Mathematical modelling the formation and evolution of melt ponds on sea ice

A dissertation submitted for the degree of
Doctor of Philosophy

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October 2003
Abstract

Sea ice is formed in the polar oceans by the freezing of seawater. It is a sensitive component of the global climate, and its influence on climate is significant through feedbacks, such as the albedo-feedback mechanism. Melt ponds are pools of melt water formed annually on the Arctic sea ice surface. Melt ponds form because of preferential absorption of solar radiation compared to bare sea ice and snow.

This thesis investigates the evolution of melt ponds on the surface of sea ice, primarily through numerical modelling, and also through simple studies of individual processes that are deemed to be important in the evolution of melt ponds.

The theoretical model of melt ponds upon sea ice developed in this thesis is based upon the mushy layer equations and a relatively simple two-stream radiation model. A sea ice mushy layer consists of a solid ice matrix surrounded by brine. The two-stream radiation model has been used in previous sea ice studies, and incorporates a summertime parameterisation of optical properties based upon SHEBA field data.

Stationary solutions of the model, without melt ponds or snow, are analysed, showing potentially two stationary ice thicknesses for a given set of forcing data. However, a linear stability analysis reveals that only the larger of the two solutions is stable. Fundamental summertime processes are investigated and discussed. These include melt pond thermal stability, melt pond drainage, and the evolution of sea-ice lenses (ice formed at the interface of fresh drained water and the salty ocean). The full melt-pond–sea-ice thermo-radiative model is forced using primarily SHEBA data. Model sensitivity to processes and important parameters is investigated.

The model simulates the evolution of ponds well and demonstrates the importance of radiative effects on summertime evolution. The potential application of the model to GCMs is discussed.
Acknowledgements

I would like to thank my supervisor Daniel Feltham for the opportunity that he gave me to study sea ice at CPOM and his continued support throughout the course of my thesis. He has always given me the encouragement necessary to keep motivated.

I would also like to thank my family for always providing me with books when I wanted them (and when I didn’t) and for encouraging me to go to university.

Without the love, support, and generosity of Gerry, Jo, and Anna I would not have been able to complete my thesis. I am eternally grateful.

Finally, I dedicate this thesis to Alice who inspires me to believe in myself. Her love, kindness, and support are inspirational and always motivate me to achieve my fullest potential.
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Chapter 1

Background

1.1 Introduction to sea ice

Sea ice is formed by the freezing of seawater in the high latitudes of the Northern and Southern Hemispheres. Arctic sea ice exhibits annual variability of areal extent of approximately 7 million square kilometres ranging from about 8 to 15 million square kilometres (Serreze et al. [113]), and Antarctic sea ice exhibits annual variability of areal extent of approximately 14 million square kilometres ranging from about 3 to 17 million square kilometres (Zwally et al. [143]). The thickness of sea ice ranges from a few tens of centimetres in young ice to several metres in dynamically ridged ice. Sea ice acts as a membrane between the atmosphere and the ocean, through which heat, moisture, and momentum are transported.

Sea ice is composed of a relatively pure solid ice matrix surrounded by brine. As seawater freezes to form sea ice, salt is expelled from the growing ice platelets (Weeks and Ackley [129]), although some salt remains trapped in brine pockets within the sea ice. The salt that is rejected into the ocean increases the density of the surface water. The salt enriched surface water can lead to unstable density profiles causing convective overturning that transports oxygen and dissolved carbon dioxide down into the ocean (Wadhams [126]).

Besides brine pockets other sea ice inclusions include air bubbles, sediment from
the ocean and biota, which are photosynthetically active organisms, such as marine algae and diatoms (Ackley [2]).

The sea ice cover evolves through thermodynamic and dynamic effects. Thermodynamic effects cause the sea ice to grow or melt due to variations in atmospheric or ocean forcing and also affect physical properties of the sea ice. Dynamic effects cause the sea ice to move and deform in response to the atmosphere and ocean. For example, the ice cover can diverge leaving areas of open water called leads.

1.2 Importance of sea ice

The polar regions are especially sensitive to changes in climate (Stanhill [119]), it is therefore particularly important to understand the physical processes occurring in these regions. Sea ice plays a significant role in the heat balance (Eicken and Lemke [30]) and salt balance (Aagaard and Carmack [1]) of the polar regions. For example, snow covered sea ice reflects up to ninety percent of the incoming solar energy (its albedo is 0.9), whereas the ocean only reflects about ten percent of the incoming solar energy (its albedo is 0.1). This difference leads to the albedo-feedback mechanism, where a decrease (increase) in ice areal extent leads to more (less) energy being absorbed by the oceans, leading to further decreases (increases) in sea ice extent (Eicken and Lemke [30]). The formation and ablation of sea ice also influences the global thermohaline circulation (Aagaard and Carmack [1]; Wadhams [126]).

Changes in sea ice cover are one of the largest uncertainties in predicting global temperature increase (Carson [17]). Global Circulation Models (GCMs) predict that in the next 100 years there will be a warming in the Arctic of 2–3 times the global mean, and that the thermohaline circulation will weaken (Intergovernmental Panel on Climate Change (IPCC) [53]). The UK Hadley Centre Climate Model (HADCM3) predicts a decrease in Arctic winter ice thickness from approximately 5 m to 1 m for a quadrupling of CO₂ (Gregory and Oerlemans [40]). Consistent with this prediction, recent analysis of submarine track data suggests that the mean
draft of Arctic sea ice decreased from about 3 m to 1.8 m in the second half of the last century (Rothrock et al. [106]). However, such reports, and associated claims that the thinning is due to anthropogenic factors, are difficult to quantify due to fundamental uncertainty in the regional, seasonal and inter-annual variability of Arctic sea ice thickness (McLaren et al. [78]). Understanding the response of sea ice to climate variation is therefore vital in making accurate predictions of potential climate change in the future.

1.3 Melt ponds

Melt ponds are the most distinctive summertime feature of Arctic sea ice, with estimated sea ice coverage ranging from 5–50% (Eicken et al. [28]). Melt ponds are pools of water that form on the surface of sea ice due to increased short-wave radiation. This is in contrast to Antarctic sea ice, where melt ponds are relatively rare (Andreas and Ackley [4]), although recently there is localised observational evidence, from the NW Weddell Sea, to the contrary (Wadhams [126]).

Melt ponds are important because they influence the summertime energy and mass balance through the albedo-feedback mechanism (Ebert and Curry [25]), alter the physical and optical properties of sea ice (Cox and Weeks [21]; Maykut [72]; Perovich et al. [96]; Untersteiner [125]), and are an important factor influencing Arctic summertime ecology (Ferguson et al. [34]; Gradinger [39]). Recent observations suggest that inter-annual variability of sea ice thickness is controlled by summertime melting rather than wind and ocean forcing (Laxon et al. [59]). Therefore, a detailed understanding of summer melt processes is important.

The albedo of pond-covered sea ice is less than bare sea ice and so melt ponds preferentially absorb short-wave radiation. Typical melt pond albedos are in the range of 0.15 to 0.45 (Fetterer and Untersteiner [35]), while typical sea ice/snow albedos are in the range of 0.52 to 0.87 (Perovich [91]). The preferential absorption of short-wave radiation by ponds leads to an increase in their size. This is another example of
the albedo-feedback mechanism (melting leading to lower albedo resulting in more melting). This difference in solar energy absorption between ponds and bare sea ice combined with the natural and forced convection within the ponds explains why the melt rate beneath melt ponds is 2-3 times that of bare sea ice (Bogorodskii [11]; Fetterer and Untersteiner [35]).

The importance of melt ponds during the summer is clear when we consider area-averaged albedos, averaged across different surface types, such as bare sea ice, melt ponds, snow, and leads. An estimate of the area-averaged albedo $\alpha_{avg}$ can be obtained using $\alpha_{avg} = (1-P)\alpha_{ice} + P\alpha_{pond}$, where $P$ is the pond fraction, $\alpha_{ice}$ is the sea ice albedo, and $\alpha_{pond}$ is the melt pond albedo. The Collaborative Interdisciplinary Cryospheric Experiment measured albedo over first-year sea ice in the Wellington Channel (75°N, 94°W) and used these measurements with aircraft videography to determine regional albedo during the melt season (Hanesiak [47]). The regional average was $\alpha_{avg} = 0.55$, with pond fractions reaching $P = 0.35$. These were determined using an airborne albedometer and video camera respectively. Pond fractions should be relatively accurate, since pond albedo correlates well with naked eye observations. However, the estimate of the average albedo is more susceptible to error due to variability in solar illumination, instrument stability, and measurement error. Assuming $\alpha_{ice} = 0.52-0.7$ (Perovich [91]) the average albedo implies $\alpha_{pond} = 0.27-0.61$. The resulting values for $\alpha_{pond}$ are quite different to observed values (e.g. Fetterer and Untersteiner [35]), which is because these estimates are sensitive to the pond fraction $P$ as this term affects the pond albedo non-linearly.

Similar results were obtained using 5 km data from the Advanced Very High Resolution Radiometer (AVHRR) Polar Pathfinder project, which were analysed for the same duration as the Surface Heat Budget of the Arctic (SHEBA) field experiment to determine the regional albedo (SHEBA Satellite Remote Sensing Group [115]). However, measurements determined from satellite data contain errors since they can only measure certain wavelength radiation, have a limited field of view, and must make several assumptions concerning extinction of radiation within the atmosphere and within the ice itself. The estimated area-averaged albedos of about 0.55 are
significantly different to those of bare sea ice, and it is the low melt pond albedos that lower the area-averaged albedo (Maykut [72]).

The role of melt ponds in the evolution of sea ice is important. As well as albedo variations already discussed, they affect salinity variation in sea ice and can drain, releasing fresh water into the upper layers of the ocean. Melt pond drainage is the primary source of desalination of sea ice during summer (Cox and Weeks [21]; Untersteiner [125]) and can lead to the formation of under-ice melt ponds. Draining meltwater alters the energy and mass budget of the sea ice since it can alter the ocean-to-ice heat flux (Notz et al. [86]). Melt ponds are therefore an important factor affecting the mass and energy of sea ice, through their influence on albedo and heat fluxes, and can also affect the salt and heat budget of the ocean mixed layer (Eicken et al. [29]).

1.4 Summertime evolution of Arctic melt ponds

Although the Arctic night (permanent darkness) is broken in February (for a latitude of about 75°), summertime in the Arctic does not begin until late May. By this time the short-wave radiation, and ice and snow temperatures have been raised enough to allow melting to occur (Sturm et al. [120]). Figure 1.1 shows the incoming short-wave radiation from the SHEBA field experiment for 1998. Initial melting of the snow and ice occurs at about day 148 (May 29), and occurs when the incoming short-wave radiation has almost peaked.

The increase in temperature of the sea ice increases the overall internal brine content of the sea ice (Fetterer and Untersteiner [35]). This significantly affects the optical properties of the ice, allowing more absorption and less scattering, and so the albedo decreases (Perovich et al. [96]). However, while the sea ice is still snow covered the albedo remains relatively high. The increasing short-wave radiation melts the surface snow cover leading to meltwater runoff and the initial formation of melt ponds. This meltwater has low/zero salinity since it is of meteorological origin. The
albedo-feedback mechanism causes the melt ponds to increase in size both laterally and vertically.

The rapid development of melt ponds leads to a significant change in the surface albedo. This is demonstrated in figure 1.2, which shows the temporal evolution of three Arctic surfaces at the SHEBA field site (after Perovich et al. [96]): White ice (bare sea ice with its surface above sea level); a light pond (a pond with a high albedo in the optical range); and a dark pond (a pond with a low albedo in the optical range). The figure demonstrates the rapid variation in albedo for the melt ponds: a reduction of 0.5 in 10 days for the snow covered ice transforming to melt ponds; and the comparatively slow variation from snow covered to bare sea ice.

The geometry and coverage of ponds is dictated to some extent by the sea ice type on which they form (Fetterer and Untersteiner [35]). Sea ice can be categorised into two main types: first-year ice; and multi-year ice (Wadhams [126]). First-year ice is ice that has not undergone the summer melt season, whereas multi-year ice is ice that has survived at least one summer melt season. First-year sea ice is relatively smooth whereas multi-year sea ice is hummocky. This difference is because multi-year ice has undergone significant deformation. A further difference is that ponds on
first-year ice are generally more extensive and are generally linear in shape, whereas those on multi-year ice are more circular and separated (Fetterer and Untersteiner [35]).

Since melt ponds have low salinities (Eicken et al. [27]), which are much less than 24.7 parts per thousand (ppt), they have a temperature of maximum density that is greater than their equilibrium freezing temperatures. This means that warm melt pond water will tend to sink, while cool melt pond water will tend to rise. This can lead to an unstable density profile within the pond and turbulent convection can occur, contributing to increased melt rates (Bogorodskii [11]). This natural convection is enhanced by the wind forcing at the surface, and leads to scalping at the sides of the ponds (Fetterer and Untersteiner [35]).

Sea ice is a porous medium and so the pressure head caused by the melt ponds leads to drainage through the sea ice (Eicken et al. [29]). Melt pond drainage is important because it is the main source of multi-year sea ice desalination and is called flushing (Cox and Weeks [21]; Untersteiner [125]). Melt pond drainage can also lead to the development of under-ice melt ponds. This is because the warm, low salinity, meltwater that drains through the ice to the ocean can become supercooled and freeze. The meltwater that is trapped beneath the ice is called the under-ice


CHAPTER 1

1.6 Thermodynamic sea ice models

Sea ice influences and is influenced by the climate. Previous research has formulated models to describe the thermodynamic response of the sea ice cover to imposed forcing. To a certain degree, sea ice is horizontally homogenous, and the standard methodology for simulating the thermodynamics of sea ice is with one-dimensional...
thermodynamic sea ice models.

Existing one-dimensional thermodynamic models of sea ice include those described in Maykut and Untersteiner [77], Semtner [111], Mellor and Kantha [80], and Ebert and Curry [25]. However, most present day thermodynamic sea ice models used in process studies are derivations and extensions of the thermodynamic sea ice model of Maykut and Untersteiner [77].

Maykut and Untersteiner's [77] one-dimensional thermodynamic sea ice model uses a nonlinear diffusion equation to model heat transport within the sea ice. It also includes internal heating due to penetration of solar radiation. The specific heat and thermal conductivity of the sea ice are prescribed functions of temperature and salinity. Melt pond formation is not included, and any surface melting is assumed to instantaneously run-off. The model was used to simulate the annual evolution of sea ice in the Central Arctic. Model results generally agreed well with field observations (Maykut and Untersteiner [77]). However, the uncertainty in the parameters of any sea ice model means that changing individual parameters is almost always sufficient to allow reasonable simulations to be achieved (Shine and Henderson-Sellers [116]).

Semtner [111] simplified the Maykut and Untersteiner [77] model by removing internal heating due to solar radiation, setting the specific heat and thermal conductivities to be constants, and also set the number of gridpoints in the numerical model to three, to increase computational efficiency for climatic simulation. The simplified model was compared to Maykut and Untersteiner's [77] model and found to produce similar equilibrium ice thicknesses, although the seasonal variation is found to be larger (Semtner [112]). The model was applied to simulations of the Arctic and Antarctic using specified climatological data by Washington et al. [128].

Semtner [111] also outlined a zero-layer formulation, which is where the model only has ice thickness, snow thickness and surface temperature as the prognostic variables. The zero-layer formulation has been adopted by GCMs (e.g. HADCM3), due to its simplicity. However, although the annual average ice thickness is reasonably reproduced, the zero-layer model has been shown to introduce errors in the seasonal
cycle of sea ice (Semtner [112]).

Mellor and Kantha [80] coupled a sea ice model to an ocean model. The sea ice model is an extension of the 3-layer model of Semtner [111], which includes dynamic effects such as ice divergence, and a quasi-melt-pond parameterisation that consists of a layer of standing water. Including this standing water layer (as compared to allowing all of the surface melt to run into the ocean) increases the mean ice thickness, because of the increased availability of meltwater at the end of summer, which can readily be frozen. However, the standing water layer does not influence albedo, which is the primary influence of melt ponds due to the albedo-feedback mechanism.

The model proposed by Ebert and Curry [25] is an extension of Maykut and Untersteiner's [77] thermodynamic model. The heat transport within the sea ice is identical to Maykut and Untersteiner [77], however this model includes a surface-dependent albedo parameterisation, an explicit melt pond parameterisation and a lead parameterisation. The albedo parameterisation is based upon four spectral bands whose albedo varies independently for alternative surface scenarios: dry snow; melting snow; bare sea-ice; melt ponds; and open water. The melt pond parameterisation is based on a simple energy balance based entirely on absorption of solar radiation in the pond and the surface of the sea ice. The model's sensitivity was examined and used to determine important feedbacks that are present in the model. These include two positive feedbacks: the albedo-feedback; and the conduction-feedback, where thinner ice conducts heat from the ocean to the atmosphere more rapidly, leading to earlier onset of melting, which leads to thinner ice.

Ebert and Curry [25] compared the melt pond component of their model to observations. The fractional area of melt ponds was an imposed function chosen to match a reasonably large data set, whereas the pond depth was compared to just two data points of observed pond depths (Ebert and Curry [25]), which was not an accurate assessment of the validity of pond depths for their model. Furthermore, since the pond depth in their model is limited to be less than a maximum depth (0.8 m) the examination of feedbacks for large pond depths may be inappropriate using this
Figure 1.3: Evolution of average melt pond depth of Ebert and Curry's [25] model and SHEBA [95] data. January 1 is given by day 0 model.

Figure 1.3 depicts average pond depths calculated using the Ebert and Curry [25] model against recent observations from the SHEBA field experiment (Perovich et al. [95]). It can be seen that Ebert and Curry [25] overestimated the rate of pond growth and did not account for feedbacks occurring later in the melt season that retard the rate of melting.

Another limitation of Ebert and Curry's [25] model is that it does not consider the time dependency of the sensible heat stored within the melt ponds and the corresponding influence of melt ponds on the surface energy budget. Also, the model does not account for the turbulent convection that occurs within melt ponds, which alters the melt rate beneath the melt pond and the turbulent heat fluxes at the surface (Bogorodskii [11]).

1.7 Aims and preview of model

The physical and optical properties of sea ice are complicated. This is due to the complex composition of sea ice and the intricate morphology of ice, brine, air, solid
salt and other inclusions. Optical properties of sea ice can change dramatically, both spatially and temporally (Perovich [89]). To extrapolate from data it is necessary to use theoretical models, and to gain an intuitive understanding of the important processes, a simple theoretical model is useful, so that the relative influence of different parameters can be assessed.

The central aim of this thesis is to develop a new one-dimensional, thermodynamic melt-pond–sea-ice model that is based on physical laws to represent sea ice that has melt ponds on its surface. The sensitivity of the model to candidate parameters will be examined and the model will also be used to highlight processes that are important in physical models of melt ponds.

Instead of basing the governing equation for temperature within the sea ice on the nonlinear diffusion equation of Maykut and Untersteiner [77], I base the governing equation for temperature on the mushy layer equations. Mushy layers describe binary alloys, which consist of a solid matrix surrounded by its melt. For sea ice the solid matrix is composed of effectively pure ice, and the melt is composed of brine. Utilising the mushy layer equations incorporates an extra parameter, the solid fraction, which describes the local proportion of the ice that is solid. This parameter ensures that energy is conserved at melting interfaces. Previous models such as those developed by Maykut and Untersteiner [77] and Ebert and Curry [25] do not conserve energy at their uppermost melting interfaces (Bitz and Lipscombe [9]). Under the assumptions used in this thesis, the mushy layer equation describing conservation of heat is transformed into a nonlinear diffusion equation, analogous to the nonlinear diffusion equation that Maykut and Untersteiner [77] used to describe conservation of heat.

The melt-pond–sea-ice model replaces the commonly used Beer's law representation of radiative transfer in sea ice with the slightly more complicated two-stream radiation model. This model is advantageous in that the albedo can be calculated from the optical properties and ice thickness, whereas for Beer's law formulations albedo is specified as an external parameter. In the presence of melt ponds a parameterisation of the optical properties is used to simulate the variation caused by
morphological changes to the sea ice during summer.

Important processes included in the sea ice model are: snow cover, simple drainage, and turbulent convection in the melt pond. In the simulation results, these processes are neglected, and compared to simulations that include these processes, to demonstrate their effect on the model. In addition to the melt-pond–sea-ice model simulations, I also address melt season related processes that are of importance to the model. These include a detailed analysis of the stationary behaviour of the sea ice only component of the sea ice model, an estimate of the critical Rayleigh number for the onset of thermal convection, an investigation of drainage, and an investigation of the evolution of sea-ice lenses.

The melt-pond–sea-ice model is primarily focussed on Arctic sea ice, because the forcing data describe Arctic conditions, however, it is also applicable to melt ponds in the Antarctic. With some straightforward modifications, the model could also be applied to other geophysical surface melt processes such as surface melting of glaciers.

1.8 Structure of thesis

In chapter 2, I describe existing thermodynamic and optical theory necessary to understand later chapters. In chapter 3, I apply the two-stream radiation model to a melt pond on sea ice and SHEBA data is used to develop a parameterisation for the optical properties of the sea ice during the summer. In chapter 4, I develop a new thermo-radiative melt-pond–sea-ice model and describe the limitations that are inherent in the model. In chapter 5, I neglect melt ponds and analyse the stationary solutions of the sea ice only component of the melt-pond–sea-ice model to obtain a fuller understanding of the model. It is found that there can be two stationary solutions, however, a stability analysis on a simple conducting solid slab model indicates that only one solution is typically stable. This result is confirmed for the full model by numerical simulation. In chapter 6, I analyse specific processes that
are related to melt pond and sea ice evolution during the summer, namely: thermal stability of a melt pond; drainage rates; and the evolution of sea-ice lenses formed from drainage of melt ponds. In chapter 7, the results of numerical simulations using the melt-pond–sea-ice model are presented. Important physics of melt ponds are highlighted, by showing the results of model runs for which they are neglected. The sensitivity of the melt-pond–sea-ice model to candidate parameters is also examined. In chapter 8, I conclude the thesis with a summary of results, implications, and suggestions for future work.
Chapter 2

Existing Theory

2.1 Introduction

The thermo-radiative melt-pond-sea-ice model that I formulate and examine in this thesis comprises a thermodynamic component that describes how heat is transported within sea ice and melt ponds and a radiative component that describes the radiation field within sea ice and melt ponds. To understand these principal components, I now present background theory for mushy layers and sea ice, which form the basis of the thermodynamic component, and optical properties of sea ice and radiation models used in sea ice research, which form the radiative component.

2.2 Mushy layers and sea ice

A mushy layer formed from a binary alloy is composed of a rigid matrix of solid bathed in its impure melt. Mushy layers form due to morphological instabilities that arise from constitutional supercooling ahead of freezing interfaces (Worster [141]). Figure 2.1 shows the actual temperature and equilibrium temperature profiles ahead of a freezing interface of a solution undergoing constitutional supercooling. Line $T_I$ is the liquidus curve, which describes the equilibrium freezing temperature for the solution at a given concentration, and line $T^*_L$ is the actual temperature of the so-
Constitutional supercooling (grey shading) occurs when $T_a < T_l$. Constitutional supercooling at a freezing interface is caused by differences in the diffusion rate of heat and solute. Solutes typically diffuse much more slowly than heat and so excess solute near the interface depresses the local freezing point and thus causes constitutional supercooling. This almost always causes the instability of a solid–liquid interface leading to the intricate geometry of the mushy layer (Worster [141]).

Examples of alloys in which mushy layers form include sodium-chloride solution, and steel (iron–carbon) (Feltham and Worster [33]). For extensive reviews of mushy layers including solidification and convection see Worster [139, 140, 141].

### 2.2.1 Conservation equations

The equations that describe heat and solute transport within a mushy layer assume that the mushy layer can be described as a continuum. The local temperature and solute concentration are averages across several dendrites. The local proportion of mushy layer that is solid is given by the solid fraction $\phi(z, t)$ at position $z$ and time $t$.
Local conservation of energy within a mushy layer is given by
\[
(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_l U \cdot \nabla T = \nabla \cdot (k_m \nabla T) + \rho_s L \frac{\partial \phi}{\partial t} + \frac{\partial E^R(z,t)}{\partial t},
\]
(2.1)
where \(T(z,t)\) is the local temperature, \(\phi(z,t)\) is the local solid fraction, and \(U\) is the 'Darcy' velocity, defined to be the volume flux of brine per unit of perpendicular, cross-sectional area. Subscripts \(m\), \(l\) and \(s\) represent mushy layer, liquid and solid properties respectively. \((\rho c)_x\) represents the effective volumetric specific heat capacity of component \(x\) and \(k_x\) is the effective thermal conductivity of component \(x\). \(\rho_s L\) is the latent heat of fusion of the solid per unit volume. \(E^R(z,t)\) is a source of internal heating (e.g. penetrating short-wave radiative energy per unit volume).

The equation for conservation of energy (2.1) shows the rate of change of temperature within a control volume at fixed position (first term), due to advection of heat (second term), thermal diffusion (third term), latent heat released (or absorbed) due to a change in brine volume (fourth term), and internal heating (fifth term).

Local conservation of solute is given by
\[
(1 - \phi) \frac{\partial C}{\partial t} + U \cdot \nabla C = \nabla \cdot (D_m \nabla C) + (C - C_s) \frac{\partial \phi}{\partial t},
\]
(2.2)
where \(C\) is the (volumetric) concentration of solute in the liquid, \(C_s\) is the concentration of solute incorporated into the solid, and \(D_m \approx (1 - \phi) D_l\) is the effective solutal diffusion coefficient of the mushy layer, where \(D_x\) is the solutal diffusion coefficient of component \(x\).

Equation (2.2) describes the rate of change of concentration within the local liquid region (first term) due to advection of solute (second term), solutal diffusion (third term), and internal phase change (fourth term).

The most abundant salt in seawater is sodium chloride, although seawater also contains small amounts of other salts such as mirabilite and hydrohalite (Weeks and Ackley [129]). The sodium-chloride–water solution is the binary component alloy most similar to seawater (Weeks and Ackley [129]; Wettlaufer et al. [132]).
Wettlaufer et al. [132] found negligible differences between sea ice grown from seawater and sea ice grown from aqueous solutions of sodium chloride. Maykut and Light [73] also found that radiatively, at temperatures above $-8.2\, ^\circ C$, there is little difference between sea ice formed from seawater and sea ice formed from sodium-chloride–water solution. Since both the thermal and radiative properties of sea ice are well represented by the sodium-chloride–water system, I assume that seawater can be approximated by the binary alloy, sodium-chloride–water solution at a salt concentration of 35 ppt.

The equations describing conservation of heat (2.1) and solute (2.2) are coupled by the assumption that the mushy layer is in local thermodynamic equilibrium. For local thermodynamic equilibrium to be maintained the solute must diffuse much faster than the diffusion of variations in sensible and latent heat fluxes through the mushy layer depth. If $\delta$ is the dendrite (platelet) spacing then the interstitial solute transport timescale is $\delta^2/D_l$, where $D_l$ is the diffusivity of the solute. Therefore, to maintain local thermodynamic equilibrium, $\delta^2/D_l$ is required to be much less than the timescale of heat transport through the mushy layer (Feltham and Worster [33]). In sodium chloride solutions the interstitial solute transport timescale is about $1000\, s$, which is less than the typical timescale of thermodynamic variations in sea ice such as diurnal forcing (Feltham and Worster [33]). When sea ice is in local thermodynamic equilibrium, it can be assumed that the temperature and brine concentration lie on the liquidus curve in the phase equilibrium diagram for seawater, which I will briefly describe.

**Equilibrium Phase Diagrams**

Equilibrium phase diagrams depict the different phases that coexist at thermodynamic equilibrium at different temperatures and pressures in a system with a known bulk composition (Weeks and Ackley [129]). It is assumed that the freezing temperature is insensitive to pressure changes through the depth of sea ice.

Figure 2.2 is a schematic equilibrium phase diagram for a binary alloy. A binary
Figure 2.2: Simplified binary alloy phase equilibrium diagram. Temperature $T_E$ is the eutectic temperature. Curves $l_1$ and $l_2$ are the liquidus curves. Curves $s_1$ and $s_2$ are the solidus curves. For a binary alloy at temperature $T^*$ and bulk concentration $C^*$, the alloy will be a mush (solid and liquid) with solid at concentration $C_s(T^*)$ and liquid at concentration $C_l(T^*)$.

The alloy consists of two components, the solvent and the solute. The eutectic temperature, $T_E$, is the temperature below which the alloy is solid. The curves $l_1$ and $l_2$ are the liquidi and the curves, $s_1$ and $s_2$ are the solidi. The liquidi give the temperature at which the alloy melts/freezes as a function of concentration. The solidi give the temperature at which the alloy may form a solid solution as a function of concentration.

Suppose the liquidi are given by $T = T_L(C)$ and the solidi are given by $T = T_S(C)$ (with inverse relationships given by $C = C_L(T)$ and $C = C_S(T)$ respectively), then at a point $(C^*, T^*)$ the binary alloy will be composed of liquid at concentration $C_L(T^*)$ in thermodynamic equilibrium with solid at concentration $C_S(T^*)$ (see figure 2.2).

For most aqueous solutions the solidi are almost vertical, which means that almost no salt is incorporated into solid solution (Weeks and Ackley [129]). Therefore, almost no salt is incorporated directly into lattice sites in the atomic structure of the ice crystal, and so the line $s_1$ is almost vertical and coincident with $C = 0$. 

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The equilibrium phase diagram for seawater is shown in figure 2.3. It was published by Assur [6], based upon the work of Nelson and Thompson [85] (Schwerdtfeger [110]). The nonlinearity occurs due to precipitation of different salts (e.g. mirabilite $\text{Na}_2\text{SO}_4 \cdot 10\text{H}_2\text{O}$ at $-8.2^\circ\text{C}$). Since it is assumed that the sea ice is in local thermodynamic equilibrium the temperature and brine concentration lie on the liquidus.

2.2.2 Boundary conditions

The boundary of a mushy layer is difficult to define since the mushy layer is a composition of dendrites and interstitial fluid. Because mushy region quantities are averaged over several dendrites the boundary of the mushy region is described by a region of similar thickness to the dendrite spacing, with finite extent (Worster...
CHAPTER 2

SECTION 2.2

[141]). The boundary conditions between a mushy layer and a liquid region are then described by suitable jump conditions across the finite interface, so that continuity is not guaranteed \textit{a priori} (Worster [141]).

Conservation of heat across an interface between a mushy region and a liquid region is given by

\[ \rho_s \mathcal{L} \phi \frac{dh}{dt} = k_m \mathbf{n} \cdot \nabla T_m - k_l \mathbf{n} \cdot \nabla T_l, \]

where subscripts \( m \), \( l \), and \( s \), denote quantities of the mushy layer, liquid, and solid phases respectively, \( \rho_s \) is the density, \( \mathcal{L} \) is the latent heat of fusion, \( \phi \) is the solid fraction, \( dh/dt \) is the velocity of the boundary in the direction of the outward pointing normal \( \mathbf{n} \), \( k_x \) is the thermal conductivity of component \( x \), and \( T \) is the temperature.

Conservation of solute across an interface between a mushy region and a liquid region is given by

\[ (C - C_s) \phi \frac{dh}{dt} = D_m \mathbf{n} \cdot \nabla C_m - D_l \mathbf{n} \cdot \nabla C_l, \]

where \( C \) is the concentration of the liquid phase, \( C_s \) is the concentration of the solid phase, and \( D_x \) is the solutal diffusivity of component \( x \).

2.2.3 Application of mushy layers to sea ice

In chapter 1, I briefly described sea ice and some of its processes in relation to climate. Here, I describe how sea ice forms (highly detailed accounts are available in Ackley [2], Wadhams [126], or Weeks and Ackley [129]) and then describe some applications of mushy layers to sea ice.

Sea ice forms at the surface of the polar oceans due to cooling from the atmosphere. The surface cooling creates an unstable density profile within the Arctic surface waters (down to the pycnocline, a density discontinuity) and convective overturning occurs. This convection transports cool dense surface water downwards which is replaced by warmer up-welling water. In brackish water (water with a salinity
less than 24.7 ppt), the temperature of maximum density is above the freezing point and vertical mixing occurs only for temperatures greater than this (Wadhams [126]). Once the surface water has cooled to the freezing point of seawater (from $-1.65^\circ\text{C}$ (32 ppt) to $-1.8^\circ\text{C}$ (35 ppt)) ice crystals can begin to form. The ice crystals that form are almost pure ice, since the equilibrium solute partition coefficient (molar ratio of solute in solid phase to solute in liquid phase) is less than $10^{-4}$ (Weeks and Ackley [129]), and are generally in the form of discs (2–3 mm diameter) (Wadhams [126]). Therefore, for sea ice, $C_s$ is effectively zero. The discs grow laterally and are morphologically unstable forming radial hexagonal crystals, which can break up due to the action of the ocean. This collection of broken crystals is known as frazil ice (or grease ice).

If the sea is calm then the frazil ice can quickly freeze together to form a thin sheet of young ice called nilas (Wadhams [126]). If the sea is rough then the frazil ice can freeze together during compressions within the wave field (Wadhams [126]) to form a coherent slush. As these coherent regions increase in size they form pancake ice, which is the drained slush with a rim formed by accretion of frazil ice. The layer of pancake ice dampens the wave field.

Once a thin sheet of nilas or consolidated pancakes has formed, the ice grows by congelation growth. Congelation growth is where the crystals that are in contact with the ocean grow down into the ocean. The crystal structure of sea ice is hexagonal, with the oxygen atoms arranged so that they are at the centre and vertices of a tetrahedron. This arrangement concentrates the oxygen atoms along planes called basal planes (Weeks and Ackley [129]). The c-axis is the direction perpendicular to the basal planes, and the a-axes are parallel to the basal plane and in the direction of the vertices of the hexagonal arrangement. Crystal growth is preferential in the direction of the a-axes so that the general alignment of the ice crystals occurs with the c-axis horizontal (Wadhams [126]). This kind of crystal growth, with the c-axis horizontal, results in long vertical columnar crystals and is known as congelation or columnar ice.

Pancake ice is typically found within the Antarctic because the strength of the wave
field of the Southern Ocean makes it difficult to form nilas. In the Arctic, conditions are generally more quiescent and lead to more columnar growth originating from nilas. However, in the Odden Ice Tongue in the Greenland Sea, pancake ice is the most prevalent ice type due to the local hydrography.

There have been several empirical and theoretical studies relating mushy layers to sea ice. Wettlaufer et al. [131, 132] conducted experiments that simulated the initial formation of sea ice using sodium-chloride solutions cooled from above. The thin sea ice exhibited critical behaviour: for ice thinner than a critical thickness the dense brine could not overcome the dynamic resistance provided by the ice crystals and the dense brine was wholly incorporated between the interstices of the sea ice; for ice thicker than a critical ice thickness the dense brine was rejected into the melt. This critical behaviour can be explained in terms of the critical conditions for compositional convection within the sea ice (Worster and Wettlaufer [142]). It was also found that as ice increases past the critical thickness the depth averaged solid fraction becomes independent of the initial concentration and is a function only of depth. Wettlaufer et al. [131] relate this last finding to work by Winnebrenner et al. [135], and state that the volume scattering within sea ice is principally determined by the brine fraction. Whilst this is true (see next section), the optical properties are highly dependent on the distribution of the brine within the sea ice, which depends on growth history (Perovich [91]). Therefore, the relationship between solid fraction in thin ice and ice depth can be used to determine the optical properties of the sea ice, but only if the growth history is also known.

Schulze and Worster [108] analysed two-dimensional steady convection in a mushy layer during solidification. Regions of negative solid fraction occurred for which the model is inappropriate. However, these regions of negative solid fraction represent the formation of chimneys and were in fact found to initialise from within the mushy layer, rather than from the boundary. This is in contrast to the study by Maksym and Jeffries [69] who modelled the formation of snow-ice in Antarctica, using a two-dimensional numerical model of a mushy sea ice layer and overlying mushy snow-ice layer. In this study the formation of chimneys was found to occur from the surface.
of the mushy sea ice layer down into its interior, resembling brine drainage channels.

Feltham and Worster [32] investigated the response of a porous mushy layer model to an external parallel flow in its melt, and applied it to the case of sea ice responding to a laminar ocean flow. The flow in the melt over incipient corrugations of the mush–melt interface induced a flow in the mushy layer due to the Bernoulli effect. It was found that the laminar ocean flow induced instability at the mush–melt interface due to the deformation of isotherms by the induced flow in the mushy layer. The instability was compared to an instability investigated by Gilpin et al. [38] that was caused by variations in turbulent heat flux into pure ice from pure water over corrugations of the ice–ocean boundary. Feltham and Worster [32] showed that their instability mechanism requires an ocean shear approximately 125 times smaller than that of Gilpin et al. [38], and anticipate corrugations at the sea-ice–ocean boundary with wavelength of the same magnitude as the sea ice depth.

2.3 Optical properties and radiative models of sea ice

Short-wave radiation plays a critical role in the heat balance of the sea ice cover during summer (Grenfell and Maykut [45]; Maykut and Perovich [74]; Maykut and Untersteiner [77]). Short-wave radiation is distinguished from long-wave radiation by its wavelength. Short-wave radiation has wavelengths less than 4000 nm, whereas long-wave radiation has wavelengths greater than 4000 nm (Maykut [71]).

Observations of sea ice have revealed that the optical properties of sea ice are complex. These properties can change dramatically, both spatially and temporally. Observations show that temperature, brine volume (Buckley and Trodahl [15]; Perovich and Grenfell [93]) and sea ice surface conditions (Grenfell and Maykut [45]) all affect the optical properties. There are also changes in refractive index with temperature, with increased scattering as the temperature decreases (Maykut and Light [73]).
In the sea ice literature there are three main types of optical model that have been examined. These are: the Beer's law model (e.g. Maykut and Untersteiner [77]), which is also referred to as the Bouguert-Lambert law model or Bouguert law model; the two-stream model (e.g. Perovich [89]); and the multi-stream Discrete Ordinates model (e.g. Grenfell [43]). These models have been applied to a variety of case studies ranging from calculations of sea ice surface melting and sea ice bottom surface ablation (Maykut and Untersteiner [77]) to estimates of photosynthetically active radiation (Arrigo et al. [5]).

2.3.1 Optical properties

Incident short-wave radiation on sea ice consists of a direct component (from the Sun) and a diffuse component (from the atmosphere and clouds). Depending on the sky and surface conditions, some of the incoming short-wave radiation is reflected by Fresnel reflection because of differences in the real part of the refractive index of air and sea ice, some is reflected after being scattered internally within the sea ice, some is absorbed inside the sea ice, and some is transmitted through the sea ice into the ocean.

Absorption of radiation is the conversion of radiation into other forms of energy (heat) and is measured by the absorption coefficient, with units per metre. Scattering of radiation results in a change in direction, frequency, or polarisation and is caused either by discontinuities within a medium or by interactions at the atomic level (Telecoms Glossary 2000 [121]). Scattering is measured by the scattering coefficient, with units per metre. In sea ice, scattering usually results from differences in the real part of the indices of refraction of ice, $n_{ice} \approx 1.31$, and internal inclusions such as brine pockets and air bubbles (Perovich [91]). The real part of the index of refraction of a particular medium corresponds to the ratio of speed of light in a vacuum to the speed of light within that medium.

Absorption by air inclusions in the sea ice is negligible, therefore absorption by brine and ice are the most significant factors in determining the absorption coefficient of
sea ice (Perovich [91]). However, there could always be contributions from other impurities, which are generally strongly absorbing and weakly scattering (Perovich [91]). For example, impurities such as sediment and biota are strongly absorbing and lead to the formation of cryoconite holes on the base of melt ponds (Podgorny and Grenfell [102]). Cryoconite holes are localised melt regions on the base of melt ponds that melt faster than the surrounding ice.

Air bubbles, in contrast to their role in absorption, are more strongly scattering than brine pockets. This is clearly demonstrated by the observation on breaking waves – as waves break, air is entrained into the breaking wave resulting in a bright white foam indicating increased scattering compared to the smooth ocean surface. However, the brine pockets are significantly more complicated in their role in scattering, because the real part of their refractive index is dependent upon temperature ($n_{br} \approx 1.397$ at $-32.0$ °C, $n_{br} \approx 1.341$ at $-2$ °C at 589 nm, Maykut and Light [73]).

Further, at cold temperatures precipitated salts form and are also strong scatterers (e.g. mirabilite $Na_2SO_4 \cdot 10H_2O$ precipitates from $-8.2$ °C and hydrohalite $NaCl \cdot H_2O$ precipitates from $-22.9$ °C, see figure 2.3). Studies by Maykut and Light [73] found that refractive indices of sodium-chloride–water solutions and seawater brines were the same for their sub-zero experiments; however, below $-8.2$ °C the bulk properties will be different due to precipitation of the various salts contained in seawater. The scattering coefficient not only depends upon the amount of brine and air inclusions, but also on their distribution in the ice (Perovich [91]). The wavelength dependence of the real part of the refractive index of ice, brine and air is very weak at optical wavelengths (< 4000 nm) and is typically assumed constant with wavelength (Grenfell [42, 43]).

### 2.3.2 Radiative models

The optical models used in sea ice thermodynamic models are important as the mass balance is highly sensitive to the incoming radiation (Fetterer and Untersteiner [35]). I now look in more detail at the three main types of optical model to assess their
Beer's (Bouguert-Lambert) law

Beer's law assumes that the change in radiance at a certain position as radiation traverses through a medium is proportional to the radiance at that position. The radiance is the intensity of the radiation per unit area per unit wavelength per unit solid angle in a particular direction and has units $\text{W m}^{-2}\text{nm}^{-1}\text{sr}^{-1}$. The constant of proportionality is called the extinction coefficient and has units $\text{m}^{-1}$.

Beer's law for a homogenous, plane-parallel medium at wavelength $\lambda$ is therefore given by

$$I_B(z) = I_B(z_0) \exp(-\kappa(z - z_0)),$$

(2.5)

where $I_B(z)$ is the down-welling radiance at position $z$ at wavelength $\lambda$, $z_0$ represents the position of the surface, and $\kappa$ is the extinction coefficient at wavelength $\lambda$. In Beer's law the down-welling radiance must be specified at the surface, so that the albedo must be specified as an external parameter when this model is applied to sea ice.

Maykut and Untersteiner [77] incorporated Beer's law into their thermodynamic model of sea ice. The fraction of incident radiation that is transmitted through the surface layer of sea ice and does not contribute to mass (sublimation/evaporation) or temperature changes at the surface is denoted by $i_0$. However, the value of $i_0$ is far from certain. Maykut and Untersteiner [77] used a value of 17%, even though data from Untersteiner [124] indicated that about 32% of incident radiation did not play a part in surface melting (Maykut and Untersteiner [77]). Grenfell and Maykut [45] measured $i_0$ to be 18–35% of the incident radiation in white ice (ice with its surface above freeboard). This percentage of penetrating solar radiation depends upon the relative proportion of diffuse to direct radiation, since on sunny days the incident radiation has a proportionally larger long-wave component, which is attenuated more rapidly, lowering $i_0$ compared to cloudy days (Ebert and Curry [25]; Perovich [91]). It also clearly depends on the surface characteristics of the sea
ice. For example, if the surface is drained and bubbly with increased scattering, \( i_0 \) will be reduced.

**Two-stream models**

The two-stream photometric model calculates the up-welling and down-welling irradiances in a translucent medium. The irradiance at a specific wavelength is defined to be the radiance at that wavelength integrated over a hemisphere with units \( \text{W m}^{-2} \text{nm}^{-1} \). The two-stream model in the context of polar optics was formulated by Dunkle and Bevans [24], based on the work of Schuster [109], to calculate the up-welling and down-welling irradiances in a snow pack. The two-stream model, in the context of sea ice, was introduced by Grenfell and Maykut [45] to take into consideration the finite depth of sea ice. It was improved upon by Grenfell [41] who considered a two-layer system of snow and sea ice and also included (hitherto disregarded) Fresnel reflection at the air–ice/air–snow interface. More recently, Perovich [89, 90] utilised an \( n \)-layer model similar to Grenfell [41] to account for vertical variations in the optical properties of sea ice.

The two-stream model can be derived directly from the equation of radiative transfer (Meador and Weaver [79]). Since this is the model I will use later, I briefly outline this below.

The equation of radiative transfer equation for diffuse radiation in a plane-parallel medium is given by (Chandrasekhar [18])

\[
\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\mu, \phi; \mu', \phi') I(\tau, \mu', \phi') d\phi' d\mu' \\
- \frac{1}{4} F p(\mu, \phi; -\mu_0, \phi_0) e^{-\tau/\mu_0},
\]

under the assumption of time-independence, elastic scattering (no frequency conversion), no internal sources, and sufficient distance between scattering inclusions, where \( \theta = \cos^{-1} \mu \) and \( \phi \) are spherical polar co-ordinates measured with respect to the outward surface normal and refer to the direction of an incident beam of radiation of intensity \( I \), at optical depth \( \tau = (k + r)z \) (where \( k \) is the absorption
coefficient and \( r \) is the scattering coefficient), \( \pi F \) is the incident flux parallel to the direction of the incident beam of radiation defined by \( \mu_0 = -\cos \theta_0 \) and \( \phi_0 \), and \( p(\mu, \phi; \mu', \phi') \) is the phase function for radiation scattered from \((\mu', \phi')\) into \((\mu, \phi)\).

The phase function \( p(\mu, \phi; \mu', \phi') \) describes the angular distribution of scattered radiation. It gives the rate at which energy is being scattered in direction \((\mu, \phi)\), from direction \((\mu', \phi')\), per unit solid angle.

Equation (2.6) describes the change of intensity of diffuse radiation in direction \((\mu, \phi)\) with respect to the vertical direction (first term) due to absorption (second term), scattering (third term), and direct (unscattered) radiation (fourth term).

Normalising the phase function with \( 4\pi \omega_0 \) (where \( \omega_0 = r/(k + r) \) is the single scattering albedo, which represents the fraction of incident radiation that is scattered), integrating equation (2.6) azimuthally and then integrating hemispherically over the upward and downward hemispheres respectively yields (Meador and Weaver [79])

\[
\frac{dI^-}{d\tau} = -\int_0^1 I(\tau, -\mu)d\mu + \frac{1}{2} \int_0^1 \int_{-1}^1 p(\mu, -\mu')I(\tau, \mu')d\mu'd\mu + \pi F \omega_0 (1 - \beta_0)e^{-\tau/\mu_0} \quad \text{and} \\
\frac{dI^+}{d\tau} = \int_0^1 I(\tau, \mu)d\mu - \frac{1}{2} \int_0^1 \int_{-1}^1 p(\mu, \mu')I(\tau, \mu')d\mu'd\mu - \pi F \omega_0 \beta_0 e^{-\tau/\mu_0},
\]

where

\[
I^\pm = \int_0^1 \mu I(\tau, \pm\mu)d\mu
\]

and

\[
\beta_0 = \frac{1}{2\omega_0} \int_0^1 p(\mu_0, -\mu')d\mu'.
\]

By assuming the \( \mu \) dependence of \( I \) (e.g., the Eddington assumption assumes \( I(\tau, \mu) = A(\tau)\mu + B(\tau) \)) and approximating the integrals, equations (2.7) and (2.8) may be written

\[
\frac{dI^-}{d\tau} = \gamma_2 I^+ - \gamma_1 I^- + \pi F \omega_0 \gamma_4 e^{-\tau/\mu_0} \quad \text{and} \\
\frac{dI^+}{d\tau} = \gamma_1 I^+ - \gamma_2 I^- - \pi F \omega_0 \gamma_3 e^{-\tau/\mu_0},
\]

where
where $\gamma_1$, $\gamma_2$, $\gamma_3$, and $\gamma_4$ are determined from the approximations of the integrals and are independent of the optical depth $\tau$.

In the approach taken by Schuster [109] and Dunkle and Bevans [24] the equation of radiative transfer is not the basis for the development of their two-stream model. They consider a single slab of material that is finite in vertical extent but infinite in horizontal extent. It is also assumed that for each wavelength $\lambda$: the medium is homogenous and of constant optical density; the individual particles that compose the medium are such that the radiation within the medium is perfectly diffuse; the scattering coefficient, $r$, is constant; and the absorption coefficient, $k$, is also constant.

Then the equations that describe the variation of up-welling ($F_u$) and down-welling ($F_d$) irradiance for the two-stream model are given by

$$F_u' = -(k + r)F_u + rF_d, \text{ and}$$  \hspace{1cm} (2.13)

$$F_d' = (k + r)F_d - rF_u, \hspace{1cm} (2.14)$$

where prime denotes differentiation with respect to $z$, $k$ is the absorption coefficient at a given wavelength, and $r$ is the scattering coefficient at a given wavelength (see figure 2.4).

Equation (2.13) describes the change in the down-welling irradiance (first term) because of loss due to absorption and scattering of the down-welling stream (second and third terms) and gain due to scattering of the up-welling stream (fourth term). Equation (2.14) describes the change in the up-welling irradiance (first term) because of loss due to absorption and scattering of the up-welling stream (second and third terms) and gain due to scattering of the down-welling stream (fourth term). Note that the sign change occurs since the up-welling stream is traversing the medium in the negative $z$ direction.

We can derive the two-stream equations of Dunkle and Bevans [24] (equations 2.13 and 2.14) directly from the two-stream equations derived from the equation of radiative transfer (equations 2.11 and 2.12). The phase function is taken to be isotropic (equal scattering in all directions), and the incident radiation is assumed to be dif-
Figure 2.4: Scattering (r m\(^{-1}\)) and absorption (k m\(^{-1}\)) across an infinitesimally thin layer in the two-stream radiation model. Down-welling radiation \(F_1(z)\) is scattered \((rF_1(z)\delta z)\) at depth \(z\), and absorbed across the layer \((kF_1(z)\delta z)\). The resulting down-welling radiation is \(F_1(z + \delta z)\), and includes the backscattered up-welling radiation \(rF_1(z + \delta z)\delta z\). Up-welling radiation \(F_1(z + \delta z)\) is scattered \((rF_1(z + \delta z)\delta z)\) at depth \(z + \delta z\), and absorbed across the layer \((kF_1(z + \delta z)\delta z)\). The resulting up-welling radiation is \(F_1(z)\), and includes the backscattered down-welling radiation \(rF_1(z)\delta z\)
fuse (propagating with equal intensity in all directions) so that there is no incident direct component. These assumptions correspond to: $\gamma_1 = 1; \gamma_2 = \omega_0, \gamma_3 = 0; \text{ and } \gamma_4 = 0$, which can be seen by substituting $p = 4\pi\omega_0, F = 0, \text{ and } I = I(\tau)$ into the equation of radiative transfer (2.6). Identifying $I^+$ with $F_1$ and $I^-$ with $F_1$, and using the definition of the optical depth yields equations (2.13) and (2.14).

Differentiation of equations (2.13) and (2.14) leads to the decoupled equations

$$F''_1 = \kappa^2 F_1 \text{ and } (2.15)$$

$$F''_1 = \kappa^2 F_1, \text{ where } (2.16)$$

$$\kappa = (k^2 + 2kr)^{1/2} . \quad (2.17)$$

is identified as the extinction coefficient. The general solution of equations (2.15) and (2.16) is

$$F_1 = Ae^{\kappa z} + Be^{-\kappa z}, \text{ and } (2.18)$$

$$F_1 = Ce^{\kappa z} + De^{-\kappa z}, \quad (2.19)$$

where $A, B, C$ and $D$ are coefficients dependent upon optical properties and depth.

Multi-stream Discrete Ordinate models

The Discrete Ordinates Method of Chandrasekhar [18] uses Gaussian quadrature and a Legendre expansion of the phase-function coupled with a Fourier cosine expansion of the radiant intensity to break up the integro-differential equation that is the equation of radiative transfer (2.6) into a system of linear differential equations. Gaussian quadrature splits up the radiant intensity field into ‘streams’ defined by zeros of Legendre polynomials and assigns to each stream an appropriate weighting determined by Gauss’ formula.

This model has been applied to cloudy and hazy atmospheres by Liou [61]. Perovitch and Grenfell [94] applied a four-stream model to a single slab of sea ice that involved the use of energy conservation at the atmosphere–ice boundary to overcome inaccuracies that arose from considering refraction.
In 1983, Grenfell [42] applied a 16-stream model to a single slab of sea ice. Grenfell [42] also developed a correlation between the microstructure of the ice to the optical properties using estimates and empirical formulations. A four-stream model was introduced by Grenfell [43] that was based on his earlier 16-stream model. Instead of a single layer, a multi-layer formulation was used. Across the internal layer boundaries refraction was included and found to cause significant alterations to albedo estimates.

Recently there have been attempts to unify the microstructure of sea ice to its optical properties (e.g. Grenfell [42, 43]; Mobley et al. [81]). Grenfell [42] used an evolving model of the microstructure together with empirical relationships to determine the optical properties of first-year sea ice. Grenfell [43] did not utilise the evolving model for first-year ice, but related brine volume, gas volume and solid salt content to density, temperature and salinity in each layer of his multi-layer model. Mobley et al. [81] used experimental data to derive the optical properties from physical properties.

### 2.3.3 Suitability of radiative model

Sea ice optical models have evolved over the past 40 years, beginning with the simple Beer's law single layer models (e.g. Maykut and Untersteiner [77]) through to the more complicated multi-stream, multi-layer formulation (e.g. Grenfell [42, 43]). The development of these models resulted from significant advances in the atmospheric optics field (Joseph et al. [54]; Liou [61]) and also in the snow optics field (Bohren and Barkstrom [12]; Dunkle and Bevans [24]; Warren [127]; Wiscombe and Warren [137]).

Until recently there have been insufficient data to validate the more advanced models. Early field experiments only calculated albedo, absorption, extinction and transmission coefficients for a variety of ice types (e.g. Grenfell [41]; Grenfell and Maykut [45]). Recent field experiments such as Electro-Magnetic Properties Of Sea Ice (EMPOSI, 1994) and SHEBA (1998) have gathered larger amounts of data that are useful
to validate the more complex models (Mobley [81]).

The Beer's law model has the major advantage of being fast to compute, and easy to implement. Only a qualitative description of the ice is needed, rather than a detailed description of scattering inclusions required by more sophisticated models. Also, it is straightforward to incorporate a multi-layer formulation to represent vertical variations of optical properties (e.g. Cheng et al. [20]). Beer's law assumes infinite thickness of a homogeneous medium, and so poorly represents thin ice (Perovich [91]). It is therefore not a good approximation except for a single uniform thick layer of sea ice. Since sea ice exhibits large thickness and physical variation temporally, Beer's law representations have limited validity in large-scale descriptions of sea ice.

The two-stream model (e.g. Grenfell [41]; Grenfell and Maykut [45]; Perovich [89]) has a significant advantage over Beer's law formulations, in that it can account for thin ice (Perovich and Grenfell [93]). It also enables albedo estimates to be determined from the model, whereas Beer's law models must prescribe albedo. Another advantage of the two-stream model is that only a qualitative description of the ice is needed, rather than a detailed description required by more complex models. However, scattering is assumed to be isotropic, even though scattering in sea ice is highly anisotropic (Buckley and Trodahl [14]). Also, it is assumed that the incident radiation is perfectly diffuse. However, during the Arctic summer, low Arctic clouds are prevalent, forming about 70% coverage and making up most of the cloud cover (Makshtas et al. [68]), so that the incoming radiation can be considered diffuse (Perovich [89]). With the lack of information concerning the angular distribution of the radiation field, it is not possible to account for those times when the cloud cover is diminished, so that direct radiation is incident on the surface. However, the influence of the direct radiation on the overall energy budget shouldn't be significant.

The assumption of horizontal homogeneity is also valid, since the horizontal dimension of sea ice floes range from metres to several kilometres (Perovich et al. [99]). Topographic variation exists within the sea ice cover in the form of ridges, although the minimum separation distance between adjacent ridges is of the order of tens of metres (Mai et al. [67]). Therefore, the assumption of horizontal homogeneity is
still valid away from these ridged regions.

The multi-stream Discrete Ordinates Method gives a good description of the radiant intensity field (depending on the number of streams) and can account for refraction and the optical properties of the inhomogeneities in sea ice. There are difficulties when treating refraction (Perovich and Grenfell [94]) and also in specifying the phase function. Advances have been made to try to model the strongly anisotropic phase function (e.g. $\delta$-Eddington method, Joseph et al. [54]; $\delta$-M method, Wiscombe [136]), and these have been found to improve albedo estimates using fewer streams.

Beer’s law is unsuitable for modelling radiative transfer in a melt pond because it assumes that the medium is infinitely thick and there is the requirement of specifying the albedo. The two-stream approach on the other hand can include the effects of multiple reflections from the bottom of the pond, and thin sea ice, and allows one to determine the albedo from the model. Furthermore, its computational simplicity is a significant advantage over the Discrete Ordinates model, especially when incorporating it into a thermodynamic sea ice model. For these reasons the optical model that I will use for the melt-pond–sea-ice model is the two-stream radiation model that has been described in this chapter.
Chapter 3

Application of two-stream radiation model to melt-pond–sea-ice model

3.1 Introduction

In this chapter I apply the two-stream model to sea ice in a similar way to Perovich [89]. Perovich [89] utilised the two-stream model discussed in chapter 2 in a multi-layer model of sea ice. The layers were used to represent vertical inhomogeneity of optical properties. Since I wish to model melt ponds during the summer, and also melt ponds that are freezing at their surfaces, I use a three-layer version of the two-stream model, the three layers being refrozen sea ice, internal melt or melt pond, and sea ice (see figure 3.1).

First, I present some simplifications that can be made to the two-stream radiation model. Second, I describe the boundary conditions that are appropriate for the three-layer radiation model that I will use. Third, I present the actual optical coefficients for the three layer model, and use them to determine the albedo. Fourth, I describe a spectral model, which I will utilise in chapter 5 when determining stationary solutions of a sea-ice-only model. Fifth, I present a parameterisation
of the optical properties of the sea ice, which depends on the pond depth and uses actual SHEBA field data, to enable time dependency in the sea ice optical properties for summertime evolution. Finally, I discuss the relevance and implications of the results of this chapter.

3.2 Simplification of the radiation model

The two-stream radiation model is governed by equations (2.15) and (2.16), with general solution

\[ F_i = A e^{\kappa z} + B e^{-\kappa z}, \text{ and} \]

\[ F_\uparrow = C e^{\kappa z} + D e^{-\kappa z}, \]  

(3.1)

(3.2)

where \( F_i \) and \( F_\uparrow \) are the down-welling and up-welling irradiances, \( \kappa \) is the extinction coefficient, and \( A, B, C, \) and \( D \) are optical coefficients dependent upon optical properties and thickness.

Substituting equations (3.1) and (3.2) into the governing equation for the down-welling radiation (equation 2.13) and equating coefficients yields

\[ (\kappa + k + \tau)A = \tau C, \text{ and} \]

\[ (-\kappa + k + \tau)B = \tau D, \]  

(3.3)

(3.4)

under the restriction \( \tau \neq 0 \). If \( \tau = 0 \) then there is no scattering and the medium through which the radiation travels is purely absorbing. This has been shown to be a valid approximation for melt ponds with a depth less than 1 m (Podgorny and Grenfell [102]). In this purely absorbing case \( \kappa = k \) and the general solution takes the form

\[ F_i = B e^{-\kappa z}, \text{ and} \]

\[ F_\uparrow = C e^{\kappa z}, \]  

(3.5)

(3.6)

which can be found immediately by setting \( \tau = 0 \) in equations (2.13) and (2.14).
Returning to the scattering case \((r \neq 0)\), we can eliminate the scattering coefficient \(r\) between equation (2.17) and equation (3.3) to obtain

\[
\frac{\kappa + k}{\kappa - k} A = C. \tag{3.7}
\]

Let \(s = (\kappa - k)/(\kappa + k)\) then we have that \(A = sC\). Similarly equation (3.4) yields \(sB = D\). The same relationships between the coefficients \(A, B, C,\) and \(D\) can be obtained by starting with the governing equation for the up-welling radiation (equation 2.16). The parameter \(s\) clearly satisfies \(0 \leq s \leq 1\), since the absorption coefficient must be less than or equal to the extinction coefficient by equation (2.17). Typically, for sea ice, the absorption coefficient is much less than the extinction coefficient, and values of \(s\) are usually greater than 0.5 (Grenfell [41]; Grenfell and Perovich [46]).

The general solution for the scattering case, using these relationships, is thus

\[
F_i = sCe^{\kappa z} + \frac{D}{s}e^{-\kappa z}, \quad \text{and} \tag{3.8}
\]

\[
F_1 = Ce^{\kappa z} + De^{-\kappa z}. \tag{3.9}
\]

These facts immediately reduce the order of the problem by a factor of two, since we only need to determine half of the original set of coefficients.

### 3.3 Boundary conditions appropriate to sea ice

The boundary conditions that are used in the three-layer two-stream radiation model are identical to those used by Perovich [89]. Denote the layer closest to the surface (the refrozen melt pond surface) by the suffix 0, the melt pond or internal melt layer by the suffix 1, and the lower sea ice layer by the suffix 2. The position \(z_i\) within each layer is measured relative to the surface of that layer \((i = 0, 1, 2)\). The thickness of each layer is \(h_i\), and the optical properties of each layer are expressed in terms of the two independent parameters, which are the extinction coefficient \(\kappa_i\), and the parameter \(s_i = (\kappa_i - k_i)/(\kappa_i + k_i)\), for \(i = 0, 1, 2\) (see figure 3.1).
At the atmosphere-ice interface both the incident short-wave radiation and the up-welling irradiance $F_{\text{io}}(0)$ can have a Fresnel reflection component, $R_0$ (Perovich [89]). This occurs only at the surface due to the difference between the real part of the index of refraction of air ($n_{\text{air}} \approx 1.0$) and ice ($n_{\text{ice}} \approx 1.31$).

Between the internal layers of ice and internal melt we assume that the difference in the real part of index of refraction of ice and water ($n_{\text{br}} \approx 1.33$) is negligible. Also across each layer boundary the up-welling and down-welling fluxes are assumed to be continuous, so that energy is conserved.

Similarly at the ice-ocean interface it is assumed that there is no Fresnel reflection. Finally, at the ice-ocean boundary the up-welling irradiance is considered negligible, since back-scattering by Arctic Ocean water is extremely small (Smith [118]).

Therefore, the boundary conditions appropriate to the three-layer, two-stream model that we are considering for sea ice are

\begin{align}
F_{\text{io}}(z_0 = 0) &= (1 - R_0)F_{\text{SW}} + R_0 F_{\text{io}}, \\
F_{\text{io}}(z_0 = h_0) &= F_{\text{io}}(z_1 = 0), \\
F_{\text{io}}(z_0 = h_0) &= F_{\text{io}}(z_1 = 0), \\
F_{11}(z_1 = h_1) &= F_{12}(z_2 = 0), \\
F_{11}(z_1 = h_1) &= F_{12}(z_2 = 0), \text{ and} \\
F_{12}(z_2 = h_2) &= 0,
\end{align}

where $F_{\text{SW}}$ is the incident irradiance at a specific wavelength (assumed to be diffuse), and $R_0$ is the Fresnel reflection coefficient (approximately 0.05, Perovich [89]).

### 3.4 Three-layer two-stream model overview

Figure 3.1 shows the system of equations that govern the three-layer, two-stream radiation model when it is applied to the sea ice system we wish to consider (refrozen sea ice/internal melt/sea ice). Substitution of the general solution (3.8) and (3.9) for each layer into equations (3.10)–(3.15) yields six equations in six unknowns, which
### ATMOSPHERE

\[ F_{10}(0) = R_0 F_{10}(0) + (1 - R_0) F_{SW} \text{ at } z_0 = 0 \]

#### ICE (0)

\[ F_{10} = s_0 C_0 e^{\kappa_0 z_0} + D_0 e^{-\kappa_0 z_0} / s_0 \]
\[ F_{10} = C_0 e^{\kappa_0 z_0} + D_0 e^{-\kappa_0 z_0} \]

\[ F_{10}(h_0) = F_{11}(0) \]
\[ F_{10}(h_0) = F_{11}(0) \text{ at } z_0 = h_0, \ z_1 = 0 \]

### POND /

#### INTERNAL MELT (1)

\[ F_{11} = B_1 e^{-\kappa_1 z_1} \]
\[ F_{11} = C_1 e^{\kappa_1 z_1} \]

\[ F_{11}(h_1) = F_{12}(0) \]
\[ F_{11}(h_1) = F_{12}(0) \text{ at } z_1 = h_1, \ z_2 = 0 \]

### ICE (2)

\[ F_{12} = s_2 C_2 e^{\kappa_2 z_2} + D_2 e^{-\kappa_2 z_2} / s_2 \]
\[ F_{12} = C_2 e^{\kappa_2 z_2} + D_2 e^{-\kappa_2 z_2} \]

\[ F_{12}(h_2) = 0 \text{ at } z_2 = h_2 \]

### OCEAN

**Figure 3.1:** Summary of three-layer two-stream model at a specific wavelength, showing up-welling and down-welling radiation and boundary conditions
can be immediately solved to yield

\[ C_0 = -AY_0^2(s_2X_2Y_1^2 - s_0Z_2)/D; \]  
\[ D_0 = As_0(s_0s_2X_2Y_1^2 - Z_2)/D; \]  
\[ B_1 = AY_0Z_2(s_0^2 - 1)/D; \]  
\[ C_1 = As_2X_2Y_0Y_1^2(s_0^2 - 1)/D; \]  
\[ C_2 = A(1 - s_0^2)s_2Y_0Y_1^2/D; \]  
\[ D_2 = As_2(s_0^2 - 1)Y_0Y_1/D, \text{ where} \]  
\[ s_i = (\kappa_i - k_i)/(\kappa_i - k_i); \]  
\[ X_i = (1 - \exp(-2\kappa_ih_i)); \]  
\[ Z_i = (1 - s_i^2\exp(-2\kappa_ih_i)); \]  
\[ Y_i = \exp(-\kappa_ih_i); \]  
\[ A = (1 - R_0)F_{sw}; \]  
\[ B = 1/s_0 - R_0; \]  
\[ U = s_0 - R_0, \text{ for } i = 0, 1, 2, \text{ and} \]  
\[ D = Bs_0(s_0s_2X_2Y_1^2 - Z_2) + UY_0^2(s_0Z_2 - s_2X_2Y_1^2). \]

For simplicity I assume that the optical properties of both sea ice layers (layer 0 and layer 2) are identical. Therefore, \( \kappa_0 = \kappa_2, k_0 = k_2, \text{ and } s_0 = s_2. \) To avoid overflow errors in numerical computations of the radiation model it is sensible to modify the form of the general solution for the up-welling and down-welling irradiance so that the exponentials with positive exponents are expressed in such a way that their exponent is negative (Thomas and Stamnes [122]). For example, \( \exp(\kappa_2z) \) is expressed as \( \exp(\kappa_2(z - h_2))/\exp(-\kappa_2h_2). \)

### 3.5 Albedo and energy

The albedo is an important parameter in terms of sea ice thermodynamics, as it determines how much short-wave energy can penetrate into the ice. It has a signifi-
cant influence on the mass balance in thermodynamic sea ice models (Fetterer and Untersteiner [35]; Maykut and Untersteiner [77]). For completeness, I now express the albedo and absorbed energy in terms of the two-stream model.

### 3.5.1 Albedo

The spectral albedo, \( \alpha \), is the total reflectance for solar radiation at a specific wavelength and is made up of the energy that undergoes Fresnel reflection at the surface plus the energy scattered upward through the surface from within the sea ice. Therefore, in terms of the two-stream model as defined, the spectral albedo is given by

\[
\alpha = R_0 + \frac{(1 - R_0)F_{10}(z_0 = 0)}{F_{SW}}.
\]  

(3.22)

Substituting for the up-welling surface irradiance \( F_{10}(z_0 = 0) \) using the three-layer two-stream model yields the spectral albedo as

\[
\alpha = R_0 + \frac{(1 - R_0)^2(s_2X_0Y_1^2(s_0^2 - Y_0^2) - s_0Z_2X_0)}{B_0(s_0s_2X_0Y_1^2 - Z_2) + UY_0^2(s_0Z_2 - s_2X_0Y_1^2)}.
\]  

(3.23)

The total albedo \( \alpha_{tot} \) is the mean spectral albedo weighted with the spectral incoming irradiance given by

\[
\alpha_{tot} = \frac{\int_0^\infty \alpha F_{SW}(\lambda) d\lambda}{F_{SW(tot)}},
\]  

(3.24)

where \( F_{SW(tot)} = \int_0^\infty F_{SW}(\lambda) d\lambda \), \( F_{SW}(\lambda) \) is the spectral short-wave irradiance, and \( \lambda \) is the wavelength of the short-wave radiation. Therefore, a change in the incident spectral distribution, because of clouds for example, can lead to changes in the total albedo (Perovich [91]). Note that if there is no spectral variation then equation (3.24) implies that \( \alpha_{tot} = \alpha \).

### 3.5.2 Energy

The rate at which radiative energy per unit volume, \( E^R \), is absorbed by the ice depends upon the depth dependence of the net irradiance. The net irradiance at depth \( z_i \) of layer \( i \) at wavelength \( \lambda \) is given by \( F_{net(i)}(z_i, \lambda) = F_{i1}(z_i) - F_{i1}(z_i) \), for
$i = 0, 1, 2$. At a particular depth the rate of change of radiative energy per unit volume is given by the change with depth of the total net irradiance integrated over all wavelengths,

$$
\left( \frac{\partial E^R}{\partial t} \right)_{z_i} = -\frac{\partial}{\partial z} \left( \int_{0}^{\infty} F_{{\text{net}(i)}(z_i, \lambda)} \, d\lambda \right).
$$

(3.25)

In a thermodynamic formulation equation (3.25) is used as the source function in the equation describing conservation of heat (e.g. as the source function in the mushy layer heat equation 2.1).

### 3.6 Spectral considerations

A substantial difficulty when modelling radiative transfer in sea ice is the spectral variation of the optical properties (Perovich et al. [100]) and the incident short-wave radiation (Grenfell [41]). I describe here the methodology for including spectral variation.

I assume that we may divide the short-wave radiation field into $n$ bands of arbitrary bandwidth, across which the optical properties of the sea ice remain constant, and that all wavebands are adjacent. Then the total net short-wave irradiance over all wavelengths in layer $i$ ($= 0, 1, 2$), $F_{{\text{net}(i)}(z_i)}$, becomes

$$
F_{{\text{net}(i)}(z_i)} = \sum_{j=1}^{n} F_j^{ij} \Lambda_j,
$$

(3.26)

where $F_j^{ij}(z_i)$ is the net short-wave flux per unit wavelength of band $j$ with bandwidth $\Lambda_j$ in layer $i$ ($j = 1, \ldots, n$).

The two-stream equations and coefficients for three layers are then also indexed over $j = 1, \ldots, n$ so that

$$
P_j^{ij} = s_j^{ij} C_j^{ij} e^{\kappa_j z_i} + D_j^{ij} e^{-\kappa_j z_i} / s_j^{ij},
$$

(3.27)

$$
P_j^{ti} = C_j^{tj} e^{\kappa_j z_i} + D_j^{tj} e^{-\kappa_j z_i},
$$

(3.28)

and the total albedo (equation 3.24) is therefore given by

$$
\alpha_{\text{tot}} = \frac{\sum_{j=1}^{n} \alpha_j F_{SW(j)} \Lambda_j}{\sum_{j=1}^{n} F_{SW(j)} \Lambda_j},
$$

(3.29)
where $F_{SW(j)}$ is the short-wave radiation per unit wavelength in band $j$, the spectral albedo of band $j$ is given by $\alpha_j = R_0 + (1 - R_0)F_{10}^j(z_0 = 0)/F_{SW(j)}$, for $i = 0, 1, 2$ and $j = 1, \ldots, n$.

3.7 Parameterisation of sea ice optical properties during ponding using SHEBA field data

As melt ponds develop on the surface of sea ice during summer, the underlying sea ice undergoes significant physical changes affecting the optical properties and physical properties such as permeability (Maykut [72]). The three-layer two-stream radiation model in its current form relies on optical properties that do not vary temporally. This is inadequate for summertime.

To allow variation of the sea ice optical properties during summer, I develop a parameterisation in the presence of melt ponds by comparing the two-stream model with spectrally averaged properties to field data from the SHEBA experiment.

3.7.1 SHEBA melt pond data

Total albedo and mass balance results were routinely collected throughout the summer at SHEBA along a 200 m survey line called the Albedo Line. Albedo calculations were determined using a pyranometer and mass balance was determined using thickness stakes and hot-wire mass balance gauges (Perovich et al. [95]). There were approximately 7 ponds along the Albedo line, although some were actually connected regions of larger ponds. Figures 3.2 and 3.3 show melt pond albedos plotted against melt pond depth from the SHEBA data along the Albedo Line for the combined data from all 7 ponds and from one individual pond (50 m along the Albedo Line) respectively. The individual pond located 50 m along the Albedo line was approximately 8 m in diameter with a relatively uniform depth. The albedo data were determined from separate temporal datasets for albedo and mass balance.
Figure 3.2: Combined observations of total albedo and depth at 7 ponds on the Albedo Line at SHEBA obtained from Perovich et al. [95] and were taken from the approximate midpoints of the ponds. The measurement errors in the albedo are greater than ±0.01, since this value is the accuracy of the optical equipment under idealised conditions. In the field, there will be other factors such as icing of the optical collecting surfaces, shadowing from objects near the measuring equipment, and potential error due to direct radiation being incorrectly accounted for. Errors in pond depth are reported to be of the order of ±2 cm because of sagging in the string which was the reference level (Perovich et al. [95]), although this may be an underestimate due to the effect of surface waves on the melt pond due to the wind or field personnel. Also shown in the figures are approximate best-fit curves that were determined using the two-stream radiation model, by assuming that the optical properties of the sea ice are fixed, the thickness of the ice is infinite, and then determining the least square error by varying the extinction coefficient of the pond. The range of pond depths was 1 cm to 52 cm, and albedos ranged from 0.07 to 0.5. The combined pond data (figure 3.2) clearly shows a very large inter-pond variability. The individual pond (figure 3.3) also shows a large degree of variability. This scatter is due to many reasons, such as inclusions within the pond and sea ice, variation in properties of the sea ice, and clouds affecting the incident spectral distribution thus affecting the total albedo.
Figure 3.3: Observations of total albedo and depth at one pond on the Albedo Line at SHEBA

3.7.2 Melt pond optical property parameterisation

The major uncertainty in summertime albedos comes from the significant variations in the optical properties of the ice underlying melt ponds, with spectral albedos of ice underlying ponds ranging from 0.2 to 0.7 (Podgorny and Grenfell [102]). Furthermore, as summer progresses the increased brine volumes in the sea ice beneath ponds lowers the extinction coefficient (Grenfell and Maykut [45]). This variation must be taken into consideration when modelling the melt pond evolution, otherwise the energy partitioning between the ice and the pond will be incorrect. To this end, I introduce a parameterisation for the optical properties of the sea ice based upon the pond thickness and albedo data from SHEBA.

For an infinitely thick sea ice layer without a melt pond, the two-stream model albedo (equation 3.23) is given by

$$\alpha_\infty = R_0 + \frac{(1 - R_0)^2 s_{2\text{winter}}}{1 - R_0 s_{2\text{winter}}}.$$  \hfill (3.30)

since in the limit $h_0 \to 0$, $h_1 \to 0$ and $h_2 \to \infty$ we have that $X_0 \to 0$, $Y_0 \to 1$, $X_1 \to 0$, $Y_1 \to 1$, $X_2 \to 1$, $Z_2 \to 1$, and $Y_2 \to 0$. The parameter $s_{2\text{winter}}$ is the constant value of the parameter $s_2$ used in the sea ice layers when there is no melt pond. For sea ice thicker than 0.8 m the albedo does not vary significantly with
thickness (Perovich et al. [98]), and so if it is assumed that there is no spectral variation, equation (3.30) can be rearranged to determine the parameter $s_2^{\text{winter}}$. If we assume an appropriate value of the albedo at large thickness $\alpha_\infty$, then $s_2^{\text{winter}}$ can be determined using equation (3.30) when there is no pond and the brine volume is relatively low, so that

\[ s_2^{\text{winter}} = \frac{\alpha_\infty - R_0}{R_0 \alpha_\infty + 1 - 2R_0}. \]  

Knowledge of $s_2^{\text{winter}}$ provides a relationship between the extinction coefficient of sea ice $\kappa_2$ and the absorption coefficient $k_2 = \kappa_2(1 - s_2^{\text{winter}})/(1 + s_2^{\text{winter}})$.

For an infinitely thick sea ice layer with a finite melt pond on its surface, the variable $s_2$ can be derived in a similar way to the constant wintertime value $s_2^{\text{winter}}$ (3.31), by assuming $h_1 \neq 0$ and rearranging the two-stream albedo (3.23), and is given by

\[ s_2 = \frac{\alpha_{\text{pond}} - R_0}{(1 - 2R_0 + \alpha_{\text{pond}} R_0) Y_1^2}. \]  

The SHEBA data for the melt pond at 50 m along the Albedo Line are the melt pond depth and the albedo. Assuming a bulk value for the extinction coefficient of the pond (which would be expected to exhibit far less variation than the extinction coefficient of the sea ice), we can determine $s_2$ for a given pond depth from each data point. The reason for just considering an individual pond with a relatively high albedo is that it exhibits less variation due to factors such as dirt or cryoconite holes on the melt pond base, and sea ice albedo.

As the extinction coefficient decreases, $s_2$ decreases. There is a correlation between pond depth and brine volume in sea ice beneath melt ponds, which can be deduced from the observation of increasing brine volume through the summer melt season (Fetterer and Untersteiner [35]) and the observation of increasing average depth of melt ponds through the summer melt season (Perovich et al. [95]). Therefore, I make the assumption that the relationship between melt pond depth and $s_2$ is given by

\[ s_2(h_{\text{pond}}) = s_2^{\text{winter}} \exp(-\tau h_{\text{pond}}), \]  

for some parameter $\tau > 0$. Although this relationship is arbitrary, it is reasonable...
since larger pond depths are associated with lower extinction coefficients within the sea ice (lower values of $s_2$), because of increased brine volume. Also, $s_2$ remains bounded above by its wintertime (pond-free) value and below by 0, for all melt pond depths.

By minimising the square of the difference between the values of $s_2$ derived from SHEBA data (equation 3.32) and the values of $s_2$ determined from equation (3.33), the optimal parameter $\tau$ can be determined. The spectral albedo of melt ponds decay to their Fresnel reflection component for wavelengths greater than 700 nm (Grenfell and Maykut [45]). Therefore, the bulk optical properties of the melt pond are chosen to be similar to values less than 700 nm. I used the 500 nm value of extinction as derived by Hale and Querry [66], so that $\kappa_1 = 0.025 \text{ m}^{-1}$. Using $s_2^{\text{winter}} = 0.73$ so that the infinite ice only albedo is about 0.73 and corresponds to frozen multi-year white ice (Grenfell and Maykut [45]), and $R_0 = 0.05$, the optimal value of $\tau$ is found to be 4.03.

### 3.8 Discussion

In this chapter I developed the three-layer, two-stream radiation model that will be incorporated into the melt-pond–sea-ice thermodynamic model in the next chapter. This model was chosen since it was the simplest model that allowed for the different optical properties of the layers of ice and molten regions and allowed the albedo to be determined from internal parameters.

I have presented new simplifications that reduce the number of unknown coefficients in each scattering layer (sea ice) from 4 to 2, so that it is only necessary to apply the boundary conditions to fully determine the radiation field using the two-stream formulation. Perovich [89] presented a two-stream radiation model for a sea ice layer that contained multiple layers to represent vertical variation of the optical properties within the sea ice. This formulation for $n$-layers resulted in $4n$ equations in $4n$ unknowns. Using the methodology from this chapter the resulting set of $4n$ equa-
ations is reduced to only $2n$ equations. This immediately reduces the computational overhead by a factor of 2, leading to more rapid calculations using the two-stream model in this form.

Explicit formulae were presented for the optical coefficients and the albedo for the three-layer, two-stream radiation model. The methodology for a spectral radiation model was also described, because the optical properties of sea ice vary considerably with wavelength (Perovich [89]). The difficulty with incorporating a multiple-band radiation model, however, into a thermodynamic model is in the uncertainty in each of the spectral band's optical coefficients, and in the uncertainty in the distribution of the incident radiation field. There is also a significant computational overhead caused by the addition of extra spectral bands.

I choose to utilise an 8-band radiation model only in the numerical solution of a steady state problem of a single layer of sea ice described in chapter 5. The other sections of the thesis utilise a single band formulation where the optical properties have to be considered as bulk properties of sea ice.

Finally, I presented a parameterisation of the optical properties of the sea ice in the presence of melt ponds that accounts for summertime variation in optical properties of sea ice. This was formulated and justified using SHEBA data.

The most sophisticated albedo parameterisation previously used in thermodynamic models is that of Ebert and Curry [25], which assumes that the sea ice albedo does not vary during the summer and only depends on ice thickness. Observations by Podgorny and Grenfell [102] indicate that the sea ice albedo beneath melt ponds is highly variable. Varying the optical properties of the sea ice in an explicit way ensures that the albedo of melt ponds on sea ice varies with both melt pond thickness and the optical properties of the sea ice due to morphological changes.
Chapter 4

Formulation of melt-pond–sea-ice model

4.1 Introduction

In this chapter, I formulate the new melt-pond–sea-ice model. In sections 4.2–4.4, I describe the governing equations for heat transport within the sea ice, melt pond and internal melt region, and snow. In section 4.5, I describe the boundary conditions for the different configurations of the model. The simple implementation of drainage is explained in section 4.6. The method of solution is explained in section 4.7. The forcing data used for the Arctic simulations is described in section 4.8. Limitations of the model are described in section 4.9. Finally, the chapter is concluded with a summary in section 4.10.

Since I assume that seawater can be described by a sodium-chloride–water solution (see section 2.2), the fundamental equation describing heat transfer in the sea ice component of the model is given by the mushy layer heat equation. The model is one-dimensional and does not incorporate any horizontal parameterisations (e.g. meltwater run-off, ice divergence). The annual cycle of melt ponds and the physical processes that can occur were discussed in chapter 1.

The model is one-dimensional with z-axis pointing downward. There are six model
configurations: (a) sea ice only; (b) sea ice and snow layer; (c) sea ice and melt pond; (d) sea ice, internal melt region, and refrozen upper ice layer; (e) sea ice, internal melt region, refrozen upper ice layer and snow layer; and (f) ocean only. The standard annual cycle is shown schematically in figure 4.1. Also shown are the equation numbers of the governing equations that describe heat transport within the ice, snow and melt regions, and the equation numbers of the boundary conditions at the interfaces. In winter, snow covers the sea ice (b). Snow fall occurs in a prescribed manner identical to the methodology of Maykut and Untersteiner [77] (section 4.5.4). The snow–atmosphere boundary is located at \( z = h_{\text{snow}} \), the snow–ice interface is located at \( z = h_{\text{surface}} \), and the ice–ocean interface is at \( z = h_{\text{ice}} \). The motion of the ice–ocean interface is governed by a Stefan condition (section 4.5.1). Heat within the snow is governed by a simple diffusion equation (section 4.4), whereas heat within the sea ice is governed by the mushy layer heat equation (section 4.2). Melting snow in summer (section 4.5.4) leads to initial melt pond formation (c). The pond–ice interface is located at \( z = h_{\text{pond}} \), and once the snow covered sea ice forms a melt pond, the snow–atmosphere and the snow–ice interfaces are collocated at the melt pond’s surface. Melt ponds transport heat by turbulent convection or diffusion depending on the time dependent Rayleigh number (section 4.3). At the end of summer, melt ponds can refreeze from their surface down (d) and in this case the ice–internal melt region interface is located at \( z = h_{\text{intupper}} \), and the atmosphere–ice interface is located at \( z = h_{\text{surface}} \). The model does not consider the situation of refrozen ponds that collapse due to the weight of the snow cover (e.g. Fetterer and Untersteiner [35]). Snow begins to fall in the autumn, and so the model can consist of snow covered refreezing ponds (e). Eventually the ponds completely refreeze and the model becomes snow covered sea ice again (b).

Various other situations can occur depending on the physical parameters (e.g. rapid drainage can prevent melt ponds forming). The ocean-only configuration (f) is grossly simplified, but ensures that the ice can reform from open-ocean if the sea ice completely melts through. The model is forced in most simulations using data from the Atmospheric Flux Group of the SHEBA field experiment (section 4.8).
Figure 4.1: Schematic diagram of evolution of boundaries of melt-pond–sea-ice model. Equation numbers of governing equations and boundary conditions also shown.
4.2 Heat transport within sea ice

Heat transport within the sea ice is described by the one-dimensional mushy layer heat equation, c.f. equation (2.1),

\[ (\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_l U \frac{\partial T}{\partial z} = \frac{\partial}{\partial z} \left( k_m \frac{\partial T}{\partial z} \right) + \rho_s L \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z} F_{\text{net}}(z), \]  

(4.1)

where \((\rho c)_m\) is the volumetric specific heat capacity of the sea ice, \(T\) is the temperature within the sea ice, \(t\) is time, \(U\) is the vertical Darcy velocity of brine flow within the sea ice, \(z\) is the vertical spatial co-ordinate (pointing downwards), \(k_m\) is the thermal conductivity of the mushy layer, \(\rho_s L = 3.0132 \times 10^8 \text{ J/m}^3\) (Schwerdtfeger [110]) is the volumetric latent heat of fusion of the solid phase (fresh ice), \(\phi\) is the solid fraction (the local volume fraction of sea ice that is solid), and \(F_{\text{net}}(z)\) is the irradiance within the sea ice layer of the incoming short-wave radiation at depth \(z\) integrated across the short-wave spectrum.

I assume that the volumetric specific heat capacity of the sea ice \((\rho c)_m\) and the thermal conductivity of the sea ice \(k_m\) are determined by the mixture relations

\[ (\rho c)_m = (\rho c)_s \phi + (\rho c)_l (1 - \phi), \]  

(4.2)

\[ k_m = k_s \phi + k_l (1 - \phi), \]  

(4.3)

where \((\rho c)_s = 1.883 \times 10^6 \text{ J/m}^2\text{K}\) is the volumetric specific heat capacity of the solid phase, \((\rho c)_l = 4.185 \times 10^6 \text{ J/m}^3\text{K}\) is the volumetric specific heat capacity of the liquid phase (brine), \(k_s = 2 \text{ W/mK}\) is the thermal conductivity of the solid phase, and \(k_l = 0.5 \text{ W/mK}\) is the thermal conductivity of the liquid phase, which are all assumed to be constant with values obtained from Schwerdtfeger [110]. It has been shown that \(k_m \leq k_s \phi + k_l (1 - \phi)\), with equality achieved when the solid and liquid layers are aligned parallel to the direction of heat flow (Batchelor [7]). In columnar sea ice, the ice platelets are mainly aligned parallel to the direction of heat diffusion and so the mixture relation (4.3) is a good approximation to the thermal conductivity (Wettlaufer, Worster and Huppert [133]). However, this is not necessarily a good approximation at the surface of first year sea ice, because the surface is mainly composed of randomly oriented frazil crystals. For simplicity I
assume that the solid and liquid densities are identical \( \rho_s = \rho_l \), so that dynamic
effects due to volumetric changes during phase change are ignored.

I assume that the sea ice is in local thermodynamic equilibrium, so that the tem­
perature and concentration are related by the liquidus curve (see section 2.2). Close
to the freezing temperature of pure water the liquidus curve is linear. I assume that
this relationship is valid for all temperatures, even though seawater exhibits nonlin­
earity due to precipitation of different dissolved salts at a variety of temperatures
(see figure 2.3). Therefore,

\[
T = T_L(C) = -\Gamma C + T_L(0),
\]

where \( T_L(C) \) is the equilibrium freezing temperature for a concentration of solute
\( C \), and \( T_L(0) \) is the equilibrium freezing temperature at zero concentration. Since I
assume that seawater can be modelled as a sodium-chloride–water solution with a
concentration of 35 ppt the equilibrium freezing temperature at 35 ppt is 271.2 K
so that \( T_L(0) = 273 \) K and \( \Gamma = 0.0514 \) K/ppt.

The local bulk salinity is defined to be

\[
C_{\text{bulk}}(z) = \phi C_s + (1 - \phi)C,
\]

which is the local concentration of salt per unit volume of sea ice, and I assume that
\( C_s = 0 \) ppt (Weeks and Ackley [129]). The local bulk salinity is the concentration of
the melt that is obtained by totally melting a local volume of sea ice. For simplicity,
it is assumed that the local bulk salinity is constant (6 ppt, Kovacs [57]) throughout
the sea ice so that the solid fraction can be expressed in terms of temperature by
rearranging equation (4.5) and using the liquidus relationship (4.4). Replacing the
time derivative of the solid fraction in equation (4.1) and replacing the solid fraction
in the mixing relations (4.2) and (4.3) yields

\[
(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_l U \frac{\partial T}{\partial z} = \frac{\partial}{\partial z} \left( k_m \frac{\partial T}{\partial z} \right) - \rho_s L \frac{\partial T}{\partial t} \frac{T_s - T_{\text{bulk}}}{(T_s - T)^2} - \frac{\partial}{\partial z} F_{\text{net}},
\]

with

\[
(\rho c)_m = ((\rho c)_s - (\rho c)_l) \frac{T_{\text{bulk}} - T}{T_s - T} + (\rho c)_l \text{ and}
\]

\[
k_m = (k_s - k_l) \frac{T_{\text{bulk}} - T}{T_s - T} + k_l,
\]
where \( T_s = T_L(C_s) = 273 \text{ K} \) is the equilibrium freezing temperature of fresh ice, and \( T_{\text{bulk}} = T_L(C_{\text{bulk}}) = 272.69 \text{ K} \) is the equilibrium freezing temperature at the bulk salinity.

### 4.3 Heat transport within melt pond and internal melt region

Melt pond temperatures are observed to be a few tenths of a degree above freezing (Eicken et al. [27]), and can reach more than a degree on a sunny day (Eicken et al. [29]). Observations of salinities of melt ponds from the ARCTIC'91 cruise ranged from 0.02 ppt to about 30 ppt, with modal value of about 1 ppt (Eicken et al. [27]). The largest salinities were few in number and are a result of entrainment of seawater within the ponds (Eicken et al. [27]). The maximum salinity neglecting observations of order 30 ppt was about 10 ppt. Melt ponds with high salinities are typically highly stratified, with low salinity water near the surface, and are associated with highly permeable sea ice allowing a connection between the pond and the ocean (Tucker III and Perovich [123]). Figure 4.2 shows the temperature of maximum density of water for different salinities (plotted using the UNESCO international equation of state formula for seawater density, IES 80). For salinities reported above (0–10 ppt), the temperature of maximum density is greater than observed melt pond temperatures and therefore, for typical melt pond temperatures (less than 274.8 K), increased temperature corresponds to increased density.

For melt ponds with low salinities (< 10 ppt), where the surface is being heated, 'warm' surface water will be more dense than cool water at the base of the melt pond, and so there will be a natural propensity for the surface water to sink and be replaced by water at depth. This overturning is inhibited by frictional losses due to the melt pond’s viscosity. Turbulent convective motions take place when the melt pond becomes dynamically unstable, which occurs once the time dependent Rayleigh number \( \text{Ra}(t) \) exceeds the critical Rayleigh number \( \text{Ra}_{\text{crit}} \) (see section 6.2). The time
Figure 4.2: Variation of temperature of maximum density with salinity of water

dependent critical Rayleigh number is given by

$$Ra(t) = \frac{g \alpha H_{\text{pond}}^3 \max(\Delta T)}{\nu l \kappa l}, \quad (4.9)$$

where $g$ is the magnitude of acceleration due to gravity, $\alpha = 5 \times 10^{-5}$ K$^{-1}$ is the coefficient of thermal expansion of the water in the melt pond (which is estimated using the UNESCO formula for seawater density at salinity 6 ppt and is assumed constant in the temperature range I am interested in), $\max(\Delta T)$ is the maximum temperature difference $\Delta T$ in the pond (typically across the whole depth), $\kappa l = k_l/(\rho c)_l \simeq 10^{-7}$ m$^2$/s is the thermal diffusivity of the melt pond, $\nu l \simeq 10^{-6}$ m$^2$/s is the kinematic viscosity, and $H_{\text{pond}}$ is the depth of the melt pond. I define the critical Rayleigh number $Ra_{\text{crit}}$ to be 630, from a numerical study later in this thesis (see section 6.2).

A simple order of magnitude estimate of the Rayleigh number of a 0.1 m melt pond assuming a 2 K temperature difference across the entire pond depth is

$$Ra(t) \simeq \frac{10 \text{ m/s}^2 \cdot 5 \times 10^{-5} \text{ K}^{-1} \cdot 2 \text{ K} \cdot (0.1 \text{ m})^3}{10^{-6} \text{ m}^2/\text{s} \cdot 10^{-7} \text{ m}^2/\text{s}} \simeq 10^7, \quad (4.10)$$

using equation (4.9). Such large Rayleigh numbers lead to turbulent convective motion.

In the non-turbulent regime, $Ra(t) < Ra_{\text{crit}}$, the equation describing conservation of
heat is

\[ (\rho c)_1 \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial z} \right) = k_1 \frac{\partial^2 T}{\partial z^2} - \frac{\partial}{\partial z} F_{\text{net}}(z), \]  

(4.11)

where \( u \) is the velocity of the fluid in the pond due to drainage. Equation (4.11) describes the rate of change of temperature in a control volume at fixed position (first term), subject to advection of heat (second term), thermal diffusion (third term), and radiative heating (fourth term).

Turbulent convection leads to mixing of the melt pond, except near the boundaries where diffusive effects are dominant. As the Rayleigh number becomes larger the temperature gradient steepens near the boundaries and flattens in the core of the melt pond. This is shown in figure 4.3 for the evolution of the average horizontal temperature for turbulent convection between fixed boundaries at fixed boundary temperatures for varying Rayleigh number (Kuo [58]).

In the turbulent case, I seek the evolution of the mean core temperature of the pond \( \bar{T}(t) \). For simplicity I assume that the pond salinity is constant and that the mixing of the pond leaves the salinity unchanged, even when the pond is formed initially from a snow layer on the surface of the sea ice. This is reasonable since pond salinities are generally low (Bogorodskii [11]; Eicken et al. [27]). However, a layered structure, with a thermally-convecting fresh upper layer overlying a diffusive stable-stratified lower salty layer, may exist in the melt pond if the molten sea ice is relatively saline.

Heat transfer across the melt-pond–sea-ice and internal-melt–sea-ice boundaries due to turbulent convective overturning is modelled using the four-thirds law for turbulent convection. The four-thirds law arises when the heat flux across a fluid layer is assumed to be independent of the depth of a fluid layer, so that the Nusselt number, which is the ratio of the convective to diffusive heat flux, is proportional to the Rayleigh number to the power of one third (Linden [60]). Then the heat flux at the boundary of a liquid region (melt pond or internal melt), directed outward, is given
Figure 4.3: Distribution of mean horizontal temperature against height for standard thermal convection between fixed boundaries at constant temperature (after Kuo [58]). Label next to each curve is value of ratio of Rayleigh number to critical Rayleigh number by

\[ F_c(T^*) = \begin{cases} 
\text{signum}(\bar{T} - T^*) (\rho c)_t J |\bar{T} - T^*|^{4/3} & \text{if } Ra(t) \geq Ra_{\text{crit}} \\
-k_l \frac{\partial T}{\partial z} & \text{if } Ra(t) < Ra_{\text{crit}} \text{ and } z = h_p, \\
k_t \frac{\partial T}{\partial z} & \text{if } Ra(t) < Ra_{\text{crit}} \text{ and } z = h_s,
\end{cases} \]

(4.12)

where \( T^* \) is the boundary temperature, \( \bar{T} \) is the mean temperature of the convecting region, \( h_p \) is the location of the pond–ice interface \( h_{\text{pond}}, h_s \) is the location of the atmosphere–pond interface \( h_{\text{surface}} \) or upper-ice–internal-melt interface \( h_{\text{intupper}}, \) and

\[ J = \gamma \left( \frac{g \alpha \kappa_l^2}{\nu_l} \right)^{1/3}, \]

(4.13)

where \( \gamma \) is a dimensionless number, which is taken to be 0.1 (Huppert [51]).

For the turbulent case conservation of energy in the well-mixed region of the melt pond yields

\[ (\rho c)_t (h_p - h_s) \frac{\partial \bar{T}}{\partial t} = -F_c(T_0) - F_c(T_p) - \int_{h_s}^{h_p} \frac{\partial}{\partial z} F_{\text{net}}(z) \, dz, \]

(4.14)

where I assume that the radiative energy that would be absorbed across the thin boundary layers is transferred instantaneously into the turbulent core, and that the
boundary layers are always much thinner than the convective region, which is valid for developed melt ponds since $Ra/Ra_{crit}$ is typically $O(10^4)$ (using equation 4.10). Also, since melt ponds form rapidly as meltwater flows to low lying regions from the early snow melt (Fetterer and Untersteiner [35]), the boundary layers should become thinner than the turbulent convective core quickly. Equation (4.14) equates the change in total heat of the turbulent core (first term) to the heat flux in/out from the surface (second term) minus the heat flux out of the base (third term) plus the total heat gain from internal heating due to incoming short-wave radiation (fourth term).

### 4.4 Heat transport within snow

Since the snow cover is highly complex and is not the main focus of this thesis, I choose to use a simple model of snow that is almost identical to that used by Maykut and Untersteiner [77]. It is also the same snow evolution model that is used in most of the later thermodynamic sea ice models (e.g. Ebert and Curry [25]; Flato and Brown [36]). Heat transport within the snow is governed by the diffusion equation,

$$\left(\rho c\right)_{snow} \frac{\partial T}{\partial t} = k_{snow} \frac{\partial^2 T}{\partial z^2},$$  \hspace{1cm} (4.15)

where $(\rho c)_{snow} = 6.9 \times 10^5$ J/m$^3$K is the constant volumetric specific heat capacity of snow, and $k_{snow} = 0.31$ W/m K is the constant thermal conductivity of snow (Bitz and Lipscomb [9]). Radiation is neglected within the snow cover since most of the short-wave energy is scattered near the surface (Wiscombe and Warren [137]). This leads to high albedos (greater than 0.8), so that only a relatively small fraction of incident radiation penetrates the snow and leads to internal heating. The influence of this small amount of radiation should only influence the timing of the start of the melt season, and is unlikely to produce qualitatively different results during the melt season.

There have been thermodynamic sea ice models that have incorporated radiation schemes within the snow layer (e.g. Cheng et al. [20]), however, the radiation
schemes that have been implemented, have been variations on Beer's law formulations. There have also been thermodynamic snow models that incorporate two-stream radiation models (e.g. Koh and Jordon [56]; Liston et al. [62]; Schlatter [107]). However, these models focussed on Antarctic land ice, and so the models were only concerned with the evolution of a single layer. An interesting feature amongst these models was the observation of subsurface melting, which I discuss in relation to the melt-pond–sea-ice model in section 7.2.2.

4.5 Boundary conditions

4.5.1 Sea ice only

Long-wave radiation $F_{LW}$ is radiated by the Earth and the atmosphere and typically has wavelengths between 4000 nm and 50000 nm (Maykut [71]). It enters as a boundary term since it only penetrates a few millimetres into the ice (Grenfell [41]). Short-wave radiation $F_{SW(tot)}$ comes directly from the Sun (or is reflected without loss of frequency) and has wavelengths less than 4000 nm (Maykut [71]). Since it can penetrate deeply within the ice it contributes to internal phase change as well as to the surface boundary condition.

A blackbody absorbs and emits electromagnetic radiation at all wavelengths. The emitted power from the blackbody is proportional to the fourth power of its absolute temperature $T$, given by the Stefan-Boltzmann law, $P = \sigma T^4$, where $\sigma = 5.67 \times 10^{-8}\, \text{W/m}^2\text{K}^4$ is the Stefan-Boltzman constant. A greybody absorbs a fraction of the incident radiation at all wavelengths, and so the emitted radiation is given by that fraction multiplied by Stefan-Boltzmann law. The radiation that is emitted from the sea ice, a greybody, is therefore given by $\varepsilon \sigma T_0^4$, where $T_0$ is the surface temperature and $\varepsilon$ is the emissivity.

The only other way that energy can be transported from the sea ice to the atmosphere is by turbulent convection in the atmospheric boundary layer. The sensible
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Heat flux $F_{\text{sens}}$ results from the difference in temperature of the sea ice and the atmosphere. The latent heat flux $F_{\text{lat}}$ results from the latent heat transfer associated with evaporation, condensation, and sublimation. The turbulent heat fluxes depend upon surface roughness, wind speed, boundary layer stability, temperature gradients and water vapour gradients (Maykut [71]).

I use the bulk parameterisations for the sensible and latent heat fluxes given by

$$F_{\text{sens}} = \rho_a c_a C_T v (T_a - T_0)$$

and

$$F_{\text{lat}} = \rho_a L^* C_T v (q_a - q_0),$$

where $\rho_a = 1.275 \text{ kg/m}^3$ is the density of dry air, $c_a = 1005 \text{ J/kgK}$ is the specific heat capacity of dry air, $T_a$ (K) is the air temperature at some reference height (usually 10 m, obtained from data), $T_0$ (K) is the surface temperature, $v$ (m/s) is the wind speed at the reference height (obtained from data), $L^* = 2.501 \times 10^6 \text{ J/kg}$ is the latent heat of vaporisation, $q_0$ is the specific humidity (mass of water vapour per unit mass of air) at the surface, $q_a$ is the specific humidity at the reference level (obtained from data), and $C_T$ is a stability dependent bulk transfer coefficient following Ebert and Curry [25]. This is given by

$$C_T = \begin{cases} 
C_{T_0} \left(1 - \frac{2v\lambda}{1+\lambda \sqrt{Ri}}\right), & \text{Ri} < 0, \\
C_{T_0} (1 + \beta \lambda \text{Ri})^{-2}, & \text{Ri} \geq 0,
\end{cases}$$

where $\lambda$ is the bulk Richardson number given by

$$Ri = \frac{g(T_a - T_0) \Delta z}{T_a u_a^2},$$

with $\Delta z$ being a height of 10 m. The Richardson number, which determines whether convection is free or forced, is given by the ratio of the buoyancy force to inertia. The parameter $c = 1961b/C_{T_0}$ for a roughness length of $1.6 \times 10^{-4}$ m over ice (Ebert and Curry [25]), and $b' = 20$ is a parameter used in fitting the above flux approximation with measurements (see Businger et al. [16] and Louis [65]). The parameter $C_{T_0}$ is a turbulent bulk transfer coefficient, which is assumed equal for heat and moisture and equal to $1.3 \times 10^{-3}$ for sea ice (Ebert and Curry [25]). Since there have not been any measurements of sensible or latent heat fluxes over melt
ponds to date (E.L. Andreas, principal investigator of the SHEBA Atmospheric Surface Flux Group, personal communication, December 3, 2002), I assume that the value of the transfer coefficient $C_T$ for melt ponds is identical to that used for leads by Ebert and Curry [25] ($1.0 \times 10^{-3}$). The specific humidity at the surface is estimated using the assumption that the air at the surface is saturated, so that

$$q_0 \approx \frac{0.622p_v}{p_{atm} - 0.378p_v},$$

where $p_v \approx 2.53 \times 10^{8}\exp(-5420/T_0)$ (kPa) is the partial pressure of water vapour (Rogers and Yau [104]), and $p_{atm}$ (kPa) is the atmospheric pressure obtained from data.

When the surface state of the model is sea ice, so that the model configuration is either ice only (a) or ice with internal melt (d), the net energy at the upper surface ($z = h_{surface}$) is given by

$$(F_{net})_{ice}^0 = k_m \frac{\partial T}{\partial z} + F_{LW} - \epsilon_{ice} \sigma T_0^4 + (1 - i_0)(1 - \alpha_{tot})F_{SW(tot)} - F_{sens} - F_{lat}, \quad (4.20)$$

where $\epsilon_{ice} = 0.99$ is the emissivity of sea ice (Ebert and Curry [25]), $T_0$ is the temperature at the surface of the sea ice, and $i_0$ is the fraction of radiation that is not absorbed near the surface of the sea ice.

The parameter $i_0$ represents the fraction of incident radiation that passes through the surface into the interior of the ice or pond and does not contribute to energy changes at the surface. The surface energy fluxes (e.g. long-wave radiation, and turbulent fluxes) are transferred over small lengthscales (e.g. absorption depth, viscous molecular sublayer). Therefore, the $i_0$ parameter can be interpreted physically as the fraction of incident radiation that is absorbed over a small lengthscale at the surface. To conserve energy within the sea ice, the radiation within the sea ice model, predicted by the two-stream radiation model, is attenuated by $i_0$ of its initial value.

In numerical modelling, there are three issues concerning the $i_0$ parameter that mean the physical interpretation must be modified. Firstly, in a numerical model energy is only ‘measured’ at the location of numerical gridpoints. Therefore, $i_0$
must represent the fraction of energy that passes through the surface of the model to the first interior gridpoint. Secondly, the extinction coefficient decreases by up to two orders of magnitude near the surface of sea ice (Grenfell and Maykut [45]). Unless the optical model used reproduces the correct variation between the first two gridpoints the error introduced must be accounted for by modifying the magnitude of \( i_0 \). Thirdly, the radiation model that I use neglects spectral variation. Certain wavelengths of radiation are absorbed more effectively than others, and in general, longer wavelength radiation is absorbed more rapidly.

Previous studies (e.g. Ebert and Curry [25]; Maykut and Untersteiner [77]) had coarse resolutions and used simple estimates of \( i_0 \). However, in this thesis I use a fixed number of gridpoints at a much higher resolution (641 gridpoints), and don’t have spectral variation or vertical variations of optical properties. Therefore, for the sea ice surface I define \( i_0 \) as

\[
\begin{align*}
\text{\( i_0 = \min \left( 0.4, \frac{F_{\text{net(first interior point)}}}{F_{\text{net(surface)}}} \right). \quad (4.21) \end{align*}
\]

The second argument of the minimum function in equation (4.21) is the fraction of incident radiation that isn’t reflected by the sea ice at the first interior gridpoint. However, since vertical variations of the extinction are not taken into consideration, I limit \( i_0 \) to be at most 0.4 chosen from sensitivity studies. This ensures that subsurface melting for the case of melting bare sea ice is avoided (see section 7.2.2). Subsurface melting is not allowed since the three-layer two-stream radiation model inadequately models the optical properties of the ice at the beginning of the melt season. To incorporate subsurface melting would require a parameterisation of the optical properties of the sea ice near the surface that was dependent on temperature. For coarse resolutions (e.g. gridspacings of 10 cm) the definition of \( i_0 \) given by equation (4.21) corresponds closely to the cloudy values for white ice (0.35) described by Grenfell and Maykut [45].

For numerical simplicity, I assume that the melting temperature of the upper surface of the sea ice \( T_L(C_{\text{pond}}) \) is marginally less than the equilibrium freezing temperature of the bulk salinity of the ice, so that an explicit Stefan condition can be used. If
it were assumed that the ice melted at the equilibrium freezing temperature of the 
bulk salinity of the ice, then the solid fraction is zero at the interface and an implicit 
condition would be required to determine the motion of the melting interface (see 
Feltham and Worster [32], for example). However, since the melt rate is relatively 
slow (even during ponding), to leading order the implicit condition is still satisfied.

If the surface temperature, \(T_0\), is less than the melting temperature then the net en­
ergy at the upper surface of the sea ice must be zero. If the surface temperature is at 
the melting temperature (\(T = T_L(C_{pond})\)) then the net energy at the surface (\(F_{\text{net}}\)) must balance the latent heat required to melt the mushy layer, \(\rho_s L \phi dh_{\text{surface}}/dt\), 
where \(h_{\text{surface}}\) is the position of the surface. Note that the latent heat released as the 
mushy layer melts is scaled by the solid fraction. Using the mushy layer equations 
the ablation is automatically modelled in a way that conserves energy at melting 
interfaces, unlike previous thermodynamic sea ice models (e.g. Ebert and Curry, 
[25]; Maykut and Untersteiner, [77]).

Since the melt-pond–sea-ice model does not contain any description of the mixed 
layer of the ocean, I assume, in common with previous models (e.g. Bitz and Lib­
scomb [9]; Ebert and Curry [25]), that the ocean has constant salinity and is at its 
equilibrium freezing temperature. Then, any surplus or deficit of salt due to phase 
change at the sea-ice–ocean interface is assumed instantly removed to the mixed 
layer of the ocean. At the sea-ice–ocean interface \(z = h_{\text{ice}}\) the sea ice is at the 
freezing temperature of the ocean (assuming that it has salinity 35 ppt), so that

\[
T = T_L(C_{\text{ocean}}). \tag{4.22}
\]

The velocity of the ice–ocean boundary is given by the Stefan condition,

\[
\rho_s L \phi \frac{dh_{\text{ice}}}{dt} = k_m \frac{\partial T}{\partial z} - F_{\text{ocean}}, \quad (z = h_{\text{ice}}), \tag{4.23}
\]

where \(F_{\text{ocean}}\) is the heat flux from the ocean directed into the base of the ice. Equation
\(4.23\) states that the excess/deficit of heat at the melting interface balances 
the latent heat released/absorbed at the melting interface. It should be noted that 
in reality the flux of salt from the ocean to the moving sea-ice–ocean interface con­trols the growth/ablation rate (Notz et al. [86]), since salt diffuses two orders of
magnitude more slowly than heat. However, the error introduced by neglecting the rate limiting effect of the salt flux is small for reasonably thick sea ice, but can be significant for very thin ice (see section 6.4).

If the model were coupled to an ocean model then condition (4.22) would be replaced by the condition that the ice–ocean interface is at the equilibrium freezing temperature dictated by the salinity at the interface. In addition to the Stefan condition (4.23), there would be a further condition required to close the problem to express conservation of salt at the freezing/melting interface (see Mellor and Kantha [80], for example).

### 4.5.2 Melt pond

When the surface state of the model is a melt pond, the net energy at the surface \( z = h_{\text{surface}} \) is given by

\[
(F_{\text{net}})_{\text{pond}}^0 = F_c(T_0^0) + F_{\text{LW}} - \epsilon_w \sigma T_0^4 + (1 - i_0)(1 - \alpha_{\text{tot}}) F_{\text{SW}}(\text{tot}) - F_{\text{sens}} - F_{\text{lat}},
\]

where \( \epsilon_w = 0.97 \) (Ebert and Curry [25]) is the emissivity of the pond.

For melt ponds, most of the short-wave radiation beyond 700 nm is absorbed very effectively (Perovich et al. [95]). This is the reason why observed melt pond albedos correlate with naked eye estimates of melt pond albedo (the naked eye’s optical range is about 380 nm to 780 nm). The rapidly absorbed radiation is assumed to contribute to the surface energy balance via the \( i_0 \) parameter. The uncertainty of the amount of short-wave radiation in this range is largely due to the variation in cloud cover. From observations of cloudy sky incident radiation (Grenfell [41]), up to 40% of the incident short-wave radiation field is beyond the 700 nm range. Therefore, the parameter \( i_0 \) in this case is assumed constant and is set equal to 0.6.

At the pond–sea-ice interface, the temperature \( T \) is identical to the equilibrium freezing temperature of the melt pond, so that

\[
T = T_L(C_{\text{pond}}),
\]

(4.25)
and the velocity of the boundary is given by the Stefan condition,

\[ \rho_s C_p \frac{dh_{\text{pond}}}{dt} = k_m \frac{\partial T}{\partial z} + F_c(T_c(C_{\text{pond}})), \quad (z = h_{\text{pond}}). \quad (4.26) \]

At the ice–ocean interface we have the same boundary conditions as the ice-only configuration.

**Melt Pond Melt Through**

The thermodynamic formulation of melt ponds that has been introduced cannot melt through the sea ice layer. This can be seen by considering the Stefan conditions at the surface and the base of the sea ice, equations (4.23) and (4.26), in the limit of small sea ice thickness. If \((h_{\text{ice}} - h_{\text{pond}})/H \ll 1\), where \(H\) is a typical sea ice thickness, such that the magnitude of the conductive flux within the ice is much greater than the magnitude of the external heat fluxes (this can always be made true by taking the sea ice thickness small enough), then the Stefan conditions become

\[ \rho_s C_p \frac{dh_{\text{pond}}}{dt} \sim k_m \frac{\partial T}{\partial z} \quad \text{at } z = h_{\text{pond}}, \quad \text{and} \quad (4.27) \]

\[ \rho_s C_p \frac{dh_{\text{ice}}}{dt} \sim k_m \frac{\partial T}{\partial z} \quad \text{at } z = h_{\text{ice}}. \quad (4.28) \]

In this limiting case the temperature profile is linear, and \(\partial T/\partial z < 0\), hence the pond–ice interface must start to freeze and the ice–ocean boundary must start to melt (c.f. Martin and Kauffman [70]). For there to be the potential of melting the rate of melting at the ice–ocean interface must be greater than the rate of freezing at the pond–ice interface, which is only true if

\[ \left. \frac{k_m}{\phi} \right|_{z=h_{\text{ice}}} > \left. \frac{k_m}{\phi} \right|_{z=h_{\text{pond}}}. \quad (4.29) \]

This inequality is not satisfied for the parameters that I use in the numerical model.

From observations, however, it is required that melt ponds should have the potential to disappear through both thermodynamic and mechanical mechanisms: melt ponds are weak regions of the sea ice cover, and the forces exerted on the ice beneath ponds due to dynamical movement of sea ice can cause their disintegration (Hanesiak et
al. [47]). Also, thin layers of sea ice beneath melt ponds are highly permeable and can allow seawater to entrain into the ponds, which increases the melt rate and can result in complete ablation of ice beneath melt ponds (Perovich et al. [95]).

For these reasons, I specify the minimum sea ice thickness for which a growing pond can exist on the surface. From numerical tests, I set this to be 0.1 m, once the pond depth is greater than 0.2 m. I then assume that if the ice thickness becomes less than 0.1 m (due to pond growth) the ice melts through entirely. Assuming that the temperature profile within the ice is linear and accounting for the variation of solid fraction the energy required to melt the sea ice layer is $Q$. The energy $Q$ is determined by integrating the volumetric specific heat $(\rho c)_m$, which depends on temperature, with respect to temperature (from the current temperature at a point to the melting temperature) and then with respect to depth (from the surface to the base), to yield the amount of energy required to raise the sea ice to its melting temperature, and then adding on the amount of latent heat required to perform phase change. The total energy is given by

$$Q = \frac{T_b - T_p}{T_s - T_p} \rho_s C H_i - (\Delta \rho c) \frac{T_b - T_s}{T_o - T_p} H_i \left( (T_s - T_o) \log \left( \frac{T_s - T_o}{T_s - T_p} \right) + T_o - T_p \right) + \frac{1}{2} (\rho c)_s H_i (T_p - T_o),$$

where $T_o = T_{\text{ocean}}$, $T_p = T_L(C_{\text{pond}})$, $H_i = h_{\text{ice}} - h_{\text{pond}}$, $\Delta \rho c = (\rho c)_s - (\rho c)_i$, and I assume that during this melting process all of the forcing fluxes remain constant and the thickness remains fixed. Then the time taken to melt the sea ice layer $T$ is given by

$$T = \frac{Q}{F_c(T_L(C_{\text{pond}})) + F_{\text{ocean}} + \Delta F_{\text{net}}},$$

where $\Delta F_{\text{net}}$ is the radiation that is absorbed across the sea ice layer.

### 4.5.3 Internal melt region

If a melt pond is not particularly saline then constitutional supercooling and hence the onset of instability leading to the formation of a mushy layer could be delayed or may not even occur. This is because the rejected solute ahead of the freezing
interface may be insufficient to induce the instability that causes the mushy layer to form (see Mullins and Sekerka [83]).

For numerical simplicity I assume that the refreezing surface of a melt pond forms a mushy layer of the same constant bulk salinity as the lower sea ice layer, and with freezing temperature identical to the upper surface of the lower sea ice layer. This enables the sea ice to reform as a continuous block once the internal melt region has completely refrozen. An alternative would be to treat the region as a separate layer, however, this would require additional computational overhead to track another interface and to compute internal temperatures.

Therefore, at the upper-ice-internal-melt interface \( z = h_{\text{upper}} \) the temperature is at the melting temperature of the pond, so that

\[
T = T_L(C_{\text{pond}}),
\]

and the velocity of the boundary is given by the Stefan condition

\[
\rho_s C_p \frac{dh_{\text{upper}}}{dt} = k_m \frac{\partial T}{\partial z} - F_c(T_L(C_{\text{pond}})).
\]

The sea-ice–atmosphere boundary condition \( z = h_{\text{surface}} \) for the internal melt region case is identical to the surface boundary condition for sea ice only when there is no melting.

At the internal-melt–sea-ice interface \( z = h_{\text{pond}} \) the boundary conditions are identical to the boundary conditions of the melt pond at the melt-pond–sea-ice interface. The temperature at the interface is at the melting temperature (equation 4.25), and the velocity of the boundary is given by the Stefan condition (equation 4.26). At the ice–ocean interface the boundary conditions are the same as the ice-only configuration.

### 4.5.4 Snow layer

Snow covers in the Arctic are generally spatially uniform (Maykut and Untersteiner [77]). The accretion of snow onto the ice surface in the model is identical to Maykut
and Untersteiner [77]. The accumulation of snow is assumed to fall linearly between specific dates. Between August 20 and October 30, 30 cm falls. From November 1 to April 30 there is a further 5 cm accretion. Between May 1 and May 31, 5 cm more falls, so that by the beginning of the melt season there is 40 cm of snow. If the melt season begins before May 31, then snow fall ceases. If the melt season finishes after August 20, then the onset of snowfall is delayed until freeze-up, and then 30 cm of snow falls by October 30. This approximate pattern of snowfall used by Maykut and Untersteiner [77] is also found to be similar to recent observational data from SHEBA (Sturm et al. [120]).

At the snow-ice interface $z = h_{\text{surface}}$ it is assumed that there is no melting until a melt pond forms. When there is no melting at the snow-atmosphere interface $z = h_{\text{snow}}$ the heat flux is assumed to be continuous across the snow-ice boundary, and so

$$k_{\text{snow}} \frac{\partial T}{\partial z} = k_m \frac{\partial T}{\partial z}, \quad (z = h_{\text{surface}}).$$

(4.33)

It is also assumed when there is no melting that the temperature is continuous across the snow-ice interface. The evolution of the snow-ice interfacial temperature is then determined implicitly from equation (4.33) using an iterative method (see section 4.7).

When the ice is snow covered, which is either when there is a single layer of ice or ice with an internal melt region beneath the snow, the net energy at the upper surface of the snow ($z = h_{\text{snow}}$) is given by

$$(F_{\text{net}})_{\text{snow}}^{\text{snow}} \equiv k_{\text{snow}} \frac{\partial T}{\partial z} + F_{\text{LW}} - \epsilon_{\text{snow}} \sigma T_0^4 + (1 - i_0)(1 - \alpha_{\text{tot}})F_{\text{SW}}(t) - F_{\text{sens}} - F_{\text{lat}},$$

(4.34)

where $T_0$ is the temperature at the surface of the snow layer, and $\epsilon_{\text{snow}} = 0.99$ (Ebert and Curry [25]) is the emissivity of the snow. If the surface temperature of the snow, $T_0$, is less than its melting temperature (273 K) then the net energy at the upper surface must be zero.

The melting of snow is complex, with meltwater percolation and refreezing, and rapid densification of the snow cover (Maykut and Untersteiner [76]). Since the primary
interest here is in the role of melt ponds, I follow Maykut and Untersteiner's [77] treatment.

As soon as the surface temperature of the snow reaches the freezing point, the snow is allowed to melt. If the surface temperature is at the melting temperature then the net energy at the surface \( (F_{\text{net}})_0 \) must balance the latent heat required to melt the snow layer, so that

\[
\rho_\text{snow} \cdot L_\text{snow} \frac{dh_{\text{snow}}}{dt} = (F_{\text{net}})_0, \\
\]

where \( \rho_\text{snow} \) (kg/m\(^3\)) is the density of the snow, and \( L_\text{snow} = 332424 \) J/kg (Ebert and Curry [25]) is the latent heat of the snow.

The initial quantity of melting is determined by equating the energy required to make the snow layer isothermal at its melting temperature and the energy required to raise the surface of the sea ice to its melting temperature (between the first two gridpoints) with the latent heat required to melt the undetermined quantity of snow.

Since heat transport within the snow cover is described by the diffusion equation and the rate of accretion at the snow surface is small, the quasi-steady assumption is valid and the temperature profile within the snow is linear (Worster [141]). The temperature profile between the first two gridpoints within the sea ice can also be assumed to be approximately linear, because the number of gridpoints is so large (641).

Since the snow layer and uppermost part of the sea ice have linear temperature profiles, the quantity of melting \( (h_{\text{surface}} - h_{\text{snow}}^*) \) that is required to make the snow isothermal at its melting temperature and bring the surface of the sea ice to its melting temperature is determined from the equation

\[
\rho_\text{snow} \cdot L_\text{snow} (h_{\text{surface}} - h_{\text{snow}}^*) = \int_{h_{\text{snow}}}^{h_{\text{surface}}} \int_{T_{\text{snow}(1)}}^{T_{\text{snow}(2)}} (\rho c)_{\text{snow}} dT' dz' \\
+ \int_{h_{\text{surface}}}^{h_{\text{gridpoint}}} \int_{T_{\text{ice}(1)}}^{T_{\text{ice}(2)}} (\rho c)_m(T') dT' dz', \\
\]

where \( T_{\text{ice, snow}(1)} \) is the temperature profile before melting has occurred, \( T_{\text{ice, snow}(2)} \) is the temperature profile after melting has occurred, and \( h_{\text{gridpoint}} \) is the position.
of the first interior gridpoint. Assuming a linear temperature gradient inside the snow yields for the first integral

$$\frac{(pc)_{\text{snow}}(h_{\text{surface}} - h_{\text{snow}})(T_0 - T_{\text{ice}})}{2},$$

where $h_{\text{snow}}^*$ is the position of the snow–atmosphere interface after the melting has occurred, $c_{\text{snow}} = 2092.2 \text{ Jm/kgK}$ is the specific heat capacity of snow and is assumed constant, $T_0$ is the surface temperature of the snow, and $T_{\text{ice}}^*$ is the snow–ice interface temperature. Assuming a linear temperature gradient between the first two gridpoints inside the ice and because the volumetric heat capacity of the sea ice is less than or equal to $(pc)_i$ the second integral is less than or equal to

$$\frac{(pc)_i(h_{\text{intgridpoint}} - h_{\text{surface}})(T_{\text{ice}}^* - T_{\text{ice}}')}{2},$$

where $T_{\text{ice}}'$ is the temperature of the first interior gridpoint of the sea ice. However, since the distance between the first interior gridpoint of the sea ice and surface of the sea ice is small compared to the thickness of the snow (about 3 mm for 2 m of sea ice compared to 0.4 m of snow), the second integral can be neglected. Therefore, the amount of melting at the surface can be approximated by

$$ (h_{\text{surface}} - h_{\text{snow}}^*) = \frac{c_{\text{snow}}(h_{\text{surface}} - h_{\text{snow}})(T_0 - T_{\text{ice}}^*)}{2c_{\text{snow}}}.$$

Once this quantity of melting has occurred the snow is assumed instantaneously to be isothermal at its melting temperature (273 K) and the ice surface is brought to its melting temperature ($T_L(C_{\text{pond}})$). The density of the snow is also assumed to increase instantaneously to a limiting value, so that $\rho_{\text{snow}}$ increases to 450 kg/m$^3$ from 330 kg/m$^3$ (Maykut and Untersteiner [77]), to represent rapid densification. The snow is then allowed to melt according to equation (4.35) until a melt pond forms, with the initial height of the melt pond being given by $(h_{\text{surface}} - h_{\text{snow}}^*)\rho_{\text{snow}}/\rho_i$ (the snow water equivalent). For simplicity I assume that once the depth of the snow surface melts to the initial pond height the melting snow instantaneously becomes a melt pond. During this melting period the density of the snow is assumed to increase linearly to the density of the pond.
Snow albedos range from about 0.74 for melting snow to 0.87 for new snow (Perovich [91]). However, since only a simple model of snow is being used I assume the snow albedo to be constant (0.84) during winter. As the snow layer melts the albedo varies linearly from an initial melting value (0.74) to its final melt pond value, which is dictated by the two-stream radiation model.

**4.5.5 Open ocean**

Since I do not incorporate any ocean model, I use a grossly simplified model to represent the ocean without sea ice. This model is introduced so that the full cycle of seasonal sea ice can be simulated without the necessity of incorporating input parameters for new sea ice formation and is not supposed to be realistic in detail.

As the model can form ice from the ocean, it could be used to investigate the growth of first year ice. However, results from these investigations should be treated with caution as frazil formation and sea ice genesis processes are not correctly incorporated into the model.

To simulate the cooling of the surface mixed layer during summer into autumn, I assume that the temperature of the surface mixed layer is above the melting temperature until day 275 (Flato and Brown [36]). After day 275, I assume that the ocean remains at its freezing temperature (271.2 K). Flato and Brown [36] incorporated a highly simplified model of the upper mixed layer of the ocean, however, the depth of this layer was tuned to match dates for the onset of freezing at the ocean surface. Therefore, there seems to be little advantage in incorporating a highly simplified mixed layer as compared to assuming the date for the onset of freezing.

The net energy at the upper surface \((z = h_{surface})\) is given by

\[
(F_{\text{net}})_{\text{ocean}} = F_{\text{ocean}} + F_{\text{LW}} - \varepsilon_w \sigma T_{\text{ocean}}^4 + (1 - \varepsilon_0)(1 - \alpha_{tot}) F_{SW(tot)} - F_{\text{sens}} - F_{\text{lat}}. \tag{4.40}
\]

If the net energy at the upper surface is less than zero then latent heat is released, and the surface freezes, so that the model is then described by the sea-ice-only component.
4.6 Drainage during ponding

Since summer sea ice is porous (Eicken et al. [29]), drainage occurs within melt ponds that are above sea level at a rate dictated by the pressure head that they create (see section 6.3). This is an important mechanism, as it is believed to be the primary source of desalination (flushing) of multi-year sea ice (Cox and Weeks [21]; Untersteiner [125]).

Drainage within each pond, however, is influenced significantly by the floe geometry, which determines the freeboard height, the melt rate of the catchment area of the pond, and the permeability and saturation of the sea ice (see section 6.3). These effects are two-dimensional, which I do not wish to consider. Therefore, unless floe geometry (and therefore sea level) and local melt rate are prescribed, it is not possible to determine the drainage rate.

In section 6.3, I demonstrate that rapid drainage can cause melt ponds to have their surfaces fixed at the sea surface, so that the drainage rate is identical to the change in freeboard. If the area of the pond is assumed to be much less than the area of the sea ice floe, then changes in freeboard are only governed by changes in mass of the floe rather than the pond.

If it is assumed that the sea ice floe without melt ponds has a constant melt rate \( \dot{h}^* \), then this simple approximation implies that the melt ponds have a constant drainage rate, which is identical to the sea ice floe surface melt rate \( \dot{h}^*_s \). Melt rates of sea ice floes vary due to synoptic events, but typically they are of the order of centimetres per day (Perovich et al. [95]). I assume that the drainage rate is constant and set it to be 2 cm/day from sensitivity tests. Since the model is one-dimensional the drainage rate is the same as the Darcy velocity in the sea ice \( U \), and the fluid velocity in the melt pond \( u \). Later, I examine the sensitivity of the numerical simulations to drainage rate (section 7.2.1).
4.7 Method of solution

The sets of equations to be solved within each domain are decoupled, in the sense that the interfacial temperatures at the boundaries are fixed, except when a snow layer is present that is not melting. The main difficulties in solving the heat equation in each domain are, therefore, the moving boundaries and nonlinearity in the heat equation.

The governing equations and boundary conditions were non-dimensionalised using a lengthscale $L$, a thermal diffusive timescale $\tau = L^2/k_t$, and a fluxscale $F$. To eliminate the moving boundaries the governing equations for each layer (snow layer, upper ice layer, melt pond/internal melt, lower ice layer) were transformed from the $(z,t)$ co-ordinate system to the $(\xi, \zeta)$ co-ordinate system, by the transformation

$$\xi = \frac{z - h_a}{h_b - h_a}, \quad \zeta = t,$$

(4.41)

where $h_a < h_b$ are the positions of the boundaries of the respective layers. The moving domain $[h_a, h_b]$ is then transformed onto the fixed domain $[0, 1]$. The moving-boundary problem is therefore reformulated as a fixed-boundary problem with additional nonlinear terms in the heat equation. A major advantage of this method is that interpolation of temperatures at the moving boundaries are not required, and that the number of gridpoints in the numerical scheme remains fixed. A further advantage of this method is that library routines from the Numerical Algorithms Group (NAG) can be used to solve the governing equations.

The governing equations in this new co-ordinate system are first-order, parabolic partial differential equations (PDEs), which I solve using the NAG library routine D03PCF. D03PCF is a general first-order linear/nonlinear parabolic PDE solver. The spatial discretisation is obtained by finite differences. However, there is no error estimate or error control in D03PCF and so the mesh selection is important (Dew and Walsh [23]). The method of lines reduces each PDE to an ODE. The resulting ODE is solved using a backward differentiation formula (see the NAG Fortran Library Routine Document D03PCF).
To solve the coupled snow–ice layers using D03PCF an iterative scheme is used. The temperature gradients at each side of the interface are approximated using a three-point backward(snow)/forward(ice) difference approximation. D03PCF is integrated, at each iteration of a single timestep, using an estimate of the interfacial temperature at the end of the timestep (the estimate is calculated from the current rate of change of the interfacial temperature). The interfacial condition on the heat flux forms a quadratic equation for the interfacial temperature (given the two sets of two gridpoints on each side of the interface). The temperature of the non-boundary gridpoints at the end of the integration of D03PCF are used in the quadratic equation to determine what the expected interfacial temperature should be that would satisfy the heat flux condition. If the difference between the initial estimate and the expected interfacial temperature is greater than a prescribed tolerance (0.001 in non-dimensional temperature units) then the expected interfacial temperature is used as the estimate in the next iteration. If the difference is less than the prescribed tolerance then the iterations are stopped. Convergence usually occurs within two iterations, since the rate of change of the snow–ice interfacial temperature is used to determine the initial value of the iterative process.

During each timestep, the forcing fluxes \( F_{SW}, F_{LW}, F_{sens}, F_{lat} \) and \( F_{ocean} \) and the thickness of each layer are assumed fixed. If the temperature becomes greater than the melting temperature after integrating the temperatures in the ice forward in time, then the newly molten region is interpolated to estimate its initial thickness (either internally or at the surface boundary). Newly molten regions then change the model boundary conditions for the next timestep. At the end of each timestep the forcing fluxes are updated, and the thickness of each layer is updated. See appendices A and B for more detail on the numerical method, algorithm, and code.

### 4.8 Forcing data

The model input forcing parameters are: incoming short-wave radiation; incoming long-wave radiation; air temperature at 10 m (used in determining the sensible heat
Figure 4.4: Predicted (white line) and observed (black) seasonal evolution of short-wave radiation ($F_{SW}(tot)$). January 1 1997 is given by day 0

For simplicity the model forcing was assumed to have no diurnal variation, and varied continuously throughout the annual cycle, repeating itself every 365 days. Atmospheric SHEBA data were obtained from the online CODIAC system provided by JOSS/UCAR [55] and are derived from data from the SHEBA Surface Flux Group (see the SHEBA Surface Flux Group Home Page [114]). An assumed profile of the form $A_{lw} + B_{lw} \cos(2\pi t/365)$ was fitted to the long-wave data using the method of least squares (where $t$ is measured in days), and an assumed profile of the form $A_{sw} + B_{sw} \sin(2\pi(t - 249)/365)$ was fitted to the short-wave data. The data points of the incoming short-wave radiation that were zero were neglected in the least-squares fit, since they would lead to spurious results. If the estimated functional form of the short-wave radiation produced a negative value it was set to zero. The best-fit coefficients for the short-wave flux are $A_{sw} = 29.25$ and $B_{sw} = -240.59$. The best-fit coefficients for the long-wave flux are $A_{lw} = 214.27$ and $B_{lw} = -73.81$. The predicted and observed short- and long-wave fluxes are shown in figures 4.4 and 4.5.
Figure 4.5: Predicted (thick black line) and observed (thin black line) seasonal evolution of long-wave radiation ($F_{LW}$). January 1 1997 is given by day 0.

<table>
<thead>
<tr>
<th>Model/Observation</th>
<th>Short-wave ($10^8$ J/m$^2$)</th>
<th>Long-wave ($10^8$ J/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>This study</td>
<td>28.94</td>
<td>67.57</td>
</tr>
<tr>
<td>Maykut and Untersteiner [77]</td>
<td>31.55</td>
<td>69.45</td>
</tr>
<tr>
<td>Ebert and Curry [25]</td>
<td>31.86</td>
<td>67.91</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of total annual short- and long-wave radiation between this study, Maykut and Untersteiner [77], and Ebert and Curry [25] (after Ebert and Curry [25]).

Table 4.1 shows the total annual short-wave and long-wave radiation for this thesis compared to two other studies (after Ebert and Curry [25]). The total annual short-wave radiation used here is less than other models, although the long-wave radiation is similar to that used in other models.

To estimate the ocean heat flux entering the base of the sea ice would require coupling to an ocean model. However, since the aim is not to analyse the interaction between ocean and sea ice, the ocean heat flux is prescribed as an external parameter. For simplicity the ocean heat flux was set to be a constant 2 W m$^{-2}$ (Maykut and Untersteiner [77]).
The sensible and latent heat fluxes are estimated using bulk parameterisations (section 4.5.1). The parameterisations utilise the air temperature, saturation specific humidity, air pressure, and wind speed. These are estimated from the SHEBA Surface Flux Group data. The data were averaged for each month and the average monthly values were linearly interpolated. Since the data for October were sparse the averaged September and November value was used. The annual wind speed monthly average was 4.90 m s\(^{-1}\) (c.f. 5 m s\(^{-1}\), Ebert and Curry [25]) with standard deviation 0.27 m s\(^{-1}\), and so a constant value of 4.90 m s\(^{-1}\) is used. The forcing fluxes used in the model are shown in table 4.2.
<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short-wave (W m(^{-2}))</td>
<td>0</td>
<td>0</td>
<td>66.7</td>
<td>182.4</td>
<td>254.0</td>
<td>266.4</td>
<td>125.5</td>
<td>112.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>SHEBA</td>
</tr>
<tr>
<td>Long-wave (W m(^{-2}))</td>
<td>142.9</td>
<td>162.4</td>
<td>192.7</td>
<td>231.6</td>
<td>264.7</td>
<td>285.1</td>
<td>286.1</td>
<td>267.5</td>
<td>234.03</td>
<td>196.34</td>
<td>162.4</td>
<td>143.2</td>
<td>SHEBA</td>
</tr>
<tr>
<td>Air Temp. (K)</td>
<td>243.7</td>
<td>241.4</td>
<td>250.4</td>
<td>256.0</td>
<td>263.7</td>
<td>272.2</td>
<td>273.1</td>
<td>271.7</td>
<td>269.0</td>
<td>260.6</td>
<td>252.3</td>
<td>241.1</td>
<td>SHEBA</td>
</tr>
<tr>
<td>Atmos. Press. (kPa)</td>
<td>102.9</td>
<td>102.1</td>
<td>101.7</td>
<td>101.5</td>
<td>101.8</td>
<td>101.7</td>
<td>101.6</td>
<td>100.5</td>
<td>101.5</td>
<td>101.2</td>
<td>100.9</td>
<td>101.8</td>
<td>SHEBA</td>
</tr>
<tr>
<td>Sat. Spec. Hum.</td>
<td>0.29</td>
<td>0.23</td>
<td>0.56</td>
<td>0.89</td>
<td>1.79</td>
<td>3.33</td>
<td>3.68</td>
<td>3.42</td>
<td>2.73</td>
<td>1.71</td>
<td>0.68</td>
<td>0.21</td>
<td>SHEBA</td>
</tr>
<tr>
<td>Ocean Heat (W m(^{-2}))</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>Maykut &amp; Untersteiner</td>
</tr>
</tbody>
</table>

**Table 4.2:** Mid-monthly forcing data used in melt-pond-sea-ice model
4.9 Some model limitations

The melt-pond–sea-ice model has several limitations, which are due to deficiencies in the physics of the model. In this section I outline the main limitations and describe their implications.

The model of the snow cover of sea ice is quite basic and has been utilised by the sea ice community ever since its introduction by Maykut and Untersteiner [77]. Developments of the model have resulted in only minor changes such as using temperature dependent thermal properties (Ebert and Curry [25]). A limitation of previous simulations using this model is that their numerical solutions were based on finite-difference methods applied at approximately 0.1 m spaced intervals with the moving boundary interpolated between gridpoints. As the snow cover is typically less than or equal to 0.4 m, there can be only 5 gridpoints at most. This limitation led to poor accuracy of the solution. The numerical method used here uses a (large) fixed number of gridpoints at all times, and is more accurate.

During snowmelt the downward heat transport of melting snow due to surface melt, percolation, and refreezing is parameterised using a simple energy balance. The major difference between the model and reality is that the model temperature at the ice–snow interface will be too cold for a few days and then will jump to the respective melting temperatures of the snow and ice (Maykut and Untersteiner [76]). The snow cover will not affect the qualitative behaviour of melt ponds, and quantitatively should only influence the timing of the onset of the melt season.

The assumption that there is a constant ocean heat flux at the base of the sea ice is a reasonable assumption away from the summer melt season, but is inaccurate during the melt season itself because of the influence of drainage of meltwater through the ice (Perovich and Elder [92]) and because of the effect of solar heating of the ocean during summer (Maykut and Perovich [75]). However, incorporating a variable ocean heat flux will only influence the rate of ablation or accretion at the ice–ocean boundary, which should make little difference to relatively thick sea ice.
The radiation model that is used in the melt-pond-sea-ice model is deficient mainly because of the lack of spectral resolution. This will result in inaccuracies in the estimate of the absorption and transmission of radiation during summer, because different wavelengths can be absorbed more effectively than others. Also, sea ice can often be affected by impurities such as soot and biota (Grenfell et al. [44]; Perovich [91]), which is not represented in the model. However, the optical model improves on previous models because it is the optical properties of the ice that are parameterised, and the albedo is predicted from the radiation model.

The effect of salinity on the physical properties of the ice and salinity redistribution are not modelled in the melt-pond-sea-ice model. Salinity affects the density, thermal conductivity, latent heat, and permeability (Maykut and Untersteiner [76]). The influence of salinity is most pronounced in first-year ice, before it has experienced summer melt, and so the model is expected to be most appropriate in representing melt processes on multi-year sea ice. To incorporate salinity into the model would require coupling the equation describing heat transfer within a mushy layer to the equation describing conservation of solute within a mushy layer (e.g. Maksym and Jeffries [69]; Schulze and Worster [108]).

The melt pond salinity is not accurately modelled by the melt-pond-sea-ice model, because the surface snow cover is assumed to not affect the melting temperature at the surface of the sea ice. However, the pond that initially forms from the snow cover will be undergoing turbulent convection so will rapidly transport heat to the sea ice and initiate melting. In the early stages the freezing temperature of the pond–ice interface will be underestimated, however, once sufficient melting has occurred the difference between the pond salinity and that of the sea ice will be minimal.

The initial formation of an ice cover and the subsequent processes that occur in the early stages of sea ice growth are not modelled accurately. The assumption that sea ice will begin to form on an open ocean from day 275 is based on observations at a single location and is not representative of the Arctic as a whole. However, previous attempts to incorporate simple mixed layer models to predict the onset of surface freezing were forced to fit observational data (e.g. Flato and Brown [36]),
so that the assumption that freezing occurs at a specific date is just as accurate as estimates from a simple mixed layer model. The sea ice when it is thin is also not well represented by the model. This is because at small ice thickness, sea ice is mechanically weak (Hanesiak [47]), and if it is newly formed it will have a different salinity structure to ice of larger thickness (Wettlaufer et al. [131]). The result of this deficiency is that the expected ice thickness will be in error the season after it has formed.

4.10 Summary

In this chapter I have described a new melt-pond-sea-ice model. The basis of the sea ice component of the model is the equation describing heat transport within a mushy layer. The sea ice mushy layer is coupled via boundary conditions to an overlying melt pond, which can either be turbulent or diffusive, depending on the time-dependent Rayleigh number. The melt pond component of the model cannot melt through completely for the given forcing. However, this possibility has been incorporated via an energy conservation relationship, which activates under certain conditions. Previous models of sea ice during the melt season are also unable to melt but since the simulations used timesteps of the order 1 day, numerical inaccuracy allowed the ice to melt completely over the space of a single timestep.

A simplified snow model used in previous studies (Bitz and Lipscomb [9]; Ebert and Curry [25]; Maykut and Untersteiner [77]) that neglects internal heating due to radiation is utilised. The boundary condition between the snow and the sea ice requires an iterative scheme to determine the evolution of the interfacial temperature. The iterative scheme reduces the computational efficiency of the numerical model, since repeated calls to the partial differential equation solver are necessary for each timestep.

The forcing fluxes that are required to drive the model are the total incoming long-wave radiation, the total incoming short-wave radiation, the wind speed, the air
temperature, the saturation specific humidity, the atmospheric pressure, and the oceanic heat flux. These terms enter the model via the boundary conditions at the surface and the base, and the incoming short-wave radiation also influences the model through internal heating.

Freeze-up of the model is achieved from the surface down, with the addition of another layer, which is assumed to reform as a mushy layer immediately. This approximation is used for computational efficiency, so that the mushy layer can reform into a unified single block in autumn.

The model presented here is the first to use a thermodynamic formulation for the formation and evolution of melt ponds. The only advanced previous model (Ebert and Curry [25]) used an energy conservation relationship based upon the incoming solar radiation to determine the melt rate of the melt pond. However, this is inadequate since turbulent heat transfer processes are an important factor in melt pond evolution (Fetterer and Untersteiner [35]).

Since melt ponds typically have low salinities (Eicken et al. [27]) the melt pond turbulent parameterisation should be a reasonable approximation. If a pond is more saline, however, the pond may become stably-stratified (Tucker III and Perovich [123]) and a diffusive heat transfer process may be more appropriate.

The model that has been described in this chapter is utilised in chapter 5 to investigate stationary solutions of the model in the absence of melt ponds without a snow cover. Three separate processes related to the model are investigated in chapter 6. The full melt-pond–sea-ice model is then investigated in chapter 7. Finally, in chapter 8, I describe the conclusions of this thesis, its implications, and future work.
Chapter 5

Application of sea ice model formulation without melt ponds

5.1 Introduction

Before considering the melt pond problem in detail it is instructive to look at the sea ice only case to gain greater understanding of the interaction of the melt-pond–sea-ice model and the two-stream radiation model. I investigate the stationary behaviour of the mushy layer model of sea ice, without a melt pond or brine drainage, as described in chapter 4. Stationary solutions are time independent solutions, so that the model is forced with constant data. When variations of the ice cover are slow, such as during winter and spring in the Arctic, the sea ice evolution is quasi-steady. Then the temperature profile within the sea ice is well approximated by the stationary temperature profile (Worster [141]). Therefore, the study of stationary solutions describes the quasi-steady evolution of the ice cover.

Incorporation of the two-stream radiation model is seen to allow for the development of two physically possible stationary ice thicknesses for the same forcing data. These two solutions occur both for a single band radiation model and also a discretised spectral radiation model. Investigation of the related stability problem for a non-mushy-layer ice layer coupled to a two-stream radiation model leads me to postulate
Atmosphere
\[ z = 0 \quad k_m \frac{\partial T}{\partial z} + F_{LW} - \varepsilon_\text{ice} \sigma T_0^4 + (1 - i_0)(1 - \alpha_{\text{tot}}) F_{SW(tot)} - F_{\text{sens}} - F_{\text{lat}} = 0 \]

Sea Ice
\[ z + \downarrow \quad (\rho c)_m \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left( k_m \frac{\partial T}{\partial z} \right) + \rho_i \mathcal{L} \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z} F_{\text{net}} \]
\[ T = T_L(C) = T_L(C_0) - \Gamma C \]
\[ z = h_{\text{ice}} \quad \phi \rho_i \mathcal{L} \frac{dh_{\text{ice}}}{dt} = k_m \frac{\partial T}{\partial z} - F_{\text{ocean}} \]
\[ T = T_L(C_{\text{ocean}}) \]

Ocean

Figure 5.1: Overview of time-dependent governing equations and boundary conditions for sea ice only case with no surface melting

that the larger stationary ice thickness is stable to infinitesimal perturbations, while
the smaller stationary ice thickness is unstable to infinitesimal perturbations. This
is verified to be true for the full mushy layer model using the numerical model
described in chapter 4.

The single slab model is the sea ice only component of the melt-pond-sea-ice model
for the case of no melting as described in chapter 4. The mushy layer equations
are used to describe conservation of heat and mass within the sea ice, and I assume
that there is no fluid flow within the sea ice, and that the bulk salinity is constant.
The surface of the sea ice is at \( z = 0 \), and the base of the sea ice is at \( z = h_{\text{ice}} \),
with \( z \) pointing vertically downwards. At the upper surface there is assumed to be
no melting, so that the net flux is zero (equation 4.20). At the ice–ocean boundary
the evolution of the interface is described by the Stefan condition (equation 4.23).
The temperature at the ice–ocean boundary is also assumed to be at the equilibrium
freezing temperature of the ocean (equation 4.22). The thermal conductivity \( k_m \) and
the volumetric specific heat capacity \( (\rho c)_m \) are assumed to satisfy the mixture rela-
tions (equations 4.2 and 4.3). Figure 5.1 shows an overview of the time-dependent
governing equations and boundary conditions for the sea ice only model.
Stationary solutions are formally those solutions of the system of equations that are independent of time (i.e. all time derivatives are zero). They represent the state of the sea ice subjected to persistent or slowly varying forcing. In order for stationary solutions to be physically realisable it is required that: the thickness of the ice is physically real \( h_{\text{ice}} > 0 \); the temperature of the ice is physically real (that is \( T > 0 \) K and \( T \) is less than the local freezing temperature); and the temperature profile has no extrema \( \frac{dT}{dz} \neq 0 \) except at an inflection point or on the boundaries. If the temperature profile had internal extrema then there would be a convergence or divergence of sensible heat within the ice, which would contradict the stationary hypothesis. A further requirement is that the stationary solutions are stable to infinitesimal perturbations. If they were not then they are unlikely to observed except in perfectly idealised situations.

Stationary solutions of the single slab model consist of a stationary temperature profile in the ice \( T(z) \) and a stationary ice thickness \( h_{\text{ice}} \). The stationary governing equation of temperature is a second-order nonlinear differential equation and to obtain a solution for any ice thickness I apply the two boundary conditions at the base of the sea ice \( z = h_{\text{ice}} \). The stationary ice thickness must satisfy the surface boundary condition. Therefore, the surface boundary condition can be used to determine the stationary ice thickness.

First, I reformulate the problem to determine the stationary temperature profile for any ice thickness. Second, I define the transcendental equation to determine the stationary ice thickness, which I call the Thickness Equation. Third, I consider the problem without any short-wave radiation and determine the stationary behaviour in this case. The case of no short-wave radiation corresponds to the wintertime case when there is permanent darkness, but could equally correspond to a situation where all of the incoming energy is being scattered very effectively over a very short lengthscale (e.g. in highly scattering white ice) and can be assumed to be absorbed at the surface. Fourth, I consider the problem with short-wave radiation. I consider the case with and without spectral variation and find that there can be two stationary solutions for a wide range of forcing data. Finally, I perform a simple
linear stability analysis on a non-mushy-layer model of sea ice to demonstrate that when there are two stationary solutions the smaller of the two solutions is unstable. This is then verified numerically for the full mushy layer model using the model described in chapter 4.

5.2 The governing equation of temperature

Assuming that stationary solutions do exist and there is no brine flow, the equation describing conservation of heat in the stationary case is

\[
\frac{d}{dz} \left( k_m \frac{dT}{dz} \right) = \frac{d}{dz} F_{net}(z). \tag{5.1}
\]

The boundary condition at \( z = h_{\text{ice}} \) is given by the stationary version of the Stefan condition so that

\[
k_m \frac{dT}{dz} - F_{\text{ocean}} = 0. \tag{5.2}
\]

Integrating equation (5.1) from \( h_{\text{ice}} \) to \( z \), and using the boundary condition at \( z = h_{\text{ice}} \) (equation 5.2), it is found that

\[
k_m \frac{dT}{dz} = G(z), \tag{5.3}
\]

where

\[
G(z) = F_{net}(z) - F_{net}(h_{\text{ice}}) + F_{\text{ocean}}. \tag{5.4}
\]

Assuming that the bulk salinity is constant we can determine the solid fraction, in the same way as in section 4.2, which is

\[
\phi(z) = \frac{T_L(C_{\text{bulk}}) - T}{T_L(C_s) - T}. \tag{5.5}
\]

The mixture relation for the thermal conductivity is \( k_m = \phi k_s + (1 - \phi) k_i \) so that substituting for \( k_m \) and \( \phi \) in equation (5.3) yields

\[
\frac{du}{dz} = \frac{G(z)}{k_s} (u - \tau_1), \tag{5.6}
\]
where

\[ u(z) = T(z) + \tau \]  \hspace{1cm} (5.7)

is the temperature translated by a constant,

\[ \tau = -\left( \frac{(k_s - k_l)T_L(C_{bulk}) + k_l T_L(C_s)}{k_s} \right), \quad \text{and} \]

\[ \tau_1 = \tau + T_L(C_s) = \frac{(k_s - k_l)(T_L(C_s) - T_L(C_{bulk}))}{k_s}. \]  \hspace{1cm} (5.9)

The constant temperature boundary condition (4.22) at \( z = h_{\text{ice}} \) for this first order nonlinear differential equation becomes

\[ u(h_{\text{ice}}) = T_L(C_{\text{ocean}}) + \tau, \]  \hspace{1cm} (5.10)

since it is assumed that any salt expelled at the ice–ocean interface is immediately advected into the ocean.

Equations (5.6) and (5.10) are the transformed governing equation for temperature and corresponding boundary condition. In its most general form, equation (5.6), together with its boundary condition (5.10), does not have an analytical solution and must therefore be solved numerically.

The nonlinearity of equation (5.6) is entirely due to the temperature dependency of the thermal conductivity. Assuming that the thermal conductivity is constant \((k_s = k_l)\) yields the far simpler expression

\[ k_s \frac{dT}{dz} = G(z), \]  \hspace{1cm} (5.11)

which implies that the stationary solutions are: monotonic increasing if \( G(z) > 0 \) in the sea ice domain; non-physical if \( G(z^*) = 0 \) for some \( z^* \in (0, h_{\text{ice}}) \); and monotonic decreasing if \( G(z) < 0 \) in the sea ice domain, since \( G(z) \) is a monotonic decreasing function of \( z \). The temperature profile is linear when there is no internal heating due to short-wave radiation.
5.3 The Thickness Equation

The temperature inside the ice is governed by equation (5.6) subject to the boundary condition at the lower boundary \( z = h_{\text{ice}} \) given by equation (5.10). However, for the temperature profile to be stationary we must also satisfy the surface boundary condition (section 4.5.1). The conductive heat flux in the surface boundary condition can be replaced by \(-G(0)\), using equation (5.3). To close the stationary problem and determine the stationary ice thickness we have the transcendental condition for the ice thickness that I call the Thickness Equation, given by

\[
F_{\text{net}}(h_{\text{ice}}) - F_{\text{net}}(0) - (1 - \epsilon_0)(1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) = \Gamma(h_{\text{ice}}),
\]

where \(\Gamma(h_{\text{ice}}) = F_{\text{ocean}} + F_{\text{lw}} - \epsilon_{\text{ice}}\sigma T_0^4 - F_{\text{sens}} - F_{\text{lat}}\), or equivalently

\[
F_{\text{net}}(h_{\text{ice}}) - (1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) = \Gamma(h_{\text{ice}}),
\]

since the net short-wave radiation at the surface corresponds to the radiation that is not reflected from the ice, and is transmitted through the boundary into the interior of the ice. Equation (5.13) states that the radiative energy absorbed by the whole sea ice system (incoming radiation - transmitted radiation) balances all the other heat fluxes incident on the sea ice system. Note that \(\Gamma(h_{\text{ice}})\) is the net flux on the sea ice excluding the short-wave radiation.

When there is short-wave radiation, the left-hand side of the Thickness Equation is clearly negative, since \(F_{\text{net}}(z)\) is a decreasing function of \(z\). Therefore, we may deduce that if any solutions are to exist in this case, it is necessary that \(\Gamma(h_{\text{ice}}) < 0\).

When there is no short-wave radiation the Thickness Equation reduces to the much simpler form

\[
\Gamma(h_{\text{ice}}) = 0.
\]

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5.4 Summary of the stationary problem

For reference, I summarise the stationary problem for the single slab model of sea ice. To determine the stationary solutions of the single slab model it is necessary to solve

\[ \frac{d u}{dz} = \frac{G(z)}{k_s} (u - \tau_1) \quad \text{subject to} \quad (5.15) \]

\[ u(h_{\text{ice}}) = T_L(C_{\text{ocean}}) + \tau \quad \text{and} \quad (5.16) \]

\[ \Gamma(h_{\text{ice}}) = F_{\text{net}}(h_{\text{ice}}) - (1 - \alpha_{\text{tot}}) F_{SW}(tot), \quad (5.17) \]

where

\[ u(z) = T(z) + \tau < T_L(C_{\text{bulk}}) + \tau < 0, \]

\[ G(z) = F_{\text{net}}(z) - F_{\text{net}}(h_{\text{ice}}) + F_{\text{ocean}}, \]

\[ \tau = -((k_s - k_L)T_L(C_{\text{bulk}}) + k_LT_L(C_s))/k_s < 0, \]

\[ \tau_1 = \tau + T_L(C_s) > 0, \]

and

\[ \Gamma(h_{\text{ice}}) = F_{\text{ocean}} + F_{\text{LW}} - \epsilon_{\text{ice}} \sigma T_0^4 - F_{\text{sens}} - F_{\text{lat}}. \]

5.5 No short-wave radiation

In this section, I neglect short-wave radiation completely. I analyse the behaviour of the stationary temperature profile (by analysing the translated temperature \( u \)) under different forcing regimes and determine an implicit solution. Using the Thickness Equation I set out the conditions necessary for stationary solutions to exist. For Arctic wintertime conditions it is seen that stationary solutions are only likely for positive ocean heat flux. The implicit formula for the temperature profile is used in conjunction with the Thickness Equation to determine an example equilibrium thickness for specific forcing data.

5.5.1 Analysis of translated temperature

In the case of no short-wave radiation \( F_{SW}(tot) = 0 \) so that \( G(z) = F_{\text{ocean}} \), a constant. The governing equation of temperature is then given by

\[ \frac{d u}{dz} = \frac{F_{\text{ocean}}}{k_s} (u - \tau_1), \quad (5.18) \]
with boundary condition (5.16). Equation (5.18) has implicit solution given by

\[ z = h_{\text{ice}} + \frac{k_s}{F_{\text{ocean}}} \left( u - u(h_{\text{ice}}) + \tau_1 \log \left( \frac{u - \tau_1}{u(h_{\text{ice}}) - \tau_1} \right) \right), \tag{5.19} \]

which has a turning/inflection point at \( u = 0 \), since \( dz/du = 0 \) has solution \( u = 0 \).

Under the assumption that \( k_s > k_l \) and \( T_L(C_s) > T_L(C_{\text{bulk}}) > T_L(C_{\text{ocean}}) \), which is true for sea ice, it is straightforward to show that \( \tau_1 > 0 \) and \( u(h_{\text{ice}}) < 0 \). Since it is required that the sea ice is below its equilibrium freezing temperature \( T_L(C_{\text{bulk}}) \) and since the corresponding translated equilibrium freezing temperature \( u_{\text{bulk}} = T_L(C_{\text{bulk}}) + \tau \) is less than zero, implicit solutions that could potentially have turning/inflection points and could therefore be undefined for the whole sea ice domain (i.e. those that contain \( u = 0 \)) can be neglected.

There are then three cases:

1. If \( F_{\text{ocean}} > 0 \) then \( du/dz > 0 \) and hence \( T(z) \) is monotonic increasing in \( z \).
   We therefore require, for physically real solutions to exist, that \( 0 \, \text{K} < T(z) \leq T_L(C_{\text{ocean}}) \).

2. If \( F_{\text{ocean}} = 0 \), then the solution of equation (5.18) is \( u(z) = T_L(C_{\text{ocean}}) + \tau \), the constant solution.

3. If \( F_{\text{ocean}} < 0 \) then \( du/dz < 0 \) and hence \( T(z) \) is monotonic decreasing in \( z \).
   We require that the temperature is less than the local freezing temperature \( T_L(C_{\text{bulk}}) \), where \( T_L(C_{\text{bulk}}) \leq -\tau \), so that there is no molten region within the sea ice. So we are more restricted and therefore we require, for physically real temperature profiles to exist, that \( T_L(C_{\text{ocean}}) \leq T(z) \leq T_L(C_{\text{bulk}}) \leq -\tau \), with equality between \( T_L(C_{\text{bulk}}) \) and \( -\tau \) occurring only when \( T_L(C_{\text{bulk}}) = T_L(C_s) \).

Hence, for no short-wave radiation there are three distinct cases of the stationary temperature profile for \( F_{\text{ocean}} > 0 \), \( F_{\text{ocean}} = 0 \), and \( F_{\text{ocean}} < 0 \). In these cases, \( T(z) \) is monotonic increasing, constant, and monotonic decreasing, respectively.
5.5.2 Stationary solutions with no short-wave radiation

The Thickness Equation for the case of no radiation is

\[ \Gamma(h_{ice}) = 0, \quad (5.20) \]

where \( \Gamma(h_{ice}) = F_{ocean} + F_{LW} - \epsilon_{ice} \sigma T_0^4 - F_{sens} - F_{lat} \).

For \( F_{ocean} > 0 \) we have a monotonic increasing temperature profile (section 5.5.1), which means that for the temperature \( T(z) \) to be physically real, it must satisfy

\[ F_{LW} + F_{ocean} - F_{sens} - F_{lat} \geq 0, \]

using the Thickness Equation (5.20) and the requirement that \( T_0 > 0 \) K. This condition is easily satisfied for wintertime conditions in the Arctic, since \( F_{LW} \approx 170 \) W m\(^{-2}\), \( F_{ocean} \approx 2 \) W m\(^{-2}\), \( F_{sens} \approx -20 \) W m\(^{-2}\), and \( F_{lat} \approx 0 \) W m\(^{-2}\) (Maykut and Untersteiner [76]).

For \( F_{ocean} < 0 \) we have a monotonic decreasing temperature profile, which means that for the temperature \( T(z) \) to be physically real, it must satisfy

\[ \epsilon_{ice} \sigma T_L(C_{ocean})^4 \leq F_{LW} + F_{ocean} - F_{sens} - F_{lat} \leq \epsilon_{ice} \sigma T_L(C_{bulk})^4, \]

using the Thickness Equation and the fact that \( T_L(C_{ocean}) \leq T_0 \leq T_L(C_{bulk}) \). This condition is highly restrictive and is unlikely to be satisfied in the Arctic wintertime, since the range of the inequality is given by \((303.7 \) W m\(^{-2}\), \(310.7 \) W m\(^{-2}\)).

If \( F_{ocean} = 0 \) then the Thickness Equation implies that we only have a solution if

\[ F_{LW} + F_{ocean} - F_{sens} - F_{lat} - \epsilon_{ice} \sigma T_L(C_{ocean})^4 = 0. \]

The stationary thickness in the case of zero oceanic heat flux is not unique because the temperature profile corresponds to the constant solution, and does not vary with depth.

It is straightforward in the case of no short-wave radiation to determine the stationary ice thickness analytically. The Thickness Equation (5.20) for the case of no
Table 5.1: Parameter values used with no short-wave radiation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_L(C_{\text{ocean}}))</td>
<td>271.2 K</td>
</tr>
<tr>
<td>(T_L(C_{\text{bulk}}))</td>
<td>272.6 K</td>
</tr>
<tr>
<td>(T_L(C_s))</td>
<td>273 K</td>
</tr>
<tr>
<td>(F_{\text{ocean}})</td>
<td>5 W m(^{-2})</td>
</tr>
<tr>
<td>(F_{\text{LW}})</td>
<td>220 W m(^{-2})</td>
</tr>
<tr>
<td>(F_{\text{sens}})</td>
<td>-5 W m(^{-2})</td>
</tr>
<tr>
<td>(F_{\text{lat}})</td>
<td>1.7 W m(^{-2})</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>(5.67 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4})</td>
</tr>
<tr>
<td>(\epsilon_{\text{ice}})</td>
<td>0.99</td>
</tr>
<tr>
<td>(k_s)</td>
<td>2.0 W m(^{-1}) K(^{-1})</td>
</tr>
<tr>
<td>(k_l)</td>
<td>0.5 W m(^{-1}) K(^{-1})</td>
</tr>
</tbody>
</table>

short-wave radiation can be solved immediately to determine the unique positive surface temperature. Setting \(z = 0\) in equation (5.19) and substituting for the surface temperature yields an expression for the equilibrium ice thickness. This equilibrium ice thickness is unique, since the surface temperature is unique, so equation (5.19) can be used to prove that there can only be one possible ice thickness.

An example stationary temperature profile and solid fraction profile is presented in figures 5.2 and 5.3, respectively, for the parameter values in table 5.1. The salinity of the ocean, the bulk salinity, and the salinity of the solid ice component of sea ice are assumed to be 35 ppt, 6 ppt, and 0 ppt, respectively. The atmospheric forcing fluxes were the values from April used by Maykut and Untersteiner [77]. The ocean heat flux is highly variable both spatially and temporally (Perovich and Elder [92]).

I choose the ocean heat flux to be a very approximate value for late spring from Perovich and Elder [92] to coincide with the atmospheric forcing data. The thermal conductivities are approximated from Schwerdtfeger [110], but assumed constant. The stationary ice thickness is \(h_{\text{ice}} = 7.239\) m. The corresponding temperature and solid fraction profiles were determined by solving the governing equation for the temperature numerically using the mathematical software Mathematica version 3.0 and are shown in figures 5.2 and 5.3 respectively. The surface temperature is 252.5 K with corresponding solid fraction \(\phi = 0.985\). The solid fraction at the ice–ocean interface is \(\phi = 0.827\).
**Figure 5.2:** Temperature profile inside sea ice with no short-wave radiation. Equilibrium ice thickness = 7.239 m

**Figure 5.3:** Solid fraction profile inside sea ice with no short-wave radiation. Equilibrium ice thickness = 7.239 m
5.6 Short-wave radiation

Short-wave radiation is now considered. In a similar way to the case with no short-wave radiation the behaviour of the temperature is considered. The case without any spectral variation is analysed and the assumption that $\Gamma(h_{ice})$ is constant is used to simplify analysis. For Arctic forcing data it can be shown that there is the potential for two stationary solutions under this simplification. Dropping the assumption that $\Gamma$ is constant again permits two stationary solutions and an example pair are presented. The case with spectral variation is considered numerically and two stationary solutions are found for the forcing data. Finally, the sensitivity of the stationary solutions in the full spectral model to the forcing data is determined and two stationary solutions are shown to exist for a wide range of forcing data.

5.6.1 Analysis of translated temperature

The governing equation of temperature when there is short-wave radiation is given by (5.15) with boundary condition (5.16).

Analogous to the case of no short-wave radiation we can relate the sign of $\frac{du}{dz}$ to the sign of $G(z)$. Since $F_{\text{net}}(z)$ is monotonic decreasing in $z$ we can deduce immediately, from the definition of $G(z)$, and given that $F_{SW}(tot) \neq 0$, that $G(z)$ is also monotonic decreasing in $z$.

As for no short-wave radiation, there are three separate cases:

1. If there exists $z^* \geq h_{ice}$ such that $G(z^*) = 0$, then $F_{ocean} \geq 0$ (by the definition of $G(z)$) and $G(z) \geq 0$ for all $z$ satisfying $0 \leq z \leq h_{ice}$. Therefore, by continuity, $\frac{du}{dz} \geq 0$ for $z$ satisfying $0 \leq z \leq h_{ice}$ and hence $T(z)$ is monotonic increasing in $z$. Then we require that the temperature satisfies $0 \, K < T(z) \leq T_L(C_{ocean})$.

2. If there exists $0 < z^* < h_{ice}$ such that $G(z^*) = 0$, then if $u(z) < 0$ for all $0 \leq z \leq h_{ice}$ then there is a turning point in $u$ at $z = z^*$ (and an internal
molten region if \( u(z^*) > -\tau \). This would correspond to a point in the ice where sensible heat could build up and contradict the stationary hypothesis.

3. If there exists \( z^* \leq 0 \) such that \( G(z^*) = 0 \), then \( G(z) \leq 0 \) for all \( z \) satisfying \( 0 \leq z \leq h_{ice} \), and there exists a unique, critical negative oceanic heat flux \( F_{ocean}^* \) that corresponds to \( G(0) = 0 \). Therefore, by continuity, \( du/dz \leq 0 \) for \( z \) satisfying \( 0 \leq z \leq h_{ice} \) and hence \( T(z) \) is monotonic decreasing in \( z \). As before (section 5.5.1), we require that \( T_L(C_{ocean}) \leq T(z) \leq T_B(C_{bulk}) \).

Therefore, when the short-wave radiation is non-zero, there are two possible cases of the stationary temperature profile for \( F_{ocean} \geq 0 \) and \( F_{ocean} \leq F_{ocean}^* \). In these cases \( T(z) \) is monotonic increasing and decreasing, respectively.

### 5.6.2 Thickness Equation with no spectral variation

**Constant \( \Gamma(h_{ice}) \)**

For simplicity, I will assume that \( \Gamma(h_{ice}) \) is constant and show that in this case there can be two physically real solutions (i.e. \( h_{ice} > 0 \)) to the Thickness Equation. In full, using the two-stream formulation and abbreviations from section 3.4, the Thickness Equation under the assumption that \( \Gamma(h_{ice}) = \Gamma \) a constant becomes

\[
F_{SW}(1 - R_0)(1 - s_2)(-1 + i_0(1 + s_2)Y_2^{-1} - s_2Y_2^{-2}) + \Gamma(-1 + s_2^2Y_2^{-2} + R_0s_2(1 - Y_2^{-2})) = 0, \tag{5.21}
\]

where \( Y_2 = \exp(-\kappa_2h_{ice}) \), and the 2 subscripts are a legacy of the three-layer two-stream model, referring to ice properties.

For a physically real ice thickness solution, \( h_{ice} \), of the Thickness Equation it is required that \( h_{ice} > 0 \), which means that \( e^{\kappa_2h_{ice}} > 1 \) must be true. Equation (5.21) is equivalent to the quadratic in exponentials

\[
\alpha e^{2\kappa_2h_{ice}} + \beta e^{\kappa_2h_{ice}} + \gamma = 0, \text{ where} \tag{5.22}
\]
\[ \alpha = -F_{SW}(1 - R_0)(1 - s_2) - \Gamma(1 - R_0 s_2), \quad (5.23) \]
\[ \beta = F_{SW} i_0 (1 - R_0)(1 - s_2^2), \quad (5.24) \]
\[ \gamma = s_2 (\Gamma s_2 - R_0) - F_{SW} (1 - R_0)(1 - s_2). \quad (5.25) \]

There are potentially two solutions (since we are dealing with a quadratic equation), but it is not clear if both can be physically real solutions for the ice thickness \( h_{\text{ice}} \) (i.e. both positive). I analyse the solutions and determine conditions for when they are physically real, which is equivalent to \( e^{\gamma h_{\text{ice}}} > 1 \), and show that for parameters appropriate to the Arctic there can be two physically possible stationary ice thicknesses.

Consider any quadratic equation of the form \( \alpha x^2 + \beta x + \gamma = 0 \), with solutions \( x_1 = (-\beta + \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha \) and \( x_2 = (-\beta - \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha \). The conditions for the solutions, \( x_1 \) and \( x_2 \), to be greater than 1 (assuming a non-negative discriminant) are summarised in table 5.2 (see appendix C). For example, referring to table 5.2, if \( \alpha > 0 \) and \( \beta > 0 \), then it is only possible to have one solution greater than 1, and this is so when \( \alpha + \beta + \gamma < 0 \). This information also tells us whether a solution to a quadratic in exponentials such as equation (5.22) is positive (i.e. \( h_{\text{ice}} > 0 \) since this is equivalent to \( e^{\gamma h_{\text{ice}}} > 1 \)).

The fundamental parameters of the quadratic equation that determine whether a solution is greater than one are \( \alpha, \beta, 2\alpha + \beta \) and \( \alpha + \beta + \gamma \). Assume that a solution, \( h_{\text{ice}} \), of (5.22) is real valued (possibly negative) so that the discriminant is greater than, or equal to, zero. By the physical definitions of the parameters \( F_{SW}, i_0, R_0, \) and \( s_2 \), we have that \( F_{SW} > 0, i_0 > 0, (1 - R_0) > 0, \) and \( (1 - s_2^2) > 0 \), so that \( \beta > 0 \). Therefore only half of the cases in table 5.2 are applicable to equation (5.22), so it is only necessary to consider the parameters \( \alpha, 2\alpha + \beta \) and \( \alpha + \beta + \gamma \).

The aim is to use the information about arbitrary quadratics in table 5.2 to deduce specific information about equation (5.22). Specifically I will show that for parameters commensurate with sea ice, there is the possibility of two stationary solutions. I will therefore determine what the equivalent inequalities are to \( \alpha < 0, 2\alpha + \beta > 0, \)
CHAPTER 5

SECTION 5.6

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<th>(x_2 &gt; 1)</th>
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<td>&lt; 0</td>
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<td>= 0</td>
<td>Not Possible</td>
<td>Automatically True</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>Not Possible</td>
<td>(α + β + γ &gt; 0)</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>≤ 0</td>
<td>N/A</td>
<td>Not Possible</td>
<td>(α + β + γ &gt; 0)</td>
</tr>
</tbody>
</table>

Table 5.2: Conditions for solutions \((x_1, x_2)\) of a quadratic equation \((αx^2 + βx + γ = 0)\) to be greater than one

and \(α + β + γ < 0\), in terms of \(F_{SW}, Γ, i_0, R_0\), and \(s_2\), to see when two solutions are possible. By means of straightforward algebra it can be shown that

\[
\begin{align*}
α < 0 & \text{ is equivalent to } F_{SW} > ΓX^*, \\
2α + β > 0 & \text{ is equivalent to } F_{SW} < ΓY^*, \\
\text{and } α + β + γ < 0 & \text{ is equivalent to } F_{SW} > ΓZ^*,
\end{align*}
\]

where

\[
\begin{align*}
X^* &= -\frac{(1 - R_0 s_2)}{(1 - R_0)(1 - s_2)}, \\
Y^* &= \frac{2(1 - R_0 s_2)}{(1 - R_0)(1 - s_2)(-2 + i_0 + i_0 s_2)}, \\
\text{and } Z^* &= -\frac{1}{(1 - i_0)(1 - R_0)}.
\end{align*}
\]

The relative position of the lines \(F_{SW} = X^*Γ\), \(F_{SW} = Y^*Γ\) and \(F_{SW} = Z^*Γ\) in \(Γ-F_{SW}\) space are important because the position of these lines relative to one another determines the number of solutions to the Thickness Equation. I analyse the relative positions of these lines by considering their gradients, \(X^*, Y^*\) and \(Z^*\).
The restrictions that must be satisfied for there to be two stationary solutions are

\[ \Gamma Y^* > F_{SW} > \Gamma X^* \quad \text{and} \quad \Gamma Y^* > F_{SW} > \Gamma Z^*. \]  

(5.26) \hspace{1cm} (5.27)

I assume that \( s_2 = 0.73 \) is a typical wintertime value, which corresponds to frozen multi-year white ice (Grenfell and Maykut [45]), \( R_0 = 0.05 \), and \( i_0 = 0.4 \). The gradient \( X^* \) is greater than the gradient \( Y^* \) if and only if \( i_0(1 + s_2) > 0 \), which is always true for physically real parameters.

It is straightforward to show that for possible values of \( s_2 \), the gradient \( Z^* \) is greater than the gradient \( Y^* \) if and only if

\[ i_0 < \frac{2s_2(1 - R_0)}{1 - 2R_0s_2 + s_2^2}. \]

For typical sea ice values such as those above the right hand side of this inequality is 0.94 (\( s_2 = 0.73 \), \( R_0 = 0.05 \), \( i_0 = 0.4 \)) and so the gradient \( Z^* \) is greater than the gradient \( Y^* \) in this case.

The previous two statements imply that there are potentially two stationary solutions (depending on the incoming short-wave radiation and the value of \( \Gamma \)) to the Thickness Equation. Finally, the relationship between \( X^* \) and \( Z^* \) is that the gradient \( X^* \) is greater than the gradient \( Z^* \) if and only if \( s_2 < i_0/(1 - R_0 + i_0R_0) \). For the values given above (\( s_2 = 0.73 \), \( R_0 = 0.05 \), \( i_0 = 0.4 \)) the inequality is false, so that the gradient \( X^* \) is typically less than the gradient \( Z^* \).

Figure 5.4 shows the \( \Gamma - F_{SW} \) plane with the boundaries of the inequalities that are necessary to determine if there are two stationary solutions to the Thickness Equation. The parameters used are \( s_2 = 0.73 \), \( R_0 = 0.05 \), and \( i_0 = 0.4 \). It is clear that there is a significant range of the \( \Gamma - F_{SW} \) plane (shaded grey) that possess two solutions.

**Variable \( \Gamma(h_{ice}) \)**

I now consider the Thickness Equation (5.17) with a single band of radiation and variable \( \Gamma(h_{ice}) \). I will consider the behaviour of both sides of the Thickness Equation
Two stationary solutions

\[ F_{SW} = Y \Gamma \]
\[ F_{SW} = X \Gamma \]
\[ F_{SW} = Z \Gamma \]

Figure 5.4: Net fluxes on sea ice excluding short-wave radiation (\( \Gamma \)) against short-wave radiation (\( F_{SW} \)), showing region with two stationary solutions. The three lines are the boundaries of the inequalities that determine the number of stationary solutions

individually and then consider the actual solution to the complete equation.

The monotonic decreasing function \( G(z) \) (equation 5.4) in the governing equation of temperature makes the variation of surface temperature with ice thickness more complicated. The function \( \Gamma(h_{ice}) \) is defined in terms of the surface temperature \( T_0 = T(z = 0) \) and therefore to describe its variation with ice thickness \( h_{ice} \) it is necessary to determine the variation of \( T_0 \) with \( h_{ice} \). I will show that, for \( F_{ocean} \geq 0 \), \( T_0 \) is a strictly decreasing function of \( h_{ice} \).

Let the \( h_{ice} \) dependence of \( u \) (the translated temperature) be written explicitly into the argument of \( u \) (i.e. \( u(z) = u(z, h_{ice}) \)). Suppose that \( h_1 < h_2 \) and suppose \( u(z^*, h_1) = u(z^*, h_2) \) for some \( 0 < z^* < h_1 \), where

\[ z^* = \max \{ \tilde{z} : u(\tilde{z}, h_1) = u(\tilde{z}, h_2), \tilde{z} \in [0, h_1] \}. \]

Since \( u \) is monotonic increasing in \( z \), for \( F_{ocean} \geq 0 \), and \( u(h_1, h_1) = u(h_2, h_2) \) (see section 5.2) it follows from continuity that \( du/dz(z^*, h_1) \geq du/dz(z^*, h_2) \). However, since \( G(z, h_1) < G(z, h_2) \) for all \( 0 < z < h_1 \) (see appendix D), we must have that \( du/dz(z^*, h_1) < du/dz(z^*, h_2) \) (using equation 5.6) and we have a contradiction.
must therefore conclude that \( u(z^*, h_1) = u(z^*, h_2) \) is never true. Hence, \( u(z, h_1) > u(z, h_2) \), and \( T(z, h_1) > T(z, h_2) \), for all \( h_1 < h_2 \), and thus, when \( F_{\text{ocean}} \geq 0 \), \( T \) is a monotonic decreasing function of \( h_{\text{ice}} \) (for fixed \( z \)). We can therefore deduce that \( T_0 \) is a monotonic decreasing function of \( h_{\text{ice}} \), for \( F_{\text{ocean}} \geq 0 \).

From the definition of \( \Gamma(h_{\text{ice}}) \) it is clear that \( \Gamma(h_{\text{ice}}) \) is a decreasing function of surface temperature for positive temperatures. Therefore, for positive oceanic heat fluxes \( \Gamma(h_{\text{ice}}) \) is an increasing function of ice thickness.

When \( F_{\text{ocean}} < 0 \) the surface temperature cannot be shown to be monotonic with ice thickness, although numerical observations appear to demonstrate that, for sea ice, it is typically the case (the only ice thicknesses that have temperature profiles defined for the whole ice domain are generally small and have approximately linear temperature profiles).

I now analyse the radiative terms of the Thickness Equation (equation 5.17) and determine their limiting values. Denote the right hand side of equation (5.17) by \( a \), which corresponds physically to the negative of the short-wave radiation absorbed by the sea ice.

The turning points of \( a \) (with respect to \( h_{\text{ice}} \)) satisfy

\[
i_0(1 - R_0 s_2) - 2s_2(1 - R_0)Y_2^{-1} + i_0 s_2(s_2 - R_0)Y_2^{-2} = 0, \tag{5.28}
\]

which is determined by differentiating \( a \) with respect to \( h_{\text{ice}} \) and setting equal to zero. Again this is a quadratic in exponentials, but this time

\[
\alpha = i_0(1 - R_0 s_2), \tag{5.29}
\]
\[
\beta = -2s_2(1 - R_0), \quad \text{and} \tag{5.30}
\]
\[
\gamma = i_0 s_2(s_2 - R_0). \tag{5.31}
\]

If there are any turning points at all (so that there are real solutions to equation (5.28) then table 5.2 can be used to determine how many turning points there are for positive ice thicknesses. It can be shown that for physically real parameters, it is not possible to obtain two turning points for positive ice thicknesses (see appendix E). Therefore, there are only one or zero turning points for positive ice thickness.
I now consider the limiting values of the radiative term. Substituting for the two-stream radiative model yields

\[ a = \frac{F_{SW}(1 - R_0)(1 - s_2)((-1 + i_0)(1 + s_2)Y_2^{-1} - s_2 Y_2^{-2})}{1 - s_2^2 Y_2^{-2} - R_0 s_2 (1 - Y_2^{-2})}, \]

with \( \lim_{h_{\text{ice}} \to 0} a = -F_{SW}(1 - R_0)(1 - i_0) \) and \( \lim_{h_{\text{ice}} \to \infty} a = -\frac{F_{SW}(1 - R_0)(1 - s_2)}{1 - R_0 s_2} \).

Clearly, \( a \leq 0 \) always, and \( \lim_{h_{\text{ice}} \to 0} a < \lim_{h_{\text{ice}} \to \infty} a \) if and only if

\[ s_2 > \frac{i_0}{(1 - R_0 + R_0 i_0)}, \]

which is typically true for sea ice values. For example, if \( s_2 = 0.73 \), \( i_0 = 0.4 \), and \( R_0 = 0.05 \), then \( i_0/(1 - R_0 + R_0 i_0) = 0.412 < 0.73 \), so that the inequality (5.33) is true.

Solutions of the Thickness Equation correspond geometrically to the intersection of
Table 5.3: Additional parameter values used for a single band of short-wave radiation

<table>
<thead>
<tr>
<th>$F_{SW} = 190 \text{ W m}^{-2}$</th>
<th>$R_0 = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_2 = 0.73$</td>
<td>$\kappa_2 = 3.0 \text{ m}^{-1}$</td>
</tr>
</tbody>
</table>

$\Gamma_0 > \lim_{h \to 0} a$; \hspace{1cm} (5.34)

$\lim_{h \to 0} a < \lim_{h \to \infty} a$; \hspace{1cm} (5.35)

there is at least one solution (not a turning point of $a$); and \hspace{1cm} (5.36)

$F_{\text{ocean}} > 0$. \hspace{1cm} (5.37)

These conditions are typically true for wide range of sea ice values. Therefore, if a solution can be found numerically (relating to condition 5.36) for positive oceanic heat fluxes (condition 5.37) then if conditions (5.34) and (5.35) are satisfied almost certainly there is a second solution. The situation that this would not be the case is when the numerical solution corresponds to a turning point in $a$ with respect to ice thickness.

The Thickness Equation for a single band of radiation and positive oceanic heat flux was solved numerically for values in tables 5.1 and 5.3. The value for the extinction coefficient is approximately that for interior white ice with wavelength approximately 700 nm and was obtained from Perovich [89] (originally from Grenfell and Maykut [45]). For simplicity, the absorption coefficient of bubble free ice was used in determining $s_2$ (since brine volumes are low), and was obtained from Grenfell and Perovich [46], and was also taken at a wavelength approximately 700 nm. The Thickness Equation yielded two genuine stationary solutions (0.0573 m and 1.160 m). Figure 5.5 shows the graph of $a$ and $\Gamma(h_{\text{ice}})$ against $h_{\text{ice}}$. It can be clearly seen that $\Gamma(h_{\text{ice}})$ is increasing with ice thickness and that the radiative term, $a$, possesses a single turning point in the positive range. The temperature profiles and corresponding solid fraction profiles are shown in figures 5.6-5.9. It is found that
5.6.3 Thickness Equation with spectral variation

This is the only section of this thesis for which there will be spectral variation. If I assume that the net short-wave flux and albedo are all defined using a one-layer n-band two-stream radiative model (see section 3.4), the Thickness Equation (5.17) can be written as

\[ \sum_{j=1}^{n} a_j = \Gamma(h_{\text{ice}}), \]

where \( a_j = (F_{\text{net}(2)}(h_{\text{ice}}) - (1 - \alpha_{\text{tot}})F_{\text{SW}(j)}) \Lambda_j \), and \( j = 1, \ldots, n \) denotes the waveband, using the notation from section 3.6 for the spectral radiation model (the 2 subscript again corresponds to legacy notation denoting the lowermost ice layer in the three-layer two-stream radiation model).

Analytical analysis of the n-band model is prohibitively cumbersome. Figures 5.10 and 5.11 show the discretised spectral short-wave radiation and spectral extinction coefficient, respectively, for an 8-band model. Both the incident short-wave radiation spectrum and the extinction coefficients are approximated from Grenfell [41]. The
**Figure 5.7:** Solid fraction profile inside sea ice with single band of short-wave radiation. Equilibrium ice thickness = 0.0573 m

**Figure 5.8:** Temperature profile inside sea ice with single band of short-wave radiation. Equilibrium ice thickness = 1.160 m
Figure 5.9: Solid fraction profile inside sea ice with single band of short-wave radiation. Equilibrium ice thickness = 1.160 m

short-wave spectrum is that of a cloudy sky, and was scaled uniformly across all wavelengths to produce the required incoming short-wave flux identical in magnitude to the single-band case. The extinction coefficients correspond to interior white ice.

It is not obvious that the $\sum a_j$ term of the Thickness Equation has the same properties as its individual components. Using the data from tables 5.3 and 5.4, the Thickness Equation was solved numerically for an 8-band model using the mathematical software Mathematica version 3.0. With these parameters, the Thickness Equation yields two solutions at 0.021 m and 2.755 m. Figure 5.12 shows the graph of $\sum_{j=1}^{8} a_j$ and $\Gamma(h_{\text{ice}})$ against $h_{\text{ice}}$. Again, since $F_{\text{ocean}}$ is positive, $\Gamma(h_{\text{ice}})$ is increasing at the intersection points and so the solutions for the ice thickness are genuine stationary solutions (i.e. the temperature profile remains positive). Figures 5.13–5.16 show the corresponding temperature and solid fraction profiles.

Variation of stationary solutions with forcing data

The variation of the equilibrium ice depth with the various energy fluxes are now considered. The Thickness Equation was rearranged so that it was of the form $p(h_{\text{ice}}) = 0$, where $p$ is a continuous function of $h_{\text{ice}}$. For a given set of forcing data,
Figure 5.10: Discrete spectral short-wave radiation ($F_{SW(j)}$)

Figure 5.11: Discrete spectral extinction coefficient ($\kappa_2^j$)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{SW(j)}^j$ (W m$^{-2}$nm$^{-1}$)</td>
<td>0.109</td>
<td>0.636</td>
<td>0.372</td>
<td>0.310</td>
<td>0.225</td>
<td>0.116</td>
<td>0.116</td>
<td>0.016</td>
</tr>
<tr>
<td>$\kappa_2^j$ (m$^{-1}$)</td>
<td>1.266</td>
<td>1.236</td>
<td>1.546</td>
<td>4.2</td>
<td>14.32</td>
<td>49.32</td>
<td>122.17</td>
<td>140</td>
</tr>
<tr>
<td>$s_2^j$</td>
<td>0.865</td>
<td>0.925</td>
<td>0.856</td>
<td>0.780</td>
<td>0.737</td>
<td>0.788</td>
<td>0.714</td>
<td>0.757</td>
</tr>
<tr>
<td>$\Lambda_j$ (nm)</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 5.4: Additional parameter values used for 8 spectral bands of short-wave radiation
Figure 5.12: Negative short-wave radiation absorbed in sea ice ($\sum_{j=1}^{8} a_j$ W/m$^2$) and net flux on sea ice excluding short-wave radiation ($\Gamma$ W/m$^2$) against ice thickness. Intersections of $\sum_{j=1}^{8} a_j$ and $\Gamma$ represent equilibrium ice thicknesses ($h_{\text{ice}}$).

Figure 5.13: Temperature profile inside sea ice with spectrally varying short-wave radiation. Equilibrium ice thickness = 0.021 m
Figure 5.14: Solid fraction profile inside sea ice with spectrally varying short-wave radiation. Equilibrium ice thickness = 0.021 m

Figure 5.15: Temperature profile inside sea ice with spectrally varying short-wave radiation. Equilibrium ice thickness = 2.755 m
the function $p$ was determined at several depths. Solutions $h_{\text{ice}}$ of the Thickness Equation occur between adjacent depths across which $p$ changes sign. If more than one solution occurred between two adjacent depths then this would not be known. The zeros were found using an optimised root-finding algorithm to an accuracy of three significant figures.

Figure 5.17 shows the variation of $h_{\text{ice}}$ with $F_{\text{LW}}$ for parameters given by tables 5.3 and 5.4. Clearly there is a significant range of $F_{\text{LW}}$ for which there exist two equilibrium ice thicknesses for the same forcing data.

Figure 5.18 shows the variation of equilibrium ice thickness $h_{\text{ice}}$ for both positive and negative ocean heat flux $F_{\text{ocean}}$. As for the long-wave radiation, it can be seen that for positive values of $F_{\text{ocean}}$, less than $35.3 \text{ W m}^{-2}$, there are two possible equilibrium ice thickness solutions.

For negative values of ocean heat flux only one physically real stationary ice thicknesses can occur and these can only occur for $F_{\text{ocean}} < F^*_{\text{ocean}} = -0.26 \text{ W m}^{-2}$. Solutions with negative oceanic heat flux greater than $F^*_{\text{ocean}}$ have turning points in their temperature profiles due to the change in sign of $G(z)$ in the interior of the sea ice and so do not satisfy the stationary hypothesis. The reason that there is only
one stationary solution is that the temperature is monotonic decreasing, so that larger ice thicknesses are likely to possess temperature profiles that contain molten regions, turning points, or are undefined for the whole domain.

As the ocean heat flux tends to zero from above the larger stationary ice thickness tends to infinity. This is because the surface temperature becomes decoupled from the ice thickness. At large ice thicknesses the radiation only influences the surface of the ice. For zero ocean heat flux this corresponds to a decoupling of the relationship between ice thickness and surface temperature, since at large depths, where shortwave radiation cannot penetrate, the temperature gradient will be zero and will therefore have constant temperature. The surface temperature in the case of zero ocean heat flux can then be determined from the Thickness Equation (5.17), since $F_{\text{net}}(h_{\text{ice}}) \to 0$ as $h_{\text{ice}} \to \infty$, and so the surface temperature can be obtained from the relationship

$$T_0^4 = (1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) + F_{\text{LW}} - F_{\text{sens}} - F_{\text{lat}}.$$  \hspace{1cm} (5.39)

For large ice depths, and non-zero ocean heat flux, the surface temperature depends on the ice thickness and so the ice thickness and surface temperature are still coupled in this case.

**Figure 5.17:** Variation of stationary ice thickness with incoming long-wave flux
5.7 Linear stability analysis of stationary solutions of a simplified model

The two-stream radiation model leads to potentially two stationary solutions for the single layer of sea ice, with constant imposed forcing. The existence of the second stationary solution is due wholly to the two-stream radiation model and not because of the nonlinearity introduced by using the mushy layer formulation. If the stationary states are stable, then perturbations will decay to zero, and if the stationary states are unstable, then the perturbations will grow. Linear stability analysis can be used to examine the stability of the system to infinitesimal perturbations. By linearising the system about its stationary state and examining the evolution of the linearised system one can determine if the system is stable or unstable.

In this section, I investigate the stability to infinitesimal perturbations of the simpler system of a solid medium with constant thermal conductivity coupled to the two-stream radiation model, using the same boundary conditions as the full mushy-layer model. Essentially, the internal latent heat release due to phase change is neglected.
I assume that the results of these analyses generalise to the case using the mushy layer model. It will be seen that when there are two solutions, the larger solution is stable to infinitesimal perturbations and the smaller solution is unstable to infinitesimal perturbations. Finally, I verify this result for the full mushy layer model using the numerical model described in chapter 4.

5.7.1 Non-dimensional governing equations

Consider a translucent solid material of similar properties to sea ice that has constant thermal conductivity (of the same magnitude as sea ice), and constant optical properties that can be described using the two-stream radiation model. The material is assumed to be infinite and homogeneous in horizontal extent, and finite and homogeneous in vertical extent with domain \([0, h]\). The imposed boundary conditions are the same as the sea ice model described in chapter 4.

I non-dimensionalise lengths with a typical sea ice depth, \(L\), times with the thermal diffusive time-scale \((\rho c)_l L^2/k_l\), temperature with a typical temperature difference, \(\Delta T = (T_L(C_s) - T_L(C_E))\), where \(C_E\) is the eutectic composition, and boundary fluxes with a typical short-wave radiative flux \(F\). The surface boundary condition (equation 4.20) is then

\[
\theta' = \varepsilon(\theta + \hat{A})^4 + \mathcal{F}\hat{G} - \mathcal{F}(1 - i_0)(1 - \alpha_{tot})F_{SW}(tot), \quad (z = 0), \quad (5.40)
\]

where the prime denotes differentiation with respect to \(z\), \(\theta\) is the non-dimensional temperature, and \(\hat{A} = T_L(C_s)/\Delta T\), \(\hat{G} = F_{LW} - F_{sens} - F_{lat}\),

\[
\varepsilon, \quad F = \frac{FL}{k_l \Delta T}, \quad S = \frac{\rho_s L}{(\rho c)_l \Delta T},
\]

are non-dimensional parameters.

Heat transport within the solid ice is described by the diffusion equation with source term determined from the two-stream radiation model. Expressed in non-dimensional notation, which can be obtained from the mushy layer heat equation (4.1) by setting the solid fraction to unity and assuming that the volumetric specific
heat capacity and the thermal conductivity are constant, this is

\[ \dot{\theta} = \theta'' - \mathcal{F}_{\text{net}}, \]

(5.41)

where \( \dot{\cdot} \) is used to denote differentiation with respect to time. The Stefan condition (4.23) describes the evolution of the lower boundary and is given by

\[ \mathcal{S} h = \theta' - \mathcal{F}_{\text{ocean}}, \quad (z = h), \]

(5.42)

and it is also assumed that the temperature of the lower boundary is at the non-dimensional equilibrium freezing temperature of the ocean (4.22), so that

\[ \theta = \theta_{\text{ocean}}, \quad (z = h). \]

(5.43)

Figure 5.19 shows the stationary solutions of this model for varying oceanic heat flux using the forcing fluxes and optical properties for the one-band radiation model in tables 5.1 and 5.3. The thermal conductivity of the ice was taken to be constant and equal to the value for pure ice (2 W/mK).

### 5.7.2 Perturbation equations

I consider infinitesimal perturbations to the stationary solution \( \{\theta_{ss}(z), h_{ss}\} \), which consists of the stationary temperature profile \( \theta_{ss}(z) \) (see appendix F) and stationary
ice thickness $h_{ss}$, and examine their initial evolution. The perturbed temperature
and height are given by

$$\theta = \theta_{ss}(z) + \hat{\theta}(z)e^{\sigma t}, \quad \text{and}$$

$$h = h_{ss} + \hat{h}e^{\sigma t},$$

where $\hat{\theta}$ and $\hat{h}$ are the initial amplitudes of the perturbations (assumed to be small) and $\sigma$ can be complex.

Upon substitution of the perturbed solution into the equation describing conservation of heat (5.41), using Taylor series expansions about the stationary ice thickness and neglecting products of small terms, we have

$$\sigma\hat{\theta} = \frac{\partial^2 \hat{\theta}}{\partial z^2},$$

having used the stationary relation $\theta''_{ss} = F'_{\text{net}}$ to remove the source term. Since this equation is diffusive, and the forcing data are independent of time, the transition from stability to instability is not expected to occur via oscillatory solutions so that $\sigma$ is anticipated to be real.

Similarly, the temperature condition (5.43) at the base of the sea ice becomes

$$\hat{\theta} = -\hat{h} \left( \frac{\partial \theta_{ss}}{\partial z} + \frac{\partial \theta_{ss}}{\partial h} \right), \quad (z = h_{ss}),$$

where I have used the temperature condition at the base of the ice to eliminate $\theta_{ocean}$. The Stefan condition (5.42) and the surface condition (5.40) become

$$S \sigma \hat{h} = \hat{h} \left( \frac{\partial^2 \theta_{ss}}{\partial z^2} + \frac{\partial \theta_{ss}}{\partial h} \right) + \frac{\partial \hat{\theta}}{\partial z}, \quad (z = h_{ss}),$$

and

$$\hat{h} \frac{\partial^2 \theta_{ss}}{\partial z \partial h} + \frac{\partial \hat{\theta}}{\partial z} = 4\mathcal{E}(\theta_{ss}(h_{ss}) + \hat{\theta}) \left( \hat{h} \frac{\partial \theta_{ss}}{\partial h} + \hat{\theta} \right) + F(1-i_0)F_{SW}(\text{tot})\hat{h} \frac{\partial \alpha_{\text{tot}}}{\partial h}, \quad (z = 0).$$

The definition of $\theta_{ss}$ implies that

$$\frac{\partial \theta_{ss}}{\partial z} + \frac{\partial \theta_{ss}}{\partial h} = 0, \quad (z = h_{ss}),$$

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and

\[ \frac{\partial^2 \theta_{ss}}{\partial z^2} + \frac{\partial^2 \theta_{ss}}{\partial h^2} = 2F \frac{\partial F_{\text{net}}(z, h)}{\partial h}, \quad (z = h_{ss}), \]  

(5.51)

(see appendix F). Therefore, equations (5.47) and (5.48) become

\[ \hat{\theta} = 0, \quad (z = h_{ss}), \]  

(5.52)

and

\[ \mathcal{S} \sigma \hat{h} = \hat{h} \left( 2F \frac{\partial F_{\text{net}}(z, h)}{\partial h} \right) + \frac{\partial \hat{\theta}}{\partial z}, \quad (z = h_{ss}). \]  

(5.53)

The evolution of the perturbation is therefore governed by equation (5.46), subject to boundary condition (5.49) at \( z = 0 \), and (5.52) and (5.53) at \( z = h_{ss} \).

5.7.3 Solutions of the perturbation equations

The equation for the perturbed temperature (5.46) can be solved analytically using the Stefan condition (5.53) and the temperature condition (5.52) at \( z = h_{ss} \) to obtain

\[ \hat{\theta}(z) = \frac{2F \theta_{\text{net}}(h_{ss}) + \sigma \mathcal{S} \hat{h}}{\sqrt{\sigma}} \sinh(\sqrt{\sigma}(z - h_{ss})). \]  

(5.54)

To determine the initial linear evolution of the perturbation we substitute for \( \hat{\theta} \) from equation (5.54) into equation (5.49) and solve for \( \sigma \).

The difficulty in determining \( \sigma \) from equation (5.49) results from the relationship between the stationary ice thickness and the ice depth, which is strongly coupled via the two-stream radiation model. For oceanic heat flux of 5 W m\(^{-2}\), two physically real stationary ice thicknesses are found and are 0.275 m and 5.409 m. Figure 5.20 shows a contour plot of the map \( \Psi : \mathcal{C} \to \mathcal{R} \), which satisfies

\[ \Psi \equiv \left| \frac{\partial^2 \theta_{ss}}{\partial z \partial h}(0) + \frac{\partial \hat{\theta}}{\partial z}(0) - 4\mathcal{E}(\theta_{ss}(h_{ss}) + \hat{A})^3 \left( \hat{h} \frac{\partial \theta_{ss}}{\partial h}(0) + \hat{\theta}(0) \right) \right| 

- \mathcal{F}(1 - i_0) F_{\text{SW(tot)}} \hat{h} \frac{\partial \alpha_{\text{tot}}}{\partial h} \right|, \]  

(5.55)

derived from equation (5.49), for the equilibrium ice thickness 5.409 m. The zeros of the real-valued function \( \Psi \) correspond to the growth rate \( \sigma \). Darker shading
Figure 5.20: Contour plot of the real-valued positive function $\Psi(\sigma)$ (equation 5.55). Darker shading indicates lower value. Intersection of dashed line is location of maximum zero indicates lower values of $\Psi$. The zeros (darkest regions) align along the real axis as anticipated, and as the magnitude of the imaginary component increases, the absolute value of $\Psi$ increases rapidly. The growth rates in figure 5.20 are seen to be real and negative, and so the stationary ice thickness is stable.

Using a secant method to determine the maximum growth rate of the perturbations, the growth rates are found to be 3.751 (0.0275 m) and $-0.324$ (5.409 m) in non-dimensional period units (c.f. figure 5.20). Clearly, the smaller stationary ice thickness is unstable (the perturbations don’t decay) and the larger stationary ice thickness is stable (the perturbations decay). Varying the oceanic heat flux and repeating the analysis revealed that the lower branch of figure 5.19 was unstable, and the upper branch was stable.

### 5.7.4 Stationary solutions of full model

The full model described in chapter 4 can be configured to run with fixed forcing fluxes and no snow cover. As a test of convergence of the numerical solutions, the number of gridpoints was doubled for each of a series of test runs until the stationary
Equilibrium ice thickness obtained was within 0.0013 m of the analytical solution. This level of convergence was achieved with 641 gridpoints. The reason that such a large number of gridpoints are necessary is that the numerical algorithm D03PCF does not have any error estimate or error control (Dew and Walsh [23]). This number of gridpoints is the model standard for all other configurations.

The model forcing was set identical to that used in the stationary solutions of section 5.6.2 (tables 5.1 and 5.3). Figure 5.21 compares the temperature profile of the larger equilibrium ice thickness from the theoretical analysis (figure 5.8) to the numerical calculation of the stationary solution of the full model and shows good agreement.

Running the full model from a variety of initial conditions allows the stability results of the simplified model to be compared to the full mushy layer model. Figures 5.22-5.24 show the evolution of the full model with stationary forcing for initial thicknesses 1.22 m, 1.00 m, and 0.04 m respectively. For simplicity the initial temperature profiles were linear, and the initial surface temperatures were those of the larger equilibrium ice thickness, for the two larger initial ice thicknesses, and that of the smaller equilibrium ice thickness, for the smaller initial ice thickness.
**Figure 5.22:** Evolution of ice depth in full model with stationary forcing from initial depth = 1.22 m

**Figure 5.23:** Evolution of ice depth in full model with stationary forcing from initial depth = 1.00 m
Figure 5.24: Evolution of ice depth in full model with stationary forcing from initial depth $= 0.04$ m

It is found that for all ice thicknesses greater than the smaller ice thickness, the sea ice tends towards the larger equilibrium thickness, and for ice thicknesses less than or equal to the smaller ice thickness, the ice melts entirely. The same evolutionary behaviour occurs for varying oceanic heat flux. Therefore, the lower branch of the equilibrium ice thickness against ocean heat flux diagram, figure 5.25, (for a single band of radiation) is unstable, with the ice melting completely for ice thicknesses less than the smaller equilibrium ice thickness.
Figure 5.25: Stationary ice thickness for varying oceanic heat flux showing the stable (solid) and unstable (dashed) branches. Evolution of numerical model from different initial conditions shown by arrows

5.8 Discussion

The stationary solutions of the sea ice only case of the melt-pond–sea-ice model have been analysed. When there is no short-wave radiation the stationary ice thickness can be determined analytically. For typical wintertime forcing values it was shown that stationary solutions are only possible for positive ocean heat flux. When there are stationary solutions they are unique.

When short-wave radiation was introduced it was shown that there is potential for two stationary solutions for the same forcing data. This was entirely due to the two-stream radiation model, which leads to a quadratic structure in exponential functions in the Thickness Equation (equation 5.17). For typical Arctic conditions in a simplified case it was shown that two stationary solutions are possible. Stationary solutions were presented for a single band radiation model and a full spectral model. The sensitivity of the stationary ice thickness to the long-wave flux and the ocean heat flux was examined for the full spectral model. For both parameters there was a large range that permitted two-stationary solutions and a maximum value above which no stationary solutions were possible. For negative ocean heat flux between
The stability of a simplified system of pure ice, with constant thermal conductivity and a single-band two-stream short-wave radiation scheme, was analysed. Again there were two stationary solutions for a large range of forcing data, and it was found that the larger stationary ice thickness was stable, whereas the lower stationary ice thickness was unstable. The full model, which uses the mushy layer equations, can be forced with constant forcing data and no snow cover, and solved numerically for direct comparison against theoretical results. The theoretical stationary solutions for the single band case were used to test convergence of the numerical solutions and make an appropriate selection of the number of gridpoints. The stability results of the simplified model agreed with the results of the numerical model.

The stationary solutions examined represent the temperature of the ice in a quasi-steady regime. Quasi-steady evolution occurs when the rate of solidification is small, and the temperature profile for each time is approximated well by the stationary temperature profile (Worster [141]). Only the larger of the two stationary solutions is stable and there is a minimum ice thickness that can exist for each set of forcing
parameters, given by the smaller stationary ice thickness. Ice that is thinner than
the minimum thickness will completely melt; ice that is thicker than the minimum
thickness will tend towards the larger stationary solution. The reason for this is
that the albedo is more sensitive to ice thickness variations when the ice thickness
is small (see figure 5.26). A perturbation of a large equilibrium ice thickness will
result in little variation in the incoming energy (due to albedo changes), whereas a
perturbation of a small equilibrium ice thickness will result in a relatively large vari­
ation in the incoming energy. If the perturbation of a smaller ice thickness solution
is such that the ice thickness is increased, the albedo will be increased, leading to
albedo-feedback that makes the ice tend to its larger equilibrium thickness. If the
perturbation of a smaller ice thickness is such that the ice thickness is decreased,
the albedo will decrease, so more energy will be absorbed, which tends to melt the
ice completely.
Chapter 6

Melt ponds – special processes

6.1 Introduction

In this chapter, I investigate three phenomena that are important in relation to melt ponds. These are convection, drainage, and under-ice melt ponds. These particular studies were motivated during the course of investigating the full melt-pond–sea-ice model described in chapter 4.

In section 6.2, I perform a linear stability analysis to investigate the critical Rayleigh number at which thermal instability occurs within an idealised melt pond. The sensitivity of the critical Rayleigh number to forcing parameters is also investigated. Comparisons are made between the estimate of the critical Rayleigh number and the only other estimate I am aware of in the literature.

In section 6.3, I consider drainage of melt ponds. Using simple models I demonstrate that melt ponds rapidly drain to sea level, and examine the sensitivity of the drainage rate to permeability. It is found that although the drainage rate is fast, it is insufficient to violate the condition of local thermodynamic equilibrium within the sea ice.

In section 6.4, I analyse a simple model of sea-ice lenses, which is the ice that forms between fresh drained meltwater and the ocean beneath sea ice. The mushy
layer formulation is shown to exhibit qualitatively different behaviour to that of a constant diffusive formulation. The limitations of the model are then examined quantitatively, by comparison with a recent model by Notz et al. [86]. Finally, the model is used to estimate heat fluxes from observation of sea-ice lens migration, and to estimate the magnitude of the discrepancy accounting for the model limitations.

6.2 Convection

6.2.1 Introduction

In chapter 4, I discussed the potential for convection to occur within a melt pond and explained how this is modelled in the melt-pond–sea-ice model. In this section, I determine the critical Rayleigh number for an idealised melt pond due to thermal instability, including the effects of radiation as described by the two-stream radiation model. The methodology for determining the critical Rayleigh number follows that of Chandrasekhar [19] for thermal stability between two horizontal planes, with modified boundary conditions appropriate for melt ponds. The critical Rayleigh number is found by solving a sixth order linear ordinary differential equation, with boundary conditions imposed at two boundaries. This is solved numerically using a method due to Whitehead and Chen [134]. I outline the main steps in the following sections, and relegate extraneous material to the appendix. Finally, the sensitivity of the critical Rayleigh number is examined.

6.2.2 Melt pond governing equations

I suppose that the Boussinesq approximation of the Navier-Stokes equations is a valid description of fluid flow within the melt pond. The Boussinesq approximation uses the assumption that the coefficient of expansion of the fluid is small, so that density variations can be ignored except in the body force term (although the variation in density is small, the resulting body force can be large compared with
the inertial term, Chandrasekhar [19]). It is interesting to note that for the ther­mal stability problem the resulting perturbation equations are identical if either the full Navier-Stokes equations or the Boussinesq approximation are utilised (Chandrasekhar [19]). The influence of salinity variation on the equation of state for the melt pond is neglected (Bogorodskii [11]). This assumption is valid because melt ponds are relatively fresh, and in a wholly diffusive regime heat diffuses two orders of magnitude faster than solute. I also neglect the influence of surface winds creating large-scale circulations within the melt pond. This is a reasonable assumption if the surface of the melt pond is below the level of the surrounding floe and the diameter of the pond is relatively small.

The equations describing the fluid flow within the melt pond are then

$$\frac{Du}{Dt} = -\nabla p + X + \frac{\delta \rho}{\rho_0} X + \nu \nabla^2 u,$$

(6.1)

where $D/Dt$ is the total derivative (derivative following a fluid element), $u$ is the fluid velocity within the melt pond, $p$ is the pressure, $X = (0,0,-gY)$ is the body force due to gravity and $\delta \rho$ is the departure of the density from a fixed reference density $\rho_0$. Since $X$ is conservative we can write it as a potential, $X = \nabla Y$, where $Y = -gz$. Note that for this section the $z$-axis points vertically upwards. Equation (6.1) describes the acceleration of a fluid element following the fluid (first term) subject to the pressure gradient (second term), gravitational acceleration (third and fourth term) and viscosity (fifth term).

Melt pond temperatures are typically less than their temperature of maximum density (Eicken et al. [27]) and so I approximate the equation of state using the linear relationship,

$$\rho = \rho_0 (1 + \alpha (T - T_0)),$$

(6.2)

where $\alpha = (1/\rho_0)(\partial \rho/\partial T) \approx 5 \cdot 10^{-5}$ K$^{-1}$ is the coefficient of expansion appropriate for moderately saline water near the freezing temperature (section 4.3). The positive sign before the coefficient of expansion is opposite to that in linear approximations to the equation of state of water used at higher temperatures, since in this case warmer water is denser.
The equation of state (6.2) implies that
\[
\frac{\delta p}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \alpha(T - T_0),
\]  
and introducing the reduced pressure \( p^* = p - \rho_0 Y \), which is the pressure minus the hydrostatic component, equation (6.1) can be rewritten as
\[
\frac{Du}{Dt} = -\frac{\nabla p^*}{\rho_0} + \alpha(T - T_0)X + \nu_l \nabla^2 u.
\]  
In the melt pond the equation describing conservation of heat is
\[
\frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa_l \nabla^2 T + \frac{R(z)}{(\rho c)_l},
\]  
where \( R(z) \) is the radiative source term which is assumed to be independent of horizontal position.

Finally, the continuity equation (conservation of mass) is given by
\[
\nabla \cdot u = 0,
\]  
where density variations are neglected in the Boussinesq approximation as they are small.

**Boundary conditions**

The melt pond is assumed to have infinite horizontal extent and have horizontal upper and lower boundaries. The upper boundary is assumed free while the lower boundary is assumed fixed. Since flow between the porous sea ice and melt pond is much smaller than convective flow in the melt pond, \( u \cdot n = 0 \) at the lower boundary (\( n \) is the unit normal). We also apply no-slip at the lower boundary, \( u \cdot \tau = 0 \) (\( \tau \) is the unit tangent) and thus \( u = 0 \) (the no-slip condition). It is also assumed that the temperature at the melt pond’s lower boundary is fixed at the equilibrium freezing temperature \( T_L(C_{pond}) \). Since the no-slip condition must be satisfied for all \( x \) and \( y \) on the boundary it follows from the equation of continuity (6.6) that \( \partial w/\partial z = 0 \) on the boundary.
Suppose that $P_{ij}$ is the stress (force per unit area) acting in the direction of $i$ per unit area on an element of surface normal to $j$ ($i,j = x, y, z$). At the upper boundary ($z = H_{pond}$) it is required that the tangential stress is zero, since the pond surface is a free surface. The condition that the tangential stress at the upper surface is zero is

$$P_{xz} = 0 \quad \text{and} \quad P_{yz} = 0.$$  \hspace{1cm} (6.7)

Since the isotropic component of the stress (the pressure) has no transverse component, the conditions given by equation (6.7) are equivalent to the vanishing of the viscous stress tensor,

$$p_{zz} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \quad \text{and} \quad p_{yz} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0,$$ \hspace{1cm} (6.8)

where $p_{ij}$ ($i,j = x, y, z$) is the viscous stress tensor, $\mu$ is the coefficient of viscosity, $u = (u, v, w)$ and where I have used the assumption of incompressibility (equation 6.6). Since $w$ vanishes everywhere on the upper boundary, equations (6.8) imply that

$$\frac{\partial u}{\partial z} = 0 \quad \text{and} \quad \frac{\partial v}{\partial z} = 0$$ \hspace{1cm} (6.9)

on the upper boundary. Differentiating the continuity condition (6.6) with respect to $z$ yields the condition for the free surface,

$$\frac{\partial^2 w}{\partial z^2} = 0.$$ \hspace{1cm} (6.10)

At the upper boundary there is also the surface energy balance, which is

$$k_l \frac{\partial T}{\partial z} + F_{LW} + (1 - i_0)(1 - \alpha_{tot})F_{SW(tot)} - F_{sens} - F_{lat} - \epsilon_w \sigma T_0^4 = 0.$$ \hspace{1cm} (6.11)

### 6.2.3 Non-dimensionalisation

I non-dimensionalise temperature with a temperature scale $\Delta T$ so that $\theta = (T - T_L(C_s))/\Delta T$ is the non-dimensional temperature, lengths with a typical lengthscale $L$, time with the thermal diffusive timescale $L^2/\kappa_l$ and pressure with $\rho_0 \kappa_l^2 / L^2$. 

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Since the temperatures at the boundaries are not fixed \textit{a priori} as in the standard thermal convection problem, the temperature scale $\Delta T$ is not as straightforward to obtain (Whitehead and Chen [134]). Since the absorption of radiation by the pond is relatively small (compared to the sea ice) the stationary temperature profile for physically real forcing parameters will typically be monotonic increasing (from the base of the pond). I therefore define the temperature scale $\Delta T$ to be given by the temperature difference across the melt pond for the stationary regime, which is straightforward to determine, and the lengthscale $L$ to be the depth of the pond. This is appropriate under the assumption of a monotonic stationary temperature profile, since the instability is due to the total depth of the melt pond with temperature difference $\Delta T$.

Equation (6.4) can then be written in non-dimensional form
\begin{equation}
\frac{Du}{Dt} = -\nabla p^* - \Pr Ra \theta k + \Pr \nabla^2 u. \tag{6.12}
\end{equation}
where all of the variables are now non-dimensional, $k$ is the unit vector in the $z$-direction, $\Pr = \nu_1 / \kappa_1$ is the Prandtl number, and $Ra = \alpha(\Delta T)gL^3 / \nu_1 \kappa_1$ is the Rayleigh number. The heat equation (6.5) can be rewritten as
\begin{equation}
\frac{D\theta}{Dt} = \nabla^2 \theta + \mathcal{F} R^*(z), \tag{6.13}
\end{equation}
where $\mathcal{F} = FL / \kappa_1 \Delta T$, $R^*(z) = R(z)L/F$, and $F$ is a typical short-wave radiative flux. The continuity equation (6.6) becomes
\begin{equation}
\nabla \cdot u = 0. \tag{6.14}
\end{equation}

**Boundary conditions**

Upon non-dimensionalisation the lower boundary conditions at $(z = 0)$ are given by
\begin{equation}
\theta = \theta_{\text{pond}}, \quad w = 0 \quad \text{and} \quad \frac{\partial w}{\partial z} = 0. \tag{6.15}
\end{equation}

The upper boundary conditions are
\begin{equation}
\frac{\partial \theta}{\partial z} + \mathcal{F} F_{\text{in}} - \mathcal{E} \left( \theta + \frac{T_0}{\Delta T} \right)^4 = 0, \quad w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial z^2} = 0, \tag{6.16}
\end{equation}
where \( F_{\text{in}} = (1 - n_0)(1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) + F_{\text{LW}} - F_{\text{sens}} - F_{\text{lat}} \) is the prescribed net incoming flux, \( \mathcal{F} = FL/k_t \Delta T \) and \( \mathcal{E} = \epsilon_w \sigma (\Delta T)^3 L/k_t \) are non-dimensional parameters.

### 6.2.4 Linear stability analysis

In a similar way to section 5.7, I now examine the evolution of an infinitesimal perturbation of the stationary state. In the undisturbed state of no motion, the temperature \( \theta_s(z) \), where the subscript \( s \) denotes stationary, must satisfy the stationary version of equation (6.13)

\[
0 = \nabla^2 \theta_s + \mathcal{F} R^*(z) . \tag{6.17}
\]

Taking note of the boundary conditions, the heat equation is integrated first from \( z \) to 1 with respect to \( z \) and then secondly from 0 to \( z \) to obtain the stationary temperature profile

\[
\theta_s(z) = \theta_{\text{pond}} + \left\{ \mathcal{E} \left( \theta_s(1) + \frac{T_0}{\Delta T} \right)^4 - \mathcal{F} F_{\text{in}} \right\} z + \mathcal{F} \int_0^z \int_{z_2}^1 R^*(z_1) \mathrm{d}z_1 \mathrm{d}z_2 . \tag{6.18}
\]

This implicit equation for \( \theta_s(1) \) can be solved to determine the full stationary solution.

Perturb the system slightly so that

\[
\theta(x,t) = \theta_s(z) + \theta_1(x,t),
\]

\[
p^*(x,t) = p^*_s(z) + p^*_1(x,t), \text{ and}
\]

\[
u(x,t) = u_1(x,t),
\]

where \( \theta_1(x,t) \ll 1 \), \( p^*_1(x,t) \ll 1 \) and \( u_1(x,t) \ll 1 \). Then the momentum equation (6.12) upon neglecting products of perturbation terms and use of the stationary state becomes

\[
\frac{\partial u_1}{\partial t} = -\nabla p^*_1 - Pr \text{Ra} \theta_1 k + Pr \nabla^2 u_1 . \tag{6.19}
\]

Similarly the heat equation becomes

\[
\frac{\partial \theta_1}{\partial t} = -w_1 \frac{\partial \theta_s}{\partial z} + \nabla^2 \theta_1 . \tag{6.20}
\]
where \( w_1 \) is the vertical component of \( \mathbf{u}_1 \). Finally, the continuity equation becomes

\[
\nabla \cdot \mathbf{u}_1 = 0. \tag{6.21}
\]

Taking the curl twice of equation (6.19), operating on the \( z \)-component of the resulting equation with \( (\partial/\partial t - \nabla^2) \), and using equation (6.20) yields

\[
\left( \frac{\partial}{\partial t} - \Pr \nabla^2 \right) \left( \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w_1 = \Pr \text{Ra} \frac{\partial}{\partial z} \nabla_H^2 w_1, \tag{6.22}
\]

where \( \nabla_H \) is the horizontal Laplacian. Since derivatives with respect to \( x \) and \( y \) only appear in the form of the horizontal Laplacian there is no preferred horizontal direction in the problem (Chandrasekhar [19]). This permits separable solutions of the form

\[
w_1 = W(z)f(x,y)e^{\sigma t} \tag{6.23}
\]

provided that

\[
f_{xx} + f_{yy} + a^2 f = 0,
\]

where \( \sigma \) can be complex and \( a \) is the wavenumber.

Substituting equation (6.23) into equation (6.22) yields

\[
(\text{Pr} (D^2 - a^2) - \sigma) ((D^2 - a^2) - \sigma) (D^2 - a^2) W = -\text{Pr} \text{Ra} \frac{\partial}{\partial z} a^2 W. \tag{6.24}
\]

### 6.2.5 Solution of the marginal stability problem

It can be shown that \( \sigma \) is real so that the marginal states (the perturbations neither grow nor decay) are characterised by \( \sigma = 0 \) (see appendix G) and the transition from stability to instability occurs via a stationary state. This is known as the principle of exchange of stability (Chandrasekhar [19]). The marginal state is governed by the perturbation equations with \( \sigma \) set to zero. Specifically,

\[
(D^2 - a^2)^3 W = -\text{Ra} \frac{\partial\theta_s}{\partial z} a^2 W, \tag{6.25}
\]

subject to boundary conditions

\[
W = 0, \quad D^2 W = 0, \quad (D^2 - a^2)^2 DW = 4\epsilon \left( \frac{\theta_s + T_0}{\Delta T} \right)^3 (D^2 - a^2)^2 W, (z = 1), \tag{6.26}
\]
and
\[ W = 0, \quad DW = 0, \quad (D^2 - a^2)^2 W = 0, \quad (z = 0), \quad (6.27) \]
which are obtained by linearising the boundary conditions (6.15) and (6.16).

Equations (6.25)–(6.27) form a linear sixth-order ODE two-point boundary value problem, which is a characteristic value problem for \( \text{Ra} \). Therefore, for a given \( a^2 \) there will only be particular values of \( \text{Ra} \) for which the equations admit non-zero solutions. The minimum characteristic value with respect to \( a^2 \) is then the critical Rayleigh number at which instabilities set in.

The numerical method of solution of equations (6.25)–(6.27) is based on that of Whitehead and Chen [134]. Since equation (6.25) is linear, we can consider the solution to be a superposition of three independent solutions of 'half' of the boundary value problem, and determine the appropriate combination that will yield a non-zero solution to the characteristic value problem. Let \( W_i (i = 1, 2, 3) \) be a solution of equation (6.25) subject to the boundary conditions (6.26) and any three further independent linear boundary conditions at \( z = 0 \), such that \( W_i \neq kW_j (i \neq j, k \text{ real}) \). Consider \( W = AW_1 + BW_2 + CW_3 \). Since equation (6.25) is linear, \( W \) is also a solution of the governing equation (6.25) and the boundary conditions (6.26) at \( z = 0 \). The linearity of the boundary conditions (6.25) and the boundary conditions (6.26) at \( z = 0 \) means that \( W \) satisfies the upper boundary conditions if and only if

\[
\begin{align*}
AL_1W_1 + BL_1W_2 + CL_1W_3 &= 0, \quad (6.28) \\
AL_2W_1 + BL_2W_2 + CL_2W_3 &= 0, \quad (6.29) \\
AL_3W_1 + BL_3W_2 + CL_3W_3 &= 0, \quad (6.30)
\end{align*}
\]

where \( L_i (i = 1, 2, 3) \) are the linear operators representing the three linear boundary conditions of the form \( L_iW_j = 0 \) at \( z = 1 \).

Let the matrix \( \mathbf{L} = \left(l_{ij}\right) = L_iW_j \) and the vector \( \mathbf{b} = (A, B, C)^T \) then equations (6.28) – (6.30) can be written as the matrix equation

\[
\mathbf{Lb} = \mathbf{0}. \quad (6.31)
\]

Equation (6.31) has non-zero solutions if and only if \( \det(\mathbf{L}) = 0 \). Therefore, for a
Figure 6.1: Marginal stability curve for melt pond showing the variation of the Rayleigh number against the wavenumber of the perturbation. The critical Rayleigh number ($R_{a_{\text{crit}}}$) = 633.45 is the minimum value of the marginal stability curve and occurs at wavenumber = 2 m$^{-1}$.

<table>
<thead>
<tr>
<th>$F_{SW}(\text{tot})$</th>
<th>$250$ W m$^{-2}$</th>
<th>$F_{LW}$</th>
<th>$250$ W m$^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{sens}$</td>
<td>$10$ W m$^{-2}$</td>
<td>$F_{lat}$</td>
<td>$5$ W m$^{-2}$</td>
</tr>
<tr>
<td>Ice depth</td>
<td>$2$ m</td>
<td>Pond depth</td>
<td>$0.025$ m</td>
</tr>
</tbody>
</table>

Table 6.1: Parameters used in stability analysis

given $a^2$, the minimum Rayleigh number such that the marginal stability problem has a non-zero solution, occurs at the smallest value of $Ra$ for which $\det(L) = 0$.

The numerical method outlined above was coded up using the mathematical software Mathematica version 3.0. Figure 6.1 shows the marginal stability curve for a thin melt pond of thickness 0.025 m for parameters appropriate to the beginning of the summer melt season (see table 6.1). The minimum Rayleigh number for these parameters is found to be 633.435 with corresponding wavenumber 2.00.

The sensitivity of the critical Rayleigh number to the short-wave radiation, long-wave radiation and the pond depth was examined, by altering each of the parameters in turn whilst keeping all other parameters fixed at their initial values (table 6.1). The critical Rayleigh number was relatively insensitive to the short-wave radiation,
The critical Rayleigh number was calculated as the long-wave radiation varied between 200 W m\(^{-2}\) and 290 W m\(^{-2}\). As the long-wave radiation increases, the surface temperature of the stationary temperature decreases. This is the result of the balance between the only free parameters in the surface energy balance (6.11), which are the conductive flux, \( -k_l \partial T / \partial z \), and the outgoing long-wave radiation, \( \epsilon_w \sigma T_0^4 \). The value 290 W m\(^{-2}\) of the long-wave radiation represents the approximate limit at which the surface temperature becomes less than the temperature of the base of the pond, contradicting the stationary hypothesis. The critical Rayleigh number was relatively insensitive to the long-wave radiation across most of the range except as the long-wave radiation approached 290 W m\(^{-2}\): from \( F_{LW} = 200 \) W m\(^{-2}\) to \( F_{LW} = 280 \) W m\(^{-2}\) the critical Rayleigh number increased by 2.88 from 631.89 to 634.77; and from \( F_{LW} = 280 \) W m\(^{-2}\) to \( F_{LW} = 290 \) W m\(^{-2}\) the critical Rayleigh number increased by 6.52 to 641.29 (see figure 6.3).

The critical Rayleigh number is more sensitive to typical variations in long-wave radiation than short-wave radiation because less short-wave energy enters the pond. This is because the short-wave energy can penetrate into the sea ice, and is also reflected at the melt pond surface, so that the variation of short-wave radiation
Figure 6.3: Sensitivity of critical Rayleigh number to long-wave radiation absorbed in the pond is less than the long-wave radiation. Increased radiation leads to increased stability, because as radiation increases, the surface temperature of the stationary temperature profile decreases. For the relatively small pond thicknesses considered the temperature profile is monotonic increasing, therefore, there is less potential energy available at the onset of instability.

The critical Rayleigh number was calculated as pond depths were varied from 0.00001 m to 0.07 m. The critical Rayleigh number ranged from 669.07 to 517.61, decreasing nonlinearly with pond depth (see figure 6.4). Deeper ponds have higher surface temperatures and therefore the potential energy at the onset of instability is larger, which leads to decreased critical Rayleigh numbers. Caution must be exercised when interpreting the critical Rayleigh number at small depths (a few millimetres), because it has been shown that instabilities caused by variations in surface tension with temperature are significant (Pearson [88]). It has in fact been shown that the original experiments on thermal convection by Benard, with a free upper surface, are likely to be exhibiting instability due to surface tension variations instead of density variations (Pearson [88]).
6.2.6 Discussion

I have performed a standard stability analysis for thermal convection, but applied to the situation of an idealised melt pond, with more complicated boundary conditions and an equation of state that means warmer fluid sinks. Under the assumption that the temperature profile is monotonic increasing (from its base) the Rayleigh number was defined using the depth of the melt pond and the temperature difference of the stationary temperature profile across the entire pond depth. Since the absorption of radiation within the pond is low (Grenfell and Maykut [45]), steady temperature profiles are unlikely to possess turning points, so that penetrative convection can be neglected. Penetrative convection occurs when convective motion within an unstably stratified region of fluid infiltrates surrounding stably stratified regions.

I have shown numerically that the critical Rayleigh number for an idealised melt pond is insensitive to the short- and long-wave radiation (figures 6.2 and 6.3). The critical Rayleigh number is more sensitive to pond depth, but even in this case it is relatively insensitive (figure 6.4). For use in the numerical model, I choose to use a constant value of the Rayleigh number. Therefore, I choose the value 630, which is of the same order as the estimates made with a 0.025 m melt pond in the previous section.
To my knowledge, there has only been one other investigation into melt pond stability that determined a critical Rayleigh number, by Bogorodskii [11]. Bogorodskii [11] attempted to determine the critical Rayleigh number by neglecting radiation, using a simplified surface boundary condition, and assuming that the long-wave modes will be most unstable. To determine the critical Rayleigh number he performed a perturbation expansion for the marginal state about the smallest modes. The problems with this methodology are that the critical Rayleigh number must be determined in terms of the wavenumber, and the long-wave modes aren’t necessarily most unstable. These facts can be clearly seen from the marginal stability curve in figure 6.1. It is also clear for the standard case of thermal stability between two free boundaries of fixed temperature and no radiation since the Rayleigh number at a given wavenumber, $a$, is given by $Ra = (\pi^2 + a^2)^{3/2}/a^2$, with corresponding critical Rayleigh number (the minimum Rayleigh number) $27\pi^4/4 \approx 657.5$ (Chandrasekhar [19]). The estimate of the critical Rayleigh number that Bogorodskii proposed was 288, significantly lower than my estimate of 630. This demonstrates that the longest wavelengths are unlikely to be most unstable.

## 6.3 Drainage

### 6.3.1 Introduction

Drainage of surface meltwater is an important component of the summertime evolution of sea ice. The flushing of this meltwater, driven by gravity, redistributes salt within the ice and expels salt from the sea ice base. It is also responsible for the formation of under-ice melt ponds: low salinity surface water which drains through the sea ice to the high salinity ocean becomes supercooled and freezes (see section 6.4).

Melt ponds can typically be categorised into at-sea level ponds and above-sea level ponds (Perovich et al. [97]). At-sea level ponds are ponds that have their bases below sea level and are observed to have their surfaces fixed at sea level, since the
pond will adjust to hydrostatic equilibrium on a relatively fast timescale of the order a few minutes. Above-sea level ponds (referred to as ‘Alpine’ ponds by Perovich et al. [97]) are ponds that have their bases above sea level. These ponds do not have the same constraint as the at-sea level ponds, but are subject to net gains or losses due to meltwater in the pond catchment area increasing their depth, and drainage through the base and sides of the melt pond decreasing their depth. Both types of melt pond were observed during the SHEBA field experiment (Perovich et al. [97]).

There is also heuristic evidence from SHEBA of a correlation between the depths of melt ponds and their position above sea level (Perovich et al. [95]) – higher ponds were observed to have smaller depths. The reason for this, is that higher ponds will have lower net gains from the local catchment area than lower ponds.

Acton et al. [3] considered one-dimensional drainage of a finite initial mass of fluid overlying a porous medium. Drainage was modelled using Darcy’s law. Darcy’s law is a constitutive relation for flow in a porous medium that states that the volume flux is proportional to the pressure gradient – where the volume flux is averaged over several pores (Ockendon and Ockendon [87]). This type of one-dimensional drainage is directly applicable to above-sea level ponds in the limit of small aspect ratio – i.e. when the pond depth is much less than the minimum pond diameter. Using the result of Acton et al. [3] the velocity of the pond surface \( v \) is given by

\[
v = \frac{\Pi \rho_l g \mu}{(1 + H_{\text{pond}}/L)},
\]

(6.32)

where \( v \) is positive downwards, \( \Pi \) is the permeability (m\(^2\)), which ranges from \( 10^{-8} \) m\(^2\) to \( 10^{-12} \) m\(^2\) in sea ice (Eicken et al. [29]) and is dependent on the void fraction \((1 - \phi)\) (Phillips [101]), \( \rho_l \) is the density of the meltwater (kg/m\(^3\)), \( g \) is gravitational acceleration (m/s\(^2\)), \( \mu \) is the dynamic viscosity (kg/m s), \( H_{\text{pond}} \) is the depth of the melt pond (m), and \( L \) is the distance the percolating meltwater front has penetrated under the pond (m).

If the sea ice has a drained freeboard prior to the onset of substantial drainage of meltwater there is the potential for the formation of lenses. Lenses are inclusions within a multi-phase flow (e.g. air–water) in porous media that can alter the perme-
ability (e.g. Bear [8]). The structure of sea ice is especially prone to the formation of lenses from air inclusions, since brine pockets are aligned along the plate boundaries of the ice crystals in sea ice. If a meltwater front is percolating towards sea level, any air trapped between the meltwater front and sea level will reduce the permeability. If the void structure of the sea ice connecting the freeboard surface to sea level is akin to vertical pipes then the permeability would become zero as the meltwater blocked off the pipes. The draining surface would then behave as a gravity current in a porous medium such as that described by Huppert and Woods [52]. As the current spread, contact phenomena (e.g. surface tension) would become important for large times, and a slow moving current would eventually lead to filling of the pipes and a corresponding increase of the permeability.

For the porosity structure of sea ice, the occurrence of air lenses would lead to a decrease in permeability (not necessarily to zero) in a thin region between the sea level depth and the draining meltwater. This type of gravity current has been described mathematically by Pritchard et al. [103]. The layer of decreased permeability (with its air inclusions) is not observed in melt ponds that have melted below sea level. However, it is observed in ice-cores of refrozen above-sea level melt ponds as a region of increased air inclusions (Shokr and Sinha [117]), since the freeboard is not static and percolation may not evolve as a continuously draining front.

In this section, I formulate simple one-dimensional models to determine drainage rates of at-sea level ponds whose surfaces initially are above sea level, and determine the dependency of drainage rate on the permeability of sea ice.

First, I formulate a one-dimensional model for drainage of a pond with its base below sea level. A leading order composite approximation for the position of the melt pond surface is determined, and is used to estimate the drainage rate of the pond. Second, I formulate a one-dimensional model for drainage of a pond with its base above sea level, assuming that the ice beneath the pond is saturated with fluid. The model is then used to estimate the drainage rate of the pond and the variation of the drainage rate with permeability. Finally, the implications of the results of these models of drainage are discussed.
6.3.2 Simple drainage model of sea ice: pond base below sea level

I now consider the simple situation of an above-sea level melt pond, which has its upper surface above sea level and pond base below sea level. The surface of the melt pond is at $z = h_s$, the pond–ice interface is at $z = h_p$, the ice–ocean interface at $z = h_i$. Let the draft $H_d = h_i - h_o$, where $z = h_o$ is the position of sea level and the $z$-axis points vertically downward. The draft is complicated and depends on sea ice floe geometry, permeability, saturation, snow depth, and ocean and wind forcing. However, the rate of melting is much less than the drainage rate of a melt pond whose surface is above sea level so the time dependency of the draft, pond–ice interface and ice–ocean interface can be neglected. Therefore, the only time-dependent height will be $h_s$. A schematic diagram of this simple drainage model is shown in figure 6.5.

For simplicity, I consider a one-dimensional melt pond, which is valid since the horizontal lengthscale of a typical melt pond is much larger than its vertical dimension and the vertical pressure gradients are typically much greater than horizontal pressure gradients (neglecting surface winds). I also neglect any differences in density
of meltwater and brine and those due to temperature and salinity, and any vertical variations in porosity so that the permeability is constant and the continuity equation in the melt pond and the sea ice is simply

$$\frac{\partial w}{\partial z} = 0, \quad (6.33)$$

where \( w \) represents the fluid velocity in the melt pond, and the pore velocity in the sea ice. The pore velocity is the actual fluid velocity within the pores and is related to the Darcy velocity \( W \) (the volume flux of brine per unit perpendicular cross-sectional area) by \( W = (1 - \phi)w \).

One-dimensional flow implies (by the continuity equation) that the flow is only time dependent. Flow in the melt pond is described by the Navier-Stokes equations (c.f. equation 6.1), which in one dimension are given by

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_i} \frac{\partial p}{\partial z} + g. \quad (6.34)$$

Flow within the sea ice is described by Darcy’s Law given by

$$W = -\frac{\Pi}{\mu} \frac{\partial p}{\partial z} + \frac{\Pi \rho_i g}{\mu}. \quad (6.35)$$

At the surface of the pond the pressure is fixed at atmospheric pressure, \( p_{\text{atm}} \). Assuming the ocean is hydrostatic, the pressure at the lower boundary is determined by Archimedes’ principle. The pressure due to the seawater displaced by the sea ice floe is \( p_{\text{atm}} + \rho_w (h_i - h_o) \), where the seawater density is assumed identical to the meltwater density.

Equations (6.34) and (6.35), under the assumption of constant permeability, imply that \( p_{zz} = 0 \) in both the sea ice and the melt pond. Therefore, the pressure is a linear function of position, dependent on time, in both the pond and the ice.

The interfacial condition at the melt-pond–sea ice interface are continuity of mass flux, which means that \( W = w \), and continuity of pressure, which means that \( p = p(t) \), at \( z = h_p \).

The linear pressure gradients can be written down in terms of the pond–ice interfacial pressure \( p(t) \) and substituted into the Navier-Stokes equation and Darcy’s law. This
yields

\[ \begin{align*}
\dot{h}_s &= \frac{1}{\rho_i h_p - h_s} p(t) + g, \text{ and } \\
\dot{h}_s &= -\frac{\Pi}{\mu} \left( \frac{\rho_i g (h_i - h_o) - p(t)}{h_i - h_p} \right) + \frac{\Pi \rho_i g}{\mu},
\end{align*} \]  

(6.36) (6.37)

where I have used \( \partial w/\partial t = \dot{h}_s \) and \( W = w = \dot{h}_s \) and assumed that there is no accretion of fluid at the surface of the pond due to lateral drainage of the sea ice floe.

Let \( H_p = h_p - h_s \) and \( H_i = h_i - h_p \), then eliminating the interfacial pressure between equations (6.36) and (6.37) yields the following equation for the pond depth

\[ \rho_i H_p \ddot{H}_p + \frac{\mu}{\Pi} H_i \dot{H}_i + \rho_i g H_p = \rho_i g (H_d - H_i). \]  

(6.38)

I non-dimensionalise lengths with \( H_d - H_i \), the final pond depth, and time with \( \mu H_i / \Pi \rho_i g \), the time taken for fluid to flow through the sea ice under its own weight, so that equation (6.38) expressed in non-dimensional variables is given by

\[ T H_p \ddot{H}_p + \dot{H}_p + H_p = 1, \]  

(6.39)

where \( T \) is a non-dimensional parameter such that

\[ T = \frac{\Pi^2 \rho_i^2 g (H_d - H_i)}{\mu^2 H_i^2} \approx \frac{(10^{-9} \text{ m}^2)^2 \cdot (10^3 \text{ kg/m}^3)^2 \cdot 10^1 \text{ m/s}^2 \cdot 10^{-1} \text{ m}}{(10^{-3} \text{ kg/m} \text{s})^2 \cdot (10^0 \text{ m})^2} = 10^{-6} \ll 1, \]

for typical sea ice values.

Let \( t^* = T^\gamma t \) be a rescaling of time, then the distinguished limits (Nayfeh [84]) of equation (6.39) assuming that \( H_p = \text{ord}(1) \), are given by \( \gamma = 0 \) so that

\[ \dot{H}_p + H_p = 1, \]  

(6.40)

and \( \gamma = 1 \) so that

\[ H_p \ddot{H}_p + \dot{H}_p = 0, \]  

(6.41)

where derivatives are now with respect to the rescaled variable \( t^* \). I use superscript \( o \) (outer) to denote solutions of equation (6.40) and superscript \( i \) (inner) to denote solutions of equation (6.41).
I assume that the initial conditions for the problem are $H_p = H_0$, and $\dot{H}_p = w_0$ at $t = 0$. I now perform a simple asymptotic analysis to leading order to demonstrate that the inertial term (the first term) in equation (6.39) is unimportant.

Equation (6.40) has solution

$$H_p^o(t) = Ae^{-t} + 1,$$

and so $H_p^o \to 1$ as $t \to \infty$. Therefore, in the case of no accretion any pond with its base below sea level will always become an at-sea level pond as time tends to infinity.

Equation (6.41) is more complicated. Under the assumption of zero initial velocity ($w_0 = 0$), equation (6.41) simply has the trivial solution

$$H_p^i(t) = H_0,$$

satisfying the initial conditions at $t = 0$. If the initial velocity is assumed non-zero, then equation (6.41) has solution

$$H_p^i(t) = B \operatorname{Li}^{-1} \left( \operatorname{Li} \left( \frac{H_0}{B} \right) - \frac{t^*}{B} \right),$$

where $B = H_0 \exp(w_0)$, $\operatorname{Li}$ is the Logarithmic Integral given by

$$\operatorname{Li}(x) = \int_0^x \frac{dt}{\log(t)},$$

and $\operatorname{Li}^{-1}(x)$ is the Inverse Logarithmic Integral defined as one of two single valued continuous functions depending on the initial velocity (see figure 6.6). At $t = 0$, $H_p^i(0) = H_0 = B \operatorname{Li}^{-1} \left( \operatorname{Li} \left( \frac{H_0}{B} \right) \right)$ and it is required that $H_p^i(t)$ is continuous. The argument of the Inverse Logarithmic Integral of $H_p^i(t)$ (equation 6.44) is a decreasing function of time, since $B > 0$. Therefore, if $w_0 > 0$ then $\operatorname{Li}^{-1}$ is defined by the continuous map $\operatorname{Li}^{-1} : \mathbb{R}^+ \to [0,1)$, and if $w_0 < 0$ then $\operatorname{Li}^{-1}$ is defined by the continuous map $\operatorname{Li}^{-1} : \mathbb{R} \to (1, \infty)$ (see figure 6.6).

Note that the first integral of equation (6.41) is straightforwardly determined since $\dot{H}_p^i = -\log(H_p^i/B)$ is the anti-derivative of $\dot{H}_p^i$, found by rearranging equation (6.41). This solution satisfies the initial conditions at $t = 0$, and it can be seen that in the
Figure 6.6: Graph of the multi-valued Inverse Logarithmic Integral $\text{Li}^{-1}(x)$. Single valued functions are obtained as either the negative real axis mapping onto $[0, 1)$ or the whole real axis mapping onto $(1, \infty)$. These occur when the initial velocity of the pond depth ($w_0$) is negative or positive respectively.

Limit $t^* \to \infty$, $H_p^i \to B$. If $w_0 > 0$ then the pond depth tends to a limit greater than its initial depth, whereas if $w_0 < 0$ then the pond depth tends to a limit less than its initial depth. Therefore, over a timescale of order $T$ the perturbation of the initial condition is of the order $B - H_0 = H_0(\exp(w_0) - 1)$. If the initial velocity in dimensional units is taken to be $w_0^* = 2$ cm/day, then the initial velocity in non-dimensional units is approximately

$$w_0 = \frac{w_0^* \mu H_i}{(H_d - H_i) \Pi \rho g} \approx \frac{2 \times 10^{-7} \text{m/s} \cdot 10^{-3} \text{kg/m/s} \cdot 1 \text{m}}{10^{-1} \text{m} \cdot 10^{-9} \text{m}^2 \cdot 10^2 \text{kg/m}^3 \cdot 10 \text{m/s}^2} = 2 \times 10^{-4}. \quad (6.46)$$

Therefore, the maximum perturbation from the initial state due to inertial effects will be small, and so inertial effects can be neglected.

To match the inner solutions (6.43) and (6.44) to the outer solution (6.42), I use an intermediate variable $\eta = t/T^\alpha = t^*T^{(1-\alpha)}$, where $0 < \alpha < 1$ (Hinch [50]). Considering $T \to 0+$ with $\eta$ fixed, implies that $t \to 0$ and $t^* \to \infty$. Matching the leading order expressions in the inner and outer solutions as $t \to 0$ yields

$$A = H_0 - 1,$$  \quad (6.47)
for zero initial velocity, and

\[ A = H_0 e^{w_0} - 1, \quad (6.48) \]

for non-zero initial velocity.

A composite approximation is a non-unique uniform asymptotic approximation valid over the entire domain (Hinch [50]). In terms of the inner and outer solutions the composite approximation \( H_p^c(t) \) is given by

\[ H_p^c(t) = H_p^i(t) + H_p^o(t) - \lim_{t \to 0} H_p^o(t). \]

The composite approximation valid over the entire domain is therefore given by

\[ H_p^c(t) = (H_0 - 1)e^{-t} + 1, \quad (6.49) \]

for zero initial velocity, and

\[ H_p^c(t) = (H_0 e^{w_0} - 1)e^{-t} + 1 + H_0 e^{w_0} Li(-1) \left( Li(e^{-w_0}) - \frac{te^{-w_0}}{TH_0} \right) - H_0 e^{w_0}, \quad (6.50) \]

for non-zero initial velocity. As stated before, the initial inertial effects will be small and so equation (6.49) can be used to estimate the drainage through the ice.

Figure 6.7 shows the evolution in dimensional variables of an initially 0.3 m pond draining through ice with \( H_d = 1.8 \) m, so that the average floe thickness is about 2 m, and ice thickness beneath the pond \( H_i = 1.7 \) m. The permeability is \( 10^{-9} \) m\(^2\), the density is 1000 kg/m\(^3\), and the viscosity is 0.00179 kg/m s. The time taken to drain the pond to within 1 cm of the sea level is 930 seconds. As anticipated the timescale of drainage of an at-sea level pond is of the order of a few minutes, which is short compared to the timescale of thermodynamic variation of the melt-pond-sea-ice system.

### 6.3.3 Simple drainage model of sea ice: pond base above sea level

The analysis for the case when sea level is below the base of the melt pond (see figure 6.8) is similar to the previous section. From the previous section it can be assumed
that any initial inertial effects are negligible as the pond drains. I also assume that
the sea ice between the base of the melt pond and sea level is saturated, so that
the effect of air inclusions in this region is neglected. Early drainage of the pond,
prior to it completely draining, is described by the previous section. Specifically,
since inertial effects are negligible, it is described by the non-dimensional composite
approximation given by equation (6.49).

The time taken for complete drainage of the pond (to its base $h_p$) in non-dimensional
units is then given by

$$t = \log(1 - H_0),$$

noting that $H_0 < 0$ since the lengthscale $H_d - H_i$ is now negative.

The equation describing the subsequent evolution of the draining ice, by Darcy’s
law, is

$$\dot{h}_s = -\frac{\Pi}{\mu} \left( \frac{\rho g(h_i - h_0)}{h_i - h_s} \right) + \frac{\Pi \rho g}{\mu}. \quad (6.51)$$

Using the same lengthscale and timescale as in the previous subsection, the non-
dimensional version of equation (6.51) is

$$\dot{H}_p + \frac{H_d - H_i}{H_i} H_p \dot{H}_p + H_p = 1, \quad (6.52)$$

Figure 6.7: Evolution of surface of at-sea level pond: pond base below sea level.
Pond surface initially at 0.3 m, sea level at 0.1 m, and base of pond at 0 m
Figure 6.8: Cartoon of drainage model: pond base above sea level. Position of pond surface = $h_s$. Position of pond–ice interface = $h_p$. Position of sea ice floe base = $h_i$. Position of sea level = $h_o$. Pond depth = $H_p = (h_p - h_s)$. Ice depth below pond = $H_i = (h_i - h_p)$. Sea ice draft = $H_d = (h_i - h_o)$

which has an implicit solution for $H_p$ given by

$$t - \log(1 - H_0) = -\frac{H_d - H_i}{H_i}H_p - \frac{H_d}{H_i} \log(1 - H_p),$$

(6.53)

where $\log(1 - H_0)$ is the time at which the pond becomes completely drained.

Figure 6.9 shows the evolution in dimensional variables of an initially 0.05 m pond draining through ice with $H_d = 1.8$ m, so that the average floe thickness is about 2 m, and ice thickness beneath the pond $H_i = 1.9$ m. The permeability is $10^{-9}$ m$^2$, the density is 1000 kg/m$^3$, and the viscosity is 0.00179 kg/m s. The time taken to drain the pond to within 1 cm of sea level is 111 seconds.

Figure 6.10 shows the variation of time taken to drain the 0.05 m pond to within 1 cm of sea level against permeability. The range of permeabilities shown are typical values observed on sea ice throughout the year (Eicken et al. [29]). Since the timescale scales with $\mu H_i/\Pi \rho \Phi g$ the logarithm of the drainage time is proportional to the logarithm of the permeability, which is why a straight line is observed in figure 6.10. The drainage timescale has a significant variation with permeability and so accurate permeability measurements are necessary if model predictions of the drainage rate of sea ice are to be believed.
Figure 6.9: Evolution of surface of at-sea level pond: pond base above sea level.
Pond surface initially at 0.05 m, sea level at -0.1 m, and base of pond at 0 m.

Figure 6.10: Variation of pond drainage time against permeability for pond with base above sea level (plotted on logarithmic axes). Pond surface initially at 0.05 m, sea level at -0.1 m, and base of pond at 0 m. Pond drainage time defined as time taken to drain pond to within 0.01 m of sea-level.
6.3.4 Discussion

The drainage timescales of the two types of ponds discussed in the previous two subsections are of the order of minutes, which are clearly fast. However, in the above calculations I have ignored lateral drainage from the sea ice floe into the pond.

To incorporate lateral drainage the surface velocity becomes \( h_s = u - m \), where \( m \) is the rate of accumulation at the surface of the pond. Equations (6.36) and (6.37) become

\[
\frac{1}{\rho_l} \left( \frac{\rho g (h_i - h_o) - p(t)}{h_i - h_p} \right) + \frac{\rho_l g}{\mu}.
\]

To maintain a steady melt pond depth above sea level equations (6.54) and (6.55) show that

\[
m = \frac{\rho_l g}{\mu} \left( \frac{h_o - h_s}{h_i - h_p} \right),
\]

which for the examples of at-sea level ponds, described in the last two sections, correspond to accumulation rates of 55.65 m/day and 37.34 m/day, respectively. The order of magnitude difference is due to the fact that the pond with its base above sea level has a smaller meltwater head to maintain than the pond with its base below sea level.

Lateral drainage into the melt pond from the surrounding ice determines the accumulation rate \( m \), and depends upon the average melt rate of the catchment area \( v \), and the ratio of the area of the catchment area to the area of the pond \( R_{c/p} \). The accretion rate is then given by \( m = v R_{c/p} \), so that if the catchment area is large compared to the area of the pond, then the relative accretion rate will also be large. Assuming that the melt rate of the catchment area is about 2 cm/day (Perovich et al. [95]), then for the two examples (pond base below sea level, figure 6.7; pond base above sea level, figure 6.9) the area ratios are \( R_{c/p} = 2783 \) and \( R_{c/p} = 1867 \) to maintain a steady pond depth. This suggests steady melt pond depths are unlikely to be observed for large developed ponds, but could be observed in small ponds.
The mushy layer model assumes that the sea ice is in local thermodynamic equilibrium, so that the temperature and salinity are related by the liquidus relationship. For local thermodynamic equilibrium to be maintained the drainage timescale must be greater than the timescale of interstitial solute transport, $\delta^2/D_t$ (see section 2.2.1), which is approximately 1000 s (Feltham and Worster [32]). I define the drainage timescale using the dimensional version of the composite approximation (6.49) for the pond depth determined in section 6.3.2. I define the drainage timescale to be the depth of the sea ice beneath the pond divided by the initial drainage velocity as estimated using the composite approximation (6.49) and is given in dimensional units by

$$\frac{\mu H_t^2}{\Pi \rho_I g (H_0 - H_d + H_i)} \text{ seconds.} \quad (6.57)$$

For the two drainage examples the drainage timescales are approximately equal to 2650 s for the pond with its base below sea level and 4400 s for the pond with its base above sea level. Since the meltwater heads are extreme values (demonstrated by the estimates of $R_{c/p}$), the drainage timescale for melt ponds can be assumed to be greater than the interstitial solute transport timescale. Therefore local thermodynamic equilibrium should still be maintained within the sea ice with drainage.

In this section, I have shown that drainage is an important and highly variable component of sea ice. However, the drainage rate is dictated by the amount of lateral drainage from the surrounding catchment area. With a small at-sea level pond with the pond surface above sea level, the meltwater head can be maintained so long as there is sufficient flow from the surrounding region. With larger ponds, however, much larger quantities of meltwater are required to maintain a head above sea level. Therefore, larger ponds are likely to have their surfaces fixed at sea level. In numerical modelling, to simulate this process would require knowledge of the geometry, permeability, saturation, and melt rate of the entire sea ice floe, so that the sea ice freeboard is known. If these parameters were known, then the position of the melt pond surface could be calculated. The melt-pond-sea-ice model in this thesis is only one-dimensional, and so the constant rate of drainage that is used to describe drainage is only a first approximation.
6.4 Migration of sea-ice lenses

6.4.1 Introduction

Under-ice melt ponds are related to surface melt ponds because they form on the underside of sea ice through which has drained low salinity surface meltwater (Eicken [26]). The relatively warm, low salinity surface water drains through the sea ice to the ocean, becomes supercooled and may freeze. The area-extent of under-ice ponds is relatively uncertain, although Eicken [26] suggests they are extensive in the Arctic. Early field studies also found under-ice ponds to be extensive, with some estimates as large as 50% of the area of the summer pack ice (Hanson [49]). If under-ice ponds do cover a large area of the underside of Arctic ice their influence on the heat and mass budget (they store sensible heat and freshwater), would be significant. Under-ice melt ponds are also important with regard to marine ecology since they have been found to be a significant contributing factor to the Arctic marine food web (Gradinger [39]). Gradinger [39] found that under-ice ponds contained much higher algal pigment concentrations than in sea ice, with an algal composition that was distinct from other compositions found in the Arctic.

A sea-ice lens is the ice layer that forms between an under-ice melt pond and the
ocean (see figure 6.11). The aim of this section is to formulate a simple model to describe the migration of sea-ice lenses and use the model to make predictions of the ocean heat flux from sea-ice lens temporal data. This study was motivated by numerical solutions of the melt-pond–sea-ice model for which the melt pond had melted through significantly, leaving a thin layer of ice, which then froze at its surface and melted at its base, steadily migrating upwards (see figure 7.9, page 199), c.f. Martin and Kauffmann [70].

Sea ice ablation occurs via dissolution rather than melting (Notz et al. [86]). Dissolution is the process that maintains the ice–ocean interface in thermodynamic equilibrium (Woods [138]). For example, suppose the ocean was slightly superheated then the ice–ocean interface warms but remains in thermodynamic equilibrium as the ice dissolves, lowering the salt concentration and increasing the equilibrium freezing temperature. This process is called dissolution, and the rate of diffusion of salt determines the rate of movement of the sea-ice–ocean interface (Woods [138]). During freezing, however, the sea ice dendrites extend past the solutal boundary layer in the ocean, and the interfacial temperature is well approximated by the equilibrium freezing temperature of the ocean (Worster [141]). In this section, I continue to utilise the assumption that the ice–ocean interface is at the equilibrium freezing temperature of the ocean, and examine the implications of this assumption.

The model I consider is similar to that considered by Notz et al. [86]. The main difference is in the use of a mushy layer instead of a solid layer and heat fluxes are impinging on both upper and lower surfaces. The mushy layer formulation is seen to lead to a different qualitative behaviour of the sea-ice lens evolution. Also, the model formulated in this section is used to determine ocean heat fluxes from mass balance measurements, whereas Notz et al. [86] utilised their model to determine mass balance from estimates and observations of ocean temperature and salinity.

First, a one-dimensional model of a sea-ice lens is formulated, similar to the sea ice component of the melt-pond–sea ice model described in chapter 4. Second, the evolution of the lens thickness for constant heat fluxes is determined. The model is contrasted to a similar model with identical thermal conductivity and latent heat
Figure 6.12: Schematic diagram of sea-ice lens model. Position of upper surface = $h_0$. Position of lower boundary = $h_1$. Thickness of lens = $H = (h_1 - h_0)$. Temperature of upper boundary = $T_0$. Temperature of lower surface = $T_1$. Conductive flux in sea-ice lens into upper boundary = $k_0 \frac{\Delta T}{H}$, where $\Delta T = (T_1 - T_0)$. Conductive flux in sea-ice lens into lower boundary = $-k_1 \frac{\Delta T}{H}$. Heat flux from under-ice pond into upper boundary = $F_0$. Heat flux from ocean into lower boundary = $F_1$ of fusion at its boundaries and is seen to exhibit different behaviour. Third, the limitations of the model are examined, by comparison with the model of Notz et al. [86], which incorporates a parameterisation for the interfacial temperature between the lens and the ocean that depends upon the far-field temperature and salinity. Finally, the model is used to infer heat fluxes incident on a sea-ice lens estimated for data from ice station Charlie (Hanson [49]). Estimates of the corrections due to solutal transport for the ice-ocean interfacial temperature and ocean heat flux are also determined.

6.4.2 Sea-ice lens model

I do not consider the formation of the sea-ice lens. Instead, I only consider the situation when a thin layer of ice has formed between the dense ocean water and the fresher water above. I assume that the ice layer can be described by the equations
describing a mushy layer, with constant bulk salinity and no fluid flow. I also assume
that the ice layer is thin and the overlying ice and under-ice pond are thick enough
so that radiative heating is negligible (see figure 6.11). The two-stream model with
a 0.5 m pond and 0.5 m ice cover allows approximately 10% of the incident radiation
to the sea-ice lens, however, the under-ice pond will contain growing dendrites (Notz
et al. [86]), which will further decrease the amount of radiation at the sea-ice lens.
Since the Stefan number is large ($S \approx 40$) the rate of solidification is small, so that
the quasi-steady assumption is valid and the temperature profile is linear (Worster
[141]).

The model is one-dimensional and the $z$-axis is assumed to point vertically down­
ward. Subscript 0 represents terms at the fresh (upper) side of the sea-ice lens, and
subscript 1 represents terms at the ocean side of the ice lens (see figure 6.12).

Given sufficient thermal energy the internal fresh layer (under-ice pond) undergoes
turbulent convection (see section 4.3) and should therefore be well mixed. The ocean
is assumed to be convecting and mixing the fluid adjacent to the lens rapidly enough
to keep the salinity at the lower boundary constant. Under these assumptions the
effects of solutal diffusion from the surrounding liquid layers are neglected.

The main influence on the rate of ice growth/decay are the turbulent heat fluxes
assumed to be directed into the ice layer, $F_0$ and $F_1$. The absolute position of the
upper interface is $h_0$ and the absolute position of the lower interface is $h_1$. The lens
thickness $H = h_1 - h_0$. The solid fraction at the upper and lower interfaces is given
by $\phi_0$ and $\phi_1$ respectively. The upper and lower interfaces are at the respective
freezing temperatures of the upper and lower layers of fluid, $T_0$ and $T_1$.

It is assumed that explicit Stefan conditions describe the movement of the upper
and lower interfaces and are given by

$$\phi_0 \rho_s L \frac{dh_0}{dt} = k_0 \frac{T_1 - T_0}{H} + F_0 \quad \text{and} \quad \tag{6.58}$$
$$\phi_1 \rho_s L \frac{dh_1}{dt} = k_1 \frac{T_1 - T_0}{H} - F_1, \quad \tag{6.59}$$

where $\rho_s$ is the density of the fresh ice in the mushy layer, $L$ is the latent heat of
fusion, $k_0$ and $k_1$ are the thermal conductivities at the upper and lower interfaces,
The thermal conductivity $k_m(z)$ of the ice layer is assumed, as in section 4.2, to be given by the mixture relation

$$k_m = \phi \rho_s + (1 - \phi) \rho_l,$$

where $\phi = (T_{bulk} - T)/(T_s - T)$ is the local solid fraction of the ice layer, and $\rho_l$ is the density of the brine.

Subtracting equation (6.58) from (6.59) and rearranging yields the following equation describing the rate of change of the thickness of the ice layer

$$\frac{d}{dt} (H^2) = \alpha + \beta H,$$

where

$$\alpha = \frac{2(T_1 - T_0)}{\rho_s \mathcal{L}} \left( \frac{k_1}{\phi_1} - \frac{k_0}{\phi_0} \right)$$

and

$$\beta = -\frac{2}{\rho_s \mathcal{L}} \left( \frac{F_1}{\phi_1} + \frac{F_0}{\phi_0} \right).$$

The equilibrium solution $H_{eq}$, a time-independent solution, for the ice layer is immediate from equation (6.61) and is given by

$$H_{eq} = -\frac{\alpha}{\beta} = \frac{(T_1 - T_0)(k_1 \phi_0 - k_0 \phi_1)}{F_1 \phi_0 + F_0 \phi_1}.$$  

Although the thickness is in equilibrium the actual boundaries may not be, so that a constant thickness sea-ice lens may migrate. The velocity of the lens, whose thickness is in equilibrium, is given by either equation (6.58) or equation (6.59).

The equilibrium ice thickness is only physically realisable if it is non-negative and bounded. These requirements imply that for a physically possible equilibrium ice thickness we must have that (by considering equation 6.62) $F_1 \phi_0 + F_0 \phi_1 \neq 0$ so that the equilibrium thickness is bounded. Thus, if $F_0 \geq 0$ and $k_1/k_0 > \phi_1/\phi_0$ then we require $-F_1/F_0 > \phi_1/\phi_0$, since $T_1 - T_0 < 0$ by assumption; if $F_0 \geq 0$ and $k_1/k_0 < \phi_1/\phi_0$ then we require $-F_1/F_0 < \phi_1/\phi_0$; if $F_0 < 0$ and $k_1/k_0 > \phi_1/\phi_0$ then we require $-F_1/F_0 < \phi_1/\phi_0$; and if $F_0 < 0$ and $k_1/k_0 < \phi_1/\phi_0$ then we require $-F_1/F_0 > \phi_1/\phi_0$. For sea ice it will typically be the case that $F_0 \geq 0$ and

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that \( k_1/k_0 < \phi_1/\phi_0 \) so that we require \(-F_1/F_0 < \phi_1/\phi_0\) for a physically realisable equilibrium ice thickness. For the special case of \( F_0 = 0 \) this requirement becomes \( F_1 > 0 \).

Equation (6.61) is separable (assuming \( F_0 \) and \( F_1 \) are independent of time), and it can be integrated readily to obtain an implicit solution for \( H \) given by

\[
t = \frac{2\alpha}{\beta^2} \log_e \left( \frac{\exp(\frac{a}{\alpha}(H - H_0))(\alpha + \beta H_0)}{\alpha + \beta H} \right),
\]

(6.63)

where \( \log_e \) is the natural logarithm and \( H_0 \) is the initial thickness of the ice layer.

Equation (6.61) can be satisfied explicitly using the ProductLog function\(^1\). The solution for \( H \) is given by

\[
H(t) = -\frac{\alpha}{\beta} \left( 1 + \text{ProductLog} \left( -\frac{\exp\left(-1 - \frac{\beta(H_0 + \beta H)}{\alpha}\right)(\alpha + \beta H_0)}{\alpha} \right) \right),
\]

(6.64)

however, as was the case for the Inverse Logarithmic Integral (section 6.3), the ProductLog function is multi-valued, so care must be taken in choosing its definition to obtain a continuously differentiable function that satisfies the initial conditions (see figure 6.13). The appropriate choice of branch of the multi-valued ProductLog function depends on the sign of the equilibrium ice thickness, which can be seen by considering equation (6.64) at \( t = 0 \). If the equilibrium ice thickness is positive then the upper branch is the appropriate choice for the ProductLog function. In this case as \( t \to \infty, H \to -\alpha/\beta \). If the equilibrium ice thickness is negative then the lower branch is the appropriate choice for the ProductLog function. In this case, at finite time \( t_{melt} \) the lens melts (\( H = 0 \)) and \( t_{melt} \) can be determined by equation (6.63).

If the thermal conductivity of the sea-ice lens and latent heat required to melt the sea-ice lens is assumed to be identical at the boundaries (not necessarily constant), so that \( \phi_1 k_0 = \phi_0 k_1 \), then this analysis no longer holds. In this case the sea-ice lens thickness is given by

\[
H(t) = -\frac{(F_1 + F_0)}{\rho_s L (\phi_1 - \phi_0)} t + H_0,
\]

(6.65)

for constant heat fluxes. This case is equivalent to the lens model considered by Notz et al. [86], because they considered the case of a sea-ice lens of uniform

\(^1\)The ProductLog Function, \( W(z) \), is a generalised logarithm that satisfies \( z = W \exp(W) \).
thermal conductivity and identical latent heat at each boundary. In this case there
is no equilibrium thickness, and the resulting case is either complete melt, or infinite
thickness depending on the sign of \((F_1 + F_0)/(\phi_1 - \phi_0)\).

The time-dependent solution for the thickness of the sea-ice lens (equation 6.64) can
be used to determine the time-dependent solution of the positions of the boundaries
of the sea-ice lens analytically. The boundaries satisfy

\[
  h_x(t) = \frac{1}{a_1^2} \left( a_1 \beta^2 h_x(0) - (b_1 - c_1) \log_e \left( \frac{g(0)}{g(t)} \right) + c_1 (g(0) - g(t)) \right),
\]

where \(h_x(t)\) is the boundary with subscript \(x = 0, 1\), \(a = \phi_x \rho_x c, b = k_x(T_1 - T_0)\),
\(c = F_0\) when \(x = 0\) and \(c = -F_1\) when \(x = 1\), and

\[
  g(t) = \frac{H(t)}{H_{eq}} - 1.
\]

An example equilibrium sea-ice lens is shown in figure 6.14. The initial lens thick­
ness is 0.1 m and the equilibrium lens thickness is 0.76 m, which is achieved after
approximately 500 days. The sea-ice lens thermal properties and forcing data are
contained in table 6.2.
Figure 6.14: Time dependent evolution of sea-ice lens thickness and boundary positions with constant forcing

6.4.3 Model limitations: comparison with salinity incorporating model

Notz et al. [86] developed a model of sea-ice lens ablation that accounted for the effects of solutal dissolution at the ice–ocean interface. The model assumed a linear temperature profile. The sea-ice lens was not modelled as a mushy layer and the properties of the lens were assumed constant. Conservation of heat at the ice–ocean boundary was described by a Stefan condition that utilised a turbulent parameterisation of the ocean heat flux. Conservation of solute at the ice–ocean interface utilised a turbulent parameterisation of the solutal flux. The model was closed using the constraint of local thermodynamic equilibrium at the ice–ocean interface, so that the temperature and concentration are related by the liquidus relationship. The model could then be used to estimate the velocity of the lower boundary, the interfacial temperature and salinity, which then allows the ocean heat flux to be estimated. The model was used to investigate basal ablation and was applied to Arctic Ice Dynamics Joint Experiment (AIDJEX) and SHEBA data.

To gauge the difference between the model of Notz et al. [86] and the model in this section, I reformulate the model from this section to be similar to the model
of Notz et al. [86] and analyse the effect on the sea-ice lens–ocean boundary by parameterising the turbulent heat flux from the ocean in the same way as Notz et al. [86]. I refer to the mushy layer model from this section as the salt-neglecting model, meaning that the effects of solutal transport to the interface are neglected, and the reformulated mushy layer model as the salt-incorporating model, meaning that solutal transport to the interface is included.

The turbulent parameterisation of the ocean heat fluxes are based upon the Reynolds averaged turbulent heat flux, and using dimensional analysis can be expressed as

\[ F_1^* = \rho_s c \phi_1 \alpha_h u^* \delta T, \tag{6.67} \]

where \( c \) is the specific heat capacity of water, \( \alpha_h \) is a bulk transfer coefficient, \( u^* \) is the friction velocity, and \( \delta T = \hat{T} - T_1 \), where \( T_1 \) is the far-field (ocean) temperature, and \( \hat{T} \) is the temperature at the ice–ocean interface, which is different to the far-field temperature since the salt cannot diffuse as quickly as heat from the ice–ocean boundary (Notz et al. [86]).

The equation describing the evolution of the lower boundary in this salt-incorporating case is given by (c.f. equation 6.59)

\[ \phi_1 \rho_s \mathcal{L} \frac{dh_1}{dt} = k_1 \frac{T_1 - T_0}{H} + k_1 \frac{\delta T}{H} - F_1^*, \tag{6.68} \]

where \( F_1^* \) is the ocean heat flux directed into the ice, parameterised by equation (6.67). If the motion of the lower boundary \((dh_1/dt)\) is assumed to be identical between the salt-incorporating model (6.68) and the salt-neglecting model (6.59), the apparent heat flux in the salt-neglecting model is identical to the modified heat flux in the salt-incorporating model, and so

\[ F_1 = -k_1 \frac{\delta T}{H} + F_1^*. \tag{6.69} \]

Equation (6.69) is an explicit relationship between the heat fluxes estimated by the two models. The relative percentage error in the ocean heat flux in the salt-neglecting model compared to the salt incorporating model is given by

\[ 100 \left( \frac{F_1 - F_1^*}{F_1^*} \right) = 100 \left( \frac{-k_1}{H \rho_s c \phi_1 \alpha_h u^*} \right), \tag{6.70} \]
CHAPTER 6  
SECTION 6.4

Lens thickness (m)

0.1
0.2

0.15
0.05

Figure 6.15: Relative error of ocean heat flux for salt-neglecting model compared to salt-incorporating model for varying lens thickness

where I have used equation (6.67) and (6.69).

The equation describing the relative percentage error (6.70) due to neglecting the influence of salt is plotted in figure 6.15 as a function of sea-ice lens thickness using the representative values $\alpha_h = 0.0095$, $u^* = 0.005$ (Notz et al. [86]) and the values from table 6.2. Neglecting the rate limiting effect of salt always results in an underestimate of the magnitude of the ocean heat flux as indicated by the negative values shown in figure 6.15.

It is clear that error is only important for small ice thicknesses, or alternatively very small friction velocities. The data from ice station Charlie exhibits a minimum sea-ice lens thickness of 0.04 m (see figure 6.16), which would yield an error, because of neglecting salt, of about 30%. Notz et al. [86] state that errors of up to an order of magnitude can occur for ice less than 0.1 m thick, however, from the calculation of the error (using low friction velocity) it is seen that only for very small ice thicknesses (less than 1 cm) are the errors this extreme. In the regime of a very small ice thickness it is likely that both types of model will break down, because of dynamic effects such as brine flow, which would affect the salinity of the under ice pond.

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Table 6.2: Representative values appropriate for simple sea-ice lens model

<table>
<thead>
<tr>
<th>Property</th>
<th>Upper (0)</th>
<th>Lower (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (°C)</td>
<td>-0.35</td>
<td>-1.8</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.111</td>
<td>0.827</td>
</tr>
<tr>
<td>$k$ (W/m°C)</td>
<td>0.667</td>
<td>1.741</td>
</tr>
<tr>
<td>$F$ (W/m²)</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>$T_{bulk}$ (°C)</td>
<td>-0.311</td>
<td></td>
</tr>
<tr>
<td>$\rho_s$ (kg/m³)</td>
<td>900</td>
<td></td>
</tr>
<tr>
<td>$L$ (J/kg)</td>
<td>334800</td>
<td></td>
</tr>
</tbody>
</table>

6.4.4 Inference of forcing fluxes

Table 6.2 contains a representative set of values for the physical properties of sea-ice lenses for sea ice. The equilibrium thickness in this case is 0.68 m. Assuming that the parameters are correct, the model can be used to infer oceanic heat flux $F_1$ and internal heat flux $F_0$ for a given sea-ice lens evolution (c.f. Wettlaufer [130]). Such sea-ice lens evolution was measured during the field experiment at U.S. drifting station Charlie in 1959 (Hanson [49]). Figure 6.16 shows the evolution of a sea-ice lens at ice station Charlie for part of summer 1959. Also shown are the heat fluxes estimated from the sea-ice lens model that would simulate the observed sea-ice lens variation at ice station Charlie. Equations (6.58) and (6.59) are used to infer the heat flux at the upper and lower boundaries of the under ice ponds, and the heat fluxes are assumed to remain constant for each period between observations.

Between 22 August and 9 September the base of the sea-ice lens ($h_1$) is melting and the surface of the sea-ice lens ($h_0$) is freezing, and the incident ocean heat fluxes are within the variability of those observed during summertime in the Arctic (Perovich and Elder [92]). However, between 9 September and 16 September the rate of melting at the base decreases significantly and results in much larger heat fluxes incident on the sea-ice lens to balance the large conductive flux from the ice.

From the assumptions that were made earlier for the salt-incorporating model of sea
Figure 6.16: Evolution of sea-ice lens at ice station Charlie (after Hanson [49]).

Also shown are the estimated average heat fluxes on the sea-ice lens that are necessary to produce the variation in lens position and thickness observed.

Ice derived from Notz et al. [86], it is possible to estimate corrected forcing fluxes at the base of the sea-ice lens, which account for the difference in salinity between the ocean and the ice–ocean interface. It is also possible to make estimates of the actual interfacial salinity.

The salt-neglecting heat flux $F_1$ is related explicitly to the salt-incorporating heat flux $F_1^*$ by equation (6.70). Therefore, in terms of the salt-neglecting model the corrected heat flux is given by

$$F_1^* = F_1 \left(1 - \frac{k_1}{H \rho_s c_\phi \alpha_k u^*}\right)^{-1}. \quad (6.71)$$

The interfacial temperature can be determined from equation (6.69) immediately so that

$$\hat{T} = T_1 + \frac{H}{k_1} (F_1 - F_1^*). \quad (6.72)$$

Table 6.3 shows the salt-neglecting basal heat fluxes, the corrected basal heat fluxes, the percentage error, and the estimated ice–ocean interfacial temperatures.

Including the estimated corrections due to the flux of salt at the interface results in an increase in the magnitude of the heat flux at the interface. Again, the first two epochs exhibit values within natural summertime variability, however, the final
### Table 6.3: Estimated corrected heat flux, relative error, interfacial temperature, and salinity at ice station Charlie

<table>
<thead>
<tr>
<th>date</th>
<th>$F_1^*$ (Wm$^{-2}$)</th>
<th>$F_1$ error (%)</th>
<th>$\hat{T}$ (°C)</th>
<th>$\hat{C}$ (ppt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22/8/1959</td>
<td>29.37</td>
<td>-8.53</td>
<td>-1.60</td>
<td>31.14</td>
</tr>
<tr>
<td>31/8/1959</td>
<td>43.56</td>
<td>-22.20</td>
<td>-1.51</td>
<td>29.28</td>
</tr>
<tr>
<td>9/9/1959</td>
<td>-73.77</td>
<td>-28.70</td>
<td>-2.30</td>
<td>44.69</td>
</tr>
</tbody>
</table>

The estimated correction is larger than the expected summertime variability. This is because the melt rate of the lens-ocean interface is small, and it is likely that the timescale of turbulent solutal transfer between the ocean and ice is less than the timescale of melt and so the model neglecting solutal transfer is more appropriate in this case.

#### 6.4.5 Discussion

In this section, I have formulated a simple model of sea-ice lens evolution. The evolution of the sea-ice lens was determined quasi-analytically under the assumption of fixed forcing fluxes, and can be expressed using a multi-valued function (the ProductLog function), with the appropriate value of the multi-valued function being defined by the initial conditions in the same way as the Logarithmic Integral in section 6.3.

The thickness of the mushy-layer sea-ice lens possesses a stationary solution. In the case of equal thermal conductivity and latent heat of fusion at the upper and lower boundary, the evolution of the ice thickness was shown to possess no stationary solutions and to exhibit evolutionary behaviour that is qualitatively different to the mushy layer formulation. The evolution of the ice thickness for equal thermal conductivity at the boundaries under constant forcing was linear with time, which meant that the ice either melted entirely or froze without limit.

I utilised early mass balance measurements of a sea-ice lens at ice station Charlie
(Hanson [49]) to determine the heat fluxes on both the upper and lower surfaces of the sea-ice lens. Since the mass balance is known, a simple parameterisation of the turbulent ocean heat flux (6.67) could be used to estimate the error in the heat flux due to neglect of solute transport at the ice–ocean interface, and the ice–ocean interfacial temperature from the assumption about the far-field ocean temperature.

The method of determining the ocean heat flux from the ice–ocean Stefan condition has been used previously for sea ice floes (e.g. Wettlaufer [130]). Notz et al. [86] also determined the ocean heat flux from the Stefan condition for their sea-ice lens model. However, their model assumed that the heat flux from the under-ice pond was zero, and calculated ice thickness. Therefore, if the heat flux was non-zero, which is likely for the thin overlying ice observed at ice station Charlie, and the sea-ice lens thickness estimate is in error, then the estimates of heat flux at the ice–ocean interface would be inaccurate.

The magnitudes of the estimated oceanic heat fluxes were large because of the rapid release of latent heat at the boundaries (see figure 6.16) and also exhibited significant temporal variability. This behaviour is consistent with recent observations of ocean heat flux at SHEBA (Perovich and Elder [92]). During summer the input of solar radiation into the ocean enhances the heat flux into the sea ice (Perovich and Elder [92]).

The main limitation of the sea-ice lens model is its neglect of solute transport at its interfaces. Since salt diffuses much slower than heat, sea ice ablation is rate limited by salt transport at the boundary, so that it dissolves instead of melts (Notz et al. [86]). The model assumes that mixing is rapid enough to ensure that the salt diffuses to the boundary fast enough, so that the salinity of the boundaries of the sea-ice lens are the same as the adjacent fluid. To gauge the typical errors introduced by neglecting solute transport, the model was compared to a recent model by Notz et al. [86]. The estimated error was determined using a turbulent parameterisation of heat transport from the ocean to the ice as used in the sea-ice lens model of Notz et al. [86]. The error is inversely proportional to the ice thickness and the friction velocity. Larger friction velocities allow salt to be transported more readily to the
Taking the friction velocity to be $0.005 \text{ ms}^{-1}$, yielded errors of about 30% or above for thicknesses less than 0.04 m. However, typical sea-ice lens thicknesses are greater than 0.04 m so that 30% is an estimate for the upper bound of the error introduced by neglecting salt transport.

### 6.5 Summary of melt pond process studies

In this chapter, I have examined a variety of processes that are relevant to melt pond evolution. First, I performed a numerical study to determine the critical Rayleigh number for a melt pond. Since turbulent convection is a more effective heat transfer mechanism than thermal diffusion, it is vital to be able to estimate the conditions for the onset of thermal instability.

The fact that stationary temperature profiles of the melt pond were monotonic, because the absorption coefficient of the pond is relatively small, enabled the Rayleigh number to be defined by the temperature difference across the whole depth of the melt pond, consistent with previous research (e.g. Whitehead and Chen [134]). The critical Rayleigh number was found to be insensitive to the forcing parameters, so that a constant value (630) is prescribed for use in the full melt-pond-sea-ice model.

Second, I examined the nature of summertime drainage in sea ice. Simple models were proposed that demonstrated the rapid drainage of melt ponds to sea level and quantified the effects of permeability on drainage. Although the timescale for drainage is rapid, the sea ice remains in local thermodynamic equilibrium. Therefore, the assumption of local thermodynamic equilibrium in the melt-pond-sea-ice model remains valid.

It was shown that to maintain large heads of meltwater requires large fluxes from the surrounding floe, which are only possible for smaller melt ponds. The incorporation of drainage into the melt-pond-sea-ice model, even such a simple model as utilised in this chapter, would require knowledge of the floe geometry, permeability, and melt rate.
Third, I analysed a simple model of sea-ice lens migration. The model was shown to have a quasi-analytical solution for the sea-ice lens thickness in terms of a multi-valued function. The model exhibited qualitatively different behaviour than the simplified case in which the model was assumed to have the same thermal conductivity at its boundaries. This qualitative difference suggests care must be taken in Stefan problems with two moving boundaries when justifying assumptions regarding constant thermal conductivity and latent heat of fusion.

The assumption that the boundaries are at the equilibrium freezing temperature of the far-field ocean was examined by utilising parameterisations of the turbulent transport of heat from the ocean. The relative error of the ocean heat flux was compared to the sea-ice lens model of Notz et al. [86]. Errors were relatively small for reasonable sea-ice lens thickness. The model can be used to predict impinging heat fluxes on a sea-ice lens whose evolution is known.
Chapter 7

Melt-pond–sea-ice model simulations

7.1 Introduction

In this chapter, I present the results of simulations of the annual cycle of the melt-pond–sea-ice system using the model described in chapter 4. First, I describe the baseline case and discuss the quality of the simulation by comparing it to observations from SHEBA and to previous theoretical research. I investigate and discuss the sensitivity of the maximum melt pond depth to the drainage rate, and assess the importance of snow cover, the parameter $i_0$ of the sea ice, and the turbulent convective parameterisation, to the model simulation. Second, I investigate the sensitivity of the model to optical properties, the parameter $i_0$ of the melt pond, the snow albedo, and the sensible and latent heat flux when there is a melt pond. For each parameter, I explain the physical reasons for the nature of the sensitivity observed within the model. Finally, I summarise the results and describe the conclusions of the simulation results.

All model runs in this chapter are initialised with a linear temperature gradient in the sea ice and the snow and a surface temperature of $-30$ °C. In the snow covered case the snow–ice interfacial temperature is determined simply using the condition
of conservation of conductive flux at this boundary (equation 4.33). In model runs that do not have a snow cover the initial temperature gradient is linear also. The initial ice thickness is set to 2 m unless otherwise stated, and the snow depth begins at 0.32 m consistent with the snow cover formulation described in chapter 4.

The likely long-term fate of ice that forms melt ponds will be that it melts completely in summer, with new ice formation through the next season. Therefore, following previous work by Flato and Brown [36] and Gabison [37], the melt-pond–sea-ice model is allowed to run for one summer melt season from 1 January to 31 December and is then terminated.

### 7.2 Standard case

The standard case was chosen to be the simulation with model parameters that gave a reasonable approximation to the expected summertime evolution of melt ponds. The parameter with the most uncertainty is the drainage rate (see section 7.2.1). In section 4.6, I argued that the constant drainage rate is a proxy for the surface melt rate of the entire floe because melt ponds of sufficient diameter drain so fast that typically their surfaces should remain at sea level. Typical surface melt rates can be up to a few centimetres per day (Perovich et al. [95]), and the standard case drainage rate was chosen to be 2 cm/day.

Figure 7.1 shows the evolution of the positions of the boundaries of the melt-pond–sea-ice system. Also shown is the evolution of the albedo for the annual cycle (uppermost line). The figure is therefore a side profile of the sea ice for the year, from the perspective of an observer standing on the surface of the sea ice during winter.

During winter the evolution is slow, with minor variations at the snow surface due to precipitation and at the ice–ocean interface due to melting or freezing. As the short-wave energy increases (see figure 7.3), the temperature of the snow increases until it is warm enough to begin melting, which occurs on day 171 (June 21). When
Figure 7.1: Annual cycle for standard case. Lines indicate motion of boundaries through the year, relative to wintertime snow–ice interface. Uppermost line is the evolution of the albedo through the year. January 1 is given by day 0 melting begins, the snow albedo drops from its wintertime value (0.84) to its melting value, which is determined from the linearly interpolated value, based on initial snow thickness and expected initial pond depth. Variations in the linearity of the albedo decrease during this stage are due to changes in ice thickness that affect the linearly interpolated estimate of albedo. The ice is thickening slightly which leads to the convex albedo profile. The time taken to melt the snow and thus form an initial melt pond is about 9 days, so that on day 180 (June 30) there is a melt pond of depth 0.127 m with albedo 0.45. The melt pond initially becomes shallower, resulting in an increase in the albedo. However, as summer continues the pond deepens and the albedo decreases. The temperature of the pond is in good agreement with observations of pond temperature of a few tenths above freezing (Eicken et al. [27]), with maximum surface temperature 0.54 °C and maximum core temperature 0.10 °C. On day 217 (August 6) the melt pond is at its maximum depth (0.43 m) having melted through 1.04 m of ice, and the albedo is at its minimum value (0.166). On day 224 (August 13) the melt pond has decreased in depth (0.39 m) and surface ablation has melted 1.14 m of ice. However, it is at this time that refreezing at the melt pond surface occurs, leading to a rapid increase in albedo up to 0.721. The reason for this rapid increase is that the wintertime optical properties of the
radiation model are incorporated at this point so that a thin layer of sea ice is highly effective at reflecting the incident radiation. The constant drainage rate yields a net mass loss at the surface of 0.740 m. After only 7 days, on day 231 (August 20), snow begins to fall, and the albedo rises again to its constant wintertime value (0.84). The internal molten region remains as a store of sensible and latent heat, but refreezes by day 259 (September 17). On day 322 (November 19) the sea ice has cooled sufficiently and begins to refreeze. By the end of December the ice has refrozen so that it is 1.864 m thick, which implies a net annual loss of 0.136 m of sea ice.

The data available from the SHEBA field experiment provide a necessary check on the simulations. SHEBA melt pond mass balance data from Perovich et al. [95] from 26 gauges indicated that the average maximum pond depth was 0.36 m, with maximum 0.54 m. The amount of surface ablation for all the gauges averaged 0.75 m, with minimum 0.4 m and maximum 1.09 m. The standard case exhibited maximum pond depth 0.43 m, and 1.14 m of surface ablation. Therefore, the standard case is exhibiting behaviour that is at the upper end of the expected behaviour.

Figure 7.2 compares the evolution of average pond depths observed at SHEBA (Perovich et al. [95]), with the evolution of pond depths from the Ebert and Curry [25] model, and the evolution of pond depths from this study. The start of the melt season for the standard case begins late compared to SHEBA data, a feature common among numerical simulations (e.g. Ebert and Curry [25]; Maykut and Untersteiner [77]), but not as late as the Ebert and Curry [25] model. The standard case underestimates the pond depth at the beginning of the melt season and overestimates the pond depth toward the end of the melt season. The most likely reason for this is the specification of a constant drainage rate. Melt rates at the surface of sea ice typically wax and wane with the summer season (Perovich et al. [95]), so that the constant drainage rate is an overestimate at the beginning and end of the summer melt season.

Freeze-up in the standard case occurs in mid-August just as in the SHEBA data. Since the surface forcing data are averaged SHEBA data, this similarity between freeze-up dates is evidence to support the use of convective heat transport terms
This study

Figure 7.2: Comparison of melt pond depth evolution: melt-pond–sea-ice model (standard case); Ebert and Currys’ [25] model; and SHEBA (Perovich et al. [95]) data. January 1 is given by day 0

in the surface energy balance. This is because the onset of freeze-up is dictated by the surface energy balance. The close correspondence between the model and the observations indicates that the sensible heat in the pond is a good approximation and the heat flux at the surface is a good approximation, and leads to similarities in the freeze-up date. As a comparison between the Ebert and Curry [25] model and the standard case, I determined the average deviation of melt pond depth from the SHEBA data (Perovich et al. [95]). For the Ebert and Curry [25] model the average deviation was 3.16 cm and for the standard case the average deviation was 1.31 cm. However, part of the difference is likely the result of the different forcing data used, and the fact that the evolution of melt pond depth in the Ebert and Curry [25] model was fitted to just two data points from another data set.

Figure 7.3 shows the evolution of the partitioning of the short-wave radiation between the sea ice and melt pond, the ocean, and the atmosphere. Shown on the diagram are the incoming short-wave radiation, the short-wave radiation that is reflected, the short-wave radiation that is absorbed within the whole melt-pond–sea-ice system, and the short-wave radiation that is transmitted through to the ocean. The short-wave radiation peaks on day 156 (June 6) at 269.8 W/m², so that
Figure 7.3: Evolution of reflected, absorbed, transmitted, and total ($F_{SW(tot)}$) short-wave radiation of melt-pond–sea-ice system for the standard case. January 1 is given by day 0 when melt ponds form (day 180) the short-wave radiation is decreasing. However, when a melt pond has formed the absorbed short-wave radiation rapidly triples in value reaching a maximum of 147.0 W/m² on day 204 (July 24) as compared to 43.17 W/m² on day 156 when the incoming short-wave radiation is at a maximum but the sea ice is still snow covered. Table 7.1 shows the annual radiative energy budget split into ponded and unponded periods. For the period when a melt pond is present, the short-wave energy absorbed is 67.7% of the annual absorbed energy and this is for a period of just 43 days. The total transmitted short-wave radiation for the year is just 0.001% of the total annual incoming short-wave radiation. However, the amount of short-wave radiation transmitted in the presence of melt ponds is 91.74% of the annual transmission to the ocean.

Energy balance measurements from SHEBA indicate that there is an energy deficit of about 60 MJ/m² over a year, part of which has been suspected of being transmitted energy through the ice and melt ponds (Perovich and Elder [92]; Perovich et al. [97]). However, the annual estimate (table 7.1) of just 2 MJ/m² is far short of the missing 60 MJ/m². This may be because of the overestimate of the ice thickness of the model at the beginning of the year compared to observations at SHEBA,
since the mean ice thickness at SHEBA was 1.5 m in October 1997 (Perovich et al. [97]). It may also be because more energy was transmitted through the ice than is estimated in this model due to differences in the parameterisation of the sea ice optical properties and the optical properties at SHEBA. Summertime optical properties of the sea ice are highly variable (Podgorny and Grenfell [102]), and so this effect may be important.

To test the effect of ice thickness on the optical properties, the simulation of the standard case was repeated with initial ice thickness reduced to 1.75 m. The maximum pond depth was found to be 0.426 m, with 1.063 m surface ablation at that time. The maximum surface ablation in this case was 1.167 m.

Intuitively it would be expected that the albedo-feedback mechanism would accelerate melting of the ice due to lower albedo for thinner ice. However, the maximum pond depth is less than the standard case. The reason for this is that the albedo is relatively insensitive to changes in ice thickness when the ice thickness is greater than 0.8 m (Perovich et al. [98]). The reason that the maximum surface ablation is greater than the standard case is because the onset of melt is earlier. Onset of melt is earlier because the conductive flux through the depth of the sea ice is increased prior to the melt season. The reason the maximum melt pond depth is less is also because of the conductive flux of the sea ice. In the presence of melt ponds, heat is transported away from the pond–ice interface toward the ocean, and thinner ice means that heat can be transported away more rapidly. Therefore, the melt rate is

<table>
<thead>
<tr>
<th></th>
<th>Ponds Present (10^8 J/m²)</th>
<th>Ponds Absent (10^8 J/m²)</th>
<th>Whole Year (10^8 J/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflected</td>
<td>2.40</td>
<td>17.51</td>
<td>19.91</td>
</tr>
<tr>
<td>Absorbed</td>
<td>5.05</td>
<td>3.96</td>
<td>9.01</td>
</tr>
<tr>
<td>Transmitted</td>
<td>0.02</td>
<td>0.0018</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>7.46</strong></td>
<td><strong>21.48</strong></td>
<td><strong>28.94</strong></td>
</tr>
</tbody>
</table>

Table 7.1: Annual energy budget for the standard case
Figure 7.4: Sensitivity of maximum pond depth to initial sea ice depth at standard drainage rate

marginally less rapid, which results in a smaller maximum pond depth. This modifies the ‘conduction-feedback’ loop described by Ebert and Curry [25]. Ebert and Curry [25] found that thinner ice with larger conductive fluxes to the surface began to melt earlier and was found to melt more than thicker ice. However, their model did not include the effect of melt ponds on the conductive flux of the sea ice during summer.

As the initial ice thickness becomes much thinner (less than 0.75 m), the albedo-feedback mechanism becomes more effective than the conduction-feedback in the summer and so the maximum pond depth increases (see figure 7.4). However, simultaneously there is physically less ice available to melt and so the maximum surface ablation decreases (see figure 7.5). This demonstrates that the presence of melt ponds increases the complexity of feedback-mechanisms that affect the mass balance of the sea ice.

Table 7.2 shows the annual radiative energy budget for the thinner 1.75 m initial ice thickness split into ponded and unponded periods. The lower albedo in the presence of melt ponds, described above, leads to less reflected short-wave radiation and more short-wave energy being absorbed. The most significant change is in the transmitted radiation, which almost triples for the whole year as compared to the 2 m initial ice
Figure 7.5: Sensitivity of maximum surface ablation to initial sea ice depth at standard drainage rate

Table 7.2: Annual energy budget for initial ice thickness 1.75 m

<table>
<thead>
<tr>
<th></th>
<th>Ponds Present (10^8 J/m²)</th>
<th>Ponds Absent (10^8 J/m²)</th>
<th>Whole Year (10^8 J/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflected</td>
<td>2.41</td>
<td>17.50</td>
<td>19.91</td>
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<tr>
<td>Absorbed</td>
<td>5.02</td>
<td>3.98</td>
<td>9.00</td>
</tr>
<tr>
<td>Transmitted</td>
<td>0.03</td>
<td>0.004</td>
<td>0.034</td>
</tr>
<tr>
<td>Total</td>
<td>7.46</td>
<td>21.48</td>
<td>28.94</td>
</tr>
</tbody>
</table>

thickness. However, the annual total is still only about 3.4 MJ/m², which is far short of the 60 MJ/m² deficit determined from SHEBA data. Therefore, if the parameter assumptions are valid, it is unlikely that the observed energy deficit is solely due to radiation transmitted through the sea ice cover. Other potential sources for the missing energy are heat advected from the edge of the summer sea ice cover, which was only 100 km from the SHEBA camp in the height of summer, and deep warmer water reaching the underside of the ice to contribute to melting (Perovich et al. [97]).

A typical temperature profile for Autumn (day 250) of the standard case is shown in figure 7.6. The model in this configuration is the most complex since it has
four separate layers that are tracked at each timestep. The snow depth is 0.078 m and the internal melt region is 0.142 m. The air temperature has cooled from its summertime value, of around 0 °C, to −3.3 °C. The sensible and latent heat fluxes are directed into the ice and are about 6 W/m² and 3 W/m² respectively.

![Figure 7.6: Temperature profile inside melt-pond–sea-ice model at day 250. Snow–ice interface is at 0 m](image)

Figure 7.6: Temperature profile inside melt-pond–sea-ice model at day 250. Snow–ice interface is at 0 m

Figure 7.7 shows the evolution of temperature inside the ice from day 0 to day 365 (c.f. figure 7.1). The upper and lower boundaries are not represented exactly on the diagram. The oscillations in the temperature contours near the snow–ice interface are a result of the software used to construct the plot. The temperatures observed in the sea ice of the standard case are similar to observed temperature profiles from SHEBA (see Perovich et al. [95]) and are generally lower than those of Maykut and Untersteiner [77]. The general evolution pattern is the same as both observations from SHEBA (Perovich et al. [95]) and theoretical results presented in Maykut and Untersteiner [77] with the ice steadily becoming isothermal as summer approaches. The melt pond and internal melt region is contained within the −0.8 °C isotherm shown in the figure, until the sudden jump in the contours at day 259 when the internal melt region refreezes and the data are interpolated back to a single grid.

Maykut and Untersteiner's [77] model predicts a more rapid cooling of the sea ice during the autumn than observed. Their model begins to refreeze at the base of
Figure 7.7: Contour plot of annual evolution of temperature (°C) inside sea ice. January 1 is given by day 0.

the sea ice at the end of October, whereas the melt-pond-sea-ice model begins to refreeze at its base in late November. As the internal melt region cools and refreezes it releases sensible and latent heat to the surrounding sea ice, which diffuses away from the melt region. This extra heat reduces the rate of cooling of the sea ice in the autumn, consistent with observations, and reduces the temperature gradient in the lower half of the sea ice, leading to less wintertime growth.

Previous research has assumed that the influence of melt ponds on the overall mass balance of sea ice is relatively small since they refreeze in autumn (Maykut and Untersteiner [77]; Mellor and Kantha [80]). However, my calculations show that the presence of melt ponds affect the mass balance through the albedo-feedback mechanism, drainage, and through the delayed onset of refreezing at the ice-ocean interface caused by sensible and latent heat stored in melt ponds at the end of summer.

7.2.1 Drainage

As drainage is potentially the most uncertain parameter in the model (see section 6.3) it is useful to consider the model sensitivity to drainage. The most informative
Figure 7.8: Sensitivity of maximum pond depth to drainage rate for standard case parameter to consider on the model runs is the maximum melt pond depth. This is because it can be directly compared to observations and is a parameter that is routinely measured in studies of melt ponds (e.g. Eicken et al [27]; Morassutti and LeDrew [82], Perovich et al. [95]). The pond depth is also crucial in determining the albedo of melt pond covered sea ice and is utilised in parameterisations of melt pond albedo (Morassutti and LeDrew [82]).

Figure 7.8 shows the sensitivity of the maximum pond depth to the drainage rate. At low drainage rates (0–1.5 cm/day), the maximum pond depths are large, whereas at higher drainage rates (1.5–2.5 cm/day), the maximum pond depths are small. The reason for this trend is that low drainage rates promote the albedo-feedback mechanism, because the pond depth increases rapidly, reducing the albedo, increasing melting. At larger drainage rates (> 2.5 cm/day) the drainage rate is so rapid that initial melt ponds formed from snow melt just drain completely.

Qualitatively there are three types of behaviour that are observed as the drainage rate varies:

1. At low drainage rates (0–1.5 cm/day), the sea ice melts through significantly. However, the sea ice does not melt through entirely because freeze-up occurs before the ice can melt through. The sea ice between the melt pond and the
ocean acts like a sea ice lens, with the upper surface freezing and the lower surface melting (see section 6.4). Simultaneously, the melt pond refreezes from its surface down. At the end of the year, there is still enough thermal energy for the internal molten region to persist within the ice, which is due to the large melt pond at the end of the summer. An example of this type of behaviour is shown in figure 7.3 which shows the evolution of the snow, melt pond, and sea ice thickness for the case of no drainage. The melt pond melts through most of the ice and sea-ice lens evolution is then observed for the ice adjacent to the ocean.

2. At intermediate drainage rates (1.5–2.5 cm/day), the sea ice does not melt through as much as with lower drainage rates. The resulting melt ponds are of typically observed thickness (< 1 m) at the time of refreezing. The sea ice between the melt pond and the ocean does not behave like a sea-ice lens, and the melt ponds refreeze before late autumn. An example of this type of behaviour is shown in figure 7.1 which shows the evolution of the standard case (2 cm/day).

3. At large drainage rates (> 2.5 cm/day), the drainage rate is so rapid that the initial melt pond drains entirely. Typically, at each timestep the surface energy balance indicates that a melt pond forms. However, since the drainage rate is so rapid it immediately drains away. An example of this type of behaviour is shown in figure 7.10, which shows the evolution of the snow, melt pond, and sea ice thickness for the case of drainage at 3.5 cm/day. The rapidly draining initial melt pond is observed as the small wedge shape in figure 7.10 around day 180.

The results of the simulations at intermediate drainage rates give information about freeze-up rates of melt ponds. The day of the year that the ponds began to freeze, their depth at this time, the day they had completely frozen, and the time taken for them to freeze completely are shown in table 7.3. These data can be used to estimate the time taken for melt ponds to freeze in the autumn for an arbitrary
Figure 7.9: Annual cycle for standard case with no drainage. Lines indicate motion of boundaries through the year, relative to wintertime ice-atmosphere interface. January 1 is given by day 0.

Figure 7.10: Annual cycle for standard case with 3.5 cm/day drainage rate. Lines indicate motion of boundaries through the year, relative to wintertime ice-atmosphere interface. January 1 is given by day 0.
maximum melt pond depth. The number of days taken to refreeze the melt pond for a given maximum melt pond depth was fitted to a quadratic equation of the form $az^2 + bz$, where $x$ is the depth of the pond (m), by the method of least squares. The optimal solution was

$$\text{Time} = 31.643 \cdot \text{Depth}^2 + 65.127 \cdot \text{Depth}. \quad (7.1)$$

The reason that the quadratic form does not have an additional constant is because I require a zero thickness pond to be instantly refrozen. The estimate is valid for snow-covered ice with similar snow depths to the standard case. This is because the refreezing is mainly due to cooling at the surface, which can be seen in figure 7.1 as the surface of the internal melt region is refreezing faster than the base. If the snow depth was greater than the model then it would be expected that the time taken to refreeze would be increased.

The simple relationship given in equation (7.1) provides a straightforward way to estimate how long individual ponds will take to freeze. For example, suppose that the freeze-up begins on day 226 (August 15), then for a melt pond to completely freeze by the beginning of November (day 304) will require the melt pond to be 0.857 m at the start of freeze-up. Since melt ponds are typically less than 0.8 m (Ebert and Curry [25]), this indicates that the melt ponds should not take until November to freeze. The maximum pond depth observed at the mass balance gauges from SHEBA was 0.54 m (Perovich et al. [95]), and so the time taken to refreeze this pond is approximately 44.4 days, by equation (7.1). Therefore, it is estimated that melt ponds at SHEBA should have refrozen by about September 28.

Previous estimates of the time taken for melt ponds to refreeze suggest that ponds should be frozen by November (Fetterer and Untersteiner [35]). At the SHEBA experiment melt ponds were found to not be completely frozen by the beginning of October (Perovich et al. [97]). The difference between the observations at SHEBA and the estimate is likely due to differences in the depth of the snow cover and melt ponds between the model and the observations.
### Table 7.3: Time taken for melt ponds to refreeze for the standard case

<table>
<thead>
<tr>
<th>Drainage rate (cm/day)</th>
<th>Pond depth at freeze-up (m)</th>
<th>Day freeze-up begins</th>
<th>Day freeze-up ends</th>
<th>Days to freeze</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.335</td>
<td>227</td>
<td>373</td>
<td>146</td>
</tr>
<tr>
<td>1.5</td>
<td>0.991</td>
<td>226</td>
<td>316</td>
<td>90</td>
</tr>
<tr>
<td>2.0</td>
<td>0.385</td>
<td>224</td>
<td>259</td>
<td>35</td>
</tr>
</tbody>
</table>

### Figure 7.11: Annual cycle for standard case without a snow cover at standard drainage rate. Lines indicate motion of boundaries through the year, relative to wintertime ice-atmosphere interface. January 1 is given by day 0

#### 7.2.2 Role of the snow cover

In order to investigate the role of the snow cover, I compare model runs without a snow cover with the standard case. The effects of the snow cover are to insulate the sea ice from the atmosphere and so retard the rate of warming or cooling of the sea ice, to prevent most radiation from penetrating the sea ice, and to lower the surface temperature relative to bare sea ice (Sturm et al. [120]).

It is found that without a snow cover, the sea ice begins melting on day 170, which is 10 days earlier than the snow covered case. The melt water drains more rapidly than the ice melts so that a pond doesn’t form (see figure 7.11).
The sensitivity of the maximum pond depth of the snow-free model to the drainage rate is shown in figure 7.12. The same qualitative behaviour as the snow covered case (figure 7.8) is observed, with increased drainage rates leading to a decrease in the maximum depth of the pond. However, the snow-free case is more sensitive and the pond becomes completely drained for drainage rates greater than 1 cm/day. The increased sensitivity to drainage is because the initial pond depth in the snow-free case is 0 m, whereas the initial pond depth in the snow covered case is about 0.12 m. A non-zero initial pond depth allows the melt process to get underway even if the drainage rate causes the pond to initially become thinner. When there is no pond initially, the specified constant drainage rate prevents the pond from forming as rapidly.

The annual evolution of the snow-free model with drainage rate 0.75 cm/day is shown in figure 7.13. A pond forms but does not completely melt the ice. It can be seen that as melting begins (day 170) a pond forms but drains completely and it is not until about day 190 that the pond can actually form.

Figure 7.14 shows the evolution of the upper-surface temperature of the snow covered standard case and snow-free case with drainage rate 0.75 cm/day. It can be clearly seen that the surface temperature of the snow covered case is less than the
Figure 7.13: Annual cycle for standard case without a snow cover at 0.75 cm/day drainage rate. Lines indicate motion of boundaries through the year, relative to wintertime ice–atmosphere interface. January 1 is given by day 0

surface temperature of the snow-free case, and post-melt season, the snow cools more rapidly, since the temperature gradient is much less than the ice-only case, resulting in lower surface temperatures, by the surface energy balance (equation 4.34).

The snow cover has been shown to have a significant influence on the annual evolution of the sea ice. During the winter a lack of snow cover can be seen to allow more growth at the ice–ocean interface because of its insulating properties. This difference in ice growth can be seen by comparing figure 7.1 and figure 7.11. The presence of a snow cover was also shown to lower the surface temperature relative to bare sea ice (figure 7.14). Melt ponds are shown to be more sensitive to drainage without a snow cover, because the initial depth of the melt ponds in this case is zero.
Figure 7.14: Mid-monthly surface temperatures for snow covered and snow-free sea ice. January 1 is given by day 0

Sensitivity of $i_0$ parameter: subsurface melting

The parameter $i_0$ for the sea ice, which represents the fraction of short-wave energy that passes through the surface and contributes to internal heating, is observed to be a controlling mechanism for the potential formation of subsurface melt regions. Subsurface melting occurs when the rate of heating due to solar radiation is greater than the rate of cooling by conduction (Brandt and Warren [13]), and has been observed both experimentally and theoretically (Boggild et al. [10]; Koh and Jordan [56]; Liston et al. [62]). However, the results of theoretical models were derived from single-phase models (i.e. ice or snow only without any phase change), which could not model the evolution of the subsurface melt region. The surface energy balance (equation 4.20) for sea ice is given by

$$\langle F_{\text{net}} \rangle^\text{ice}_0 = k_m \frac{\partial T}{\partial z} + (1 - i_0)(1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) + F_{\text{LW}} - F_{\text{sens}} - F_{\text{lat}} - \epsilon_{\text{ice}} \sigma T_0^4. \quad (7.2)$$

In the case of no melting ($\langle F_{\text{net}} \rangle^\text{ice}_0 = 0$ and the surface energy balance can be interpreted as an expression for the conductive flux, $-k_m \partial T/\partial z$, assuming all other parameters are known.

Figure 7.15 shows the sign of the conductive flux for the atmospheric forcing data used in the melt-pond–sea-ice model for the year against the value of $i_0$. For sim-
Figure 7.15: Annual evolution of sign of conductive flux \(-k_m \frac{\partial T}{\partial z}\) at atmosphere-ice interface for varying fraction of short-wave radiation that penetrates into the interior of the sea ice \(i_0\). Shaded region denotes region where conductive flux is positive.

plicity, prescribed sensible and latent heat fluxes are taken from Maykut and Untersteiner [77], which are shown in table 7.4, and the surface temperature is assumed to be \(-0.35^\circ C\), so that the ice is nearly melting at all times. The grey region denotes times for which the conductive flux at the surface is positive. If the conductive flux is negative then the temperature gradient at the surface is positive. During summer when the ice has heated sufficiently a negative conductive flux will correspond to internal melt regions. This is because a negative conductive flux corresponds to positive temperature gradient, so that the temperature increases with depth inside the ice, resulting in internal regions that are above the melting temperature of the ice. If the surface temperature reaches the melting temperature at the end of May (which is when snow melt is typically observed in the Arctic) then the transition point for the change in sign of the conductive flux is approximately \(i_0 = 0.45\) (see figure 7.15).

In the full melt-pond-sea-ice model the \(i_0\) parameter for the sea ice is limited to be at most 0.4 to prevent internal melt regions from developing in the snow-free case (see section 4.5.1). If the restriction on \(i_0\) in the full melt-pond-sea-ice model is relaxed,
<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensible (W/m²)</td>
<td>-19.06</td>
<td>-12.28</td>
<td>-11.63</td>
<td>-4.68</td>
<td>7.27</td>
<td>6.3</td>
</tr>
<tr>
<td>Latent (W/m²)</td>
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<td>0.32</td>
<td>0.48</td>
<td>1.45</td>
<td>7.42</td>
<td>11.29</td>
</tr>
<tr>
<td>Month</td>
<td>Jul</td>
<td>Aug</td>
<td>Sep</td>
<td>Oct</td>
<td>Nov</td>
<td>Dec</td>
</tr>
<tr>
<td>Sensible (W/m²)</td>
<td>4.85</td>
<td>6.46</td>
<td>2.74</td>
<td>-1.61</td>
<td>-9.04</td>
<td>-12.76</td>
</tr>
<tr>
<td>Latent (W/m²)</td>
<td>10.32</td>
<td>10.64</td>
<td>6.28</td>
<td>3.06</td>
<td>0.16</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 7.4: Sensible and latent heat fluxes used by Maykut and Untersteiner [77].
Fluxes directed away from sea ice surface

Figure 7.16: Annual cycle for standard case without a snow cover and fraction of short-wave radiation that penetrates into the interior of the sea ice $i_0 \simeq 1$. Lines indicate motion of boundaries through the year, relative to wintertime ice–atmosphere interface. January 1 is given by day 0
then $i_0 \approx 1.0$ for the snow-free case. The reason for this is that $i_0$ is defined as the fraction of the incoming short-wave energy at the first internal gridpoint (equation 4.21). Since the number of gridpoints is large (641), this ratio is almost unity.

Figure 7.16 shows the annual evolution of the sea ice if this restriction on $i_0$ is lifted. Subsurface melting occurs at depth 0.187 m on day 181 (July 1), and the sea ice develops a large internal melt region during summer. The upper surface of the internal melt region melts at about 0.5 cm/day until day 201 (July 21), when the upper surface of the internal melt region reaches depth 0.088 m. After day 201, the upper surface of the internal melt region begins to freeze, since the incoming radiation is decreasing (see figure 7.3). The lower surface of the internal melt region continues to melt until day 251 (September 9), after which it begins to freeze. On day 293 (October 21) the internal melt region refreezes entirely at depth 0.77 m. For this case, the conductive flux never becomes positive as one would expect from figure 7.15. The reason for this is that the sea ice surface temperature is less than the melting temperature for the entire year, so that the conductive flux required to balance the surface energy balance (equation 4.20) is smaller and remains negative.

This internal melt region evolution does not melt through to the atmosphere–ice boundary because the two-stream radiation model does not account for the optical property variation in the upper layer of sea ice in the summer. The albedo remains high throughout the year and ranges from 0.724 to 0.734. To accurately model this phenomenon requires a more accurate description of the interaction of radiation with sea ice. In particular, the variation of optical properties with temperature of the ice above the internal melt region.

### 7.2.3 Diffusive melt pond

I now consider the results of the model for the case when turbulent convection is neglected and heat transfer within the melt pond is governed by diffusion only. Diffusive melt ponds could be realised for the case of a saline melt pond since these are typically highly stratified (Tucker III and Perovich [123]), which would inhibit
Figure 7.17: Annual cycle for standard case without a snow cover and no turbulent convection within melt pond and zero drainage rate. Lines indicate motion of boundaries through the year, relative to wintertime ice–atmosphere interface. January 1 is given by day 0

Convection. This can occur if there is infiltration of seawater through a thin melt pond, or by infiltration through the sides of a sea ice floe, such as in the formation of snow-ice (Maksym and Jeffries [69]).

Figure 7.17 shows the annual evolution of a diffusive melt pond with no snow cover and no drainage. Qualitatively, figure 7.17 looks similar to the evolution of melt ponds in the turbulent case. However, the slow transport of heat by diffusion as compared to turbulent convection means that the surplus of heat at the pond–ice interface responsible for melting is not due to the transport of heat from the melt pond into the ice, but is in fact due to heat transport from within the sea ice to the pond–ice interface.

Figure 7.18 shows the temperature of the melt pond from figure 7.17 at day 199 (July 19) using 11 sample points. At this time the pond is about 0.45 m. The conductive flux into the pond–ice boundary from the pond is 0.03 W m\(^{-2}\), however the conductive flux into the pond–ice boundary from the sea ice is 12.098 W m\(^{-2}\), so that the melt rate is dictated solely by the conductive flux from the sea ice.
Figure 7.18: Temperature profile inside melt pond with no turbulent convection at day 199 using 11 sample points. Surface of pond is at 0 m. Pond depth is 0.45 m.

The reason for this is that the pond–ice interface is fixed at the melting temperature, but the radiation absorbed near the surface of the sea ice increases its temperature so that after about 8 days it is greater than the melting temperature. Therefore, the temperature gradient is positive at the surface of the sea ice and heat is transported to the pond–ice interface from within the sea ice. Since the melt pond at day 199 is quite deep (0.45 m) the temperature at the surface of the melt pond is seen to be reacting to the surface energy balance. The pond’s low thermal conductivity as compared to the sea ice means that the ice is insulated to a large degree from the atmospheric forcing and responds solely to radiative heating within the sea ice.

The region of increased temperature beneath the pond–ice interface may lead one to think that an internal melt region should form. However, this is not the case and in reality a ‘mushy’ zone forms between the melting interface and the sea ice beneath (Crank [22]). The mushy zone in this context is a region that has attained the melting temperature, but requires the addition of sufficient heat to perform phase change. The latent heat is added by radiative heating, which persists throughout this mushy zone, but is small in the melt pond because of the difference in absorption coefficients between the pond and the sea ice.

The sea-ice–melt-pond model cannot reproduce this phenomenon, and its temper-
ature can become greater than the melting temperature within the sea ice. This results in a region of negative solid fraction beneath the surface, which results in a heat flux towards the surface which drives melting. This result is not valid. To deal with the mushy zone would require a reformulation of the numerical method using an enthalpy formulation, in which case the solution over all phases is a weak solution (see Crank [22]).

7.3 Model sensitivity

The melt-pond-sea-ice model contains several components that have a large degree of uncertainty. In this section, I investigate the sensitivity of the simulation to specific parameters. The parameters that I investigate are: the optical properties of sea ice and melt ponds ($\kappa_2$, $s_2$, and $\kappa_1$); the $i_0$ parameter of the melt pond; the wintertime snow albedo; and the bulk turbulent heat transfer coefficient ($C_{T0}$) for the sensible and latent heat flux in the presence of melt ponds.

To examine sensitivity I perform simulations with different values of the parameter under investigation while holding all other parameters constant. The dependent variable that I use to assess model sensitivity is the same as that used to examine the sensitivity of the drainage rate of the standard case, i.e. the maximum pond depth. Since the simulations were shown to be particularly sensitive to drainage rate and this is relatively uncertain, I consider the sensitivity of the maximum pond depth to drainage rate for each of the parameters under investigation.

In addition to the maximum pond depth, I also present results for the maximum surface ablation. The maximum surface ablation represents the quantity of surface melting that occurs during the summer. For the standard case, maximum ablation occurs soon after freeze-up depending on the thermal inertia of the melt pond.
Figure 7.19: Sensitivity of maximum pond depth to sea ice extinction coefficient $\kappa_2$ at standard drainage rate

7.3.1 Optical properties

Since the optical properties of sea ice are highly variable (Perovich [91]), it is useful to analyse the sensitivity of the model to optical properties. The optical properties of the melt-pond–sea-ice model are described by two coefficients ($\kappa_2$ and $s_2$) for the ice and one coefficient ($\kappa_1$) for the pond.

The extinction coefficients for the ice and pond are straightforward to interpret because they represent the amount of attenuation (absorption and scattering) of the radiation as it passes through the ice or pond. The parameter $s_2$ is at first glance more difficult to interpret. In fact $s_2$ does have a simple meaning: It is the albedo of the single sea ice layer when the Fresnel reflection at the upper interface is neglected in the limit of large ice thickness, which can be seen from equation (3.30). Since the albedo doesn’t vary significantly for ice greater than 0.8 m thickness (Perovich et al. [98]), $s_2$ can be thought of as a proxy for the ice albedo.
Figure 7.20: Sensitivity of maximum surface ablation to sea ice extinction coefficient $\kappa_2$ at standard drainage rate

Figure 7.21: Sensitivity of maximum pond depth to sea ice extinction coefficient $\kappa_2$ and drainage rate
Optical properties of sea ice

Figures 7.19 and 7.20 show the sensitivity of the maximum pond depth and maximum surface ablation to the extinction coefficient of the sea ice ($\kappa_2$) at the standard drainage rate, respectively. A lower sea ice extinction coefficient corresponds to proportionally less energy absorbed by the sea ice near the surface than the interior. This allows more sensible heat to be stored within the interior of the ice, so that less energy can contribute to surface melting, in agreement with the results of Maykut and Untersteiner [77]. The result of this is that ponds are shallower and the amount of surface ablation is less. The maximum pond depth and maximum surface ablation exhibit most sensitivity for sea ice extinction coefficients between 2.5 m$^{-1}$ and 3 m$^{-1}$.

Figure 7.21 shows the sensitivity of the maximum pond depth for varying drainage rates and varying sea ice extinction coefficients. As for the standard case ($\kappa_2 = 3$ m$^{-1}$) all the results exhibit the same behaviour as the drainage rate varies: increasing drainage rate leading to decreasing pond depths, with ponds just disappearing entirely for drainage rates beyond 2.5 cm/day. However, at the standard case the maximum pond depth is most sensitive, indicated by the large spacing between the 2.5 m$^{-1}$ (triangle) and 3.0 m$^{-1}$ (cross) extinction coefficient.

Figures 7.22 and 7.23 show the sensitivity of the maximum pond depth and maximum surface ablation to the sea ice albedo proxy $s_2$, respectively. Lower sea ice albedo $s_2$ corresponds to larger pond depths and increased surface ablation. Lower ice albedo reduces the albedo of the whole system, allowing more energy into the melt pond and sea ice, so that more melting can occur. The same qualitative behaviour as for the extinction coefficient is found as the drainage rate varies, which is shown in figure 7.24.
Figure 7.22: Sensitivity of maximum pond depth to sea ice optical parameter $s_2$

at standard drainage rate

Figure 7.23: Sensitivity of maximum surface ablation to sea ice optical parameter $s_2$ at standard drainage rate
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Optical properties of the pond

The optical properties of the melt pond are more straightforward than those of the ice because the pond does not have the same temporally evolving morphological complexity as the ice. However, the optical properties of the pond can be affected by biological and particulate inclusions (e.g. soot), which can either be in suspension or accumulate at the base of ponds. The effect of suspended inclusions is to increase the absorption coefficient of the pond.

Figures 7.25 and 7.26 show the sensitivity of the maximum pond depth and the maximum surface ablation to the extinction coefficient of the melt pond for standard drainage, respectively. The extinction coefficient ranges from typical clear water values (0.025 m\(^{-1}\)) to highly turbid suspensions (0.4 m\(^{-1}\)). As the absorption coefficient of the pond increases the pond can convert more short-wave radiation into heat, and also there is a decrease in albedo. Both of these effects result in larger pond depths and more sea ice ablation as the extinction coefficient increases.

Figure 7.27 shows the sensitivity of the maximum pond depth to the drainage rate for the range of extinction coefficients used in figure 7.25. The same qualitative behaviour is found as before, with low drainage rates corresponding to deeper ponds,

Figure 7.24: Sensitivity of maximum pond depth to sea ice optical parameter \(s_2\) and drainage rate
Figure 7.25: Sensitivity of maximum pond depth to melt pond extinction coefficient $\kappa_1$ at standard drainage rate

Figure 7.26: Sensitivity of maximum surface ablation to melt pond extinction coefficient $\kappa_1$ at standard drainage rate
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\[ A \sim (m) \quad 0.025 \quad 2.5 \quad A \quad g \quad 2 \quad 1 \quad 1.5 \quad A \quad I \quad 0.5 \]

\[ \bullet \quad 0.075 \quad 3 \quad 3.5 \quad 0.5 \]

\[ \boxed{\text{Figure 7.27: Sensitivity of maximum pond depth to melt pond extinction coefficient } \kappa_1 \text{ and drainage rate}} \]

and larger drainage rates resulting in ponds that drain immediately as they form. However, care must be taken when interpreting this figure, since the absorption coefficient does not vary by equal amounts across its range.

7.3.2 Melt pond \( i_0 \) parameter

The \( i_0 \) parameter in the case of melt ponds determines the amount of radiation that passes through the surface of the pond and contributes to internal heating. This is highly uncertain because the amount of radiation that can pass through the surface depends on the spectral composition of the incident radiation. Clouds absorb longer wavelength radiation more strongly than shorter wavelength radiation (Perovich [91]). Therefore, when it is cloudy the \( i_0 \) parameter for the melt pond is larger than with clear skies.

Figures 7.28 and 7.29 show the sensitivity of the maximum pond depth and maximum surface ablation to a wide range of values of the \( i_0 \) parameter of the melt pond, respectively. As \( i_0 \) increases, the radiative heating of the melt pond and sea ice increases, however, increasing \( i_0 \) decreases the magnitude of the turbulent convective heat flux from the atmosphere to the pond, which can be seen from the surface

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**Figure 7.28:** Sensitivity of maximum pond depth to fraction of short-wave radiation penetrating into melt pond and sea ice $i_0$ at standard drainage rate

**Figure 7.29:** Sensitivity of maximum surface ablation to fraction of short-wave radiation penetrating into melt pond and sea ice $i_0$ at standard drainage rate
Figure 7.30: Sensitivity of maximum pond depth to fraction of short-wave radiation penetrating into melt pond and sea ice $i_0$ and drainage rate energy balance (4.24). The turbulent convective heat flux into the pond is more sensitive to variations in $i_0$ than the radiative heating of the pond due to short-wave radiation, so that increasing $i_0$ leads to a decrease in the maximum pond depth and an increase in the amount of surface ablation.

Figure 7.30 shows the sensitivity of the maximum pond depth to the drainage rate for several values of $i_0$ for the pond. The same qualitative behaviour of the variation of maximum pond depth with drainage rate is observed as before. The maximum pond depth is also seen to exhibit increased sensitivity as $i_0$ varies at the standard drainage rate.

7.3.3 Snow albedo

The simple snow cover that was used has a constant winter albedo of 0.84 in the standard case. This is at the upper end of the range of observed snow albedos (Perevich [91]). The effect of varying snow albedo is to influence the onset of melt and to change the amount of energy entering the sea ice system (Maykut and Untersteiner [77]).

Figures 7.31 and 7.32 show the sensitivity of the maximum pond depth and max-
Figure 7.31: Sensitivity of maximum pond depth to snow albedo at standard drainage rate

Figure 7.32: Sensitivity of maximum surface ablation to snow albedo at standard drainage rate
Figure 7.33: Sensitivity of snow melt onset to snow albedo. January 1 is given by day 0

Figure 7.34: Sensitivity of maximum pond depth to snow albedo $\alpha_{\text{snow}}$ and drainage rate
imum surface ablation to the snow albedo, respectively. As the snow albedo decreases the amount of energy that enters the snow and sea ice before the onset of melt increases, which increases the amount of surface ablation and increases the pond depths. The maximum pond depth and maximum surface ablation are most sensitive about the standard case snow albedo (0.84). As the snow albedo decreases below 0.82 the maximum pond and maximum surface ablation becomes far less sensitive, since in these cases the surface ablation is large and the pond almost melts through the ice entirely.

The reason for the dramatic sensitivity of the pond depth to the snow albedo is because of the sensitivity of the onset of melt to the snow albedo. Figure 7.33 shows the sensitivity of the onset of snow melt to snow albedo. The relationship is nonlinear and ranges from day 162 to 180 for snow albedos from 0.76 to 0.86, respectively. The rapid variation of melt onset date between 0.82 and 0.86 demonstrates why the maximum pond depth is sensitive in this region. The surface energy balance (equation 4.34) implies that variations in snow albedo result in variations in the conductive flux and the surface temperature (and the sensible and latent heat fluxes to some extent), which result in the observed nonlinearity.

Figure 7.34 shows the sensitivity of the maximum pond depth to the drainage rate for different snow albedos. The qualitative behaviour is the same as the other sensitivity tests. The nonlinear sensitivity of the maximum pond depth to the snow albedo is clearly visible for fixed drainage rate, and is shown by the increased spacing between observed values as the maximum pond depth decreases.

### 7.3.4 Turbulent heat flux

The parameterisation for the sensible and latent heat flux in the presence of melt ponds is derived from Ebert and Curry [25]. However, the bulk transfer coefficient $C_{T_0}$ is assumed identical to that used for leads by Ebert and Curry [25]. This is because to date there have been no direct measurements of sensible and latent heat fluxes over melt ponds (E.L. Andreas, principal investigator of the SHEBA Atmo-
Figure 7.35: Sensitivity of maximum pond depth to turbulent transfer coefficient of pond $C_{T0}$ at standard drainage rate

Figure 7.36: Sensitivity of maximum surface ablation to turbulent transfer coefficient of pond $C_{T0}$ at standard drainage rate
Melt ponds usually occur in low lying areas of sea ice floes (Fetterer and Untersteiner [35]) and so local dynamic effects due to the unique topography of melt ponds may mean that the bulk transfer coefficient is different from that of leads.

Figures 7.35 and 7.36 show the sensitivity of the maximum pond depth and maximum surface ablation to the bulk turbulent transfer coefficient of the melt pond, respectively. As the transfer coefficient increases the maximum pond depth and the amount of surface ablation decreases. This is because increased transfer coefficients lead to increases in the outgoing sensible and latent heat flux during summer. Consideration of the surface energy balance indicates that the turbulent heat flux into the pond decreases, leading to smaller pond depths. The maximum pond depth is relatively sensitive to bulk transfer coefficients less than 0.00125 with an almost linear dependence. For bulk transfer coefficients greater than 0.00125, once the ponds have drained fully the outgoing sensible and latent heat flux are too large to allow ponds to reform. In this case, if the incoming short-wave radiation is large enough, internal melt regions can form temporarily.

Figure 7.37 shows the sensitivity of the maximum pond depth to the drainage rate for varying turbulent transfer coefficient. The maximum pond depth exhibits the same
behaviour for varying drainage rate as before, with lower drainage rates leading

to larger pond depths. Once again, the maximum sensitivity is observed at the

standard drainage rate.

### 7.4 Summary

#### Standard case

The melt-pond–sea-ice model can reproduce the formation and evolution of melt

ponds during the summer melt season. The standard case is representative of the

upper end of the spectrum of melt pond evolution, with surface ablation corre­

sponding to the largest observed values at the SHEBA field site. Comparison with

averaged melt pond data from SHEBA showed that the temporal evolution of pond

depth was simulated well.

The model also reproduced a reasonable albedo evolution, which decreases rapidly as

the melt pond forms, and increases rapidly as the melt pond freezes over, consistent

with temporal observations of albedo from SHEBA (Perovich et al. [96]). The

similarity in albedo is achieved through a combination of the optical model used

and the optical properties parameterisation for melt ponds described in chapter 3.

The model demonstrated the importance of melt ponds in terms of their effect on

the absorption of radiation (figure 7.3) and was used to estimate the energy budget

for the case of pond-covered sea ice in the Arctic (table 7.1). Using the standard

optical parameters the energy deficit of 60 MJ/m² observed at SHEBA (Perovich

and Elder [92]; Perovich et al. [97]) could not be attributed to radiative energy

transmitted into the ocean. However, transmission estimates could be in error if the

optical properties of the sea ice were not parameterised accurately during summer.

Intuitively, one might expect that the albedo-feedback mechanism would be expected
to cause thinner ice to have lower albedo, resulting in increased melting and deeper

melt ponds. However, it was found that decreasing the sea ice thickness in the
standard case lead to increased surface ablation and smaller pond thickness. This is explained by the increased rate of conduction of heat through the sea ice from the pond–ice interface, reducing the rate of melting, but additionally the onset of melt is earlier so that the total amount of melting can increase. This result provides for the first time a quantitative estimate for the albedo-feedback and conduction-feedback in relation to melt ponds. It is therefore, a combination of radiation and thermodynamics that determine the mass balance of the sea ice in the presence of melt ponds.

The melt-pond–sea-ice model improves on the previous models with respect to the estimated rate of cooling of the sea ice in the autumn. The melt-pond–sea-ice model releases sensible and latent heat into the sea ice as the melt ponds refreeze ensuring the ice cools more slowly as observations suggest (Maykut and Untersteiner [77]). In reality, heat would also diffuse horizontally from the melt ponds, although the anisotropy of the thermal conductivity (Weeks and Ackley [129]) means that horizontal diffusion of heat would be weaker than vertical diffusion. Therefore, heat from melt ponds would reduce the rate of cooling in the surrounding ice, and the delay in cooling would be effective for the whole ice cover. The result of this extra heat in the ice is to reduce the amount of growth during winter, leading to thinner ice, which would therefore be more susceptible to pond growth in the next melt season (Fetterer and Untersteiner [35]).

Variation of drainage rate was found to result in qualitatively different types of model behaviour. At low drainage rates (< 1.5 cm/day), the melt ponds would become deep and grow very rapidly, often growing so much that they almost melted the ice. At large drainage rates (> 2.5 cm/day), melt ponds cannot form readily and result in initial melt ponds draining away completely. Large drainage rates correspond to rapid variations in freeboard as melt water drains laterally into the melt pond. Intermediate drainage rates (~ 2 cm/day) correspond to melt ponds that evolved into reasonable size ponds, and didn’t drain too rapidly or melt too quickly compared to observations.

Ebert and Curry [25] utilised the run-off fraction in their sea ice model as a free
parameter, in a similar way in which Maykut and Untersteiner [77] used the sea ice albedo to tune their model. It is a common feature of sea ice models that there is such a large degree of uncertainty within model parameters that they can be tuned within error estimates of parameters to look like reality (Shine and Henderson-Sellers [116]). In the melt-pond-sea-ice model the inherent uncertainty and obvious variability of the drainage parameter meant that this could legitimately be used for tuning. Future research should re-evaluate the assumption of the constant drainage rate and the estimate used in the standard case.

Previous thermodynamic models have neglected the mass transfer mechanism of drainage through the sea ice cover. However, it has been pointed out (Eicken et al. [29]) that the hydrology of the ice affects the temporal evolution of sea ice albedo. This follows because the depth of the pond is negatively correlated with its albedo. The significant sensitivity to the drainage parameter in the melt-pond–sea-ice model chimes well with the requirement of incorporating hydrology into a determination of sea ice albedo.

Without a snow cover, the sea ice exhibited greater seasonal variation in temperature and the melt ponds were more sensitive to the drainage rate. The reason for the change in sensitivity is because the melt ponds that formed from snow-free sea ice initially had zero thickness, whereas the melt ponds that formed from snow covered ice were initiated by the melting of the snow.

Subsurface melting was demonstrated to be possible with the snow-free model if the restriction that the parameter $i_0$ of the sea ice is less than 0.4 was relaxed. However, the radiation model for the three-layer case is insufficient to model the evolution during the early part of the summer, since simulations show that melt ponds could not form. To incorporate subsurface melt would require investigation into the variation of optical properties of the ice near the surface prior to the onset of melt.

If turbulence in the melt pond is replaced by heat transfer by diffusion, simulations revealed reasonable evolution rates. However, analysis of the conductive flux at
Figure 7.38: Schematic diagram of interactions of sensitivity parameters and their resulting influence on the maximum pond depth. Solid line indicates increase in first quantity results in increase in second quantity. Dashed line indicates increase in first quantity results in decrease in second quantity. Fraction of short-wave radiation penetrating into melt pond and sea ice = $i_0$. Turbulent transfer coefficient for melt pond = $C_{T0}$. Extinction coefficient of sea ice = $\kappa_2$. Sea ice optical parameter = $s_2$ (proxy for sea ice albedo). Extinction coefficient of melt pond = $\kappa_1$. Albedo of snow = $\alpha_{\text{snow}}$. Albedo of melt pond = $\alpha_{\text{pond}}$

the pond-ice interface revealed that melting was caused by a region of excess heat beneath the pond-ice interface. This is artificial and results from radiative heating and the assumption that there is a sharp interface between the sea ice and the melt pond. In reality a mushy zone forms between the sea ice and the melt pond, which is at the melting temperature, but is not fully liquid or solid.

Summary and discussion of model sensitivity

The qualitative results of the sensitivity analysis of the preceding section are summarised schematically in figure 7.38. The parameters that were varied are contained within the ovals. The quantities in the dashed boxes are influenced when the pa-
rameters are varied and determine the observed direction of the variation of the maximum pond depth (solid box). The diagram is split into three distinct components: the sea ice; the pond; and the surface energy balance. The arrows between the different components represent the influence of the source component on the target component. If the arrow is a solid line, then an increase in the source component leads to an increase in the target component. If the arrow is a dashed line, then an increase in the source component leads to a decrease in the target component.

The turbulent fluxes referred to in the melt pond are those due to turbulent convection, which are represented by the four-thirds law in the melt-pond-sea-ice model. The reason that $q$ and $C$ affect this component of the model is through their effect on the surface energy balance, the result of which affects the turbulent heat flux at the surface of the pond. The energy referred to in the melt pond is the sensible heat of the melt pond, which is effectively proportional to core temperature. The 'energy near surface' in the sea ice is the sensible heat near the surface that influences the rate of melting. The conductive flux in the sea ice is the conductive heat flux at the pond–ice interface that is directed into the interface. Increased conductive flux into the interface results in increased melting at the interface.

As an example, consider the extinction coefficient of the sea ice ($\kappa_2$). If the extinction coefficient of the sea ice decreases then less energy is absorbed near the surface of the sea ice. Less energy absorbed near the surface of the sea ice results in a smaller conductive flux into the pond–ice interface. The smaller conductive flux results in a lower melt rate and hence the maximum pond depth is reduced.

The melt-pond–sea-ice model presented in this thesis is the first model to explicitly model melt ponds as an extra phase of a sea ice model and it is important to understand the model’s sensitivity. The parameters varied in the study were chosen because they are uncertain, either they are naturally highly variable (e.g. the absorption coefficient of sea ice) or their physical nature is not well understood (e.g. $i_0$ parameter of melt pond). I define a measure of model sensitivity $\delta_{\text{sens}}(x)$, where $x$ is the parameter being considered, as the non-dimensionalised partial derivative of the maximum pond depth with respect to parameter $x$. The scales used for
Table 7.5: Sensitivity parameter ($\delta_{\text{sens}}$) for variables whose sensitivity was examined. Larger magnitude of $\delta_{\text{sens}}$ indicates increased sensitivity. Sign of $\delta_{\text{sens}}$ indicates direction of sensitivity.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\delta_{\text{sens}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drainage</td>
<td>-4.052</td>
</tr>
<tr>
<td>Snow albedo</td>
<td>-28.571</td>
</tr>
<tr>
<td>$i_0$ pond</td>
<td>-8.881</td>
</tr>
<tr>
<td>Melt pond absorption</td>
<td>0.236</td>
</tr>
<tr>
<td>Sea ice absorption</td>
<td>3.387</td>
</tr>
<tr>
<td>$s_2$</td>
<td>-9.605</td>
</tr>
<tr>
<td>$C_{T0}$</td>
<td>-2.235</td>
</tr>
</tbody>
</table>

non-dimensionalisation are determined by the standard case values of the maximum pond depth and the parameter being considered. The partial derivative is determined using a central finite-difference scheme. The model sensitivity is

$$ \delta_{\text{sens}}(x) = \frac{< x >}{< H_{\text{pond}(\text{max})} >} \frac{\partial H_{\text{pond}(\text{max})}}{\partial x}, $$

(7.3)

where angle brackets $< >$ denote the value at the standard case, and $H_{\text{pond}(\text{max})}$ is the maximum pond depth. The value of $\delta_{\text{sens}}$ is therefore a measure of sensitivity in the vicinity of the standard case parameters.

Table 7.5 shows the value of the sensitivity measure $\delta_{\text{sens}}$ for the parameters investigated in section 7.3. If $\delta_{\text{sens}}(x)$ is positive then increases in parameter $x$ result in increases in the maximum pond depth, and if $\delta_{\text{sens}}(x)$ is negative then increases in parameter $x$ result in decreases in the maximum pond depth. The magnitude of the sensitivity parameter indicates the strength of the sensitivity, with larger magnitudes corresponding to increased sensitivity. The most influential parameter was the snow albedo. The reason for this is because the effect of altering the snow albedo accumulates through the year as the energy entering the system is affected. Decreased snow albedos lead to earlier onset of melt, which leads to increased melt and deeper melt ponds. The next most influential parameter was the sea ice albedo
proxy $s_2$ and then the $i_0$ parameter of the melt pond. The least influential parameter was the melt pond absorption coefficient. This is because the total amount of radiation absorbed by the pond is relatively small in comparison to the turbulent heat fluxes at the boundaries (equation 4.14). However, it should be noted that the sensitivity parameter is only a local estimate of sensitivity. Since the model is nonlinear, the sensitivity parameter may vary across parameter space.

As expected the interaction of radiation with sea ice, snow, and melt ponds is fundamental to controlling summertime melt, and it is the interaction of radiation with sea ice that is seen to be most important in the presence of melt ponds.
Chapter 8

Conclusions, implications and future work

In this thesis I have developed a new physically-based, thermodynamic model of sea ice that is covered by melt ponds during summer. This model is consistent with previous theoretical models and observational results. The thermodynamic components of the model are based upon the equation describing heat transport in a mushy layer for the sea ice and a turbulent convective parameterisation for the melt pond. Previous thermodynamic sea ice models do not accurately model summertime surface melt, since it is either assumed that melt ponds do not form or they are parameterised only simplistically. The melt-pond–sea-ice model developed here, more accurately models the melting at the surface of sea ice beneath melt ponds than previous models, as shown by comparison with observations.

Turbulent convection of a melt pond was shown to occur for a critical Rayleigh number of about 630 (section 6.2), which was relatively insensitive to the imposed atmospheric forcing. Model simulations show that typical Rayleigh numbers are much greater than this critical value and ponds should therefore be turbulently convecting (section 4.3).

The radiative component of the model is based upon the two-stream optical model as applied to sea ice, and includes a new parameterisation of the optical properties to
simulate the changes that occur in the sea ice in the presence of melt ponds (section 3.7). The construction of the model improved previous representations by halving the number of unknown coefficients (section 3.2). This magnitude of improvement is highly significant in a thermodynamic sea ice model, because at each timestep the radiation field may have to be calculated several times at each gridpoint within the sea ice.

Previous thermodynamic models of sea ice have prescribed albedo as an external variable, e.g. Bitz and Lipscomb [9]; Cheng et al. [20]; Ebert and Curry [25]; Maykut and Untersteiner [77]. The use of the two-stream model has, for the first time in a thermodynamic sea ice model, allowed albedo to be determined by the model. However, further work is necessary to allow the optical properties to vary through the year so that the annual variation of optical properties is simulated accurately. The marginal increase in complexity of the spectrally averaged two-stream model as compared to Beer’s law means that it is suitable for use in GCMs, which I outline later.

The equation describing heat transport in a mushy layer used in this thesis was demonstrated to be analogous to the equation describing heat transport in sea ice in other sea ice models, e.g. Ebert and Curry [25]; Maykut and Untersteiner [77]. However, the mushy layer formulation gives the distribution of solid fraction with depth in the sea ice from an assumption that the bulk salinity of the sea ice is constant.

The solid fraction has the potential in future research to quantify morphological changes and hence optical properties of the sea ice. The local solid fraction could be used to reproduce the annual variation of optical properties that occur in the melt-pond–sea-ice model. However, further work would be required to determine the influence of the solid fraction on the optical properties throughout the depth of the ice.

The melt-pond–sea-ice model was seen to be sensitive to the drainage rate, and exhibited qualitatively different types of behaviour as the drainage rate varied. Simple
drainage models were formulated (section 6.3) and using a simple matched asymptotic expansion, the drainage rate was shown to be completely dominated by the viscous stresses in the sea ice and the meltwater head as opposed to the inertia of the pond. For permeabilities appropriate to sea ice, the drainage rate was seen to be fast, although the drainage timescale was still longer than the timescale of local interstitial solute transport. The result is that even with relatively large drainage rates the assumption of local thermodynamic equilibrium is still valid. Local thermodynamic equilibrium is assumed in the melt-pond–sea-ice model and is implicitly assumed in other thermodynamic sea ice models through the empirical salinity and temperature dependent physical properties.

The computer code that is used to solve the equations describing the model is flexible enough to be applied to a variety of situations. It is straightforward to implement new routines for forcing data and the radiation model, and can be initialised with arbitrary initial conditions (see appendix B).

The flexibility of the model was demonstrated by its ability to incorporate the phenomenon of subsurface melting (section 7.2.2). Further research into the variation of optical properties of the surface of the ice during summer is necessary to correctly simulate this phenomenon. If a correlation of solid fraction with optical properties is developed, it should be possible to model the sea ice surface optical properties directly and simulate melt pond formation on bare ice that initiates from subsurface melting. Note that such a correlation will be highly variable due to variations in air bubbles, pollutants, etc. Alternatively, using a different set of optical properties in the surface layer of sea ice as compared to the lower layer of sea ice should lead to the initiation of melt ponds from subsurface melting.

As with all thermodynamic sea ice models there are limitations (see section 4.9), and the results of the model must be interpreted with respect to these limitations. However, the ability of the melt-pond–sea-ice model to reproduce a reasonable simulation of melt ponds lends confidence to future explicit inclusion of melt ponds in thermodynamic sea ice modelling.
The model was shown to be most sensitive to the radiative model parameters, especially the snow albedo, the sea ice albedo proxy $s_2$, and the $i_0$ parameter of the melt pond (section 7.3). This was to be expected since the formation of melt ponds is driven by the increase in summertime short-wave radiation. Recently, it has been shown that current simulations of summertime melting of Arctic sea ice contain errors, due to either radiative forcing or the parameterisation of absorption of short-wave radiation (Laxon et al. [59]). The strong sensitivity of the melt-pond–sea-ice model to the radiative parameters concurs with this result, since minor inaccuracies in radiative properties will lead to large differences in the solution.

Turbulence in some melt ponds may be suppressed by a strong, stable salinity stratification. Simulations revealed different methods of melting at the melt pond–sea ice interface depending on the convective state of the melt pond. The rapid melt that occurs during turbulent convection means that there is a sharp interface between the pond and the sea ice, which melts because of the imbalance in heat flux at the interface, even though the sea ice is being heated relatively strongly by short-wave radiation. In the slow diffusive melt regime there is not a sharp interface, and melting occurs mainly because of heating due to the absorption of short-wave radiation in the sea ice. Melting proceeds as a two-stage transition from sea ice into an unstructured mixture of solid and liquid and then into pure liquid. The mixture region is a mushy zone that has insufficient enthalpy to be fully molten. Further research into this form of melting is necessary to compare the convective and diffusive rates of melting. Since melting in the diffusive case is due to radiative heating of the sea ice, rather than heat transfer from the surface, the diffusive case should be more sensitive to the optical properties of the sea ice and the albedo of the melt pond. However, the number of diffusive ponds should be less than the number of convective ponds since the salinity of melt ponds are generally low (Eicken et al. [27]).

The surface energy balance is a critical component of the sea ice system (Hanesiak et al. [48]). For example, the $i_0$ parameter of the melt pond, which affects the amount of incoming short-wave radiation, also influenced the turbulent heat transfer at the upper boundary of the melt pond, with the somewhat counter-intuitive result that
increased $i_0$ leads to decreased melt pond depths and surface ablation.

The stationary behaviour of the sea ice component of the melt-pond–sea-ice model was examined in detail in chapter 5. There are two stationary solutions for given forcing data. This was due to the two-stream radiation model. A linear stability analysis of a simplified model revealed the smaller of the two stationary solutions to be unstable. This result was verified numerically for the full model (and also allowed the convergence of the numerical scheme to be tested).

Previous investigations of stationary behaviour of sea ice have utilised simpler radiation schemes (i.e. Beer’s law) and do not contain multiple stationary solutions. The stability of the stationary solutions was controlled by the dependency of albedo on sea ice thickness. The sea ice albedo decreases rapidly as the thickness decreases below about 0.8 m (Perovich et al. [98]), so that perturbations of small ice thickness stationary solutions yield greater perturbations in the surface energy balance than perturbations to larger ice thickness stationary solutions. The result is that the smaller stationary ice thickness is unstable.

The implication for sea ice is that sea ice greater in thickness than the smaller stationary solution will evolve towards the larger equilibrium ice thickness, thinning or thickening as necessary. Sea ice that is thinner than the smaller stationary solution will completely melt.

The albedo-feedback mechanism in the presence of melt ponds due to sea ice thickness variation is only important for thinner ice (less than 0.8 m), with the conduction-feedback being more important for thicker ice. The effect of the conduction-feedback described by Ebert and Curry [25], which causes the onset of melt to be earlier, was modified in this thesis to include the conduction-feedback during the summer in the presence of melt ponds, which reduces the melt rate beneath melt ponds (section 7.2). The net effect of this combined conduction-feedback was that decreasing ice thickness led to ponds that were shallower, although the total amount of surface ablation was greater. The melt-pond–sea-ice model demonstrates that several feedbacks operate simultaneously with some being more important than others for
certain ranges of parameters.

In section 6.4, it was shown that there are qualitative differences in the equilibrium solution of an evolving sea ice lens depending on the description of the thermal conductivity and latent heat. This has implications for zero-layer sea ice models when there is melting at both interfaces. Many GCMs use a zero-layer formulation of sea ice, so that a linear temperature gradient and constant thermal conductivity is assumed inside the ice, using the formulation proposed by Semtner [111]. However, some zero-layer models use a constant latent heat of fusion as well as a constant thermal conductivity (e.g. Lohmann and Gerdes [64]). During summer, when the ice is evolving at both upper and lower interfaces, fundamentally different solutions will be observed depending on the description of the thermal conductivity and latent heat of fusion. This result has important implications for the behaviour of solutions of any Stefan problem with two moving boundaries for which the temperature gradient is assumed linear.

The melt-pond–sea-ice model can be used to quantify predictions of long term behaviour of melt ponds subject to specific averaged forcing. It can also be used to determine fundamental physical parameters useful to the global energy budget, such as the quantity of radiation reflected, absorbed and transmitted by the ponds, and also how the melt ponds influence the storage of heat within the sea ice. The model could be reconfigured to test alternative radiation schemes and parameterisations of albedo in the presence of melt ponds, which are of fundamental interest to climate modellers (Morassutti and LeDrew [82]).

Previous sea ice models either fail to incorporate melt ponds (Maykut and Untersteiner [77]) or fail to explicitly model melt ponds as an extra phase. Furthermore, previous models of melt ponds (e.g. Ebert and Curry [25]) just use a simple parameterisation, which does not fully account for the influence of different feedbacks on the melt-pond–sea-ice system. Fetterer and Untersteiner [35] pointed out that the reason previous sea ice models have very thin equilibrium ice thickness, when observed summertime average albedos of about 0.55 are used, is because these average summertime albedos are an average of three different ice types (bare ice and
snow, melt ponds, and open ocean). The melt-pond–sea-ice model is the missing melt pond component of these sea ice models during summertime.

**Alternative parameterisations of sea ice drainage**

The drainage rate of melt ponds is highly variable (Eicken et al. [29]) and depends on factors such as local permeability, meltwater head, floe geometry, saturation, and floe melt rate. The constant drainage rate (2 cm/day) used in the melt-pond–sea-ice model can be considered to be a first approximation. It would be useful to examine the qualitative effect of different drainage parameterisations on the melt-pond–sea-ice model simulation.

There are SHEBA data available for surface melt rates of sea ice without melt ponds. These data could be used to develop a more sophisticated parameterisation of drainage using the simple assumption that the drainage rate is equivalent to the surface melt rate. Surface melt data, from the SHEBA field experiment (Perovich et al. [95]), for individual gauges in sea ice without melt ponds exhibit approximately sinusoidal behaviour through the melt season. Therefore, a second approximation to pond drainage of at-sea level ponds could be a sinusoidal variation of the drainage rate with time, whose total drainage matches observations. However, this kind of approximation is still fairly primitive and makes the assumption that the pond surface is at sea level. A sophisticated model would require information on the whole sea ice floe, and would require an estimate of freeboard and permeability. However, these are variable through the summer (Eicken et al. [29]) and would be challenging to estimate from existing large-scale sea ice models.

**Sea ice thickness based optical parameterisations**

The two-stream radiation model that was utilised in this thesis has the potential to be applied directly to current GCMs (in the absence of melt ponds) to develop simple parameterisations of the albedo and transmission through sea ice that depend on
ice thickness, snow thickness and optical properties of the sea ice. Annual variation of optical properties could be related to sea ice temperatures.

A two-layer, two-stream radiation model can be used to describe snow cover on top of sea ice. The albedo can be directly obtained from equation (3.23) and the transmission is simple to calculate as it is the ratio of the down-welling irradiance at the base of the sea ice to the incoming short-wave radiation.

The radiation model will need to be compared to spectral albedo and transmission data to determine a reasonable approximation for the whole annual cycle, which were measured at SHEBA. Specifically, the choice of optical parameters will need to be optimised using the data for different times of year and correlated with the surface temperature.

**Application of melt-pond–sea-ice model to GCMs**

The success of the melt-pond–sea-ice model in reproducing seasonal evolutionary behaviour lends confidence to its extension to GCMs. Most GCMs utilise very simplified thermodynamics of sea ice, which can be traced back to the simplification of Maykut and Untersteiners’ [77] model by Semtner [111]. For example, in the current UK Met Office GCM (HadCM3) the thermodynamic component consists of a zero-layer sea ice model (Semtner [111]), with variations of sea ice albedo relating linearly to surface temperature. Another common model is the three-layer model (Semtner [111]), which assumes that there are a fixed number of gridpoints within the ice, with a linear temperature profile between each internal gridpoint. The temperature of the ice and snow is then determined using simplifications of the surface boundary condition and neglects radiation inside the ice.

Given the significance of melt ponds to the energy and mass budget of sea ice and the lack of any melt pond parameterisations within GCMs, I propose that the melt pond model developed in this thesis be incorporated into current GCMs.

Modern GCMs, e.g. HADCM3 and the Los Alamos CICE model, consider an ice
thickness distribution that describes probabilistically the variation of ice thickness over a given area (Rothrock [105]). The probability distribution of ice thicknesses is split into discrete bins so that there are only a finite number of thickness categories. Each ice thickness category evolves through thermodynamic and dynamic forcing. To incorporate a melt pond scheme into an ice thickness distribution model would require a melt pond parameterisation for each ice thickness category. Linearisation of the four-thirds law for turbulent convection in the melt pond would allow analytical expressions to be found for the mean pond temperature and pond surface temperature. Therefore, the implementation of a melt pond scheme would at least double the memory requirements of the sea ice model. Without loss of generality, I will describe the melt pond parameterisation for only one of the ice thickness categories.

Neglecting other surface phenomena such as leads I define a pond fraction \( P \) to be the time-dependent fractional surface area covered by melt ponds. Then the area-averaged total albedo is given by

\[
\alpha_{\text{tot}} = (1 - P) \alpha_{\text{ice}} + P \alpha_{\text{pond}}.
\]  

(8.1)

Since the horizontal development of melt ponds is uncertain, the rate of change of the pond area fraction \( \dot{P} \) must be specified using previous observations (e.g. Ebert and Curry [25]; El Naggar et al. [31]). Differences exist in the area evolution of melt ponds on first-year and multi-year ice (El Naggar et al. [31]; Fetterer and Untersteiner [35]).

The melt-pond–sea-ice system is represented by large-scale averages of two ice-types: bare ice; and melt pond covered ice. Before melting, the sea ice model will only consider one ice thickness so that the ice covered in melt ponds will be redundant. When summertime melting begins, the model considers the two cases as separate systems (sea ice without a melt pond and sea ice with a melt pond).

The bare ice is simply a standard sea ice model, which does not allow a melt pond to form at its surface. The bare ice has a specified floe run-off fraction \( F_{\text{RO}} \) (the fraction of surface melt water that drains completely from the ice floe, or into another ice
thickness category) and a specified melt pond run-off fraction $F_{mRO}$ (the fraction of surface melt water that drains into the melt pond of this thickness category). Due to their uncertainty these parameters need to be tuned. Previous work of Ebert and Curry [25] specified $F_{mRO}$ to be 0.85 following sensitivity tests.

The melt pond component consists of a sea ice model including radiation, which melts at its surface because of the presence of melt ponds. The melt pond is assumed to be turbulently convecting, since the presence of a melted snow cover should ensure the time dependent Rayleigh number is greater than the critical Rayleigh number. The only additional parameters to the temperatures within the sea ice in the melt pond component are the average core temperature of the melt pond and the surface temperature of the melt pond.

The drainage of the melt ponds is specified as an external parameter. This could be estimated from the model if it is assumed that the pond surface remains at sea level. If the pond becomes fully drained, because the bottom of the pond is above sea level, then evolution of the melt pond component is prescribed in the same way as the sea ice only component. If a fully drained pond melts beneath sea level then a pond can form there. This model does not take into consideration ponds that are above sea level. These could be incorporated once the hydrology of sea ice is better understood.

Autumn freeze-up will occur at different times for the two ice types. When the pond begins to freeze the heat stored within the pond must be released to the rest of the ice. It is straightforward to have an additional surface layer of ice and then allow the melt pond component of the model to freeze consistently.

The two components (bare sea ice and melt pond covered sea ice) run independently until the point at which the melt pond has completely refrozen. At this point the two ice types must be averaged to form a single ice type that will be used throughout wintertime in a way that conserves mass and energy.

One of the main drawbacks of the proposed model is that it neglects the effects of lateral melting of melt ponds. Also, there is the uncertainty in the fraction of
radiation that is transmitted through the surface of the melt pond in the model, $i_0$. It is necessary to include the effects of radiation within the sea ice, since it has been shown in this thesis that radiation is an important component that determines the evolution of melt ponds. The proposed model incorporates the influence of melt ponds on the albedo and redistribution of heat and meltwater through the ice. Area averaged albedo estimates of the model could be validated and tuned using data that are already available from satellite observations (e.g. AVHRR, SHEBA Satellite Remote Sensing Group [115]).
Appendix A

Transformation of governing equations and boundary conditions to fixed domain

The governing equations and boundary conditions described in chapter 4 are transformed to fixed domains so that they can be solved using the NAG routine D03PCF. In this appendix I describe how these equations are non-dimensionalised and then how they are transformed into a body-fitted co-ordinate system.

A.1 Non-dimensionalisation

The temperature $T$ is non-dimensionalised using

$$\theta = \frac{T - T_L(C_s)}{T_L(C_s) - T_L(C_{ocean})} = \frac{T - T_L(C_s)}{\Delta T},$$

where $\theta$ is the non-dimensional temperature, $T_L(C_s) = 273$ K is the equilibrium freezing temperature of the solid component of sea ice (pure ice), $T_L(C_{ocean}) = 271.2$ K is the equilibrium freezing temperature of the ocean, and $\Delta T = T_L(C_s) - T_L(C_{ocean}) = 1.8$ K is the temperature difference between the equilibrium freezing temperature of the solid component of sea ice and the ocean.
Lengths are non-dimensionalised using a typical ice thickness \( L = 1 \) m, and times are non-dimensionalised using the thermal diffusive timescale \( t^* = L^2 (\rho c)_l / k_l \approx 8.4 \times 10^6 \) s. Fluxes of energy in the boundary conditions are non-dimensionalised with a typical short-wave flux \( F = 200 \) W/m\(^2\).

### A.1.1 Governing equations

The mixture relations for \((\rho c)_m\) and \(k_m\) (equations 4.2 and 4.3) and the definition of the solid fraction \(\phi\) under the assumption that the bulk salinity is constant are given by

\[
(\rho c)_m = \left(\frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}\right) \mathcal{R} + 1 \quad (\rho c)_l, \tag{A.1}
\]

\[
k_m = \left(\frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}\right) \mathcal{K} + 1 \quad k_l, \tag{A.2}
\]

\[
\phi = \frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}, \tag{A.3}
\]

where \(\theta_{\text{bulk}}\) is the non-dimensional equilibrium freezing temperature at the bulk salinity, \(\theta_s\) is the non-dimensional equilibrium freezing temperature of the solid component of sea ice, \(\mathcal{R} = (\rho c)_s / (\rho c)_l - 1\) is a non-dimensional parameter, and \(\mathcal{K} = k_s / k_l - 1\) is a non-dimensional parameter.

The non-dimensionalised version of the equation describing conservation of heat within the sea ice (equation 4.6) is then given by

\[
\left\{ \left(\frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}\right) \mathcal{R} + 1 \right\} \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left\{ \left[ \left(\frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}\right) \mathcal{K} + 1 \right] \frac{\partial \theta}{\partial z} \right\} - \left\{ \left(\frac{\theta_{\text{bulk}} - \theta}{\theta_s - \theta}\right) \mathcal{R} + 1 \right\} U \frac{\partial \theta}{\partial z} - S \frac{\partial \theta}{\partial t} \frac{\theta_s - \theta_{\text{bulk}}}{(\theta_s - \theta)^2} - \mathcal{F} \frac{\partial}{\partial z} F_{\text{net}},
\]

where \(S = \rho_s L/((\rho c)_l \Delta T)\) is the Stefan number, \(\mathcal{F} = FL/(k_l \Delta T)\) is a non-dimensional parameter, and \(z, U, t\) and \(F_{\text{net}}\) are non-dimensional.

The non-dimensionalised version of the equation describing conservation of heat within the melt pond or internal melt region when there is no turbulent convection...
(equation 4.11) is given by
\[\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - u \frac{\partial \theta}{\partial z} - F \frac{\partial}{\partial z} F_{\text{net}}, \quad (A.5)\]
where \(u\) is non-dimensional.

The non-dimensionalised version of the equation describing conservation of heat within the melt pond or internal melt region when there is turbulent convection (equation 4.14) is given by
\[(h_l - h_u) \frac{\partial \theta}{\partial t} = F \left\{ \pm |\theta_u - \theta|^{4/3} - |\theta - \theta_{\text{pond}}|^{4/3} \right\} + F \left\{ F_{\text{net}}(h_l) - F_{\text{net}}(h_u) \right\}, \quad (A.6)\]
where \(F = F/((\rho c)_t J (\Delta T)^{4/3})\) is a non-dimensional parameter, \(J\) is the turbulent heat flux factor defined by equation (4.13), \(\theta_u\) is the temperature of the uppermost boundary of the melt region and the heat transport can be directed either into or out of this boundary so that the sign can be positive or negative, \(\theta_{\text{pond}}\) is the non-dimensional equilibrium freezing temperature of the fluid in the melt region, and \(h_u\) and \(h_l\) are the non-dimensional freezing temperatures of the upper and lower boundaries of the melt region, respectively.

The non-dimensionalised version of the equation describing conservation of heat within the snow (equation 4.15) is given by
\[
\frac{(\rho c)_{\text{snow}}}{(\rho c)_t} \frac{\partial \theta}{\partial t} = \frac{k_{\text{snow}}}{k_l} \frac{\partial^2 \theta}{\partial z^2}. \quad (A.7)
\]

### A.1.2 Boundary conditions

The non-dimensional surface boundary condition, located at \(z = h_{\text{surface}}\) when there is no snow and \(z = h_{\text{snow}}\) when there is snow, in all cases (e.g. equation 4.20), except for a turbulently convecting pond, is of the form
\[
k_z \frac{\partial \theta}{\partial z} + F (F_{\text{LW}} + (1 - i_0)(1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot}) - F_{\text{sens}} - F_{\text{lat}}) - \mathcal{E} \left( \theta + \frac{T_L(C_s)}{\Delta T} \right)^4 = 0, \quad (A.8)
\]
where \(k_z\) is the thermal conductivity of the sea ice, snow, or melt pond, and \(\mathcal{E} = \epsilon \sigma (\Delta T)^3 L/k_l\) is a non-dimensional parameter. If the surface is snow and is melting
then the net energy at the surface, given by the left hand side of equation (A.8), is identical to \((\rho_{\text{snow}} c_{\text{snow}} / \rho_{s} c_{s}) S \Delta h_{\text{snow}} / dt\).

In the case of a turbulently convecting pond the surface boundary condition becomes

\[
\mp |\theta_{u} - \theta|^{4/3} + G \{ F_{LW} + (1 - i) (1 - \alpha_{\text{tot}}) F_{SW} (\text{tot}) - F_{\text{sens}} - F_{\text{lat}} \} = 0,
\]

where \(I = \varepsilon \sigma (\Delta T)^{8/3} / ((\rho c)_{j} J)\) is a non-dimensional parameter.

The Stefan condition at the melt-region–sea-ice interface \((z = h_{\text{pond}})\) is given by

\[
S \phi \frac{d h_{\text{pond}}}{dt} = \frac{\partial \theta}{\partial z} - \frac{k_{m}}{k_{1}} \frac{\partial \theta}{\partial z}, \tag{A.10}
\]

when there is no turbulent convection in the melt region and

\[
S \phi \frac{d h_{\text{pond}}}{dt} = \frac{F}{\varepsilon} |\theta - \theta_{p}|^{4/3} + \frac{k_{m}}{k_{1}} \frac{\partial \theta}{\partial z}, \tag{A.11}
\]

when there is turbulent convection in the melt region. The Stefan condition at the melt-region–upper-sea-ice-layer interface is similar to equations (A.10) and (A.11). At the melt-region–sea-ice interface, it is also assumed that the temperature is at the equilibrium freezing temperature of the melt region so that

\[
\theta = \theta_{\text{pond}}. \tag{A.12}
\]

When there is refreezing at the surface the Stefan condition at the upper boundary of the melt region takes similar form to equations (A.10) and (A.11) and the temperature at this boundary is identical to equation (A.12).

The Stefan condition at the sea-ice–ocean interface \((z = h_{\text{ice}})\) is given by

\[
S \phi \frac{d h_{\text{ice}}}{dt} = \frac{k_{m}}{k_{1}} \frac{\partial \theta}{\partial z} - FF_{\text{ocean}}, \tag{A.13}
\]

and the temperature at this interface is assumed to be at the equilibrium freezing temperature of the ocean, so that

\[
\theta = \theta_{\text{ocean}}. \tag{A.14}
\]
A.2 Transformation to a scaled domain

To fix the boundaries, the co-ordinate transformation

\[ \xi = \frac{z - a(t)}{b(t) - a(t)} \quad \eta = t, \]

is applied to each layer (sea ice, pond, snow), with \( b(t) \) identical to the position of the uppermost boundary of each layer and \( a(t) \) identical to the position of the lowermost boundary of each layer. The spatial co-ordinate of each layer is stretched by different amounts so that each region (e.g. sea ice, melt pond, snow) is always mapped to [0,1].

A.2.1 Governing equations

Equation (A.4) transforms to

\[ (a_2 \theta^2 + a_1 \theta + a_0) \frac{\partial \theta}{\partial \eta} = \frac{\dot{h}_l}{h_l - h_u}(a_2 \theta^2 + a_1 \theta + a_0) \frac{\partial \theta}{\partial \xi} \]

\[ + \frac{\dot{h}_l - \dot{h}_u}{h_l - h_u}(a_2 \theta^2 + a_1 \theta + a_0) \xi \frac{\partial \theta}{\partial \xi} \]

\[ + \frac{1}{(h_l - h_u)^2}(b_2 \theta^2 + b_1 \theta + b_0) \frac{\partial^2 \theta}{\partial \xi^2} \]

\[ + \frac{K}{(h_l - h_u)^2} c_0 \left( \frac{\partial \theta}{\partial \xi} \right)^2 \]

\[ - (a_2 \theta^2 + a_1 \theta + a_0 + S c_0) \frac{U}{(h_l - h_u)} \frac{\partial \theta}{\partial \xi} \]

\[ - \frac{F}{h_l - h_u}(d_2 \theta^2 + d_1 \theta + d_0) \frac{\partial}{\partial \xi} F_{\text{net}}(\xi, \eta), \]

where \( h_l \) is the position of the lowermost boundary and \( h_u \) is the position of the uppermost boundary, \( \cdot \) denotes differentiation with respect to \( \eta \), and \( \theta_b = \theta_{\text{bulk}} \). Equation (A.15) is valid for both sea ice regions (i.e. the sea ice adjacent to the ocean and the sea ice at the surface of the refreezing melt pond), with different \( \xi \).
appropriate for each region. The coefficients in equation (A.15) are given by

\[
\begin{align*}
    a_2 &= \mathcal{R} + 1, \\
    a_1 &= -2\theta_s - \mathcal{R}(\theta_b + \theta_s), \\
    a_0 &= S(\theta_s - \theta_b) + \theta_s^2 + \mathcal{R}\theta_b, \\
    b_2 &= \mathcal{K} + 1, \\
    b_1 &= -\mathcal{K}(\theta_s + \theta_b) - 2\theta_s, \\
    b_0 &= \theta_s^2 + \mathcal{K}\theta_b, \\
    c_0 &= \theta_b - \theta_s, \\
    d_2 &= 1, \\
    d_1 &= -2\theta_s, \\
    d_0 &= \theta_s^2.
\end{align*}
\]

Equation (A.5) transforms to

\[
\frac{\partial \theta}{\partial \eta} = \frac{1}{h_l - h_u} \left( h_u + \xi(h_l - h_u) \right) \frac{\partial \theta}{\partial \xi} + \frac{1}{(h_l - h_u)^2} \frac{\partial^2 \theta}{\partial \xi^2} + \frac{1}{(h_l - h_u)^2} \frac{\partial \theta}{\partial \xi} \frac{\partial \theta}{\partial \xi} - \frac{\mathcal{F}}{h_l - h_u} \frac{\partial \theta}{\partial \xi} F_{\text{net}}(\xi, \eta),
\]

where \( h_l \) is the lowermost boundary of the melt region, and \( h_u \) is the uppermost boundary of the melt region. In the turbulently convecting case equation (A.6) transforms to

\[
(h_l - h_u) \frac{\partial \theta}{\partial \eta} = \frac{\mathcal{F}}{\mathcal{G}} \left\{ \pm |\theta_u - \theta|^{4/3} - |\theta - \theta_{\text{pond}}|^{4/3} \right\} + \mathcal{F} \left\{ F_{\text{net}}(0) - F_{\text{net}}(1) \right\},
\]

where \( F_{\text{net}} = F_{\text{net}}(\xi, \eta) \).

Equation (A.7) transforms to

\[
\frac{(pc)_{\text{snow}}}{(pc)_l} \frac{\partial \theta}{\partial \eta} = \frac{1}{(h_{\text{surface}} - h_{\text{snow}})^2} \frac{k_{\text{snow}}}{k_l} \frac{\partial^2 \theta}{\partial \xi^2}.
\]
A.2.2 Boundary conditions

The surface boundary condition in all cases except for a convecting melt pond (equation A.8) transforms to

$$
\frac{1}{h_l - h_u} \frac{\partial \theta}{\partial \xi} + \mathcal{F}(F_{LW} + (1 - \alpha(\text{tot}))F_{SW}(\text{tot}) - F_{\text{sens}} - F_{\text{lat}}) \tag{A.19}
$$

$$
-\varepsilon \left( \theta(\xi, \eta) + \frac{T_L(C_s)}{\Delta T} \right)^4 = 0.
$$

In the case of a turbulently convecting pond the surface boundary condition equation (A.9) remains unchanged.

The Stefan condition at the melt-region–sea-ice interface (equation A.10) transforms to

$$
S(\theta_{\text{bulk}} - \theta) \frac{dh_{\text{pond}}}{d\eta} = \frac{1}{h_l - h_u} (\theta_s - \theta) \frac{\partial \theta}{\partial \xi_1} - \frac{1}{h_l - h_i} ((\theta_{\text{bulk}} - \theta)K + (\theta_s - \theta)) \frac{\partial \theta}{\partial \xi_2}, \tag{A.20}
$$

where the definition of the solid fraction has been used (equation A.3) and the equation has been multiplied through by \((\theta_s - \theta)\). The position of the uppermost surface of the melt region is given by \(h_u\) and the position of the lowermost surface of the melt region is given by \(h_i\), the position of the ice–ocean interface is given by \(h_i\). The two different values of \(\xi\) arise from obtaining derivatives in regions to which different transformations are applied. In the turbulently convecting case (equation A.11) we have that

$$
S(\theta_{\text{bulk}} - \theta) \frac{dh_{\text{pond}}}{d\eta} = \frac{\mathcal{F}(\theta_s - \theta) |\theta - \theta_l|^{4/3}}{G} - \frac{1}{h_l - h_i} ((\theta_{\text{bulk}} - \theta)K + (\theta_s - \theta)) \frac{\partial \theta}{\partial \xi_2}. \tag{A.21}
$$

The temperature at the melt-region–sea-ice interface is at the equilibrium freezing temperature of the melt, so that equation (A.12) remains unchanged with \(\theta = \theta_{\text{pond}}\).

The sea-ice–ocean Stefan condition (equation A.13) is similar to the Stefan condition at the melt-region–sea-ice interface and transforms to

$$
S(\theta_{\text{bulk}} - \theta) \frac{dh_{\text{ice}}}{d\eta} = \frac{1}{h_l - h_i} \{K(\theta_{\text{bulk}} - \theta) + \theta_s - \theta\} \frac{\partial \theta}{\partial \xi_2} - \mathcal{F}(\theta_s - \theta)F_{\text{ocean}}, \tag{A.22}
$$

with the temperature equal to the equilibrium freezing temperature of the ocean, so that equation (A.14) is unchanged with \(\theta = \theta_{\text{ocean}}\).
Appendix B

Melt-pond–sea-ice model code and algorithm

The melt-pond–sea-ice model code was developed using C++ under the SunOS 5.8 operating system using the C++ compiler from the Sun One Studio Collection. The Fortran 77 NAG library (mark 18) was used. Using the NAG library meant that more global variables were used than is ordinarily the case for C++ programs.

The model code comprises the code for the driver program and 11 additional header files and source code files for functions used in the program. The driver program is named meltpondSTD.cpp and utilised the C++ switch command to create a simple menu system. The functions that create the menus are stored in menus.cpp, with corresponding header file menus.h. The class DataFileClass is used to load and store data files into the model, so that the model can continue from a specified start time with specific initial data or store data from a certain time to continue at a later date. This is saved in the file DataFileClass.cpp, and has corresponding header file DataFileClass.h. The radiation model is also formulated as a C++ class and is called twostreamclassbodyfitted3layer, with corresponding filenames twostreamclassbodyfitted3layer.cpp and twostreamclassbodyfitted3layer.h. The functions that determine the forcing data, when there is variation in the annual cycle, are stored in forcing.cpp with header file forcing.h. When there is a convecting melt...
#!/bin/sh
# Loop over the command-line arguments
# Execute with
# sh compile.sh filename
# note: filename does not have extension (i.e not .cpp etc).

CC -compat -c twostreamclassbodyfitted3layer.cpp
CC -compat -c menus.cpp
CC -compat -c rootfind.cpp
CC -compat -c DataFileClass.cpp
CC -compat -c surfacecondition.cpp

for filename in "$@

  do
    echo "Compiling file $filename..."
    CC -compat -c 'echo $filename'.cpp
    echo "Linking to Fortran NAG library..."
    CC -compat -xlang=f77 -o 'echo $filename'.out 'echo $filename'.o
        menus.o rootfind.o twostreamclassbodyfitted3layer.o
        DataFileClass.o surfacecondition.o -L/cpnet/local/flso619d
        DataFileClass.o -L/cpnet/forte3/SUNWspolib -lnag -lsummath

  done

if [ "$#" = 0 ]
then
  echo "Use this file to compile a single .cpp with menu file and link
        it to the Fortran NAG library."
fi

---

_Figure B.1:_ Shell script used to compile melt-pond–sea-ice model program

pond, the surface energy balance is solved using functions in surfacecondition.cpp
with corresponding header file surfacecondition.h. Finally, the header file NAGheader.h
contains the prototypes for the NAG library routines that are used in the program.
The code is compiled using the UNIX shell script compile.sh (see figure B.1).

Once the code has been compiled with the command line "sh compile.sh melt-
pondSTD" the resulting executable file is called meltpondSTD.out. Running the
executable meltpondSTD.out leads to a simple user interface that allows the user
to modify options and settings. The main menu is shown in figure B.2.
APPENDIX B

hsurface   hpond   hice
0.000   0.000   2.000

Current Accuracy = 0.00000000010
Current TimeSteps = 1000000.00
View Time Interval = 86400.000
Radiation is on. Convection is on.
Pond formation is on, with drainage rate 2.000000 cm per day.
Snow formation is on.
Melting snow becomes a melt pond.

1) Change Parameters.
2) Change output filename.
3) Change initial height.
4) Radiation on/off.
5) Change accuracy.
6) Convection on/off.
7) Pond formation on/off.
8) Snow formation on/off.
9) Snow melt drain fully on/off.
10) Set drainage rate.
11) Load file.
12) RUN.
13) Quit.

Enter choice:

Figure B.2: Initial screen of melt-pond–sea ice model program
The upper half of the screen gives details of the current state of the model. These are, respectively, the positions of the interfaces (m), the accuracy level of the NAG routine D03PCF used to solve for the temperature within the model, the total number of timesteps to use in the next model run, the time interval (in terms of the model) at which to output data to file (s), whether there is short-wave radiation, whether there is turbulent convection in the pond or internal melt region, whether a pond is allowed to form and its drainage rate, whether there is snow, and if the snow melt becomes a meltpond (otherwise it is assumed to all drain away).

The lower half of the screen details options that are available to the user, some of which lead to further menus. In order to modify options, the user can simply enter the option-number and follow the on-screen instructions.

Once the user has specified the initial settings for the model run, option 12 from the main menu is selected to start the program (see figure B.2). Before the main program will run the user must answer three initialisation questions. First, the initial day of the year for the run, so that the appropriate snow level and forcing data can be applied; second, the surface temperature of the snow or sea ice; and third, the number of gridpoints that will be used in the output file. Once these have been completed the model will run and print intermediate data to the screen, which can be used for debugging. The algorithm for the model is described by:

1. Initialise variables

2. Set timesteps=0

3. Diagnose state of sea ice (snow covered ice, ice only, melt pond, internal molten region, snow covered internal molten region, ocean only)

4. Determine forcing parameters ($F_{SW}(\text{tot})$, $F_{LW}$, $F_{sens}$, $F_{lat}$, $F_{ocean}$) and initialise the radiation model for the current timestep

5. Set $i_0$ parameter

6. Set incremental timestep to be equivalent to 600 seconds
7. Integrate PDE solver (D03PCF) for current sea ice state. If the ice is snow covered the iterative algorithm is applied to determine the snow–ice interfacial temperature

8. If there is surface melting in ice only case then calculate initial size of pond

9. If there is internal melting in ice only case then calculate initial size of internal melt region

10. Calculate velocity of ice–ocean boundary

11. Calculate velocity of pond–ice or internal-melt-region–ice boundaries. If there is a convective pond or internal melt region then calculate the rate of change in core temperature

12. If there is a melt pond then check whether surface has begun to freeze. If true then determine initial thickness of ice skim

13. Calculate snow accumulation or melt

14. If there is a melt pond then calculate drainage rate of melt pond. If drainage rate will result in pond completely draining during timestep then convert to ice only

15. If file output is required then write to file

16. Update depths

17. If there is a convective pond or internal melt region then update core temperature

18. If there is an internal melt region and the upper ice has just melted (so that its depth is negative) then reset upper ice thickness to zero

19. If there is an internal melt region and the internal melt region has refrozen (so that its depth is negative) then interpolate temperatures back to single layer of ice
20. If there is a melt pond check if melt pond melt through is satisfied. If true then melt ice through and advance time to first day of refreezing

21. Increment timestep and go to 3

The resulting output data files from the melt-pond–sea-ice model are in text format for ease of viewing. The files can easily be read using Microsoft Excel as they are tab delimited and the number of data points in the data file can be limited to ensure the file contains less than 255 data entries on any one line (the maximum number of columns that Microsoft Excel can read). The first 16 lines of the output file consist of initial data, and the following lines consist of output for the model run.

Every time the model outputs data to file, it writes four lines of data, one for each possible layer of the melt-pond–sea-ice system (snow/upper sea ice/internal melt region or pond/lower sea ice). The first line is the snow layer, the second line is the upper sea ice layer, the third line is the melt pond or internal molten region, and the fourth line is the lower sea ice layer. On each line the first 19 data entries contain data relating to the state of the model or forcing data at that timestep, and all data entries after that correspond to temperature within that layer with the leftmost entry being the surface and the rightmost entry being the base. The same number of data entries for temperature occur whatever the depth of each layer.

Table B.1 shows all entries of the four lines of output data transposed for ease of presentation. The units of the data are the same as in the text of the thesis, except where explicitly stated. The reason for the change of units is for ease of interpretation. The parameter $\epsilon$ represents the emissivity of the surface depending on the current state of the sea ice, so that if the sea ice state is ice only then $\epsilon = \epsilon_{\text{ice}}$. The state parameters (state of melt-pond–sea-ice system and convective state of fluid region) are output as long integers, although in the code they are hard coded as strings using the `enumerate` function. The values for state of melt-pond–sea-ice system are 0=melt pond, 1=ice only, 2=internal melt region, 3=snow layer, 4=ocean only. The case of a snow covered internal melt region is denoted by 2, but in this case a Boolean parameter named `internalsnowlayer` is used to indicate
the presence of snow. The convective state of fluid region was also coded using the 
*enumerate* function with 0=convecting and 1=diffusing. Blank entry means that 
the space is unused (I output -99 in these cases). The positions of the boundaries 
(last 5 entries of each row) were negated so that the annual evolution plots could be 
drawn immediately using Microsoft Excel.

The source code (*.cpp, *.h) and compiled code (*.out, *.o) is attached to this thesis 
on the accompanying CD-ROM. Also contained on the CD-ROM is an example 
output data file finalmodel123temp.dat, which contains the standard annual cycle.
<table>
<thead>
<tr>
<th>Row 1</th>
<th>Row 2</th>
<th>Row 3</th>
<th>Row 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNOW</td>
<td>UPPER ICE</td>
<td>MELT</td>
<td>LOWER ICE</td>
</tr>
<tr>
<td>$i_0$</td>
<td>$i_0$</td>
<td>blank entry</td>
<td>blank entry</td>
</tr>
<tr>
<td>$F_{\text{net}(0)}(0)$</td>
<td>$F_{\text{net}(0)}(0)$</td>
<td>$F_{\text{net}(1)}(0)$</td>
<td>$F_{\text{net}(2)}(0)$</td>
</tr>
<tr>
<td>$F_{\text{net}(0)}(h_{\text{intupper}} - h_{\text{surface}})$</td>
<td>$F_{\text{net}(0)}(h_{\text{intupper}} - h_{\text{surface}})$</td>
<td>$F_{\text{net}(1)}(h_{\text{pond}} - h_{\text{intupper}})$</td>
<td>$F_{\text{net}(2)}(h_{\text{ice}} - h_{\text{pond}})$</td>
</tr>
<tr>
<td>$\alpha_{\text{snow}}$</td>
<td>$\alpha_{\text{tot}}$ (no snow)</td>
<td>$F_{\text{SW}}(\text{tot})$</td>
<td>$F_{\text{SW}}(\text{tot})$</td>
</tr>
<tr>
<td>$\alpha_{\text{snow}}F_{\text{SW}}(\text{tot})$</td>
<td>$(1 - \alpha_{\text{tot}})F_{\text{SW}}(\text{tot})$ (no snow)</td>
<td>$F_{\text{intupper}}(h_{\text{pond}})$</td>
<td>$F_{\text{intupper}}(h_{\text{ice}})$</td>
</tr>
<tr>
<td>$(1 - \alpha_{\text{snow}})F_{\text{SW}}(\text{tot})$</td>
<td>$F_{\text{ocean}}$</td>
<td>$F_{\text{SW}}(\text{tot})$</td>
<td>$F_{\text{SW}}(\text{tot})$</td>
</tr>
<tr>
<td>$\epsilon_o T_0^4$</td>
<td>$k_m \partial T / \partial z(z = h_{\text{snow}})$</td>
<td>$F_{\text{sens}}$</td>
<td>$F_{\text{SW}} - \epsilon_o T_0^4$</td>
</tr>
<tr>
<td>$k_m \partial T / \partial z(z = h_{\text{surface}})$</td>
<td>$k_m \partial T / \partial z(z = h_{\text{intupper}})$</td>
<td>$F_{\text{c}}(z = h_{\text{pond}})$</td>
<td>$F_{\text{c}}(z = h_{\text{ice}})$</td>
</tr>
<tr>
<td>$(h_{\text{surface}} - h_{\text{snow}})$</td>
<td>$(h_{\text{intupper}} - h_{\text{surface}})$</td>
<td>$(h_{\text{pond}} - h_{\text{intupper}})$</td>
<td>$(h_{\text{ice}} - h_{\text{pond}})$</td>
</tr>
<tr>
<td>$dh_{\text{snow}} / dt$ (m/day)</td>
<td>$dh_{\text{intupper}} / dt$ (m/day)</td>
<td>$dh_{\text{pond}} / dt$ (m/day)</td>
<td>$dh_{\text{ice}} / dt$ (m/day)</td>
</tr>
<tr>
<td>Time (days)</td>
<td>Time (days)</td>
<td>Time (days)</td>
<td>Time (days)</td>
</tr>
<tr>
<td>$-h_{\text{snow}}$</td>
<td>$-h_{\text{intupper}}$</td>
<td>$-h_{\text{pond}}$</td>
<td>$-h_{\text{pond}}$</td>
</tr>
<tr>
<td>$-h_{\text{surface}}$</td>
<td>$-h_{\text{intupper}}$</td>
<td>$-h_{\text{pond}}$</td>
<td>$-h_{\text{ice}}$</td>
</tr>
<tr>
<td>$-h_{\text{ice}}$</td>
<td>$-h_{\text{ice}}$</td>
<td>$-h_{\text{ice}}$</td>
<td>$-h_{\text{ice}}$</td>
</tr>
</tbody>
</table>

Table B.1: Entries for each 4 lines of data file output. Notation is as used in thesis. Units are the same as in the thesis unless noted.
Appendix C

Proofs for table 5.2

It is seen that the Thickness Equation (equation 5.13) under the assumption that the net flux on the sea ice excluding short-wave radiation ($\Gamma$) is independent of the ice thickness, and hence is constant, can be reformulated as a quadratic equation in exponential functions. Therefore, physically real solutions ($h_{\text{ice}} > 0$) correspond to exponential functions such that $e^{\kappa_2 h_{\text{ice}}} > 1$. Hence, we need to find those $e^{\kappa_2 h_{\text{ice}}} > 1$ that satisfy the Thickness Equation. Table 5.2 illustrates the conditions for the solutions of an arbitrary quadratic to have solutions greater than one. I now show how each condition in the table is determined.

The quadratic equation $ax^2 + \beta x + \gamma = 0$ has solutions $x_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha \gamma}}{2\alpha}$ and $x_2 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha \gamma}}{2\alpha}$. Assume that $\beta^2 - 4\alpha \gamma \geq 0$. I do not consider $\alpha = 0$ since this corresponds to only one possible solution for the Thickness Equation.

Suppose that $\alpha > 0$, then $x_1 > 1$ is equivalent to

$$\sqrt{\beta^2 - 4\alpha \gamma} > 2\alpha + \beta. \quad (C.1)$$

If $2\alpha + \beta \leq 0$ then $\beta < 0$ (since $\alpha > 0$) and (C.1) is automatically true. Otherwise, if $2\alpha + \beta > 0$ then (C.1) is equivalent to $\alpha + \beta + \gamma < 0$. Note that if $\beta \geq 0$ then immediately $2\alpha + \beta > 0$ and we require that $\alpha + \beta + \gamma < 0$ for $x_1 > 1$. This demonstrates the conditions necessary for $x_1 > 1$ when $\alpha > 0$. 

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We have also that, when $\alpha > 0$, $x_2 > 1$ is equivalent to
\[ \sqrt{\beta^2 - 4\alpha\gamma} < -(2\alpha + \beta). \quad (C.2) \]
If $2\alpha + \beta \geq 0$ then equation (C.2) is never true. Otherwise, if $2\alpha + \beta < 0$ then (C.2) is equivalent to $\alpha + \beta + \gamma > 0$. Note that if $\beta \geq 0$ then immediately $2\alpha + \beta > 0$ and $x_2 > 1$ is never true. This demonstrates the conditions necessary for $x_2 > 1$ when $\alpha > 0$.

Now, suppose that $\alpha < 0$, then $x_1 > 1$ is equivalent to
\[ \sqrt{\beta^2 - 4\alpha\gamma} < 2\alpha + \beta. \quad (C.3) \]
If $2\alpha + \beta \leq 0$ then (C.3) is never true. Otherwise, if $2\alpha + \beta > 0$ then (C.3) is equivalent to $\alpha + \beta + \gamma > 0$. Note that if $\beta \leq 0$ then immediately $2\alpha + \beta < 0$ and $x_1 > 1$ is never true. This demonstrates the conditions necessary for $x_1 > 1$ when $\alpha < 0$.

We have that, when $\alpha < 0$, $x_2 > 1$ is equivalent to
\[ \sqrt{\beta^2 - 4\alpha\gamma} > -(2\alpha + \beta). \quad (C.4) \]
If $2\alpha + \beta \geq 0$ then (C.4) is automatically true. Otherwise, if $2\alpha + \beta < 0$ then (C.4) is equivalent to $\alpha + \beta + \gamma < 0$. Note that if $\beta \leq 0$ then immediately $2\alpha + \beta < 0$ and we require that $\alpha + \beta + \gamma < 0$ for $x_2 > 1$. This demonstrates the conditions necessary for $x_2 > 1$ when $\alpha < 0$. 

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Appendix D

Proof that $\frac{\partial G}{\partial h} > 0$

I will show that the function $G(z, h) = F_{\text{net}}(z) - F_{\text{net}}(h) + F_{\text{ocean}}$, for the two-stream radiation model, is monotonic increasing in $h$ (the ice thickness). I show this by demonstrating that the derivative of $G$ with respect to $h$ is always positive. I will show that $G(z, h)$ is monotonic increasing in $h$ for the single band case, and this immediately generalises to a multiple band case.

Following the notation of chapter 3 and chapter 5 and dropping subscripts I define $a = e^{zh}$ and $b = e^{z(1)}$. Then $a$ and $b$ satisfy: $a > b$; $a > 1$ and $b > 1$. Substituting $a$ and $b$ in $G$, and using the relationship $s = (\kappa - k)/(\kappa + k)$, where $k$ is the absorption coefficient, it is found that

$$
\frac{\partial G}{\partial h} = \frac{-4a(-1 + R_0)F_{SW}k\Lambda\kappa[\alpha_0 k^3 + \alpha_1 k^2\kappa + \alpha_2 k\kappa^2 + \alpha_3 \kappa^3]}{b\{(1 + a^2)(1 + R_0)k^2 + 2(1 + a)k\kappa - (-1 + a)(-1 + R_0)\kappa^2\}^2},
$$

where

$$
\begin{align*}
\alpha_0 &= a(-1 + b^2)(1 + R_0); \\
\alpha_1 &= (-a(1 + b^2)(-1 + R_0) + b(1 + R_0) + a^2b(1 + R_0)); \\
\alpha_2 &= (-2b + 2a^2b - a(-1 + b^2)(1 + R_0)); \\
\alpha_3 &= (a - b - a^2b + ab^2)(-1 + R_0).
\end{align*}
$$

Clearly, the denominator of equation (D.1) and the pre-multiplying terms of the
square bracket in the numerator are positive. Therefore, I consider the terms in square brackets of the numerator and show that each term is positive.

It is obvious by inspection that $\alpha_0$ and $\alpha_1$ are positive. Hence the terms $\alpha_2$ and $\alpha_3$ must be shown to be both positive. Since $R_0 \leq 1$ it is clear that

$$\alpha_2 = -2b + 2a^2b - a(-1 + b^2)(1 + R_0) \geq -2b + 2a^2b - 2a(-1 + b^2). \quad (D.2)$$

The right-hand side of the inequality (D.2) is equivalent to $2(1 + ab)(a - b)$ and is therefore greater than zero. Hence $\alpha_2$ is positive. The term $\alpha_3$ is equivalent to $-(a - b)(ab - 1)(-1 + R_0)$ and is therefore also positive.

Each individual component of the expression in square brackets is positive and so it is true that $\partial G/\partial h$ is greater than zero (for both single and multiple bands). Hence it can be concluded that $G(z, h)$ is monotonic increasing in $h$.  

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Appendix E

Proof that there cannot be two turning points in the radiative term of the Thickness Equation

The turning points of the radiative term $a$ of the Thickness Equation (5.17) with respect to the equilibrium ice thickness $h_{ice}$ satisfy

$$i_0(1 - R_0s_2) - 2s_2(1 - R_0)Y_2 + i_0{s_2(s_2 - R_0)Y_2^2} = 0,$$

where the notation follows that of chapter 5. Equation (E.1) is determined by differentiating $a$ with respect to $h_{ice}$ and setting equal to zero. I now show that $a$ cannot have two turning points when the equilibrium ice thickness is positive.

Equation (E.1) is a quadratic in exponentials of the form

$$\alpha \exp(2\kappa_2 h_{ice}) + \beta \exp(\kappa_2 h_{ice}) + \gamma = 0,$$

where

$$\alpha = i_0(1 - R_0 s_2),$$

$$\beta = -2s_2(1 - R_0),$$

$$\gamma = i_0{s_2(s_2 - R_0)}.$$

Table 5.2 describes the conditions necessary for there to be two positive solutions of equation (E.1). For physically real parameters, $\alpha$ and $\beta$ satisfy $\alpha > 0$ and $\beta < 0$. For two positive solutions, it is therefore required that $2\alpha + \beta < 0$ and $\alpha + \beta + \gamma > 0$. 

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I will assume that \(2\alpha + \beta < 0\) is true, and then show that the second requirement
\((\alpha + \beta + \gamma > 0)\) leads to a contradiction. Therefore, it is impossible to have two
positive solutions to equation (E.1) and so there is only one or zero turning points
for positive ice thicknesses.

If \(2\alpha + \beta < 0\), then rearranging yields \(\alpha < -\beta/2\). Assume that \(\alpha + \beta + \gamma > 0\),
then \(\alpha > -\beta - \gamma\). Hence, given the two assumptions we have that \(2\gamma + \beta > 0\), by
combining the inequalities for \(\alpha\). Substituting the values of \(\gamma\) and \(\beta\) yields

\[
2\gamma + \beta = 2s_2(i_0(s_2 - R_0) - (1 - R_0)).
\]

If \((s_2 - R_0) \leq 0\), then \(2\gamma + \beta < 0\), hence we have a contradiction. If \((s_2 - R_0) > 0\),
then since \(s_2 < 1\) we have that

\[
i_0(s_2 - R_0) - (1 - R_0) < (i_0 - 1)(1 - R_0) < 0,
\]

and so \(2\gamma + \beta < 0\) and again we have a contradiction.

Therefore, since the two required inequalities \(2\alpha + \beta < 0\) and \(\alpha + \beta + \gamma > 0\) cannot
be satisfied simultaneously, there cannot be two positive solutions to equation (E.1).
Hence, there is only one or zero turning points of the radiative term of the Thickness
Equation for positive ice thicknesses.
Appendix F

Stationary solutions for the temperature profile of solid ice

I now explicitly determine the stationary temperature profile for the case of pure ice under the same boundary conditions as the sea ice only case in chapter 5. I then show that the relationships used to determine the perturbation evolution in section 5.7.2 are true.

Using the notation of section 5.7, the non-dimensional governing equation for temperature in the domain \([0,h]\) for the case of pure ice and absorbed radiation is given by

\[
\frac{\partial^2 \theta}{\partial z^2} = \mathcal{F} \frac{\partial F_{\text{net}}}{\partial z},
\]

with boundary conditions

\[
\frac{\partial \theta}{\partial z} = \mathcal{F} F_{\text{ocean}} \quad \text{and} \quad \theta = \theta_{\text{ocean}}, \text{ at } z = h.
\]

Integrating equation (F.1) twice and applying the boundary conditions yields

\[
\theta_{ss}(z, h) = \mathcal{F} \left( \int_{h}^{z} F_{\text{net}}(z^*, h) \, dz^* + (F_{\text{ocean}} - F_{\text{net}}(h, h))(z - h) \right) + \theta_{\text{ocean}}. \quad (F.4)
\]

I now show that

\[
\frac{\partial \theta_{ss}}{\partial z} + \frac{\partial \theta_{ss}}{\partial h} = 0, \text{ at } z = h. \quad (F.5)
\]
Differentiating equation F.4 with respect to $z$ and $h$ yields

$$\frac{\partial \theta_{ss}}{\partial z} = \mathcal{F}(F_{net}(z, h) + F_{ocean} - F_{net}(h, h)) \quad \text{and} \quad (F.6)$$

$$\frac{\partial \theta_{ss}}{\partial h} = \mathcal{F} \int_{h}^{z} \frac{\partial F_{net}(z^*, h)}{\partial h} \, dz + \mathcal{F}(h - z) \frac{\partial F_{net}(z, h)}{\partial h} + \mathcal{F} F_{net}(z, h)$$

$$- \mathcal{F} F_{ocean} - \mathcal{F} F_{net}(h, h). \quad (F.7)$$

Therefore, when $z = h$, equations (F.6) and (F.7) add to zero so that equation (F.5) is true.

I now show that

$$\frac{\partial^2 \theta_{ss}}{\partial z^2} + \frac{\partial^2 \theta_{ss}}{\partial h^2} = 2\mathcal{F} \frac{\partial F_{net}(z, h)}{\partial h}, \quad (z = h). \quad (F.8)$$

The second derivatives are given by

$$\frac{\partial^2 \theta_{ss}}{\partial z^2} = \mathcal{F} \frac{\partial F_{net}(z, h)}{\partial z} \quad \text{and} \quad (F.9)$$

$$\frac{\partial^2 \theta_{ss}}{\partial h^2} = \mathcal{F} \int_{h}^{z} \frac{\partial^2 F_{net}(z^*, h)}{\partial h^2} \, dz^* + \mathcal{F} \frac{\partial F_{net}(h, h)}{\partial z} - \mathcal{F}(z - h) \times$$

$$\left( \frac{\partial^2 F_{net}(h, h)}{\partial h^2} + 2 \frac{\partial^2 F_{net}(h, h)}{\partial z \partial h} + \frac{\partial^2 F_{net}(h, h)}{\partial z^2} \right), \quad (F.10)$$

therefore when $z = h$, the two equations add to $2\mathcal{F} \frac{\partial F_{net}(h, h)}{\partial z}$ so that equation (F.8) is true.
Appendix G

Principle of exchange of stabilities

I now show that growth rate $\sigma$ of the perturbation of the stationary solution for thermal stability of a melt pond (section 6.2.5) is real so that the marginal states (where the perturbation neither grows nor decays) are characterised by $\sigma = 0$.

The governing equation of the perturbation of the vertical velocity $W$ is

$$
(\text{Pr} (D^2 - a^2) - \sigma) ((D^2 - a^2) - \sigma) (D^2 - a^2) W = -\text{Pr} \text{Ra} \frac{\partial \theta_s}{\partial z} a^2 W, \quad (G.1)
$$

using the same notation as section 6.2.

The temperature can be separated in the same way as the vertical component of velocity (c.f. equation 6.23),

$$
\theta_1 = \Theta(z) g(x,y) e^{\sigma t}, \quad (G.2)
$$

and using the temperature perturbation equation (equation 6.20) the following form is obtained for the temperature perturbation in terms of the velocity perturbation

$$
(\text{Pr} (D^2 - a^2) - \sigma) (D^2 - a^2) W = -\text{Pr} \text{Ra} a^2 \Theta. \quad (G.3)
$$

Equations G.1 and G.3 can be combined to yield

$$
(\text{Pr} (D^2 - a^2) - \sigma) ((D^2 - a^2) - \sigma) (D^2 - a^2) \Theta = -\text{Pr} \text{Ra} \frac{\partial \theta_s}{\partial z} a^2 \Theta. \quad (G.4)
$$

If we substitute for the perturbation variables in the boundary conditions, it can be
shown that at the lower boundary \((z = 0)\)

\[
\Theta = 0, \quad W = 0 \quad \text{and} \quad DW = 0. \tag{G.5}
\]

At the upper boundary \((z=1)\) the boundary conditions become

\[
D\Theta - 4\varepsilon \left( \theta_s + \frac{T_0}{\Delta T} \right)^3 \Theta = 0, \quad W = 0 \quad \text{and} \quad D^2W = 0.
\]

Following Chandrasekhar [19], let

\[ G = (D^2 - a^2)W, \quad \text{and} \]
\[ F = (D^2 - a^2)(\text{Pr}(D^2 - a^2) - \sigma)W = (\text{Pr}(D^2 - a^2) - \sigma)G. \]

The equation satisfied by \(W\) (equation G.1) can be written

\[
(D^2 - a^2 - \sigma)F = -\text{Pr} \text{Ra} \frac{\partial^2_s}{\partial z^2} a^2W, \tag{G.6}
\]

and equation (G.3) can be written

\[ F = \text{Pr} \text{Ra} a^2 \Theta. \tag{G.7} \]

To prove that \(\sigma\) is real, I multiply equation (G.6) by the complex conjugate of \(F\) and integrate over the whole domain. Then I examine the imaginary component of the resulting expression. I then show that this can only be satisfied if the imaginary component of \(\sigma\) is zero. This analysis is the same as Chandrasekhar’s [19] analysis of thermal stability except that I have a variable gradient of the stationary temperature profile and different boundary conditions. The introduction of the variable temperature gradient means that I have to use bounding inequalities in the last few steps to show that the imaginary component of \(\sigma\) is zero.

Multiply equation (G.6) by \(\bar{F}\), the complex conjugate of \(F\), and integrate over the whole range of \(z\), so that

\[
\int_0^1 \bar{F}((D^2 - a^2) - \sigma)F \, dz = -\text{Pr} \text{Ra} a^2 \int_0^1 \bar{F} \frac{\partial^2_s}{\partial z^2} W \, dz. \tag{G.8}
\]
Now,
\[
\int_0^1 \bar{F} D^2 F \, dz = [\bar{F} D F]_0^1 - \int_0^1 D \bar{F} D F \, dz
\]
\[
= [\bar{F} D F]_0^1 - \int_0^1 |D F|^2 \, dz,
\]
and so equation (G.8) becomes
\[
- [\bar{F} D F]_0^1 + \int_0^1 (|D F|^2 + (a^2 + \sigma) |F|^2) \, dz = Pr Ra a^2 \int_0^1 \bar{F} \frac{\partial \theta_s}{\partial z} W \, dz. \tag{G.9}
\]
Now,
\[
\int_0^1 W \bar{F} \, dz = \int_0^1 W (Pr D^2 - a^2 - \bar{\sigma}) G \, dz
\]
\[
= \int_0^1 Pr W D^2 G \, dz - (Pr a^2 + \bar{\sigma}) \int_0^1 W G \, dz,
\]
and
\[
\int_0^1 Pr W D^2 G \, dz = -Pr \int_0^1 D W D G \, dz = Pr \int_0^1 G D^2 W \, dz, \tag{G.10}
\]
using integration by parts and the boundary conditions \(W = 0, DW = 0\) (rigid) and \(G = (D^2 - a^2) \bar{W} = 0\) (free).

Therefore
\[
\int_0^1 W \bar{F} \, dz = \int_0^1 G (Pr (D^2 - a^2) - \bar{\sigma}) W \, dz
\]
\[
= Pr \int_0^1 |G|^2 \, dz - \bar{\sigma} \int_0^1 W (D^2 - a^2) \bar{W} \, dz.
\]
Now,
\[
\int_0^1 W D^2 W \, dz = -\int_0^1 |D W|^2 \, dz,
\]
hence
\[
\int_0^1 W \bar{F} \, dz = Pr \int_0^1 |G|^2 \, dz + \bar{\sigma} \int_0^1 |D W|^2 \, dz + \bar{\sigma} a^2 \int_0^1 |W|^2 \, dz.
\]
Consider \(\int_0^1 (\partial \theta_s / \partial z) W \bar{F} \, dz\). Suppose that
\[
\min_{z \in [0,1]} \frac{\partial \theta_s}{\partial z} = m, \text{ and}
\]
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\[
\max_{\varepsilon \in [0, 1]} \frac{\partial \theta_s}{\partial z} = M.
\]

Assume that \( \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right] \geq 0 \), then since the gradient of the stationary temperature profile is necessarily single signed (in our case positive) we know that

\[
m\text{Im} \left[ \int_0^1 W \bar{F} \, dz \right] \leq \text{Im} \left[ \int_0^1 \frac{\partial \theta_s}{\partial z} W \bar{F} \, dz \right] \leq M \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right]
\]

(G.11)

Since the real and complex parts of equation (G.9) must be equal separately, we consider only the imaginary part of equation (G.9):

\[
\text{Im} \left\{ -[\bar{F} DF]_0^1 \right\} + \text{Im} (\sigma) \int_0^1 |F|^2 \, dz = \text{Pr} Ra a^2 \text{Im} \left\{ \int_0^1 \bar{F} \frac{\partial \theta_s}{\partial z} W \, dz \right\}.
\]

(G.12)

Suppose that \( \text{Im} [\bar{F} DF]_0^1 = 0 \), which we will show is true later, then equation (G.11) becomes

\[
m \text{Pr} Ra a^2 \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right] \leq \text{Im} (\sigma) \left( \int_0^1 |F|^2 \, dz \right) \leq M \text{Pr} Ra a^2 \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right],
\]

(G.13)

hence

\[
-m \text{Pr} Ra a^2 \text{Im} (\sigma) \left( \int_0^1 |DW|^2 + a^2 |W|^2 \, dz \right) \leq \text{Im} (\sigma) \left( \int_0^1 |F|^2 \, dz \right), \text{ and}
\]

\[
\text{Im} (\sigma) \left( \int_0^1 |F|^2 \, dz \right) \leq -M \text{Pr} Ra a^2 \text{Im} (\sigma) \left( \int_0^1 |DW|^2 + a^2 |W|^2 \, dz \right).
\]

(G.14)

Equations (G.14) and (G.15) can be rewritten as

\[
0 \leq \text{Im} (\sigma) \left( \int_0^1 |F|^2 + m \text{Pr} Ra a^2 (|DW|^2 + a^2 |W|^2) \, dz \right), \text{ and}
\]

\[
0 \geq \text{Im} (\sigma) \left( \int_0^1 |F|^2 + M \text{Pr} Ra a^2 (|DW|^2 + a^2 |W|^2) \, dz \right).
\]

(G.16)

(G.17)

Since the integrands are positive definite for positive \( Ra \) we deduce that \( \text{Im} (\sigma) = 0 \), under the assumption that \( \text{Im} [\bar{F} DF]_0^1 = 0 \) and \( \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right] \geq 0 \). Similarly we deduce that \( \text{Im} (\sigma) = 0 \), under the assumption that \( \text{Im} [\bar{F} DF]_0^1 = 0 \) and \( \text{Im} \left[ \int_0^1 W \bar{F} \, dz \right] \leq 0 \). Therefore, we find that under the assumption that \( \text{Im} [\bar{F} DF]_0^1 = 0 \) that the principle of exchange of stabilities is valid.
I now show that the assumption \( \text{Im} [\bar{F}DF]_0^1 = 0 \) is true.

We know that \( F = \text{Pr Ra} a^2 \Theta \) (equation G.7) for all \( z \), that \( D\Theta = 4\varepsilon (\theta_s + T_0/\Delta T)^3 \Theta \) at the upper boundary \( (z = 1) \) (equation G.5), and \( \Theta = 0 \) at the lower boundary \( (z = 0) \). Therefore,

\[
\bar{F}DF|_1 - \bar{F}DF|_0 = (\text{Pr Ra} a^2)^2 4\varepsilon (\theta_s + \frac{T_0}{\Delta T})^3 |\Theta|^2.
\]

Hence,

\[
\text{Im} [\bar{F}DF]_0^1 = 0.
\]
Bibliography


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