* of (1.1) with source term $\chi_\omega U_*$ from (g_0, g_1) to $(0, 0)$. In fact, one can show that U_* is the control function solving (1.1)–(1.2) with minimal $\|\sqrt{\chi_\omega}U\|_{L^2(\mathcal{M})}$ norm.

We will now briefly recall how the observability estimate in Theorem 1.2 proves existence of a unique minimizer for \mathcal{J} . Let us first consider the classical context where \mathcal{O} satisfies the geometric control condition and additionally that χ_ω is simply the characteristic function of the set ω . In this case, Theorem 1.2 implies that the functional \mathcal{J} is coercive and strictly convex (see, for example, [22, Theorem 2.4]). Together with the continuity of \mathcal{J} it follows that, in this setting, there exists a unique minimizer $(U_{*,0}, U_{*,1})$ in the space $L^2(\Omega) \times H^{-1}(\Omega)$.

It is, in fact, quite common in the literature to let χ_ω be the characteristic function of ω as above. In this case, the control function U_* suffers from low regularity that makes the task of numerical approximation and derivation of convergence rates challenging. Already in the seminal work [4], a theory for smoother boundary controls for the wave equation were studied. In [12, 13], the authors studied interior controls for the wave equation and, in particular, it was proved that one can construct smoother control functions by simply imposing some smoothness conditions on the initial datum (g_0, g_1) and using a sufficiently smooth cut-off function χ_ω . Let us recall their approach for the continuum problem. We need the following definition.

DEFINITION 1.3. *For each $s \in \mathbb{N}$, we say that $(y_0, y_1) \in \mathcal{D}((-\Delta)^s)$ if the following conditions are satisfied:*

- (i) $(y_0, y_1) \in H^{s+1}(\Omega) \times H^s(\Omega)$,
- (ii) $((-\Delta)^j y_0)|_{\partial\Omega} = 0$ for $j = 0, 1, \dots, \lfloor \frac{s}{2} + \frac{1}{4} \rfloor$,
- (iii) $((-\Delta)^j y_1)|_{\partial\Omega} = 0$ for $j = 0, 1, \dots, \lfloor \frac{s}{2} - \frac{1}{4} \rfloor$.

We now recall [12, Theorem 4] and [13, Theorem 1.6] to state some regularity results for the controls that are obtained when smoother cut-off functions are used.

THEOREM 1.4. *Let $s \in \mathbb{N}$. Suppose that χ_ω is a nonnegative smooth function localized in ω that maps $\mathcal{D}((-\Delta)^s)$ to itself. Assume also that the functional \mathcal{J} given by (1.6) is coercive and strictly convex. Given any initial datum $(g_0, g_1) \in \mathcal{D}((-\Delta)^s)$, let $(U_{*,0}, U_{*,1})$ denote the unique minimizer for \mathcal{J} on $L^2(\Omega) \times H^{-1}(\Omega)$. Then,*

$$(U_{*,0}, U_{*,1}) \in \mathcal{D}((-\Delta)^{s-1}).$$

Moreover, the following estimate holds:

$$(1.7) \quad \|U_*\|_{\mathcal{X}_s(\mathcal{M})} + \|u_*\|_{\mathcal{X}_{s+1}(\mathcal{M})} \leq C \|(g_0, g_1)\|_{H^{s+1}(\Omega) \times H^s(\Omega)},$$

where U_* is the unique solution to (1.3) with final datum $(U_{*,0}, U_{*,1})$, and u_* is the unique solution to (1.1) with source $\chi_\omega U_*$. Here, $C > 0$ is a constant depending only on \mathcal{M} , ω , χ_ω , and s , and $\mathcal{X}_s(\mathcal{M})$ denotes the Banach space $\mathcal{X}_s(\mathcal{M}) = \bigcap_{k=0}^s \mathcal{C}^k(0, T; H^{s-k}(\Omega))$.

Note that this theorem gives a continuum solution (u_*, U_*) to the null controllability problem (1.1)–(1.3), with smoothness properties given by (1.7). In this paper, we will need to apply Theorem 1.4 with $s = 3$. We will therefore begin by defining an admissibility condition for the set \mathcal{O} , based on the geometric control condition, followed by the admissible choices of the cut-off function χ_ω so that the assumptions of Theorem 1.4 are satisfied.

HYPOTHESIS 1.5 (admissibility condition for \mathcal{O}). *There exists $\delta > 0$ sufficiently small, such that the set $(0, T) \times \omega_\delta$ satisfies the geometric control condition, where*

$$\omega_\delta = \{x \in \omega \mid \text{dist}(x, \partial\omega \setminus \partial\Omega) > \delta\}.$$

Next, assuming that the set \mathcal{O} satisfies the admissibility condition above, we require that our cut-off function χ_ω satisfies the following three properties:

$$(1.8) \quad \begin{cases} \text{(i)} & \chi_\omega \in \mathcal{C}^\infty(\bar{\Omega}; [0, \infty)) \text{ and } \chi_\omega = 0 \text{ on the set } \Omega \setminus \omega, \\ \text{(ii)} & \chi_\omega = 1 \text{ on the set } \omega_\delta, \\ \text{(iii)} & (\partial_\nu^k \chi_\omega)|_{\partial\Omega} = 0 \text{ for } k = 1, 2, \text{ where } \nu \text{ denotes the unit normal vector to } \partial\Omega. \end{cases}$$

We will show in Appendix A that one can always construct such cut-off functions. As an example, we note that in the special case that $\partial\omega \cap \partial\Omega = \emptyset$, the cut-off function can be chosen as any function $\chi_\omega \in \mathcal{C}_c^\infty(\omega; [0, 1])$ that satisfies

$$\chi_\omega(x) = \begin{cases} 0 & \text{if } \text{dist}(x, \partial\omega) < \frac{\delta}{2}, \\ 1 & \text{if } \text{dist}(x, \partial\omega) > \delta. \end{cases}$$

In Appendix B we will show that under the Hypothesis 1.5 and given any cut-off function satisfying (i)–(iii) above, the two main assumptions of Theorem 1.4 are satisfied for $s = 3$. Therefore, this theorem applies to solve the null controllability problem (1.1)–(1.3) with the additional smoothness property that

$$u_* \in \mathcal{X}_4(\mathcal{M}) \quad \text{and} \quad U_* \in \mathcal{X}_3(\mathcal{M})$$

if the initial datum (g_0, g_1) belongs to the space $\mathcal{D}((-\Delta)^3)$. This smoothness class for the continuum solution to (1.1)–(1.3) will be important in our numerical analysis.

Before closing the section, let us emphasize that the results in this paper can also be applied to the problem of (interior) exact controllability, where the final state

$(u(T, x), \partial_t u(T, x))$ can be any pair of functions $(h_0, h_1) \in \mathcal{D}((-\Delta)^3)$. This is a consequence of the equivalence of the null and exact controllability problems for the wave equation. To illustrate this equivalence, let u_1 denote the unique solution to (1.1) with a homogeneous source term $U = 0$, but with the difference that the initial conditions are imposed at the final time $t = T$ that is to say $u_1(T, \cdot) = h_0$ and $\partial_t u_1(T, \cdot) = h_1$. This is possible due to the time-reversibility of the wave equation. Subsequently, let $(u_1|_{t=0}, \partial_t u_1|_{t=0}) = (\tilde{g}_0, \tilde{g}_1)$. Finally, let $\chi_\omega U$ be a null control that drives the system from initial data $(g_0 - \tilde{g}_0, g_1 - \tilde{g}_1)$ to $(0, 0)$. It is clear that $\chi_\omega U$ is a control that drives the solution u to (1.1) from (g_0, g_1) to (h_0, h_1) .

1.3. Previous literature. It is well known that methods based on minimizing discrete analogues of the Lagrangian (1.6) may fail to converge. This is the case, for example, when second-order central finite differences in both time and space are used to discretize (1.3), and the so-obtained discretization of (1.6) is minimized by using the conjugate gradient algorithm. This was first observed by Glowinski et al. in a series of works in the early 1990s. An excellent summary of these works is provided by Glowinski and Lions in sections 6.8–9 of [14]. It was observed that trouble lies with the high-frequency components of the discrete solution (see, e.g., section 6.8.6 of [14]) and different regularization techniques were proposed. For example, a Tikhonov type regularization procedure based on use of the biharmonic operator is discussed in detail in [14], and the efficiency of the regularization is demonstrated by numerical experiments.

The spurious modes arising at high frequencies from a finite-difference semi-discretization of the one dimensional wave equation were first rigorously analyzed in [15]. In particular, it was shown that the analogue of the estimate in Theorem 1.2 fails on the discrete level. Several numerical methods based on filtering of the spurious high frequency modes were subsequently proposed. As an early example of a result in this tradition, we mention [21] where weak convergence of a subsequence of semi-discrete approximations of a control function for the one dimensional wave equation was proven. For a thorough review of the filtering approach, we refer to the monograph [12]. There it is also shown that a semi-discrete variant of the approach has optimal convergence under the assumption that the analogue of the estimate in Theorem 1.2 is recovered on the discrete level after suitable filtering. However, it is not known if such filtered estimates hold, in general, when only the geometric control condition is assumed; see the discussion in section 5.3 of [12].

Instead of considering the control function satisfying (1.3), it is also possible to follow Russell's *stabilization implies control* principle [23]. On the continuum level, this involves an alternating iteration solving forward and backward wave equations. A suitably semi-discretized version of this scheme leads to a solution method to the null control problem with a rate of convergence exhibiting only a logarithmic loss when compared to the optimal rate [8]. However, the scheme requires that the alternating iteration is stopped after a specific number of steps, depending, for example, on the constant C in Theorem 1.2, and this stopping criterion may not be easy to implement in practice. As demonstrated in section 1.7.1.2 of [12], the iteration, in fact, diverges as the number of steps grows too large.

In a recent work [9], Cîndea and Münch formulate the controllability problem so that the wave equation (1.3) enters into the Lagrangian functional (1.6) via a Lagrange multiplier. The Lagrangian functional is further augmented with the L^2 -norm of $\square U$. In a subsequent work [20], a Lagrange multiplier is used to impose the wave equation as a first order system. The efficiency of the resulting methods is demonstrated by

numerical experiments. However, their convergence analysis is not complete as it is not known if the discrete inf-sup constants for the methods; see (39) in [9] and (6.9) in [20], respectively, are uniformly bounded from below.

Our approach is based on a Lagrangian functional where the initial conditions in (1.1) together with the final conditions in (1.2) are imposed via penalty terms, and similarly to [9], (1.1) and (1.3) are imposed via Lagrange multipliers. Instead of augmentation, we add Tikhonov type regularization terms that vanish at the correct rate as the mesh size tends to zero. This allows us to prove a discrete inf-sup property (Proposition 3.1), and subsequently an optimal convergence rate (Theorem 2.1). The present method can be seen as the continuation of our previous work in [6], where we studied numerical approximation of the dual problem to the controllability problem discussed here, that is, the data assimilation problem subject to the wave equation. A detailed comparison between these two works is given in section 6.

1.4. Outline of the paper. We start section 2 by presenting the first order finite element spaces that are used for the numerical approximation of the null controllability problem. A Lagrangian functional is then formulated in the discrete level, and the main theorem is stated (Theorem 2.1) that gives a numerical method for solving the null controllability problem. Section 3 is concerned with proving a suitable inf-sup stability estimate (Proposition 3.1) for the discrete Lagrangian. We also show the existence of a unique critical point for the discrete Lagrangian. In section 4, the inf-sup stability estimate is used together with a continuity estimate for the residual error (Lemma 4.3) to obtain a weak a priori control on the error function (Proposition 4.2). This proposition is then used to obtain an approximate version of the observability estimate in Theorem 1.2 at the discrete level (Proposition 4.6). Section 5 is concerned with the proof of Theorem 2.1. There, the key ingredients are the coercivity (Lemma 5.4), together with the approximate discrete observability estimate (Proposition 4.6). Finally, in section 6 we give a detailed comparison with our earlier work for the data assimilation problem together with some concluding remarks.

2. Discretization. Let us now present the discretization approach for (1.1)–(1.3). We will use finite differences in time and first order finite elements in space. Let $N \in \mathbb{N}$ and define $\tau = \frac{T}{N}$ to denote the uniform length of the time-steps in our numerical method. Also, let $\{t_k\}_{k=0}^N$ be defined through $t_k = k\tau$. We begin by discretizing the boundary $\partial\Omega$ and denote the resultant polyhedral domain by Ω_h . This polyhedral approximation is assumed to be sufficiently close to Ω in the sense that

$$(2.1) \quad \text{dist}(x, \partial\Omega) \leq Ch^2 \quad \forall x \in \partial\Omega_h$$

for some constant $C > 0$ that is independent of h . This is always possible since Ω has a smooth boundary (see [2], for example). Subsequently, we consider a spatial mesh \mathcal{T}_h which is a conforming quasi-uniform triangulation of the polyhedral domain Ω_h and define h_K to be the local space mesh size. We set $h = \max_{K \in \mathcal{T}_h} h_K$ to be the global mesh parameter in space and make the standing assumption that the discrete time steps τ and the spatial mesh parameter h are comparable in size, that is to say $\tau = \mathcal{O}(h)$.

We now define the spatial finite element space \mathbb{V}_h to be the space of piecewise affine continuous finite elements satisfying zero boundary condition,

$$\mathbb{V}_h = \{v \in H_0^1(\Omega_h) : v|_K \in \mathbb{P}_1(K) \forall K \in \mathcal{T}_h\}.$$

For each $u, v \in \mathbb{V}_h$, let

$$(u, v)_h = \int_{\Omega_h} u(x)v(x) dx \quad \text{and} \quad a_h(u, v) = \int_{\Omega_h} \nabla u(x) \cdot \nabla v(x) dx,$$

and

$$\|u\|_h = \sqrt{(u, u)_h}.$$

Next, we define the space-time mesh

$$\mathbb{V}_h^{N+1} = \underbrace{\mathbb{V}_h \times \mathbb{V}_h \times \cdots \times \mathbb{V}_h}_{N+1 \text{ times}}$$

and, subsequently, for each

$$u = (u^0, u^1, \dots, u^N) \in \mathbb{V}_h^{N+1},$$

we define the backward and forward discrete time differences $\partial_\tau, \tilde{\partial}_\tau$ as follows:

$$\begin{aligned} \partial_\tau u^n &= \frac{u^n - u^{n-1}}{\tau} \quad \text{for } n = 1, \dots, N, \\ \tilde{\partial}_\tau u^n &= \frac{u^n - u^{n+1}}{\tau} \quad \text{for } n = 0, \dots, N-1. \end{aligned}$$

We note in passing that the forward discrete time difference $\tilde{\partial}_\tau$ acting on $(u^n)_{n=0}^N$ can be thought of as the backward time difference for the discrete function $(u^{N-n})_{n=0}^N$. Note also that second order time differences can be written through

$$\begin{aligned} \partial_\tau^2 u^n &= \frac{u^n - 2u^{n-1} + u^{n-2}}{\tau^2} \quad \text{for } n = 2, \dots, N, \\ \tilde{\partial}_\tau^2 u^n &= \frac{u^n - 2u^{n+1} + u^{n+2}}{\tau^2} \quad \text{for } n = 0, \dots, N-2. \end{aligned}$$

Finally, consider any smooth extension of χ_ω to $\Omega \cup \Omega_h$ and define a nonnegative discrete approximate χ_h of the smooth function χ_ω such that $\chi_h \in \mathbb{V}_h$ and

$$(2.2) \quad \|\chi_\omega - \chi_h\|_{L^\infty(\Omega_h)} + h\|\chi_\omega - \chi_h\|_{W^{1,\infty}(\Omega_h)} \leq Ch^2,$$

where $C > 0$ is independent of h . Note that this is possible due to the smoothness assumption on χ_ω along with smoothness of $\partial\Omega$ and (2.1).

We now return to the null controllability problem (1.1)–(1.3). Given any $u = (u^0, \dots, u^N)$, $U = (U^0, \dots, U^N)$ in \mathbb{V}_h^{N+1} , and $z = (z^2, \dots, z^N)$, $Z = (Z^0, \dots, Z^{N-2})$ in \mathbb{V}_h^{N-1} , we define the discrete Lagrangian functional

$$\mathcal{J} : \mathbb{V}_h^{N+1} \times \mathbb{V}_h^{N+1} \times \mathbb{V}_h^{N-1} \times \mathbb{V}_h^{N-1} \rightarrow \mathbb{R}$$

through the expression

$$\begin{aligned} (2.3) \quad \mathcal{J}(u, U, z, Z) &= \mathcal{J}_0(u, U, z, Z) + \mathcal{J}_1(U), \\ \mathcal{J}_0(u, U, z, Z) &= \mathcal{G}(u, z) - \tau \sum_{n=2}^N (\chi_h U, z)_h + \mathcal{G}^*(Z, U) + \mathcal{R}(u), \\ \mathcal{J}_1(U) &= \frac{h^2}{2} \|\nabla U^N\|_h^2 + \frac{h^2}{2} \|\partial_\tau \nabla U^N\|_h^2 + \frac{h^2}{2} \|\partial_\tau \nabla U^1\|_h^2 + \frac{\tau h^2}{2} \sum_{n=1}^N \|\partial_\tau \nabla U^n\|_h^2, \end{aligned}$$

$$\begin{aligned}\mathcal{G}(u, z) &= \tau \sum_{n=2}^N \left((\partial_\tau^2 u^n, z^n)_h + a_h(u^n, z^n) \right), \\ \mathcal{G}^*(Z, U) &= \tau \sum_{n=0}^{N-2} \left((Z^n, \tilde{\partial}_\tau^2 U^n)_h + a_h(Z^n, U^n) \right), \\ \mathcal{R}(u) &= \frac{1}{2} \left[\|\nabla u^N\|_h^2 + \|\partial_\tau u^N\|_h^2 + \|\nabla(u^0 - g_0)\|_h^2 + \|\partial_\tau u^1 - g_1\|_h^2 \right].\end{aligned}$$

Let us make some remarks about the discrete Lagrangian \mathcal{J} . Here, the variables u and U should be interpreted as discrete analogues of the state variable and the control function, while z and Z are discrete variables. The terms $\mathcal{G}(u, z) - \tau \sum_{n=2}^N (\chi_\omega U, z)_h$ and $\mathcal{G}^*(Z, U)$ are weak formulations of the first equations in (1.1) and (1.3), respectively. Although in continuum, the forward and backward wave equations are completely equivalent, we use a backward discrete wave equation for the discrete control variable U . This will be important in the proof of convergence rates for our numerical analysis (see Lemma 5.3). The functional \mathcal{R} imposes the initial conditions in (1.1) as well as the final conditions (1.2). Note that the initial states z^0, z^1 for z and final states Z^{N-1}, Z^N for Z do not appear in the formulation and can be taken to be zero. Intuitively, the Lagrange multipliers are solving inhomogeneous wave equations with zero initial or final data. To summarize, \mathcal{J}_0 corresponds to (1.1)–(1.3).

We have incorporated the numerical stabilizers (also called regularizers) in the discrete level through the functional $\mathcal{J}_1(U)$. The design of these terms is driven with the goal of minimizing the errors in the numerical approximation of the null controllability problem. The first two terms in \mathcal{J}_1 correspond to the energy for the wave equation (1.3) at time $t = T$ and seem a natural inclusion, while the remaining two terms are in part motivated by our previous works for data assimilation problems for heat and wave equations [6, 7]. The regularization term in mixed derivatives also appears in [1].

Heuristically, we expect to have a critical (saddle) point in the sense that the Lagrangian attains the value

$$\inf_{(u, U) \in \mathbb{V}_h^{2N+2}} \sup_{(z, Z) \in \mathbb{V}_h^{2N-2}} \mathcal{J}(u, U, z, Z).$$

Moreover, we expect this critical point to converge to the continuum solution of the control problem (1.1)–(1.3) with the Lagrange multipliers (z, Z) converging to zero as $h \rightarrow 0$.

The Euler–Lagrange equations for the functional \mathcal{J} can be written in the form

$$\langle D_u \mathcal{J}, v \rangle + \langle D_U \mathcal{J}, V \rangle + \langle D_z \mathcal{J}, w \rangle + \langle D_Z \mathcal{J}, W \rangle = 0 \quad \forall (v, V, w, W) \in \mathbb{V}_h^{4N},$$

where D_s denotes the Fréchet derivative with respect to $s \in \{u, U, z, Z\}$. Letting

$$x = (u, U, z, Z) \in \mathbb{V}_h^{4N} \quad \text{and} \quad y = (v, V, w, W) \in \mathbb{V}_h^{4N},$$

we see that the Euler–Lagrange equations can be recast in the form

$$(2.4) \quad \mathcal{A}(x; y) = a_h(v^0, g_0) + (\partial_\tau v^1, g_1)_h \quad \forall y \in \mathbb{V}_h^{4N},$$

where $\mathcal{A} : \mathbb{V}_h^{4N} \times \mathbb{V}_h^{4N} \rightarrow \mathbb{R}$ is a bilinear form defined through

$$\mathcal{A}(x; y) = \mathcal{A}_0(u, z; v) + \mathcal{A}_1(U, z, Z; V) + \mathcal{A}_2(u, U; w) + \mathcal{A}_3(U; W),$$

with

$$\begin{aligned} \mathcal{A}_0(u, z; v) &= \mathcal{G}(v, z) + a_h(u^0, v^0) + a_h(u^N, v^N) + (\partial_\tau u^1, \partial_\tau v^1)_h + (\partial_\tau u^N, \partial_\tau v^N)_h, \\ \mathcal{A}_1(U, z, Z; V) &= \mathcal{G}^*(Z, V) - \tau \sum_{n=2}^N (\chi_h V, z)_h + h^2(\nabla U^N, \nabla V^N)_h \\ &\quad + h^2(\partial_\tau \nabla U^N, \partial_\tau \nabla V^N)_h \\ &\quad + h^2(\partial_\tau \nabla U^1, \partial_\tau \nabla V^1)_h + \tau \sum_{n=1}^N (h \partial_\tau \nabla U^n, h \partial_\tau \nabla V^n)_h, \\ \mathcal{A}_2(u, U; w) &= \mathcal{G}(u, w) - \tau \sum_{n=2}^N (\chi_h U, w)_h, \\ \mathcal{A}_3(U; W) &= \mathcal{G}^*(W, U). \end{aligned}$$

Observe, in particular, that the expressions for \mathcal{A}_2 and \mathcal{A}_3 imply that the Euler–Lagrange equations for u and U enforce discrete versions of (1.1) and (1.3). Indeed, the state variable u must solve the discrete forward wave equation with source term $\chi_h U$, while the control variable U must solve the discrete backward wave equation. We are now ready to state the main theorem in the paper as follows.

THEOREM 2.1. *Suppose that Hypothesis 1.5 holds for the set $\mathcal{O} = (0, T) \times \omega$. Let χ_ω be any function that satisfies properties (i)–(iii) in (1.8). Let $(g_0, g_1) \in \mathcal{D}((-\Delta)^3)$ and denote by (u_*, U_*) , the unique continuum solution to the interior null controllability problem (1.1)–(1.3). Then, there exists $h_0 > 0$, such that for all $0 < h < h_0$, the Euler–Lagrange equation (2.4) admits a unique solution denoted by (u_h, U_h, z_h, Z_h) . Moreover, for $n = 1, \dots, N$:*

- (i) $\|U_*^n - U_h^n\|_{L^2(\Omega)} + \|\partial_t U_*^n - \partial_\tau U_h^n\|_{H^{-1}(\Omega)} \leq C h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}$,
 - (ii) $\|u_*^n - u_h^n\|_{H^1(\Omega)} + \|\partial_t u_*^n - \partial_\tau u_h^n\|_{L^2(\Omega)} \leq C h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}$,
- where $U_*^n(\cdot) = U_*(n\tau, \cdot)$, $u_*^n(\cdot) = u_*(n\tau, \cdot)$ and $C > 0$ is a constant independent of the mesh parameter¹ h that only depends on $T, \mathcal{M}, \omega, \delta$.

See Proposition 4.2 for the estimates of z_h and Z_h .

3. Inf-sup stability estimate. This section is concerned with the study of the Euler–Lagrange equation (2.4). First, we define $\kappa, \tilde{\kappa} > 0$ to be constants such that for all $u \in \mathbb{V}_h$

$$(3.1) \quad \max\{h, \tau\} \|\nabla u\|_h \leq \kappa \|u\|_h \quad \text{and} \quad \tilde{\kappa} \|u\|_h \leq \|\nabla u\|_h,$$

where we recall that h, τ are the mesh parameters. The existence of these constants is guaranteed by the discrete inverse inequality that follows from the fact that the space-mesh is quasi-uniform (see, for instance, [5, Lemma 4.5.3]) together with the Poincaré inequality and the standing assumption that $\tau = \mathcal{O}(h)$.

¹ Recall that h and τ are assumed to be comparable, that is, $\tau = \mathcal{O}(h)$.

We introduce the following discrete norms and semi-norms:

(3.2)

$$\begin{aligned}
 |||(u, U)|||_R^2 &= \|\nabla u^N\|_h^2 + \|\partial_\tau u^N\|_h^2 + \|\nabla u^0\|_h^2 + \|\partial_\tau u^1\|_h^2 + \tau \sum_{n=1}^N \|h \partial_\tau \nabla U^n\|_h^2 \\
 &\quad + h^2 \|\nabla \partial_\tau U^N\|_h^2 + h^2 \|\partial_\tau \nabla U^1\|_h^2 + h^2 \|\nabla U^N\|_h^2, \\
 |||u|||_{F'}^2 &= \tau \sum_{n=1}^N (\|\partial_\tau u^n\|_h^2 + \|\nabla u^n\|_h^2), \\
 |||U|||_{F'}^2 &= \tau \sum_{n=2}^N (\|\partial_\tau^2 U^n\|_h^2 + \|\partial_\tau \nabla U^n\|_h^2 + \|\partial_\tau U^n\|_h^2 + \|\nabla U^n\|_h^2) \\
 &\quad + \|\partial_\tau U^1\|_h^2 + \|\nabla U^1\|_h^2, \\
 |||z|||_D^2 &= \tau \sum_{n=2}^N \|z^n\|_h^2 + \tau \sum_{n=2}^N \|\nabla \mathcal{I}z^n\|_h^2 + \|\nabla \mathcal{I}z^N\|_h^2 + \|z^N\|_h^2 \\
 |||Z|||_{D'}^2 &= \tau \sum_{n=0}^{N-2} \|Z^n\|_h^2 + \tau \sum_{n=0}^{N-2} \|\nabla \tilde{\mathcal{I}}Z^n\|_h^2 + \|\nabla \tilde{\mathcal{I}}Z^0\|_h^2 + \|Z^0\|_h^2 \\
 |||(u, U, z, Z)|||_C^2 &= |||(u, U)|||_R^2 + \tau \sum_{n=2}^N \|z^n\|_h^2 + \tau \sum_{n=0}^{N-2} \|Z^n\|_h^2.
 \end{aligned}$$

Here,

$$\mathcal{I}z^n = \tau \sum_{m=0}^n (1 + m\tau)z^m \quad \text{and} \quad \tilde{\mathcal{I}}Z^n = \tau \sum_{m=n}^N (1 + (N - m)\tau)Z^m,$$

where we have defined $z^0 = z^1 = 0$ and $Z^{N-1} = Z^N = 0$. Note that using the Poincaré inequality we have the following:

$$\|\nabla \mathcal{I}z^n\|_h \geq C \|\mathcal{I}z^n\|_h \quad n = 2, \dots, N,$$

for some $C > 0$ independent of h , with an analogous estimate holding for $\tilde{\mathcal{I}}Z$ as well.

The above norms and semi-norms have the following interpretations. The $|||(\cdot, \cdot)|||_R$ semi-norm captures the stability properties of the bilinear form \mathcal{A} due to the regularization terms in \mathcal{J}_1 and the data fitting terms in \mathcal{R} . The norms $|||\cdot|||_F$, $|||\cdot|||_{F'}$, $|||\cdot|||_D$, and $|||\cdot|||_{D'}$ quantify stability properties of the discrete wave equations for u, U, z , and Z given by $\mathcal{G}(u, z)$ and $\mathcal{G}^*(Z, U)$. There is a delicate counterbalance in the strength of the norms $|||\cdot|||_F, |||\cdot|||_{F'}$ (in terms of the Sobolev scales) for the functions u, U compared to that of $|||\cdot|||_D, |||\cdot|||_{D'}$ for the Lagrange multipliers z, Z . For instance, in the continuum limit $\tau, h \rightarrow 0$, the $|||\cdot|||_F$ norm is reminiscent to $\|\cdot\|_{H^1(\mathcal{M})}$, while the $|||\cdot|||_D$ norm is analogous to $\|\cdot\|_{H^{-1}(0,T;H^1(\Omega))} + \|\cdot\|_{L^2(\mathcal{M})}$. The $|||(\cdot, \cdot)|||_C$ semi-norm quantifies continuity of \mathcal{A} in the dual variables from above and below; see Proposition 3.1 below and Lemma 4.3.

The rest of this section is concerned with the following proposition.

PROPOSITION 3.1. *There exists $h_0, C > 0$ such that for all $h \in (0, h_0)$ and all $x = (u, U, z, Z) \in \mathbb{V}_h^{4N}$ there exists $y = (v, V, w, W) \in \mathbb{V}_h^{4N}$ satisfying*

$$\mathcal{A}(x; y) \gtrsim |||(u, U)|||_R^2 + h^2 |||u|||_{F'}^2 + h^2 |||U|||_{F'}^2 + |||z|||_D^2 + |||Z|||_{D'}^2 \gtrsim |||y|||_C^2.$$

Remark 1. Throughout the remainder of this paper we use the notation $A \gtrsim B$, to imply the existence of a positive constant $C > 0$ independent of the mesh parameter h , such that $A \geq CB$.

We can use Proposition 3.1 to show that the Euler–Lagrange equation (2.4) admits a unique solution $x_h = (u_h, U_h, z_h, Z_h) \in \mathbb{V}_h^{4N}$. Indeed, let N_h denote the dimension of \mathbb{V}_h . Equation (2.4) is a linear system governed by a $4N_h N \times 4N_h N$ matrix. Existence and uniqueness of a discrete solution x_h will follow, if we can show that the kernel of this matrix is trivial. But this follows immediately from Proposition 3.1.

Before presenting the proof of Proposition 3.1, we state a few lemmas, the first of which is trivial.

LEMMA 3.2. *Let $x = (u, U, z, Z) \in \mathbb{V}_h^{4N}$. If $y = (u, U, -z, -Z)$, then*

$$\mathcal{A}(x; y) = \|(u, U)\|_R^2.$$

The estimates in the next lemma are discrete analogues of energy estimates for the wave equation corresponding to various Sobolev norms. The energy estimates for u and U will be stronger in the Sobolev scale but eventually rescaled by h^2 and this will be balanced by weaker Sobolev spaces with no scaling on the dual variables z, Z . For the proof, we refer the reader to [6, Remark 1 and Lemmas 3.4 and 3.5].

LEMMA 3.3. *Let $(u, U, z, Z) \in \mathbb{V}_h^{4N}$. We define $z^0 = z^1 = Z^{N-1} = Z^N = 0$. Define the test function $y = (v, V, w, W)$ through*

$$\begin{aligned} v^n &= \mathcal{I}z^n \quad \text{for } n = 0, \dots, N, \\ V^n &= \tilde{\mathcal{I}}Z^n \quad \text{for } n = 0, \dots, N, \\ w^n &= (2T - n\tau)\partial_\tau u^n \quad \text{for } n = 2, \dots, N, \\ W^n &= \tilde{\partial}_\tau^2 U^n + (2T - (N - n)\tau)\tilde{\partial}_\tau U^n \quad \text{for } n = 0, \dots, N - 2. \end{aligned}$$

The following estimates hold:

$$(3.3) \quad \begin{aligned} \mathcal{G}(u, w) &\gtrsim \|u\|_F^2 - \|(u, 0)\|_R^2, & \mathcal{G}(v, z) &\gtrsim \|z\|_D^2, \\ \mathcal{G}^*(h^2 W, U) &\gtrsim h^2 \|U\|_{F'}^2 - \|(0, U)\|_R^2, & \mathcal{G}^*(Z, V) &\gtrsim \|Z\|_{D'}^2, \end{aligned}$$

where the constants in the inequalities only depend on T, Ω .

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $x = (u, U, z, Z)$ and define $y = (\hat{v}, \hat{V}, \hat{w}, \hat{W}) \in \mathbb{V}^{4N}$ through

$$(3.4) \quad \begin{aligned} \hat{v}^n &= u^n + \gamma v^n, & \hat{V}^n &= U^n + \alpha V^n, \\ \hat{w}^n &= -z^n + \alpha h^2 w^n, & \hat{W}^n &= -Z^n + \gamma h^2 W^n, \end{aligned}$$

where $\gamma > \alpha > 0$ and v, V, w, W are chosen as in Lemma 3.3. Recalling the definition of the linear form $\mathcal{A}(x; y)$ together with Lemma 3.2, we write

$$\mathcal{A}(x; y) = \|(u, U)\|_R^2 + \gamma \mathcal{A}_0(u, z; v) + \alpha \mathcal{A}_1(U, z, Z; V) + \alpha h^2 \mathcal{A}_2(u, U; w) + \gamma h^2 \mathcal{A}_3(U; W).$$

By Lemma 3.3, there exists $C_1, C_2 > 0$ only depending on T, Ω such that

$$(3.5) \quad \begin{aligned} &\alpha h^2 \mathcal{G}(u, w) + \gamma \mathcal{G}(v, z) + \alpha \mathcal{G}(Z, V) + \gamma h^2 \mathcal{G}(W, U) \\ &\geq C_1 (h^2 \gamma \|U\|_{F'}^2 + \alpha h^2 \|u\|_F^2 + \gamma \|z\|_D^2 + \alpha \|Z\|_{D'}^2) - C_2 \gamma \|(u, U)\|_R^2. \end{aligned}$$

We now set $\alpha = \alpha_0\gamma$ for a fixed $0 < \alpha_0 < \min\{1, \frac{3}{4}\tilde{\kappa}^2C_1^2, \frac{C_1^2}{4T^2}\}$ and show that the proposition holds for this choice of $y \in \mathbb{V}_h^{4N}$ when γ is sufficiently small independent of h . First, note that

$$a_h(u^0, v^0) = a_h(\partial_\tau u^1, \partial_\tau v^1) = a_h(U^N, V^N) = a_h(\partial_\tau U^N, \partial_\tau V^N) = 0.$$

We use the Cauchy–Schwarz inequality to obtain the following bounds for the remaining (possibly) negative terms in $\mathcal{A}(x; y)$:

$$\begin{aligned} |a_h(u^N, v^N)| &\leq \frac{1}{C_1} \|\nabla u^N\|_h^2 + \frac{C_1}{4} \|\nabla \mathcal{I}z^N\|_h^2, \\ |(\partial_\tau u^N, \partial_\tau v^N)_h| &\leq \frac{(1+T)^2}{C_1} \|\partial_\tau u^N\|_h^2 + \frac{C_1}{4} \|z^N\|_h^2, \\ h^2 |(\partial_\tau \nabla U^1, \partial_\tau \nabla V^1)_h| &\leq h^2 \frac{\kappa^2(1+T)^2}{C_1} \|\partial_\tau \nabla U^1\|_h^2 + \frac{C_1}{4} \|Z^0\|_h^2, \\ \tau \sum_{n=1}^N |(\tau \partial_\tau \nabla U^n, \tau \partial_\tau \nabla V^n)_h| &\leq \frac{\kappa^2(1+T)^2}{C_1} \tau \sum_{n=1}^N \|\tau \partial_\tau \nabla U^n\|_h^2 + \frac{C_1}{4} \tau \sum_{n=0}^{N-2} \|Z^n\|_h^2, \\ h^2 \left| \tau \sum_{n=2}^N (\chi_h U^n, w^n)_h \right| &\leq \frac{4T^2}{C_1} \tau \sum_{n=2}^N h^2 \|U^n\|_h^2 + \frac{C_1}{4} \tau \sum_{n=2}^N h^2 \|\partial_\tau u^n\|_h^2, \\ \left| \tau \sum_{n=2}^N (\chi_h V, z)_h \right| &\leq \frac{1}{\tilde{\kappa}^2 C_1} \tau \sum_{n=2}^N \|z^n\|_h^2 + \frac{C_1}{4} \tau \sum_{n=0}^{N-2} \|\nabla \tilde{\mathcal{I}}Z^n\|_h^2. \end{aligned}$$

Combining these bounds we deduce that

$$\begin{aligned} \mathcal{A}(x; y) &\geq \| (u, U) \|_R^2 - C_3\gamma \| (u, U) \|_R^2 + \alpha \frac{3C_1}{4} \| Z \|_{D'}^2 + \alpha \frac{3C_1}{4} h^2 \| u \|_F^2 \\ &\quad + \left(\frac{3C_1\gamma}{4} - \frac{\alpha}{\tilde{\kappa}^2 C_1} \right) \| z \|_D^2 + \left(C_1\gamma - \frac{4T^2\alpha}{C_1} \right) h^2 \| U \|_{F'}^2, \end{aligned}$$

where $C_3 = C_2 + 2(1 + \kappa^2)(1 + T)^2 \frac{1}{C_1}$. The first claimed inequality then follows for γ sufficiently small. To prove the second inequality in the proposition, we use the Cauchy–Schwarz inequality to obtain the following bounds for $y \in \mathbb{V}_h^{4N}$:

$$\begin{aligned} (3.6) \quad \| (\hat{v}, 0) \|_R^2 &\lesssim \| (u, 0) \|_R^2 + \| z \|_D^2, \quad \| (0, \hat{V}) \|_R^2 \lesssim \| (0, U) \|_R^2 + \| Z \|_{D'}^2, \\ \tau \sum_{n=2}^N \|\hat{w}^n\|_h^2 &\lesssim \tau \sum_{n=2}^N \|z^n\|_h^2 + h^2 \|u\|_F^2, \quad \tau \sum_{n=0}^{N-2} \|\hat{W}^n\|_h^2 \lesssim \tau \sum_{n=2}^N \|Z^n\|_h^2 + h^2 \|U\|_{F'}^2. \end{aligned}$$

This completes the proof. □

4. A weak a priori error estimate and an approximate discrete observability estimate for the error. Throughout this section, we will let (u_h, U_h, z_h, Z_h) denote the unique solution to (2.4). The main goal here is to prove an approximate discrete analogue of the continuum observability estimate.² This will be done in several steps. We start by proving a weak preliminary error estimate (Proposition 4.2), and then use this estimate to prove Proposition 4.6 that we call an approximate discrete observability estimate for the error function. This proposition will subsequently be used as a key ingredient to prove the main theorem.

² Not to be confused with discrete observability estimates in the literature.

In what follows, we will let (u_*, U_*) denote the continuum solution to (1.1)–(1.3). Let us observe that since $(g_0, g_1) \in \mathcal{D}((-\Delta)^3)$, it follows from Theorem 1.4 that

$$\|u_*\|_{\mathcal{X}_4(\mathcal{M})} + \|U_*\|_{\mathcal{X}_3(\mathcal{M})} \leq C\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)},$$

where $C > 0$ only depends on $\mathcal{M}, \omega, \delta$ and $\mathcal{X}_3(\mathcal{M}) = \bigcap_{k=0}^3 \mathcal{C}^k(0, T; H^{3-k}(\Omega))$.

Due to the mismatch between Ω and Ω_h , we extend the functions in \mathbb{V}_h to all of $\Omega_h \cup \Omega$ by setting them to zero on the set $\Omega \setminus \Omega_h$. We will also utilize the extension operator (see [24]) $E : H^s(\Omega) \rightarrow H^s(\Omega \cup \Omega_h)$, $s \geq 0$ to define the extended functions u_*^e and U_*^e on the set $(0, T) \times (\Omega \cup \Omega_h)$ through

$$u_*^e(t, \cdot) = Eu_*(t, \cdot) \quad \text{and} \quad U_*^e(t, \cdot) = EU_*(t, \cdot) \quad t \in [0, T].$$

We will slightly abuse the notation by dropping the superscript e when there is no confusion.

Let us recall the definition of the H^1 projection interpolator $\pi_h : H_0^1(\Omega) \rightarrow \mathbb{V}_h$ defined through

$$(4.1) \quad a_h(\pi_h u, v) = a_h(u, v) \quad \forall v \in \mathbb{V}_h.$$

We have the following lemma. For the proof, we refer the reader to [6, Lemma 4.2].

LEMMA 4.1. *Let $u \in H_0^1(\Omega)$. Then*

$$\|Eu - \pi_h u\|_{L^2(\Omega_h)} \lesssim h\|u\|_{H_0^1(\Omega)},$$

and if, additionally, $u \in H^2(\Omega)$, then

$$\|Eu - \pi_h u\|_{H^1(\Omega_h)} \lesssim h\|u\|_{H^2(\Omega)}.$$

We define

$$(4.2) \quad \tilde{u}_h = u_h - \pi_h u_*, \quad \tilde{U}_h = U_h - \pi_h U_*, \quad \text{and} \quad x_h = (\tilde{u}_h, \tilde{U}_h, z_h, Z_h).$$

We have the following proposition.

PROPOSITION 4.2. *There exists $h_0 > 0$ such that for all $0 < h < h_0$, the following estimate holds:*

$$\|z_h\|_D + \|Z_h\|_{D'} + h\|\tilde{u}_h\|_F + h\|\tilde{U}_h\|_{F'} + \|(\tilde{u}_h, \tilde{U}_h)\|_R \lesssim h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}.$$

Let us remark that as an immediate consequence of this proposition, the Lagrange multipliers (z_h, Z_h) converge to zero with a rate that is proportional to the space-time mesh parameter h . Moreover, we have

$$\mathcal{R}(u_h) \lesssim h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)},$$

implying that the initial and final states of the discrete solution u_h converge to the desired values at the optimal rate. In order to prove this proposition, we need the following lemma.

LEMMA 4.3. *Let $x_h \in \mathbb{V}_h^{4N}$ be as in (4.2). Then*

$$\mathcal{A}(x_h; y) \lesssim h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}\|y\|_C \quad \forall y \in \mathbb{V}_h^{4N}.$$

Proof. Let $y = (v, V, w, W)$. We can use the Euler–Lagrange equation (2.4) to write

$$\mathcal{A}(x_h; y) = a_h(v^0, g_0) + (\partial_\tau v^1, g_1)_h - \mathcal{A}((\pi_h u_*, \pi_h U_*, 0, 0); y) = S_1 + S_2 + S_3 + S_4 + S_5,$$

where

$$S_1 = -\mathcal{G}(\pi_h u_*, w) + \tau \sum_{n=2}^N (\chi_\omega U_*^n, w^n)_h,$$

$$S_2 = -\mathcal{G}^*(W, \pi_h U_*),$$

$$S_3 = \tau \sum_{n=2}^N (\chi_h \pi_h U_*^n - \chi_\omega U_*^n, w^n)_h,$$

$$S_4 = -a_h(\pi_h u_*^N, v^N) + a_h(g_0 - \pi_h u_*^0, v^0) - (\partial_\tau \pi_h u_*^N, \partial_\tau v^N)_h + (g_1 - \partial_\tau \pi_h u_*^1, \partial_\tau v^1)_h,$$

$$S_5 = -\mathcal{A}_1((\pi_h U_*, 0, 0); V).$$

For the term S_1 , we first observe that u_* satisfies

$$\tau \sum_{n=2}^N \int_\Omega (\partial_t^2 u_*^n \cdot w^n + \nabla u_*^n \cdot \nabla w^n) dx = \tau \sum_{n=2}^N \int_\Omega \chi_\omega U_*^n \cdot w^n dx,$$

where $u_*^n = u_*(n\tau, \cdot)$ and $U_*^n = U_*(n\tau, \cdot)$ and we are identifying u_*, U_* with their extensions u_*^e, U_*^e . The test function w^n is extended to $\Omega \cup \Omega_h$ by setting it equal to zero outside Ω_h . Note that since $w \in H_0^1(\Omega_h)$ the extended function w^n belongs to $H_0^1(\Omega_h \cup \Omega)$. Together with the definition of the interpolator π_h , we can write

$$S_1 = \underbrace{\tau \sum_{n=2}^N ((1 - \pi_h) \partial_\tau^2 u_*^n, w^n)_h}_{I_1} + \underbrace{\tau \sum_{n=2}^N (\partial_t^2 u_*^n - \partial_\tau^2 u_*^n, w^n)_h}_{I_2} + \tau \sum_{n=2}^N (\varsigma_E^n, w^n)_{\Omega_h \setminus \Omega},$$

where $\varsigma_E u_*^n = E(\chi_\omega U_*^n) - \square E u_*^n$. We utilize [6, Lemma 3.1] to write

$$|(\varsigma_E^n, w^n)| \leq \|\varsigma_E^n\|_{\Omega_h \setminus \Omega} \|w^n\|_{\Omega_h \setminus \Omega} \lesssim (\|\chi_\omega U_*^n\|_{L^2(\Omega)} + \|\partial_t^2 u_*^n\|_{L^2(\Omega)} + \|u_*^n\|_{H^2(\Omega)}) h^2 \|\nabla w^n\|_h.$$

Thus using Theorem 1.4 for u_*, U_* and the inverse discrete inequality in (3.1) for w^n , we can write

$$\begin{aligned} \left| \tau \sum_{n=2}^N (\varsigma_E^n, w^n)_{\Omega_h \setminus \Omega} \right|^2 &\lesssim h^2 (\|U_*\|_{\mathcal{C}(0,T;L^2(\Omega))}^2 + \|u_*\|_{\mathcal{C}^2(0,T;L^2(\Omega))}^2) \\ &\quad + \|u_*\|_{\mathcal{C}(0,T;H^2(\Omega))}^2 \left(\tau \sum_{n=2}^N \|w^n\|_h^2 \right) \\ &\lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \| (0, 0, w, 0) \|_{\mathcal{C}}^2. \end{aligned}$$

Using the same analysis as that of the terms I_1, I_2 in [6, Proposition 4.3] we obtain

$$\begin{aligned} |I_1|^2 &\lesssim h^2 \left(\int_0^T \|\nabla \partial_t^2 u_*\|_{L^2(\Omega)}^2 dt \right) \left(\tau \sum_{n=2}^N \|w^n\|_h^2 \right), \\ |I_2|^2 &\lesssim \tau^2 \left(\int_0^T \|\partial_\tau^3 u_*\|_{L^2(\Omega)}^2 dt \right) \left(\tau \sum_{n=2}^N \|w^n\|_h^2 \right). \end{aligned}$$

We can therefore use Theorem 1.4 to conclude that

$$|S_1| \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|(0, 0, w, 0)\|_C.$$

For the term S_2 , we first note that

$$\tau \sum_{n=0}^{N-2} \int_{\Omega} (W^n \cdot \partial_t^2 U_*^n + \nabla W^n \cdot \nabla U_*^n) \, dx = 0,$$

where we recall the notation $U_*^n(\cdot) = U_*(n\tau, \cdot)$. Therefore, using the definition of the interpolator π_h we can write

$$S_2 = \tau \sum_{n=0}^{N-2} (W^n, (1 - \pi_h) \tilde{\partial}_\tau^2 U_*^n)_h + \tau \sum_{n=0}^{N-2} (W^n, \partial_t^2 U_*^n - \tilde{\partial}_\tau^2 U_*^n)_h + \tau \sum_{n=0}^{N-2} (\zeta_E^n, W^n),$$

where $\zeta_E^n = -\square E U_*^n$. Analogous to the term S_1 we obtain the bound

$$|S_2| \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|(0, 0, 0, W)\|_C,$$

where we have used Theorem 1.4 again. For the term S_3 , we write

$$\begin{aligned} |S_3| &= \left| \tau \sum_{n=2}^N ((\chi_h - \chi_\omega) \pi_h U_*^n, w^n)_h + \tau \sum_{n=2}^N (\chi_\omega (\pi_h U_*^n - U_*^n), w^n)_h \right| \\ &\lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|(0, 0, w, 0)\|_C, \end{aligned}$$

where we have used the bound (2.2) together with Lemma 4.1 for the first term and Lemma 4.1 for the second term. For S_4 , we use the the fact that $u_*(0, \cdot) = g_0$, $\partial_t u(0, \cdot) = g_1$, and (1.2) with the approximation properties of π_h to deduce that

$$|S_4| \leq 2\sqrt{\mathcal{R}(\pi_h u_*) \mathcal{R}(v)} \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|(v, 0)\|_R,$$

where we have used the fact that $\pi_h u_*^N = 0$ and Theorem 1.4 to obtain the following bounds:

$$\begin{aligned} \|\partial_\tau \pi_h u_*^N\|_h &= \|(\pi_h - 1) \partial_\tau u_*^N + \partial_\tau u_*^N - \partial_t u_*^N\|_h \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}, \\ \|\partial_\tau \pi_h u_*^1 - g_1\|_h &= \|(\pi_h - 1) \partial_\tau u_*^1 + \partial_\tau u_*^1 - g_1\|_h \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}, \\ \|\nabla(\pi_h u_*^0 - g_0)\|_h &= \|\nabla(\pi_h - 1) u_*^0\|_h \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}. \end{aligned}$$

Finally, for the term S_5 , we write

$$|S_5| \leq 2\sqrt{\mathcal{J}_1(\pi_h U_*) \mathcal{J}_1(V)} \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|(0, V)\|_R,$$

where we have used the following bounds for $\mathcal{J}_1(\pi_h U_*)$:

$$\begin{aligned} \tau \sum_{n=1}^N \|\tau \nabla \partial_\tau \pi_h U_*^n\|_h^2 &\lesssim \tau^2 \|U_*\|_{H^1(0, T; H^1(\Omega))}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2, \\ \|h \nabla \partial_\tau \pi_h U_*^N\|_h^2 &\lesssim h^2 \|U_*\|_{H^2(0, T; H^1(\Omega))}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2, \\ \|h \nabla \partial_\tau \pi_h U_*^1\|_h^2 &\lesssim h^2 \|U_*\|_{H^2(0, T; H^1(\Omega))}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2, \\ \|h \nabla \pi_h U_*^N\|_h^2 &\lesssim h^2 \|U_*\|_{H^1(0, T; H^1(\Omega))}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2. \end{aligned}$$

Combining the estimates yields the claim. □

Proof of Proposition 4.2. Let x_h be as in (4.2). Using Proposition 3.1, there exists $y \in \mathbb{V}_h^{4N}$ such that

$$\mathcal{A}(x_h; y) \gtrsim (\|(\tilde{u}_h, \tilde{U}_h)\|_R + h\|\tilde{u}_h\|_F + h\|\tilde{U}_h\|_{F'} + \|z_h\|_D + \|Z_h\|_{D'})\|y\|_C.$$

Combining this estimate with Lemma 4.3 yields the claim. □

Lemma 4.1 can be used together with Theorem 1.4 to obtain the following corollary.

COROLLARY 4.4.

$$\|u_h\|_F + \|U_h\|_{F'} \lesssim \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}.$$

We are now ready to state two key ingredients of the proof of Theorem 2.1. The first estimate is regarding the error function \tilde{u}_h . We recall that for each n , \tilde{u}_h^n is extended to $\Omega \cup \Omega_h$ by setting it to be zero outside Ω_h .

LEMMA 4.5. *Let \tilde{u}_h be as defined in (4.2). For $n = 1, \dots, N$, the following estimate holds:*

$$\|\tilde{u}_h^n\|_{H^1(\Omega)}^2 + \|\partial_\tau \tilde{u}_h^n\|_{L^2(\Omega)}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2.$$

Proof. Note that Proposition 4.2 implies that

$$\|\tilde{u}_h^0\|_{H^1(\Omega)} + \|\partial_\tau \tilde{u}_h^1\|_{L^2(\Omega)} \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}.$$

Recall that u_h satisfies the discrete wave equation

$$\tau \sum_{n=2}^N ((\partial_\tau^2 u_h^n, w^n)_h + a_h(u_h^n, w^n)) = \tau \sum_{n=2}^N (\chi_h U_h^n, w^n)_h,$$

and that u_* solves the wave equation (1.1). Thus standard discrete energy estimates for the wave equation apply to derive the claimed inequality (see, for example, [11, Lemma 6]). □

Next, we state the following approximate observability estimate for the error function \tilde{U}_h defined in (4.2). We remind the reader that for each n , U_h^n is extended to $\Omega \cup \Omega_h$ by setting it to zero outside Ω_h .

PROPOSITION 4.6. *Let \tilde{U}_h be as defined in (4.2). For $n = 1, \dots, N$, the following estimate holds:*

$$\|\tilde{U}_h^n\|_{L^2(\Omega)}^2 + \|\partial_\tau \tilde{U}_h^n\|_{H^{-1}(\Omega)}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2.$$

Proof. We begin by defining the continuous piecewise affine function

$$\hat{U}_h(t, \cdot) = U_h^{n-1}(\cdot) + (t - t_{n-1})\partial_\tau U_h^n(\cdot) \quad \text{for } t \in [t_{n-1}, t_n],$$

for $n = 1, \dots, N$. Let $\mathcal{E} = \hat{U}_h - U_*$ and define the bounded linear functional R through

$$(4.3) \quad \langle R, W \rangle = \int_0^T \int_\Omega (-\partial_t \mathcal{E} \cdot \partial_t W + \nabla \mathcal{E} \cdot \nabla W) dx dt \quad \forall W \in H_0^1(\mathcal{M}).$$

For the remainder of this proof, we will identify U_* with its E -extension to $(0, T) \times (\Omega \cup \Omega_h)$. Applying Theorem 1.2 with $\tilde{O} = (0, T) \times \omega_\delta$ and using the fact that $\chi_\omega = 1$ on ω_δ , we obtain

$$(4.4) \quad \sup_{t \in [0, T]} (\|\mathcal{E}(t, \cdot)\|_{L^2(\Omega)}^2 + \|\partial_t \mathcal{E}(t, \cdot)\|_{H^{-1}(\Omega)}^2) \lesssim \|\sqrt{\chi_\omega} \mathcal{E}\|_{L^2(\mathcal{M})}^2 + \|R\|_{H^{-1}(\mathcal{M})}^2 + \|\mathcal{E}\|_{L^2((0, T) \times \partial\Omega)}^2.$$

We proceed to prove the following bounds:

$$(4.5) \quad \|\mathcal{E}\|_{L^2((0, T) \times \partial\Omega)} \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)},$$

$$(4.6) \quad |\langle R, W \rangle| \lesssim h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|W\|_{H^1(\mathcal{M})} \quad \forall W \in H_0^1(\mathcal{M}),$$

$$(4.7) \quad \|\sqrt{\chi_\omega} \mathcal{E}\|_{L^2(\mathcal{M})}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2.$$

The proposition then follows by writing $\tilde{U}_h^n(\cdot) = \mathcal{E}(n\tau, \cdot) + \mathcal{E}_*(n\tau, \cdot)$ with $\mathcal{E}_*(n\tau, \cdot) = U_*(n\tau, \cdot) - \pi_h U_*^n(\cdot)$. The term $\mathcal{E}_*(n\tau, \cdot)$ may then be bounded by $h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}$ using Lemma 4.1 and Taylor development in time followed by Theorem 1.4.

First, we prove the estimate (4.5). Recalling that $U_*|_{(0, T) \times \partial\Omega} = 0$, we write

$$\|\mathcal{E}\|_{L^2((0, T) \times \partial\Omega)}^2 = \|\hat{U}_h\|_{L^2((0, T) \times \partial\Omega)}^2 \lesssim \tau \sum_{n=0}^N \|U_h^n\|_{L^2((0, T) \times \partial\Omega)}^2.$$

Applying [6, Lemma 4.1] and Corollary 4.4, we deduce that

$$\tau \sum_{n=0}^N \|U_h^n\|_{L^2((0, T) \times \partial\Omega)}^2 \lesssim \tau \sum_{n=0}^N h^2 \|\nabla U_h^n\|_h^2 \lesssim h^2 \| \|U_h\| \|_{F'}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2.$$

Let us now consider the estimate (4.6). We introduce the notation \bar{W}^n through

$$(4.8) \quad \bar{W}^n(\cdot) = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} W(t, \cdot) dt \quad \text{for } n = 1, \dots, N.$$

Note that using the Poincaré and Cauchy–Schwarz inequalities, we have

$$(4.9) \quad \begin{aligned} \tau \sum_{n=1}^{N-1} \|W^n - \bar{W}^n\|_{L^2(\Omega)}^2 &\lesssim \tau^2 \|W\|_{H^1(0, T; L^2(\Omega))}^2, \\ \|\nabla \bar{W}^n\|_{L^2(\Omega)}^2 &\lesssim \tau^{-1} \|W\|_{L^2(0, T; H^1(\Omega))}^2 \quad \text{for } n = 1, \dots, N, \\ \|\bar{W}^N\|_{L^2(\Omega)}^2 &\lesssim \tau \|W\|_{H^1(0, T; L^2(\Omega))}^2. \end{aligned}$$

Since $\square U_* = 0$ on \mathcal{M} and $U_h = 0$ on $(0, T) \times (\Omega \setminus \Omega_h)$, we have

$$\begin{aligned}
 \langle R, W \rangle &= \int_0^T (-\partial_t \hat{U}_h, \partial_t W)_h + a_h(\hat{U}_h, W) \\
 &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} [-(\partial_\tau U_h^n, \partial_t W)_h + a_h(U_h^{n-1}, W) + (t - t_{n-1})a_h(\partial_\tau U_h^n, W)] dt \\
 &= \tau \sum_{n=1}^{N-1} (\partial_\tau^2 U_h^{n+1}, W^n)_h + \tau \sum_{n=1}^N a_h(U_h^{n-1}, \bar{W}^n) \\
 &\quad + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (t - t_{n-1})a_h(\partial_\tau U_h^n, W) dt \\
 &= \tau \underbrace{\sum_{n=1}^{N-1} (\partial_\tau^2 U_h^{n+1}, W^n - \bar{W}^n)_h}_{\text{I}} + \underbrace{\tau a_h(U_h^{N-1}, \bar{W}^N)}_{\text{II}} \\
 &\quad + \tau \underbrace{\sum_{n=0}^{N-2} ((\tilde{\partial}_\tau^2 U_h^n, \bar{W}^{n+1})_h + a_h(U_h^n, \bar{W}^{n+1}))}_{\text{III}} \\
 &\quad + \underbrace{\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (t - t_{n-1})a_h(\partial_\tau U_h^n, W) dt}_{\text{IV}}.
 \end{aligned}$$

We will first proceed to bound each of the terms I, III, and IV and then treat the term II. For the term I, we use the Cauchy–Schwarz inequality to write

$$\begin{aligned}
 |\text{I}|^2 &\leq \left(\tau \sum_{n=2}^N \|\partial_\tau^2 U_h^n\|_h^2 \right) \left(\tau \sum_{n=1}^{N-1} \|W^n - \bar{W}^n\|_{L^2(\Omega)}^2 \right) \\
 &\lesssim \tau^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \|W\|_{H^1(\mathcal{M})}^2,
 \end{aligned}$$

where we have used the first bound in (4.9) followed by Corollary 4.4. For the term III, we first note that U_h satisfies

$$\tau \sum_{n=0}^{N-2} \left((\tilde{\partial}_\tau^2 U_h^n, \pi_h \bar{W}^{n+1})_h + a_h(U_h^n, \pi_h \bar{W}^{n+1}) \right) = 0,$$

which, together with the definition of the interpolator π_h and Corollary 4.4, implies that

$$\begin{aligned}
 |\text{III}|^2 &= \left| \tau \sum_{n=0}^{N-2} (\tilde{\partial}_\tau^2 U_h^n, (\pi_h - 1)\bar{W}^{n+1})_h \right|^2 \\
 &\leq \left(\tau \sum_{n=2}^N \|\partial_\tau^2 U_h^n\|_h^2 \right) \left(\tau \sum_{n=0}^{N-2} \|(\pi_h - 1)\bar{W}^{n+1}\|^2 \right) \\
 &\lesssim \tau^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \|W\|_{H^1(\mathcal{M})}^2.
 \end{aligned}$$

For the term IV, we use the Cauchy–Schwarz inequality to write

$$|\text{IV}|^2 \lesssim \tau^2 \left(\tau \sum_{n=1}^N \|\partial_\tau \nabla U_h^n\|_h^2 \right) \|W\|_{H^1(\mathcal{M})}^2 \lesssim \tau^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \|W\|_{H^1(\mathcal{M})}^2,$$

where we have used Corollary 4.4 again. It remains to bound the term II. Here, we use the fact that U_h solves the Euler–Lagrange equation (2.4) again to write

$$\tau a_h(U_h^{N-1}, \bar{W}^N) = \tau a_h(U_h^{N-2}, \bar{W}^N) + \tau^2 a_h(\partial_\tau U_h^{N-1}, \bar{W}^N).$$

But using Corollary 4.4 we have the bound $\|\nabla \partial_\tau U_h^{N-1}\|_h \lesssim \tau^{-\frac{1}{2}} \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}$, which together with the bound (4.9) implies that

$$\tau a_h(\tau \partial_\tau U_h^{N-1}, \bar{W}^N) \lesssim \tau \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|W\|_{H^1(\mathcal{M})}.$$

For the remaining term, we observe that

$$\begin{aligned} \tau a_h(U_h^{N-2}, \bar{W}^N) &= \tau a_h(U_h^{N-2}, \pi_h \bar{W}^N) \\ &= \underbrace{-\tau(\tilde{\partial}_\tau^2 U_h^{N-2}, (\pi_h - 1)\bar{W}^N)_h}_{S_1} - \underbrace{\tau(\tilde{\partial}_\tau^2 U_h^{N-2}, \bar{W}^N)_h}_{S_2}. \end{aligned}$$

To bound the term S_1 , we use Lemma 4.1 and the second bound in (4.9) to obtain

$$|S_1| \lesssim \tau \|\partial_\tau^2 U_h^N\|_h \tau^{\frac{1}{2}} \|W\|_{H^1(\mathcal{M})} \lesssim \tau \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|W\|_{H^1(\mathcal{M})},$$

where we have used Corollary 4.4 to write the bound

$$\|\partial_\tau^2 U_h^N\|_h \lesssim \tau^{-\frac{1}{2}} \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}.$$

Finally, for the term S_2 , we write

$$|S_2| \lesssim \tau \|\partial_\tau^2 U_h^N\|_h \|\bar{W}^N\|_h \lesssim \tau \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} \|W\|_{H^1(\mathcal{M})},$$

where we have used the third bound in (4.9) together with Corollary 4.4 in the last step. This completes the proof of bound (4.6).

To prove (4.7), we first define the piecewise constant time interpolant π_0 as follows:

$$\pi_0 v^n = v(t_n) \quad \text{for } t \in (t_{n-1}, t_n] \quad \text{and } n = 1, \dots, N.$$

This interpolant satisfies the bound

$$\|\pi_0 v - v\|_{L^2(0,T)} \lesssim \tau \|v\|_{H^1(0,T)}.$$

Note that by adding and subtracting χ_h and using (2.2) we have

$$\begin{aligned} \|\sqrt{\chi_\omega} \mathcal{E}\|_{L^2(\mathcal{M})}^2 &\leq \|\chi_\omega - \chi_h\|_{L^\infty(\Omega)}^2 \|\mathcal{E}\|_{L^2(\mathcal{M})}^2 + \|\sqrt{\chi_h} \mathcal{E}\|_{L^2(\mathcal{M})}^2 \\ &\lesssim h^2 \int_0^T (\|U_*\|_{L^2(\Omega)}^2 + \|\hat{U}_h\|_{L^2(\Omega)}^2) dt + \|\sqrt{\chi_h} \mathcal{E}\|_{L^2(\mathcal{M})}^2. \end{aligned}$$

Observe that

$$h^2 \int_0^T \|\hat{U}_h\|_{L^2(\Omega)}^2 dt \lesssim h^2 \|\tilde{U}_h\|_{F'}^2.$$

Using Corollary 4.4 and Theorem 1.4 it follows that

$$\|\sqrt{\chi_\omega} \mathcal{E}\|_{L^2(\mathcal{M})}^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \|\sqrt{\chi_h} \mathcal{E}\|_{L^2(\mathcal{M})}^2.$$

Furthermore,

$$\|\sqrt{\chi_h} \mathcal{E}\|_{L^2(\mathcal{M})}^2 \leq C(h^2 + \tau^2) \|U_*\|_{H^1(\mathcal{M})}^2 + \int_0^T \|\sqrt{\chi_h} \pi_0 \pi_h U_* - \sqrt{\chi_h} \hat{U}_h\|_h^2 dt.$$

To bound the second term we write

$$\begin{aligned} \int_0^T \|\sqrt{\chi_h} \pi_0 \pi_h U_* - \sqrt{\chi_h} \hat{U}_h\|_h^2 dt &\lesssim \int_0^T \|\sqrt{\chi_h} \pi_0 \pi_h U_* \\ &\quad - \pi_0 \sqrt{\chi_h} \hat{U}_h\|_h^2 dt + \int_0^T \|\pi_0 \hat{U}_h - \hat{U}_h\|_h^2 dt \\ &= \tau \sum_{n=1}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\pi_0 \hat{U}_h - \hat{U}_h\|_h^2 dt. \end{aligned}$$

It suffices to bound the second term of the right-hand side. Using the piecewise linearity of \hat{U}_h we observe that

$$\begin{aligned} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\pi_0 \hat{U}_h - \hat{U}_h\|_h^2 dt &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|(t - t_n) \partial_\tau U_h^n\|_h^2 dt \leq \tau \sum_{n=1}^N \|\tau \partial_\tau U_h^n\|_h^2 \\ &\lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2, \end{aligned}$$

where we have used Corollary 4.4 in the last step. This completes the proof of the bound (4.7). \square

5. A strong a priori error estimate and proof of the main theorem.

This section is concerned with the proof of the main theorem. The idea is to use the approximate discrete observability estimate together with an improved coercivity estimate to produce a stronger error estimate, stated as follows.

PROPOSITION 5.1. *Let \tilde{U}_h be as defined in (4.2). The following residual estimate holds:*

$$\tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2.$$

The proof of the main theorem follows from Proposition 5.1. Indeed, note that the first claimed inequality in Theorem 2.1 follows from combining Propositions 4.6 and 5.1, while the second claimed inequality follows from combining Lemma 4.5 and Proposition 5.1. We proceed to prove Proposition 5.1. This will be divided into parts. We define the refined test function

$$(5.1) \quad \hat{y} = (\hat{v}, \hat{V}, \hat{w}, \hat{W})$$

through the expressions

$$\begin{aligned} \hat{v} &= \tilde{u}_h + \gamma v, & \hat{V} &= \tilde{U}_h + \alpha V, \\ \hat{w} &= -z_h - \gamma \tilde{U}_h + \alpha h^2 w, & \hat{W} &= -Z_h + \gamma h^2 W, \end{aligned}$$

where $\gamma > \alpha > 0$ and v, V, w, W are chosen as in Lemma 3.3 in terms of the discrete functions z_h, Z_h, \tilde{u}_h and \tilde{U}_h , respectively. Let us also define a norm on \mathbb{V}_h^{4N} through the expression

$$(5.2) \quad \begin{aligned} \|(u, U, z, Z)\|_S^2 &= \tau \sum_{n=2}^N \|\sqrt{\chi_h} U^n\|_h^2 + \|(u, U)\|_R^2 + h^2 \|u\|_F^2 \\ &\quad + h^2 \|U\|_{F'}^2 + \|z\|_D^2 + \|Z\|_{D'}^2, \end{aligned}$$

We have the following three lemmas. These will be subsequently used to prove Proposition 5.1.

LEMMA 5.2. *Let x_h, \hat{y} be defined as in (4.2) and (5.1) respectively. The following estimate holds:*

$$\|\hat{y}\|_C \lesssim h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} + \|x_h\|_S.$$

Proof. Recall that

$$\|\hat{y}\|_C^2 = \|(\hat{v}, 0)\|_R^2 + \|(0, \hat{V})\|_R^2 + \tau \sum_{n=2}^N \|\hat{w}^n\|_h^2 + \tau \sum_{n=0}^{N-2} \|\hat{W}^n\|_h^2.$$

We proceed to bound each of the four terms appearing on the right-hand side. Indeed, using (5.1) and the Cauchy–Schwarz inequality we have the bounds

$$\begin{aligned} \|(\hat{v}, 0)\|_R^2 &\lesssim \|(\tilde{u}_h, 0)\|_R^2 + \|\nabla v^N\|_h^2 + \|\partial_\tau v^N\|_h^2 + \|\nabla v^1\|_h^2 + \|\partial_\tau v^1\|_h^2 \\ &\lesssim \|(\tilde{u}_h, 0)\|_R^2 + \|\nabla v^N\|_h^2 + \|\partial_\tau v^N\|_h^2 \lesssim \|(\tilde{u}_h, 0)\|_R^2 + \|z_h\|_D^2, \\ \|(0, \hat{V})\|_R^2 &\lesssim \|(0, \tilde{U}_h)\|_R^2 + h^2 \tau \sum_{n=1}^N \|\partial_\tau \nabla V^n\|_h^2 + h^2 \|\partial_\tau \nabla V^1\|_h^2 \\ &\lesssim \|(0, \tilde{U}_h)\|_R^2 + \tau \sum_{n=0}^{N-2} \|Z_h\|_h^2 + \|Z_h^0\|_h^2 \lesssim \|(0, \tilde{U}_h)\|_R^2 + \|Z_h\|_{D'}^2, \\ \tau \sum_{n=0}^{N-2} \|\hat{W}^n\|_h^2 &\lesssim \tau \sum_{n=0}^{N-2} \|Z_h^n\|_h^2 + h^2 \|\tilde{U}_h\|_{F'}^2, \\ \tau \sum_{n=2}^N \|\hat{w}^n\|_h^2 &\lesssim \tau \sum_{n=2}^N \|z_h^n\|_h^2 + h^2 \|\tilde{u}_h\|_F^2 + \tau \sum_{n=2}^N \|\tilde{U}_h\|_h^2, \end{aligned}$$

Applying Proposition 4.6 in the last bound and using the definition of the $\|\cdot\|_S$ norm yields the claim. \square

LEMMA 5.3. *The following estimate holds:*

$$|\mathcal{G}(\tilde{u}_h, \tilde{U}_h)| \lesssim h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} (h\|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} + \|x_h\|_S).$$

Proof. We begin by using the discrete version of Leibniz rule to write

$$\begin{aligned} \mathcal{G}(\tilde{u}_h, \tilde{U}_h) &= \mathcal{G}^*(\tilde{u}_h, \tilde{U}_h) - \underbrace{(\tilde{u}_h^{N-1}, \partial_\tau \tilde{U}_h^N)_h + (\partial_\tau \tilde{u}_h^N, \tilde{U}_h^N)_h + (\tilde{u}_h^0, \partial_\tau \tilde{U}_h^1)_h - (\partial_\tau \tilde{u}_h^1, \tilde{U}_h^1)_h}_{\text{I}} \\ &\quad - \underbrace{\tau a_h(\tilde{u}_h^0, \tilde{U}_h^0) - \tau a_h(\tilde{u}_h^1, \tilde{U}_h^1) + \tau a_h(\tilde{u}_h^{N-1}, \tilde{U}_h^{N-1}) + \tau a_h(\tilde{u}_h^N, \tilde{U}_h^N)}_{\text{II}}. \end{aligned}$$

Now using the fact that $\mathcal{G}^*(W, U_h) = 0$ for all W (see (2.4)), we obtain

$$\mathcal{G}^*(\tilde{u}_h, \tilde{U}_h) = -\mathcal{G}^*(\tilde{u}_h, \pi_h U_*),$$

which is identical to the term S_2 in Lemma 4.3. Therefore, we have the bound

$$\begin{aligned}
 |\mathcal{G}^*(\tilde{u}_h, \pi_h U_*)|^2 &\lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \left(\tau \sum_{n=0}^N \|\tilde{u}_h^n\|_h^2 \right) \\
 (5.3) \quad &\lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \left(h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \right. \\
 &\quad \left. + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 \right),
 \end{aligned}$$

where we have used Lemma 4.5 in the last step. To analyze the terms I and II, we first observe that

$$\begin{aligned}
 \|\nabla \tilde{u}_h^1\|_h &= \|\nabla \tilde{u}_h^0 + \tau \nabla \partial_\tau \tilde{u}_h^1\| \leq \|\nabla \tilde{u}_h^0\|_h + \kappa \|\partial_\tau \tilde{u}_h^1\|_h. \\
 \|\nabla \tilde{u}_h^{N-1}\|_h &= \|\nabla \tilde{u}_h^N - \tau \nabla \partial_\tau \tilde{u}_h^N\| \leq \|\nabla \tilde{u}_h^N\|_h + \kappa \|\partial_\tau \tilde{u}_h^N\|_h.
 \end{aligned}$$

Now, for the term I, we can use Proposition 4.2 with Proposition 4.6 to deduce that

$$(5.4) \quad |\text{I}|^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \left(h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 \right).$$

For the term II, we use the bounds

$$\tau \|\nabla \tilde{U}_h^k\|_h \leq \kappa \|\tilde{U}_h^k\|_h \quad \text{for } k = 0, 1, \dots, N,$$

and write

$$|\text{II}|^2 \lesssim \|(\tilde{u}_h, 0)\|_R^2 (\|\tilde{U}_h^0\|_h^2 + \|\tilde{U}_h^1\|_h^2 + \|\tilde{U}_h^{N-1}\|_h^2 + \|\tilde{U}_h^N\|_h^2).$$

We can apply Proposition 4.2 with Proposition 4.6 again to obtain

$$|\text{II}|^2 \lesssim h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 \left(h^2 \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)}^2 + \tau \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 \right).$$

Combining this with inequalities (5.3) and (5.4) completes the proof. \square

LEMMA 5.4. *Let x_h, \hat{y} be defined as in (4.2), (5.1), respectively. The following estimate holds:*

$$\mathcal{A}(x_h; \hat{y}) \geq C \|x_h\|_S^2 - C' h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} (h \|(g_0, g_1)\|_{H^4(\Omega) \times H^3(\Omega)} + \|x_h\|_S),$$

where $C, C' > 0$ are constants independent of the parameter h .

Proof. This proof mirrors the proof of Proposition 3.1. We start by writing

$$(5.5) \quad \mathcal{A}(x_h; \hat{y}) = \|(\tilde{u}_h, \tilde{U}_h)\|_R^2 + \tau \gamma \sum_{n=2}^N \|\sqrt{\chi_h} \tilde{U}_h^n\|_h^2 - \gamma \mathcal{G}(\tilde{u}_h, \tilde{U}_h) + \mathcal{A}(x_h; \hat{y}),$$

where $\tilde{y} = (\gamma v, \alpha V, \alpha h^2 w, \gamma h^2 W)$ with v, V, w , and W defined as in (5.1). The analysis of the last term on the right-hand side is exactly as in the proof of Proposition 3.1. Therefore, using the same bounds as in that proof, we deduce that

$$(5.6) \quad \mathcal{A}(x_h; \hat{y}) \geq -C_1 \gamma \|(\tilde{u}_h, \tilde{U}_h)\|_R^2 + C_2 \alpha (h^2 \|\tilde{u}_h\|_F^2 + h^2 \|\tilde{U}_h\|_{F'}^2 + \|z_h\|_D^2 + \|Z_h\|_{D'}^2)$$

for some $C_1, C_2 > 0$. Finally, combining (5.5)–(5.6) and applying Lemma 5.3 yields the claim. \square

Proof of Proposition 5.1. We choose \hat{y} as in (5.1). Lemma 5.4 applies and we have

$$\mathcal{A}(x_h; \hat{y}) \geq C \| \|x_h\|_S^2 - C' h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} (h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} + \| \|x_h\|_S).$$

On the other hand, Lemma 4.3 applies, and together with Lemma 5.2, we write

$$\begin{aligned} \mathcal{A}(x_h, \hat{y}) &\lesssim h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} \| \hat{y} \|_C \\ &\lesssim h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} (\| \|x_h\|_S + h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)}). \quad \square \end{aligned}$$

Combining these bounds, we note that the following inequality holds:

$$\| \|x_h\|_S^2 \lesssim h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} (h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)} + \| \|x_h\|_S).$$

This implies that

$$\| \|x_h\|_S \lesssim h \| (g_0, g_1) \|_{H^4(\Omega) \times H^3(\Omega)}.$$

6. Further remarks.

6.1. A comparison with the data assimilation problem. We start this section with a comparison with our earlier work for the dual problem to null controllability for the wave equation, that is the data assimilation (DA) problem.

Let us briefly recall the DA problem as follows. Let $\omega \subset \Omega$ and consider a solution u to the wave equation (1.1) without the a priori knowledge of the initial data (g_0, g_1) . The DA problem reads as follows: determine the solution u , given the additional piece of data $q = u|_{(0,T) \times \omega}$.

To solve DA, one can study the critical points for the Lagrangian

$$\mathcal{L}(u, z) = \frac{1}{2} \| \|u - q\|_{L^2(\mathcal{O})}^2 + \int_0^T \int_{\Omega} (\partial_t^2 u \cdot z + \nabla u \cdot \nabla z) \, dx \, dt,$$

where the wave equation is imposed on u through the Lagrange multiplier z . Similar to the theory of controllability, existence of a unique minimizer for this functional is guaranteed by an observability estimate on the set $\mathcal{O} = (0, T) \times \omega$.

In [6], we considered an approach based on finite element method for numerically solving the DA problem, using first order finite elements in space and finite differences in time. Analogous to the current work, this method was based on defining a discrete analogue for the Lagrangian \mathcal{L} that additionally incorporates numerical stabilizers in u . Optimal convergence rates were proven under the assumption that the geometric control condition is satisfied on the set \mathcal{O} (see [6, Theorem 4.6]).

Although our approach to solving the null controllability problem here draws similarities to that in [6], in the sense that similar numerical stabilization terms are used, we outline three of the key differences that makes the control problem more challenging.

First, for the DA problem, the data fitting term $\| \|u - q\|_{L^2(\mathcal{O})}$, incorporated in the Lagrangian functional on the discrete level, gives optimal error bounds on the set \mathcal{O} . Then the error bound on \mathcal{O} , combined with the stability properties for the wave equation (obtained through discrete energy estimates) and the continuum observability estimate (Theorem 1.2), gives optimal error bounds for the DA problem. Notice, however, that for the null controllability problem, proving error bounds in the set \mathcal{O} is much harder as the control function U is a priori unknown. In fact, error

bounds for the control function U in the set \mathcal{O} are obtained in the final section of the paper (see Proposition 5.1). In the case of the null controllability problem, it is not a priori clear how the observability estimate appears in the convergence rates. To retrace the steps of the proof, we recall that we first derived a weak a priori error estimate (Proposition 4.2). We then used this estimate combined with the observability estimate to obtain an approximate version of the observability estimate at the discrete level (Proposition 4.6). Finally, using a key *hidden* coercivity estimate in the Lagrangian \mathcal{J} (see Lemma 5.4) combined with Proposition 4.6, we were able to obtain the optimal error estimates.

The second key difference is the fact that both the state variable u and the control function U are unknowns in the controllability problem whereas in the data assimilation problem the only unknown is the state variable. A closer inspection of our work in [6], together with the previous literature on the Lagrangian formulations of the control problem, suggests that a discrete version of the more commonly studied continuum Lagrangian

$$\begin{aligned} \mathcal{L}(u, U) = & \frac{1}{2} \int_0^T \int_{\Omega} \chi_{\omega} |U|^2 dt dx - (g_1, U(0, \cdot))_{L^2(\Omega)} + \langle g_0, \partial_t U(0, \cdot) \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} \\ & - \int_0^T \int_{\Omega} \partial_t^2 u \cdot U dt dx - \int_0^T \int_{\Omega} \nabla u \cdot \nabla U dt dx \end{aligned}$$

may yield a numerical method, as long as the correct stabilization terms are incorporated. However, we were unable to derive optimal error estimates for this formulation, mainly due to the inconvenient feature that the critical point for the Lagrange multiplier U in this formulation represents the control function and will not be zero. The Lagrangian formulation employed in this paper introduces two Lagrange multipliers (z, Z) , which makes the analysis complete.

Finally, let us emphasize that the data assimilation problem has nice features on the continuum level that the controllability problem is lacking. The former problem has a unique solution whereas for the latter the solution is unique only under additional constraints such as (1.3). The question of existence and smoothness for the latter problem are quite trivial. Indeed, existence is guaranteed as long as the data q comes from an actual solution to the wave equation, while smoothness follows by requiring that the data q comes from a smooth solution. However, for the control problem, existence is a consequence of the deep result by Bardos, Lebeau, and Rauch [4] while smoothness also requires additional assumptions (see Theorem 1.4)

6.2. Concluding remarks. We have designed a fully discrete finite element method for the numerical approximation of the interior null controllability problem subject to the wave equation. The first order case was considered, using piecewise affine finite element approximation in space and a first order finite difference formula in time. A Tikhonov type regularization was applied to the control function at initial and final times, but the regularization parameter was chosen to scale with h in such a way that the perturbation due to regularization vanishes at a suitable rate. This allowed us to prove error estimates that are optimal compared to interpolation error, for the state variable and suboptimal with one order in h for the control variable. Observe, however, that the convergence rate of the latter is determined by the norms in the left-hand side of the observability estimate of Theorem 1.2 and the convergence rate of the residual quantities of the scheme evaluated in the norms of the right-hand side (4.5)–(4.7). The former cannot be improved. Since also the bound (4.6) is optimal

for piecewise affine approximation it appears that the error in the control variable is optimal if the continuum stability properties and the numerical approximation properties are both taken into account.

Let us also remark that no regularization was applied to the Lagrange multiplier variables z and Z , leading to a system where (u, U) and (z, Z) are only weakly coupled allowing for solution algorithms using the classical forward-backward solving approach. Finally, it bears pointing out that the approach using weakly consistent regularization, discrete inf-sup stability, and the observability estimate is not limited to the first order case, but can be extended to high order methods, using the modus operandi designed herein. This requires the introduction of suitable residual based regularization terms that are weakly consistent to the right order, which appears to be most feasible in the space-time framework. This is a topic for future work.

Appendix A. In this section, we proceed to construct an example of a smooth nonnegative cut-off function that satisfies properties (i)–(iii) in (1.8).

Let us first recall the boundary normal coordinates near $\partial\Omega$ in Ω , that are given by the locally smooth diffeomorphism $F : \mathbb{R} \times \partial\Omega \rightarrow \Omega$ defined through

$$F(x', x_n) = \gamma_{x'}(x_n).$$

Here, x' is a point on $\partial\Omega$ given in local coordinates by (x_1, \dots, x_{n-1}) , and $\gamma_{x'}(\cdot)$ denotes the normal line to the boundary $\partial\Omega$ with $\gamma_{x'}(0) = x'$, parametrized in terms of its arc-length. This map gives a local coordinate system such that the points $(x_1, \dots, x_{n-1}, 0)$ are on the boundary.

Let ω_δ be defined as in Hypothesis 1.5 and define $\Gamma = \partial\omega \cap \partial\Omega$ and $\Gamma_\delta = \partial\omega_\delta \cap \partial\Omega$. Let Γ'_δ be a small open neighborhood of Γ_δ , and let Γ''_δ be a small open neighborhood of Γ'_δ such that $\overline{\Gamma''_\delta} \subset \Gamma$. Let $\varepsilon > 0$ and choose a smooth $\psi(x')$ on $\partial\Omega$ such that $\psi = 1$ on Γ'_δ and such that $\text{supp } \psi \subset \Gamma''_\delta$. Subsequently, define a smooth function Ψ in the boundary normal coordinates by $\Psi(x', x_n) = \psi(x')$. Next, choose $\eta \in C^\infty(\overline{\Omega}; [0, 1])$ such that

$$(A.1) \quad \eta(x) = \begin{cases} 1 & \text{if } \text{dist}(x, \partial\Omega) < \frac{\varepsilon}{2}, \\ 0 & \text{if } \text{dist}(x, \partial\Omega) > \varepsilon. \end{cases}$$

Finally, let $\Phi \in C^\infty(\overline{\Omega}; [0, 1])$ be such that $\Phi = 1$ on ω_δ and $\Phi = 0$ on $\Omega \setminus \omega$. We now define

$$\chi_\omega = \eta \Psi + (1 - \eta) \Phi.$$

Note that the above function is a globally well defined smooth nonnegative function on Ω . Indeed, the first term in this expression is supported in an ε neighborhood of the boundary $\partial\Omega$ where the boundary normal coordinates are well defined (if $\varepsilon > 0$ is sufficiently small). Let us now prove that (i)–(iii) hold. Given that $\overline{\Gamma''_\delta}$ is contained in Γ , it follows that for ε small, the first term in the definition of χ_ω vanishes on $\Omega \setminus \omega$. The second term also vanishes on $\Omega \setminus \omega$, since Φ vanishes there. Hence, (i) is satisfied. To show (ii), we note that for ε sufficiently small, $\eta\Psi = \eta$ on the set ω_δ and that $\Phi = 1$ on ω_δ . It follows that $\chi_\omega = \eta + (1 - \eta) = 1$ on ω_δ . Finally, to show (iii), we note that in an $\frac{\varepsilon}{2}$ neighborhood of the boundary, only the first term in the definition of χ_ω is nonzero. Since $\eta = 1$ there, we have $\chi_\omega = \Psi$ and it follows that $\partial_\nu^k \chi_\omega|_{\partial\Omega} = 0$ since $\partial_{x_n}^k \Psi(x', x_n) = 0$ for all $k = 1, \dots$.

Appendix B. This section is concerned with verifying the main assumptions in Theorem 1.4 for $s = 3$, given that Hypothesis 1.5 holds and that χ_ω satisfies properties (i)–(iii) in (1.8).

We start by proving that the mapping property $\chi_\omega : \mathcal{D}((-\Delta)^3) \rightarrow \mathcal{D}((-\Delta)^3)$ holds. To this end, we recall that the Laplace operator Δ has the following (well-known) expression in boundary normal coordinates near $\partial\Omega$:

$$(B.1) \quad \Delta = \partial_{x_n}^2 - a(x', x_n)\partial_{x_n} + b(x', x_n) \sum_{i,j=1}^{n-1} \partial_{x_i}(c^{ij}(x', x_n)\partial_{x_j})$$

for some smooth functions $a(x', x_n)$, $b(x', x_n)$, and $c^{ij}(x', x_n)$ near $\partial\Omega$.

Let $(y_0, y_1) \in \mathcal{D}((-\Delta)^3)$. We need to show that $(\chi_\omega y_0, \chi_\omega y_1) \in \mathcal{D}((-\Delta)^3)$. Since $\chi_\omega \in \mathcal{C}^\infty(\Omega)$, it follows that $(\chi_\omega y_0, \chi_\omega y_1) \in H^4(\Omega) \times H^3(\Omega)$ and the first property in Definition 1.3 is satisfied. Let us now show that $\chi_\omega y_0$ satisfies (ii). First, observe that since $y_0 = 0$ on $\partial\Omega$, we have $\chi_\omega y_0 = 0$ on $\partial\Omega$. Next, using (B.1), together with the fact that $y_0(x', 0) = 0$, we obtain

$$\Delta(\chi_\omega y_0)(x', 0) = (\partial_{x_n}^2(\chi_\omega y_0))(x', 0) - a(x', 0)(\partial_{x_n}(\chi_\omega y_0))(x_n, 0).$$

Since $\partial_{x_n}^k \chi_\omega(x', 0) = 0$ for $k = 1, 2$, this reduces to

$$\Delta(\chi_\omega y_0)(x', 0) = \chi_\omega(x', 0)((\partial_{x_n}^2 y_0)(x', 0) - a(x', 0)(\partial_{x_n} y_0)(x', 0)).$$

Using again the fact that $y_0(x', 0) = 0$, this can be recast as

$$\Delta(\chi_\omega y_0)(x', 0) = \chi_\omega(x', 0)(\Delta y_0)(x', 0) = 0,$$

where the last step uses the fact that $\Delta y_0|_{\partial\Omega} = 0$. This shows that property (ii) in Definition 1.3 also holds for $\chi_\omega y_0$. Analogously, we can show that property (iii) in Definition 1.3 holds for $\chi_\omega y_1$. Thus χ_ω maps $\mathcal{D}((-\Delta)^3)$, and subsequently, the first assumption in Theorem 1.4 is verified.

Let us now show that \mathcal{J} is coercive and strictly convex (following [22, Theorem 2.4]). Indeed, since $(0, T) \times \omega_\delta$ satisfies the geometric control condition, Theorem 1.2 applies to obtain the bound

$$(B.2) \quad \|U\|_{C([0,T];L^2(\Omega))} + \|\partial_t U\|_{C([0,T];H^{-1}(\Omega))} \leq C_0 \|U\|_{(0,T) \times \omega_\delta} \leq C_0 \|\sqrt{\chi_\omega} U\|_{L^2(\mathcal{M})}$$

for all U that solve the wave equation (1.3), where $C_0 > 0$ is a constant depending on $T, \Omega, \omega, \delta$. In the last inequality above, we have used the facts that χ_ω is nonnegative and that $\chi_\omega = 1$ on ω_δ .

Applying the Cauchy–Schwarz inequality we write

$$\mathcal{J}(U_0, U_1) \geq \frac{1}{2} \|\sqrt{\chi_\omega} U\|_{L^2(\mathcal{M})}^2 - \|(g_0, g_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} \|(U_0, U_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}.$$

Now applying the estimate (B.2), it follows that

$$\mathcal{J}(U_0, U_1) \rightarrow \infty \quad \text{as} \quad \|(U_0, U_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \rightarrow \infty,$$

and therefore, by definition, \mathcal{J} is coercive.

To show strict convexity, note that given any pair (U_0, U_1) and (V_0, V_1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and any $\lambda \in [0, 1]$ we have

$$\begin{aligned} \mathcal{J}(\lambda(U_0, U_1) + (1-\lambda)(V_0, V_1)) &= \lambda \mathcal{J}(U_0, U_1) + (1-\lambda) \mathcal{J}(V_0, V_1) \\ &\quad - \frac{1}{2} \lambda(1-\lambda) \int_0^T \int_\Omega \chi_\omega |U - V|^2 dt dx, \end{aligned}$$

where U, V solve (1.3) with final data (U_0, U_1) and (V_0, V_1) , respectively. Now, using (B.2) again, we deduce that for $(U_0, U_1) \neq (V_0, V_1)$, there holds

$$\mathcal{J}(\lambda(U_0, U_1) + (1 - \lambda)(V_0, V_1)) < \lambda \mathcal{J}(U_0, U_1) + (1 - \lambda) \mathcal{J}(V_0, V_1).$$

Therefore, the functional \mathcal{J} is strictly convex on $L^2(\Omega) \times H^{-1}(\Omega)$.

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