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Essays on Auction Theory

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Abstract

This Ph.D. Thesis consists of three contributed papers. In the first paper we study multiunit common value auctions with informed and less informed bidders. We show that bidders with less information can bid very aggressively and do surprisingly well in terms of probability of winning and expected revenue. We also show that the degree of aggressiveness and success of bidders with less information is positively related to the number of units for sale. We explain these phenomena in terms of the balance of the winner's curse and the loser's curse and their different effect on bidders with different quality of information.

In the second paper we model a situation in which an auctioneer puts up for sale several identical units that have the property of common value for the bidders. One of the bidders, the incumbent, has better information about this common value, than the other bidders, the entrants. We show that in this situation an open ascending auction can give strictly higher expected utility to the entrants, and strictly higher expected revenue to the auctioneer. We provide an intuition for these results based on the different effect of the winner's curse on bidders that have different quality of information.

In the last paper we analyse a multistage game of competition among auctioneers. In this game, auctioneers commit to some publicly announced reserve prices, in a first stage, and bidders choose to participate in one of the auctions, in a second stage. We show that the set of Nash equilibrium is non-empty. We also show that one property of the equilibrium set is that when the number of auctioneers and bidders tends to infinity, almost all auctioneers with production cost low enough to trade announce a reserve price equal to their production costs. Our paper confirms previous results for some "limit" versions of the model by McAfee (1993), Peters (1997), and Peters and Severinov (1997).

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Chapter 1

Introduction

The theoretical study of auctions has developed enormously in the last two decades. The successful introduction of the Game Theory tools into the economic analysis introduced the study of auction design in the Economic research agenda. A field that had been studied mostly by engineers and experts in mathematics of operation research.

The first exhaustive analysis of an auction from a Game Theory perspective was conducted by Vickrey (1961), a paper whose path-breaking contributions accounted for Vickrey winning the Nobel price in 1996. But it was not until the 80's when the theoretical analysis of auctions developed and the basis of what nowadays is called Auction Theory were set up. Papers like those of Myerson (1981), Milgrom and Weber (1982), and Riley and Samuelson (1981) defined the basic analytical tools and the key results. In these first papers the problem was defined in the most simple set-up: one single auctioneer wants to sell one unit to a well defined set of bidders. These bidders were supposed to be risk neutral, ex ante symmetric and to act non co-operatively.

Very soon, real life problems showed that these basic models were insufficient. Thus, risk aversion (for instance, Maskin and Riley (1984)), collusion among bidders (Robinson (1985)), multiunit sales (Maskin and Riley (1989)), externalities among bidders (Jehiel and Moldovanu (1996)), bidders with budget constraints (Che and Gale (1998)), bidders with a costly entry decision (Levin and Smith (1994)), asymmetries among bidders (Maskin and Riley (2000)), or competition among auctioneers (McAfee

(1993)) were considered.

We shall focus in this Ph.D. thesis on some particular problems related to the last two topics, the presence of asymmetries among bidders and competition among auctioneers. In the following paragraphs we shall provide a very short overview of the theoretical knowledge on these two topics. We start with asymmetries among bidders to finish with auction competition.

The first and obvious point is the precise meaning of asymmetric bidders, or even of symmetric bidders. The most obvious asymmetries arise in models of complete information set-up, then, it means that bidders put different willingness to pay in the object. Under incomplete information the meaning is less obvious. Note that many auction problems are modelled naturally under incomplete information assumptions. This usually captures the intuition that auctions are used when there are problems of asymmetric information, mainly between the auctioneer and the bidders, but also among bidders.

The standard Game Theory analysis of models with private information follows Harsanyi proposal of converting games of incomplete information in games of imperfect information. This argument assumes that a fictitious player called Nature draws in an initial movement each bidders' private information, that we call types, and communicates them privately to each bidder. The distributions used by the player Nature represent beliefs that are common knowledge among the bidders about the private information of each bidder, i.e. the distribution of the bidders' types. In this set-up, asymmetric bidders refers to situations in which bidders' common beliefs give different types' distributions to different bidders.

The literature has distinguished between two main assumptions about the meaning of the bidders' types: the private value model and the common value model.

In the private value model, we assume that the value that each bidder puts on the object is determined exclusively by her type which will differ in general from other bidders' types. Hence, asymmetric bidders means in this case that the common knowledge beliefs about bidders' valuations give different distributions to the private preferences of each bidder. Under this assumption, we can consider, for instance, situations in which it is commonly known that a bidder's utility from winning the object is on average higher than another bidder's utility.

Consider next the common value model. Then, we assume that the value of the object is common to all the bidders but it is usually unknown. The bidders instead have noisy estimates of this common value. In this case, we shall refer to the bidder's type as the noisy estimate about the common value of the object that each bidder has. One interesting case of asymmetries is when the precision of the bidders' estimates differs, i.e. it is commonly known that some of the signals are more informative about the value of the good than the others. One good example is the auction of oil tract leases. In this case, one possible source of asymmetries is the presence of a bidder (or more than one) that is already exploiting an adjacent tract.

Note that we have given above examples where the presence of asymmetries among bidders was something exogenous to the economic problem of interest. But there are also other economic situations in which asymmetries are expected to arise endogenously. This is for instance the case if there is open acquisition of information before the auction, sequential auctions of complementary (or substitutive) goods, or pre-auction investments publicly observable that enhances the bidders willingness to pay. This point reinforces the need of understanding auctions with asymmetric bidders.

The study of models with asymmetric bidders has not developed in correspondence with their interest. The problem is that the analytical treatment is quite complex. For instance, the equilibrium bid functions of a first price auction does not have a closed form solution in general. Even the proof of existence of an equilibrium (see for instance, Lebrun (1996), or Athey (2000)) or uniqueness of the solution (see Lizzeri and Persico (1995)) requires a serious analytical effort. There are even less papers that provide comparative static results. Among them the most prominent is Maskin and Riley (2000) under the private value assumption.

If we restrict to second price auctions, the analysis is usually more tractable. For instance, under the private value assumption these auction formats have a straightforward solution because there is a unique weakly dominant strategy for each bidder. This is to bid her true value of the object. Under the common value assumption there some additional complications because bidders also infer information about the expected value of the good from the other bidders' behaviour. There are few papers that provide analytical results for common value auctions with asymmetric bidders (one example of such papers is Jewitt (forthcoming)).

We study in Chapter 2 and Chapter 3 of this Ph.D. Thesis simple models of common value auctions with asymmetric bidders. These models have received little attention due to the difficulty of their solution, one exception is Jewitt (forthcoming). We have gone around many of the analytical difficulties by restricting to generalisations of the second price auctions. We shall provide under these assumptions some unexpected results.

The starting point of Chapter 2 is a result by Milgrom (1981). Milgrom shows that under some assumptions bidders without relevant information lose with probability one against informed bidders. This captures the intuition that more information is beneficial for the bidder in an auction. We challenge this intuition by showing that bidders with less information can bid very aggressively and do surprisingly well in terms of probability of winning and expected revenue. We also show that the degree of aggressiveness and success of bidders with less information is positively related to the number of units for sale.

In Chapter 3 we explore this idea a bit further and compare two auction formats, a sealed bid uniform price auction and an open ascending auction, in a similar framework. These auctions formats are such that they generalise respectively the second price and the English auction to multiunit sales. In this case, we show that if there is one better informed bidder and several less informed bidders the open ascending auction can give higher expected utility to the less informed bidders, and higher expected revenue to the auctioneer than the sealed bid auction.

These two results show how misleading can be arguments based on the symmetric model where the revenue equivalence theorem, see Myerson (1981), establishes the equivalence of many auction formats in terms of auctioneer's revenue and bidders' expected utility. These analysis suggest that there is still much to be done in the area in order to understand what is the optimal selling mechanism in the presence of asymmetries. From a different perspective it also suggests that the variations of the auction format can also induce different levels of entry and acquisition of information. Finally, papers like Daripa (1998) show that we can apply intuitions form asymmetric common value models to understand better models finance in which insider trading is an issue.

In the last chapter we study a different topic, competition among auctioneers. Basically, these models extend the notion of price competition to more complex selling mechanisms like auctions. The literature of auction competition is still in a very preliminary state. The basic model was proposed by McAfee (1993). This model analyses a multistage game of auction competition. In a first stage each auctioneer commits publicly to an auction mechanism. Then, bidders in a second stage choose one auction, if any, to enter. In a final stage, each of the auctioneers runs his announced auction mechanism among all the bidders that have chosen his auction.

Subsequent papers in the area, like Peters and Severinov (1997), Peters (1997a), and also our model in Chapter 4, have provided more rigorous foundations to McAfee's model. For instance, in our model we consider a version of McAfee's model with a finite number of bidders and auctioneers and we show that the model is well behaved in the following sense. In the reduced game computed by substituting in the first stage auctioneers' objective functions, the Nash equilibrium of the following stages, the auctioneers' payoffs are continuous. This result allows us to prove the existence of an equilibrium in the whole game using standard game theory theorems. We also show that the limit of the set of equilibria converge in some sense when the numbers of auctioneers and bidders go to infinity to the equilibrium that McAfee proposed.

Within McAfee's (1993) model there are still many assumptions that can be relaxed to get a deeper understanding of the model. For instance, Epstein and Peters (1999) consider a much wider set of general mechanisms that allows for such mechanisms as price matching offers. Moreover, McAfee's (1993) concept of competition does not exhaust the possibilities. For instance, the widespread of internet auctions has shown that the competition among auctioneers does not only happens at the stage of participating in an auction, as it is modelled by McAfee. It is usually the case that bidders can participate simultaneously in various auctions and they can even adjust their bids in all the auctions they participate according to the evolution of the different auctions. This happens in auctions like those run by eBay or Amazon, see Ockenfels and Roth (2000) for a description of the auction formats of these two companies. This introduces a different competition pattern than that studied by McAfee (1993).

Another aspect of McAfee's (1993) model goes deeper in the foundations of competition among sellers. McAfee's model does not limit to consider how to choose optimally a given parameter in a selling mechanism under the competition of other sellers. McAfee goes further by allowing the sellers to chose from a wide range of mechanisms that for instance include price posting, and he shows that in equilibrium sellers would opt for auctions. In this sense, this basic model focuses not only on the question of the level of competition but also in the institutional outcome of competition.

McAfee's (1993) has been contested by Peters (1994). Peters shows that in a model with many sellers and buyers, the sellers can prefer in equilibrium a fix price mechanisms to a more complex auction mechanism. Mainly, Peters' argument follows because unlike McAfee, he assumes that the sellers offers are not observable by all the buyers. Instead, Peters assumes that buyers approach to the seller at a given random rate. Hence, if the seller has a discount rate less than one, waiting for more buyers to run a multilateral mechanism as an auction will be costly for the seller. This justifies why in some instances the seller can prefer a price offer to a single buyer to an auction among some buyers.

The idea that transaction costs can make auctions less attractive than simple price posting has also been suggested in the monopoly case, see for instance Wang (1993). Nevertheless, perhaps transaction costs are not the only disadvantage of auctions with respect to other mechanisms. In the monopoly case Harris and Raviv (1981) have suggested that if the number of potential buyers is smaller than the number of units that the seller has, the seller prefers fixing a price to running an auction. Note that arguments of this kind can be specially relevant when more sellers are introduced in the picture or when production is endogeneous. Then, the proportion of buyers to units per seller is an endogenous parameter.

Models that study which forms of competition will prevail in a market has not only theoretical importance. Recently we have assisted to a swing towards the use of auction mechanisms in different markets. Probably the more spectacular has been those markets based on internet. Another example is the increasing allocation of resources by governments through auctions, for instance in tendering processes, like the Private Finance Initiative in the United Kingdom, or in the sale of licenses to operate in natural monopolies, for instance spectrum auctions. Also big companies have started to choose retailers using auction procedures. We believe that the understanding of the forces that are leading this movement plays a crucial role to complete our knowledge of the modern economic problems.

Chapter 2

Successful Uninformed Bidding

2.1 Introduction

This paper studies multiunit, common value auctions in which some bidders have better information than others. In equilibria of such auctions the *worse* informed bidders can bid very aggressively and do surprisingly well. We display this effect in a sequence of models and discuss when it arises, and when it does not. We also argue that the correct intuitive explanation for these results relies on the balance of the *winner's curse* and the *loser's curse* effects.

The theoretical study of multiunit, common value auctions is important because a number of real life auctions have at least some similarity to such auctions. Examples are auctions of oil and gas leases, treasury bill auctions, and auctions of parts of the radio spectrum. It is important to understand how poorly informed or uninformed bidders behave in these auctions because their actions can influence the efficiency of the auction outcome as well as the expected revenue of the auctioneer. Their presence can then also affect the optimal auction design.

For the case that bidders have unit-demand, and that the number of units for sale is smaller than the number of well-informed bidders, Milgrom (1981) has displayed an equilibrium of a second price auction in which bidders without relevant private information lose out to better informed bidders with probability one. In this paper, we focus on the opposite case, that there are at least as many units for sale as there are well-informed bidders. In practice, for example in the auctions cited in the previous paragraph, it often seems realistic that well-informed bidders form only a small fraction of the total market.

We show that Milgrom's result is reversed, and that the uninformed or poorly informed bidders can win with positive probability. In fact, we find a kind of monotonicity: The more units are for sale, the more aggressively is the bid behaviour of uninformed or poorly informed bidders, and the more likely it is that they win. In extreme cases, this probability can become one. We also show that the unconditional expected utility of the informed bidders may be less than that of the completely uninformed bidders.

It is important to emphasise that, although we consider multiunit auctions, like Milgrom we maintain the assumption that each bidder individually demands only one unit. Thus, our results are unrelated to the difficult problems arising in auctions in which bidders are allowed to submit multiunit-demands. Because we maintain the unit-demand assumption, it is also obvious how the second price auction needs to be defined in the multiunit case, say with k units for sale: the bidders with the k highest bids win and pay the $k+1$-th highest bid.

The observation that uninformed bidders may win auctions is not original to this paper. In fact, Engelbrecht-Wiggans, Milgrom, and Weber (1983) showed that this may happen in the single unit case if the format is a first price auction. Engelbrecht-Wiggans, Milgrom and Weber's result was extended by Daripa (1998) to a multiunit set-up, using a generalisation of the first price auction. The auction format of Daripa is more difficult to analyse than ours. His analysis is also complicated by the fact that he allows for multiunit-demand. As a consequence, we obtain a more clear-cut analysis than Daripa. For example, we do not face as severe problems of multiplicity of equilibria as Daripa does.

Another reason for our interest in the second price format is that it allows us to develop particularly clearly the intuition for our findings. We explain the relatively good performance of poorly informed or uninformed bidders with respect to informed bidders in terms of the differential effect of the winner's curse and the loser's curse on the incentives to bid of bidders with different quality of information.

In the (generalised) second price auction a bidder will want to raise his bid by a small amount, say from b to $b+\epsilon$, if the expected value of a unit, conditional on its price being $p \in (b, b+\epsilon)$, is larger than p. The price is p if and only if the k-th highest bid of the other bidders is p. This event is the intersection of two events, one of which implies good news whereas the other implies bad news for the bidder. The good news is that at least k other bidders have been willing to bid p or more. If these bidders had any private information at all, it must have been favourable. This is good news. This effect has been called the "loser's curse" as a bidder who neglects this effect will regret losing. The bad news is that at least $m-k$ other bidders (where m denotes the total number of bidders) have bid p or less, and hence, if they had any private information at all, this must have been unfavourable. This effect has been called the "winner's curse" as a bidder who neglects this effect will regret winning.[1]

The winner's curse reduces the incentives to bid higher, whereas the loser's curse raises the incentives to bid higher. Both effects are stronger for less informed bidders. The reason is that the average quality of the information of the other bidders is higher from the point of view of a poorly informed bidder than from the point of view of a well-informed bidder. If the loser's curse is sufficiently strong in comparison to the winner's curse we can expect that in equilibrium bidders with less information win more often than bidders with more information. Moreover, we can also expect that the stronger the loser's curse in comparison to the winner's curse, the more often less informed bidders win. This explains the monotonicity of the behaviour of uninformed or poorly informed bidders with respect to the number of units. The more units there are for sale, the more winners and the fewer losers there are in the auction, thus the loser's curse will be stronger and the winner's curse will be weaker.

Note that when there is only one unit for sale the good news of the loser's curse are completely dominated by the bad news of the winner's curse. In this case we

[1] The "winner's curse" is well-known in the auction literature, see for instance the survey by Milgrom (1989). The concept of "loser's curse" is less stablished. It was first used by Holt and Sherman (1994) in the context of a bargaining model. The concept was introduced in auction models by Pesendorfer and Swinkels (1997). They also presented a formal definition of the meaning of the winner's curse and the loser's curse in the spirit of that given in our paper.

can say that the loser's curse plays no role. At the opposite extreme is the case when the number of units for sale equals the number of bidders minus one. Then the winner's curse is completely offset by the loser's curse. The winner's curse thus plays no role, and it can only be the loser's curse that affects the incentives to bid higher. Consequently, when the number of units for sale equals the number of bidders minus one and there is more than one unit for sale, less informed bidders bid more aggressively than better informed bidders.

One possible application of our results concerns the case in which the auctioneer can choose into how many "lots" to divide what he has for sale. For the case that the auctioneer is committed to a generalised second price auction, our results indicate that he should choose a large number of "lots" if, for some reason, the success chances of poorly informed bidders is important to him.

The most closely related papers are those of Milgrom (1981), Engelbrecht-Wiggans, Milgrom, and Weber (1983) and Daripa (1998) which were already discussed above. Another related study is that of Pesendorfer and Swinkels (1997). This paper, like ours, studies the generalisation of the second price auction to the multiunit case when bidders have unit-demand. Pesendorfer and Swinkels (1997) differs from our paper in two respects. Firstly, they assume that all bidders have signals of equal informativeness, whereas our focus is on the case that some bidders have more informative signals than others. Secondly, they focus on the case that the number of units for sale and the number of bidders are large. By contrast, our focus is on the case of a fixed, finite number.

This paper is structured as follows: In Section 2, we study a basic model in which there are one bidder with relevant, although potentially incomplete information, and several other, completely uninformed bidders. Section 3 extends the model and analyses a case in which there are several bidders who hold relevant information whereas other bidders are completely uninformed. In Section 4, we extend the model of Section 2 into a different direction, and allow the bidders who were uninformed in Section 2 to hold some pieces of information. We only assume that their information is less significant than that of the well-informed bidder. We show that the equilibria in this set-up converge in an appropriate sense to the equilibrium in Section 2 as the significance of the less informed bidders' signals tends to zero.

2.2 An Auction with One Informed and Many Uninformed Bidders

An auctioneer puts up for sale through auction k indivisible units of a good. There are $n+1$ bidders, $n \geq 2$.[2] Each bidder can bid for one or zero units of the good.[3] We assume that the number of bidders is greater than the number of units for sale, $n+1 > k$.

Each bidder obtains a von Neumann Morgenstern utility of $v - p$ if she obtains one unit of the good, and she obtains a von Neumann Morgenstern utility of zero if she obtains no unit. The value v is common to all bidders. One bidder, the *informed* bidder, receives privately a signal s, whereas the other bidders, the *uninformed* bidders, do not receive any signal. For simplicity we assume that $v = s$.[4] The signal s is drawn from the interval $[\underline{s}, \overline{s}]$ (where $0 \leq \underline{s} < \overline{s}$) with a continuous distribution function $F(s)$. This distribution is assumed to have support $[\underline{s}, \overline{s}]$.

The auction used is a uniform price auction. We assume that there are neither a reserve bid nor an entry fee. All bidders submit simultaneously non-negative bids. The bidders who make the k highest bids win one unit each. The price which they have to pay is the $k+1$-th highest bid. If the k-th highest bid and the $k+1$-th highest bid have the same value b, then the price in the auction is b, all bidders who make a bid strictly higher than b get one unit with probability 1, and the remaining winners are randomly selected among all bidders who have made bid b, whereby all such bidders have the same probability of being selected.

To analyse equilibrium bidding in this auction we begin with the following obser-

[2] In the case $n = k = 1$ the auction game which we are considering has very many equilibria. Since an analysis of these equilibria would distract from the main point of this paper, we restrict attention to the case $n \geq 2$.

[3] Equivalently we could assume that a perfectly divisible good is for sale. All bidders have constant marginal utility. The auctioneer splits the good into k identical lots and allows each bidder to bid for at most one of these lots.

[4] Engelbrecht-Wiggans, Milgrom, and Weber (1983) show that this assumption is equivalent to assume that v and s are two affiliated random variables.

vation:

Proposition 2.1. *The informed bidder has a weakly dominant strategy: $b_I^*(s) = s$ for all $s \in [\underline{s}, \overline{s}]$.*

Proof. This follows from the standard argument that is used to show that in single object, private value, second price auctions bidding one's true value is a dominant strategy. ■

Given Proposition 2.1 we can focus on the behaviour of the uninformed bidders. We shall assume that all uninformed bidders play the same pure or mixed strategy. We shall describe this mixed strategy by its distribution function $G_U^* : [\underline{s}, \overline{s}] \rightarrow [0, 1]$. Notice that we rule out bids which are not in the interval $[\underline{s}, \overline{s}]$. Such bids are weakly dominated. We shall call a strategy of the uninformed bidders an *equilibrium* strategy if together with the weakly dominant strategy of the informed bidder it constitutes a Bayesian Nash equilibrium of the auction game.

We consider first two cases that allow for an analysis specially clear-cut. The first of these is when the number of units for sale is only one. Then, the only effect on the incentives to bid is the winner's curse as it was already suggested in the introduction. In this case, due to the absence of the loser's curse on the incentives to win, these are always higher for the informed than for the uninformed bidders. As expected, the next proposition states that there is a unique equilibrium where the uninformed bidders bid lower than any type of the informed bidder.

Proposition 2.2. *If there is only one unit for sale, $k = 1$, there is only one equilibrium strategy for the uninformed bidders, to bid $\underline{s}$ with probability one.*

Proof that the proposed strategy is an equilibrium strategy: In the proposed equilibrium the uninformed bidders get utility zero. The only possible deviation for uninformed bidders is to raise their bids. If all uninformed bidders except one bid $\underline{s}$, and one uninformed bidder raises her bid to some value $b > \underline{s}$, then this uninformed bidder wins if and only if the informed bidder's bid is between $\underline{s}$ and b. Moreover, the price which the uninformed bidder has to pay is exactly the informed bidder's bid which equals the true value of one unit. Therefore, the expected utility from raising the bid is zero. Thus, there is no strict incentive for uninformed bidders to raise their bids.

Proof that there are no other equilibrium strategies: Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids above $\underline{s}$. Then each uninformed bidder can gain by changing her strategy, and bidding $\underline{s}$ with probability 1. To see this distinguish the following two events: (i) the highest of all other uninformed bidders' bids is greater than the informed bidders' bid; and (ii) the highest of all other uninformed bidders' bids is less than or equal to the informed bidders' bid. Observe that both events occur with positive probability. In event (ii) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (i), however, there is a strict incentive to be among the losers of the auction, this is, there is a winner's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is among the winners. Thus, she can strictly gain by deviating to $\underline{s}$. ■

Remark 2.1. *If $k = 1$: (i) The price is completely uninformative, since it is always equal to $\underline{s}$. (ii) The informed bidder wins with probability 1 the unique unit for sale. (iii) The informed bidder has positive expected utility whereas the uninformed bidders have expected utility zero.*

The other specially simple case is when the number of units for sale equals the number of uninformed bidders. As we suggested in the introduction, then the winner's curse plays no role, it is always dominated by the loser's curse. In this case, the incentives to win of the uninformed bidders are always greater than those of the informed bidder. As a consequence the uninformed bidders bid higher than any type of the informed bidder:

Proposition 2.3. *If there are n units for sale, $k = n$, there is only one equilibrium strategy for the uninformed bidders, to bid $\overline{s}$ with probability one.*

Proof that the proposed strategy is an equilibrium strategy: In the proposed equilibrium the uninformed bidders have utility zero. This is because they all win with probability one, but the price equals the bid of the informed bidder, i.e. the value of the good. If an uninformed bidder lowers her bid, she loses the auction whenever the informed bidder's bid is above her lower bid. Otherwise she wins, but at a price which equals the informed bidder's bid. Hence her expected utility is again zero. Thus, no uninformed bidder can gain by deviating.

Proof that there are no other equilibrium strategies: Suppose all uninformed bidders choose the same mixed strategy, and assume that this strategy assigns positive probability to bids below $\bar{s}$. Then each uninformed bidder can gain by changing her strategy, and bidding $\bar{s}$ with probability 1. To see this distinguish the following two events: (i) the lowest of all other uninformed bidders' bids is greater than or equal to the informed bidders' bid; and (ii) the lowest of all other uninformed bidders' bids is less than the informed bidders' bid. Observe that both events occur with positive probability. In event (i) all bids give expected utility zero, thus the change in bidding strategy has no effect. In event (ii), however, there is a strict incentive to be among the winners of the auction, this is, there is a loser's curse. If the bidder adopts the same mixed strategy as all other uninformed bidders, there is a positive probability that she is not among the winners. Thus, she can strictly gain by deviating to $\bar{s}$. ■

Remark 2.2. *If $k = n$: (i) The price reveals the true value. (ii) With probability 1 all units are won by uninformed bidders. (iii) All bidders have expected utility zero.*

In other cases, namely when $1 < k < n$, both the winner's curse and the loser's curse effects can affect the equilibrium outcome. The study of the interaction of these two effects requires a slightly different analysis than that of the previous cases. This analysis is done in the next proposition:

Proposition 2.4. *If $1 < k < n$, then there exists a unique equilibrium strategy for the uninformed bidders:*[5]

$$G_U^*(b) = \frac{F(b)(n-k)\,(b - E\,[s|\, s \leq b])}{F(b)(n-k)\,(b - E\,[s|\, s \leq b]) + (1 - F(b))(k-1)\,(E\,[s|\, s \geq b] - b)},$$

for all $b \in [\underline{s}, \bar{s}]$.

[5] Here and in the following $E\,[.|\,.]$ denotes the expected value of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.

Proof. This proof is broken down into two steps.

Step 1. In the first step we consider mixed strategies of the uninformed bidder that have a continuous distribution function. A necessary condition for such strategies to be an equilibrium is that each uninformed bidder is indifferent between all the bids in the support, if she takes as given that all the other uninformed bidders adopt the proposed strategy, and that the informed bidder plays her weakly dominant strategy. This is just the standard indifference condition characterising Nash equilibria in mixed strategies, extended to the case of infinite strategy spaces.

This indifference condition is satisfied only if each uninformed bidder gets zero expected utility. To see why notice that the number of units for sale is less than the number of uninformed bidders, thus the lowest bid in the support of uninformed bidders' strategy must lose with probability one. Thus, she gets zero expected utility.

To apply this condition we distinguish two events under which an uninformed bidder can win the auction: (i) the price in the auction equals the bid of the informed bidder, and (ii) the price in the auction equals the bid of another uninformed bidder. Under event (i), the expected utility of winning is trivially zero, the price equals the value of the good. Hence, the expected utility of winning must also be zero under event (ii) for all bids of the uninformed bidders that are in the support of G^*_U. Or equivalently, the expected utility of winning under event (ii) and conditional on the price equals b must be zero almost surely.

To formalise the last necessary condition, we introduce for an arbitrary b in the support of the equilibrium mixed strategy of the uninformed bidders, the notation $\mathbb{P}(b)$. This stands for the probability that the informed bidder's bid, s, is greater than b, conditional on the following event: there are exactly $k-1$ bids above b among $n-2$ uninformed bidders' bids and the informed bidder's bid. This is the probability that an uninformed bidder suffers a loser's curse at price b. Similarly, $1-\mathbb{P}(b)$ is the probability that an uninformed bidder suffers a winner's curse at price b. Using this notation, we can write our necessary condition as:

$$\mathbb{P}(b)E[s|s \geq b] + (1-\mathbb{P}(b))E[s|s < b] - b = 0, \tag{2.1}$$

almost surely. Here $\mathbb{P}(b)$ equals by definition:

$$\frac{\binom{n-2}{k-2}[1-F(b)][1-G_U^*(b)]^{k-2}G_U^*(b)^{n-k}}{\binom{n-2}{k-2}[1-F(b)][1-G_U^*(b)]^{k-2}G_U^*(b)^{n-k}+\binom{n-2}{k-1}F(b)[1-G_U^*(b)]^{k-1}G_U^*(b)^{n-k-1}}.$$

The unique G_U^* that solves the above necessary condition for a given b must be as defined in the proposition. Since this function is continuos, strictly increasing and satisfies $G_U^*(\underline{s}) = 0$ and $G_U^*(\overline{s})$=1, the unique candidate for an equilibrium continuous distribution function must be that in the proposition.

It only remains to be shown that this distribution function is in fact an equilibrium strategy. This follows since we have already shown that the expected utility of each uninformed bidder given that the other uninformed bidders play the proposed strategy, and that the informed bidder plays her weakly dominant strategy, is zero for all bids in $[\underline{s}, \overline{s}]$.

Step 2. In this second step we study mixed strategies that have a discontinuous distribution function. Assume that G_U^* is one of such strategies with an atom at $\hat{b}$. We focus on the incentives to deviate of an uninformed bidder, say bidder l. For the sake of simplicity we introduce the following notation. Let $b_{(k)}$ be the k-th highest bid of all the bidders but l. Define the event "$\hat{b}$ wins" to be the event in which bidder l when making a bid $\hat{b}$ wins one unit, and the event "$\hat{b}$ loses" the complement of "$\hat{b}$ wins", this is the event in which bidder l when making a bid $\hat{b}$ loses the auction.

We begin by arguing that we must have: $E[v|b_{(k)} = \hat{b}$ and $\hat{b}$ wins$] \geq \hat{b}$. Suppose instead $E[v|b_{(k)} = \hat{b}$ and $\hat{b}$ wins$] < \hat{b}$. If this were the case, then bidder l could gain by shifting all probability mass that is placed on $\hat{b}$ to some bid $\hat{b} - \epsilon$ where $\epsilon > 0$ is close to zero. This change would obviously make no difference to player l's utility in the case that $b_{(k)} > \hat{b}$, nor would it affect l's utility in the case that $b_{(k)} = \hat{b}$ and $\hat{b}$ loses. Finally, it would obviously also not make any difference in the case that $b_{(k)} < \hat{b} - \epsilon$. In the event that $b_{(k)} = \hat{b}$ and $\hat{b}$ wins, which has positive probability, the change in strategy would lead to a strict increase in player l's utility. Finally, the probability of the event that $\hat{b} - \epsilon \leq b_{(k)} < \hat{b}$ can be made arbitrarily small by choosing a sufficiently small ϵ, so that it does not affect the advantageousness of the proposed deviation.

In a similar way it can be argued that we must have $E[v|b_{(k)} = \hat{b}$ and $\hat{b}$ loses$] \leq \hat{b}$.

If $\hat{b} = \overline{s}$, the event $b_{(k)} = \hat{b}$ means that the bid of the informed bidder is below $\overline{s}$. As a consequence the first of the conditions above cannot be satisfied. Similarly, it can be shown that $\hat{b} = \underline{s}$ violates the second of the conditions above.

We can complete our indirect proof by arguing that if $\underline{s} < \hat{b} < \overline{s}$, then $E[v|b_{(k)} = \hat{b}$ and $\hat{b}$ wins$] < E[v|b_{(k)} = \hat{b}] < E[v|b_{(k)} = \hat{b}$ and $\hat{b}$ loses$]$, this is, that there is a winner's and a loser's curse at price $\hat{p}$. This last inequality obviously contradicts the other two inequalities. Suppose you knew that $b_{(k)} = \hat{b}$, but you did not know whether the informed bidder is bidding above or below $\hat{b}$. If you learned that the informed bidder is bidding above $\hat{b}$, then the probability that $\hat{b}$ wins would drop. Hence, $\hat{b}$ wins has strictly negative correlation with the event that the informed bidder is bidding above $\hat{b}$, conditional on $b_{(k)} = \hat{b}$. This implies that whenever $\hat{b}$ wins it is ex post more likely that the informed bidder is bidding below $\hat{b}$, and vice versa when $\hat{b}$ loses. ■

Remark 2.3. *If $1 < k < n$: (i) The price contains information about the true value, but it is an imperfect signal. (ii) All bidders have positive probability of winning. (iii) The informed bidder has positive expected utility, but the uninformed bidders have expected utility zero.*

In order to state the next result of the paper, we introduce the following definition:

Definition: We say that the uninformed bidders bid *relatively more aggressively* than the informed bidder the more units there are for sale if and only if for every s in $(\underline{s}, \overline{s})$ the probability that the bid of an uninformed bidder is above $b_I^*(s)$ increases when the number of units for sale increases.

We can now state next corollary:

Corollary 2.1. *The uninformed bidders bid relatively more aggressively than the informed bidder the more units there are for sale.*

Proof. The corollary follows trivially from Proposition 2.2 and Proposition 2.3 when the starting number of units for sale is 1 or when the final number of units for sale is n respectively. In other cases since the bid of the informed bidder is $b_I^*(s) = s$, it is enough to show that $1 - G_U^*(b)$ increases with k. This follows directly from the definition of G_U^* given in Proposition 2.4. ■

This corollary can be explained in terms of the winner's and loser's curse. Increasing the number of units for sale increases the relative strength of the loser's curse with respect to the winner's curse. This implies that the uninformed bidders' incentives to win increase more than those of the informed bidder. As a consequence, each uninformed bidder bids relatively more aggressively than the informed bidder.

Figure 2.1 shows the plot of the density of the distribution of the equilibrium mixed strategy of the uninformed bidders for $k = \{2, 3, 4, 5\}$ given that F is a uniform distribution function on $[0, 1]$ and $n = 6$. Since the bid of the informed bidder does not change with k, this graph illustrates Corollary 2.1.

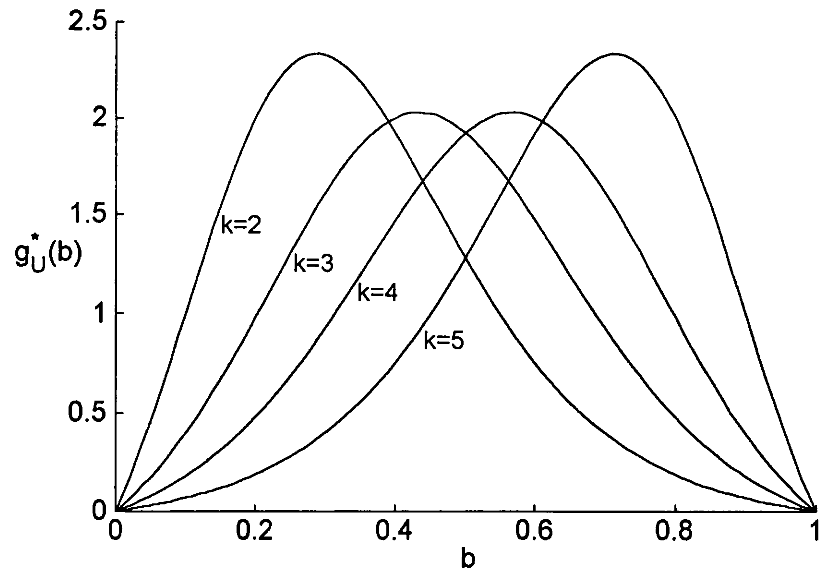

Figure 2.1: Plot of the density (g_U^*) of G_U^* for $n = 6$.

2.3 An Auction in Which Uninformed Bidders have Positive Expected Utility

In the model of the previous section the uninformed bidders can win with positive probability but they will always receive zero expected utility. The purpose of this section is to construct a model in which the uninformed bidders can win, and their expected payoff is strictly positive.

As before, we assume that there are k units of the same good for sale. The number of bidders is now assumed to be $n_I + n_U$, where $n_I + n_U > k$ and $n_I < n_U$. Among the bidders, n_I bidders are called *informed* bidders. Each of these bidders receives privately a signal s_i. The other n_U bidders are called *uninformed* bidders. They receive no signal. The value of the good, v, equals $\frac{\sum_{i=1}^{n_I} s_i}{n_I}$. We assume that the signals s_i are independently drawn from the set $[\underline{s}, \overline{s}]$ $(0 \leq \underline{s} < \overline{s})$ according to the same continuous distribution function F with support $[\underline{s}, \overline{s}]$. Bidders' preferences and the auction game are the same as in Section 2.

As in Section 2, we shall focus on symmetric equilibria. In this section, this will mean that all informed bidders play the same strategy, and all uninformed bidders play the same strategy. For simplicity, we shall focus on equilibria in pure strategies instead of allowing for mixed strategies as in Section 2. We shall denote by $b_I^* : [\underline{s}, \overline{s}] \to R^+$ the strategy of the informed bidders and by $b_U^* \in R^+$ the bid of the uninformed bidders. We shall further simplify our arguments by assuming that the informed bidders play a continuous and strictly increasing strategy.

Some of the results of the previous section generalise in natural ways to the model of the current section. For example, in the case $k \leq n_I$, it can be proved that in the unique equilibrium outcome the uninformed bidders lose with probability one. Such equilibria generalise the equilibrium in Proposition 2. For the case $n_I < k < n_U$ one can show that there is no equilibrium in pure strategies. In this respect this case is similar to the case of Proposition 4.

We shall not deal explicitly in this paper with the two cases mentioned in the previous paragraph. We shall also omit the rather special case $k = n_U$. Instead, we shall focus on the case that $k > n_U$. This case yields for our purposes the most interesting result. The result is similar to Proposition 3. We use the symbol $s_{(r)}$ to refer to the r-th highest signal of the informed bidders.

Proposition 2.5. *Suppose $k > n_U$. Then the bid functions (b_I^*, b_U^*) constitute an equilibrium if and only if:*

$$b_I^*(s) = E\left[v|\, s_{(q)} = s_{(q+1)} = s\right]$$
$$b_U^* \geq E\left[v|\, s_{(q)} = s_{(q+1)} = \overline{s}\right].$$

Here, we define $q \equiv k - n_U$.

These conditions are such that the uninformed bidders win with probability one in equilibrium.

We first provide an example of equilibrium bid functions. In this example we assume $n_I = 6$, $n_U = 8$, $k = 10$ and F to be a uniform distribution function with support $[0, 1]$. Then $b_I^* = 2/3s + 1/4$ and $b_U^* = 11/12$ satisfy the conditions given in Proposition 2.5. A plot of these equilibrium bid functions appears in figure 2.2.

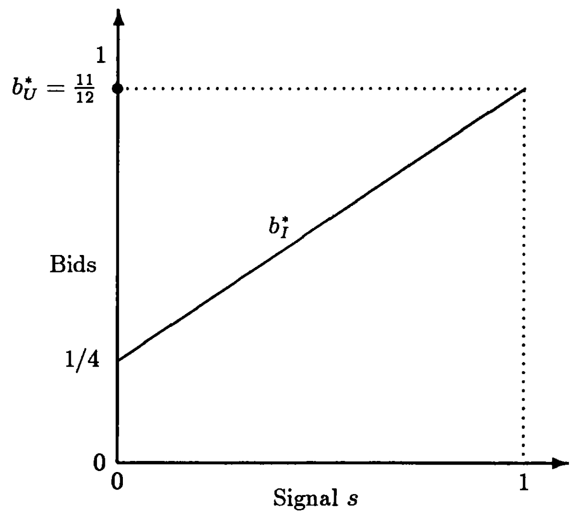

Figure 2.2: Equilibrium bid functions with $n_I = 6$, $n_U = 8$ and $k = 10$.

Proof that the conditions in the Proposition are sufficient: The strategy of the informed bidders is the same as in the standard symmetric equilibrium in an auction of q

units and no uninformed bidders. The arguments used for the standard symmetric equilibrium by Milgrom (1981) to show that the bidders do not have incentives to deviate also explain why the informed bidders do not have incentives to deviate if the conditions in the proposition are satisfied.

To see that the uninformed bidders do not have incentives to deviate suppose that all the bidders stick to some strategies that satisfy the proposed conditions. Under this assumption we can state the following arguments. First, each uninformed bidder does not have incentives to arise her bid since she is already winning with probability one. Second, if one uninformed bidder lowers her bid below b_U^* she does not improve her payoffs. This is because by deviating the bidder loses when the price in the auction is between her deviating bid and b_U^*, and the expected value of winning at a given price is positive for all the equilibrium prices. To see why, take an arbitrary price of the auction p. This price p must correspond to the equilibrium bid of a type s of the informed bidders. Then, the expected utility of winning at price p is the difference between the expected value of the good conditional on the $q+1$-th highest signal equals s, and the price p. Since the price p equals the expected value of the good conditional on the q-th and $q+1$-th highest signals equal s by definition of s, the difference is positive.

Proof that the conditions in the Proposition are necessary: Assume that the bid of the uninformed bidders is above all the bids of the informed bidders, then the informed bidders' equilibrium strategy must be an equilibrium strategy of the same auction game with q units for sale, n_I informed bidders and no uninformed bidders. In this last case, we can use a proof similar to that by Harstad and Levin (1986) to show that there is a unique equilibrium strategy. This equilibrium strategy is such that each informed bidder bids the expected value of the good given that the q and the $q+1$ highest signal of the informed bidders equal her own private signal. This is the equilibrium bid function that appears in the proposition.

In order to complete the proof it only remains to be shown that the uninformed bidder cannot lose the auction with positive probability in equilibrium. This proof follows a similar structure and notation to *Step 2* of the proof of Proposition 2.4.

Assume that the uninformed bidders' bid, b_U^*, is below the maximum bid of the bid function of the informed bidders. We focus on the incentives to deviate of an uninformed bidder, say bidder l. We reintroduce the notation $b_{(k)}$ for the k-th highest bid of all the bidders but l. Define the event "l wins" to be the event in which bidder l wins when making the bid b_U^* and the event "l loses" as its complement, this is, the event in which bidder l loses when making the bid b_U^*.

We can use the same arguments as in *Step 2* of the proof of Proposition 2.4 to show that two necessary conditions of equilibrium are: $E[v|b_{(k)} = b_U^* \text{ and } l \text{ wins}] \geq b_U^*$ and $E[v|b_{(k)} = b_U^* \text{ and } l \text{ loses}] \leq b_U^*$.

We start showing that these inequalities cannot be met simultaneously if b_U^* is between the minimum and the maximum bid of the informed bidders. We prove this using a random variable $\tilde{I}$[6] that stands for the number of informed bidders that bid above b_U^*. Our indirect proof is to show that the conditional distribution of $\tilde{I}$ shifts in the sense of strictly first order stochastic dominance, upwards when l loses and downwards when l wins. Since we restrict attention to equilibria in which the informed bidders' bid function is strictly increasing, this is sufficient for our claim.

By definition:[7]

$$\begin{aligned} \Pr\left(l \text{ wins}|\, \tilde{I} = I, k - n_U < \tilde{I} < k\right) &= \tfrac{k-I}{n_U} \\ \Pr\left(l \text{ loses}|\, \tilde{I} = I, k - n_U < \tilde{I} < k\right) &= \tfrac{n_U - k + I}{n_U}, \end{aligned}$$

where $k - n_U < \tilde{I} < k$ is the same event than $b_{(k)} = b_U^*$. If $\tilde{I}$ is less than k but more than $k - n_U$, the k-th highest bid of the other bidders is the bid of one of the uninformed bidders, all of which bid b_U^*.

Hence,

$$\frac{\Pr\left(l \text{ wins}|\, \tilde{I} = I, b_{(k)} = b_U^*\right)}{\Pr\left(l \text{ loses}|\, \tilde{I} = I, b_{(k)} = b_U^*\right)}$$

decreases strictly with I. Therefore, l wins and l loses, conditional on $b_{(k)} = b_U^*$ can be interpreted as a pair of signals that satisfy the Monotone Likelyhood Ratio Property.

[6] We use the standard notation $\tilde{I}$ for the random variable and I for its realisation.

[7] Here and in the following Pr(. | .) denotes the expected probability of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line.

Consequently, the distribution of $\tilde{I}$ conditional on l loses and $b_{(k)} = b_U^*$ strictly first order stochastically dominates the distribution of I conditional on l wins and $b_{(k)} = b_U^*$.

We complete our proof with the case in which b_U^* is equal or below the minimum bid of the informed bidders. In this case, since the events l wins, l loses and $b_{(k)} = b_U^*$ are uninformative of v, the two inequalities above simplify to $b_U^* = E[v]$. Moreover, since the number of informed bidders is less than the number of units for sale, this means that the price in the auction is $E[v]$ with probability one. Hence, an informed bidder with a signal low enough has incentives to deviate and bid below b_U^*. Bidding above means winning with probability one at price $E[v]$ and bidding below losing with probability one. ■

Proposition 2.6. *If $k > n_U$, then the expected utility of each uninformed bidder is strictly positive and strictly greater than the unconditional expected utility of each informed bidder in equilibrium.*

Proof. Each uninformed bidder wins with probability one and pays the $q+1$ highest bid of the informed bidders. This difference is strictly positive because the expected value of the good given that the $q+1$ highest bid of the informed bidders equals the price is higher than the price for all potential prices. On the other hand, from an ex ante point of view, an informed bidder wins if and only if her signal is among the q highest signals of the informed bidders. This is with probability q/n_I. Conditional on winning, the informed bidder gets one unit of the good and pays the $q+1$ highest bid of the informed bidders, this is, she gets the same utility as each uninformed bidder. Since each uninformed bidder wins with probability one and each informed bidder only with probability q/n_I the proposition follows. ■

It is remarkable that, contrary to the previous section, competition among the uninformed bidders does not dissipate all the uninformed bidders' rents. The reason for this is that their demand (n_U units) is less than their supply (k units).

2.4 An Auction with One Informed and Many Poorly Informed Bidders

In this section we extend the analysis of Section 2 to a model where there is one informed bidder and some "poorly" informed bidders. The purpose of this extension is double. First, to show that similar results to those in Section 2 also hold in this more general set-up. Second, to prove that the equilibria in this extended model converge in an appropriate sense to the equilibria in the model of Section 2 when the informativeness of the poorly informed bidders' signal goes to zero.

In this section we keep all the assumptions of Section 2 except the information structure. This is modified to allow for less informative signals.

More precisely, we assume that the value of the good v is a weighted average of one signal s and n signals s_i^P $(i = 1, 2, ...n)$. Each signal s_i^P is less informative about v than s in the sense of a smaller weight in the former average. Formally, $v = \frac{s+\lambda\sum_{i=1}^{n} s_i^P}{1+n\lambda}$, with $0 < \lambda < 1$. We assume that the signal s and all the signals s_i^P are independently drawn from the set $[\underline{s}, \overline{s}]$ $(0 \leq \underline{s} < \overline{s})$ according to the same continuous distribution function F with support $[\underline{s}, \overline{s}]$. We shall assume that one bidder (the *informed* bidder) receives privately the signal s, whereas each of the other bidders (the *poorly informed* bidders) receives privately a different signal s_i^P.

An equilibrium of the game is a bid function $b_I^* : [\underline{s}, \overline{s}] \longrightarrow \mathbb{R}^+$ for the informed bidder and a bid function $b_P^* : [\underline{s}, \overline{s}] \longrightarrow \mathbb{R}^+$ for the poorly informed bidders, such that (b_I^*, b_P^*) is a Bayesian Nash equilibrium of the game.

For the sake of simplicity we restrict attention to equilibria in continuous and strictly increasing strategies. Moreover, only equilibria in which all the bidders have an unconditional positive probability of winning the auction are analysed. This constraint rules out some strange equilibria that exist in the case $k = 1$ and $k = n$.[8]

[8] For instance, if $k = 1$, there exists a set of equilibria where the informed bidder bids high enough and the poorly informed bidders bid low enough. In this case, the informed bidder does not have incentives to deviate because she wins with probability one at a very low price. On the other hand, a poorly informed bidder could win only if she would bid above the very high bid of the informed bidder. Since in this case winning would mean paying the very high bid of the informed bidder, the poorly informed bidders do not have incentives to deviate. Equilibria of this type have been called in a different set-up *degenerate* by Bikhchandani and Riley (1991).

Next, we introduce an assumption that simplifies the analysis of equilibria.

Assumption 2.1. *$\sigma - E[s|\, s \le \sigma]$ and $E[s|\, s \ge \sigma] - \sigma$ are respectively strictly increasing and strictly decreasing in σ.*

Assumption 2.1 is satisfied by many distribution functions. If F has a continuously differentiable density, Bikhchandani and Riley (1991) show (Lemma 3) that a sufficient condition for the first part of the assumption is that $F(s)$ is strictly log-concave. Similarly, if F has a continuously differentiable density, a sufficient condition for the second part is that $1 - F(s)$ is strictly log-concave, this is that $F(s)$ has an increasing hazard rate.

For the analysis of this section we use a function $\phi : [\underline{s}, \overline{s}] \to [\underline{s}, \overline{s}]$. This function assigns to each type s^P of the poorly informed bidders the type $\phi(s^P)$ of the informed bidder who, in equilibrium, makes the same bid b as s^P does. This function will be implicitly defined by an equation $\Psi(\phi(s^P), s^P) = 0$, where $\Psi : [\underline{s}, \overline{s}]^2 \to \mathbb{R}$ is the difference between the expected value of the good given that there are $k-1$ bidders bidding above b, two poorly informed bidders bidding b and all the other bidders bidding below b; and the expected value of the good given that there are $k-1$ poorly informed bidders bidding above b, the informed bidder and one poorly informed bidder bidding b and all the other bidders bidding below b. In order to express this condition formally we project the first expected value on the events $s \ge \phi$ and $s < \phi$:

$$\begin{aligned}\Psi(\phi, s^P) = \mathbb{P}^*(\phi, s^P)&E[v|s \ge \phi, s^P_{(k-1)} = s^P_{(k)} = s^P] \\ &+ (1 - \mathbb{P}^*(\phi, s^P))E[v|s < \phi, s^P_{(k)} = s^P_{(k+1)} = s^P] \\ &\qquad - E[v|s = \phi, s^P_{(k)} = s^P], \quad (2.2)\end{aligned}$$

where $\mathbb{P}^*(\phi, s^P)$ is the probability that the bid of the informed bidder is above b given that there are $k-1$ bidders bidding above b among the bid of the informed bidder and the bids of $n-2$ poorly informed bidders. This is $\mathbb{P}^*(\phi, s^P) = 0$ if $k = 1$, $\mathbb{P}^*(\phi, s^P) = 1$ if $k = n$ and if $1 < k < n$:

$$\mathbb{P}^*(\phi, s^P) = \frac{\binom{n-2}{k-2}[1-F(\phi)][1-F(s^P)]^{k-2}F(s^P)^{n-k}}{\binom{n-2}{k-2}[1-F(\phi)][1-F(s^P)]^{k-2}F(s^P)^{n-k} + \binom{n-2}{k-1}F(\phi)[1-F(s^P)]^{k-1}F(s^P)^{n-k-1}}.$$

Lemma 2.1. *There exists a unique $\phi(s^P)$. This function ϕ is continuous and strictly increasing in s^P.*

See the proof in the Appendix.

The next proposition makes use of the function ϕ to characterise the set of equilibrium bid functions.

Proposition 2.7. *The pair of bidding strategies (b_I^*, b_P^*) is an equilibrium, if and only if:*

$$b_P^*(s^P) = b_I^*(\phi(s^P)) = E\left[v \middle| s = \phi(s^P), s_{(k)}^P = s^P\right], \tag{2.3}$$

for all $s^P \in [\underline{s}, \overline{s}]$, and:

- *If $k = 1$:*

$$\int_{\phi(\overline{s})}^{s} (E[v|\tilde{s} = \nu, s_{(1)}^P = \overline{s}] - b_I^*(\nu))f(\nu)d\nu \leq 0, \tag{2.4}$$

for all s in $[\phi(\overline{s}), \overline{s}]$.

- *If $k = n$:*

$$\int_{s}^{\phi(\underline{s})} (E[v|\tilde{s} = \nu, s_{(n)}^P = \underline{s}] - b_I^*(\nu))f(\nu)d\nu \geq 0, \tag{2.5}$$

for all s in $[\underline{s}, \phi(\underline{s})]$.

See the proof in the Appendix.

Next lemma shows that if $1 < k < n$, equation (2.3) determines uniquely not only b_P^* but also b_I^*. In the other cases, Proposition 2.7 does not determine a unique b_I^*.

Lemma 2.2.

(i) If $k = 1$, then $\phi(\underline{s}) = \underline{s}$ and $\phi(\overline{s}) < \overline{s}$.

(ii) If $k = n$, then $\phi(\underline{s}) > \underline{s}$ and $\phi(\overline{s}) = \overline{s}$.

(iii) If $1 < k < n$, then $\phi(\underline{s}) = \underline{s}$ and $\phi(\overline{s}) = \overline{s}$.

See the proof in the Appendix.

In Figures 2.3, 2.4 and 2.5 there appear some examples of equilibrium bid functions that illustrate Lemma 2.2. All these examples are done assuming that F is a uniform distribution function with support $[0, 1]$.

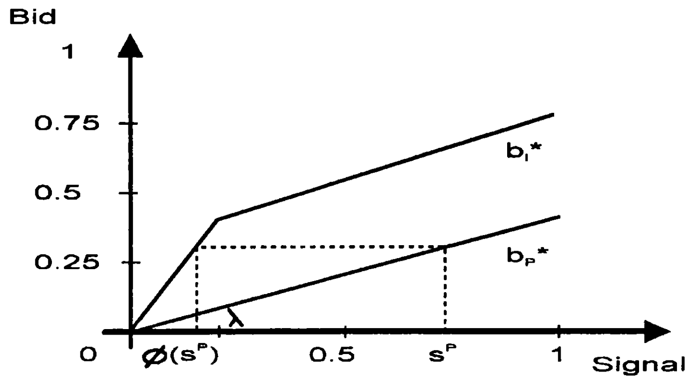

Figure 2.3: Equilibrium bid functions with $k = 1$, $n = 5$ and $\lambda = 0.2$.

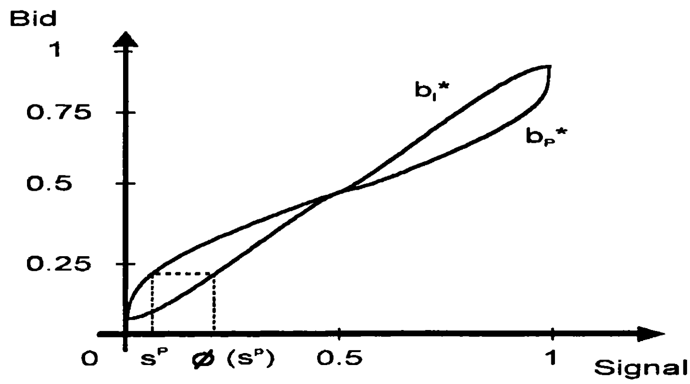

Figure 2.4: Equilibrium bid functions with $k = 3$, $n = 5$ and $\lambda = 0.2$.

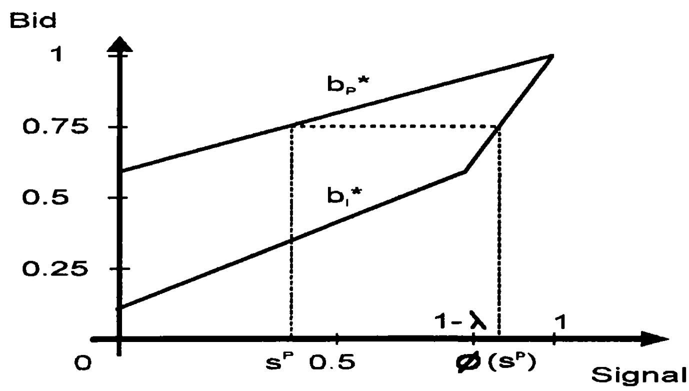

Figure 2.5: Equilibrium bid functions with $k = 5$, $n = 5$ and $\lambda = 0.2$.

We apply Proposition 2.7 to show that we can state a similar result to Corollary 2.1 in Section 2 in this new framework. We start with the following definition:

Definition: We say that the poorly informed bidders bid *relative more aggressively* than the informed bidder the more units are for sale if and only if for every s in $(\underline{s}, \overline{s})$ the probability that the bid of a poorly informed bidder is above $b_I^*(s)$ increases when the number of units for sale increases.

We can now state the next corollary:

Corollary 2.2. *Each poorly informed bidder bids relatively more aggressively than the informed bidder the more units there are for sale.*

See the proof in the Appendix.

The last point of this section is to provide a robustness test for the equilibrium outcomes given in Section 2. The next proposition accomplishes this task.

Proposition 2.8. *When λ goes to zero:*

(i) If $k = 1$, then the equilibrium bid function of the poorly informed bidders converges point-wise to $\underline{s}$. Moreover, in the limit each poorly informed bidder loses with probability one.

(ii) If $k = n$, then the equilibrium bid function of the poorly informed bidders converges point-wise to $\overline{s}$. Moreover, in the limit each poorly informed bidder wins with probability one.

(iii) If $1 < k < n$, then the equilibrium bid function of the informed bidder converges point-wise to $b_I^*(s) = s$, the equilibrium bid function of the informed bidder in Section 2. The equilibrium distribution of bids of the poorly informed bidders converges to the equilibrium distribution of bids of the uninformed bidders given in Proposition 2.4, Section 2.

See the proof in the Appendix.

2.5 Conclusions

In this paper we have studied several models in which there were one or more well informed and some other bidders with either no information or worse information. We have shown in this set-up that the relative performance of the more informed-less informed bidders depends on the interaction of two effects, the winner's curse and the loser's curse, and its differential effect on bidders with different quality of information. Basically, we showed that the leading effect is the loser's curse if the number of units for sale is large and the winner's curse if it is small. We based on these arguments to explain the surprising result that when there are several units for sale, bidders with less information can do better than bidders with better information in terms of probability of winning and expected revenue.

Our analysis leaves several extensions for further research. We distinguish two profitable ways of complementing our analysis. The first one is the study of other auction procedures, for instance, in next chapter we shall study the open ascending auction. This popular auction is similar to the (generalised) second price auction studied in this paper. The main difference to our concern is that the losers' bids are revealed along the auction and so the winner's curse is very much alleviated. Thus, we could expect that the loser's curse dominates (if there is more than one unit for sale) and consequently, that less informed bidders do better than more informed bidders. This extension is provided in Chapter 3 of this Ph.D. dissertation. The second branch of extensions is the study of the consequences for auction design of our analysis on the performance of less informed bidding. For instance, this should be a major worry if the auctioneer concern is to promote the entry[9] of poorly informed or uninformed bidders, or if he is interested in providing incentives to acquire information.

2.6 Appendix

In this appendix we provide the proofs of the lemmas and propositions in Section 4. To follow our arguments more easily, it is useful to notice that Ψ simplifies to:

[9] As we assume in Chapter 3 of this Ph.D. dissertation.

$$\Psi(\phi, s^P) =$$
$$\frac{(k-1)(1-F(\phi))F(s^P)}{A}\left[E\left[s|\, s \geq \phi\right] - \phi - \lambda\left(E\left[s|\, s \geq s^P\right] - s^P\right)\right] -$$
$$\frac{(n-k)F(\phi)(1-F(s^P))}{A}\left[\phi - E\left[s|\, s \leq \phi\right] - \lambda\left(s^P - E\left[s|\, s \leq s^P\right]\right)\right], \quad (2.6)$$

where $A \equiv (k-1)(1-F(\phi))F(s^P) + (n-k)F(\phi)(1-F(s^P))$.

Proof of Lemma 2.1. We start the proof showing by contradiction that $E[s|s \geq \phi]-\phi-\lambda\left(E[s|s \geq s^P] - s^P\right)$ and $\phi - E[s|s \leq \phi] - \lambda\left(s^P - E[s|s \leq s^P]\right)$ must be non negative when $\Psi(\phi, s^P) = 0$. Assume that one is negative. Then $\Psi(\phi, s^P) = 0$ implies that they are both negative. But under Assumption 1, if the first expression is negative, $\phi > s^P$, and if the second expression is negative, $\phi < s^P$.

As a consequence, under Assumption 1, Ψ must be strictly decreasing in its first argument and strictly increasing in its second when $\Psi(\phi, s^P) = 0$. Hence, the lemma follows as Ψ is continuous. ∎

Proof of Proposition 2.7.
Necessary proof. A local necessary condition that must be satisfied by the equilibrium bid functions almost everywhere is the following: if marginal changes in a bidder's bid change marginally her probability of winning, then her conditional bid must be equal to the expected value of the good conditional on the her private information and on the k-th highest bid of the other bidders equals her bid.

Our proof starts with the study of a set of bids B where the condition above binds. This set B is defined as the intersection of the interior of the range of the bid function of the informed bidder and the interior of the range of the bid function of the poorly informed bidders.

The first step is to show that B is not empty. Since the functions b_I^* and b_P^* are by assumption continuous and strictly increasing, B can be empty if and only if either $b_I^*(\underline{s}) \geq b_P^*(\overline{s})$ or $b_P^*(\underline{s}) \geq b_I^*(\overline{s})$. We only need to check that none of these two possibilities can happen in equilibrium.

If $k = 1$ then it can be neither $b_I^*(\underline{s}) \geq b_P^*(\overline{s})$ nor $b_P^*(\underline{s}) \geq b_I^*(\overline{s})$. The reason is that in these cases there are bidders that lose the auction with probability one, and by assumption we rule out this possibility in equilibrium. If $k = n$ the case $b_P^*(\underline{s}) \geq b_I^*(\overline{s})$ is ruled out because of identical reasons.

If $k > 1$ it cannot be that $b_I^*(\underline{s}) \geq b_P^*(\overline{s})$. To see why, assume that this is the case. Then, the informed bidder gets one unit with probability one and the poorly informed bidders compete for the $k-1$ units left. As a consequence, the equilibrium strategies of the poorly informed bidder must be equilibrium strategies of an auction with n poorly informed bidders, $k-1$ units for sale and without informed bidder. In this last case, we can use a proof similar to that by Harstad and Levin (1986) to show that there is a unique symmetric equilibrium strategy and this is such that: $b_P^*(s^P) = E[v|s_{(k-1)}^P = s_{(k)}^P = s^P]$. But given this bid function, types of the informed bidder with a very low signal have incentives to bid below $b_P^*(\overline{s})$. This is because the expected utility of winning the auction of a type of the informed bidder s' conditional on the price close enough to $b_P^*(\overline{s})$ is arbitrary close to the difference between the expected value conditional on $b_{(k)} = b_P^*(\overline{s})$ $(E[v|s = s', s_{(k)}^P = \overline{s}])$ and $b_P^*(\overline{s})$ $(E[v|s = s', s_{(k)}^P = s_{(k+1)}^P = \overline{s}])$, that is strictly negative for s' close enough to $\underline{s}$.

Similarly, it can be shown that if $1 < k < n$, it cannot be that $b_P^*(\underline{s}) \geq b_I^*(\overline{s})$. In this case, types of the informed bidder with signals close enough to $\overline{s}$ have incentives to rise their bids.

We continue our argument considering a generic bid b in B. The local necessary condition implies that the type of the informed bidder that bids b must be such that b equals the expected value of the good conditional on the private information of this type and on the k-th highest bid of the poorly informed bidders equals the bid b. If we define $\sigma(b)$ and $\sigma^P(b)$ as the inverse bid functions of the informed bidder and of the poorly informed bidder respectively, we can formalise this condition as:

$$b = E\left[v|\, s = \sigma(b), s_{(k)}^P = \sigma^P(b)\right]. \tag{2.7}$$

Similarly, the local necessary condition implies that the type of the poorly informed bidders that bids b must be such that b equals the expected value of the good conditional on the information of this type and on the k-th highest bid of the other bidders equals the bid b. In order to simplify this condition we distinguish two different events: either (i) the k-th highest bid of the other bidders is the bid of the informed bidder, or (ii) the k-th highest bid of the other bidders is the bid of another poorly informed bidder. Both events happen with strictly positive probability. The local necessary condition of the poorly informed bidders is identical to the local necessary condition of the informed bidder under event (i). This implies that the local necessary condition of the poorly informed bidders must also be satisfied under event (ii). We formalise this last condition projecting on two events: the event the informed bidder's bid is above b and the event the informed bidder's bid is below b:

$$\begin{aligned} b = \mathbb{P}^*(\sigma, \sigma^P) E \left[v \,|\, s \geq \sigma(b), s^P_{(k-1)} = s^P_{(k)} = \sigma^P(b) \right] + \\ (1 - \mathbb{P}^*(\sigma, \sigma^P)) E \left[v \,|\, s \leq \sigma(b), s^P_{(k)} = s^P_{(k+1)} = \sigma^P(b) \right]. \quad (2.8) \end{aligned}$$

Using simultaneously equations (2.7) and (2.8) we get the condition: $\Psi(\sigma(b), \sigma^P(b)) = 0$. Thus, $\sigma(b) = \phi(\sigma^P(b))$ for all b in B.

We next show that $B = (b^*_P(\underline{s}), b^*_P(\overline{s}))$. Suppose that the infimum of B is not $b^*_P(\underline{s})$. Since b^*_I and b^*_P are continuous and strictly increasing, $b^*_I(\underline{s})$ must be strictly greater than $b^*_P(\underline{s})$, but this contradicts that ϕ is strictly increasing and that $\phi(\underline{s}) \geq \underline{s}$ according to its definition. Similarly, we can show that the supremum of B must be $b^*_P(\overline{s})$.

To complete this part of the proof it only remains to be shown that the inequality (2.4) is necessary if $k = 1$ and the inequality (2.5) is necessary if $k = n$. We only consider the case $k = 1$, the other case can be proved in a symmetric way. Suppose that there is an s in $[\phi(\overline{s}), \overline{s}]$ such that the inequality (2.4) is not satisfied. Then, it can be shown that the types of the poorly informed bidders arbitrarily close to $\overline{s}$ are better off bid $b^*_I(s)$ than their equilibrium bid.

Sufficient Proof. We start with two remarks that we use to show that no type of the bidders has incentives to lower her bid. The first remark is that a type of one bidder does not have incentives to lower her bid if lower types do not have incentives to do so. The reason is that higher types of a bidder put higher value on winning than lower types, they expect to pay the same price but they give a higher expected value to the good. The second remark is that the necessary proof conducted above showed that the conditions of the proposition assure that the type $\underline{s}$ of the informed bidder and of the poorly informed bidders do not have incentives to lower her bid and that no type of the bidders has incentives to reduce her bid locally. Since we restrict attention to continuous and increasing bid functions, the two remarks above are enough to prove that no type of no bidder has incentives to lower her bid.

Similarly, we can show that no type of no bidder has incentives to rise her bid. ■

Proof of Lemma 2.2.

(i) Define $\eta(s) = s - E[\tilde{s}|\, \tilde{s} \leq s]$. Assumption 2.1 assures η is strictly increasing for all $s \in [\underline{s}, \overline{s}]$. If $k = 1$, then $\phi(s^P) = \eta^{-1}(\lambda\eta(s^P))$. Hence, $\phi(\overline{s}) = \eta^{-1}(\lambda\eta(\overline{s})) < \eta^{-1}(\eta(\overline{s})) = \overline{s}$.

(ii) Define $\mu(s) = E[s|\, s \geq s] - s$. Assumption 2.1 assures μ is strictly decreasing for all $s \in [\underline{s}, \overline{s}]$. If $k = n$, then $\phi(s^P) = \mu^{-1}(\lambda\mu(s^P))$. Hence, $\phi(\underline{s}) = \mu^{-1}(\lambda\mu(\underline{s})) > \mu^{-1}(\mu(\underline{s})) = \underline{s}$.

The other claims are trivial since $\Psi(\phi, s^P)$ has a straight forward unique solution in those cases. ■

Proof of Corollary 2.2. To prove the corollary it is enough to show that the type of the informed bidder that bids the same bid as a given type of the poorly informed bidders in equilibrium increases when the number of units increases. Since by Proposition 2.3 $b_P^*(s^P) = b_I^*(\phi(s^P))$, the statement before follows if $\phi(s^P)$ shifts upwards when we increase k. We can use the same arguments than in the proof of Lemma 2.1 to show that Ψ is strictly decreasing in its first argument and shifts upwards when we increase k around points such that $\Psi(\phi, s^P) = 0$. This completes the proof. ■

Proof of Proposition 2.8.

(i) We use the function η defined in the proof of Lemma 2.2. Since $\eta(\underline{s}) = 0$, $\lim_{\lambda\to 0} \phi(s) = \lim_{\lambda\to 0} \eta^{-1}(\lambda\eta(s)) = \eta^{-1}(0) = \underline{s}$.

(ii) We use the function μ defined in the proof of Lemma 2.2. Since $\mu(\bar{s}) = 0$, $\lim_{\lambda \to 0} \phi(s) = \lim_{\lambda \to 0} \mu^{-1}(\lambda \mu(s)) = \mu^{-1}(0) = \bar{s}$.

(iii) The first part follows directly from $\lim_{\lambda \to 0} b_I^*(s) = s$ because of Proposition 2.3. For the second part, notice that Ψ is continuous in λ and has unique solution to $\Psi(\phi, s^P) = 0$ for all $s^P \in [\underline{s}, \bar{s}]$ when $\lambda = 0$. Then $\phi(s^P)$ must satisfy in the limit $\Psi(\phi(s^P), s^P) = 0$. Given that in the limit $b_I^*(s) = s$, we can write the condition before as $\Psi(b, \phi^{-1}(b)) = 0$. This implies that in the limit: $F(\phi^{-1}(b)) = G_U^*(b)$ for all $s \in [\underline{s}, \bar{s}]$, where $G_U^*(b)$ is as defined in Proposition 2.4.

■

Chapter 3

Multinunit Auctions with a Well Informed Incumbent

3.1 Introduction

Klemperer (2000) has suggested that two important problems, collusion and entry deterrence, have not received enough emphasis in auction theory. We provide a natural set-up where entry deterrence can be an important issue and provide some considerations about auction formats.

Our set-up consists of an auction in which an auctioneer puts up for sale several units of a homogeneous good that has the pure common value property among the bidders, this is that all the bidders put exactly the same utility on this object. We assume that there are two types of bidders: an *incumbent* and some *entrants.* We assume that the incumbent has an advantage with respect to the entrants in terms of the quality of her information about the actual common value of the good for sale. More formally, the incumbent receives a noisy signal about the common value of the object that is more informative about the common value of the object than the noisy signal of the entrants.

The fact that there exists a better informed incumbent could deter the entry of the less informed entrants. We show that under these circumstances an open ascending auction gives higher expected utility to entrants than a sealed bid auction for a fixed number of entrants in the auction. It is in this sense that we say that an open ascending auction should attract more entrants than a sealed bid auction. Although we do not provide a proper entry model, it is straightforward to provide models of entry in auctions in which our results imply that the open ascending auction attracts more entrants than the sealed bid auction.[1]

We complement this result by showing that an open ascending auction can also give higher expected revenue to the auctioneer than a sealed bid auction with a fixed number of entrants and under some specific assumptions. Moreover, we shall show that under the same specific assumptions, an open ascending auction implements the expected revenue maximising mechanism among all the incentive compatible mechanisms that always sell all the units. These specific assumptions are that either the entrants are almost uninformed about the value of the object or the distribution of their signals is uniform.

The intuitions of our model are based on the different strength of the winner's curse in different auction formats, and its asymmetric effect on the incentives to raise the bid on bidders that have different quality of information. We also talk about a symmetric effect to the winner's curse, the loser's curse, that complements the explanations.

Quite informally,[2] the winner's curse refers to the bad news about the value of the good that the event "winning" conveys, namely, the fact that there are some other bidders that are bidding below the price. Hence, if they have any information about the value of the good, it must be pessimistic. The other important effect is the loser's curse. This effect refers to the good news about the value of the good that the event "winning" conveys. In this case, if there is more than one unit for sale, winning means that there are some other bidders that are bidding above the price. Thus, if they have any information about the value of the good, it must be optimistic. Consequently a

[1] For instance, suppose that the incumbent's decision to participate is not an issue. Then we could consider a two stage model in which in a first stage, the entrants take a costly entry decision, and then, in a second stage, those entrants that have decided to participate compete with the incumbent in the auction. Consider the case in which the entry decision is taken before each entrant receives her private signal. Suppose that a pure strategy equilibrium of the entry game is played or when entry is sequential. Then, our result on entrants' expected utility translates directly into higher levels of entry in the open ascending auction than in the sealed bid auction. We conjecture that this is also true when all the bidders randomise between entering or not.

[2] A more formal analysis is provided in Section 3.5.

relatively strong winner's curse should induce more conservative bidding, whereas a relatively strong loser's curse should induce more aggressive bidding.

We consider two auction set-ups that offer a clear-cut analysis. The kind of sealed bid auction that we model is a generalisation of the second price auction in the case of multiunit sales and where bidders can demand no more than one unit,[3] whereas the open ascending auction is a generalisation of an English auction[4] in the same circumstances.

To our concern the basic difference between these two auction formats is the amount of information that is revealed during the game. In the open ascending auction the bids of the losers are revealed along the auction. Hence, if the equilibrium strategies of the game are monotonic with respect to the bidders' private information, this private information should become public along the equilibrium path. On the other hand, in the sealed bid auction no information is revealed until the end of the game. This means that the open ascending auction has a smaller winner's curse than the sealed bid auction. Note, however, that the loser's curse is the same in both auction formats.

The reduction of the winner's curse in the open ascending auction increases the incentives to bid higher with respect to the sealed bid auction. However, in a model in which bidders are symmetric, it does not change the allocation (a symmetric equilibrium in strictly increasing strategies exists in both the sealed bid auction and the open ascending auction). Hence, under our assumption that signals are statistically independent, the revenue equivalence theorem assures that the auctioneer's expected revenue and the bidders' expected utility are the same in the sealed bid auction than in the open ascending auction, see Myerson (1981).

But in the case of asymmetric bidders, like in our case, the allocation among bidders

[3]There are several real life examples in which each bidder cannot bid for more than one unit of the good. This was for instance the case of the auction of licences of third generation mobile telephones that was conducted in the UK. From a broader perspective we also see this assumption as a simplification that gets around the complexities of the multi-demand auctions, mainly due to the multiplicity of equilibria.

[4]The auction we model is actually closer to what is usually called a Japanese auction. According to Milgrom and Weber (1982):

> [In the Japanese auction], the price is raised continuously, and a bidder who wishes to be active at the current price depresses a button. When he releases the button, he has withdrawn from the auction.

can change because both the winner's curse and the loser's curse have a stronger effect on the incentives of bidders with less precise information (the entrants in our model) than on bidders with more precise information (the incumbent in our model). The reason is that both effects refer to the information of the rivals, and on average, this must be of higher quality from the point of view of the less informed bidders than from the point of view of more informed bidders. This point was originally suggested in Chapter 2 of this Ph.D. dissertation.

Think of an extreme example in which one incumbent is perfectly informed about the value of the object and several entrants have no private information about the value of the object. In this case, the incumbent does not suffer neither winner's curse nor loser's curse. There are two reasons for this, first, the incumbent has all the relevant information, and second, the entrants do not have any relevant information. Consequently, the incumbent does not update her expected valuation of the object with any information that comes from the entrants' bidding behaviour. On the other hand, the entrants have a strong winner's curse and loser's curse. The fact that the incumbent is bidding above or below the price would changes the entrants' expectations about the value of the good.

Consequently, the entrants' incentives to increase the bid relative to the incumbent's incentives are higher in the open ascending auction than in the sealed bid auction. This means that the entrants are more willing to outbid the incumbent in the open ascending auction than in the sealed bid auction. In equilibrium, the entrants win more often in the open ascending auction than in the sealed bid auction. This explains why they also get higher expected utility.

Moreover, the increase in aggressiveness of the bid behaviour of the entrants explains why the expected revenue of the auctioneer can increase. This is not a general result since the incumbent tends to bid relatively less aggressively in an open ascending auction.

The optimality in terms of revenue of the open ascending auction follows from the fact that this auction induces the optimal allocation rule between the incumbent and the entrants under some specific assumptions. This contrasts with the sealed bid auction that under the same assumptions tends to allocate the good to the incumbent too often. In Section 3.6 we provide an intuition why the open ascending auction is the maximum revenue auction among all the auctions that always sell the good when the entrants are completely uninformed.

This paper also reinforces the point, originally stated in Chapter 2 of this Ph.D. dissertation, that less informed bidders can do surprisingly well with respect to more informed bidders in multiunit auctions. In fact, as we show with our results, these arguments are stronger in the open ascending auctions than in the sealed bid auction.

We believe that our model captures some of the aspects involved in the design of the auctions of licenses for third generation mobile telephones that have taken place in Europe. Typically, in these auctions there were some incumbents that were already operating in related markets, like mobile telephones of second generation or fixed line telephones. Hence, these incumbents probably had a more accurate information about the value of a licence than outsiders. Moreover, in line with the traditional worries of competition policy, one of the concerns in the auction design was the number of entrants.

Nevertheless, the pure common value assumption can seem excessive in the above example and probably in many other interesting problems. We believe that this case, as some others, can be modelled approximately using the pure common value assumption. Other cases are treasure bill auctions and oil tract leases. It also happens in these two examples that it is commonly known that the common value of the object for sale is better known by some bidders than others. Moreover, we also see our approach as a way of providing an stylised analysis of the effect of differences in quality of information without mixing with other considerations.

Note that the sense in which our model can represent situations close to common value is very subtle. Klemperer (1998) has shown that in the case of common value models in which one of the bidders has a small private value advantage, an open ascending auction can give very low expected revenue to the auctioneer and very low expected utility to the disadvantaged bidders. Thus, if we apply our perspective that the entrants are usually bidders with some kind of disadvantage, this result says exactly the opposite to what our model says.

Nonetheless, there is one important difference between Klemperer's (1998) paper and our paper. Klemperer (1998) assumes that there is only one unit for sale and one advantaged bidder. It is actually *only* under the assumption that the number of units for sale equals the number of advantaged bidders when small private value advantages can have the dramatic effects that we mention above. For instance, Bulow and Klemperer (1999) show that if there is more than one unit for sale and only one bidder with a small private value advantage the effect on the auction outcome is also small. We assume that the number of units for sale is greater than the number of advantaged bidders, in our case only one. We thus believe that the introduction of small private value perturbations that benefit the incumbent should have a corresponding small effect on the outcome of our models. It is for this reason that we believe that our model is a good proxy model of almost common value situations in which the number of units for sale is greater than the number of incumbents, or in general advantaged bidders.

In our analysis we shall only give limited uniqueness results. We shall show that the equilibrium that we give for the sealed bid auction is unique in a broad class of equilibria. These are the equilibrium in continuous and strictly increasing strategies and in which all the entrants play the same strategy, i.e. symmetric. However, we shall not provide any uniqueness proof for the open ascending auction.

The consequence of our lack of general uniqueness results is the following. There could be other equilibria of the open ascending auction in which this auction does very badly in the two dimensions that we consider, entrants' expected utility and auctioneer's expected revenue. The problem of multiplicity of equilibria is not particular to our model. The same problem appears in other papers that study open ascending auctions, even under the assumption that bidders are ex ante symmetric, for instance Milgrom and Weber (1982). We solve the problem of multiplicity of equilibria following the standard approach taken in the literature.

The fact that we restrict attention to symmetric equilibrium of the sealed bid auction has some consequences. We model two kind of auctions, a sealed bid auction that generalises the second price auction, and an open ascending auction that generalises the English auction. In both types of auctions there are always asymmetric equilibria in which a number of bidders equal to the number of units for sale bid very high and all the other bidders bid very low, see for instance Bikhchandani and Riley (1991). This kind of equilibria exist independently of the existence of asymmetries among the bidders.[5] Using this kind of equilibria we can provide examples in which both auction formats produce the same expected price and the same expected revenue for the entrants. However, these asymmetric equilibria require a lot of co-ordination among the bidders.

The most closely related papers are those that we have already explained by Klemperer (1998), and Bulow and Klemperer (1999). These two papers study situations in which there are some kind of asymmetries in almost common value models. Jewitt (forthcoming) has also worked on asymmetric common value auctions under the assumption that there are only two bidders and one unit for sale. The difference with our approach is that we shall focus on multiunit sales with many bidders.

Our comparison between the sealed bid auction that we propose and the open ascending auction makes less sense when we assume the private model. The reason is that in both auction formats it is weakly dominant for the bidders to bid the true value. Under the private value assumption, Maskin and Riley (2000) provide a different comparison. They show that under some assumptions a weak bidder can prefer a first price sealed bid auction to a second price sealed bid auction.

Note that our revenue ranking is similar to that provided by Milgrom and Weber (1982). They prove that an open ascending auction, the English auction, gives higher expected revenue than a sealed bid auction, the second price auction (and than other standard auctions like the first price sealed bid auction, or the Dutch auction). In their result, the fact that the open ascending auction has a smaller winner's curse than the sealed bid auction also plays an important role. Nevertheless, the underlying intuition is different. In our model the key is the different effect of the winner's curse in bidders with different quality of information, whereas in Milgrom and Weber's model

[5] These equilibria exist even in the private value case. However, in the private value case they are in weakly dominated strategies.

the important thing is that the private information of a bidder gives information about the private information of the other bidders. In fact, in order to avoid the interference of the effect pointed out by Milgrom and Weber (1982), we assume that the bidders' signals are statistically independent. Under this assumption Milgrom and Weber's model shows that these different auction formats are revenue equivalent.

Our model also gives a different prediction than Milgrom and Weber's model. Entrants get higher expected utility in the open ascending auction than in the sealed bid auction conditional and unconditional on the type. Whereas in Milgrom and Weber's model, it is easy to show that all the bidders get smaller expected utility unconditional on the type in the open ascending auction than in the sealed bid auction.

We shall start this paper with a description of the basic set-up in Section 3.2. Then, we shall solve first the sealed bid auction, Section 3.3, and then the open ascending auction in Section 3.4. We compare these two auction set-ups according to the entrant's expected utility in Section 3.5. In Section 3.6 we analyse the auctioneer's expected revenue and compute the optimal auction. Section 7 concludes.

3.2 The Model

We assume that one auctioneer wants to sell k ($k > 1$) units of a homogeneous good to a pool of $n+1$ ($n+1 > k$) risk neutral bidders. We restrict attention to the pure common value case. According to this paradigm, the value of the good, that we call v, is common to all bidders, but the bidders have only imperfect information about this value. In order to reflect this uncertainty we assume that v is a weighted average of signals. More precisely, we assume that $v = \frac{s+\lambda\sum_{i=1}^{n} s_i^E}{1+\lambda n}$, where $\lambda \in (0,1)$. In this sense, we say that s is more informative of v than each of the signals s_i^E. Note that if λ equals 1 each of the signals s_i^E is as informative about v as s is, and if λ equals zero, each signal s_i^E is completely uninformative about v. We assume that s and all the signals s_i^E are identically and independently distributed according to a distribution function F with density f and support $[\underline{s}, \overline{s}]$. By assuming independence of the signals we avoid that our result mixes with the revenue results provided by Milgrom and Weber (1982).

We assume that there is one bidder,[6] that we call the *incumbent*, that receives privately the signal s, whereas each of the other bidders, that we call the *entrants*, receives a different signal s_i^E.

We also assume that the number of units for sale is strictly more than one. Our approach does not carry over naturally to the single unit case. The main difficulty comes from the fact that the open ascending auction is degenerate in the following sense. There exists a continuum of equilibria that are robust to the equilibrium selection that we and other aucthors, eg Milgrom and Weber (1982), have used.[7]

3.3 The Sealed Bid Auction

In this section we assume that the incumbent and n entrants compete in a generalised second price sealed bid auction. In this auction set-up, all bidders submit simultaneously one bid each. The k bidders that have submitted the highest bids get one unit each at the price of the $k+1$ highest bid, i.e. the highest losing bid. If the k-th highest bid and the $k+1$-th highest bid have the same value b, then the price in the auction is b, all bidders who make a bid strictly higher than b get one unit each with probability 1, and the remaining winners are randomly selected among all bidders who have made bid b, whereby all such bidders have the same probability of being selected. For the sake of simplicity we do not allow neither for reserve prices nor for entry fees.[8]

We shall assume that the number of entrants plus the incumbent is strictly greater than the number of units for sale plus one, i.e. $n > k$. This assumption assures that there is more than one loser. Otherwise, as we explain in Section 3.4, the sealed bid

[6] Assuming that there is more than one single incumbent makes the model much more complex, and it is not possible to give the clear-cut analysis that we give with only one incumbent.

[7] The nature of these multiplicity of equilibria is not a peculiar issue of the asymmetric assumption on the bidders types. Milgrom (1981) shows in the symmetric case with one unit for sale and two bidders that for whatever surjective, increasing function h, with domain $[\underline{s}, \overline{s}]$ there exists an equilibrium of the open ascending auction in which one of the bidders conditional on having a type s bids the same bid as the other bidder with type $h(s)$. Nevertheless, in the symmetric bidders game an obvious equilibrium selection criteria is the symmetric equilibrium. But, this criterion does not help in asymmetric situations.

[8] Note that if there were an entry fee or a reserve price the effect on bidder's entry would be similar to the assumption that entry is costly. If we assume that the incumbent always enters and that the entry decision is taken before nature communicates the bidder her private type, the reasoning suggested in footnote 1 for entry costs should apply.

auction and the open ascending auction are strategically equivalent.

The strategies on this game are going to be bid functions that map types into bids. We shall restrict to symmetric Nash equilibria in the sense that we shall assume that all the entrants use the same bid function in equilibrium. Hence, the equilibrium behaviour will be characterised by two bid functions $b_I^s, b_E^s : [\underline{s}, \overline{s}] \rightarrow \mathbb{R}^+$.

We shall further restrict to equilibria in continuous and strictly increasing strategies. Nevertheless, at this stage, it is not clear that there exists such equilibria. We show below that the next assumption will assure that it is the case.

Assumption 3.1. *$\sigma - E[s| s \leq \sigma]$ and $E[s| s \geq \sigma] - \sigma$ are respectively strictly increasing and strictly decreasing in σ.*

Assumption 3.1 is satisfied by many distribution functions. If F has a continuously differentiable density, Lemma 3 in Bikhchandani and Riley (1991) shows that a sufficient condition for the first part of the assumption is that F is strictly log-concave. Similarly, if F has a continuously differentiable density, a sufficient condition for the second part is that $1-F$ is strictly log-concave, this is that F has an increasing hazard rate.

In order to analyse the game we propose some strategies for the bidders. Basically, these strategies are such that each bidder's bid conditional on her type equals the expected value of the good conditional on: the bidders' type, and the information that the bidder infers in equilibrium from the event that the k-th highest bid of the other bidders equals her own bid. Note that this condition is a direct generalisation of the condition that is satisfied by the bid function of the symmetric equilibrium used by Milgrom and Weber (1982) with ex ante symmetric bidders. Note that if a bidder follows a strategy that satisfies this condition, she is actually indifferent between winning and losing in the event that the price equals her bid and she wins.

Our model has one difficulty that does not appear when all the bidders are symmetric and we look for a symmetric equilibrium. In order to state the indifference conditions that we suggested in the former paragraph, we shall need to determine a function that relates the type of the incumbent that submits the same bid as a given type of the entrants. We solve this problem proposing a function and later we show that this function is consistent with the strategies that it induces.

Let $\phi^s : [\underline{s}, \overline{s}] \to [\underline{s}, \overline{s}]$ be a function such that the following two conditional expected values are equal. The first one is the expected value of the good conditional on the k-th highest signal of the entrants' signals equals s and the incumbent's signal equals $\phi^s(s^E)$. The second one is the expected value of the good conditional on the event that either the k-th and the $k+1$-th highest signals of the entrants' signals equal s and the incumbent's signal is below $\phi^s(s^E)$, or $k-1$-th and the k-th highest signals of the entrants' signals equal s^E and the incumbent's signal is above $\phi^s(s^E)$. We formalise this condition below. Note that we drop the dependence of ϕ^s on s^E to simplify the notation.[9]

$$\begin{aligned} E[v|s = \phi^s, s^E_{(k)} = s^E] = \mathbb{P}(\phi^s, s^E)E[v|s \geq \phi^s, s^E_{(k-1)} = s^E_{(k)} = s^E] \\ + (1 - \mathbb{P}(\phi^s, s^E))E[v|s < \phi^s, s^E_{(k)} = s^E_{(k+1)} = s^E], \quad (3.1) \end{aligned}$$

where $\mathbb{P}(\phi^s, s^E)$ is the probability that the bid of the incumbent is higher than ϕ^s given that either the k-th and the $k+1$-th highest signals of the entrants' signals equal s and the incumbent's signal is below ϕ^s, or $k-1$-th and the k-th highest signals of the entrants' signals equal s^E and the incumbent's signal is above ϕ^s. This is:

$$\mathbb{P}(\phi^s, s^E) = \frac{\binom{n-2}{k-2}[1-F(\phi^s)][1-F(s^E)]^{k-2}F(s^E)^{n-k}}{\binom{n-2}{k-2}[1-F(\phi^s)][1-F(s^E)]^{k-2}F(s^E)^{n-k} + \binom{n-2}{k-1}F(\phi^s)[1-F(s^E)]^{k-1}F(s^E)^{n-k-1}}.$$

We next check that the function ϕ^s as defined in equation 3.1 exists, is unique and verifies some properties that we shall need later.

Lemma 3.1. *Equation (3.1) defines a unique function ϕ^s. This function has domain $[\underline{s}, \overline{s}]$ and range $[\underline{s}, \overline{s}]$ and it is continuous and strictly increasing in s^E.*

Proof. Lemma 2.1 in Chapter 2 of this Ph.D. dissertation proves that the function ϕ^s implicitly defined in equation (3.1) is unique, continuous and strictly increasing. To

[9] Here and in the following $E[.|.]$ denotes the expected value of the random variable in front of the vertical line, conditional on the event which is defined after the vertical line, and $s^E_{(k)}$ denotes a random variable that equals the k-th highest signal of the entrants' signals.

prove that the domain and the range of ϕ^s equal $[\underline{s}, \overline{s}]$ it suffices to show that $\phi^s(\underline{s}) = \underline{s}$, and $\phi^s(\overline{s}) = \overline{s}$ due to the monotonicity of the function. Simple computations on equation (3.1) show that these two conditions hold. ■

The next step is to propose some bid functions. We shall show later that these functions characterise the unique symmetric equilibrium of the game.

Consider then the following bid functions,

$$b_I^s(\phi) = E[v|s = \phi, s_{(k)}^E = (\phi^s)^{-1}(\phi)], \tag{3.2}$$

and,

$$b_E^s(s^E) = E[v|s = \phi^s(s^E), s_{(k)}^E = s^E]. \tag{3.3}$$

Note that the properties of ϕ^s stated in Lemma 3.1 assure that both the incumbent's strategy and the entrants' strategy are well defined by these conditions for all the bidders' types, and that they are continuous and strictly increasing. From the definition of these functions is trivial that $b_I^s(\phi^s(s^E)) = b_E^s(s^E)$. Hence, ϕ^s is consistent with the bid functions that it induces.

Trivially, the incumbent's bid is the expected value of the good conditional on winning and tying with the k-th highest bid of the other bidders. The entrants' bid is the expected value of the good conditional on winning and tying with the k-th highest bid of the other bidders, when this bid is submitted by the incumbent. The definition of ϕ^s in equation (3.1) assures that this expected value is also equal to the expected value of the good conditional on winning and tying with the k-th highest bid of the other bidders, when this bid is submitted by another entrant. Hence, the entrants' bid is the expected value of the good conditional on winning and tying with the k-th highest bid of the other bidders

Next proposition shows that these strategies are the unique symmetric equilibrium strategies.

Proposition 3.1. *The bidding strategies defined in equation (3.2) and equation (3.3), plus the definition of ϕ^s in equation (3.1) characterise the unique symmetric equilibrium of the sealed bid auction in continuous and strictly increasing strategies.*

Proof. See Proposition 2.7 in Chapter 2 of this Ph.D. dissertation.[10] ■

Finally, we provide a restatement of equation (3.1) that we shall use later.

$$(k-1)(1-F(\phi^s))F(s^E)\left[E\left[s|\,s\geq\phi^s\right]-\phi^s-\lambda\left(E\left[s|\,s\geq s^E\right]-s^E\right)\right]=$$
$$(n-k)F(\phi^s)(1-F(s^E))\left[\phi^s-E\left[s|\,s\leq\phi^s\right]-\lambda\left(s^E-E\left[s|\,s\leq s^E\right]\right)\right]. \quad (3.4)$$

3.4 The Open Ascending Auction

In this section we assume a different selling mechanism, the open ascending auction. More precisely, we assume that the auction procedure is as follows, at every moment of time there are two types of bidders: active bidders and inactive bidders. Bidders are active until they manifest that they want to become inactive. Once a bidder has decided to become inactive her decision is irreversible. The identity of the active bidders is publicly observable during the auction. During the auction the price is publicly observable and increases continuously from zero. At each moment in time bidders can decide to become inactive. The price stops increasing whenever the number of active bidders is less than or equal to the number of units for sale. In this case, each of the active bidders gets one unit. The rest of the units are randomly allocated (with equal probability) among the bidders that quit at the last price. The price paid by all the winners is the last price at which bidders quit.[11] As in the sealed bid auction we shall assume that there is neither an entry fee nor a reserve price.

[10] In Chapter 2 uniqueness is proved under an additional assumption. This additional assumption is that all the bidders have a strictly positive unconditional probability of winning the auction. However, this assumption is not needed to prove uniqueness in the case $1 < k < n$.

[11] Another variant of the Japanese auction assumes additionally that the price is stopped every time a bidder quits. Then the auctioneer offers the remaining bidders the possibility of quitting at that price. When no more bidders want to quit then the auctioneer starts to increase the price continuously from where the price was last stopped. This set-up is analysed in a different problem by Bulow and Klemperer (1999). The equilibrium that we present in this paper is robust to such modification of the auction procedure.

Note that this auction procedure releases a lot of information. Namely, at every moment of time, bidders know the identity of all the active bidders, and the quitting prices and identity of all the inactive bidders. There are some real life instances where this is actually the case, for instance in the third generation of mobile telephones auction in the UK. In cases in which the auction procedure does not reveal such detailed information, we would expect the solution of the game to be between the results in the sealed bid auction and the results in the open ascending auction.

Remember that we did not provide in Section 3.3 any results for the sealed bid auction when $k = n$. The next remark shows that our results for the open ascending auction are sufficient.

Remark 3.1. *If the number of entrants in the sealed bid auction equals the number of units for sale, this is $k = n$, the sealed bid auction and the open ascending auction are strategically equivalent.*

To see why, note that when $k = n$ the only release of information in the open ascending auction happens when the game finishes. This is because the number of bidders equals the number of units for sale plus one. Hence, the space of bidders' strategies is the same in both auction formats. The remark follows since the winners and the price is determined in the same form in both auction formats.

We shall also use Assumption 3.1. This assumption plays a similar role to that in the former section.[12]

A strategy for a bidder specifies a price level at which this bidder will quit if the auction reaches or has reached that price as a function of bidder's type and the prices at which each of the inactive bidders has left the auction. We shall provide two strategies, one for the incumbent and another one for the entrants, and later we shall show that these two strategies constitute a perfect Bayesian-Nash equilibrium of the game.

Basically, our proposed strategies are such that at each informational set, each bidder stays in the auction until the price reaches the expected value of the good conditional on: (i) her private information; (ii) the information that she can have

[12] In this case, we only need to assume that $E[s|s \geq \sigma] - \sigma$ is strictly increasing.

inferred from the price level at which the inactive bidders have quit; and (iii), the event that the final price in the auction equals her proposed bid and she wins. This is the basic idea behind the strategies proposed by Milgrom and Weber (1982). Note that if a bidder follows a strategy that satisfies this condition, she is indifferent between winning and losing in the event that the price equals her bid and she wins.

As in the sealed bid auction, the fact that we have two different bid functions, one for the entrants and another one for the incumbent, makes the analysis more complex than in Milgrom and Weber (1982). To define the bid functions using the indifference condition that we suggest in the former paragraph we shall need to determine a function that relates the types of the incumbent and the entrants that submit the same bid at each information set. As in the previous section, we shall first propose such a function, then we shall use it to determine the bid strategies, and later we shall show that the function we propose is consistent with the strategies that it induces.

Let $\phi^o : [\underline{s}, \overline{s}] \to [\underline{s}, \overline{s}]$ be a function such that the following conditional expected values are equal. The first one is the expected value of the good conditional on the $k-1$-th and the k-th highest signal of the entrants' signals are equal to s^E and the incumbent's signal is greater than or equal to $\phi^o(s^E)$. The second one is the expected value of the good conditional on the k-th highest signal of the entrants' signals is equal to s^E and the incumbent's signal equals $\phi^o(s^E)$. We formalise this condition below. Again we drop the dependence of ϕ^o on s^E to simplify the notation.

$$E[v|s \geq \phi^o, s^E_{(k-1)} = s^E_{(k)} = s^E] = E[v|s = \phi^o, s^E_{(k)} = s^E]. \tag{3.5}$$

Note the following two remarks which we shall show later are related. The first one is that we have defined ϕ^o in such a way that it is independent of the information set in which the game is. We shall see later that in the equilibrium that we propose, the relation between the types of the incumbent and the entrants that leave the auction at the same price is independent of the information set. The second one is a consequence of our assumptions that the signals are independent and the value of the good is convex combination of the bidders' signals. If the left hand side expected value equals the right hand side expected value, then these two expected values are also equal conditional on some additional events that depend on entrants' signals that are weakly below s^E.

We simplify the equation (3.5) to make the proof of the following lemma easier.

$$\lambda(E[s|s \geq s^E] - s^E) = E[s|s \geq \phi^o(s^E)] - \phi^o(s^E). \tag{3.6}$$

Lemma 3.2. *The function ϕ^o is uniquely defined in the domain $[\underline{s}, \overline{s}]$ by equation (3.6). This function is continuous and strictly increasing. Moreover, $\phi^o(\underline{s}) > \underline{s}$ and $\phi^o(\overline{s}) = \overline{s}$.*

Proof. Define $\eta(s) = E\left[\tilde{s} \middle| \tilde{s} \geq s\right] - s$, where $\tilde{s} \sim F$. Assumption 3.1 assures η is strictly decreasing for all $s \in [\underline{s}, \overline{s}]$, moreover η is trivially continuous, and $\eta(\underline{s}) = E[s]$ and $\eta(\overline{s}) = 0$. Since $\phi^o(s^E) = \eta^{-1}(\lambda\eta(s^E))$, ϕ^o is trivially well defined for all $s^E \in [\underline{s}, \overline{s}]$, and it must be continuous and strictly increasing. Finally, $\phi^o(\underline{s}) = \eta^{-1}(\lambda\eta(\underline{s})) = \eta^{-1}(\lambda E[s]) > \underline{s}$ and $\phi^o(\overline{s}) = \eta^{-1}(\lambda\eta(\overline{s})) = \eta^{-1}(0) = \overline{s}$. ■

We next construct bid functions that specify at which price the bidder quits for the different informational sets of the game using the function ϕ^o that we have defined above. We shall proceed recursively.

Consider first the information set in which no bidder has left the auction yet. Then, let the incumbent's bid function be equal for types $\phi \geq \phi^o(\underline{s})$ to:

$$b^o_{I,0}(\phi) \equiv E[v|s = \phi, s^E_{(k)} = \dots = s^E_{(n)} = (\phi^o)^{-1}(\phi)], \tag{3.7}$$

and if $\phi < \phi^o(\underline{s})$, to:

$$b^o_{I,0}(\phi) \equiv E[v|s = \phi, s^E_{(k)} = \dots = s^E_{(n)} = \underline{s}]. \tag{3.8}$$

Let also the entrants' bid function be equal to:

$$b^o_{E,0}(s^E) \equiv E[v|s = \phi^o(s^E), s^E_{(k)} = \dots = s^E_{(n)} = s^E], \tag{3.9}$$

for all s^E.

We define the bid functions in other information sets recursively. Consider that the incumbent and m entrants are still active, and $n - m$ entrants have already left the auction at prices $p_1 \leq ... \leq p_{n-m}$. Then, the incumbent's bid function equals for types $\phi \geq \phi^o(\underline{s})$,

$$b^o_{I,n-m}(\phi|p_1,...,p_{n-m}) \equiv E[v|s=\phi, s^E_{(k)} = ... = s^E_{(m)} = (\phi^o)^{-1}(\phi), \mathcal{H}(p_1,...,p_{n-m})], \tag{3.10}$$

and if $\phi < \phi^o(\underline{s})$,

$$b^o_{I,n-m}(\phi|p_1,...,p_{n-m}) \equiv E[v|s=\phi, s^E_{(k)} = ... = s^E_{(m)} = \underline{s}, \mathcal{H}(p_1,...,p_{n-m})], \tag{3.11}$$

and, the entrants' bid function, for all s^E,

$$b^o_{E,n-m}(s^E|p_1,...,p_{n-m}) \equiv E[v|s=\phi^o(s^E), s^E_{(k)} = ... = s^E_{(m)} = s^E, \mathcal{H}(p_1,...,p_{n-m})], \tag{3.12}$$

where $\mathcal{H}(p_1,...,p_{n-m}) \equiv \{b^o_{E,0}(s^E_{(n)}) = p_1, ..., b^o_{E,n-m-1}(s^E_{(m+1)}|p_1,...,p_{n-m-1}) = p_{n-m}\}$. And in case, for some l, $p_l \notin \text{range}(b^o_{E,l-1}(s^E_{(n-l+1)}|p_1,...,p_{n-l+1}))$, we just eliminate the corresponding equation from $\mathcal{H}$.

Note that $\mathcal{H}$ is a function that formalises the beliefs about the private information of the inactive bidders. These beliefs are consistent with the bid functions along the equilibrium path. For out of equilibrium paths, we have determined in an arbitrary way the bidders' beliefs. This is reflected in the assumption that if $p_l \notin \text{range}(b^o_{E,l-1}(s^E_{(n-l+1)}| p_1,...,p_{n-l+1}))$, we just eliminate the corresponding equation from $\mathcal{H}$. These beliefs are, however, immaterial for the equilibrium. Had we defined these beliefs in another arbitrary way, we would have not upset the equilibrium. The reason is that once the bidder has left the auction, she gets zero expected utility independent of the final outcome of the auction.[13]

And finally, we consider an iterative definition of the bid function for information sets in which the incumbent has left the auction at a price p^I at a information set in which there were m' entrants active and $n-m'$ entrants that had left the auction at

[13] Had we assumed that the auctioneer stops the price every time that a bidder leaves the auction, we should be more careful with the out of equilibrium beliefs. The reason is that in this case, it is possible that other bidders leave the auction exactly at the same price as the deviator. Then, it can be the case that the deviator wins the auction with positive probability.

prices $p_1 \leq ... \leq p_{n-m'}$. Then, the entrants' bid function equals,

$$\bar{b}^o_{E,n-m}(s^E|p_1,...,p_{n-m};p^I) \equiv$$
$$E[v|s^E_{(k+1)} = ... = s^E_{(m)} = s^E, b^o_{I,n-m'}(s|p_1,...,p_{n-m'}) = p^I, \mathcal{H}(p_1,...,p_{n-m};p^I)], \quad (3.13)$$

where m is the number of entrants still active, and $p_1 \leq ... \leq p_{n-m}$ is the price at which the $n-m$ inactive entrants have left the auction. We have also defined $\mathcal{H}$ in these information sets extending in the same way the definition we give above. We leave the reader the details of this extension.

Note that the properties of ϕ^o assure that both the incumbent's strategy and the entrants' strategy are well defined, and that they are continuous and strictly increasing in the type. These functions trivially satisfy that an entrant with type s^E leaves the auction at the same price level as the incumbent with type $\phi^o(s^E)$ in all information sets. Note also that if all the bidders follow the proposed strategies, the transition between information sets is smoothed in the following sense: The prices at which the types of the bidders that are still active leave the auction according to the proposed strategies are above the price at which the transition between information sets happens. Hence, there is no "rush" after one bidder leaves the auction.

We next interpret the bidders' strategy under the assumption that all the bidders follow the proposed strategies. Suppose for instance that no bidder has left the auction yet. Consider an incumbent with type ϕ and suppose that the price has reached the level $b^o_{I,0}(\phi)$. If $n-k+1$ entrants were to quit at this price level, then the incumbent could infer from their behaviour that $x^E_{(k)} = x^E_{(k+1)} = ... = x^E_{(n)} = (\phi^o)^{-1}(\phi)$. In that case, the incumbent would estimate that the value of the good equals $E[v|s = \phi, x^E_{(k)} = x^E_{(k+1)} = ... = x^E_{(n)} = (\phi^o)^{-1}(\phi)]$. Note that this is the actual value of the incumbent's bid with a type ϕ. Hence, the incumbent stays active until a price level at which she is indifferent between winning and losing. We can apply a similar argument to information sets in which some of the bidders have already left the auction. In that case, the only modification is that we consider the information that is inferred from the price at which the inactive bidder left the auction. We can give a similar explanation for the entrants' strategy. The only difference appears in information sets in which the incumbent is still active. The condition that we have given only assures the indifference of the entrant between winning and losing when the incumbent is one of the bidders that leave the auction. The definition of ϕ^o in equation (3.6) guaranties that the entrant is also indifferent between winning and losing when the incumbent is not within those bidders that leave the auction.

The incumbent's bid function for types below $\phi^o(\underline{s})$ satisfies an arbitrary condition. If all the bidders follow the proposed strategies, these bids play no role except when no other bidder has left the auction and the number of entrants equals the number of units for sale. The reason is that the bid of these types is never the price in the auction.

When the number of entrants equals the number of units for sale ($n = k$) and all the bidders follow the equilibrium strategies, the bid of types of the incumbent below $\phi^o(\underline{s})$ can determine the final price in the auction. We have determined this bid in such a way that if the price is fixed by these bids, then an entrant with type $\underline{s}$ gets zero expected utility. If instead of this bid function we substitute it by another bid function that is below the one we propose, we can show that the strategies we propose with this change still constitute an equilibrium. The only difference in the auction outcome is that the minimum type of the entrants gets strictly positive expected utility, and all the other entrants' types get higher expected utility. Note that the case $n = k$ is the case in which both auction formats are strategically equivalent. Hence, it is irrelevant for our auction comparison the definition of the bid function for these types as far as we are consistent in the criteria we use for each auction format.

We conclude this section showing that the strategies we have proposed are equilibrium strategies of the game.

Proposition 3.2. *The set of incumbent's bid functions $\{b^o_{I,l}\}_{l=0}^{n-k}$ and entrants' bid functions $\{b^o_{E,l}\}_{l=0}^{n-k}$ and $\{\bar{b}^o_{E,l}\}_{l=0}^{n-k-1}$ are a perfect Bayesian-Nash equilibrium of the game.*

Proof. We first show that if all the bidders follow the proposed strategies, the bidders get non negative expected utility conditional on the type and conditional on winning at a price, for all the prices at which they can win in the equilibrium path. This will assure that bidders do not have incentives to deviate by leaving the auction before what their proposed strategy indicates.

Consider, for instance, a type s of the incumbent such that $s \geq \phi^o(\underline{s})$, otherwise the proof is trivial. Suppose that the incumbent with type s wins and that the last entrant that leaves the auction, this is the one that fixes the price, has a type s^E. Since the bid functions are strictly increasing, there is a type $\phi^o(s^E) \leq s$ of the incumbent that would leave the auction at the same price as an entrant with type s^E in this information set. Such type by definition of the strategies is indifferent between winning or losing in that information set and at that price, i.e. gets zero expected utility. Since s is better news than $\phi^o(s^E)$, the incumbent with type s gets non negative expected utility. The proof is similar for the entrants. The only difference is when $k = n$ and the price is fixed by an incumbent with type $s < \phi^o(\underline{s})$. We have already explained why in this case the entrants get non negative expected utility.

We next show that a bidder does not have incentives to deviate staying longer than her proposed strategy indicates if all the other bidders follow the proposed strategies. Consider, for instance, that a type s of the incumbent stays in the auction after the price has reached her quitting price and she wins at a price fixed by an entrant with type s^E. Since the bid functions are strictly increasing, there is a type $\phi^o(s^E) \geq s$ of the incumbent that would leave the auction at the same price as an entrant with type s^E in the same information set. Such type, by definition of the strategies, is indifferent between winning or losing in that information set and at that price, i.e. gets zero expected utility. Since s is worse news than $\phi^o(s^E)$, the incumbent with type s gets non positive expected utility. The proof is identical for the entrants. ■

The equilibrium strategies that we propose have two nice properties. The first one is that when λ tends to one, i.e. when our model tends to a model with symmetric bidders, the strategies converge to the direct generalisation of Milgrom and Weber's (1982) equilibrium strategies for the English auction to multiunit, unidemand auctions. The second nice property is that when λ tends to zero the incumbent's bid function converges to the unique weakly dominant strategy for the limit game defined when $\lambda = 0$. Note that in this limit game, the incumbent is perfectly informed about the value of the good. Hence, using the same logic that in the private value case, we can show that it is weakly dominant for the incumbent to bid the true value of the good. Bidding above only allows winning additionally when the price is above the value of the good and bidding below does not reduced the price conditional on winning but it means losing at some prices at which it is profitable to win.

We shall not provide uniqueness results in this section. The reason is that to provide uniqueness[14] we should need to introduce additional restrictions in the equilibrium set with no clear intuitive meaning. Note that as Bikhchandani and Riley (1991) suggest there exists multiplicity of symmetric equilibrium in continuous and strictly increasing bid functions, even when all the bidders are ex ante symmetric.[15]

3.5 Entrants' Expected Utility

In order to compare the open ascending auction and the sealed bid auction we follow an indirect approach. We first appeal to Myerson's (1981) results to translate the problem of comparing expected utilities across incentive compatible mechanisms to sufficient conditions on the probabilities of winning the auction. Later, we show that these conditions allow for a straightforward comparison between the open ascending auction and the sealed bid auction.

Lemma 3.3. *Consider an incentive compatible mechanism where $Q^I(s)$ and $Q_i^E(s^E)$ are respectively the probability of winning one unit for the incumbent with type s and for an entrant i with type s^E. Then, $Q^I(s)$ and $Q_i^E(s^E)$ must be weakly increasing and:*

- *the conditional expected utility of the incumbent equals:*

$$U^I(s) \equiv \frac{1}{1+\lambda n} \int_{\underline{s}}^{s} Q^I(\tilde{s}) d\tilde{s} + U^I(\underline{s}), \tag{3.14}$$

where $U^I(\underline{s})$ is the expected utility of the incumbent with type $\underline{s}$.

[14] For instance, we can construct an equilibrium in which the incumbent quits at very low prices according to a strictly monotone strategy. The entrants stay active till the incumbent leaves the auction. Once the incumbent has left the auction, the entrants play a symmetric strategy similar to that of the symmetric equilibrium of the game with symmetric bidders. The equilibrium is completed fixing the out of equilibrium beliefs such that when the incumbent deviates it is common knowledge among the entrants that the incumbent's type is $\underline{s}$.

[15] In the symmetric case this multiplicity is less severe because it does not change neither the final allocation nor the final price in the auction.

- *the conditional expected utility of the entrants equals:*

$$U_i^E(s^E) \equiv \frac{\lambda}{1+\lambda n}\int_{\underline{s}}^{s^E} Q_i^E(\tilde{s})d\tilde{s} + U_i^E(\underline{s}), \tag{3.15}$$

where $U_i^E(\underline{s})$ is the expected utility of entrant i with type $\underline{s}$.

Proof. Direct using the arguments in the proof of Lemma 2 in Myerson (1981). ∎

Trivially the expected utility of the type $\underline{s}$ in both auction formats, and for both the incumbent and the entrants, is zero. Hence, in order to show that the expected utility of the entrant is higher in an open ascending auction than in a sealed bid auction, we only need to show that the probability that a given type of the entrant wins the auction is higher in the open ascending auction than in the sealed bid auction, for all types. Similarly, if we want to show that the expected utility of the incumbent is lower in the open ascending auction than in the sealed bid auction, we only need to show that the probability that a given type of the incumbent wins is lower in the open ascending auction than in the sealed bid auction, for all types.

Next, we study the probability of winning for the incumbent and the entrants in both auction formats.

Proposition 3.3. *Suppose that $1 < k < n$ and suppose that the bidders play the equilibrium given in Proposition 3.2 in the open ascending auction and the unique symmetric equilibrium in the sealed bid auction (see Proposition 3.1), then:*

A given type s^E ($s^E \in (\underline{s}, \overline{s})$) of an entrant has strictly higher probability of winning in an open ascending auction than in a sealed bid auction. Conversely, a given type s ($s \in (\underline{s}, \overline{s})$) of the incumbent has strictly lower probability of winning in a sealed bid auction than in an open ascending auction.

Proof. One feature of the symmetric equilibria is that all the entrants bid according to a common bid function. This means that a given type of the entrant always beats those entrants that have a lower type than her type. Hence, the only way an entrant could win more often in the open ascending auction than in the sealed bid auction is by beating the incumbent more often. Due to the monotonic properties of the bid functions, this follows if the type of the incumbent that submits the same bid as a given type of the entrants is higher in the open ascending auction than in the sealed bid auction. Hence we need to show that $\phi^o(s^E) > \phi^s(s^E)$ for all $s^E \in (\underline{s}, \overline{s})$.

We start by proving that $E[s|s \geq \phi^s(s^E)] - \phi^s(s^E) > \lambda\left(E[s|s \geq s^E] - s^E\right)$. Suppose the contrary. Then, equation (3.4) implies that $\phi^s(s^E) - E[s|s \leq \phi^s(s^E)] \leq \lambda\left(s^E - E[s|s \leq s^E]\right)$. But according to the monotonic properties stated in Assumption 3.1 and since $\lambda < 1$, the first inequality implies that $\phi^s(s^E) > s^E$, whereas the second inequality implies that $\phi^s(s^E) < s^E$, that it is a contradiction.

The above inequality implies together with equation (3.6) that $E[s|s \geq \phi^s(s^E)] - \phi^s(s^E) > E[s|s \geq \phi^o(s^E)] - \phi^o(s^E)$. Hence, Assumption 3.1 implies that $\phi^s(s^E) < \phi^o(s^E)$, that completes the first claim of the proposition.

The second claim ca be proved similarly. ■

As we claim in the introduction, the key to understand this result is to study the incentives to increase one's bid in the different auction formats. Consider first the sealed bid auction. A bidder will want to raise his bid by a small amount, say from b to $b + \epsilon$, if the expected value of a unit, conditional on its price being $p \in (b, b + \epsilon)$, is larger than p. The price is p if and only if the k-th highest bid of the other bidders is p. This event can be expressed as the intersection of two events, one of which implies good news, and the other one that implies bad news. The good news is that at least k of the other bidders have been willing to bid p or more. If these bidders had any private information at all, it must have been favourable. This is good news. This effect has been called the *loser's curse* as a bidder who neglects this effect will regret losing. The bad news is that at least $N - k$ of the other bidders (where N denotes the total number of bidders, in our case $n + 1$) have bid p or less, and hence, if they had any private information at all, this must have been unfavourable. This effect has been called the *winner's curse* as a bidder who neglects this effect will regret winning.

Consider next the local incentives to change the strategy in the open ascending auction. Suppose that in a given information set in which there are still M active bidders the price reaches the level b, and a bidder is considering whether to remain active a bit longer, say until $b + \epsilon$, or not. She will stay active if the expected value of the good conditional on the event that the final price in the auction is equal to $p \in (b, b+\epsilon)$ is larger than p, otherwise she will quit immediately. The price is p if and only if two events happen simultaneously. Similarly to the sealed bid auction, one event reflects good news and the other one bad news. We also refer to them as the loser's curse and the winner's curse. In this case, the loser's curse is that at least k of the other bidders were active until price p. Hence, if they have any private information about the value of the good it must be favourable. The winner's curse in the open ascending auction is that at least $M-k$ of the other bidders have quit between b and p. In principle, if these bidders had any private information about the value of the good, it must be unfavourable.

Note that the event loser's curse has the same good information than in the sealed bid auction. On the other hand, the bad news associated to the winner's curse is less "bad" in the open ascending auction than in the sealed bid auction in two senses. First, the event winner's curse only contains information about $M-k$ bidders whereas in the sealed bid auction it contains information about $N-k$ bidders. In general M will be smaller than N because some bidders will have quit before the game moves to the current information set. Note that in an equilibrium with strictly monotone strategies the private information of the $N-M$ inactive bidders becomes common knowledge. Second, the extent of the bad news associated to the winners curse is limited by the fact the $M-k$ bidders that quit between b and p were still willing to buy at a price b.

The decrease in the strength of the winner's curse makes the good news of the loser's curse relatively more influential. As we reason in the introduction, the incentives of less informed bidders are more sensitive to the news that come from the winner's curse and the loser's curse. Hence, when we move from a sealed bid auction to an open ascending auction, the incentives of less informed bidders to increase one's bid should shift upwards relatively more than the incentives of more informed bidders. This explains why the entrants win more often and the incumbent less often in the open ascending auction than in the sealed bid auction.

One direct consequence of Proposition 3.3 is the following corollary.

Corollary 3.1. *Suppose that* $1 < k < n$ *and suppose that the bidders play the equilib-*

rium given in Proposition 3.2 in the open ascending auction and the unique symmetric equilibrium in the sealed bid auction (see Proposition 3.1), then:

- *The expected utility of an entrant conditional on the type s^E ($s^E \neq \underline{s}$) is strictly higher in the open ascending auction than in the sealed bid auction.*
- *The expected utility of the incumbent conditional on the type s^E ($s \neq \underline{s}$) is strictly lower in the open ascending auction than in the sealed bid auction.*

3.6 The Auctioneer's Expected Revenue

The comparison of auctioneer's expected revenue is a difficult exercise. On the one hand, we could expect a larger expected revenue in the open ascending auction than in the sealed bid auction because of the greater aggressiveness in the bidding behaviour of the entrants. But, on the other hand, the incumbent will tend to bid relatively more conservatively. Hence, it is not obvious what the comparison will give us.

These difficulties preclude us to provide general results about the comparison of expected revenues. Nevertheless, we shall show that under some specific assumptions, the open ascending auction gives higher expected revenue to the auctioneer than the sealed bid auction. Moreover, we shall show that under the same assumptions, the sealed bid auction is optimal within a set of selling mechanisms. More precisely, we shall consider the set of incentive compatible mechanisms in which all the units are always sold to the bidders (one unit each) and in which the minimum type of all the bidders gets non negative expected utility. This last constraint captures the individual rationality constraint of the bidders. Note that the sealed bid auction and the open ascending auction are two mechanisms in this set.

In general, the auctioneer's expected revenue is the difference between the expected surplus generated in the mechanism and the sum of the expected utilities of all the bidders. Since the mechanisms that we consider always allocate all the units of the good among the bidders, the pure common value assumption implies that the expected surplus generated in the auction is always constant and equal to the expected value of the good, $\int_{\underline{s}}^{\overline{s}} s\,dF(s)$, times the number of units for sale, k. Hence our optimal mechanism is the one that minimises the sum of the expected utilities of the bidders.

According to Lemma 3.3, the expected utility of the incumbent equals:

$$E[U^I(s)] = \frac{1}{1+\lambda n}\int_{\underline{s}}^{\overline{s}}\int_{\underline{s}}^{s} Q^I(\tilde{s})\,d\tilde{s}dF(s) =$$
$$\frac{1}{1+\lambda n}\int_{\underline{s}}^{\overline{s}}(1-F(s))Q_I(s)\,ds = \frac{1}{1+\lambda n}\int_{\underline{s}}^{\overline{s}}\frac{1-F(s)}{f(s)}Q_I(s)\,dF(s),$$

where we have taken $U^I(\underline{s})$ to be equal to zero. The reason is that this condition is the one that assures that the individual rationality constraint of all the types of the incumbent is satisfied at the minimum cost for the auctioneer.[16]

We can write $Q^I(s) = \int_{[\underline{s},\overline{s}]}\cdots\int_{[\underline{s},\overline{s}]} d^I(s, s_1^E, ..., s_n^E)\,dF(s_1^E)...dF(s_n^E)$, where $d^I(s, s_1^E, ..., s_n^E)$ indicates the probability that the incumbent gets the object given the vector of types of all the bidders $(s, s_1^E, ..., s_n^E)$. Hence,

$$E[U^I(s)] = \frac{1}{1+\lambda n}E_{s,s_1^E,...,s_n^E}[\frac{1-F(s)}{f(s)}d^I(s, s_1^E, ..., s_n^E)].$$

Similarly, for entrant i,

$$E[U_i^E(s_i^E)] = \frac{\lambda}{1+\lambda n}E_{s,s_1^E,...,s_n^E}[\frac{1-F(s_i^E)}{f(s_i^E)}d_i^P(s, s_1^E, ..., s_n^E)],$$

where $d_i^E(s, s_1^E, ..., s_n^E)$ indicates the probability that entrant i gets the object given the vector of types of all the bidders $(s, s_1^E, ..., s_n^E)$.

Hence, the optimal auction is characterised by the allocation induced by some functions $(d^I, d_1^E, ..., d_n^E)$ which minimise:

$$E_{s,s_1^E,...,s_n^E}\left[\frac{1-F(s)}{f(s)}d^I(s, s_1^E, ..., s_n^E) + \lambda\sum_{i=1}^{n}\frac{1-F(s_i^E)}{f(s_i^E)}d_i^P(s, s_1^E, ..., s_n^E)\right], \qquad (3.16)$$

subject to the feasibility constraint that $d^I(s, s_1^E, ..., s_n^E) + \sum_{i=1}^{n} d_i^P(s, s_1^E, ..., s_n^E) = k$, the constraint that each bidder can get no more than one unit, this is $d^I, d_i^P \in [0,1]$

[16] We shall not be more explicit on how to derive this result because it follows naturally from the techniques in Myerson (1981).

for $i = 1, 2, ..., n$, and the incentive compatibility constraint that d^I, d_i^P must be such that Q^I and Q_i^E are weakly increasing for $i = 1, 2, ..., n$, see Lemma 3.3.

Suppose that the hazard rate $f(s)/(1 - F(s))$ is strictly increasing, a normal assumption in auction theory, then the above expression is minimised allocating the k units of the good to the k-th entrants with highest signals if the $k - th$ highest signal of the entrants, say $s^E_{(k)}$, verifies $1 - F(s)/f(s) \geq \lambda(1 - F(s^E_{(k)}))/f(s^E_{(k)})$; and otherwise, giving one unit to the incumbent and one units to each of the entrants with the $k - 1$ highest signals. This can be captured with a function ϕ^* implicitly defined by the following equation:

$$\frac{1 - F(\phi^*(s^E))}{f(\phi^*(s^E))} = \lambda \frac{1 - F(s^E)}{f(s^E)}. \tag{3.17}$$

Note that under the assumption of strictly increasing hazard rate, this function is uniquely defined in the former equation and it is strictly increasing. Now, we use the definition of ϕ^* to prove the first part of the following proposition.[17]

Proposition 3.4. *Suppose that $1 < k < n$ and suppose that the bidders play the equilibrium given in Proposition 3.2 in the open ascending auction and the unique symmetric equilibrium in the sealed bid auction (see Proposition 3.1), then:*

- *If the distribution function of the signals, F, is uniform, then the open ascending auction gives higher expected revenue than the sealed bid auction.*
- *There exists a $\bar{\lambda} > 0$ such that for all $\lambda < \bar{\lambda}$ the open ascending auction gives higher expected revenue than the sealed bid auction.*

Moreover, the open ascending auction implements the optimal allocation in the first case and in the limit when λ goes to zero.

Proof. In the case of the uniform distribution function, the implicit definition of ϕ^o in equation (3.6) is $1 - \phi^o(s^E) = \lambda(1 - s^E)$. This means that for this distribution function $\phi^o = \phi^*$. The revenue equivalence theorem assures that two auctions that allocate the

[17]We could also use this approach to prove the second part of the proposition. Instead, we use a different approach that conveys better the underlying intuitions.

good in the same fashion (and in which the minimum type gets the same expected utility) give the same expected revenue to the auctioneer. Hence, the open ascending auction implements the maximum expected revenue auction among all the auctions that always sell all the units. Moreover, since the sealed bid auction implements a different allocation rule, the first result of the proposition follows.

For the second case, note that when λ tends to zero: (i) the incumbent bids the true value of the object in the equilibrium of the open ascending auction given in Proposition 3.2; and (ii), all the entrants remain active until the incumbent quits and then quit simultaneously. This means that the price equals the true value of the object with probability that tends to one when λ goes to zero. Hence, the auctioneer achieves full surplus extraction in the limit when λ goes to zero. On the other hand, in Chapter 2 of this Ph.D. dissertation, we have shown that in the sealed bid auction the limit when λ tends to zero of the incumbent bid function in the unique symmetric equilibrium also equals the true value of the object, but the entrants randomise among all the bids between $[\underline{s}, \overline{s}]$ (Proposition 2.4 and Proposition 2.8). One characteristic of this randomisation is that the entrants get zero expected utility, and the incumbent strictly positive expected utility. Hence, the auctioneer does not get full surplus extraction. ■

The case when λ is close to zero has an intuitive explanation. If λ is close to zero then, as we have already explained, the incumbent will remain in the auction until a value arbitrarily close to the true value of the object. Suppose that the entrants' strategy is such that the entrants leave the auction with a positive probability before the incumbent. One entrant that leaves the auction gets zero utility. However, if she stays active, two things can happen. Either the incumbent quits, and then the entrants' best action is to quit immediately, which gives a expected utility close to zero, or a number of entrants high enough to finish the auction quit. Thus, the incumbent's bid, and hence the value of the good must be above the price and this gives a strictly positive expected utility to the entrant.

Consequently, the entrants have incentives to stay in the auction until the incumbent quits. This explains why in equilibrium the entrants' strategy puts probability close to one to the action of staying in the auction until the incumbent leaves the auction. The consequence is that the price will be arbitrarily close to the incumbent's bid, this is the true value of the object. The auctioneer thus gets almost full surplus extraction, and hence, in the limit when λ goes to zero, our open ascending auction implements the maximum expected revenue mechanism.

The case when the distribution of the signals is uniform is more complex and it seems to depend on the symmetry properties of the uniform distribution function.

3.7 Conclusions

We have analysed in this paper a multiunit common value auction in which there are two types of bidders: an incumbent with accurate information about the common value of the object, and some entrants with more noisy information than the incumbent about the common value of the object.

We have shown that an open ascending auction can give higher expected utility to the entrants than a sealed bid auction. This result gives shows that if one of the concerns of the auctioneer is how to attract more entrants into an auction where a well informed incumbent participates, the auctioneer could prefer an open ascending auction. We have also shown in this model that the open ascending auction also does better than the sealed bid auction in terms of expected revenue under some specific assumption. In fact, under the same specific assumptions the open ascending auction implements the maximum expected revenue mechanism among the mechanisms that always sell all the units.

It remains unclear how other auction formats such as the sealed bid pay your bid auctions will do in the same set-up. The main difficulty to extend the analysis is that other auction formats is that usually they do not have a closed form solution.

We do not provide any result for the case of the auction of one single unit. There are two difficulties in this case: the first one is that the English auction, i.e. an open ascending auction when there is one unit for sale, has under our assumptions a severe problem of multiplicity of equilibria, and it is not clear which equilibria to choose. Second, Klemperer (1998) has shown that small private value differences can have huge effects in the auction outcome of the English auction with one unit for sale and one advantaged bidder. We believe that incumbents usually have not only better information than entrants but also private value advantages. Hence, a more accurate model for the single unit for sale and single incumbent should take into account these private value differences.

Chapter 4

Competition among Auctioneers

4.1 Introduction

In this paper, we study a multistage game of competition among auctioneers. In the fist stage auctioneers compete for a common pool of bidders by means of credible announcements of the minimum price accepted in a second price auction. In a second stage, that we call the *entry game* each bidder chooses an auction, if any, to participate. Finally, in the last stage, each of the announced auctions takes place among all those bidders that have chosen it.

We provide two results in this model. The first one is that we prove that there always exists an equilibrium of the whole game. The second one is that in the limit of the equilibrium set when the numbers of auctioneers and bidders go to infinity, almost all the auctioneers with types low enough to trade announce a reserve price equal to their respective production costs with probability arbitrary close to one.

Our limit result is better understood considering first the single auctioneer case. Myerson (1981) shows that if there is one single auctioneer it is optimal for him to fix a reserve price above his production cost with generality. This strategy means that the auctioneer does not trade with some bidders with valuation between the reserve price and the auctioneer's production cost. But the losses of not trading with these types are dominated by the increase in the price that bidders with valuation above the reserve price pay.

Two features of our model explain why this result does not hold when the numbers of auctioneers and bidder goes to infinity. First, changes in a single reserve price have a negligible effect on the price that bidders expect to pay in the other auctions when the numbers of auctioneers and bidders approach to infinity. Second, the unique equilibrium of the entry game is such that bidders expect to pay the same price conditional on winning in all the auctions which they enter. As a consequence, when an auctioneer increases his reserve price the effect on the expected price that the bidders with valuation above the reserve price expect to pay conditional on winning should vanish as the market increases to the limit. This means that the positive effect of increasing the reserve price above the production cost disappears in the limit. However, the negative effect of losing profitable trades still remains. Hence, the auctioneer's incentives to distort trade by announcing reserve prices above his production cost disappear in the limit.

Note, however, that when there is a finite number of auctioneers and bidders the auctioneer still can have incentives to increase the reserve price above his production cost. In this case, an increase of the reserve price means that some bidders move to other auctions and, hence, it increases the expected price in these other auctions. This increase in the general level of prices in the other auctions also means an increase in our auctioneer's expected price. This explains why auctioneers can fix a reserve price above the production cost. Burguet and Sákovics (1999) have shown that this is the case when there are two auctioneers. Consequently, our result implies that this monopolist distortion vanishes as the numbers of auctioneers and bidders tend to infinity.

Our limit result is aligned with those by McAfee (1993), Peters (1997a), and Peters and Severinov (1997). They show that in a game of competition among auctioneers similar to ours but with infinite numbers of auctioneers and bidders,[1] there exist an equilibrium in which each auctioneer always fixes a reserve price equal to his production cost. But, there are two differences with respect to our model. The first one is

[1] McAfee (1993) does not assume that the numbers of auctioneers and bidders are infinite. Instead, he assumes that an auctioneer does not take into account that when he changes his mechanism, the expected utility that bidders can get in other auction mechanisms changes. He justifies this assumption conjecturing that it should be true in the limit with infinite number of auctioneers and bidders.

that we provide a limit result based on the equilibrium of the whole game and for a finite number of agents. They instead compute an equilibrium of the auctioneers' game defined by some payoff functions computed under infinite number of agents assumptions.

Second, we provide a kind of uniqueness equilibrium prediction. We show that the limit of the equilibrium of the game characterises the outcome of the game up to a negligible fraction of auctioneers. In these other papers, only existence results are provided. One exception is Peters and Severinov (1997). They also provide uniqueness results although they limit to equilibria in which all the auctioneers announce the same reserve price whereas we also consider equilibria in which different auctioneers announce different reserve prices.

Peters (1998) and Peters (2000a) also look to related problems under some infinite number of agents assumptions. The first of the papers deals with the private value assumption with correlated private types, and the second with the common value assumption. Our model only covers the private value assumption with independent private types.

The underlining core of our approach is the characterisation of the equilibrium of the entry game through a set of tractable equations. These equations provide a tractable way of computing the limit payoffs in the reduced game of competition among auctioneers that we use to prove our convergence result.

This approach also allows us to show that the equilibrium of the entry game is unique among the equilibria in which all the bidders use the same strategy, and that this equilibrium is continuous with respect to the vector of reserve prices announced by the auctioneers. These two points are important since from them it is obvious how to prove the existence of an equilibrium in the whole game. Uniqueness means that we can define in a straightforward manner the reduced game of competition among auctioneers: evaluating the auctioneers' payoff functions at the unique symmetric equilibrium of the entry game. Continuity assures that the auctioneers' payoff functions of the reduced game are continuous, and hence, we can apply standard theorems to prove existence of an equilibrium for the reduced game of competition among auctioneers.

But, the analysis of the entry game is a complex technical task. Peters and Severinov (1997) have characterised the equilibrium of the entry game when either all the auctioneers announce the same reserve price or when all the auctioneers but one announce the same reserve price. Nevertheless, most of the difficulties arise when we allow for more than two different reserve prices.

Actually, these difficulties are not exclusive of the model of competition that we analyse. Peters (2000b) studies a similar multistage game of competition but in which each seller commits to a price instead of conducting an auction with a reserve price. Peters (2000b) also considers the entry game when the sellers' offers are heterogeneous, but he restricts to the case in which all the buyers are identical. Related difficulties have also arisen in other models of decentralised trade as in Peters (1997b). In this case, Peters (1997b) also avoids these problems by assuming that there are heterogeneous agents only in one side of the market.

Nonetheless, we solve our model by allowing heterogeneity in both sides of the market at the entry game. This makes sense even if we consider that all the auctioneers are identical, as Peters and Severinov (1997) assumes. The reason is that the reduced game of competition among auctioneers with finite number of agents does not have a Nash equilibrium in which all the auctioneers announce the same reserve price. This is true even when all the auctioneers are identical and the number of agents is large. We also extend Peters and Severinov's analysis by allowing also for differences in the production costs of the auctioneers.

Quite interestingly, our model offers a very natural way of studying decentralised trade. Each auctioneer announces a supply curve for their only unit that it is characterised by the reserve price of a second price auction and then the bidders choose trading partner. Note that each seller by himself constitutes a *local market* in which his supply curve is crossed with the demand curve that it is formed with the bids of the bidders that participate in that auction to determine the price in this local market.

One of the features of the second price auction is that to announce the true value of the good is an equilibrium strategy for the bidders, i.e. to announce their true demand. Moreover, we show in the paper that as the numbers of auctioneers and bidders go to infinity, the fraction of auctioneers with production cost low enough to trade that announce a reserve price different to their production cost tends to zero, i.e. in the limit auctioneers announce their true supply curve. This means that in the limit the local market converges to a competitive outcome.

Another interesting question is whether the *global market*, this is the market of all the auctioneers and bidders, converges to a competitive outcome or not. In this sense, we can say that the global market is not competitive in two senses. The first one is that the price does not converge to a competitive price. The price in each of the auctions is a random variable that it is not directly determined by the global demand and supply curves but rather by the random entry strategy of bidders. The second is that the allocations are not competitive. Since the equilibrium involves all the bidders randomising entry it could happen that some auctions do not receive any bidder, even if they have a relative low production cost. Moreover, it could also be that some other auctions receive several high valuation bidders. Then, only one of the bidders wins the auction whereas the others are rationed.

These inefficiencies in the global market have been already pointed out by Peters (1997a) for a limit version of our model with infinite numbers of auctioneers and bidders. On the other hand, Satterthwaite and Williams (1989), and Williams (1991) have shown that the incentives to misrepresent the preferences in a double auction disappear when the numbers of auctioneers and bidders tend to infinity. This suggests that the frictions that motivate the failure of convergence to a competitive outcome are related to the fact that in our model trade is decentralised.

From a different perspective, our results also relate to the literature on Bertrand competition, and more precisely, to the variation suggested by Edgeworth in which firms are capacity constrained. Remember that in our case each auctioneer has only one unit. One characteristic of Bertrand-Edgeworth competition is that it does not imply that the competitive price prevails in equilibrium. Instead, only when the numbers of auctioneers and bidders tend to infinity, the market price and the allocation converge respectively to the competitive price with probability one, see for instance Allen and Hellwig (1986).

Note, however, that the frictions of decentralised trade are usually minimised in the Bertrand-Edgeworth set-up by the assumption that if a buyer is rationed by one seller she can turn to other sellers. In fact, we can conclude from the analysis of Peters (2000b) and Burdett, Shi, and Wright (2000) that once we introduce our assumption that buyers can attend to no more than one buyer, the same inefficiencies as in our model arise, even with an infinite number of buyers and sellers.

Finally, notice that our paper also relates to another branch of papers, those that deal with mechanism design under common agency, see for instance Stole (1997). In these papers, several principals design simultaneously an optimal mechanism for the same agent. Although mechanism design is a more general set-up that includes auctions, in order to allow for such generality these models only consider one single agent. Our model differs in that we allow for more than one agent, this is we allow for more than one bidder.

We start with a description of the model in Section 2. In order to solve the game we proceed backwards. In Section 3, we solve the second stage, the entry game. We use the solution of the entry game to compute the reduced game of competition of auctioneers. We study this reduced game in Section 4. Section 5 provides the limit results of our model. Section 6 concludes.

4.2 The Model

There are $J \in \mathbb{N}$ auctioneers and $kJ \in \mathbb{N}$ bidders. We shall later consider the limit $J \to \infty$. When doing this, we shall keep the ratio $k > 0$ of bidders to auctioneers fixed.

Each auctioneer has the ability to produce a single indivisible unit of output. We assume that each auctioneer j observers his own production cost w_j before the beginning of the game, whereas the other auctioneers (and bidders) only know that it is drawn independently from the set $[0, 1]$ according to a probability distribution function H which is the same for all auctioneers.

Each bidder wishes to purchase exactly one unit of the commodity. Each bidder i observes her reservation prices x_i privately before the beginning of the game. All other players only know that reservation prices are independently drawn from the set $[0, 1]$ according to the same distribution function F with a density f and support[2] $[0, 1]$.

If an auctioneer j with production cost w_j trades with a bidder i with type x_i at a price p, they are assumed to obtain a von Neumann Morgenstern utility of $p - w_j$ and of $x_i - p$ respectively. In the case that there is no trade, both the auctioneer and the bidder get a von Neumann Morgenstern utility of 0. Notice that this assumption implies that the production occurs, and production costs are incurred, only once a trade has been agreed. The production cost could also be seen as an opportunity cost.

We consider a three stage game. In the first stage, each auctioneer announces an auction rule. For most of the paper we assume that auctioneers can only choose second price auctions without entry fees. Their only choice variable is the reserve price in their auction. Auctioneers make these choices simultaneously. Once each auctioneer has chosen his reserve price the choices are made public.

In the second stage, that we call the *entry game*, each bidder can either pick one and only one auction[3] in which she wants to participate, or she can choose to participate in no auction. In the final stage those bidders who have chosen to participate in some particular auction make their bids in that auction.

Notice that it is a weakly dominant strategy in the final stage to bid one's true value. This is independent of the number of other bidders in the auction. Therefore, it is unimportant whether the outcome of the second stage is observed before the third stage begins.

The most obvious restrictive assumption in our model is that auctioneers can only choose second price auctions without entry fees. We make that assumption for sim-

[2]The assumption that the support of F equals $[0, 1]$ implies that we do not consider situations in which the production cost of an auctioneer is below the minimum valuation of the bidders. We believe that our model could be extended to cover this case. The only required modification would be that in the limit, as J tends to infinity, auctioneers with production costs below the lower bound of the support of F would set reserve prices between their production costs and this lower bound, rather than equal to their production costs. This fact has already been mentioned by Peters (1997a).

[3]We believe that our results could be easily extended to the case in which bidders can participate in more than one auction under the following additional assumptions. Each bidder has a constant marginal utility for a finite number of units and zero for additional units. The number of units from which the bidder obtains strictly positive utility is greater than the number of auctions in which she can participate. Under these assumptions it is still true that it is weakly dominant for the bidder to bid her true value of the good. If these assumptions are not met then there is no straightforward solution for the bidding game, and hence, we cannot extend easily our analysis.

plicity. However, we shall show later that our results extend to the case in which auctioneers cannot only choose second price auctions but also first price auctions. Note that in this case it might matter whether the outcome of the second stage is observable or not. This is because optimal bidding behaviour in a first price auction depends on the number of other bidders participating in that auction. Hence, we shall also consider that each auctioneer can choose whether the number of bidders in his auction becomes common knowledge before the third stage begins or not. We shall explain why the main results which we show for the basic version of our model also hold for this extended version.

Obviously, it would be desirable to analyse a model in which the auctioneers' strategy space is even larger. For example, one would like to allow the auctioneers to announce other standard auctions which treat all bidders symmetrically, such as all pay auctions, or second price auctions with entry fees, for instance as McAfee (1993) and Peters (1997a) do. In addition, one could allow auctioneers to choose auctions which treat bidders asymmetrically, for example by allowing only some but not all bidders to participate. Finally, it is potentially important to consider mechanisms which condition on the mechanism choice by other auctioneers, for example by including rules which are similar to "price matching clauses", see for instance Epstein and Peters (1999). We do not know whether our results extend to the case in which auctioneers are allowed to choose from these more general classes of mechanisms.

The reason why it is easy to introduce first price auctions into the auctioneers' strategy space, but difficult to extend the strategy space further, is somewhat subtle. If a second price auction with reserve price is replaced by a first price auction with the same reserve price, then the equilibrium entry pattern and allocation rule remain unchanged.[4] Therefore, by the revenue equivalence theorem, the auctioneer's expected revenue stays the same. Now suppose that we allowed the auctioneers to choose in addition second price auctions with entry fees. We could still find an entry fee which generates the same entry pattern and allocation rule as a second price auction with

[4] If auctioneers offer symmetric mechanisms, we shall restrict attention to symmetric equilibria of the entry game and of the bidding stage. If we allowed asymmetric equilibria in either of these two stages, the revenue equivalence theorem would not even allow us to generalise our analysis to first price auctions.

reserve price, and hence yields the same expected revenue by the revenue equivalence theorem. However, the appropriate entry fee would now depend on the choices of all other auctioneers. As soon as there is uncertainty about the other auctioneer's choices, for example because of private information about their production costs, we cannot rule out that an auction with entry fee yields higher expected revenue than an auction with reserve price, and thus that the choices which constitute equilibria in the restricted strategy space are no longer equilibria in the extended strategy space. Obviously, asymmetric auctions might generate asymmetric entry patterns or allocation rules, and hence the revenue equivalence theorem does not apply, and we cannot be certain of any relation between the equilibria which we identify here, and the equilibria of a game in which auctioneers are allowed to choose asymmetric auctions.

The fact that we assume that the bidders know their own type at zero cost is also restrictive. With this assumption we disregard situations in which information acquisition is an issue. In a more general model we could distinguish two kinds of information acquisition costs: those due to an external information acquisition technology, and those which the auctioneer can influence. Peters and Severinov (1997) analyse a model of competition among auctioneers in which they allow for the latter type of information acquisition cost.

Our assumption that types are known from the beginning of the game implies that each bidder can condition her entry decision on her type. The fact that these types are privately known implies that the entry game is a game of incomplete information. The same entry game has been studied previously under the assumption that it is common knowledge that bidders are identical in the stage of choosing an auction (Peters and Severinov (1997)).

We study the game using backward induction. Since we have restricted the selling mechanisms that can be used in the third stage game to second price auctions, the solution of this game is trivial. We assume that bidders play the unique weakly dominant strategy, to bid their true value. Hence, in equilibrium the bidder with highest valuation among those that have entered the auction and bid above the auctioneers' reserve price, wins the auction and pays a price equal to the maximum of the second highest valuation and the reserve price announced by the auctioneer. This fully determines the bidders' expected utility of participating in an auction given the entry decisions of the other bidders. With these bidders' payoffs we can define the reduced game that bidders play in the second stage, the entry game. We solve this game in the next section.

4.3 The Entry Game

In this section we study the second stage game. In this game, bidders choose the auction that they will attend, if any, after observing the auctioneers' announced reserve prices. We shall show that this game has a unique symmetric Nash equilibrium and that this equilibrium is continuous in the auctioneers' reserve prices. We will use the first result to define the reduced game that the auctioneers play in the first stage in a straightforward manner and the second result will assure the continuity of the auctioneers' payoffs in this auctioneers' game. We shall also characterise the symmetric equilibrium of the entry game in a way which facilitates the proof of the convergence result in Section 4.5.

Bidders take their entry decision conditioning on the vector of reserve prices announced by the auctioneers, $\vec{r} \in [0,1]^J$, and on their private types. For notational convenience we shall assume that the elements of the vector of reserve prices are ordered increasingly. The expected utility of entering an auction given the entry decisions of the other bidders are computed assuming that the bidders bid the true value of the good. We restrict attention to equilibria in which all the bidders play the same entry strategy, possibly mixed. This means that two bidders with the same type assign in equilibrium the same probability of entering to a given auction.

Although the restriction to symmetric equilibrium is a standard practice, it is clearly restrictive in this game. To understand these restrictions it is useful to consider the following example. Assume that there are two second price auctions with no reserve price and two bidders both with the same valuation $x > 0$. It is trivial to show that this game has three Nash equilibria: a symmetric equilibrium in which each bidder enters each of the auctions with the same probability, and two asymmetric equilibria in which each bidder enters a different auction.

This example shows in particular that the symmetric equilibria of the entry game may be Pareto dominated by the asymmetric equilibria.[5] On the other hand, the asymmetric equilibria seem to require that bidders co-ordinate their entry behaviour. Therefore, by restricting attention to the symmetric equilibria of the entry game we are implicitly assuming that frictions prevent bidders from co-ordinating their entry decisions. This is probably a reasonable assumption for many markets, mainly those in which the number of auctions and bidders is large. This assumption has also been made in other papers like McAfee (1993), Peters and Severinov (1997), and Peters (1997a) that have studied similar models of competition among auctioneers.

We characterise the (possibly random) entry decision of the bidders with a function $\overline{\pi} : [0,1] \times [0,1]^J \to [0,1]^J$. This function gives a vector of probabilities of entering each of the auctions $\pi(x;\vec{r})$ for a bidder with type x given the announcement of reserve prices $\vec{r}$. We denote the j-th component of this vector by $\pi_j(x,\vec{r})$. Define the set E_j to be the closure of the interior of the set $\{x : \overline{\pi}_j(x,\vec{r}) > 0\}$. Moreover, define the cut-off valuation for a given auction j to be equal to $y_j = \min\{x : x \in E_j\}$. Then:

Lemma 4.1. *A symmetric Nash equilibrium of the entry game must satisfy for all auctions j, l:*

(a) $E_j = [y_j, 1]$.

(b) If $r_l > r_j$, then $y_l \geq y_j$ and if $r_l = r_j$, then $y_l = y_j$.

(c) If $r_l \geq r_j$. then for almost all $x \geq y_l$, $\pi_j(x,\vec{r}) = \pi_l(x,\vec{r})$.

Proof. See Lemma 2 in Peters (1997a). ∎

In the following, we shall call strategies of the type described in Lemma 4.1 "cut-off strategies". Lemma 4.1 thus says that any symmetric Nash equilibrium of the entry game must be an equilibrium in cut-off strategies. One surprising feature of cut-off strategies is that bidders who enter several auctions with positive probability always

[5] See Peters (1997a) for examples of other asymmetric equilibria when there are many bidders with different types and many auctioneers announcing different reserve prices.

randomise uniformly among these auctions. We shall provide some intuition for this feature, later, following Lemma 4.3.

Our next goal is to derive necessary and sufficient conditions for cut-off strategies to constitute an equilibrium. We begin with the following lemma:

Lemma 4.2. *If all the bidders play the same cut-off strategy, the probability that a bidder with type $x \geq y_j, y_l$ wins if she enters auction j is the same as the probability that this bidder wins if she enters auction l.*

Proof. A bidder with type x can win in equilibrium a given auction if and only if each of the other bidders either has a valuation below x, or enters any of the other auctions. The first condition is trivially the same in auction l and in auction j. The second condition holds with the same probability for both auctions given that each bidder randomises uniformly among all the auctions that she enters. ∎

Now recall the following standard result:

Lemma 4.3. *The expected utility of a bidder with type x in a second price auction is a continuous convex function (of x) which is almost everywhere differentiable with first derivative equal to the probability of winning that auction for a bidder with type x.*

Proof. See Myerson (1981) and also Lemma 1 in Peters (1997a). ∎

Suppose that all the bidders play the same cut-off strategies. Then, these two assumptions imply that if a bidder with a type $x \geq y_j, y_l$ is indifferent between auction j and auction l, she will also be indifferent between auction j and auction l for all types above x. This result depends on the bidders randomising uniformly among all the auctions that they attend. Otherwise, both the probability of winning and the expected utility will differ in auction l and auction j. This also explains why in equilibrium if bidders randomise among the auctions that they attend, they must randomise uniformly. Otherwise, bidders will have incentives to deviate.

The above paragraph suggests a way in which the task of checking whether a given cut-off strategy constitutes a symmetric Nash equilibrium can be simplified. One of the conditions which one needs to check for this is that bidders who enter different auctions with positive probability are indifferent between these auctions. The above paragraph indicates that it is sufficient to check this condition for the smallest type which enters two auctions, and that then all bidders with higher type will automatically also be indifferent.

Typically, the smallest type which enters two auctions with positive probability will be a cut-off point. In fact, we shall show in the next Lemma that necessary and sufficient conditions for a cut-off strategy to constitute a symmetric Nash equilibrium can be constructed which refer only to the incentives of bidders with cut-off types.

Our conditions will compare the expected price paid by a bidder with a type equal to an arbitrary cut-off y_j conditional on winning in auction j with the same conditional expected price in auction $j-1$. Given that Lemma 4.2 says that the probability of winning is the same in both auctions we are in fact comparing the expected utility of entering auction j and auction $j-1$.

In order to formalise these conditions we first introduce a function Ψ_{j-1}, where $\Psi_{j-1}(x, y_{j-1}, y_j, ..., y_J)$ is the expected price paid by a bidder with type x conditional on winning auction $j-1$, and given that all the other bidders play some cut-off strategies represented by $y_1, ..., y_J$. Note that we only allow Ψ_{j-1} to depend on the cut-offs $y_{j-1}, ..., y_J$. The reason is that changes in the other cut-offs do not affect entry in auction $j-1$ and hence, do not affect the expected price in that auction. We shall restrict the domain of Ψ_{j-1} to $x \geq y_{j-1}$, $y_l \geq r_l$ for all $l \geq j-1$, and $y_{j-1} \leq y_j \leq ... \leq y_J$. Other values do not make sense in an equilibrium in cut-off strategies.

In the following Lemma, the first condition has an obvious meaning. Point (ii) says that a bidder i with type $x^i = y^j$ is indifferent between auction j and auction $j-1$ if $y^j < 1$. Similarly, point (iii) says that a bidder i with type $x^i = y^j$ weakly prefers auction $j-1$ to auction j if $y^j = 1$.

Lemma 4.4. *A necessary and sufficient condition for a Nash equilibrium in cut-off strategies is that each cut-off y_j is greater than or equal to r_j and satisfies that:*

(i) If $r_j = r_1$, then $y_j = r_1$.

(ii) If $r_j \neq r_1$ *and* $y_j < 1$, *then* $r_j = \Psi_{j-1}(y_j, y_{j-1}, y_j, ..., y_J)$.

(iii) If $r_j \neq r_1$ *and* $y_j = 1$, *then* $r_j \geq \Psi_{j-1}(y_j, y_{j-1}, y_j, ..., y_J)$.

Proof. We start by showing that our conditions are sufficient. Since we impose that $y_j \geq r_j$ for all j, all bidders who enter an auction get non-negative expected utility. Hence they do not have incentives to stay out of the market. Point (i) guarantees that the minimum type that participates in any auction is r_1. Since bidders with types below r_1 cannot profitably trade in the market, point (i) assures that these types do not have incentives to deviate and enter an auction. Hence, we only need to show that points (ii), and (iii), imply that: (*) a bidder with a given type is indifferent among all the auctions which she enters with positive probability conditional on her type; and (**), a bidder with a given type does not gain from entering auctions which she does not enter with positive probability conditional on her type.

By the definition of cut-off strategies (*) says that bidder i with a type $x_i \geq y_j$ (if $y_j < 1$) must be indifferent between all auctions l such that $l \leq j$. Point (ii) implies the indifference of bidder i conditional on a type $x_i = y_j$ between auction j and auction $j-1$. Lemma 4.2 and Lemma 4.3 thus say that bidder i with type $x_i \geq y_j$ is indifferent between auction j and auction $j-1$. We can apply the same argument to show that bidder i conditional on type $x_i \geq y_j$ is indifferent between auction $j-1$ and auction $j-2$. Repeating this argument, we can show that bidder i is indifferent among all the auctions $l \leq j$.

If we take account of the definition of cut-off strategies, condition (**) says that a bidder i with type $x_i \in [y_l, y_j)$ cannot improve by deviating and entering auction j. This claim holds trivially if $x_i \leq r_j$. Consider the case $x_i > r_j$. If x_i were equal to y_j and $y_j < 1$ (the case $y_j = 1$ is considered below), the expected utility of entering auction j would be the same as the expected utility of entering auction l because of (*). Hence we only need to prove that the derivative of the expected utility of entering auction l with respect to the type is not larger than the derivative of the expected utility of entering auction j with respect to the type for bidder i with type $x_i < y_j$. If bidder i deviates and enters auction j, she cannot do better than bidding

x_i. In this case, she only wins if no other bidder enters auction j, and then she pays the reserve price r_j. This implies that the derivative with respect to the type of the expected utility that bidder i can achieve in auction j equals the probability that no other bidder enters auction j. According to Lemma 4.2 this probability equals the probability that bidder i wins auction l if she had type $x_i = y_j$. On the other hand, Lemma 4.3 says that the derivative of the expected utility of entering auction l is the probability that bidder i wins auction l with her true type x_i. Since this type is lower than y_j, the probability of winning is lower with this type. This proves that the derivatives verify the required condition.

If $y_j = 1$, then point (iii) implies that a bidder i with type $x_i = y_j$ weakly prefers auction $j-1$ to auction j. We can show as in the above paragraph that this implies that bidder i with type $x_i = y_j$ weakly prefers auction l, for $l < j$, to auction j. Hence, we can repeat the argument above.

Finally, we show that the points (i)-(iii) are necessary. Point (i) is trivial. Suppose that there is a cut-off $y_j < 1$ (and $r_j \neq r_1$) for which (ii) does not hold, this is that bidders with type y_j strictly prefer entering auction $j-1$ to entering auction j. Then, the continuity of the bidder's expected utility in the bidder's type, implied by Lemma 4.3, means that there must exist a non-empty interval of types $[y_j, y')$ that strictly prefer entering auction $j-1$ to entering auction j. Therefore, these types have incentives to deviate. We can proceed symmetrically in the case that bidders with type y_j ($y_j < 1$ and $r_j \neq r_1$) strictly prefer entering auction j to entering auction $j-1$.

We prove that (iii) is necessary in a similar fashion. Suppose that bidders with type $y_j = 1$ strictly prefer entering auction j to auction $j-1$. First, note that this can only be if $y_{j-1} < 1$, otherwise, types $y_j = 1$ would prefer auction $j-1$ because by assumption $r_{j-1} < r_j$. Then, the continuity of the bidders' expected utility guaranties that there exists a set of types $(y', 1]$, such that $y' \geq y_{j-1}$, that strictly prefer auction j to auction $j-1$. Again, these types would have incentives to deviate. ■

In order to solve the condition in Lemma 4.4 for the cut-offs, we first give an explicit formula for Ψ_{j-1} and derive some of this function's properties. We begin by introducing the following notation. Consider a bidder i who follows a cut-off strategy π, and a type x with $x \geq y_1$. Let auction l be the auction which has the highest index among all auctions in which a bidder with type x participates with positive probability. Then we denote by $z(x;\pi)$ the probability that the bidder i either does not submit a bid in auction l or that she has a type below x. The probability is given by:

$$z(x;\pi) = 1 - \frac{F(y_{l+1}) - F(x)}{J\bar{G}^J(l)} - \sum_{q=l+1}^{J} \frac{F(y_{q+1}) - F(y_q)}{J\bar{G}^J(q)} \tag{4.1}$$

where $\bar{G}^J(l)$ is the fraction of auctioneers that announce a reserve price equal or below the l-th highest reserve price, and where[6] $y_{J+1} \equiv 1$.

We can now construct the conditional distribution function of the price paid by bidder i with type x conditional on winning in auction $j-1$, supposing, of course, that $x \geq y_{j-1}$. If all bidders other than some bidder i follow the same cut-off strategy π, then the probability that bidder i with type $x \geq y_{j-1}$ wins auction $j-1$ is $z(x;\pi)^{kJ-1}$. This implies that for $x \neq 0$ and weakly above y_{j-1}, and $\tilde{x} \in [y_{j-1}, x]$ the probability that the price in auction $j-1$ is below $\tilde{x}$ given that bidder i with type x wins auction $j-1$ is $z(\tilde{x},\pi)^{kJ-1}/z(x,\pi)^{kJ-1}$. It also implies that the probability that no other bidder enters auction $j-1$ conditional on bidder i winning that auction equals $z(y_{j-1},\pi)^{kJ-1}/z(x,\pi)^{kJ-1}$. In this last case bidder i pays the reserve price r_{j-1}.

Denote by $\nu_{j-1}|_x$ the conditional distribution function of the price paid by bidder i with type $x \neq 0$ conditional on winning in auction $j-1$. Then we can summarise the arguments in the preceding paragraph with the following formal description of $\nu_{j-1}|_x$:

- If $\tilde{x} < r_{j-1}$, then $\nu_{j-1}|_x(\tilde{x}) = 0$.
- If $r_{j-1} \leq \tilde{x} < y_{j-1}$, then $\nu_{j-1}|_x(\tilde{x}) = \frac{z(y_{j-1};\pi)^{kJ-1}}{z(x;\pi)^{kJ-1}}$.
- If $y_{j-1} \leq \tilde{x} < x$, then $\nu_{j-1}|_x(\tilde{x}) = \frac{z(\tilde{x};\pi)^{kJ-1}}{z(x;\pi)^{kJ-1}}$.
- Otherwise, $\nu_{j-1}|_x(\tilde{x}) = 1$.

Hence for $x \geq y_{j-1}$:

[6] Note that the formula which we have given has on the right hand side one minus the probability of the event which is complementary to the event described in the text.

$$\Psi_{j-1}(x, y_{j-1}, y_j, ..., y_J) = \int_{-\infty}^{+\infty} \tilde{x}\, d\nu_{j-1}|_x(\tilde{x}). \tag{4.2}$$

Using this formula, we can now obtain some useful properties of Ψ_{j-1}.

Lemma 4.5. *The function Ψ_{j-1} is continuous in all its arguments, strictly increasing in x, and in all cut-offs y_j, y_{j+1}, ..., y_J, and strictly decreasing in y_{j-1}.*

Proof. In order to prove the continuity of Ψ_{j-1} with respect to a parameter that affects the distribution function $\nu_{j-1}|_x(.)$ we only need to show that this distribution function $\nu_{j-1}|_x(.)$ changes continuously with respect to the parameter of interest in all the points of continuity of the distribution function $\nu_{j-1}|_x(.)$ (Theorem 25.8, p. 335 in Billingsley (1995)). The continuity of this distribution function in these parameters follows from the continuity of F.

We prove that Ψ_{j-1} is monotonic with respect to the parameters by showing that changes in the parameters produce shifts of the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance. It is easy to see that a decrease in y_{j-1} or an increase in x shifts the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance downwards. An increase in y_l for $l > j-1$ decreases the ratio $z(\tilde{x};\pi)/z(x;\pi)$, as one can verify through differentiation, and hence it also shifts the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance downwards. ∎

It seems worthwhile to explain the intuition behind the monotonic properties in Lemma 4.5. That Ψ_{j-1} is increasing in x is self-explanatory. Next, Lemma 4.5 says that an increase in the minimum type that enter auction $j-1$, say from y_{j-1} to y'_{j-1}, keeping other things constant, decreases the price that a bidder i with type x expects to pay conditional on winning auction $j-1$. To understand this result note that the price that i pays only changes if the maximum type of the other bidders that enters auction $j-1$ with cut-off y_{j-1} is between y_{j-1} and y'_{j-1}. If the cut-off is y'_{j-1}, then the price is fixed by this maximum type of the other bidders whereas, if the cut-off is y'_{j-1}, the price equals the reserve price r_{j-1}. Since y'_{j-1} is strictly above r_{j-1}, it explains the decrease in the expected price.

The effect of an increase in a cut-off associated to another auction l to which bidders with type x enters, say $y_l < x$ to y_l', is slightly different. Then, the only difference in the price that bidder i with type x pays when she wins occurs under the following event: a bidder with type $\tilde{x} \in (y_l, y_l')$ is the bidder with maximum type among those bidders that enter auction $j-1$ when the cut-off is y_l', and this bidder enters auction l when the cut-off is y_l. This means that the price that i pays when the cut-off is y_l' is $\tilde{x}$, and the price that i pays when the cut-off is y_l is below $\tilde{x}$.

More subtle is the effect of an increase in a cut-off associated to another auction l to which bidders with type x do not enter, say y_l ($y_l \geq x$) to y_l'. A bidder i with type x does not win auction $j-1$ under the event that there is another bidder with type between y_l and y_l' that enters auction $j-1$. But, the probability of this event is higher when the cut-off is y_l' than when the cut-off is y_l. The reason is that bidders with these types enter with higher probability to auction $j-1$ when the cut-off is y_l' than when the cut-off is y_l because in the latter case these types also enter auction l. As a consequence, the probability that the other bidders have a type between y_l and y_l' conditional on the event that bidder i wins with a type x is lower when the cut-off is y_l' than when the cut-off is y_l. Hence, the probability that the other bidders have types between y_{j-1} and x conditional on the event that i wins with a type x is higher when the cut-off is y_l' than when the cut-off is y_l. This implies that moving y_l to y_l' should produce a downwards shift in the sense of first order dominance to the distribution of number of entrants in auction $j-1$ conditional on bidder i wins with a type x. This increase of entry explains why the expected price that i pays increases.

We apply the results of last lemma to show that there is a unique solution to the conditions of Lemma 4.4. We start by proving the existence of an implicit function that relates y_J and y_{J-1}.

Lemma 4.6. *If $r_J > r_1$, then for each $y_{J-1} \in [r_{J-1}, 1]$, there exists a unique function $\psi_J(y_{J-1}) \in [y_{J-1}, 1]$ such that $y_J = \psi_J(y_{J-1})$ satisfies condition (ii) and (iii). Moreover, $\psi_J(y_{J-1})$ is continuous and strictly increasing if $\psi_J < 1$, and satisfies $\psi_J(y_{J-1}) = y_{J-1}$, if $r_{J-1} = r_J$.*

Proof. Define the function $\Delta(y_J) \equiv \Psi_{J-1}(y_J, y_{J-1}, y_J) - r_{J-1}$ for a given value $y_{J-1} \in$

$[r_{J-1}, 1]$. Lemma 4.5 says that $\Psi_J(x, y_{J-1}, y_J)$ is continuous and strictly increasing in x and in y_J. This implies that $\Delta(.)$ must be continuous and strictly increasing. Since $\Psi_{J-1}(y_{J-1}, y_{J-1}, y_{J-1}) = r_{J-1} \leq r_J$, then $\Delta(y_{J-1}) \leq 0$. Hence, either: (*) $\Delta(1) > 0$ and then there exists a unique $\psi_J(y_{J-1}) \in [y_{J-1}, 1)$ such that $\Delta(\psi_J(y_{J-1})) = 0$; or (**) $\Delta(1) \leq 0$. In case (*), $y_J = \psi_J(y_{J-1})$ satisfies condition (ii), and in case (**) let $\psi_J(y_{J-1}) = 1$, then $y_J = \psi_J(y_{J-1})$ verifies condition (iii). Lemma 4.5 also shows that ψ_J is strictly increasing under case (*), this is when $\psi_J < 1$. ■

Now, assume that there exist some functions $\{\psi_l\}_{l=j+1}^{J}$ where $y_l = \psi_l(y_{l-1})$ and that have the same properties as ψ_J. The next lemma shows that then there exists a function ψ_j such that $y_j = \psi_j(y_{j-1})$ that relates y_j with y_{j-1} with the same properties.

Lemma 4.7. *SUPPOSE that there exist some functions $\{\psi_l\}_{l=j+1}^{J}$ such that ψ_l : $[r_{l-1}, 1] \to [y_{l-1}, 1]$ and that each function ψ_l gives y_l as a function of y_{l-1}. Assume also that these functions are continuous, and strictly increasing if $\psi_l < 1$.*

THEN, if $r_j > r_1$, for each $y_{j-1} \in [r_{j-1}, 1]$, there exists a unique function $\psi_j(y_{j-1}) \in [y_{j-1}, 1]$ such that $y_j = \psi_j(y_{j-1})$ satisfies condition (ii) and (iii). Moreover, $\psi_j(y_{j-1})$ is continuous and strictly increasing if $\psi_j < 1$, and satisfies $\psi_j(y_{j-1}) = y_{j-1}$, if $r_{j-1} = r_j$.

Proof. The sequence of functions $\{\psi_l\}_{l=j+1}^{J}$ defines each y_l (l>j) as a continuous and increasing function $\omega : [r_j, 1] \to [r_l, 1]$ of y_j where $\omega_l(y_l) \equiv \psi_l \circ \psi_{l-1} \circ ... \circ \psi_{j+1}(y_j)$. The properties of each function ψ_l assure that $y_J \geq y_{J-1} \geq ... \geq y_j$. Then, we can substitute these functions ω_l in the conditions (ii) and (iii), we get the following two conditions:

- If $r_j \neq r_1$ and $y_j < 1$, then $r_j = \Psi_{j-1}(y_j, y_{j-1}, \omega_{j+1}(y_j), ..., \omega_J(y_j))$.
- If $r_j \neq r_1$ and $y_j = 1$, then $r_i \geq \Psi_{j-1}(y_j, y_{j-1}, y_j, \omega_{j+1}(y_j), ..., \omega_J(y_j))$.

We can apply the arguments in the proof of Lemma 4.6 to show that these conditions define the required function with the properties stated in the lemma. ■

Remark 4.1. *The equilibrium cut-off strategy computed in the lemma above is invariant to changes in the indexes of the reserve prices of the vector $\vec{r}$.*

One direct implication of Lemmas 4.6 and 4.7 is that we can prove by induction that there exists a set of increasing functions $\psi_2, ..., \psi_J$ that give y_j as a function of y_{j-1} for all $j > 1$. Hence, according to Lemma 4.4, the first part of the next proposition follows (and so we omit the proof). The second part proves continuity of the equilibrium with respect to the auctioneers' reserve prices.

Proposition 4.1. *There exists a unique symmetric Nash equilibrium of the entry game. The associated cut-offs of this equilibrium are defined by* $y_j = \psi_j \circ \psi_{j-1} \circ ... \circ \psi_2(r_1)$. *These equilibrium cut-offs change continuously with respect to the vector of announced reserve prices* $\vec{r}$.

Proof. Take a convergent (sub-)sequence of vectors of reserve prices $\{\vec{r}_l\}$, where each $\vec{r}_l \in [0,1]^J$ and without loss of generality that the elements of each vector $\vec{r}_l$ are ordered increasingly. Call the limit of this sequence $\vec{r}$. For each of the vectors $\vec{r}_l$ and the limit vector $\vec{r}$ there exists a set of functions $\{\psi_j\}_{j=2}^J$ that determine a unique cut-off strategy characterised by the associated cut-offs. The functions $\{\psi_j\}_{j=2}^J$ are continuous in the cut-offs (Lemma 4.6 and Lemma 4.7). It is straightforward to prove that these functions changes continuously with respect to changes in the reserve prices. Hence, the unique sequence of cut-offs defined by the sequence of functions $\{\psi_j\}_{j=2}^J$ converges to the cut-offs associated to the reserve prices $\vec{r}$. This proves the continuity of the cut-offs with respect to the reserve prices. ■

The importance of this result is that shows that the continuation game that play the bidders after the auctioneers' reserve prices is well behaved. This point is crucial to obtain auctioneers' payoffs functions with nice properties. This result shows that the worries expressed by Peters (1997a) that the equilibrium selection of the entry game could have discontinuities when the number of agents is finite when auctioneers offer mechanisms from a wider class does not hold if we restrict to second price auctions with reserve prices.

From a different perspective, this result shows that the continuation game of our game in which bidders choose an auction characterises a well behaved model of decentralised trade with non-trivial strategic search. Some other papers have provided other continuation games for models of decentralised trade (Peters (1997b)). There are even some other papers that have proved that the same model we propose have nice continuity properties when either auctioneers make homogeneous offers (McAfee (1993)), bidders are homogeneous (Peters and Severinov (1997)), or with an infinite number of agents (Peters (1997a)). But the originality of our solution is that we provide a characterisation of the equilibrium of a continuation game that has nice continuity properties when there is a finite number of agents and both the offers of the seller-auctioneers and the buyer-bidders are heterogeneous.

4.3.1 First Price Auctions

In the main text of this section we have assumed that each of the auctioneers uses a second price auction to allocate the good among those buyers that match with him. We relax this assumption in this subsection and we study entry games in which some or all of the auctioneers conduct a first price auction and the other auctioneers a second price auction. Note that in the case of first price auction, it is relevant if the number of bidders that enters the auction is observable or not. The reason is that the bidder's optimal behaviour depends on the number of other bidders.

We shall show that from the point of view of both the bidders and the auctioneers, the second price auction and the first price auction, with or without observable entry, are equivalent. In order to prove this, we proceed in two steps. First, we verify that for a given entry strategies bidders get in the equilibrium associated to each auction format the same expected utility. Next, we show that the set of symmetric equilibria of the entry game is invariant to changes in the auction format of some of the auctions.

We shall refer to the different auction formats with a set $\mathcal{F} \equiv \{$second price auction, first price auction with observable entry, first price auction with unobservable entry$\}$. Let $f_j \in \mathcal{F}$ be the auction format of a generic auction j, and $\vec{f} \in \mathcal{F}^J$, the vector of auction formats of all the auctions.

Conditional on an entry strategy π that it is used by all the bidders,[7] each auction format specifies a continuation game, that we call bidding game. The strategy for a

[7] Note that we do not study continuation games induced by asymmetric entry strategies, i.e. when not all the bidders use the same entry strategy. They are technically complex since they imply asymmetric first price auctions.

bidder when auction j's format is a first price auction with unobservable entry is a bid function that maps types that enter auction j according to π into bids. If the auction format is a first price auction with observable entry, then the strategy is a bid function that maps types that enter auction j according to π, and number of bidders that enter auction j into bids. For the second price auction we shall assume that each bidder bids her true value of the good. Note that this is the unique symmetric equilibrium of a second price auction.

Lemma 4.8. *Consider the different bidding games generated in an auction j for a fixed entry strategy π, a fixed reserve price r_j, and for each $f_j \in \mathcal{F}$. Then:*

- *There exists a unique symmetric equilibrium for the induced bidding game associated to each auction format $f_j \in \mathcal{F}$. These equilibria are in strictly increasing strategies. They are such that the bidder with highest type that enters auction j wins auction j if her type is weakly above r_j. Otherwise, the auctioneer keeps the good.*
- *Suppose that all the bidders follow the unique symmetric equilibrium of each bidding game. Then, the bidder's expected utility of participating in auction j, conditional on a type x that enters with positive probability auction j according to π, is independent of j's auction format.*

Proof. The first of the points can be proved with an adaptation of the proofs given by Matthews (1995), Section 6, to our model.

We prove the second point starting with a bidder with type y_j, i.e. the infimum of the closure of the set of types that enters with positive probability auction j according to π. If y_j is less than r_j, independently of the auction format, the bidder will bid below r_j and hence, will get zero expected utility. Suppose now that $y_j \geq r_j$. As stated in point one of this lemma, the unique symmetric equilibrium associated to each auction format is in strictly increasing strategies. Hence, a bidder with type y_j only wins if no other bidder enters auction j, and then, she pays the reserve price. This remark is obvious for the second price auction and also for the first price auction with observable entry. In the case of the first price auction with unobservable entry, the result follows because the equilibrium bid of a type y_j in auction j is r_j. This is a consequence of the parallel analysis to Matthews (1995) that we suggested above. This result follows for other types because given the results above, the revenue equivalence theorem implies that the second price auction and the first price auction with observable entry are equivalent in terms of expected utility for bidders conditional on types and conditional on the number of bidders that enter the auction. The same implication holds but unconditional on the level of entry for the second price auction and the first price auction with unobservable entry. Hence, the three auctions are equivalent for the bidders unconditional on the level of entry. ■

We next show that a given entry strategy π is a symmetric equilibrium for a given vector $\vec{r}$ that describes the auctioneers' reserve prices, independently of the auction format that it is used by each auctioneer.

Lemma 4.9. *Consider a family of entry games defined by $\{\vec{r}, \vec{f^l}\}_{\vec{f^l} \in \mathcal{F}^J}$, and the associated bidding games. If bidders play the unique symmetric equilibrium associated to each auction when all the bidders play the same entry strategy, then the set of symmetric equilibrium of the entry games is invariant with respect to the auction format of each auction, i.e. with respect to $\vec{f^l}$.*

Proof. Lemma 4.8 says that the bidders' expected utility is invariant across auction formats if the bidders play a symmetric entry strategy. Hence, we only need to show that for a given symmetric entry strategy π and the symmetric equilibrium of the induced bidding games, if one bidder conditional on a type x_i deviates and enters an auction j that she does not enter according to π, the maximum payoffs in auction j that the bidder can get are independent of the auction format f_j.

The case in which $x_i < r_j$ is trivial. For the other cases, note that in a first price auction, the incentives to increase the bid for a type x_i are weakly below the incentives to increase the bid for types above x_i. Similarly, the incentives to increase the bid for a type x_i are weakly above the incentives to increase the bid for types below x. Hence, if $x_i \in [r_j, y_j]$, the optimal bid for the bidder must be between r_j and the optimal bid of y_j. The same reasoning we use in the proof of Lemma 4.8 for a bidder with type y_j can be used here to show that the optimal bid gives the same expected utility in auction j across auction formats. Finally, if $x_i > y_j$, the consequence of the above argument is that x_i's optimal bid must bid must be between the equilibrium bid of the maximum type below x_i that enters auction j with positive probability, say x_-, and the equilibrium bid of the minimum type above x_i that enters auction j with positive probability if defined, say x_+. If the auction is a first price auction, it is clear that a bidder with type x_+ will submit the same bid as a bidder with type x_-. The reason is that if x_+'s bid were above x_-'s bid, a bidder with type x_+ would have incentives to deviate and decrease her bid. This implies that x_i's optimal bid in auction j must be x_-'s bid in a first price auction. If the auction format is a second price auction x_i's optimal bid is x_i. But note that bidding x_i in a second price auction gives the same expected utility as bidding x_-. Since the revenue equivalence theorem we proof in Lemma 4.8 implies that a bidder with type x_- pays the same expected price and wins with the same probability in the three auction formats, the maximum expected utility that a bidder with type x_i can get in the three auction formats is the same. If x_+ is not defined the proof is similar. Note only that in a first price auction, a bidder with type x_i does not have incentives to bid above x_-'s bid if x_+ is not defined. ■

Corollary 4.1. *Consider a family of entry games defined by $\{\vec{r}, \vec{f^l}\}_{\vec{f^l} \in \mathcal{F}^J}$, and the continuation bidding games. If bidders play the unique symmetric equilibrium associated to each auction when all the bidders play the same entry strategy, then bidders' expected utility conditional on the type and the auctioneers' expected profits, are independent of $\vec{f^l}$ in the unique symmetric equilibrium of the entry game.*

4.4 The Auctioneers' Game

In this section, we study the reduced game of competition among auctioneers. This reduced game is defined by the auctioneers' payoffs evaluated at the unique symmetric Nash equilibrium of the entry game. This equilibrium was characterised in the previous section.

We first describe the expected profit of a generic auctioneer j. For this, we assume that the auctioneer j announces a reserve price r_j, the other auctioneers announce $\vec{r}_{-j}$, and these announcements of reserve prices generate an equilibrium in the entry game characterised by the cut-offs $\pi = \{y_j\}_{j=1}^{J}$. We distinguish three events. The first is that at least two bidders enter the auction. Then the auctioneer's profit is the difference between the value of the second highest bid in the auction and the production cost. The probability that at least two bidders enter auction j and the second highest bid is below a certain value $x \in [y_j, 1]$ equals $z(x;\pi)^{kJ} + kJ(1 - z(x;\pi))z(x;\pi)^{kJ-1} - z(y_j;\pi)^{kJ} - kJ(1-z(y_j;\pi))z(y_j;\pi)^{kJ-1}$. To see why, note that $z(x;\pi)^{kJ}+kJ(1-z(x;\pi))z(x;\pi)^{kJ-1}$ is the probability that no more than one bidder enters auction j with a type above x and $z(y_j;\pi)^{kJ} + kJ(1 - z(y_j;\pi))z(y_j;\pi)^{kJ-1}$ is the probability that no more than one bidder enters auction j. The last two probabilities follow from the fact that the probability that no bidder enters auction j with a type above x equals $z(x;\pi)^{kJ}$ and the probability that one and only one bidder enters auction j with a type above x equals $(1 - z(x;\pi))^{kJ-1}z(x;\pi)^{kJ-1}$.

The second event is that one and only one bidder enters auction j. In this case, the auctioneer's profit equals the difference between the reserve price and the production cost. This last event happens with probability $kJ(1-z(y_j;\pi))z(y_j;\pi)^{kJ-1}$. Finally, the auctioneer gets zero profit if no bidder enters the auction. This event has probability $z(y_j;\pi)^{kJ}$. Thus, auctioneer j's expected profit equals:

$$\Phi(w_j, r_j, r_{-j}) \equiv \int_{y_j}^{1} (x - w_j)\, d\left[z(x;\pi)^{kJ} + kJ(1 - z(x;\pi))z(x;\pi)^{kJ-1}\right] + (r_j - w_j)kJ(1 - z(y_j;\pi))z(y_j;\pi)^{kJ-1}. \quad (4.3)$$

Lemma 4.10. *The auctioneer's payoff function is continuous in w_j, r_j, and r_{-j}.*

Proof. The continuity with respect to w_j is trivial. Next, note that the function z is point-wise continuous with respect to changes in the cut-offs. Moreover, the equilibrium cut-offs change continuously with respect to changes in the vector of reserve prices, (Proposition 4.1). Hence, the function $z(x;\pi)$ is point-wise continuous with respect to the vector of reserve prices. It only remains to be shown that this last result is sufficient to prove continuity of the integral. Point-wise continuity of z is sufficient for set-wise continuity of the set function that generates the measure with respect to which we integrate. Thus, we can apply the generalised Lebesgue Convergence Theorem (see Royden (1988), Proposition 18, p. 270) to prove the continuity of the integral with respect to the cut-offs. ■

The reader can find this result surprising because other papers that study similar models have suggested that the auctioneers' payoff functions could be discontinuous. Peters (1997a) argues that in a game in which auctioneers are allowed to choose auctions from a wider class of mechanisms the equilibrium selection of the entry game could be discontinuous in the reserve prices. We have already discussed in the previous section why this is not the case in our model. Peters and Severinov (1997) proves that the auctioneers' payoffs are discontinuous in the limit game with infinite numbers of auctioneers and bidders. But, the discontinuity that they prove depends crucially on the assumption of infinite number of agents. In this case, even if the bidder's individual behaviour is continuous in the auctioneers' reserve prices, the aggregate behaviour can produce a discontinuity on the level of entry to the auctions. This will be the case when an infinite number of bidders change their entry behaviour with respect to a finite number of auctions.

The continuity result given above allows us to use standard theorems to prove existence of an equilibrium. For this, we consider the mixed extension of the strategy space of the auctioneers. We use Milgrom and Weber's (1985) notion of *distributional strategy.* Milgrom and Weber shows that a distributional strategy is simply another way of representing mixed strategies.[8] Let Π_H be the support of the distribution of auctioneers' types H, then j's distributional strategy is a probability measure μ_j on the set $\Pi_H \times [0, 1]$, such that the marginal distribution on Π_H is the distribution of the auctioneers' types H.

Proposition 4.2. *The auctioneers' reduced game has at least one Nash equilibrium in distributional strategies.*

Proof. We use Milgrom and Weber's (1985) existence theorem (Theorem 1). This

[8] More precisely, Aumann (1964) shows that there is a many-to-one mapping from mixed to behavioral strategies that preserves the players' expected payoffs, and Milgrom and Weber (1985) shows that there is another many-to-one payoff-preserving mapping from behavioral strategies to distributional strategies.

theorem can be used because the set of actions (reserve prices) and types (production costs) are compact metric spaces, auctioneers' types are statistically independent across auctioneers, and the auctioneer's payoff function is continuous in the auctioneer's production cost and the vector of reserve prices (Lemma 4.10). ■

With this proposition we complete the analysis of the finite game.

4.5 Limit Results

In this section we study the convergence properties of the equilibrium set of the reduced game of competition among auctioneers when the numbers of auctioneers and bidders go to infinity. As we explained in last section, this reduced game is obtained by substituting into the auctioneers' payoff functions the unique symmetric equilibrium strategies of the bidders' game.

We shall proceed in four steps. First, we compute the limit of the cut-offs that characterise the unique symmetric equilibrium of the entry game. Second, we shall use these limits to compute the limit of the auctioneers' payoff functions. Third, we show that in the limit game defined by these payoff functions, for each auctioneer the unique best response to most of the other auctioneers' announcements of reserve prices is to set a reserve price equal to the auctioneer's production cost. In fact, we shall show that this is the unique weakly dominant strategy in the game defined with the limit payoff functions.

Finally, we use the results about the limit payoff functions to deduce that we can rule out certain strategies in the finite game, provided that J is large enough. We then show that this process gives a precise equilibrium prediction: as J tends to infinity, almost all auctioneers with production costs low enough to get positive surplus from trade announce a reserve price equal to their production cost with probability arbitrary close to one.

Note that by working with the limit payoffs we avoid dealing with the more complex payoff functions of the finite game. Payoffs in the finite game are complex because the change of an auctioneer's reserve price produces not only a direct effect on the cut-off associated to this auction but also a complex indirect effect on the other cut-offs. To see why the change of one auction's reserve price should affect all the auctions' cut-offs note the following argument. When an auctioneer changes his reserve price he affects the entry decisions of some types. This change will be associated to a change in the entry decisions of the same types with respect to some other auctions. These are all the auctions with reserve prices below our auction reserve price. This has an impact on the expected price in such auctions, but it does not change the expected price in all the other auctions. Remember that a feature of the equilibrium is that bidders are indifferent among all the auctions in which they participate. Hence, if a bidder was indifferent between the auctions that have been affected by the change in our auction reserve price and the other auctions, she will no longer be indifferent between both groups of auctions after the change in the reserve price. The indifference conditions required by the equilibrium of the entry game are restored through a complex change in the level of entry to the different auctions, this is, a change in all the equilibrium cut-offs.

In the limit game, with infinite numbers of auctioneers and bidders, the indirect effect that we pointed out in the last paragraph is negligible. The change in the entry decisions of types with respect to one single auction has no effect on the level of entry in each of the other auctions.

In order to simplify the characterisation of the limit of the equilibrium cut-offs when the numbers of auctioneers and bidders go to infinity we shall discretise the auctioneer's strategy space. Under this assumption, we guarantee that in the limit when the numbers of auctioneers and bidders go to infinity there will be only a finite number of different reserve prices. We can thus use a finite number of conditions similar to conditions (i), (ii), and (iii) in Section 4.3 to characterise the limit of the equilibrium cut-off associated to each reserve price. In fact, we can show that these limit conditions are the limit of a reformulation of the original conditions (i), (ii), and (iii). This approach is more complex when we allow for a continuum of different reserve prices. Since our conditions compare the expected price in two auctions with two adjacent reserve prices, in the limit they typically turn into a complicated differential equation.

In the following we thus assume that the auctioneers choose the reserve price from a given finite subset Π of $[0, 1]$. We also assume that the distribution of the auctioneers' production cost H has support Π_H contained in the set Π. Under this assumption, we can prove that in the limit of the equilibrium of the game, the auctioneers announce reserve prices equal to their production costs with probability one. Otherwise we could only prove that the auctioneers' equilibrium randomisation puts positive probability on the two reserve prices closest to their production costs.

Our first aim is to prove that the equilibrium cut-offs converge when J goes to infinity under some conditions and to characterise their limits. For this, we consider a sequence of entry games played by the bidders in which J is the number of auctioneers and kJ is the number of bidders. Along the sequence we keep $k > 0$ fixed and let J take values in an infinite subset of the natural number, $\mathbb{N}^*$, such that if $J \in \mathbb{N}^*$ then kJ is a natural number. Then we let J tend to infinity, and consider the limit behaviour of the equilibrium entry strategies. To formalise this approach we need additional notation. Instead of referring explicitly to the vector of reserve prices chosen by J auctioneers, it is sufficient to refer to the frequency distribution of reserve prices. Lemma 4.1 shows that this frequency distribution alone determines bidders' equilibrium entry behaviour. For every $J \in \mathbb{N}^*$, we thus denote by $\mathcal{G}^J$ the set of probability distributions that can describe the announcement of reserve prices of J auctioneers. A probability distribution $G^J \in \mathcal{G}^J$ must satisfy the following conditions:

- supp $G^J \subset \Pi$, $\#$ supp $G^J \leq J$; and
- for all $x \in [0, 1]$, $G^J(x) = j/J$ for some $j = 0, 1, ...J$.

We also denote by $\mathcal{G}$ the set of probability distributions with support contained in Π. Note that each set $\mathcal{G}^J$ is a compact subset of $\mathcal{G}$ which is itself compact.

We shall initially concentrate on sequences of entry games such that in each entry game each of the reserve prices in Π is announced by at least one auctioneer. These are games in which the support of the associated distribution function G^J is Π. We define some functions for each of these games. Next, we use these functions to re-formulate in the notation of this section conditions (i), (ii), and (iii) in Lemma 4.4. Remember that these are the conditions that characterise the set of equilibrium cut-offs. We show that the re-formulated conditions converge in an appropriate sense when J tends to infinity to some limit conditions. These limit conditions will be used to prove that the equilibrium cut-offs converge and to characterise their limits. Finally, we extend our analysis and show that the limit conditions that we have provided also assure the convergence of the equilibrium cut-offs for more general sequences of entry games and also characterise their limit.

In the following we denote by R the number of elements of Π and by $\{\hat{r}_l\}_{l=1}^R$ an increasing sequence that describes Π itself. We shall focus on distributions $G^J \in \mathcal{G}^J$ that have support $\{\hat{r}_l\}_{l=1}^R$. According to Lemma 4.1, in order to describe a given cut-off strategy π for a given entry game we only need to specify two things: an increasing sequence of cut-offs $\hat{\pi} \equiv \{\hat{y}_l\}_{l=1}^R$, where $\hat{y}_l$ is the cut-off associated to auctions with reserve price $\hat{r}_l$; and the distribution of reserve prices G^J. We shall denote with $\mathcal{P}$ the set of increasing sequences of R elements in the interval $[0, 1]$.

Hence, for all $\hat{\pi} \in \mathcal{P}$ and $x \geq \hat{y}_1$, we can define the function $\tilde{z}^J(x;\hat{\pi}, G^J) \equiv z(x;\pi)$ for a given entry game described by G^J. Remember that this function specifies the probability that a given bidder either does not submit a bid in an auction with associated cut-off y_l or that she has a type below x. Hence, let $\hat{r}_l$ be the maximum reserve price of the auctions which type x enters and $\hat{y}_l$ its associated cut-off, then:

$$\tilde{z}^J(x;\hat{\pi}, G^J) = 1 - \frac{F(\hat{y}_{l+1}) - F(x)}{J\,G^J(\hat{r}_l)} - \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}) - F(\hat{y}_q)}{J\,G^J(\hat{r}_q)}, \tag{4.4}$$

where $\hat{y}_{R+1} \equiv 1$.

We also provide a new function for the expected price that a bidder pays in an auction $l-1$ conditional on winning with a type $x \geq \hat{y}_{l-1}$. We define this function as $\tilde{\Psi}^J_{l-1}(x;\hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J) \equiv \Psi_{j-1}(x; y_{j-1}, y_j, ...y_J)$, where $\hat{r}_{l-1} = r_{j-1}$ and hence $\hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R$ and G^J are sufficient to describe $y_{j-1}, y_j, ...y_J$. This function can also be computed as the integral:

$$\tilde{\Psi}^J_{l-1}(x;\hat{y}_{l-1},\hat{y}_l,...,\hat{y}_R,G^J)=\int_{-\infty}^{+\infty}\tilde{x}d\hat{\nu}^J_{l-1}|_x, \tag{4.5}$$

where $\hat{\nu}^J_{l-1}|_x(.)$ is a different way of referring to the measure $\nu_{l-1}|_x(.)$, i.e. using the new notation:

$$\hat{\nu}^J_{l-1}|_x(\tilde{x})\equiv\begin{cases}0 & \text{if } \tilde{x}<\hat{r}_{l-1}\\ \frac{\tilde{z}^J(\hat{y}_{l-1};\hat{\pi},G^J)^{kJ-1}}{\tilde{z}^J(x;\hat{\pi},G^J)^{kJ-1}} & \text{if } \hat{r}_{l-1}\leq\tilde{x}<\hat{y}_{l-1}\\ \frac{\tilde{z}^J(\tilde{x};\hat{\pi},G^J)^{kJ-1}}{\tilde{z}^J(x;\hat{\pi},G^J)^{kJ-1}} & \text{if } \hat{y}_{l-1}\leq\tilde{x}<x\\ 1 & \text{otherwise.}\end{cases} \tag{4.6}$$

We use the above functions to re-formulate in Lemma 4.4 the conditions that characterise the equilibrium cut-offs. The unique symmetric equilibrium strategy of an entry game $G^J\in\mathcal{G}^J$ where G^J has support Π is characterised by the unique sequence of cut-offs $\hat{\pi}$ such that for all $\hat{y}_l\in\hat{\pi}$:

(I) If $\hat{r}_l=\hat{r}_1$, then $\hat{y}_l=\hat{r}_1$.

(II) If $\hat{r}_l\neq\hat{r}_1$ and $\hat{y}_l<1$, then $\hat{r}_l=\tilde{\Psi}^J_{l-1}(\hat{y}_l,\hat{y}_{l-1},\hat{y}_l,...,\hat{y}_R,G^J)$.

(III) If $\hat{r}_l\neq\hat{r}_1$ and $\hat{y}_l=1$, then $\hat{r}_l\geq\tilde{\Psi}^J_{l-1}(\hat{y}_l,\hat{y}_{l-1},\hat{y}_l,...,\hat{y}_R,G^J)$.

Note that these conditions in general imply less restrictions than those imposed by conditions (i), (ii), and (iii) in Lemma 4.4. The reason is that we have eliminated those conditions that relate auctions with the same reserve prices. We can do so, because as Lemma 4.1 says, auctions with the same reserve price have the same equilibrium cut-off.

Next, we compute the limit of these conditions when J tends to infinity. For this, we consider a sequence of entry games described by a sequence of distributions of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J\in\mathcal{G}^J$) that converges to a limit distribution of reserve prices $G\in\mathcal{G}$. This will give us some limit conditions that we shall use to prove convergence of the equilibrium cut-offs and to characterise their limit. In order to state the limit of conditions (I), (II), and (III) we first define three functions. We shall show that these functions are limits of the functions $\tilde{z}^J$, $\tilde{\Psi}^J_{l-1}$, and $\hat{\nu}^J_{l-1}|_x$ in an appropriate sense.

We denote with $\underline{r}$ the minimum reserve price in the support of a given distribution $G \in \mathcal{G}$. Note that we need to define the lower bound of the support of G because we do not restrict G to have support Π. We also denote with $\underline{y}$ the cut-off associated to reserve price $\underline{r}$. Then, we define the function $\bar{z}$ for a given increasing sequence $\hat{\pi}$ with R elements in $[0,1]$ (i.e. $\hat{\pi} \in \mathcal{P}$), and a type $x \in [\hat{y}_l, \hat{y}_{l+1}]$ and $x \geq \underline{y}$, as follows:

$$\bar{z}(x; \hat{\pi}, G) \equiv e^{-k\left[\frac{F(\hat{y}_{l+1})-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G(\hat{r}_q)}\right]},$$

where recall that $\hat{y}_{R+1} \equiv 1$. We also define the function $\bar{z}$ to be equal to zero for $x \in [\hat{y}_1, \underline{y})$.

We define the function $\bar{\Psi}_{l-1}$ for a sequence $\hat{\pi} \in \mathcal{P}$, and $x > \hat{y}_{l-1}$, as follows,

$$\bar{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G) \equiv \int_{\hat{y}_{l-1}}^{x} \tilde{x}\, d\bar{\nu}_{l-1}|_x(\tilde{x}),$$

where the probability measure $\bar{\nu}_{l-1}|_x(.)$ is defined below:

$$\bar{\nu}_{l-1}|_x(\tilde{x}) \equiv \begin{cases} 0 & \text{if } \tilde{x} < \hat{r}_{l-1} \\ \frac{\bar{z}(\hat{y}_{l-1};\hat{\pi},G)}{\bar{z}(x;\hat{\pi},G)} & \text{if } \hat{r}_{l-1} \leq \tilde{x} < \hat{y}_{l-1} \\ \frac{\bar{z}(\tilde{x};\hat{\pi},G)}{\bar{z}(x;\hat{\pi},G)} & \text{if } \hat{y}_{l-1} \leq \tilde{x} < x \\ 1 & \text{otherwise,} \end{cases} \tag{4.7}$$

if $x \geq \underline{y}$, and it is defined by a single point with mass one at x for all $x < \underline{y}$.

Lemma 4.11. *Consider a sequence of entry games described by a sequence of distribution functions $\{G^J\}_{J \in \mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to $G \in \mathcal{G}$, and such that each G^J has support Π. Then, for any $\hat{\pi} \in \mathcal{P}$ and $x \in [\hat{y}_1, 1]$:*

$$\tilde{z}^J(x; \hat{\pi}, G^J)^{kJ-1} \xrightarrow{J\to\infty} \bar{z}(x; \hat{\pi}, G). \tag{4.8}$$

$$\hat{\nu}^J_{l-1}|_x(\tilde{x}) \xrightarrow{J\to\infty} \bar{\nu}_{l-1}|_x(\tilde{x}), \tag{4.9}$$

for all $\tilde{x} \in \mathbb{R}$.

$$\tilde{\Psi}^J_{l-1}(x; \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J) \xrightarrow{J\to\infty} \bar{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G), \qquad (4.10)$$

Proof. We start with the following mathematical result:[9] for any sequence $a_J \xrightarrow{J\to\infty} a$, the expression $(1 + a_J/J)^J$ converges to e^a. Thus for any sequence of cut-offs $\hat{\pi} \in \mathcal{P}$ and x such that $x \in [\hat{y}_l, \hat{y}_{l+1})$ and $x \geq \underline{y}$,

$$\left(1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^R \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}\right)^{kJ-1} \xrightarrow{J\to\infty} e^{-k\left[\frac{F(\hat{y}_{l+1})-F(x)}{G(\hat{r}_l} + \sum_{q=l+1}^R \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G(\hat{r}_q)}\right]}, \qquad (4.11)$$

where recall that $\hat{y}_{R+1} = 1$.

If $x \in [\hat{y}_1, \underline{y})$, then $1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^R \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}$ is bounded away from one, hence,

$$\left(1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^R \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}\right)^{kJ-1} \xrightarrow{J\to\infty} 0. \qquad (4.12)$$

The second convergence result in the lemma follows directly from the first result but in the case in which $x < \underline{y}$. In this last case, note that for $\tilde{x} < x$, the quotient $\tilde{z}^J(\tilde{x}, \hat{\pi}, G^J)/\tilde{z}^J(x, \hat{\pi}, G^J)$ is bounded away from one. As a consequent, $\hat{\nu}^J_{l-1}|_x(\tilde{x})$ goes to zero when J tends to infinity.

The last convergence result of the lemma follows because the second result proves convergence of the probability distribution function with respect to which we integrate. Convergence in probability distribution is sufficient for convergence in expectations (Billingsley (1995), Theorem 25.8, p. 335). ∎

[9] This result can be proved using $\left(1 + \frac{a-\epsilon}{J}\right)^J \leq \left(1 + \frac{a_J}{J}\right)^J \leq \left(1 + \frac{a+\epsilon}{J}\right)^J$, for J large enough, and $\left(1 + \frac{a}{J}\right)^J \xrightarrow{J\to\infty} e^a$ for a rational. The last result is provided for instance in White (1968), Exercise 14, p. 93. Continuity assures that the last convergence result is valid for all $a \in \mathbb{R}$.

A consequence of this lemma is that the limit of the conditions (I), (II), and (III) for a sequence of entry games that satisfy the conditions of the lemma are the following conditions:

(i') If $\hat{r}_l \leq \underline{r}$, then $\hat{y}_l = \hat{r}_l$.

(ii') If $\hat{r}_l > \underline{r}$ and $\hat{y}_l < 1$, then $\hat{r}_l = \bar{\Psi}_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G)$.

(iii') If $\hat{r}_l > \underline{r}$ and $\hat{y}_l = 1$, then $\hat{r}_l \geq \bar{\Psi}_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G)$.

Lemma 4.12. *Consider an arbitrary sequence of entry games described by the sequence of distribution functions $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) such that each G^J has support Π and that $\{G^J\}_{J\in\mathbb{N}^*}$ converges to $G \in \mathcal{G}$ when J tends to infinity. Then, the cut-offs that characterise the unique symmetric equilibrium associated to each of the games converge when J tends to infinity, and their limit is characterised by conditions (i'), (ii'), and (iii').*

Proof. We can show that these conditions are defined by functions $\bar{\Psi}_{l-1}$ that are continuous in $x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R$, and that these functions have similar monotonocities to those proved in Lemma 4.5 for Ψ_{l-1}. Hence, we can use the same method as in Section 4.3 to show that conditions (i'), (ii'), and (iii') define implicitly a unique sequence $\hat{\pi} \in \mathcal{P}$. The uniqueness of the solution of (i'), (ii'), and (iii') and the continuity with respect to $\hat{\pi}$ together with the uniqueness of the solution of conditions (I), (II), and (III) and the continuity with respect to $\hat{\pi}$ implies that the solution of conditions (I), (II), and (III), must converge and, actually, to the unique solution of conditions (i'), (ii'), and (iii'). ∎

We complete our analysis of the limit of the equilibrium cut-offs by showing that the limit of equilibrium cut-offs associated to general sequences of entry games is also characterised by the conditions (i'), (ii'), and (iii').

Lemma 4.13. *Consider a reserve price $\hat{r}_l \in \Pi$ that it is announced infinitely often in a sequence of entry games $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to $G \in \mathcal{G}$ when J tends to infinity. Then, the equilibrium cut-off associated to this reserve price converges* *when J tends to infinity and its limit is the l-th entry of the R-dimensional solution of conditions (i'), (ii'), and (iii').*

Proof. We can extend in a trivial way the method above to characterise the equilibrium cut-offs of an arbitrary entry game $G^J \in \mathcal{G}^J$. However, we cannot use the approach above to prove convergence of the equilibrium cut-offs. The problem is that our method is based on the point-wise convergence of some functions that have domain the space of sequences that can describe cut-off strategies. If the support of G^J is constant in J the domain of these functions is the same for all J and then, we can prove our convergence results. However, if we allow for general sequences of distribution functions $\{G^J\}_{J\in\mathbb{N}^*}$ with the only requirement that they converge to G, in general the support of G^J, and hence the space of sequences that describe the associated cut-offs to G^J will change with J. The trick we use is to use the same functions $\tilde{\Psi}^J_{l-1}$ but to extend their domain to the space of sequences with R elements, even if the associated distribution G^J has a support with less elements than R, the cardinality of Π. These functions together with the conditions (I), (II), and (III) will define as above a value to each reserve price in Π. These will in general give more values than equilibrium cut-offs if the support of G^J is smaller than Π. Nevertheless, we shall show that these conditions, with the extended definition of $\tilde{\Psi}^J_{l-1}$, still give us the equilibrium cut-offs associated to all the reserve prices in the support of G^J.

Next, we describe how to extend the functions $\tilde{\Psi}^J_{l-1}(x;\hat{\pi},G^J)$ for distribution functions $G^J \in \mathcal{G}^J$ that do not have support Π. In order to do so, we first extend the definition of the function $\tilde{z}^J$. Note that for $x \geq \hat{y}_i$, where $\hat{y}_i$ is the cut-off associated to the minimum reserve price in the support of G^J, the definition given above for $\tilde{z}^J$ does not depend on the fact that G^J has support π. We thus use this definition to extend the domain of $\tilde{z}^J$ to all $G^J \in \mathcal{G}^J$, and for $x \in [\hat{y}_i, 1]$. With the new definition of $\tilde{z}^J$ we define the measure $\hat{\nu}_{l-1}|_x(.)$ as above and we use this measure to extend the definition of $\tilde{\Psi}_{l-1}$ to all G^J. Note that we can only extend the domain of $\tilde{\Psi}_{l-1}$ with such an approach if $l-1 \geq i$. For $l-1 < i$ we need to evaluate $\tilde{z}^J(x;\hat{\pi},G^J)$ at $x < \hat{y}_i$ in order to construct $\tilde{\Psi}^J_{l-1}$, and we have not defined this function for such points. We thus complete the extension of the definition of $\tilde{\Psi}^J_{l-1}$ by letting $\tilde{\Psi}_{l-1}(x;\hat{\pi},G^J) \equiv x$ for all $l-1<i$. This extension makes condition (II) ($\hat{r}_l = \tilde{\Psi}^J_{l-1}(\hat{y}_l; \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R)$) consistent with condition (i) ($y_1 = r_1$) for the cut-off $\hat{y}_i$. Remember that $\hat{y}_i$ is the cut-off associated to the minimum reserve price in the support of G^J, this is the cut-off y_1 according to the notation in Section 4.3. For other values, note that for finite J they do not represent any cut-off, and for the limit, they are consistent with condition (i').

We can now apply conditions (I), (II), and (III) to an arbitrary entry game described by $G^J \in \mathcal{G}^J$. We next argue that these conditions define a unique sequence $\hat{\pi} \in \mathcal{P}$, and that this sequence is such that the elements that correspond to reserve prices in the support of G^J are in fact the equilibrium cut-offs associated to these reserve prices.

The uniqueness proof is quite similar to that of conditions (i), (ii), and (iii) given in Section 4.3. The only difference is that we only need to apply the inductive argument to construct the solution for reserve prices above $\hat{r}_i$, this is the minimum reserve price in the support of G^J, instead of reserve prices above r_1 as in Section 4.3. Note that we can repeat the arguments in Section 4.3 because for $l-1>i$, $\tilde{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R)$ is continuous in all the variables, strictly increasing in x, strictly decreasing in $\hat{y}_{l-1}$ and weakly increasing in all the other variables.

In order to prove that conditions (I), (II), and (III) applied to an arbitrary distribution function $G^J \in \mathcal{G}^J$ define the actual equilibrium cut-offs, we deduce from these conditions new conditions. We shall show that these conditions are essentially equivalent to conditions (i), (ii), and (iii) in Lemma 4.4, i.e. the conditions that characterise the equilibrium cut-offs.

Conditions (I), and (II) implied that with generality $\hat{y}_i = \hat{r}_i$, this is that the cut-off associated to auctions with the minimum reserve price in the support of G^J equals this minimum reserve price. Note that with different notation, this condition is essentially the same as condition (i).

Consider next two consecutive reserve prices in the support of G^J that are also consecutive in the increasing sequence $\{\hat{r}_l\}_{l=1}^R$ that describes Π. Then condition (II), and condition (III) applied to these reserve prices are essentially the same as conditions (ii) and (iii), respectively. The only difference is that in conditions (II) and (III) the functions $\tilde{\Psi}^J_{l-1}$ depend on values associated to all the reserve prices in Π, whereas conditions (ii) and (iii) only depend on cut-offs associated to reserve prices in the support of G^J. This does not imply any difference because the functions $\tilde{\Psi}_{l-1}$ are actually invariant with respect to changes in the values $\hat{y}_{l-1} \in \hat{\pi}$ that are associated to reserve prices out of the support of G^J. This is clear from the definition of $\tilde{z}^J$, see equation (4.4).

Finally, consider two consecutive reserve prices in the support of G^J such that there are other reserve prices between them in the increasing sequence $\{\hat{r}_l\}_{l=1}^R$ that describes Π. Then, we can substitute recursively conditions (II) and (III) for the reserve prices that are in between and get some new conditions that relate directly the two reserve prices in the support of G^J and their corresponding cut-offs. These conditions are essentially the same as conditions (ii) and (iii), respectively. This completes the proof that Conditions (I), (II), and (III) with an appropriate extension of the definition of $\tilde{\Psi}_{l-1}$ define the equilibrium cut-offs associated to an arbitrary distribution function $G^J \in \mathcal{G}^J$.

The final step is to prove convergence of conditions (I), (II) and (III) in this extended version to the limit conditions (i'), (ii'), and (iii'), when J tends to infinity. This proof is quite similar to the proof that we provide for sequences of distribution functions G^J with support Π. ■

Remark 4.2.

- *For all sequences of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ that converge to the same limit distribution of reserve prices G, the associated equilibrium cut-offs converge to the same limit values.*

- *Changes in one single reserve price in the sequence of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ do not change the limit of the associated equilibrium cut-offs.*

- *In the limit when the numbers of auctioneers and bidders go to infinity, changes in the reserve price of one single auction do not affect neither to the probability that a given bidder with type x wins in any of the other auctions nor to the price that this bidder pays conditional on winning in these other auctions. Hence, in* *the limit changes in the reserve price of one single auction do not affect to the expected utility that bidders can get in the other auctions.*

The last remark proves a property of the limit game that was conjectured by McAfee (1993) to solve his pioneer model. McAfee assumed that each auctioneer computes the payoffs of changing the design of his auction assuming that the expected utility that bidders can get in the mechanisms offered by the other auctioneers is unaffected by the change in his auction design. McAfee admits that in general this assumption is not consistent with the equilibrium analysis of the entry game of bidders when the numbers of auctioneers and bidders are finite. However, McAfee conjectures that it should be consistent with the entry game in which there are infinitely many auctioneers and bidders.

Peters and Severinov (1997) have proved this claim when auctioneers offer second price auctions and where there are no more than two different reserve prices announced by the auctioneers. They have proved McAfee's conjecture as we do. They look to the unique equilibrium of the entry game with finite number of auctioneers and bidders and compute its limit when the numbers of auctioneers and bidders go to infinity. Then, they show that the limit of the unique equilibrium verifies McAfee's conjecture. Our result supersedes Peters and Severinov analysis in the sense that we study entry games in which there are more than two different reserve prices announced by the auctioneers.

Peters (1997) also proves McAfee's conjecture for more than two reserve prices, but his analysis is quite different. Peters starts by defining the limit game that auctioneers play when there is a continuum of auctioneers and bidders. He defines this limit game with some auctioneers' payoff functions computed using some constraints. These constraints are actually the limit of some constraints that must be satisfied by a symmetric equilibrium of the entry game. He shows that these limit constraints imply that there exists a set of equilibria of the limit entry game such that McAfee's conjecture holds within that set. Our analysis improves Peters' approach in the following sense. Peters proves the conjecture based on the limit of equilibrium constraints along a concrete sequence of entry games that converge to a given limit entry game. We show that McAfee's conjecture holds for the limit of the equilibrium of the entry game for all sequences of entry games that converge to a given limit entry game.

Hence, we show that McAfee's conjecture is not only the result of a concrete limit theory, but an outcome independent of our limit approximation. On the other hand, Peters' analysis is more general than us in the sense that he allows for a continuum of different reserve prices, whereas we only consider entry games with finitely many different reserve prices. Moreover, he proves the conjecture allowing although he studies entry games in which all the auctioneers offer a second price auction with a reserve price except one auctioneer that offers a mechanism chosen from a wider set of mechanisms.

The next step is to use the limits of the equilibrium cut-offs to compute the limit of the auctioneer's expected profit. In what follows we shall denote by $\hat{\pi}^J \equiv \{\hat{y}_l^J\}_{l=1}^R$ the unique solution of conditions (I), (II), and (III) for a given entry game $G^J \in \mathcal{G}^J$. For the sake of clarity in the explanation we shall assume that G^J has support Π. We can show that all the arguments that we present next can be extended to other probability distributions in $\mathcal{G}^J$ using a similar approach to that in the proof of Lemma 4.13. Hence, each element $\hat{y}_l^J$ denotes the equilibrium cut-off associated to auctions with reserve price $\hat{r}_l$. We also denote with $\hat{\pi}^*$ the limit of $\hat{\pi}^J$ when J tends to infinity. Let also $\hat{y}_{R+1}^J \equiv 1$ and $\hat{y}_{R+1}^* \equiv 1$.

We next re-formulate the auctioneer's expected profit in order to simplify the computation of its limit. We shall introduce first one new function. This is the probability that exactly one bidder with type weakly above x ($x \geq \hat{y}_j^J$) enters a given auction with reserve price $\hat{r}_j$ given that the auctioneers' reserve prices are described by $G^J \in \mathcal{G}^J$ and that bidders play the equilibrium entry strategy $\hat{\pi}^J$. This function is as follows:

$$\Gamma^J(x; \hat{\pi}^J, G^J) \equiv kJ(1 - \tilde{z}^J(x; \hat{\pi}^J, G^J))\, \tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ-1}. \tag{4.13}$$

We next provide a re-formulation of the expected profit of an auctioneer j (without loss of generality) with production cost w_j and that announces a reserve price $\hat{r}_j \in \Pi$, given that all the auctioneers' reserve prices are described by G^J:[10]

$$\tilde{\Phi}^J(w_j, \hat{r}_j, G^J) \equiv \int_{\hat{y}_j^J}^1 (x - w_j) d\left[\tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ} + \Gamma^J(x;\hat{\pi}^J, G^J)\right] + (\hat{r}_j - w_j)\Gamma^J(\hat{y}_j^J;\hat{\pi}^J, G^J).$$

We shall define a function $\bar{\Gamma}$ that we shall use below to write the limit of the auctioneer's payoffs. We shall show that this function is actually the limit in an appropriate sense of Γ^J when J tends to infinity. This function is defined for a given limit of the equilibrium cut-offs $\hat{\pi}^* \in \mathcal{P}$, a limit distribution of reserve prices $G \in \mathcal{G}$, and a type $x \in [\hat{y}_l^*, \hat{y}_{l+1}^*]$, and $x \geq \underline{r}$, as follows,

$$\bar{\Gamma}(x;G) \equiv k\left(\frac{F(\hat{y}_{l+1}^*) - F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}^*) - F(\hat{y}_q^*)}{G(\hat{r}_q)}\right)\bar{z}(x;\hat{\pi}, G). \tag{4.14}$$

We also define $\bar{\Gamma}(x;G)$ equal to zero for all $x < \underline{r}$. This means that with generality the function $\bar{\Gamma}(x;\hat{\pi}^*, G)$ has a discontinuity in x at point $x = \underline{r}$. Remember that $\underline{r}$ is actually the limit of the equilibrium cut-off associated to $\underline{r}$.

Lemma 4.14. *Consider a given sequence of distributions of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to $G \in \mathcal{G}$, and a type $x \in (\hat{y}_1^*, 1)$ but $x \notin \{\hat{y}_l^*\}_{l=1}^R$, then:*

$$\tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ} \xrightarrow{J\to\infty} \bar{z}(x;\hat{\pi}^*, G),$$

$$\tilde{z}^J(\hat{y}_j^J;\hat{\pi}^J, G^J)^{kJ} \xrightarrow{J\to\infty} \bar{z}(\hat{y}_j^*;\hat{\pi}^*, G),$$

$$\Gamma^J(x;G^J) \xrightarrow{J\to\infty} \bar{\Gamma}(x;G),$$

[10] Note that since G^J also describes the reserve price of auctioneer j, this function must be a probability distribution function in the set $\mathcal{G}^J$ with the property that $\hat{r}_j$ is in the support of G^J. Note that this last restriction is guaranteed since we have focused in distribution functions G^J with support Π. This restriction should be explicitly considered if we consider situations in which the support of G^J is not Π.

$$\Gamma^J(\hat{y}_j^J; G^J) \xrightarrow{J\to\infty} \bar{\Gamma}(\hat{y}_j^*; G).$$

Remember that for $x < \underline{r}$, $\bar{z}(x; \hat{\pi}^*, G) = 0$, and $\bar{\Gamma}(x; G) = 0$.

Proof. We start with the case $x < \underline{r}$. In this case, $\tilde{z}_J(x; \hat{\pi}^J, G^J)$ is bounded away from 1. Then, there exists an $\eta > 0$, such that, $\lim_{J\to\infty} \tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ} \leq \lim_{J\to\infty}(1-\eta)^{kJ} = 0$. Similarly,

$$\lim_{J\to\infty} \bar{\Gamma}^J(x; G^J) = \lim_{J\to\infty} kJ(1 - \tilde{z}_J(x; \hat{\pi}^J, G^J))\tilde{z}_J(x; \hat{\pi}^J, G^J)^{kJ-1} \leq$$
$$\lim_{J\to\infty} kJ(1-\eta)^{kJ-1} = 0,$$

for an $\eta > 0$.

The last step follows from Rudin (1976), Theorem 3.20 (d), p. 57.

Consider next the case $x > \underline{r}$. According to condition (i'), (ii'), and (iii'), $\hat{y}_l^* < \hat{y}_{l+1}^*$ for all $\hat{y}_l^* < 1$. Hence,[11] if $x \in (\hat{y}_l^*, \hat{y}_{l+1}^*)$,

$$\lim_{J\to\infty} \tilde{z}^J(x; \hat{\pi}^*, G^J)^{kJ} =$$
$$\lim_{J\to\infty}\left(1 - \frac{F(\hat{y}_{l+1}^J) - F(x)}{JG^J(\hat{r}_l)} - \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}^J) - F(\hat{y}_q^J)}{JG^J(\hat{r}_q)}\right)^{kJ} =$$
$$\lim_{J\to\infty}\left(1 - \frac{\frac{F(\hat{y}_{l+1}^J)-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}^J)-F(\hat{y}_q^J)}{G^J(\hat{r}_q)}}{J}\right)^{kJ} =$$
$$e^{-k\left[\frac{F(\hat{y}_{l+1}^*)-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}^*)-F(\hat{y}_q^*)}{G(\hat{r}_q)}\right]} =$$
$$\bar{z}(x, \hat{\pi}^*, G).$$

[11]See footnote 9 for the computation of the following limit.

And using this last result in the fourth step below,

$$\lim_{J\to\infty} \Gamma^J(x;G^J) =$$

$$\lim_{J\to\infty} kJ(1-\tilde{z}_J(x;\tilde{\pi}^J,G^J))\tilde{z}_J(x;\tilde{\pi}^J,G^J)^{kJ-1} =$$

$$\lim_{J\to\infty} kJ\left(\frac{F(\hat{y}^J_{l+1})-F(x)}{JG^J(\hat{r}_l)} + \sum_{q=l+1}^{R}\frac{F(\hat{y}^J_{q+1})-F(\hat{y}^J_q)}{JG^J(\hat{r}_q)}\right)\tilde{z}_J(x;\tilde{\pi}^J,G^J)^{kJ-1} =$$

$$\lim_{J\to\infty} k\left(\frac{F(\hat{y}^J_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R}\frac{F(\hat{y}^J_{q+1})-F(\hat{y}^J_q)}{G^J(\hat{r}_q)}\right)\tilde{z}_J(x;\tilde{\pi}^J,G^J)^{kJ-1} =$$

$$k\left(\frac{F(\hat{y}^*_{l+1})-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R}\frac{F(\hat{y}^*_{q+1})-F(\hat{y}^*_q)}{G(\hat{r}_q)}\right)\bar{z}(x,\hat{\pi}^*,G) =$$

$$\bar{\Gamma}(x;G),$$

We can prove the convergence results for the sequences $\{\tilde{z}^J(\hat{y}^J_j;\tilde{\pi}^J,G^J)^{kJ}\}_{J\in\mathbb{N}^*}$ and $\{\Gamma^J(\hat{y}^J_j;G^J)\}_{J\in\mathbb{N}^*}$ in a similar way. ■

The next result gives us the limit of the auctioneer's expected profits in terms of a function $\bar{\Phi}$. This function is defined for $w_j \in \Pi_H$, $\hat{r}_j \in \Pi$, and $G \in \mathcal{G}$ as follows:

- If $\hat{r}_j < \underline{r}$, where recall that $\underline{r}$ is the minimum reserve price in the support of G, then:

$$\bar{\Phi}(w_j,\hat{r}_j,G) \equiv$$
$$\int_{\underline{r}}^{1}(x-w_j)d\left[\bar{z}(x;\hat{\pi}^*,G)+\bar{\Gamma}(x;\hat{\pi}^*,G)\right] + (\underline{r}-w_j)\left[\bar{z}(\underline{r};\hat{\pi}^*,G)+\bar{\Gamma}(\underline{r};\hat{\pi}^*,G)\right].$$

- If $\underline{r} \le \hat{r}_j < \overline{r}$, where $\overline{r}$ is the minimum reserve price among those reserve prices whose limit equilibrium cut-off equal one, then:

$$\bar{\Phi}(w_j,\hat{r}_j,G) \equiv \int_{\hat{y}_j}^{1}(x-w_j)d\left[\bar{z}(x;\hat{\pi}^*,G)+\bar{\Gamma}(x;\hat{\pi}^*,G)\right] + (\hat{r}_j-w_j)\bar{\Gamma}(\hat{y}_j;\hat{\pi}^*,G).$$

- If $r_j \geq \overline{r}$, then:

$$\bar{\Phi}(w_j, r_j, G) \equiv 0.$$

Lemma 4.15. *Consider a sequence of distribution functions $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$ and has support Π, for all $J \in \mathbb{N}^*$) that converges to $G \in \mathcal{G}$ when J tends to infinity. Then, for $w_j \in \Pi_H$, and $\hat{r}_j \in \Pi$,*

$$\tilde{\Phi}^J(w_j, \hat{r}_j, G^J) \xrightarrow{J\to\infty} \bar{\Phi}(w_j, \hat{r}_j, G).$$

Proof. For $\hat{r}_j \geq \underline{r}$, and using Lemma 4.14 and the Lebesgue convergence theorem (see Royden (1988), Theorem 16, p. 91) in the third step below:

$$\begin{aligned}
&\lim_{J\to\infty} \tilde{\Phi}^J(w_j, \hat{r}_j, G^J) = \\
&\lim_{J\to\infty} \left\{ \int_{\hat{y}_j^J}^1 (x - w_j) d\left[\tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(x; G^J)\right] + (\hat{r}_j - w_j)\Gamma^J(\hat{y}_j^J; G^J) \right\} = \\
&\lim_{J\to\infty} \Big\{ (1 - w_j) - (\hat{y}_j^J - w_j)\left[\tilde{z}^J(\hat{y}_j^J; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(\hat{y}_j^J; G^J)\right] - \\
&\qquad \int_{\hat{y}_j^J}^1 \left[\tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(x; G^J)\right] dx + (\hat{r}_j - w_j)\Gamma^J(\hat{y}_j^J; G^J) \Big\} = \\
&(1 - w_j) - (\hat{y}_j^* - w_j)\left[\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G) - +\bar{\Gamma}(\hat{y}_j^*; G)\right] - \\
&\qquad \int_{\hat{y}_j^*}^1 \bar{z}\left[(x; \hat{\pi}^*, G) + \bar{\Gamma}(x; G)\right] dx + (\hat{r}_j - w_j)\bar{\Gamma}(\hat{y}_j^*; G) = \\
&\qquad \int_{\hat{y}_j^*}^1 (x - w_j) d\left[\bar{z}(x; \hat{\pi}^*, G) + \bar{\Gamma}(x; G)\right] + (\hat{r}_j - w_j)\bar{\Gamma}(\hat{y}_j^*; G) = \\
&\qquad \bar{\Phi}(w_j, \hat{r}_j, G).
\end{aligned}$$

Note that if $\hat{y}_j^* = 1$ the expression above is zero.

Similarly, for $\hat{r}_j < \underline{r}$,

$$\lim_{J\to\infty} \tilde{\Phi}^J(w_j, \hat{r}_j, G^J) =$$
$$\lim_{J\to\infty} \left\{ \int_{\hat{y}_j^J}^1 (x - w_j) d\left[\tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(x; G^J)\right] + (\hat{r}_j - w_j)\Gamma^J(\hat{y}_j^J; G^J) \right\} =$$
$$\lim_{J\to\infty} \left\{ (1 - w_j) - (\hat{y}_j^J - w_j) \left[\tilde{z}^J(\hat{y}_j^J; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(\hat{y}_j^J; G^J)\right] - \right.$$
$$\left. \int_{\hat{y}_j^J}^1 \left[\tilde{z}^J(x; \hat{\pi}^J, G^J)^{kJ} + \Gamma^J(x; G^J)\right] dx (\hat{r}_j - w_j)\Gamma^J(\hat{y}_j^J; G^J) \right\} =$$
$$(1 - w_j) - \int_{\underline{r}}^1 \left[\bar{z}(x; \hat{\pi}^*, G) + \bar{\Gamma}(x; G)\right] dx =$$
$$\int_{\underline{r}}^1 (x - w_j) d\left[\bar{z}(x; \hat{\pi}^*, G) + \bar{\Gamma}(x; G)\right] + (\underline{r} - w_j)\left[\bar{z}(\underline{r}; \hat{\pi}^*, G) + \bar{\Gamma}(\underline{r}; G)\right] =$$
$$\bar{\Phi}(w_j, \hat{r}_j, G).$$

Note that in the third step above we have used the results that $\lim_{J\to\infty} \tilde{z}^J(x; \hat{\pi}^J, G^J) = \bar{z}(x; \hat{\pi}^*, G) = 0$ almost everywhere in x, and $\lim_{J\to\infty} \Gamma^J(x; G^J) = \bar{\Gamma}(x; G) = 0$ almost everywhere in x that were proved in Lemma 4.14. ∎

Next lemma provides a more useful statement of this limit payoff function.

Lemma 4.16.

- *If* $\hat{r}_j < \underline{r}$, *then:*

$$\bar{\Phi}(w_j, \hat{r}_j, G) =$$
$$\int_{\underline{r}}^1 \left(\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*) - w_j\right) d\bar{z}(x; \hat{\pi}^*, G) + (\underline{r} - w_j)\left[\bar{z}(\underline{r}; \hat{\pi}^*, G) + \bar{\Gamma}(\underline{r}; \hat{\pi}^*, G)\right].$$

- *If* $\underline{r} \leq \hat{r}_j < \overline{r}$, *then:*

$$\bar{\Phi}(w_j, \hat{r}_j, G) = \int_{\hat{y}_j^*}^1 \left(\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*) - w_j\right) d\bar{z}(x; \hat{\pi}^*, G).$$

- *If* $r_j \geq \overline{r}$, *then:*

$$\bar{\Phi}(w_j, r_j, G) \equiv 0.$$

Proof. We start remarking that with simple algebra it can be proved that $\bar{\Gamma}(x; G) = \bar{z}(x; \hat{\pi}^*, G) \int_x^1 \frac{1}{\bar{z}(\tilde{x}; \hat{\pi}^*, G)} \, d\bar{z}(\tilde{x}; \hat{\pi}^*, G)$. We next use this result in the fourth step below. Let $r_j \geq \underline{r}$, then:

$$\begin{aligned}
&\bar{\Phi}(w_j, \hat{r}_j, G) = \\
&\int_{\hat{y}_j^*}^1 (x - w_j) d\left[\bar{z}(x; \hat{\pi}^*, G) + \bar{\Gamma}(x; \hat{\pi}^*, G)\right] + (\hat{r}_j - w_j)\bar{\Gamma}(\hat{y}_j^*; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 (x - w_j) d\bar{z}(x; \hat{\pi}^*, G) + \int_{\hat{y}_j^*}^1 (x - w_j) d\bar{\Gamma}(x; \hat{\pi}^*, G) + (\hat{r}_j - w_j)\bar{\Gamma}(\hat{y}_j^*; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 (x - w_j) d\bar{z}(x; \hat{\pi}^*, G) - \int_{\hat{y}_j^*}^1 \bar{\Gamma}(x; \hat{\pi}^*, G) \, dx - (\hat{y}_j^* - \hat{r}_j)\bar{\Gamma}(\hat{y}_j^*; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 (x - w_j) d\bar{z}(x; \hat{\pi}^*, G) - \int_{\hat{y}_j^*}^1 \bar{z}(x; \hat{\pi}^*, G) \int_x^1 \frac{1}{\bar{z}(\tilde{x}; \hat{\pi}^*, G)} \, d\bar{z}(\tilde{x}; \hat{\pi}^*, G) \, dx - \\
&\qquad (\hat{y}_j^* - \hat{r}_j)\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G) \int_{\hat{y}_j^*}^1 \frac{1}{\bar{z}(\tilde{x}; \hat{\pi}^*, G)} \, d\bar{z}(\tilde{x}; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 (x - w_j) d\bar{z}(x; \hat{\pi}^*, G) - \int_{\hat{y}_j^*}^1 \int_{\hat{y^*}_j}^x \frac{\bar{z}(\tilde{x}; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} \, d\tilde{x} \, d\bar{z}(x; \hat{\pi}^*, G) - \\
&\qquad \int_{\hat{y}_j^*}^1 (\hat{y}_j^* - \hat{r}_j) \frac{\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} \, d\bar{z}(x; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 \left[x - \int_{\hat{y^*}_j}^x \frac{\bar{z}(\tilde{x}; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} \, d\tilde{x} - \hat{y}_j^* \frac{\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} + \hat{r}_j \frac{\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} - w_j \right] d\bar{z}(x; \hat{\pi}^*, G) = \\
&\int_{\hat{y}_j^*}^1 \left[\int_{\hat{y^*}_j}^x \tilde{x} \, d\frac{\bar{z}(\tilde{x}; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} + \hat{r}_j \frac{\bar{z}(\hat{y}_j^*; \hat{\pi}^*, G)}{\bar{z}(x; \hat{\pi}^*, G)} - w_j \right] d\bar{z}(x; \hat{\pi}^*, G) = \\
&\qquad \int_{\hat{y}_j^*}^1 \left(\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*) - w_j \right) d\bar{z}(x; \hat{\pi}^*, G).
\end{aligned}$$

The case $r_j < \underline{r}$ can be proved in a similar way. ■

Note that what this lemma says is that in the limit the auctioneer's expected profits are the limit of the expected price paid by the winning bidder in the auction. To see why remember that $\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*, G)$ is the limit of the expected price paid by a bidder with type x that wins auction j and $\bar{z}(x;\hat{\pi}^*,G)$ is the limit of the probability that the maximum type that enters auction j is below x.

Using these limit payoffs we provide the next result that is the heart of the limit result that we give at the end of the section.

Lemma 4.17. *The limit of the auctioneers' expected profit verifies,*

$$\bar{\Phi}(w^j,w^j,G) \geq \bar{\Phi}(w^j,r^j,G), \tag{4.15}$$

for all $r^j \in \Pi \setminus \{w^j\}$. Moreover, the inequality is strict but in the following cases:

(1) When $r^j < \underline{r}$ and $w^j \leq \underline{r}$.

(2) When $r^j, w^j \geq \bar{r}$.

Proof. Remark 4.2 says that changes is the reserve price r_j do not affect to the cut-offs of the other auctions. Hence, the sequence $\hat{\pi} = \{\hat{y}_i\}_{i=1}^R$ remains constant when a single auctioneer alters his reserve price. Consequently, the first part of the lemma follows since $w_j = r_j$ implies that $\bar{\Psi}_j(\hat{y}_j^*,\hat{\pi}^*,G) - w_j = \bar{\Psi}_{j-1}(\hat{y}_j^*,\hat{\pi}^*,G) - w_j = \hat{r}_j - w_j = 0$, and $\bar{\Psi}_{j-1}(x,\hat{\pi}^*,G) - w_j$ is strictly increasing in x.

It is a bit tedious, but mechanical, to show using the results in Lemma 4.15 that the inequality is strict but in the cases that we mention. ■

Corollary 4.2. *In the limit game defined by the limit payoff functions $\bar{\Phi}$, each auctioneer has a unique weakly dominant strategy to announce a reserve price equal to his production cost.*

Lemma 4.17 is, however, insufficient to characterise the convergent properties of the set of equilibria. The problem is that this lemma only establishes a kind of weak dominance that does not necessary holds for J large. We say a kind, because it gives more than the standard definition of weak dominance, it provides strict payoff comparisons up to the thresholds $\underline{r}$ and $\bar{r}$. We shall use these strict payoffs comparison to eliminate strategies in the game for J large and to prove our convergence results.

First, we explain why the strict payoff comparisons do not extend out of the thresholds $\underline{r}$ and $\overline{r}$.

The boundary $\overline{r}$ specifies the minimum reserve price that has a limit cut-off equal to one. Since the equilibrium cut-offs are an increasing function of the reserve prices, this means that all reserve prices above $\overline{r}$ attract no bidder with a probability that tends to one as J goes to infinity. This means that in the limit when J goes to infinity an auctioneer with production cost weakly above $\overline{r}$ will be indifferent among all the reserve prices above $\overline{r}$. Remember that in the limit it is weakly dominant for the auctioneer to announce a reserve price equal to his production cost. Hence, the limit payoff comparisons will not provide sufficient constraints to fix the limit of the strategies of the auctioneers with such production costs. This, however, will not matter much because the probability that such bidders trade in this market vanishes as J goes to infinity.

Definition: We say that a given production cost in the support of the distribution of the auctioneers' types is *tradable in the limit* if this production cost is strictly below the $\overline{r}$ associated to a concrete limit entry game. This limit entry game is generated by a sequence of entry games $\{G^J\}_{J\in\mathbb{N}^*}$ such that G^J converges to H, the distribution of the auctioneers' production costs.

Remark 4.3. *Proposition 4.12 implies that the set of production costs that are tradable in the limit is unique.*

The other important boundary is $\underline{r}$. This is the minimum reserve price that is announced by a positive fraction of auctioneers in the limit when J goes to infinity. The limit auctioneers' payoff function that we computed above is flat for types strictly below $\underline{r}$. This has two consequences: first, it makes the task of computing the limit of the equilibrium of the auctioneers' game more tedious; and second, it limits the reach of our results.

Proposition 4.3. *For all $\epsilon > 0$, the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost with probability greater than ϵ*

conditional on having a production cost tradable in the limit goes to zero as J tends to infinity.

See the proof in the Appendix.

4.6 Conclusions

In this paper we have analysed the multistage game of competition among auctioneers with a finite number of auctioneers and bidders. First, we have proved that the second stage game, the bidder's entry game, has a unique symmetric Nash equilibrium and we have provided a characterisation of the solution. With the unique solution of the entry game we have been able to compute the auctioneers' reduced game. We have shown that this reduced game is nice behaved and hence, we have been able to used standard game theory results to show that the game always has an equilibrium (possibly in mixed strategies).

We think that this result can have two implications. First, it can give light on how to solve similar models of decentralised trade with heterogeneity in both market sizes. Second, it can suggest either how to construct models or how to modify existing models in order to assure the existence of equilibrium even under heterogeneity in both market sides.

We have also connected our results for the finite version of the game with the limit model in which there is a continuum of auctioneers and bidders. In this sense, we have given a result in the spirit of upper-hemicontinuity of the equilibrium correspondence. More precisely, we have shown a kind of convergence of the equilibrium set when the numbers of auctioneers and bidders go to infinity to the equilibrium already computed for the limit version by Peters and Severinov (1997) and Peters (1997a). But, our result is more than a mere upper-hemicontinuity proof. We improve Peters and Severinov's (1997) model by allowing for heterogeneity on both market sides. With respect to Peter's (1997) results, his limit game is computed only for a concrete limit theory, that derived from a non generic approximation of the auctioneers payoffs, whereas we provide a generic convergence result. Moreover, none of these two paper provides a generic uniqueness result for the limit reduced game. Only Peters and Severinov (1997) give a result but for symmetric equilibria. Our analysis proves uniqueness in the limit game without restricting to symmetric equilibria.

The convergence that we have proved complements the competitive results provided by McAfee (1993), Peters (1997a), and Peters and Severinov (1997). It proves the intuitive idea that the larger is the market the less monopolistic distortions will exist. Nevertheless, these results have been provided only for a given equilibrium of the entry game, the symmetric equilibrium. It still remains unclear how robust are these results to other equilibria of the entry game.

4.7 Appendix

Proof of Proposition 4.3. We have provided in Lemma 4.15 the limit of the auctioneer's expected profits for convergence sequences of games in which each of the other auctioneers announces a reserve price. This is, however, insufficient for the proof of the Proposition because in general the equilibrium of the auctioneers' game will involve that the other auctioneers' announcements of reserve prices is given by a probability distribution. Hence, if we want to rule out some strategies from the auctioneer's strategy set we need to consider the limit of the auctioneer's expected profit for convergence sequences in which the other auctioneers randomise among different reserve prices.

Consider an infinite sequence of reduced games of competition among auctioneers defined by the sequence of payoff functions $\{\tilde{\Phi}^J(w_j, \hat{r}_j, G^J)\}_{J\in\mathbb{N}^*}$, each of which corresponds to a reduced game with J auctioneers. Let also $\mu^J \equiv \{\mu_1^J, \mu_2^J, ..., \mu_J^J\}$ be some distributional strategies for each of the auctioneers. Then, j's distributional strategy is a probability measure μ_j^J on the set $\Pi_H \times \Pi$, such that the marginal distribution on Π_H is the distribution of the auctioneers' types H. The empirical distribution of reserve prices generated by these distributional strategies in any play of the game is a random variable given by $\tilde{\mu}^J$ with expectation $\frac{1}{J}\sum_{j=1}^{J}\bar{\mu}_j^J$, where $\bar{\mu}_j^J$ is the marginal distribution of μ_j^J on Π. Let ξ^J be the probability measure that this induces on $\overline{\mathcal{G}}^J$ in the game consisting of J auctioneers. Then, if one generic auctioneer j with production cost w_j announces a reserve price $\hat{r}_j$ (without loss of generality) with probability one, then his expected payoffs equal: $\int_{G^J \in \mathcal{G}^J} \tilde{\Phi}^J(w_j, \hat{r}_j, G^J) d\xi^J(G^J)$.

Lemma 4.18. *Let $\{\mu^J\}$ be any sequence of distributional strategies having the property that $\frac{1}{J}\sum_{j=1}^{J} \bar{\mu}_j^J$ converges to some probability distribution $G \in \mathcal{G}$. Then:*

- *the probability measure ξ^J converges weakly to a measure that assigns point mass one to the distribution G.*
- *if there is one auctioneer j that plays a distributional strategy μ_j^J which marginal distribution on Π puts probability mass one in $\hat{r}_j \in \Pi$ for all $J \in \mathbb{N}^*$, then:*

$$\lim_{J \to \infty} \int_{G^J \in \mathcal{G}^J} \tilde{\Phi}^J(w_j, \hat{r}_j, G^J) d\xi^J(G^J) = \bar{\Phi}(w_j, \hat{r}_j, G).$$

Proof. The reserve prices offered by the auctioneers form a triangular system of row-wise independent random variables. Thus $\sup\left|\tilde{\mu}^J(x) - \sum_{j=1}^{J} \bar{\mu}_j^J(x)\right|$ converges almost surely to zero when J goes to infinity by an extension of the Glivenko-Cantelli theorem, see Shorack and Wellner (1986), Theorem 1, page 105. Almost surely convergence implies that the probability measure ξ^J converges weakly to a measure that assigns point mass one to the distribution G.

From Lemma 4.15, $\tilde{\Phi}^J(w_j, \hat{r}_j, G^J) \xrightarrow{J \to \infty} \bar{\Phi}(w_j, \hat{r}_j, G)$ for any sequence $\{G^J\}_{J \in \mathbb{N}^*}$ such that $G^J \xrightarrow{J \to \infty} G$. Moreover, the distribution ξ^J converges weakly to a degenerate distribution with mass point one in the distribution G. Thus,

$$\lim_{J \to \infty} \int_{G^J \in \mathcal{G}^J} \tilde{\Phi}^J(w_j, \hat{r}_j, G^J) d\xi^J(G^J) = \bar{\Phi}(w_j, \hat{r}_j, G),$$

by Billingsley (1995), Theorem 25.7, page 334. ∎

Assume next that we have an infinite convergent (sub-)sequence of reduced games of competition among auctioneers with increasing numbers of auctioneers and bidders. Each of these games must have at least one Nash equilibrium.[12] Hence, we can always take a subsequence of equilibrium distributional strategies with convergence mean as

[12] Proposition 4.2 shows that all these games have a Nash equilibrium in distributional strategies. Although this proof was done under the assumption that the strategy space is continuous some obvious modifications show that we can apply it also to a discrete strategy space. Nevertheless, given the discrete nature of the auctioneers' strategy space and the auctioneers' private types we can more naturally apply Nash (1950) existence theorem.

J tends to infinity. We shall call the limit distribution function of this mean $G \in \mathcal{G}$. With slight abuse of notation we shall call $\underline{r}$ the minimum reserve price in the support of G, and $\overline{r}$ the minimum reserve price in Π that has an associated limit equilibrium cut-off 1 given G.

Lemma 4.18 shows that in order to study the auctioneer's best response correspondence we can proceed as if the other auctioneers where playing some pure strategies that converges to limit distribution of reserve prices G. We can thus use Lemma 4.17 to show that if the other auctioneers' strategies converge to some distribution G in the above sense we can rule out certain strategies from the auctioneer' strategy set that will not be played in equilibrium. We proceed in three steps.

Step 1: *In equilibrium, if $w < \overline{r}$ and for all $w \in \Pi_H$, no auctioneer announces a reserve price above w conditional on having a production cost w for J large enough.*

Lemma 4.17 says that in the limit when J goes to infinity if an auctioneer has a production cost $w < \overline{r}$, he gets strictly higher expected utility with a reserve price w against the limit distribution of reserve prices G than with any other reserve price strictly above w and weakly above $\underline{r}$. Strictness implies that this should also be true for J large enough. Hence, in equilibrium no auctioneer with production cost w announces a reserve price weakly above $\underline{r}$ and strictly above w for J large enough. We next show that this actually implies the above statement.

The strong law of large numbers implies that when J goes to infinity, with probability one the fraction of auctioneers with production cost w equals the probability measure that a given auctioneer has a production cost w (see Billingsley (1995), Theorem 6.1, page 85). Hence, the finiteness assumption of the support of the distribution of production costs H implies that $\lim_{J\to\infty} P_w^J \geq \underline{r}$, where P_w^J is the maximum reserve price that is announced with positive probability in equilibrium by an auctioneer conditional on a production cost w. According to the paragraph above this implies that $\lim_{J\to\infty} P_w^J \leq w$.

Step 2: *Let $\underline{w}$ be the minimum production cost in the support of the distribution of production costs H. Then, for all w in Π_H such that $\underline{w} < w < \overline{r}$, all auctioneers conditional on having a production cost w announce a reserve price w in equilibrium* *and for J large enough.*[13]

Lemma 4.17 says that in the limit when J goes to infinity if $\underline{r} < w < \overline{r}$, then an auctioneer with a production cost w strictly prefers a reserve price w against the limit distribution of reserve prices G. Strictness implies that this is also true for J large enough. This means that in equilibrium, and for J large enough, all auctioneers with a production cost w such that $\underline{r} < w < \overline{r}$ announce a reserve price equal to w. Hence, we only need to show that $w > \underline{w}$ implies that $w > \underline{r}$.

Since $\underline{w} < \overline{r}$, Step 1 implies that the auctioneers with production cost $\underline{w}$ announce a reserve price below or equal to w in equilibrium and for J large enough. Due to the finiteness of the support of the distribution of production costs, the strong law of large numbers implies that when J goes to infinity, the fraction of auctioneers with production cost $\underline{w}$ is strictly positive with probability one, see Billingsley (1995), Theorem 6.1, page 85. This means that in the limit when J goes to infinity $w > \underline{r}$ for all $w > \underline{w}$.

In step 2, we rule out some strategies that involve reserve prices below the production cost mainly when the production cost is strictly above $\underline{w}$. The impossibility to compare payoffs for J large enough with the limit payoffs when J goes to infinity for reserve prices below or equal to $\underline{r}$ precludes to extend Step 2 to production costs $\underline{w}$. In the next step, we produce a weaker statement for the production cost $w = \underline{w}$ than Step 2 for $w \neq \underline{w}$. This weaker statement is, nonetheless, sufficient for the Proposition.

Step 3: *For all $\epsilon > 0$, the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost with probability greater than ϵ and conditional on a production cost $\underline{w}$ goes to zero as J tends to infinity if $\underline{w} < \overline{r}$.*

Similarly to Step 2, we only need to show that if the conditions in the statement of Step 3 are not met then $\underline{w} > \underline{r}$. Step 1 says that no auctioneer with production cost $\underline{w} < \overline{r}$ announce a reserve price above $\underline{w}$ in equilibrium and for J large enough. Suppose next that there exists an $\epsilon > 0$ such that the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost, i.e. strictly below w, with probability greater than ϵ and conditional on a production cost $\underline{w}$ goes

[13]Note that in this step and in the first step we prove more than required by the Proposition.

to $\delta > 0$ as J tends to infinity. The strong law of large numbers (see Billingsley (1995), Theorem 6.1, page 85) says that the limit of the fraction of auctioneers that announce a reserve price strictly less than $\underline{w}$ is at least $\epsilon\,\delta > 0$ wit probability one. This means that $\underline{w} > \underline{r}$.

In order to complete our proof, we only need to show that Step 1, Step 2, and Step 3 imply the Proposition. Step 1, Step 2, and Step 3 imply for all $\epsilon > 0$ the fraction of auctioneers that announce a production cost different to their production cost conditional on having a production cost below the reserve price $\overline{r}$ tends to zero as J goes to infinity. Lemma 4.17 says that auctioneers with production costs weakly above $\overline{r}$ strictly prefer to announce a reserve price weakly above $\overline{r}$ for J large enough. Hence, the distribution of reserve prices strictly below $\overline{r}$ that are observed in equilibrium when J tends to infinity equals the distribution of production costs strictly below $\overline{r}$. Note that production costs weakly above $\overline{r}$ never trade in the market and hence, they do not actually affect to the level of $\overline{r}$. This implies that the $\overline{r}$ associated to the limit of the equilibrium strategies must be actually equal to the $\overline{r}$ associated to the entry game in which the limit distribution of reserve prices coincides with the distribution of the auctioneers' production costs. This completes the proof of Proposition 4.3. ■

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