

AN INVERSE PROBLEM FOR A SEMI-LINEAR ELLIPTIC EQUATION IN RIEMANNIAN GEOMETRIES

ALI FEIZMOHAMMADI AND LAURI OKSANEN

ABSTRACT. We study the inverse problem of unique recovery of a complex-valued scalar function $V : \mathcal{M} \times \mathbb{C} \rightarrow \mathbb{C}$, defined over a smooth compact Riemannian manifold (\mathcal{M}, g) with smooth boundary, given the Dirichlet-to-Neumann map, in a suitable sense, for the elliptic semi-linear equation $-\Delta_g u + V(x, u) = 0$. We show that uniqueness holds for a large class of non-linearities when the manifold is conformally transversally anisotropic. The proof is constructive and is based on a multiple-fold linearization of the semi-linear equation near complex geometric optic solutions for the linearized operator and the resulting non-linear interactions. These interactions result in the study of a weighted integral transform along geodesics, that we call the Jacobi weighted ray transform.

CONTENTS

1. Introduction	2
1.1. Main results	3
1.2. Previous literature	5
1.3. Outline	6
2. Preliminaries	7
2.1. Direct problem	7
2.2. Multiple-fold linearization method	9
2.3. Reduction to the case $c \equiv 1$	12
2.4. A Carleman estimate	12
3. The Jacobi weighted ray transform	15
3.1. Inversion of Jacobi weighted ray transform of the first kind	16
3.2. Inversion of Jacobi weighted ray transform of the second kind	20
4. Complex Geometric Optics	23
4.1. Gaussian quasi modes	24
4.2. The remainder term	28
5. Proof of Theorem 1	29
6. Proof of Theorem 2	33
References	34

1. INTRODUCTION

Let (\mathcal{M}, g) be a smooth compact Riemannian manifold with a smooth boundary $\partial\mathcal{M}$ and $\dim \mathcal{M} := n \geq 3$. Let $\alpha \in (0, 1)$ and consider an a priori unknown function $V : \mathcal{M} \times \mathbb{C} \rightarrow \mathbb{C}$. We make the following standing assumptions.

- (i) $V(\cdot, z) \in \mathcal{C}^\alpha(\mathcal{M})$, $\forall z \in \mathbb{C}$,
- (ii) $V(x, 0) = 0$, $\forall x \in \mathcal{M}$,
- (iii) V is analytic with respect to z in the $\mathcal{C}^\alpha(\mathcal{M})$ topology,

where $\mathcal{C}^\alpha(\mathcal{M})$ is the space of Hölder continuous complex-valued functions with exponent α . By analyticity with respect to $z \in \mathbb{C}$ we mean that the following limit exists in the $\mathcal{C}^\alpha(\mathcal{M})$ topology,

$$\partial_z V(x, z) := \lim_{h \rightarrow 0} \frac{V(x, z+h) - V(x, z)}{h}.$$

As a result of analyticity, the function V admits a power series representation in the $\mathcal{C}^\alpha(\mathcal{M})$ topology given by the expression

$$(1) \quad V(x, z) = \sum_{k=1}^{\infty} V_k(x) \frac{z^k}{k!},$$

where $V_k(x) := \partial_z^k V(x, 0) \in \mathcal{C}^\alpha(\mathcal{M})$. We additionally impose the following conditions on the set of admissible functions $V(x, z)$:

- (iv) 0 is not a Dirichlet eigenvalue for the operator $-\Delta_g + V_1(x)$ on (\mathcal{M}, g) .

Here, Δ_g denotes the Laplace-Beltrami operator on (\mathcal{M}, g) given in local coordinates by the expression $\Delta_g = \sum_{j,k=1}^n \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{jk} \frac{\partial}{\partial x^k} \right)$.

In this paper, we consider the semi-linear elliptic equation

$$(2) \quad \begin{cases} -\Delta_g u + V(x, u) = 0, & \forall x \in \mathcal{M} \\ u = f \in B_{r_0}^\alpha(\partial\mathcal{M}), & \forall x \in \partial\mathcal{M} \end{cases}$$

where $B_{r_0}^\alpha(\partial\mathcal{M}) := \{h \in \mathcal{C}^{2,\alpha}(\partial\mathcal{M}) \mid \|h\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})} \leq r_0\}$. In Section 2.1, we show that, given fixed $r_0, r_1 > 0$ sufficiently small, equation (2) admits a unique solution $u \in B_{r_1}^\alpha(\mathcal{M})$. Moreover, there exists a constant $C > 0$ depending only on r_0, r_1 such that

$$(3) \quad \|u\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C \|f\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})} \quad \forall f \in B_{r_0}^\alpha(\partial\mathcal{M}).$$

We subsequently define the Dirichlet-to-Neumann (DN) map, Λ_V , for equation (2) through the expression

$$(4) \quad \mathcal{C}^{1,\alpha}(\partial\mathcal{M}) \ni \Lambda_V f := \partial_\nu u|_{\partial\mathcal{M}}, \quad \forall f \in B_{r_0}^\alpha(\partial\mathcal{M}),$$

where ν denotes the unit outward normal vector field on $\partial\mathcal{M}$. This paper is concerned with the following question.

Question 1. *Given the map Λ_V , can one uniquely determine the function V ?*

We will briefly review the history related to inverse problems for non-linear elliptic equations in Section 1.2. For now, let us recall some facts about the case where $V(x, z) \equiv V_1(x)z$. In this case, the problem reduces to a version of the Calderón conjecture [2]. This formulation of the conjecture has been extensively studied but remains open in general geometries (\mathcal{M}, g) with dimension $n \geq 3$. Uniqueness of the coefficient V_1 has been proved for analytic metrics with an analytic function V_1 [30], the Euclidean metric [34, 43] and the hyperbolic metric [20]. Beyond these cases, the most general uniqueness result is obtained in the so-called *conformally transversally anisotropic* (CTA) geometries defined as follows.

Definition 1. *Let (\mathcal{M}, g) be a compact oriented smooth Riemannian manifold with smooth boundary and dimension n . We say that (\mathcal{M}, g) is conformally transversally anisotropic, if $n \geq 3$ and the following embedding holds:*

$$\mathcal{M} \subset I^{\text{int}} \times M^{\text{int}} \quad \text{and} \quad g(x^0, x') = c(x^0, x')((dx^0)^2 \oplus g(x')),$$

where I is a finite interval, $c(x^0, x') > 0$ is a smooth function and (M, g) is a smooth compact orientable manifold of dimension $n - 1$ with a smooth boundary ∂M .

In [7] it was proved that in the linear case $V(x, z) = V_1(x)z$, the Dirichlet-to-Neumann map Λ_V uniquely determines a bounded function V_1 , under the strong assumption that the transversal manifold is *simple*, that is to say (M, g) has a strictly convex boundary and given any two points in M there exists a unique geodesic connecting them. This result was subsequently strengthened in [9] where the authors showed that Λ_{V_1} uniquely determines V_1 , if the geodesic ray transform is injective on the transversal manifold. The inversion of the geodesic ray transform is open in general, and has only been proved under certain geometrical assumptions, see for example the discussion in [9, Section 1]. For a broad review of the Calderón conjecture, and alternative formulations with the presence of non-linear coefficients, we refer the reader to survey articles [44, 45].

1.1. Main results. Let us return to Question 1. We will consider only the case where (\mathcal{M}, g) is a CTA manifold. Before stating our results let us briefly review some notations for geodesic dynamics on (M, g) . Let $SM \subset TM$ denote the unit sphere bundle on (M, g) and $\gamma(\cdot, x, \theta)$ be the unit speed geodesic with initial data (x, θ) . For all $(x, \theta) \in SM^{\text{int}}$, we define the exit times

$$(5) \quad \tau_{\pm} = \sup \{r > 0 \mid \gamma(\pm r; x, \theta) \in \partial M, \quad \dot{\gamma}(\pm r; x, \theta) \notin T\partial M\},$$

and subsequently call a geodesic γ to be maximal, if and only if $\tau_{\pm} < \infty$. Next, we define an admissibility condition on the transversal manifold (M, g) as follows.

Definition 2. *Let (M, g) be a smooth compact Riemannian manifold with boundary. We say that (M, g) is admissible if there exists a dense set of points $\mathcal{T} \subset M$ such that given any point $p \in \mathcal{T}$ there exists a non-self-intersecting maximal geodesic γ through p that contains no conjugate points to p .*

The first result in this paper can now be stated as follows.

Theorem 1. *Let (\mathcal{M}, g) be a CTA manifold such that the transversal manifold M is admissible. Suppose that $V(x, z)$ satisfies conditions (i)–(iv), that V_1 is smooth and that V_1, V_2 are a priori known. Then, the Dirichlet-to-Neumann map Λ_V uniquely determines the function V .*

The proof of this theorem relies on a multiple-fold linearization of (2) that results in the interaction of the so called complex geometric optic solutions for the corresponding linearized equation. Since V_1 is assumed to be known, the complex geometric optic solutions will be known as well. The smoothness assumption on V_1 is imposed in order to make these solutions smooth and also to simplify the task of proving suitable decay rates (see Proposition 5). Under the assumption that V_2 is assumed to be known, the non-linear interaction of the complex geometric optic solutions will result in a weighted ray transform along geodesics on the transversal manifold M . This weighted transform will be shown to be invertible along a single geodesic (see Proposition 4).

Our second main result is concerned with the recovery of the function V without imposing the assumption that the coefficient V_2 is known, in the cases where the manifold is three or four dimensional.

Theorem 2. *Let (\mathcal{M}, g) be a three or four dimensional CTA manifold such that given any point on the transversal manifold M there exists a maximal non-self-intersecting geodesic without conjugate points through that point. Suppose that $V(x, z)$ satisfies conditions (i)–(iv) and that V_1 is a priori known and smooth. Then the Dirichlet-to-Neumann map Λ_V uniquely determines the function V .*

The proof of this theorem mostly follows the same technique as the previous theorem. However, due to the weaker assumption on the coefficient V , namely that V_2 is unknown, the non-linear interaction of the complex geometric optic solutions results in a different ray transform along geodesics on the transversal manifold M . The inversion of this transform along a single geodesic is proved when the transversal manifold is two or three dimensional and left open in higher dimensions (see Proposition 3). We also refer the reader to Remark 1 in Section 3 where the restriction to three and four dimensions is discussed further.

1.2. Previous literature. The study of non-linear partial differential equations is an interesting topic in its own right, due to the complexity of the subject matter and as such, the corresponding inverse problems also carry significant mathematical interest. However, let us point out that there are applications for these inverse problems outside the realm of mathematics as well. Indeed, a large class of inverse problems for elliptic nonlinear equations can be seen as the study of stationary solutions to nonlinear equations describing physical phenomena. For example, we mention the nonlinear Schrödinger equation that arises as nonlinear variations of the classical field equations and has applications in the study of nonlinear optical fibers, planar wave guides and Bose Einstein condensates [31]. Other examples include nonlinear Klein-Gordon or Sine-Gordon equations with applications to the study of general relativity [35] and relativistic super-fluidity [47] respectively.

The majority of the literature dealing with inverse problems for non-linear elliptic equations is in the Euclidean geometry. The first uniqueness result was obtained by Isakov and Sylvester in [19] where the authors considered a Euclidean domain of dimension greater than or equal to three with non linear functions $V(x, u)$ that satisfy the homogeneity property (ii), and showed that under a monotonicity condition for V and suitable bounds on V , $\partial_u V$ and $\partial_u^2 V$, the non-linearity can be uniquely recovered on a specific subset of $\mathcal{M} \times \mathbb{R}$. There, it was also proved that under a stronger bound on V , it could be recovered everywhere. Removing the homogeneity property (ii) introduces a natural gauge for the uniqueness of the non-linearity. This was studied by Sun in [42] under similar smoothness and monotonicity assumptions. There, a similar uniqueness result as in [19] was proved (up to the natural gauge), under the additional assumption that a common solution exists.

In dimension two, the problem was first solved by Sylvester and Nachman in [18], where the authors considered a domain in two-dimensional Euclidean space with a Carathéodory type non-linearity that has a continuous bounded L^p -valued derivative in the u variable and proved unique recovery of the non-linearity. In [33] uniqueness is proved for yet another family of admissible non-linearities in two dimensional Euclidean domains. There, a connection is also made between the theoretical study of these types of semi-linear inverse problems and the physical study of semi-conductor devices and ion channels. We also mention the work of Imanuvilov and Yamamoto in [14] where the authors considered the partial data problem for the operator $\Delta u + q(x)u + V(x, u)$ on arbitrary open subsets of the boundary in two dimensions. There it was shown that if $V(x, 0) = \partial_u V(x, 0) = 0$, it is possible to uniquely recover q everywhere and also that it is possible to recover V in certain subsets of the domain, under suitable bounds on the non-linear function V .

Aside from the study of inverse problems for semi-linear equations in Euclidean geometries, let us also mention that there are several works related to inverse problems for quasi-linear elliptic equations (see for example [3, 10, 17, 32, 39, 40, 41]). It should

be emphasized that the key idea in all of these results has been a linearization technique introduced by Isakov in [15] in the context of semi-linear parabolic equations and developed further in [16, 18, 19, 39, 40]. This linearization technique together with the uniqueness results for the Calderón conjecture in Euclidean domains leads to the unique recovery of the non-linear terms.

The main novelty of this paper is to extend uniqueness results for non-linear elliptic equations to a wider class of Riemannian manifolds, known as conformally transversally anisotropic manifolds (see Definition 1). We consider local solutions about the trivial solution, but our proof is based on a multiple-fold linearization technique that differs from most of the previously mentioned works. As already discussed, the results in the Euclidean setting rely on the fact that uniqueness holds for the linearized inverse problem. This is no longer the case when \mathcal{M} is assumed to be conformally transversally anisotropic. Indeed, uniqueness results for the linearized problem rely on injectivity of the geodesic ray transform on (M, g) that is known to be true under strong geometric assumptions such as simplicity of the transversal manifold (M, g) or existence of a strictly convex foliation [46]. The strength of our results lies in removing such strong geometric assumptions. On the other hand, contrary to the Euclidean cases, the results here assume analyticity of $V(x, u)$ with respect to u .

The multiple-fold linearization technique in this paper is inspired by the study of similar types of non-linear problems for hyperbolic equations that was developed by Kurylev, Lassas and Uhlmann in [25, 26] in the context of Einstein scalar field equations and used in subsequent works in the context of semi-linear wave equations (see for example [4, 12, 28, 29, 48]). However, these works are based on the study of propagation of singularities for linear wave equations and the non-linear interactions of these singularities, making it difficult to apply them to an elliptic problem. Another key difference with all previous works in the hyperbolic setting is that we study non-linear interaction of localized solutions that correspond to a single geodesic. This will lead us to the study of a weighted transform along geodesics that we call the Jacobi ray transforms of the first and second kind. We show that it is possible to invert these transforms along a single geodesic (see Propositions 3–4).

We conclude this introductory section by remarking that while writing this paper we became aware of an upcoming preprint by Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin and Mikko Salo, which simultaneously and independently proves a similar result. We agreed to post our respective preprints to arXiv at the same time. See [27] for their preprint.

1.3. Outline. This paper is organized as follows. Section 2 is concerned with some preliminary discussions. We show that the Dirichlet-to-Neumann map Λ_V (see (4)) is well-defined. We also discuss the linearization method for solutions to equation (2)

near the trivial solution, in particular showing the appearance of what we call a non-linear interaction. The rest of Section 2 is concerned with some lemmas and notations that will be needed throughout the paper. In Section 3 we define the Jacobi weighted transform of the first and second kind along a fixed geodesic, and subsequently prove injectivity results for these two transforms, see Propositions 3–4. Section 4 starts with a review of the well-known Gaussian quasi modes for the linearized operator following [9]. In the remainder of this section we use this construction, together with a Carleman estimate to produce a family of complex geometric optic solutions for the linearized operator. In Section 5, we use an induction argument, based on the application of our linearization technique near the complex geometric optic solutions, to complete the proof of Theorem 1. Section 6 is concerned with the proof of Theorem 2.

2. PRELIMINARIES

2.1. Direct problem. In this section we prove the following proposition for the direct problem (2).

Proposition 1. $\exists r_0, r_1 > 0$ depending on (\mathcal{M}, g) , such that equation (2) admits a unique solution $u \in B_{r_1}^\alpha(\mathcal{M})$. Moreover, there holds

$$\|u\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C \|f\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})}, \quad \forall f \in B_{r_0}^\alpha(\partial\mathcal{M}),$$

for some constant C that depends on (\mathcal{M}, g) , r_0 and r_1 .

Let us define the Schrödinger operator $\mathcal{P}_{V_1} = -\Delta_g + V_1(x)$, and consider the linear equation

$$(6) \quad \begin{cases} \mathcal{P}_{V_1} u = F, & \forall x \in \mathcal{M} \\ u = f & \forall x \in \partial\mathcal{M} \end{cases}$$

where $(f, F) \in \mathcal{C}^{2,\alpha}(\partial\mathcal{M}) \times \mathcal{C}^\alpha(\mathcal{M})$. We introduce the solution operators $\mathcal{G}_{V_1}^D, \mathcal{G}_{V_1}^S$ so that the function $\mathcal{G}_{V_1}^D f$ is the unique solution to (6) subject to $F \equiv 0$ and $\mathcal{G}_{V_1}^S F$ is the unique solution to (6) subject to $f \equiv 0$. There is a constant $\kappa > 0$ (see for example [13, Chapter 4.4]) such that

$$(7) \quad \|\mathcal{G}_{V_1}^D\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M}) \rightarrow \mathcal{C}^{2,\alpha}(\mathcal{M})} + \|\mathcal{G}_{V_1}^S\|_{\mathcal{C}^\alpha(\mathcal{M}) \rightarrow \mathcal{C}^{2,\alpha}(\mathcal{M})} \leq \kappa.$$

Let us now define the function $\tilde{V}(x, z) := V(x, z) - V_1(x)z$. We have the following lemma.

Lemma 1. Let $r \in (0, 1)$. Given any $u_0, u_1 \in B_r^\alpha(\mathcal{M})$, the following estimates hold.

- (i) $\|\tilde{V}(x, u_0(x))\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq \tilde{\kappa} \|u_0\|_{\mathcal{C}^\alpha(\mathcal{M})}^2$,
- (ii) $\|\tilde{V}(x, u_1(x)) - \tilde{V}(x, u_0(x))\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq \tilde{\kappa} r \|u_1 - u_0\|_{\mathcal{C}^\alpha(\mathcal{M})}$,

where $\tilde{\kappa} > 0$ is independent of r .

Proof. First, observe that $\mathcal{C}^\alpha(\mathcal{M})$ is closed under multiplication and that there exists a constant $C > 0$, depending on (\mathcal{M}, g) , such that for any $v, w \in \mathcal{C}^\alpha(\mathcal{M})$ there holds

$$\|vw\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq C \|v\|_{\mathcal{C}^\alpha(\mathcal{M})} \|w\|_{\mathcal{C}^\alpha(\mathcal{M})}.$$

Now, using the fact that $\tilde{V}(x, 0) = \partial_z \tilde{V}(x, 0) = 0$ we write

$$\tilde{V}(x, u_k(x)) = \frac{1}{2} \int_C \partial_z^2 \tilde{V}(x, z) (u_k(x) - z) dz, \quad k = 0, 1,$$

where C is a path connecting 0 to $u(x)$ and the integral is in the sense of the $\mathcal{C}^\alpha(\mathcal{M})$ norm limits of the Riemann partial sums. Applying the \mathcal{C}^α norm we deduce that

$$\|\tilde{V}(x, u_k(x))\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq \left(C_1 \sup_{|z| \leq r} \|\partial_z^2 \tilde{V}(x, z)\|_{\mathcal{C}^\alpha(\mathcal{M})} \right) \|u_k\|_{\mathcal{C}^\alpha(\mathcal{M})}^2.$$

Similarly we have,

$$\|\tilde{V}(x, u_1(x)) - \tilde{V}(x, u_0(x))\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq \left(C_2 \sup_{|z| \leq r} \|\partial_z^2 \tilde{V}(x, z)\|_{\mathcal{C}^\alpha(\mathcal{M})} \right) r \|u_1 - u_0\|_{\mathcal{C}^\alpha(\mathcal{M})}.$$

for some $C_1, C_2 > 0$. Finally, using smoothness of $V(x, z)$ with respect to z , we deduce that

$$\sup_{|z| \leq r} \|\partial_z^2 \tilde{V}(x, z)\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq C_3,$$

for some constant $C_3 > 0$ independent of r , since $0 < r < 1$. The claim follows immediately by combining the preceding three bounds. \square

Proof of Proposition 1. We start by fixing

$$r_0 < \min \left\{ \frac{1}{1 + 3\kappa}, \frac{1}{4\tilde{\kappa}\kappa(1 + \kappa^2)}, \frac{1}{\tilde{\kappa}(2\kappa + 1)}, \frac{1}{\tilde{\kappa}(\kappa + 1)^2} \right\}, \quad r_1 = (\kappa + 1)r_0.$$

First we show existence of a solution $u \in B_{r_1}^\alpha(\mathcal{M})$. Write $u = \mathcal{G}_{V_1}^D f + \tilde{u}$ and observe that there exists a one to one correspondence between $\mathcal{C}^{2,\alpha}(\mathcal{M})$ solutions to equation (2) and solutions to the integral equation

$$(8) \quad \tilde{u} = -\mathcal{G}_{V_1}^S \left(\tilde{V}(x, \mathcal{G}_{V_1}^D f + \tilde{u}) \right) =: T_f(\tilde{u}).$$

Next noting that $r_0 < 1$, we may apply Lemma 1, and this together with the bound (7) yields

$$(9) \quad \|T_f v\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq 2\tilde{\kappa}\kappa \left(\|v\|_{\mathcal{C}^\alpha(\mathcal{M})}^2 + \|\mathcal{G}_{V_1}^D f\|_{\mathcal{C}^\alpha(\mathcal{M})}^2 \right) \quad \forall v \in B_{r_0}^\alpha(\mathcal{M}).$$

Applying (7) again and noting that $r_0 < \frac{1}{4\tilde{\kappa}\kappa(\kappa^2 + 1)}$, we deduce that T_f maps the closed set $\mathcal{B}_{r_0}^\alpha(\mathcal{M})$ to itself. Additionally, one can verify in the same way that T_f is a contraction mapping on $B_{r_0}^\alpha(\mathcal{M})$. The Banach fixed point theorem applies and

we conclude that there exists a solution $\tilde{u} \in B_{r_0}^\alpha(\mathcal{M})$ to equation (8). Observe subsequently that $u \in \mathcal{C}^{2,\alpha}(\mathcal{M})$ defined above solves (2). Applying (9) we have

$$\|\tilde{u}\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq 2\tilde{\kappa}\kappa \left(\|\tilde{u}\|_{\mathcal{C}^\alpha(\mathcal{M})}^2 + \|\mathcal{G}_{V_1}^D f\|_{\mathcal{C}^\alpha(\mathcal{M})}^2 \right) \leq \frac{1}{2} \|\tilde{u}\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} + 2\tilde{\kappa}\kappa^3 \|f\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})}^2,$$

Thus yielding the continuity estimate $\|u\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq (\kappa + 1)\|f\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})}$. This latter estimate also shows that $u \in B_{r_1}^\alpha(\mathcal{M})$.

Next we show uniqueness. Suppose for contrary that $u_1, u_2 \in B_{r_1}^\alpha(\mathcal{M})$ with $u_1 \neq u_2$ solving equation (2). Define \tilde{u}_k for $k = 1, 2$ as above and note that $\tilde{u}_k = T_f \tilde{u}_k$. Since $r_1 < 1$, Lemma 1 applies to obtain

$$\|\tilde{u}_k\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq \|u_k\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} + \|\mathcal{G}_{V_1}^D f\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq r_2,$$

where $r_2 = (1 + 2\kappa)r_0$. Finally, since $r_0 < \frac{1}{1+3\kappa}$, we can apply Lemma 1 again to deduce that T_f is a contraction mapping on the set $B_{r_2}^\alpha(\mathcal{M})$. Therefore

$$\|\tilde{u}_1 - \tilde{u}_2\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} = \|T_f \tilde{u}_1 - T_f \tilde{u}_2\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq \tilde{\kappa}(2\kappa + 1)r_0 \|\tilde{u}_1 - \tilde{u}_0\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} < \|\tilde{u}_1 - \tilde{u}_0\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})},$$

which is a contradiction. \square

2.2. Multiple-fold linearization method. We have established that the forward problem (2) is well-posed, and therefore also the DN map (4) is well-defined. In order to prove Theorems 1–2, we will work with families of Dirichlet datum f that will be arbitrarily small with respect to the $\mathcal{C}^{2,\alpha}(\partial\mathcal{M})$ norm, and are therefore only interested in the behavior of Λ_V near $f \equiv 0$. To set this idea in motion, let $m \in \mathbb{N}$ and consider a parameter $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{C}^m$ and a family of Dirichlet datum

$$(10) \quad f_\varepsilon = \sum_{k=1}^m \varepsilon_k f_k,$$

where $f_k \in \mathcal{C}^{2,\alpha}(\partial\mathcal{M})$ are fixed. Clearly, for all $|\varepsilon|$ sufficiently small, there exists a unique solution u_ε to equation (2) subject to Dirichlet data f_ε .

2.2.1. Analytic dependence on ε . Next, we prove that u_ε is analytic in a neighborhood of $\varepsilon = 0$, that is to say u_ε admits a power series representation with respect to the parameter ε in the $\mathcal{C}^{2,\alpha}(\mathcal{M})$ topology. It suffices to show that u_ε is analytic with respect to each ε_k for $k = 1, \dots, m$ (see for example [38, Theorem 1.2.25]). To this end we prove that given a fixed family $\{f_k\}_{k=1}^m$ and any fixed ε in a sufficiently small neighborhood of the origin in \mathbb{C}^m , the limit

$$(11) \quad \lim_{h \rightarrow 0} \frac{u_{\varepsilon + h e_l} - u_\varepsilon}{h}$$

exists in the $\mathcal{C}^{2,\alpha}(\mathcal{M})$ sense, where $h \in \mathbb{C}$ and e_l denotes the l^{th} unit vector in \mathbb{C}^m with $l = 1, \dots, m$.

As a first step, we proceed to prove that for all $|\varepsilon|$ small enough and all $|h| < |\varepsilon|$ there holds

$$(12) \quad \|u_{\varepsilon+he_l} - u_\varepsilon\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C|h|,$$

where $C > 0$ is independent of ε and h . We begin by observing that for $|\varepsilon|$ small and all $|h| < |\varepsilon|$, we can apply Proposition 1 to obtain the estimate

$$(13) \quad \|u_\varepsilon\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} + \|u_{\varepsilon+he_l}\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C|\varepsilon|,$$

for some constant $C > 0$ independent of ε and h . Next, we use equation (8) to write

$$\begin{aligned} u_\varepsilon &= \mathcal{G}_{V_1}^D f_\varepsilon - \mathcal{G}_{V_1}^S(\tilde{V}(x, u_\varepsilon)), \\ u_{\varepsilon+he_l} &= \mathcal{G}_{V_1}^D f_{\varepsilon+he_l} - \mathcal{G}_{V_1}^S(\tilde{V}(x, u_{\varepsilon+he_l})). \end{aligned}$$

Subtracting these two equations and applying (ii) in Lemma 1 together with (13), it follows that

$$\|u_{\varepsilon+he_l} - u_\varepsilon\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C \left(|h| \|f_l\|_{\mathcal{C}^{2,\alpha}(\partial\mathcal{M})} + |\varepsilon| \|u_{\varepsilon+he_l} - u_\varepsilon\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \right),$$

where $C > 0$ is independent of h and ε . Finally, the bound (12) follows from this estimate for ε sufficiently small.

Next, we proceed to show the main claim that the limit in (11) exists. Since 0 is not a Dirichlet eigenvalue for \mathcal{P}_{V_1} , it follows that the same is true for the operator $\mathcal{P}_{\partial_z V(\cdot, u_\varepsilon)}$, given that $|\varepsilon|$ is sufficiently small. We subsequently define $\mathcal{G}_\varepsilon^D$ and $\mathcal{G}_\varepsilon^S$ analogously to $\mathcal{G}_{V_1}^D$ and $\mathcal{G}_{V_1}^S$, corresponding to equation (6) with potential $\partial_z V(\cdot, u_\varepsilon)$ in place of V_1 . We write

$$\tilde{u}_h := u_{\varepsilon+he_l} - u_\varepsilon - h \mathcal{G}_\varepsilon^D f_l.$$

The function \tilde{u}_h satisfies the equation

$$\tilde{u}_h = -\mathcal{G}_\varepsilon^S(\tilde{V}_{\varepsilon,h}(x))$$

where

$$\tilde{V}_{\varepsilon,h}(x) = V(x, u_{\varepsilon+he_l}(x)) - V(x, u_\varepsilon(x)) - \partial_z V(x, u_\varepsilon(x))(u_{\varepsilon+he_l} - u_\varepsilon).$$

Using the smoothness of $V(x, z)$ with respect to z and analogously to Lemma 1 we deduce that there exists $C_\varepsilon > 0$ such that

$$(14) \quad \|\tilde{V}_{\varepsilon,h}(x)\|_{\mathcal{C}^\alpha(\mathcal{M})} \leq C_\varepsilon \|u_{\varepsilon+he_l} - u_\varepsilon\|_{\mathcal{C}^\alpha(\mathcal{M})}^2.$$

Thus, for h sufficiently small, by using the bounds (7), (12) and (14), we obtain

$$\|\tilde{u}_h\|_{\mathcal{C}^{2,\alpha}(\mathcal{M})} \leq C_\varepsilon |h|^2$$

for some $C > 0$ independent of h . Hence,

$$(15) \quad \lim_{h \rightarrow 0} \frac{u_{\varepsilon+he_l} - u_\varepsilon}{h} = \mathcal{G}_\varepsilon^D f_l$$

holds in the $\mathcal{C}^{2,\alpha}$ sense, proving that u_ε depends analytically in each of its parameters ε_l in a neighborhood of $\varepsilon = 0$ in \mathbb{C}^m .

2.2.2. Non-linear interaction of linearized solutions. Let us now use this linearization technique to first show that Λ_V determines the Dirichlet to Neumann map $\Lambda_{V_1}^{\text{lin}}$ associated to the linear operator \mathcal{P}_{V_1} . Of course this is a somewhat redundant argument as V_1 will be assumed to be known for us, but nevertheless this simple case will shed some light on the higher order linearization arguments. Let $m = 1$ so that $\varepsilon \in \mathbb{C}$ and write $f_\varepsilon = \varepsilon f$ for some $f \in \mathcal{C}^{2,\alpha}(\partial\mathcal{M})$. By (15) there holds $\partial_\varepsilon u_\varepsilon|_{\varepsilon=0} = \mathcal{G}_{V_1}^D f$. Moreover, since $u_\varepsilon \in \mathcal{C}^{2,\alpha}(\mathcal{M})$ and since $\partial_\nu u_\varepsilon|_{\partial\mathcal{M}}$ is determined through the map Λ_V , we can simply write

$$\partial_\nu \mathcal{G}_{V_1}^D f = \partial_\nu \partial_\varepsilon u_\varepsilon|_{\varepsilon=0} = \partial_\varepsilon \Lambda_V f_\varepsilon|_{\varepsilon=0},$$

which shows that $\Lambda_{V_1}^{\text{lin}} f = \partial_\varepsilon \Lambda_V(\varepsilon f)|_{\varepsilon=0}$.

We can also use this linearization technique to identify interactions for solutions to the linearized equation $\mathcal{P}_{V_1} u = 0$. Indeed, let us consider $\varepsilon \in \mathbb{C}^m$ with $m \geq 2$ and $\{f_k\}_{k=1}^m \subset \mathcal{C}^{2,\alpha}(\partial\mathcal{M})$. Since u_ε solves equation (2) with Dirichlet data f_ε given by (10) and since the dependence on ε is analytic, it follows that given any multi-index $\beta \in \{0, 1, \dots\}^m$ with $|\beta| = \beta_1 + \dots + \beta_m \geq 1$, the function $\partial_\varepsilon^\beta u_\varepsilon|_{\varepsilon=0}$ solves

$$\mathcal{P}_{V_1}(\partial_\varepsilon^\beta u_\varepsilon|_{\varepsilon=0}) = F_\beta$$

with homogeneous Dirichlet boundary conditions, where F_β depends on $V_1, \dots, V_{|\beta|}$ and $\partial_\varepsilon^{\beta'} u_\varepsilon|_{\varepsilon=0}$ with $|\beta'| = 1, \dots, |\beta| - 1$. Using a simple induction argument it follows that $\partial_\varepsilon^\beta u_\varepsilon|_{\varepsilon=0}$ only depends on $V_1, \dots, V_{|\beta|}$ and f_1, \dots, f_m .

We now consider a particular term in the power series expansion of u_ε near $\varepsilon = 0$ associated to the multi-index $\beta = (1, 1, \dots, 1)$ and define

$$L_{f_1, \dots, f_m} := -\frac{\partial^m}{\partial \varepsilon_1 \dots \partial \varepsilon_m} u_\varepsilon|_{\varepsilon=0}.$$

It follows that the function $L_{f_1 \dots f_m}$ satisfies the equation

$$(16) \quad \begin{cases} \mathcal{P}_{V_1} L_{f_1 \dots f_m} = V_m \prod_{k=1}^m \mathcal{G}_{V_1}^D f_k + H_{f_1, \dots, f_m}, & \forall x \in \mathcal{M} \\ L_{f_1 \dots f_m} = 0 & \forall x \in \partial\mathcal{M} \end{cases}$$

where $H_{f_1, \dots, f_m} \in \mathcal{C}^{2,\alpha}(\mathcal{M})$ is a function that only depends on V_1, \dots, V_{m-1} and $\partial_\varepsilon^{\beta'} u_\varepsilon|_{\varepsilon=0}$ with $|\beta'| = 1, \dots, m-1$. Using the argument in the previous paragraph, it follows that H_{f_1, \dots, f_m} only depends on V_1, \dots, V_{m-1} and f_1, \dots, f_m . In the particular case $m = 2$, we have $H_{f_1, f_2} \equiv 0$. Finally, let us emphasize that since the Dirichlet-to-Neumann map, Λ_V , determines $\partial_\nu u_\varepsilon|_{\partial\mathcal{M}}$, it will also uniquely determine the values $\partial_\nu L_{f_1, \dots, f_m}|_{\partial\mathcal{M}}$.

2.3. Reduction to the case $c \equiv 1$. This subsection is concerned with showing that one can without any loss in generality consider the case where $c(x^0, x') \equiv 1$ on \mathcal{M} . Let us define $\hat{g} = (dx^0)^2 + g(x)$ so that $g = c\hat{g}$. Using the transformation law of the Laplace-Beltrami operator under conformal rescalings of the metric, we write

$$(17) \quad c^{\frac{n+2}{4}}(-\Delta_g u + V(x, u)) = -\Delta_{\hat{g}} v + \hat{V}(x, v),$$

where $v = c^{\frac{n-2}{4}} u$ and $\hat{V}(x, v) = c^{\frac{n+2}{4}} V(x, c^{-\frac{n-2}{4}} v) - (c^{\frac{n-2}{4}} \Delta_g c^{-\frac{n-2}{4}}) v$. It can be easily checked that conditions (i)–(iv) also hold for the function $\hat{V}(x, z)$. Moreover, if V_1 is smooth then so is the function \hat{V}_1 .

Let $r'_0, r'_1 > 0$ and consider solutions $v \in B_{r'_1}^\alpha(\mathcal{M})$ to equation

$$(18) \quad \begin{cases} -\Delta_{\hat{g}} v + \hat{V}(x, v) = 0, & \forall x \in \mathcal{M} \\ v = f \in B_{r'_0}^\alpha(\partial\mathcal{M}) & \forall x \in \partial\mathcal{M} \end{cases}$$

It can be easily verified that for (r'_0, r'_1) small depending on (r_0, r_1) and $\|c\|_{C^3(\mathcal{M})}$, equation (18) has a unique solution given by $v = c^{\frac{n-2}{4}} u$ where u is the unique solution to equation (2) subject to Dirichlet data $c^{-\frac{n-2}{4}} f$. We can therefore uniquely determine the DN map $\Lambda_{\hat{V}}$ for equation (18) from the DN map Λ_V for equation (2) and henceforth consider the problem of determining \hat{V} from $\Lambda_{\hat{V}}$. Finally note that once uniqueness is proved for \hat{V} , we can immediately deduce uniqueness for V . Thus, without loss of generality we will make the standing assumption throughout the rest of the paper that $c \equiv 1$.

2.4. A Carleman estimate. This section is concerned with providing a right inverse for the following differential operator

$$\mathcal{L}_\lambda \cdot := e^{-\lambda x^0} (-\Delta_g + V_1) (e^{\lambda x^0} \cdot),$$

where $\lambda \in \mathbb{R}$ and $|\lambda|$ is sufficiently large, with suitable continuity estimates in $H^k(\mathcal{M})$ norm for any fixed $k \in \mathbb{N}$ (see Proposition 2).

We start by introducing some notation. Choose an arbitrarily small auxiliary extension of the manifold M into a smooth closed manifold \hat{M} without boundary, and smoothly extend the metric $g(x')$ to \hat{M} . We also extend V_1 smoothly to $\hat{T} = I \times \hat{M}$ so that $V_1 \in C_c^\infty(\hat{T})$. Here, I is the interval in Definition 1. Next, for any $m \in \mathbb{Z}$, let E be a bounded linear Sobolev extension operator $E : H^m(\mathcal{M}) \rightarrow H^m(\hat{T})$ and denote by $\{\psi_l\}_{l \in \mathbb{N}}$, the set of orthonormal eigenfunctions for the Laplace operator on (\hat{M}, g) , so that $-\Delta_g \psi_l = \mu_l \psi_l$ with $\{\mu_l\}_{l \in \mathbb{N}}$ denoting the eigenvalues.

We have the following proposition.

Proposition 2. *Let $k \in \mathbb{N}$ and suppose that (\mathcal{M}, g) is a CTA manifold as above and that $V_1 \in C^\infty(\mathcal{M})$. Then there exists $\lambda_0 > 0$, depending on (\mathcal{M}, g) , V_1 and k , such*

that for all $|\lambda| > \lambda_0$ with $\lambda^2 \notin \{\mu_n\}_{n \in \mathbb{N}}$, the equation

$$(19) \quad \mathcal{L}_\lambda r = f, \quad f \in H^k(\mathcal{M}),$$

admits some solution $u \in H^k(\mathcal{M})$, satisfying the estimate

$$\|r\|_{H^m(\mathcal{M})} \leq C \lambda^{-1} \|f\|_{H^m(\mathcal{M})} \quad \text{for } m = 0, \dots, k,$$

where the constant $C > 0$ is independent of λ .

Let us remark that in the case $k = 0$, this is well-known, see for instance [7, Proposition 4.4] and [37, Theorem 4.1]. We will present the proof here, as we need existence results with control on the $H^k(\mathcal{M})$ norm with k large. The proof here will be based on first extending to the infinite cylinder \hat{T} and then applying Fourier mode analysis with respect to the transversal manifold (M, g) . This is similar to the case $k = 0$ as presented in [37, Chapter 4]. Let us remark that it is also possible to use the Carleman estimates for the adjoint operator $\mathcal{L}_{-\lambda}$ shifted to negative Sobolev spaces (see for example [7, Lemma 4.3] and [24, Section 4]) and a standard duality argument to obtain similar right inverses. However, the estimates obtained using this approach will be in semi-classical norms. The estimates that we obtain by using the Fourier analysis approach are slightly stronger due to the fact that the operator \mathcal{L}_λ has constant coefficients with respect to the x^0 variable.

Proof of Proposition 2. We only provide the proof for the case $V_1 \equiv 0$. For the case that V_1 is smoothly supported in \hat{T} , the proof here together with the exact arguments as in [37, Theorem 4.1] yields the result. We begin by introducing an operator S_a defined for any non-zero $a \in \mathbb{R}$, and any $h \in H^k(\mathbb{R})$, as follows.

$$(S_a h)(x) = \mathcal{F}^{-1} \left(\frac{(\mathcal{F}h)(\xi)}{i\xi + a} \right),$$

where \mathcal{F} denotes the Fourier transform on \mathbb{R} . Using similar arguments as in [37, Proposition 4.4], we have that for all $m = 0, 1, \dots, k$ and all $\delta > \frac{1}{2}$:

$$(20) \quad \begin{aligned} \|S_a h\|_{H^m(\mathbb{R})} &\leq C a^{-1} \|h\|_{H^m(\mathbb{R})}, \quad \forall |a| > 1. \\ \|S_a h\|_{H_{-\delta}^m(\mathbb{R})} &\leq C \|h\|_{H_\delta^m(\mathbb{R})} \quad \forall a \neq 0, \end{aligned}$$

with C independent of the parameter a and $\|h\|_{H_\delta^m(\mathbb{R})}^2 := \sum_{j=0}^m \|(1 + |x|^2)^{\frac{\delta}{2}} \partial_x^j h\|_{L^2(\mathbb{R})}^2$.

Let $F := Ef$ with F compactly supported in \hat{T} . We begin by writing F in terms of the eigenfunctions of \hat{M} as follows

$$F(x^0, x') = \sum_{l \in \mathbb{N}} F_l(x^0) \psi_l(x').$$

In [37, Proposition 4.6], it was proved that the function

$$R(x^0, x') := \sum_{l \in \mathbb{N}} R_l(x^0) \psi_l(x'),$$

with

$$R_l = S_{\lambda + \sqrt{\mu_l}} S_{\lambda - \sqrt{\mu_l}} F_l$$

solves the equation (19) on the larger set \hat{T} and satisfies the estimate

$$\|R\|_{H^2_{-\delta}(\hat{T})} \lesssim \lambda \|F\|_{L^2_{\delta}(\hat{T})},$$

where we are using the notation

$$\|\cdot\|_{H^m_{\delta}(\hat{T})}^2 := \sum_{j=0}^m \|(1 + |x^0|^2)^{\frac{\delta}{2}} D^j \cdot\|_{L^2(\hat{T})}^2.$$

Now defining $r := \mathbb{I}_{\mathcal{M}} R$, with $\mathbb{I}_{\mathcal{M}}$ denoting the characteristic function of \mathcal{M} , it is clear that $r \in H^2(\mathcal{M})$ solves equation (19) on \mathcal{M} .

Let us proceed to prove the claimed bound in the statement of the proposition. We start by noting that given any $p = 0, \dots, k$ and $m = 0, \dots, p$, there holds

$$\begin{aligned} \left\| \sum_{l \in \mathbb{N}} (\partial_{x^0}^m R_l) \mu_l^{\frac{p-m}{2}} \psi_l \right\|_{L^2_{-\delta}(\hat{T})}^2 &= \sum_{l \in \mathbb{N}} \mu_l^{p-m} \|S_{-\lambda + \sqrt{\mu_l}} S_{\lambda + \sqrt{\mu_l}} \partial_{x^0}^m F_l\|_{L^2_{-\delta}(\hat{T})}^2 \\ &\lesssim \lambda^{-2} \sum_{l \in \mathbb{N}} \mu_l^{p-m} \|\partial_{x^0}^m F_l\|_{L^2(\hat{T})}^2, \end{aligned}$$

where in the last step, we have used the bound (20), the fact that $|\lambda - \sqrt{\mu_n}| > 0$ and that F is compactly supported in \hat{T} . Observing that

$$\partial_{x^0}^m F_l(x^0) = \int_{\hat{M}} \partial_{x^0}^m F(x^0, x') \psi_l(x') dV_g,$$

together with the fact that \hat{M} is closed, we deduce that

$$\left\| \sum_{l \in \mathbb{N}} (\partial_{x^0}^m R_l) \mu_l^{\frac{p-m}{2}} \psi_l \right\|_{L^2_{-\delta}(\hat{T})} \lesssim \lambda^{-1} \|\partial_{x^0}^m (-\Delta_g)^{\frac{p-m}{2}} F\|_{L^2(\hat{T})}.$$

To complete the proof, we write

$$\begin{aligned} \|r\|_{H^k(\mathcal{M})} &\leq \|R\|_{H^k(\hat{T})} \lesssim \sum_{p=0}^k \sum_{m=0}^p \left\| \sum_{l \in \mathbb{N}} (\partial_{x^0}^m R_l) \mu_l^{\frac{p-m}{2}} \psi_l \right\|_{L^2(\hat{T})} \\ &\lesssim \lambda^{-1} \|F\|_{H^k(\hat{T})} \lesssim \lambda^{-1} \|f\|_{H^k(\mathcal{M})}. \end{aligned}$$

□

3. THE JACOBI WEIGHTED RAY TRANSFORM

This section is concerned with the introduction of a geometrical data related to the transversal manifold (M, g) that will appear later in the proof of Theorem 1–2. Before proceeding, let us introduce some notation, following [5, Section 1.2]. Given a maximal unit speed geodesic $\gamma(t) \subset M$ with $t \in [\tau_-, \tau_+]$, we define the *orthogonal complement*, $\dot{\gamma}(t)^\perp$, at the point $\gamma(t)$ as the set

$$\dot{\gamma}(t)^\perp := \{v \in T_{\gamma(t)}M \mid g(\dot{\gamma}(t), v) = 0\}.$$

We also define the $(1, 1)$ -tensor $\Pi_\gamma(t) = \Pi_i^j(t) \frac{\partial}{\partial x^j} \otimes dx^i$ to be the projection from $T_{\gamma(t)}M$ onto $\dot{\gamma}^\perp(t)$. Finally, we say that a $(1, 1)$ -tensor $L(t)$ along γ is *transversal* if $\Pi_\gamma L \Pi_\gamma = L$. Transversal $(1, 1)$ -tensors can be viewed as linear maps from $\dot{\gamma}^\perp(t)$ to itself. Now, given such a tensor L , we consider the complex Jacobi equation

$$(21) \quad \frac{D^2}{dt^2} L(t) - K(t)L(t) = 0,$$

with the initial condition

$$L(0) = L_0, \quad \dot{L}(0) = L_1.$$

Here,

$$K = K_j^i \frac{\partial}{\partial x^i} \otimes dx^j, \quad K_j^i = g^{ik} R_{kj}$$

where R denotes the Ricci tensor. We recall from [5] that if a complex $(1, 1)$ -tensor L solves the complex Jacobi equation and L_0, L_1 are transversal, then $L(t)$ is transversal along $\gamma(t)$ for all $t \in [\tau_-, \tau_+]$.

For each maximal geodesic γ in M , we let \mathbb{Y}_γ denote the set of all transversal $(1, 1)$ -tensors $Y(t)$ that solve equation (21) subject to the additional constraint that

$$(22) \quad \begin{aligned} &Y(\tau_0) \text{ is non-degenerate, } \dot{Y}(\tau_0)Y(\tau_0)^{-1} \text{ is symmetric} \\ &\text{and } \Im(\dot{Y}(\tau_0)Y(\tau_0)^{-1}) > 0 \text{ for some } \tau_0 \in [\tau_-, \tau_+]. \end{aligned}$$

Here, \Im denotes the imaginary part and we recall that $[\tau_-, \tau_+]$ is the interval of definition associated to the maximal unit speed geodesic $\gamma(t)$ in M . We now define, for all $Y \in \mathbb{Y}_\gamma$, the Jacobi weighted ray transform of the first and second kind, $\mathcal{J}_Y^{(1)}$ and $\mathcal{J}_Y^{(2)}$, as follows.

$$(23) \quad \begin{aligned} \mathcal{J}_Y^{(1)} f &:= \int_{\tau_-}^{\tau_+} f(\gamma(t)) (\det Y(t))^{-\frac{1}{2}} dt \quad \forall f \in \mathcal{C}(M), \\ \mathcal{J}_Y^{(2)} f &:= \int_{\tau_-}^{\tau_+} f(\gamma(t)) |\det Y(t)|^{-1} dt \quad \forall f \in \mathcal{C}(M). \end{aligned}$$

The following lemma guarantees that $\mathcal{J}_Y^{(j)}$, $j = 1, 2$ is well-defined.

Lemma 2. *For all $Y \in \mathbb{Y}_\gamma$ we have that $Y(t)$ is non-degenerate, $\dot{Y}(t)Y(t)^{-1}$ is symmetric and $\Im(\dot{Y}(t)Y(t)^{-1}) > 0$ for all $t \in [\tau_-, \tau_+]$.*

We refer the reader to [21, Lemma 2.56] for the proof of this lemma. In the following subsections, we will study the injectivity of the Jacobi ray transforms of the first and second kind along a single maximal geodesic γ . Before presenting the main results and their proofs, let us give a heuristic discussion to shed some light on the approach. Recall that the matrices $Y(t)$ solve second order ODEs along the geodesics. Therefore it is possible to choose weights in the Jacobi transform that have a limiting singular behavior at any fixed point p . We will see that this singularity will also appear at all points that are conjugate to p on γ .

In the case of the Jacobi transform of the second kind, the transform will have a limiting singularity at p and its conjugate points and local information can be obtained in all dimensions under the admissibility assumption (Proposition 4). In the case of the Jacobi ray transform of the first kind, the proof is more delicate as there is no limiting singularity in the transform. Here, the imaginary part of the transform can be localized to deduce injectivity. We have presented injectivity of the Jacobi transform of the first kind along a single geodesic, only in dimensions two and three and left the higher dimensional cases open (Proposition 3). This will be further discussed in Remark 1.

3.1. Inversion of Jacobi weighted ray transform of the first kind. This subsection is concerned with the following injectivity result.

Proposition 3. *Suppose (M, g) is a two or three dimensional compact smooth Riemannian manifold with boundary. Suppose that γ is a maximal geodesic in M that contains no conjugate points. Let $f \in C(M; \mathbb{R})$. The following injectivity result holds:*

$$\mathcal{J}_Y^{(1)} f = 0, \quad \forall Y \in \mathbb{Y}_\gamma \quad \implies \quad f(\gamma(t)) = 0 \quad \forall t \in [\tau_-, \tau_+].$$

Proof of Proposition 3. For consistency of notation, we will denote the dimension of M by $n - 1$ with $n \in \{3, 4\}$ throughout this proof.

Let us first consider the case when $n = 3$. In this case the Jacobi matrices will simply be complex valued scalar functions along the geodesic, that solve (21) on the maximal interval $[\tau_-, \tau_+]$. It is in fact possible to consider this equation on a slightly larger interval $[\tau'_-, \tau'_+]$ after choosing a smooth extension of K . We consider a family of solutions $Y^\epsilon : [\tau'_-, \tau'_+] \rightarrow \mathbb{C}$ to (21) with $\epsilon > 0$, subject to the initial data

$$(24) \quad Y^\epsilon(\tau'_-) = -i\epsilon \quad \text{and} \quad \dot{Y}^\epsilon(\tau'_-) = 1.$$

Observe that $Y^\epsilon \in \mathbb{Y}_\gamma$ since condition (22) is satisfied for $\tau_0 = \tau'_-$ for all $\epsilon > 0$. Note also that

$$Y^\epsilon = X - i\epsilon Z$$

where X and Z are real-valued solutions to (21) subject to $X(\tau'_-) = 0$, $\dot{X}(\tau'_-) = 1$ and $Z(\tau'_-) = 1$, $\dot{Z}(\tau'_-) = 0$. We also record that since γ contains no conjugate points on $[\tau_-, \tau_+]$, it follows that $X(t)$ is strictly positive for $t \in [\tau_-, \tau_+]$.

Recall that by the hypothesis of the proposition,

$$\mathcal{J}_{Y^\epsilon}^{(1)} f = 0$$

for all $\epsilon > 0$. This equation reduces to

$$\int_{\tau_-}^{\tau_+} f(\gamma(t))X(t)^{-\frac{1}{2}}(1 - i\epsilon\tilde{X})^{-\frac{1}{2}} dt = 0,$$

where $\tilde{X}(t) = Z(t)X(t)^{-1}$. Applying the Taylor series approximation of $(1 - i\epsilon\tilde{X})^{-\frac{1}{2}}$ near $\epsilon = 0$ we deduce that

$$(25) \quad \int_{\tau_-}^{\tau_+} f(\gamma(t))X(t)^{-\frac{1}{2}}\tilde{X}(t)^k dt = 0, \quad \text{for } k = 0, 1, \dots$$

We claim that $\tilde{X}(t)$ is strictly decreasing on $[\tau_-, \tau_+]$. To see this, observe that

$$\dot{\tilde{X}}(t) = (\dot{Z}(t)X(t) - \dot{X}(t)Z(t))X(t)^{-2} = W_{Z,X}(t)X(t)^{-2},$$

where $W_{Z,X}(t)$ denotes the Wronskian corresponding to the Jacobi equation (21) and as such satisfies

$$W_{Z,X}(t) = W_{Z,X}(\tau'_-) = -1, \quad t \in [\tau_-, \tau_+]$$

where we are using the fact that X, Z are scalar functions that solve a second order ODE and therefore their Wronskian is constant. Using this observation, we conclude that

$$\dot{\tilde{X}}(t) < 0, \quad \forall t \in [\tau_-, \tau_+],$$

implying that \tilde{X} is strictly decreasing on $[\tau_-, \tau_+]$. We can consequently define the new variable $\tilde{t} = \tilde{X}(t)$ and rewrite (25) in terms of \tilde{t} as

$$\int_{\tilde{X}(\tau_+)}^{\tilde{X}(\tau_-)} f(\gamma(\tilde{X}^{-1}(\tilde{t}))) \frac{X(\tilde{X}^{-1}(\tilde{t}))^{-\frac{1}{2}}}{\tilde{X}'(\tilde{X}^{-1}(\tilde{t}))} \tilde{t}^k d\tilde{t} = 0, \quad \text{for } k = 0, 1, \dots$$

Finally, using the Stone-Weierstrass theorem, and since $f \in \mathcal{C}(M)$, we conclude that f must identically vanish along γ , thus concluding the proof for the case $n = 3$.

Let us now consider the case that $n = 4$. We consider an arbitrary point $p \in \gamma$ and assume without loss of generality that $p = \gamma(0)$. Let $\{v_1, v_2\} \subset \dot{\gamma}^\perp(0)$ be an orthonormal basis for $\dot{\gamma}(0)^\perp$ and for each $\epsilon > 0$, consider the unique transversal $(1, 1)$ -tensor Y^ϵ solving the Jacobi equation (21) subject to initial conditions

$$(26) \quad Y_k^\epsilon(0) = -i\epsilon v_k \quad \text{and} \quad \dot{Y}_k^\epsilon(0) = v_k \quad \text{for } k = 1, 2,$$

where Y_k^ϵ is the k^{th} column of the tensor Y^ϵ at the point p with respect to the basis $\{v_1, v_2\}$. Observe that condition (22) holds at $t = 0$ for each $\epsilon > 0$, implying that $Y^\epsilon \in \mathbb{Y}_\gamma$. By the hypothesis of the proposition, we have

$$(27) \quad \mathcal{J}_{Y^\epsilon}^{(1)} f = 0, \quad \forall \epsilon > 0.$$

We can write $Y^\epsilon = X - i\epsilon Z$ where X, Z solve (21) subject to the initial conditions

$$X_k(0) = 0, \quad Z_k(0) = v_k \quad \text{and} \quad \dot{X}_k(0) = v_k, \quad \dot{Z}_k(0) = 0, \quad \text{for } k = 1, 2.$$

Our aim now is to study the behavior of $\mathcal{J}_{Y^\epsilon}^{(1)} f$ as ϵ approaches zero. Observe that Y^ϵ with $\epsilon = 0$ is singular at $t = 0$. This singularity will allow us to obtain information about the value of f at the point p . However, Y^ϵ is analytic with respect to ϵ and studying the asymptotic behavior of \mathcal{J}_{Y^ϵ} itself will not contain any local information about f on γ . To remedy this issue, we define

$$\mathcal{S}_\epsilon f = \mathcal{J}_{Y^\epsilon}^{(1)} f - \overline{\mathcal{J}_{Y^\epsilon}^{(1)} f},$$

and note that by the hypothesis of the proposition, and since f is real valued we have $\mathcal{S}_\epsilon f = 0$. We now choose a small positive parameter ζ and assume that $0 < \epsilon < \zeta$. In what follows, we will study the limiting behavior of $\mathcal{S}_\epsilon f$ as ϵ approaches zero while ζ is fixed.

Writing $\{e_1(t), e_2(t)\}$ to denote the parallel transport of the orthonormal basis $\{v_1, v_2\}$ along γ , it is easy to see that $Y^\epsilon(t)$ can be thought of as a two by two matrix with respect to the basis $\{e_1(t), e_2(t)\}$. Next, using the Taylor series approximation for the matrix Y^ϵ near $t = 0$, we deduce that given any $|t| < \zeta$ and all $\epsilon < \zeta$, there holds

$$(28) \quad \begin{cases} |Y_{ij}^\epsilon(t)| \leq C_0 t^2 & \text{for } i \neq j \text{ and } i, j = 1, 2 \\ |Y_{jj}^\epsilon(t) - (t - i\epsilon)| \leq C_0 t^2, & \text{for } j = 1, 2, \end{cases}$$

where $C_0 > 0$ is independent of ζ and ϵ . Here, we are considering the matrix Y^ϵ with respect to the basis $\{e_1, e_2\}$. Applying these estimates to the expression for $\det Y^\epsilon$, and noting that $|t - i\epsilon| > |t|$, we deduce that there exists a constant $C_1 > 0$ independent of ϵ, ζ , such that

$$\det Y^\epsilon(t) = (t - i\epsilon)^2(1 + r^\epsilon(t)) \quad \text{with} \quad |r^\epsilon(t)| \leq C_1 |t|,$$

for all $t \in (-\zeta, \zeta)$. Consequently, there exists a constant $C_2 > 0$ independent of ϵ, ζ such that

$$(29) \quad \left| (\det Y^\epsilon)^{-\frac{1}{2}} - \overline{(\det Y^\epsilon)^{-\frac{1}{2}}} - 2i\Im((t - i\epsilon)^{-1}) \right| \leq C_2,$$

for all $t \in (-\zeta, \zeta)$.

Let us now consider the interval $[\tau_-, \tau_+] \setminus (-\zeta, \zeta)$. First, note that $\det X(0) = 0$ and that by Definition 2, no point on $\gamma(t)$ is conjugate to $\gamma(0)$ for $t \neq 0$ (see for example [6, Section 5.5]). We deduce that

$$\det X \neq 0 \quad \text{on} \quad [\tau_-, \tau_+] \setminus (-\zeta, \zeta).$$

Note also that by applying the point-wise bounds in (28) for $\epsilon = 0$, it follows that,

$$|\det X(t) - t^2| \leq Ct^3, \quad \text{for all } |t| \leq \zeta,$$

for some C independent of ζ . Together with the fact that X is non-degenerate away from the origin, we conclude that

$$\det X(t) > 0 \quad \forall t \in [\tau_-, \tau_+] \setminus \{0\}.$$

Using this observation, we write

$$\det Y^\epsilon = \det X \det(I - i\epsilon ZX^{-1}) = \det X - i\epsilon \det X \operatorname{Tr}(ZX^{-1}) + \mathcal{O}(\epsilon^2)$$

where we applied the expansion formula for the characteristic polynomial of matrices in the last step. Since $\det X$ is strictly positive away from the origin, we can conclude that there exists a constant $C_\zeta > 0$ only depending on ζ , such that

$$(30) \quad \left| (\det Y^\epsilon)^{-\frac{1}{2}} - \overline{(\det Y^\epsilon)^{-\frac{1}{2}}} \right| \leq C_\zeta \epsilon \quad \text{for } t \in [\tau_-, \tau_+] \setminus (-\zeta, \zeta).$$

Let us now analyze the limiting behavior of $\mathcal{S}_\epsilon f$ as ϵ approaches zero. To this end, we begin by writing

$$0 = \mathcal{S}_\epsilon f = A_{\zeta, \epsilon}^{(1)} + A_{\zeta, \epsilon}^{(2)} + A_{\zeta, \epsilon}^{(3)},$$

where

$$(31) \quad \begin{aligned} A_{\zeta, \epsilon}^{(1)} &= \int_{-\zeta}^{\zeta} f(\gamma(0)) \left((\det Y^\epsilon)^{-\frac{1}{2}} - \overline{(\det Y^\epsilon)^{-\frac{1}{2}}} \right) dt, \\ A_{\zeta, \epsilon}^{(2)} &= \int_{-\zeta}^{\zeta} (f(\gamma(t)) - f(\gamma(0))) \left((\det Y^\epsilon)^{-\frac{1}{2}} - \overline{(\det Y^\epsilon)^{-\frac{1}{2}}} \right) dt \\ A_{\zeta, \epsilon}^{(3)} &= \int_{[\tau_-, \tau_+] \setminus (-\zeta, \zeta)} f(\gamma(t)) \left((\det Y^\epsilon)^{-\frac{1}{2}} - \overline{(\det Y^\epsilon)^{-\frac{1}{2}}} \right) dt. \end{aligned}$$

For the term $A_{\zeta, \epsilon}^{(1)}$, we use the bound (29) together with the estimate

$$(32) \quad \int_{-\zeta}^{\zeta} \Im \left((t - i\epsilon)^{-1} \right) dt = \int_{-\zeta}^{\zeta} |\Im \left((t - i\epsilon)^{-1} \right)| dt = \pi + \mathcal{O}(\epsilon),$$

to write

$$\left| A_{\zeta, \epsilon}^{(1)} - 2\pi i f(\gamma(0)) \right| \leq C_3 \|f\|_{L^\infty} \zeta + C_4(\zeta) \|f\|_{L^\infty} \epsilon,$$

for some $C_3 > 0$ independent of ζ , ϵ and $C_4(\zeta) > 0$ independent of ϵ . Here, we are using the fact that (29) is uniformly bounded in ζ . The appearance of ζ in the first term above is due to the length of the interval of integration in $A_{\zeta, \epsilon}^{(1)}$.

For the term $A_{\zeta,\epsilon}^{(2)}$, we use the bounds (29) and (32) again to obtain

$$\left| A_{\zeta,\epsilon}^{(2)} \right| \leq C_5 (\omega_f(\zeta) + \|f\|_{L^\infty} \zeta) + C_6(\zeta) \|f\|_{L^\infty} \epsilon,$$

for some C_5 independent of ζ , ϵ and $C_6(\zeta)$ independent of ϵ , where ω_f denotes the modulus of continuity for the function f at the point $\gamma(0)$. Finally, for the term $A_{\zeta,\epsilon}^{(3)}$, we use the bound (30) to write

$$\lim_{\epsilon \rightarrow 0} A_{\zeta,\epsilon}^{(3)} = 0.$$

We now return to $\mathcal{S}_\epsilon f$, letting ϵ approach zero. Using the last three estimates for $A_{\zeta,\epsilon}^{(k)}$, $k = 1, 2, 3$, we deduce that

$$|f(\gamma(0))| \leq (C_3 + C_5) \|f\|_{L^\infty} \zeta + C_5 \omega_f(\zeta).$$

Finally, letting ζ converge to zero, it follows that $f(p) = 0$. Since $p \in \gamma$ is arbitrary, it follows that $f(\gamma(t)) = 0$ for all $t \in [\tau_-, \tau_+]$. \square

Remark 1. The restriction on the dimension is mainly due to two technical difficulties. One has to do with the parity of the dimension. Namely, when the dimension of M is odd, an equation of the form (30) will no longer be valid. This is due to the fact that $\det X$ will change sign at the origin and the imaginary part of $\det Y^\epsilon$ will not become small in ϵ away from the region $(-\zeta, \zeta)$. This is the reason that we have pursued a different approach for proving the proposition when $\dim M = 2$. Another general issue with higher dimensions seems to be the fact that the bound (29) will no longer hold.

3.2. Inversion of Jacobi weighted ray transform of the second kind. This section is concerned with the proof of the following proposition.

Proposition 4. *Suppose (M, g) is a smooth compact Riemannian manifold with boundary. Let $p \in M$ and γ be a maximal geodesic passing through p that contains no conjugate points to p . Let $f \in \mathcal{C}(M; \mathbb{C})$. The following injectivity result holds:*

$$\mathcal{J}_Y^{(2)} f = 0, \quad \forall Y \in \mathbb{Y}_\gamma \quad \implies \quad f(p) = 0.$$

Proof of Proposition 4. We will again denote the dimension of M by $n-1$ with $n \geq 3$. We consider the unit-speed parametrization $\gamma : [\tau_-, \tau_+] \rightarrow M$ with $\gamma(0) = p$. Let $\{v_1, \dots, v_{n-2}\} \subset \dot{\gamma}^\perp(0)$ be an orthonormal basis and for each $\epsilon > 0$ sufficiently small, consider the unique $Y^\epsilon \in \mathbb{Y}_\gamma$ subject to

$$(33) \quad Y_k^\epsilon(0) = -i\epsilon v_k, \quad \text{and} \quad \dot{Y}_k^\epsilon(0) = v_k,$$

where Y_k^ϵ is the k^{th} column of the tensor Y^ϵ at the point p with respect to $\{v_k\}$ with $k = 1, 2, \dots, n-2$. Writing $\{e_k(t)\}_{k=1}^{n-2}$ to denote the parallel transport of the

orthonormal basis $\{v_k\}$ along γ , it is easy to see that $Y^\epsilon(t)$ can be thought of as a $(n-2) \times (n-2)$ matrix with respect to the basis $\{e_k(t)\}$ and that

$$Y^\epsilon(t) = X(t) - i\epsilon Z(t)$$

where

$$\begin{aligned} \ddot{X} - KX &= 0, & \ddot{Z} - KZ &= 0, \\ X(0) &= 0, & Z(0) &= Id \quad \text{and} \quad \dot{X}(0) = Id \quad \dot{Z}(0) = 0. \end{aligned}$$

Here, we have realized the $(1, 1)$ -tensors $K(t)$, $X(t)$ and $Z(t)$ as matrices with respect to the basis $\{e_k(t)\}_{k=1}^{n-2}$. As in the proof of Proposition 3 we remark that since there are no conjugate points $\gamma(t)$ to $\gamma(0)$ away from the origin $t = 0$, it follows that $X(t)$ is non-degenerate on $[\tau_-, \tau_+]$ away from $t = 0$.

Let $\zeta > 0$ be a small parameter. We assume that $0 < \epsilon < \zeta$. In what follows, we will study the asymptotic behavior of $\mathcal{J}_{Y^\epsilon}^{(2)} f$ as ϵ approaches zero while ζ is fixed. We start by writing

$$(34) \quad \mathcal{J}_{Y^\epsilon}^{(2)} f = \underbrace{\int_{-\zeta}^{\zeta} f(\gamma(t)) |\det Y^\epsilon(t)|^{-1} dt}_{A_{\zeta, \epsilon}} + \underbrace{\int_{[\tau_-, \tau_+] \setminus (-\zeta, \zeta)} f(\gamma(t)) |\det Y^\epsilon(t)|^{-1} dt}_{B_{\zeta, \epsilon}}.$$

First, we analyze the term $A_{\zeta, \epsilon}$. Recall that this corresponds to the small neighborhood $(-\zeta, \zeta)$. The following point-wise estimates are analogous to (28) and hold on the set $t \in (-\zeta, \zeta)$,

$$(35) \quad \begin{cases} |Y_{ij}^\epsilon(t)| \leq C_0 t^2 & \text{for } i \neq j \text{ and } i, j = 1, 2, \dots, n-2 \\ |Y_{jj}^\epsilon(t) - (t - i\epsilon)| \leq C_0 t^2, & \text{for } j = 1, \dots, n-2, \end{cases}$$

where $C_0 > 0$ is independent of ζ and ϵ . Applying these estimates to the expression for $\det Y^\epsilon$, we deduce that

$$\left| |\det Y^\epsilon| - |t - i\epsilon|^{n-2} \right| \leq C_1 \left(\sum_{j=1}^{n-2} t^{2j} |t - i\epsilon|^{n-2-j} \right),$$

for some C_1 independent of ϵ, ζ which can be rewritten as

$$\left| 1 - |\det Y^\epsilon| |t - i\epsilon|^{-(n-2)} \right| \leq C_1 \left(\sum_{j=1}^{n-2} t^{2j} |t - i\epsilon|^{-j} \right) \leq C_1 \left(\sum_{j=1}^{n-2} |t|^j \right).$$

We deduce that for $t \in (-\zeta, \zeta)$, there holds

$$\left| 1 - |\det Y^\epsilon| |t - i\epsilon|^{-(n-2)} \right| \leq C_2 \zeta,$$

for some $C_2 > 0$ independent of ζ and ϵ . This latter bound implies that

$$(36) \quad \left| 1 - |\det Y^\epsilon|^{-1} |t - i\epsilon|^{(n-2)} \right| \leq C \zeta,$$

for all $t \in (-\zeta, \zeta)$ where $C > 0$ independent of ϵ and ζ .

We now write

$$A_{\zeta, \epsilon} = \underbrace{\int_{-\zeta}^{\zeta} f(\gamma(t)) |t - i\epsilon|^{-(n-2)} dt}_I + \underbrace{\int_{-\zeta}^{\zeta} f(\gamma(t)) |t - i\epsilon|^{-(n-2)} (-1 + |t - i\epsilon|^{(n-2)} |\det Y^\epsilon|^{-1}) dt}_{II}$$

and

$$I = f(\gamma(0)) \left(\int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt \right) + \underbrace{\int_{-\zeta}^{\zeta} (f(\gamma(t)) - f(\gamma(0))) |t - i\epsilon|^{-(n-2)} dt}_{III}$$

For the term II , we use the bound (36) to write

$$(37) \quad |II| \leq C \zeta \left(\int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt \right),$$

where the constant C is independent of ϵ, ζ . For III , we use continuity of f to write

$$(38) \quad |III| \leq C \omega_f(\zeta) \left(\int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt \right),$$

where $C > 0$ is independent of ζ, ϵ and ω_f is a modulus of continuity for f .

Next, we proceed to give a bound on $B_{\zeta, \epsilon}$ in the expression (34). To this end, recall that $X(t)$ is non-degenerate on away from $t = 0$ and therefore $|\det X|$ has a positive lower bound on $[\tau_-, \tau_+] \setminus (-\zeta, \zeta)$ that depends on ζ . Since Y^ϵ converges to X as ϵ approaches zero, we can write

$$(39) \quad |B_{\zeta, \epsilon}| < C_\zeta,$$

for some C_ζ that only depends on ζ .

We will now divide the entire expression (34) by the normalization factor

$$(40) \quad \int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt$$

and study what happens as ϵ tends to zero. Let us first observe that

$$\int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt = \int_{\tau_-}^{\tau_+} |t - i\epsilon|^{-(n-2)} dt - \int_{[\tau_-, \tau_+] \setminus (-\zeta, \zeta)} |t - i\epsilon|^{-(n-2)} dt$$

and that

$$(41) \quad \begin{cases} \int_{\tau_-}^{\tau_+} |t - i\epsilon|^{-(n-2)} \geq C_n |\log \epsilon| & \text{for } n = 3 \\ \int_{\tau_-}^{\tau_+} |t - i\epsilon|^{-(n-2)} \geq C_n \epsilon^{3-n}, & \text{for } n \geq 4. \end{cases}$$

and

$$(42) \quad \begin{cases} \int_{[\tau_-, \tau_+] \setminus (-\zeta, \zeta)} |t - i\epsilon|^{-(n-2)} \leq C'_n |\log \zeta| & \text{for } n = 3 \\ \int_{[\tau_-, \tau_+] \setminus (-\zeta, \zeta)} |t - i\epsilon|^{-(n-2)} \leq C'_n \zeta^{3-n}, & \text{for } n \geq 4. \end{cases}$$

where C_n, C'_n are positive constants that are independent of ζ and ϵ . Combining (39)–(42) we deduce that

$$\lim_{\epsilon \rightarrow 0} |B_{\zeta, \epsilon}| \left(\int_{-\zeta}^{\zeta} |t - i\epsilon|^{-(n-2)} dt \right)^{-1} = 0.$$

Thus, by dividing (34) with the normalization factor (40), using the bounds (37)–(38), and letting ϵ converge to zero, we conclude that

$$|f(\gamma(0))| \leq C(\zeta + \omega_f(\zeta)),$$

for some $C > 0$ independent of ζ . Finally, by taking the limit $\zeta \rightarrow 0$, the proposition follows. \square

4. COMPLEX GEOMETRIC OPTICS

The main aim of this section is to construct a pair of so called complex geometric optics solutions \mathcal{U}_ρ^\pm , with

$$\rho = \lambda + i\sigma, \quad \lambda > \lambda_0 > 0,$$

for the equation

$$(43) \quad \mathcal{P}_{V_1} \mathcal{U}_\rho^\pm = 0 \quad \text{on } T = I \times M.$$

Here, we have smoothly extended the known function V_1 from \mathcal{M} to the larger set $T = I \times M$ such that $V_1 \in \mathcal{C}_c^\infty(T)$. Recall from Section 2 that we have assumed without loss of generality that $c \equiv 1$. We construct solutions that take the form

$$(44) \quad \mathcal{U}_\rho^\pm(x) = e^{\pm\lambda x^0} \left(e^{i\sigma x^0} \mathcal{V}_\rho^\pm(x^0, x') + \mathcal{R}_\rho^\pm(x^0, x') \right).$$

Here, the functions \mathcal{V}_ρ^\pm are directly related to Gaussian quasi modes for the transversal manifold (M, g) and will be supported near the two dimensional sub-manifold $\mathbb{R} \times \gamma$ with γ denoting a maximal non-self-intersecting geodesic in M . It should be remarked that the Gaussian quasi mode construction is well-known and is analogous to Gaussian beams for the wave equation (see for example [1, 22, 36]). The presentation here follows [9, 23] with some modifications. The correction term \mathcal{R}_ρ^\pm will asymptotically converge to zero for any fixed non-zero $\sigma \in \mathbb{R}$, as $\lambda \rightarrow \infty$ with an arbitrary a priori fixed rate of decay s :

$$(45) \quad \|\mathcal{R}_\rho^\pm\|_{\mathcal{C}^3(\mathcal{M})} \lesssim \lambda^{-s}, \quad \forall \lambda > \lambda_0 > 0 \quad \text{and} \quad |\sigma| \leq \sigma_0.$$

These statements will be made precise in Proposition 5.

4.1. Gaussian quasi modes. Fix a unit speed maximal non-self-intersecting geodesic $\gamma(t) \in M$ with $t \in [\tau_-, \tau_+]$ and extend it as a geodesic to the larger manifold \hat{M} (see Section 2.4) so that it is defined on an interval $[\tau'_-, \tau'_+]$ with $\tau'_- < \tau_-$ and $\tau_+ < \tau'_+$. Let q be a point on $\gamma \cap (\hat{M} \setminus M)$. We define $\{v_\alpha\}_{\alpha=2}^{n-1} \subset T_q \hat{M}$ such that $\{\dot{\gamma}(q), v_2, \dots, v_{n-1}\}$ forms an orthonormal basis and denote by $\{e_\alpha(t)\}$, the parallel transport along γ of $\{v_\alpha\}$ to the point $\gamma(t)$. We define

$$y^0 := x^0 \quad \text{and} \quad y^1 := t,$$

and for each $y'' = (y^2, \dots, y^{n-1})$ define the smooth map

$$\mathcal{F}(y) = \mathcal{F}(y^0, y^1, y'') = \left(y^0, \exp_{\gamma(y^1)} \left(\sum_{\alpha=2}^{n-1} y^\alpha e_\alpha(y^1) \right) \right).$$

We use the notation $y = (y^0, y')$ with $y' = (y^1, y'')$ and recall the following lemma (see [9, Lemma 3.5]).

Lemma 3 (Fermi coordinates). *Given any sub-interval $[\tau''_-, \tau''_+]$ of (τ'_-, τ'_+) containing (τ_-, τ_+) , the coordinate system above is a smooth diffeomorphism in a neighborhood U of $I \times \gamma([\tau''_-, \tau''_+])$, and the following statements hold.*

- (i) $\mathcal{F}^{-1}(U) = I \times (\tau'_-, \tau'_+) \times B(0, \delta')$, where $B(0, \delta')$ is the ball of radius δ' centered at the origin in \mathbb{R}^{n-2} .
- (ii) $\mathcal{F}^{-1}(y^0, \gamma(y^1)) = (y^0, y^1, \underbrace{0, \dots, 0}_{n-2 \text{ times}})$ for all $y^1 \in (\tau'_-, \tau'_+)$.

Moreover, $g(y^0, y') = (dy^0)^2 + g(y')$ and $g_{jk}(y^1, 0) = \delta_{jk}$, $\frac{\partial g_{jk}}{\partial y^i}(y^1, 0) = 0$ for $1 \leq i, j, k \leq n-1$.

Let us now return to the task of constructing solutions of the form (44). Let

$$(46) \quad \mathcal{N}_\gamma = \{y \in \hat{T} \mid y^1 \in [\tau''_-, \tau''_+], |y''| < \delta\},$$

for some $0 < \delta < \delta'$. This is the neighborhood where the Gaussian quasi modes \mathcal{V}_ρ will be compactly supported. We make the ansatz

$$(47) \quad \mathcal{V}_\rho^+(y^0, y') = e^{i\rho\Theta(y')} a_\rho^+(y) \quad \text{and} \quad \mathcal{V}_\rho^-(y^0, y') = e^{-i\rho\bar{\Theta}(y')} a_\rho^-(y)$$

The functions Θ, a_ρ^\pm are called the phase and amplitude functions respectively. We observe that

$$(48) \quad \begin{aligned} \mathcal{P}_{V_1}(e^{\rho x^0} \mathcal{V}_\rho^+) &= e^{\rho(y^0 + i\Theta(y'))} \left(-\rho^2 (\mathcal{S}\Theta) a_\rho^+ - \rho \mathcal{T}^+ a_\rho^+ + \mathcal{P}_{V_1} a_\rho^+ \right), \\ \mathcal{P}_{V_1}(e^{-\bar{\rho} x^0} \mathcal{V}_\rho^-) &= e^{-\bar{\rho}(y^0 + i\bar{\Theta}(y'))} \left(-\bar{\rho}^2 (\mathcal{S}\bar{\Theta}) a_\rho^- + \bar{\rho} \mathcal{T}^- a_\rho^- + \mathcal{P}_{V_1} a_\rho^- \right), \end{aligned}$$

where

$$(49) \quad \mathcal{S}\Theta := 1 - \langle d\Theta, d\Theta \rangle_g.$$

and

$$(50) \quad \begin{aligned} \mathcal{T}^+ a_\rho^+ &:= 2\partial_{y^0} a_\rho^+ + 2i\langle d\Theta, da_\rho^+ \rangle_g + i(\Delta_g \Theta) a_\rho^+ \\ \mathcal{T}^- a_\rho^- &:= 2\partial_{y^0} a_\rho^- + 2i\langle d\bar{\Theta}, da_\rho^- \rangle_g + i(\Delta_g \bar{\Theta}) a_\rho^-. \end{aligned}$$

Here, we would like to apply the WKB method with respect to the parameter ρ in a neighborhood of $I \times \gamma \subset \hat{\mathcal{M}}$. More specifically, we start by constructing $\Theta(y')$ such that the function $\mathcal{S}\Theta(y')$ vanishes up to N^{th} order on the geodesic γ , that is to say

$$(51) \quad \left(\frac{\partial^\beta \mathcal{S}\Theta}{\partial y'^\beta} \right) (y^1, 0) = 0, \quad \forall y^1 \in (\tau''_-, \tau''_+),$$

for all multi indices $\beta \in \{0, 1, \dots\}^{n-1}$ with $|\beta| \leq N$. We make the following ansatz,

$$(52) \quad \Theta(y^1, y'') = \sum_{k=0}^N \Theta_k(y^1, y''),$$

where $\Theta_k(y^1, y'')$ is a homogeneous polynomial of degree k in the transversal variables y'' . Following [9, Section 3], we can choose

$$(53) \quad \Theta_0(y^1, y'') = y^1, \quad \Theta_1(y^1, y'') = 0, \quad \Theta_2(y^1, y'') = \sum_{i,j=2}^n \frac{1}{2} H_{ij}(y^1) y^i y^j,$$

where $H := (H_{ij})_{i,j=2}^n$ solves the following Riccati equation,

$$(54) \quad \dot{H}(t) + H(t)^2 + D(t) = 0, \quad t \in (\tau''_-, \tau''_+) \quad H(\tau_0) = H_0.$$

Here, $\tau_0 \in (\tau''_-, \tau''_+)$, $D = \frac{1}{2}(\partial_{ij}^2 g^{11}|_\gamma)_{i,j=2}^n$ and H_0 is any symmetric matrix with $\Im H_0 > 0$. The subsequent terms $\Theta_k(y')$ can be constructed by solving linear systems of ODEs and prescribing initial values at the point $t = \tau_0$ and we refer the reader to [9] for the details.

Let us now analyze the Riccati equation further as this term dictates the Gaussian type decay away from the two dimensional sub-manifold $\mathbb{R} \times \gamma$. Applying [21, Lemma 2.56]), we deduce that given any symmetric H_0 with $\Im H_0 > 0$, there exists a unique solution to equation (54). Moreover,

$$(55) \quad \Im(H(t)) > 0 \quad \forall t \in [\tau''_-, \tau''_+]$$

and there holds

$$(56) \quad \det(\Im(H(t))) \cdot |\det Y(t)|^2 = c,$$

where $c > 0$ is a constant independent of t and $Y = (Y_{ij})_{i,j=2}^n$ is the unique solution to the second order ODE

$$(57) \quad \ddot{Y}(t) + D(t)Y(t) = 0, \quad Y(\tau_0) = Y_0, \quad \dot{Y}(\tau_0) = Y_1.$$

with Y_0 any non-degenerate matrix and $Y_1 = H_0 Y_0$. We note that $H(t)$ and $Y(t)$ are related through the expression

$$H(t) = \dot{Y}(t)Y^{-1}(t).$$

Applying arguments analogous to [11, Section 3.5] one can show that the matrix $D(t)$ is equal to the (1,1)-Ricci tensor K defined in Section 3. Therefore, the matrix $Y(t)$ solving (57) above coincides with Y in Section 3, namely that it solves the complex Jacobi equation (21) (see also [5]) and satisfies the condition (22).

Next, we consider the construction of the amplitude functions a_ρ^\pm . We write

$$(58) \quad \begin{aligned} a_\rho^+(y^0, y') &= \left(v_0(y') + \rho^{-1}v_1^+(y^0, y') + \dots + \rho^{-N}v_N^+(y^0, y') \right) \chi\left(\frac{|y''|}{\delta}\right), \\ a_\rho^-(y^0, y') &= \left(\bar{v}_0(y') + \bar{\rho}^{-1}v_1^-(y^0, y') + \dots + \bar{\rho}^{-N}v_N^-(y^0, y') \right) \chi\left(\frac{|y''|}{\delta}\right), \end{aligned}$$

where δ is as in the definition of \mathcal{N}_γ and $\chi: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth non-negative function with $\chi = 0$ for $|t| > 1$ and $\chi = 1$ for $|t| < \frac{1}{2}$. We require that

$$(59) \quad \left(\frac{\partial^\beta \mathcal{T}^+ v_0}{\partial y'^\beta} \right) (y^0, y^1, 0) = 0, \quad \forall (y^0, y^1) \in I \times (\tau''_-, \tau''_+),$$

and that

$$(60) \quad \left(\frac{\partial^\beta (\mp \mathcal{T}^\pm v_k^\pm + \mathcal{P}_{V_1} v_{k-1}^\pm)}{\partial y'^\beta} \right) (y^0, y^1, 0) = 0, \quad \forall (y^0, y^1) \in I \times (\tau''_-, \tau''_+)$$

for $k = 1, \dots, N$ and all multi indices $\beta \in \{0, 1, \dots\}^n$ with $|\beta| \leq N$. The study of equation (59) is presented in [9, Section 3]. There, it is showed that if we write

$$(61) \quad v_0(y') = \sum_{j=0}^N v_{0j}(y^1, y''),$$

with v_{0j} denoting a homogeneous polynomial of degree j in y'' , then one can take

$$(62) \quad v_{00}(t) = (\det Y(t))^{-\frac{1}{2}},$$

and that the subsequent terms $v_{0j}(t, y'')$ with $j = 1, \dots, N$ can be uniquely determined by solving first order ODEs along the geodesic γ subject to some prescribed initial conditions at the point $\gamma(\tau''_-)$.

Let us now study equation (60). Here, we deviate from [9] due to the presence of y^0 -dependence in v_k , $k \geq 1$. This comes from the fact that we consider \mathcal{P}_{V_1} with a y^0 -dependent V_1 , whereas the case $V_1 = 0$ is considered in [9]. Proceeding analogously to the study of (59), we write

$$v_k^\pm(y^0, y') = \sum_{j=0}^N v_{kj}^\pm(y^0, y^1, y''),$$

where v_{kj}^\pm is a homogeneous polynomial in the y'' variables of degree j . Using the definition of \mathcal{T}^\pm , the form of the metric $g(y')$ near $y'' = 0$, (60) reduces to

$$(\partial_{y^0} + i\partial_{y^1})(\det Y(y^1)^{\frac{1}{2}} v_{kj}^\pm(y)) = Q_{kj}^\pm(y), \quad (y^0, y^1) \in I \times (-\tau''_-, \tau''_+), \quad j = 0, \dots, N$$

where the functions $Q_{kj}^\pm(y^0, y^1, y'')$ is a homogeneous polynomial in the variables y'' of degree j only depending on V_1 and the preceding terms in the expansion of the amplitude functions.

To solve for the functions v_{kj}^\pm we can proceed with an iterative process by solving at each step, an equation of the form

$$(\partial_{y^0} + i\partial_{y^1})r = F \quad \forall (y^0, y^1) \in I \times (-\tau''_-, \tau''_+).$$

To solve such an equation, we simply extend $F(y^0, y^1)$ smoothly to \mathbb{R}^2 in such a way that $F \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Let

$$\Gamma(y^0, y^1) := \frac{1}{2\pi i} (y^0 + iy^1)^{-1},$$

and pick

$$(63) \quad r = \mathbb{I}_{I \times (-\tau''_-, \tau''_+)} (\Gamma * F),$$

where \mathbb{I}_A is the characteristic function of the set A . Note that $r \in \mathcal{C}^\infty(I \times (-\tau''_-, \tau''_+))$. By using this method, we can iteratively determine the coefficients v_{kj}^\pm and thus complete the construction of the amplitude functions $a_\rho^\pm \in \mathcal{C}^\infty(T)$.

We have completed the task of constructing the Gaussian quasi modes \mathcal{V}_ρ^\pm . Let us point out that for the phase function $\Theta(y')$, we have prescribed the initial conditions for all the ODEs at the point τ_0 . This will be later exploited in Section 5. To summarize the construction, we state the following key lemma.

Lemma 4. *Let $\rho = \lambda + i\sigma$ and let $\mathcal{V}_\rho^\pm \in \mathcal{C}^\infty(\mathcal{M})$ be constructed as above. Then, for any $|\sigma| < \sigma_0$ and all $\lambda > \lambda_0$ with $\lambda^2 \notin \{\mu_n\}_{n \in \mathbb{N}}$, we have the estimates*

$$\|\mathcal{V}_\rho^\pm\|_{L^p(\mathcal{M})} \lesssim \lambda^{-\frac{n-2}{2p}}, \quad \|\mathcal{V}_\rho^\pm\|_{\mathcal{C}^k(\mathcal{M})} \lesssim \lambda^k, \quad \text{for all } p \geq 1 \text{ and } k = 0, 1, \dots$$

and

$$\|\mathcal{L}_{\pm\lambda}(e^{i\sigma x^0} \mathcal{V}_\rho^\pm)\|_{H^k(\mathcal{M})} \lesssim \lambda^{2+k-\frac{N}{2}-\frac{n-2}{4}},$$

where \mathcal{L}_λ and $\{\mu_n\}_{n \in \mathbb{N}}$ are as defined in Section 2.4.

Proof. We will show how to prove these bounds for \mathcal{V}_ρ^+ . The bounds for \mathcal{V}_ρ^- will then follow analogously. First, observe that using equations (52)–(53) together with (55), we have

$$(64) \quad |e^{i\rho\Theta(y')}| \lesssim e^{-C_0\lambda|y''|^2}, \quad \forall y' \in \mathcal{N}_\gamma,$$

where \mathcal{N}_γ is as defined in (46). Using the Fermi coordinate system together with expressions (44) and (58), it follows that

$$\|\mathcal{V}_\rho^+\|_{L^p(\mathcal{M})}^p \lesssim \int_{I \times \mathcal{N}_\gamma} e^{-C_0 p \lambda |y''|^2} dx^0 dy^1 dy'' \lesssim \lambda^{-\frac{n-2}{2}}$$

thus proving the first claim. For the second claim, we observe that

$$\|\mathcal{V}_\rho^+\|_{C^k(\mathcal{M})} \lesssim \lambda^k e^{-C_0 \lambda |y''|^2} \lesssim \lambda^k.$$

We now derive the bound for $\mathcal{L}_\lambda(e^{i\sigma x^0} \mathcal{V}_\rho^+)$. Let us first use the fact that equations (51), (59) and (60) are satisfied together with (58) to obtain the point-wise bounds on the set $I \times \mathcal{N}_\gamma$,

$$\begin{aligned} \left| \partial_y^l \left(\rho^2 (\mathcal{S}\Theta) a_\rho^+ - \rho \mathcal{T}^+ a_\rho^+ + \mathcal{P}_{V_1} a_\rho^+ \right) \right| &\lesssim \lambda^2 |y''|^{N-|l|} + \lambda |y''|^{N-|l|} + \lambda^{-N}, \\ \left| \partial_y^l e^{i\rho\Theta} \right| &\lesssim \lambda^{|l|} e^{-C_0 \lambda |y''|^2} \end{aligned}$$

for multi-indices l with $|l| = 0, 1, \dots, k$, where we are using the notation ∂_y^l to stand for derivatives of order l with respect to the Fermi coordinates. Next, we recall from (48) that

$$\mathcal{L}_\lambda(e^{i\sigma x^0} \mathcal{V}_\rho^+) = e^{i\rho\Theta} e^{i\sigma x^0} \left(-\rho^2 (\mathcal{S}\Theta) a_\rho^+ - \rho \mathcal{T}^+ a_\rho^+ + \mathcal{P}_{V_1} a_\rho^+ \right).$$

Using the previous point-wise bounds we write

$$\begin{aligned} \|\mathcal{L}_{\pm\lambda}(e^{i\sigma x^0} \mathcal{V}_\rho^\pm)\|_{H^k(\mathcal{M})}^2 &\lesssim \\ &\sum_{|l|=0}^k \int_{I \times \mathcal{N}_\gamma} \left(\lambda^2 |y''|^{N-|l|} + \lambda |y''|^{N-|l|} + \lambda^{-N} \right)^2 \lambda^{2(k-|l|)} e^{-C_0 \lambda |y''|^2} dx^0 dy^1 dy'' \\ &\lesssim \lambda^4 \lambda^{2k} \lambda^{-N-\frac{n-2}{2}}. \end{aligned}$$

□

4.2. The remainder term. In this section we complete the construction of the complex geometric optic solutions to (43) of the form (44). More specifically, we will determine the asymptotically small correction terms \mathcal{R}_ρ^\pm .

Proposition 5. *Let $s \in \mathbb{N}$, $\rho = \lambda + i\sigma$ with $|\sigma| \leq \sigma_0$, $N = 13 + \frac{n}{2} + 2s$ and consider the functions \mathcal{V}_ρ^\pm as above. There exists solutions $\mathcal{U}_\rho^\pm \in \mathcal{C}^3(\mathcal{M})$ to equation (43) of the form (44) satisfying the following estimate,*

$$\|\mathcal{R}_\rho^\pm\|_{C^3(\mathcal{M})} \lesssim \lambda^{-s},$$

for all $\lambda > \lambda_0$, $\lambda^2 \notin \{\mu_l\}_{l \in \mathbb{N}}$.

Proof. Let us start by noting that the correction term \mathcal{R}_ρ^\pm satisfies the equation

$$\mathcal{L}_\lambda \mathcal{R}_\rho^+ = -\mathcal{L}_\lambda \left(e^{i\sigma x^0} \mathcal{V}_\rho^+ \right), \quad \mathcal{L}_{-\lambda} \mathcal{R}_\rho^- = -\mathcal{L}_{-\lambda} \left(e^{i\sigma x^0} \mathcal{V}_\rho^- \right).$$

Now, combining the bounds given in Lemma 4 together with Proposition 2 we observe that there exists a solution \mathcal{R}_ρ^\pm such that

$$\|\mathcal{R}_\rho^\pm\|_{H^k(\mathcal{M})} \lesssim \lambda^{2+k-\frac{N}{2}-\frac{n-2}{4}}.$$

Now pick $k = \frac{n}{2} + 4$. Using the Sobolev embedding $\mathcal{C}^3(\mathcal{M}) \subset H^{\frac{n}{2}+4}(\mathcal{M})$, we obtain that

$$\|\mathcal{R}_\rho^\pm\|_{\mathcal{C}^3(\mathcal{M})} \lesssim \|\mathcal{R}_\rho^\pm\|_{H^k(\mathcal{M})} \lesssim \lambda^{-s}.$$

□

5. PROOF OF THEOREM 1

This section is concerned with the proof of Theorem 1. The proof will be built on an induction argument based on m , where m is the order of the linearization method discussed in Section 2.2. As the first step of induction and also to shed some light on the methodology, we start with a proposition.

Proposition 6. *Let the assumptions of Theorem 1 hold. Then the DN map Λ_V uniquely determines the function $V_3(x)$.*

Proof. We start by choosing a point $p \in \mathcal{T}$ and choose γ to be an admissible geodesic passing through p in the sense of Definition 2. Let $\rho = \lambda + i\sigma$ with σ fixed and construct a family of complex geometric optic solutions \mathcal{U}_ρ^\pm (see (44)) with $\lambda > \lambda_0$ and a decay rate for the correction terms \mathcal{R}_ρ^\pm given by an integer $s \geq \frac{n}{8}$ (see Proposition 5). Here, we have assigned the initial values for the ODEs that govern the phase function $\Theta(y')$ at the point p . Let us now define

$$f_\rho^\pm = \mathcal{U}_\rho^\pm|_{\partial\mathcal{M}}.$$

Recall that $\mathcal{U}_\rho^\pm \in \mathcal{C}^3(\mathcal{M})$ are defined by (44) and that $\mathcal{U}_\rho^\pm|_{\partial\mathcal{M}}$ is explicitly known since V_1 is assumed to be known. By using the definition of f_ρ^\pm together with the arguments in Section 2.2 we deduce that we know $\partial_\nu L_{f_\rho^+, f_\rho^+, f_\rho^-}|_{\partial\mathcal{M}}$ for $L_{f_\rho^+, f_\rho^+, f_\rho^-}$ solving the equation

$$(65) \quad \begin{cases} \mathcal{P}_{V_1} L_{f_\rho^+, f_\rho^+, f_\rho^-} = V_3 (\mathcal{U}_\rho^+)^2 \mathcal{U}_\rho^- + H_{f_\rho^+, f_\rho^+, f_\rho^-}, & \forall x \in \mathcal{M} \\ L_{f_\rho^+, f_\rho^+, f_\rho^-} = 0 & \forall x \in \partial\mathcal{M} \end{cases}$$

We note that $H_{f_\rho^+, f_\rho^+, f_\rho^-}$ is known as it only depends on V_1 and V_2 . Let $d\sigma_g$ denote the volume form on $\partial\mathcal{M}$. Applying Green's identity together with (43) and (65), we

write

$$\begin{aligned}
& - \int_{\partial\mathcal{M}} f_\rho^- (\partial_\nu L_{f_\rho^+, f_\rho^+, f_\rho^-}) d\sigma_g + \int_{\partial\mathcal{M}} (\partial_\nu \mathcal{U}_\rho^-) L_{f_\rho^+, f_\rho^+, f_\rho^-} d\sigma_g \\
(66) \quad & = \int_{\mathcal{M}} \mathcal{U}_\rho^- (\mathcal{P}_{V_1} L_{f_\rho^+, f_\rho^+, f_\rho^-}) dV_g - \int_{\mathcal{M}} (\mathcal{P}_{V_1} \mathcal{U}_\rho^-) L_{f_\rho^+, f_\rho^+, f_\rho^-} dV_g \\
& = \int_{\mathcal{M}} V_3 (\mathcal{U}_\rho^- \mathcal{U}_\rho^+)^2 dV_g + \int_{\mathcal{M}} H_{f_\rho^+, f_\rho^+, f_\rho^-} \mathcal{U}_\rho^- dV_g.
\end{aligned}$$

Thus we can conclude that for all $\lambda > \lambda_0$ with $\lambda^2 \notin \{\mu_l\}_{l \in \mathbb{N}}$, the knowledge of the DN map Λ_V uniquely determines the expression

$$\begin{aligned}
S & = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_{\mathcal{M}} V_3 (\mathcal{U}_\rho^- \mathcal{U}_\rho^+)^2 dV_g \\
& = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_{\mathcal{M}} V_3(x^0, x') e^{4i\sigma x^0} (\mathcal{V}_\rho^+ + \mathcal{R}_\rho^+)^2 (\mathcal{V}_\rho^- + \mathcal{R}_\rho^-)^2 dx^0 dV_g.
\end{aligned}$$

Applying Proposition 5 and Lemma 4 we have the bounds

$$\begin{aligned}
& \int_{\mathcal{M}} |V_3| |\mathcal{R}_\rho^+|^2 |\mathcal{R}_\rho^-|^2 dV_g \lesssim |\lambda|^{-4s} \lesssim |\lambda|^{-\frac{n-2}{2}} \lambda^{-1}, \\
& \int_{\mathcal{M}} |V_3| |\mathcal{V}_\rho^\pm|^k |\mathcal{R}_\rho^\pm|^j dV_g \lesssim \lambda^{-\frac{n-2}{2}} \lambda^{-js}.
\end{aligned}$$

We also recall the point-wise bound (64) to obtain that

$$\lambda^{\frac{n-2}{2}} \int_{I \times N_\gamma} \lambda^{-1} e^{-4\lambda \Im \Theta} dx^0 dt dy'' \lesssim \lambda^{-1}.$$

Using this bound together with the latter two bounds, the expression for S simplifies to

$$S = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_I \int_{N_\gamma} V_3(x^0, t, y'') e^{4i\sigma x^0} e^{-4\lambda \Im \Theta} e^{-4\sigma \Re \Theta} |v_0(t, y'')|^4 dx^0 dV_g$$

where we have extended V_3 to all of $I \times M$ through setting it to zero outside \mathcal{M} . Next, and by using the estimate

$$\lambda^{\frac{n-2}{2}} \int_{I \times N_\gamma} |y''| e^{-4\lambda \Im \Theta} dx^0 dt dy'' \lesssim \lambda^{-\frac{1}{2}}$$

we can further simplify S to obtain

$$S = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_{I \times N_\gamma} V_3(x^0, t, 0) e^{4i\sigma x^0} e^{-4\sigma t} e^{-4\lambda \Im \Theta} |v_{00}(t)|^4 dx^0 dt dy''.$$

Finally, applying stationary phase together with equations (56) and (62), we conclude that the Dirichlet-to-Neumann map Λ_V uniquely determines the expression

$$(67) \quad \int_{\tau_-}^{\tau_+} e^{\xi t} \mathcal{F} V_3(\xi, \gamma(t)) |\det Y(t)|^{-1} dt, \quad \text{for all } |\xi| < \frac{\sigma_0}{4},$$

with $\mathcal{F}V_3$ denoting the Fourier transform of V_3 with respect to x^0 variable. Here we have extended the function V_3 to the set $\mathbb{R} \times M$ by setting it to zero outside of \mathcal{M} . We can summarize the analysis thus far as follows. For each point $p \in \mathcal{T}$ and each admissible geodesic γ passing through p , we have shown that the Dirichlet-to-Neumann map Λ_V uniquely determines the expression (67). Now, recall from (57) that the $(n-2) \times (n-2)$ matrix Y solves the complex Jacobi equation (21) along γ subject to the initial condition

$$Y(\tau_0) = Y_0, \quad \text{and} \quad \dot{Y}(\tau_0) = Y_1 = H_0 Y_0,$$

where H_0 is symmetric and satisfies $\Im H_0 > 0$ and Y_0 is non-degenerate. This implies that $Y \in \mathbb{Y}_\gamma$.

Returning to (67) and applying Proposition 4 together with the fact that σ_0 is arbitrary, we conclude that

$$\mathcal{F}V_3(\xi, p) = 0,$$

for all $\xi \in \mathbb{R}$ and all $p \in \mathcal{T}$. The theorem now follows using the density of \mathcal{T} in M and applying the inverse Fourier transform. \square

Before presenting the proof for the more general case $m \geq 4$, we need a lemma.

Lemma 5. *Given any point $p \in M$, there exists a smooth solution W_p to the equation*

$$\mathcal{P}_{V_1} W_p = 0$$

with the additional property that $W_p(p) \neq 0$.

Proof. Let us consider $G(x; y)$ to denote the Dirichlet Green's function associated to the operator \mathcal{P}_{V_1} on \mathcal{M} . For each fixed $y \in \mathcal{M}$, $G(x; y)$ is the unique distributional solution to

$$(68) \quad \begin{cases} \mathcal{P}_{V_1} G(x; y) = \delta(x; y), & \forall x \in \mathcal{M} \\ G(x; y) = 0 & \forall x \in \partial\mathcal{M} \end{cases}$$

For each fixed y , $G(x; y)$ is smooth away from y . Moreover, since $\delta(x; y) \in H^{-\frac{n}{2}-\epsilon}(\mathcal{M})$ for any $\epsilon > 0$ and any fixed $y \in \mathcal{M}$, it follows from elliptic regularity that $G(x; y) \in H^{2-\frac{n}{2}-\epsilon}(\mathcal{M})$ for any $\epsilon > 0$. Let $h \in \mathcal{C}^\infty(\partial\mathcal{M})$ be arbitrary and choose w such that $\mathcal{P}_{V_1} w = 0$ subject to $w|_{\partial\mathcal{M}} = h$. Applying Green's identity we have

$$w(x) = \int_{\mathcal{M}} \mathcal{P}_{V_1} G(x; y) w(y) dV_g - \int_{\mathcal{M}} G(x; y) \mathcal{P}_{V_1} w(y) dV_g = - \int_{\partial\mathcal{M}} \partial_\nu G(x; y) h(y) d\sigma_g$$

If the claim fails to hold, that is $w(p) = 0$ for all $h \in \mathcal{C}^\infty(\partial\mathcal{M})$, then clearly we must have $\partial_\nu G(p; y) = 0$ for all $y \in \partial\mathcal{M}$. Since $G(p; y)$ also vanishes there, by the elliptic unique continuation principle, that $G(x; p)$ must vanish away from p implying that it is supported at the point $\{p\}$. Subsequently, it must be a linear combination of $\delta(x; p)$ and its derivatives at the point $\{p\}$. But as already mentioned in the

beginning of the proof $G(x; p)$ is smoother than $\delta(x; p)$, and whence it must vanish everywhere which is a contradiction with (68). \square

Proof of Theorem 1. We use an induction argument on $m \geq 4$ to show that V_m can be uniquely determined from the DN map Λ_V . We have already proved this for $m = 3$. Now let $m > 3$ and assume that V_j has been determined for all $j < m$. We consider a point $p \in \mathcal{T}$ with an admissible geodesic γ passing through p . Similar to the proof of Proposition 6, we pick $s \geq \frac{2}{8}$ and consider the solutions \mathcal{U}_ρ^\pm and define f_ρ^\pm analogously. We also choose the function W_p as in Lemma 5 and let h be its trace on the boundary $\partial\mathcal{M}$.

Following Section 2.2 we define

$$L_m := L_{f_\rho^+, f_\rho^+, f_\rho^-, \underbrace{h, \dots, h}_{m-3 \text{ times}}} \quad \text{and} \quad H_m := H_{f_\rho^+, f_\rho^+, f_\rho^-, \underbrace{h, \dots, h}_{m-3 \text{ times}}}.$$

Applying (16) we note that L_m solves the equation

$$(69) \quad \begin{cases} \mathcal{P}_{V_1} L_m = \tilde{V}_m (\mathcal{U}_\rho^+)^2 (\mathcal{U}_\rho^-) + H_m, & \forall x \in \mathcal{M} \\ L_m = 0 & \forall x \in \partial\mathcal{M} \end{cases}$$

where

$$\tilde{V}_m(x) = V_m(x) W_p(x)^{m-3}.$$

Recall from Section 2.2 that H_m is explicitly known as it only depends on V_1, \dots, V_{m-1} and these functions are known from the induction hypothesis. Applying Green's identity we observe that

$$(70) \quad \begin{aligned} & - \int_{\partial\mathcal{M}} f_\rho^- (\partial_\nu L_m) d\sigma_g + \int_{\partial\mathcal{M}} (\partial_\nu \mathcal{U}_\rho^-) L_m d\sigma_g \\ &= \int_{\mathcal{M}} \mathcal{U}_\rho^- (\mathcal{P}_{V_1} L_m) dV_g - \int_{\mathcal{M}} (\mathcal{P}_{V_1} \mathcal{U}_\rho^-) L_m dV_g \\ &= \int_{\mathcal{M}} \tilde{V}_m (\mathcal{U}_\rho^- \mathcal{U}_\rho^+)^2 dV_g + \int_{\mathcal{M}} H_m \mathcal{U}_\rho^- dV_g. \end{aligned}$$

We deduce that the map Λ_V uniquely determines the expression

$$\int_{\mathcal{M}} \tilde{V}_m (\mathcal{U}_\rho^+)^2 (\mathcal{U}_\rho^-)^2 dV_g.$$

Thus using the same asymptotic analysis as in the proof of Proposition 6, together with Proposition 4, we conclude that Λ_V uniquely determines $\tilde{V}_m(p)$ and consequently by Lemma 5 it determines the function V_m at all points $(x^0, p) \in \mathcal{M}$ with $p \in \mathcal{T}$. The proof is completed by noting the density of \mathcal{T} in M . \square

6. PROOF OF THEOREM 2

Proof of Theorem 2. It suffices to prove that V_2 can be uniquely reconstructed from the knowledge of Λ_V . Indeed, the recovery of the higher order derivatives V_m with $m = 3, 4, \dots$, follows by using induction as in the proof of Theorem 1.

We start by choosing an arbitrary point $p \in M$ and assume that γ is a maximal non-self intersecting geodesic passing through p . Let $\rho = \lambda + i\sigma$ with σ fixed and construct a family of complex geometric optic solutions \mathcal{U}_ρ^\pm and $\mathcal{U}_{2\rho}^\pm$ as in Section 4 with a decay rate given by an integer $s \geq \frac{n}{6}$ where we recall that $n \in \{3, 4\}$. Here, we have assigned the initial values for the ODEs that govern the phase function $\Theta(y')$ at the point p . Let us now define

$$f_\rho^\pm = \mathcal{U}_\rho^\pm|_{\partial\mathcal{M}}.$$

Recall that $\mathcal{U}_\rho^\pm \in \mathcal{C}^3(\mathcal{M})$ and that $\mathcal{U}_\rho^\pm|_{\partial\mathcal{M}}$ are explicitly known. By using the definition of f_ρ^\pm together with the arguments in Section 2.2, we deduce that we know $\partial_\nu L_{f_\rho^+, f_\rho^+}|_{\partial\mathcal{M}}$ for $L_{f_\rho^+, f_\rho^+}$ solving the equation

$$(71) \quad \begin{cases} \mathcal{P}_{V_1} L_{f_\rho^+, f_\rho^+} = V_2 (\mathcal{U}_\rho^+)^2, & \forall x \in \mathcal{M} \\ L_{f_\rho^+, f_\rho^+} = 0 & \forall x \in \partial\mathcal{M} \end{cases}$$

Let $d\sigma_g$ denote the volume form on $\partial\mathcal{M}$. Applying Green's identity we observe that

$$(72) \quad \begin{aligned} & \int_{\partial\mathcal{M}} f_{2\rho}^- (\partial_\nu L_{f_\rho^+, f_\rho^+}) d\sigma_g - \int_{\partial\mathcal{M}} (\partial_\nu \mathcal{U}_{2\rho}^-) L_{f_\rho^+, f_\rho^+} d\sigma_g \\ &= \int_{\mathcal{M}} \mathcal{U}_{2\rho}^- (\mathcal{P}_{V_1} L_{f_\rho^+, f_\rho^+}) dV_g - \int_{\mathcal{M}} (\mathcal{P}_{V_1} \mathcal{U}_{2\rho}^-) L_{f_\rho^+, f_\rho^+} dV_g \\ &= \int_{\mathcal{M}} V_2 \mathcal{U}_{2\rho}^- (\mathcal{U}_\rho^+)^2 dV_g. \end{aligned}$$

Thus we can conclude that for all $\lambda > \lambda_0$ with $\lambda^2 \notin \{\mu_l\}_{l \in \mathbb{N}}$, the knowledge of the DN map Λ_V uniquely determines the expression

$$(73) \quad S_1 = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_{\mathcal{M}} V_2 \mathcal{U}_{2\rho}^- (\mathcal{U}_\rho^+)^2 dV_g.$$

Applying Proposition 5 and Lemma 4, we have the bounds

$$\begin{aligned} & \int_{\mathcal{M}} |V_2| |\mathcal{R}_\rho^+|^2 |\mathcal{R}_{2\rho}^-| dV_g \lesssim |\lambda|^{-3s} \lesssim |\lambda|^{-\frac{n-2}{2}} \lambda^{-1}, \\ & \int_{\mathcal{M}} |V_2| |\mathcal{V}_\rho^+|^2 |\mathcal{R}_{2\rho}^-| dV_g \lesssim \lambda^{-\frac{n-2}{2}} \lambda^{-s}, \\ & \int_{\mathcal{M}} |V_2| |\mathcal{V}_{2\rho}^-| |\mathcal{V}_\rho^+| |\mathcal{R}_\rho^+| dV_g \lesssim \lambda^{-\frac{n-2}{2}} \lambda^{-s}, \\ & \int_{\mathcal{M}} |V_2| |\mathcal{V}_{2\rho}^-| |\mathcal{R}_\rho^+|^2 dV_g \lesssim \lambda^{-\frac{n-2}{2}} \lambda^{-2s}. \end{aligned}$$

Next and by using the form of the phase function and the amplitude functions given by equations (52) and (58) together with the previous four bounds, we write

$$S_1 = \lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} \int_I \int_{\mathcal{N}_\gamma} V_2 e^{4i\sigma x^0} e^{-4\lambda \Im \Theta} e^{-4\sigma \Re \Theta} |v_{00}|^2 v_{00} dx^0 dV_g$$

where it is to be understood that V_2 is extended to $I \times M$ by setting it to zero outside \mathcal{M} . Finally, applying stationary phase together with equations (56) and (62), we deduce that the Dirichlet to Neumann map determines the expression

$$(74) \quad \int_{\tau_-}^{\tau_+} e^{\xi t} \mathcal{F} V_2(\xi, \gamma(t)) (\det Y(t))^{-\frac{1}{2}} dt, \quad \forall |\xi| < \frac{\sigma_0}{4},$$

with $\mathcal{F} V_2$ denoting the Fourier transform of V_2 with respect to x^0 variable. Here, we have extended the function V_2 to the entire $\mathbb{R} \times M$ by setting it to zero outside of \mathcal{M} . We can summarize the analysis thus far as follows. For each point $p \in M$ and each admissible geodesic γ passing through p , we have shown that the Dirichlet-to-Neumann map Λ_V uniquely determines the expression (74). Repeating the same analysis, this time with the dual choices $L_{f_\rho^-, f_\rho^-}$ and $\mathcal{U}_{2\rho}^+$ we can similarly determine the expression

$$(75) \quad \int_{\tau_-}^{\tau_+} e^{\xi t} \mathcal{F} V_2(\xi, \gamma(t)) \overline{(\det Y(t))^{-\frac{1}{2}}} dt, \quad \forall |\xi| < \frac{\sigma_0}{4}.$$

Finally, combining the expressions (74)–(75) we deduce that the map Λ_V uniquely determines the expressions

$$(76) \quad S = \int_{\tau_-}^{\tau_+} e^{\xi t} f(\xi, \gamma(t)) (\det Y(t))^{-\frac{1}{2}} dt, \quad |\xi| < \frac{\sigma_0}{4},$$

where $f \in \{\Re \mathcal{F} V_2, \Im \mathcal{F} V_2\}$.

Applying Proposition 3 to the real-valued function f together with the fact that σ_0 is arbitrary, it immediately follows that $f(\xi, p)$ can be uniquely determined from Λ_V for all $\xi \in \mathbb{R}$ and all $p \in M$. We conclude that $f(\xi, x')$ can be uniquely determined from Λ_V for all $x' \in M$. Using the inverse Fourier transform, it follows that Λ_V uniquely determines the function V_2 everywhere in \mathcal{M} . \square

Acknowledgments. A.F was supported by EPSRC grant EP/P01593X/1. L.O acknowledges support from EPSRC grants EP/R002207/1 and EP/P01593X/1.

REFERENCES

- [1] V. M. Babich and V. V. Ulin, The complex space-time ray method and quasiphotons. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 117:5-12, 197, 1981. Mathematical questions in the theory of wave propagation, 12.
- [2] A. P. Calderón, On an inverse boundary value problem. In Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), pages 65-73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

- [3] C. I. Cârstea, G. Nakamura and M. Vashisth, Reconstruction for the coefficients of a quasilinear elliptic partial differential equation. arXiv e-prints, Mar 2019.
- [4] X. Chen, M. Lassas, L. Oksanen and G. Paternain, Detection of Hermitian connections in wave equations with cubic non-linearity, arXiv preprint 2019.
- [5] M. F. Dahl, Geometrization of the leading term in acoustic Gaussian beams. *J. Nonlinear Math. Phys.*, 16(1):35-45, 2009.
- [6] M.P. do Carmo, *Riemannian Geometry*. Mathematics (Boston, Mass.). Birkhäuser, 1992.
- [7] D. Dos Santos Ferreira, C. E. Kenig, M. Salo and G. Uhlmann, Limiting carleman weights and anisotropic inverse problems. *Inventiones mathematicae*, 178(1):119- 171, Oct 2009.
- [8] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, T. Liimatainen and M. Salo, The Linearized Calderón Problem in Transversally Anisotropic Geometries, *International Mathematics Research Notices*, rny234, 2018.
- [9] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas and M. Salo, The Calderón problem in transversally anisotropic geometries. *J. Eur. Math. Soc. (JEMS)*, 18(11):2579-2626, 2016.
- [10] H. Egger, J-F. Pietschmann and M. Schlottbom, Simultaneous identification of diffusion and absorption coefficients in a quasilinear elliptic problem, *Inverse Problems*, 30 (2014), 035009.
- [11] A. Feizmohammadi, J. Ilmavirta, Y. Kian and L. Oksanen, Recovery of time dependent coefficients from boundary data for hyperbolic equations, *Journal of Spectral Theory*, to appear (2020).
- [12] A. Feizmohammadi and L. Oksanen, Recovery of zeroth order coefficients in non-linear wave equations, arXiv preprint, 2019.
- [13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. *Classics in Mathematics*. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [14] O. Imanuvilov and M. Yamamoto, Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data, *Journal of Inverse and Ill-Posed Problems*, 21 (2013), 85-108.
- [15] V. Isakov, On uniqueness in inverse problems for semilinear parabolic equations. *Arch. Rational Mech. Anal.*, 124(1):1-12, 1993.
- [16] V. Isakov, Uniqueness of recovery of some systems of semilinear partial differential equations. *Inverse Problems*, 17(4):607-618, 2001.
- [17] V. Isakov, Uniqueness of recovery of some quasilinear Partial differential equations, *Comm. PDE*, 26 (2001), 1947-1973.
- [18] V. Isakov and A. I. Nachman, Global uniqueness for a two-dimensional semilinear elliptic inverse problem. *Trans. Amer. Math. Soc.*, 347(9):3375-3390, 1995.
- [19] V. Isakov and J. Sylvester, Global uniqueness for a semilinear elliptic inverse problem. *Comm. Pure Appl. Math.*, 47(10):1403-1410, 1994.
- [20] H. Isozaki, Inverse spectral problems on hyperbolic manifolds and their applications to inverse boundary value problems in Euclidean space. *American Journal of Mathematics*, 126(6):1261-1313, 2004.
- [21] A. Katchalov, Y. Kurylev and M. Lassas, *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.
- [22] A. Katchalov and M. Lassas, Gaussian beams and inverse boundary spectral problems. In *New analytic and geometric methods in inverse problems*, pages 127-163. Springer, Berlin, 2004.
- [23] C. E. Kenig and M. Salo, The Calderón problem with partial data on manifolds and applications. *Anal. PDE*, 6(8):2003-2048, 2013.

- [24] C. E. Kenig, J. Sjöstrand and G. Uhlmann, The Calderón problem with partial data, *Ann. Math.* 165 567–591 (2007).
- [25] Y. Kurylev, M. Lassas, L. Oksanen and G. Uhlmann, Inverse problem for Einstein-scalar field equations. arXiv e-prints, May 2014.
- [26] Y. Kurylev, M. Lassas and G. Uhlmann, Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Inventiones mathematicae*, 212(3):781–857, 2018.
- [27] M. Lassas, T. Liimatainen, Y. Lin and M. Salo, Inverse problems for elliptic equations with power type nonlinearities, arXiv preprint, 2019.
- [28] M. Lassas, G. Uhlmann and Y. Wang, Determination of vacuum spacetimes from the Einstein-Maxwell equations. arXiv preprint, 2017.
- [29] M. Lassas, G. Uhlmann and Y. Wang, Inverse problems for semilinear wave equations on Lorentzian manifolds. *Communications in Mathematical Physics*, 360(2):555-609, Jun 2018.
- [30] J. M. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.*, 42(8):1097-1112, 1989.
- [31] B. Malomed, Nonlinear Schrödinger equations. *Encyclopedia of Nonlinear Science*, New York: Routledge, 92(10):639-643, 2005.
- [32] C. Munoz and G. Uhlmann, The Calderón problem for quasilinear elliptic equations. arXiv e-prints, Jun 2018.
- [33] J. K. Myers, Uniqueness of source for a class of semilinear elliptic equations. *Inverse Problems*, 25(6):065008, 6, 2009.
- [34] A. I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143(1):71-96, 1996.
- [35] M. Nakamura, The cauchy problem for semi-linear Klein-Gordon equations in de Sitter spacetime. *Journal of Mathematical Analysis and Applications*, 410(1):445 -454, 2014.
- [36] J. Ralston, Gaussian beams and the propagation of singularities. In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pages 206-248. Math. Assoc. America, Washington, DC, 1982.
- [37] M. Salo, The Calderón problem on Riemannian manifolds, in *Inverse problems and applications: inside out. II*, *Mathematical Sciences Research Institute Publications*, vol. 60, Cambridge University Press, 2013, p. 167-247.
- [38] V. Scheidemann, *Introduction to Complex Analysis in Several Variables*, Birkhäuser, Basel, 2005. MR2176976
- [39] Z. Sun, On a quasilinear inverse boundary value problem. *Math. Z.*, 221(2):293-305, 1996.
- [40] Z. Sun, Inverse boundary value problems for a class of semilinear elliptic equations. *Adv. in Appl. Math.*, 32(4):791–800, 2004.
- [41] Z. Sun and G. Uhlmann, Inverse problems in quasilinear anisotropic media. *American Journal of Mathematics*, 119(4):771–797, 1997.
- [42] Z. Sun, An inverse boundary-value problem for semilinear elliptic equations. *Electron. J. Differential Equations*, pages No. 37, 5, 2010.
- [43] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.
- [44] G. Uhlmann, Electrical impedance tomography and Calderón’s problem. *Inverse Problems*, 25(12):123011, 39, 2009.
- [45] G. Uhlmann, 30 years of Calderón’s problem. *Séminaire Laurent Schwartz - EDP et applications*, pages 1-25, 2012-2013.

- [46] G. Uhlmann and A. Vasy, The inverse problem for the local geodesic ray transform, *Invent. Math.* 205 (2016), no. 1, 83–120.
- [47] O. Waldron and R. A Van Gorder, A nonlinear Klein-Gordon equation for relativistic superfluidity. *Physica Scripta*, 92(10), 2017.
- [48] Y. Wang and T. Zhou, Inverse problems for quadratic derivative nonlinear wave equations, *Communications in Partial Differential Equations*, 44:11, 1140-1158 (2019).

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, LONDON, UK-WC1E 6BT,
UNITED KINGDOM

E-mail address: a.feizmohammadi@ucl.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, LONDON, UK-WC1E 6BT,
UNITED KINGDOM

E-mail address: l.oksanen@ucl.ac.uk