Bounded control via geometrically shaping Lyapunov function derivatives

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Abstract— **This paper is two folded: in the first part a result is presented on bounded passivity-based control. This is applied on a reduced-size building model, in order to protect it from earthquakes. The second part of this paper presents an attempt to extend the same methodology towards bounded backstepping-based control. The key point here are the *Lyapunov function derivative* terms: by manipulating their geometrical shapes, feedback properties follow. On a simple one-dimensional system, we analyze performance, robustness and *bounded control*.**

Index Terms— **bounded passivity-based control, bounded backstepping-based control, earthquake engineering.**

I. INTRODUCTION

Any *real plant* has limited actuators. *Bounded control* is still, nowadays, an open subject in *control systems theory*, with enormous interest concerning industrial applications. To start with, let us distinguish two types, namely *explicit bounded control*, where a saturation block of the type

$$u_{\text{out}} = \max\{-1, \min\{1, u_{\text{in}}\}\} \tag{1}$$

is added in-between *controller* and *plant*, as opposed to *implicit bounded control*, where this saturation block should not be necessary. Let us underline that, implicit bounded control should only hold within a pre-defined framework of hypotheses. To illustrate this, let us take a simple example, often met in practice: given a known, simplified, Single-Input Single-Output (SISO) *plant* model, together with a set of constraints on both *state variables* and control law, we assume that, a class of feasible controllers can be pursued analytically. We call them *implicit bounded controllers*: a saturation block, placed in-between *controller* and simplified model is not necessary. On the other hand, if during controller synthesis constraints are disregarded, for security reasons, a saturation block should be added, prior to validating feedback behavior and effectuating an *a posteriori* feedback-loop properties analysis. The latter situation corresponds to *explicit bounded control*. Practical control engineering solutions can be pursued by using a mix on the two, mainly for safety concerns, e.g., when we test and validate any *implicit bounded controller* on the actual, *real plant*. Due to model uncertainties, it might be hard to guarantee that controller behavior should keep within desired bounds.

In this paper, we are only interested in *implicit bounded control*, which is somehow difficult to deal with analytically and to be generalized, mainly due to more involved calculations. State-of-the-art includes: Arstein–Sontag universal bounded control law in [1], [2, eq. (12.1)–(12.2)], [3, pp. 113]; universal bounded control law introduced by [4], [5, Thm. 2.8]; CLF design in [1], [5]; sufficient conditions for global stabilization for lower-triangular and feedforward nonlinear systems by means of saturated bounded feedback controllers in [6]; backstepping in [7], [5, Ex. 2.9]; forwarding in [8], [9], [6], [3, Ch. 6.2.5]; Leitmann's control in [10]; *receding horizon control* with anti-windup in [11]; model predictive control in [12]; adaptive control making use of a special projection operator in [13].

This paper is intended to be reasonably self-contained. Implicit bounded control is pursued in section II where a result is given on bounded passivity-based control, and then in section III where focus is on bounded backstepping-based control. The paper ends with conclusions in section IV.

II. PART 1. BOUNDED PASSIVITY-BASED CONTROL

A. Preliminaries: recall

In this paper we consider *autonomous*, *affine in control*, nonlinear dynamical systems:

$$\dot{x} = f(x) + g(x)u \tag{2a}$$

$$y = h(x) \tag{2b}$$

with $x \in \mathbb{R}^n$ state-space vector, $n \in \mathbb{Z}_{>0}$; $u \in \mathbb{R}^m$ is the input vector, $m \in \mathbb{Z}_{>0}$; $y \in \mathbb{R}^p$ output vector, $p \in \mathbb{Z}_{>0}$; f, h and the columns of g are continuous vector fields.

Definition 1: (ZSD in [14], [15]; [3, pp. 48]) System (2) is *zero-state detectable* (ZSD), if, for all $t \in \mathbb{R}_{\geq 0}$ and initial state $x_0 \in \mathbb{R}^n$, the following relation holds

$$y(t) \equiv 0,\ u(t) \equiv 0 \Rightarrow \lim_{t \to +\infty} x(t) = 0 \tag{3}$$

Definition 2: (ZSO in [16, pp. 604]; [15]; [3, pp. 48]) System (2) is *zero-state observable* (ZSO), if for all $t \in \mathbb{R}_{\geq 0}$ and initial state $x_0 \in \mathbb{R}^n$, the following relation holds

$$y(t) \equiv 0,\ u(t) \equiv 0 \Rightarrow x(t) \equiv 0 \tag{4}$$

In other words, the limit disappears when passing from (3) to (4). □

Theorem 1: (Passivity and stability, in [14, Thm. 2]; [15, Thm. 3.2]; [3, Thm. 2.28]) If nominal system (2), with measured output

$$y = (L_g W)^\top \tag{5}$$

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satisfies both conditions:
1. is passive with a *storage function* W of class $\mathscr{C}^1$, radially unbounded and positive definite
2. is zero-state detectable (ZSD)
then the origin $x=0$ can be globally asymptotically stabilized (GAS) by $u=-\phi(y)$; $\phi(y)$ can be any locally Lipschitz function, such that $\phi(0)=0$ and $y^{\mathrm{T}}\phi(y)>0$, for any $y\neq 0$. □

Thm. 1 is closely related to [16, Thm. 14.4]. The only difference resides in ZSD condition which is less conservative than ZSO.

By the way, on the passage we indicate a typo in [16, Thm. 14.4]: to ensure *global stability* referred to, in the body statement of that theorem and proven immediately after, the relation $y^{\mathrm{T}}\phi(y)\geq 0$ should appear instead of $y^{\mathrm{T}}\phi(y)>0$, which holds for the stronger condition of *global asymptotic stability*.

B. Main result

1) Theory: Largely inspired by [17] we show the following result.

Proposition 1: (Bounded Passivity-Based Control)

The structurally bounded, anti-symmetrical, continuous-time feedback laws:

$$u(y)=-k_1\frac{y}{k_2+y^{\top}y} \tag{6}$$

with $k_1\in\mathbb{R}_{>0}$ and $k_2\in\mathbb{R}_{>0}$, and

$$u(y)=\rho(x)\tanh\left(-\frac{1}{k_1}y\right)$$

with $\rho(x)\in(0,u_{Max}]$ arbitrarily chosen, both ensure GAS on any nonlinear, passive system (2) with measured output (5); $u_{Max}\in\mathbb{R}_{>0}$ is an application specific upper bound on the control. □

Sketch of proof: We will only analyze (6) herebelow, since the procedure is the same for both control laws. Straightforwardly apply Thm. 1. The shape of (6) is illustrated in Fig. 1. Given the definition (5) of y and (6), calculations lead to

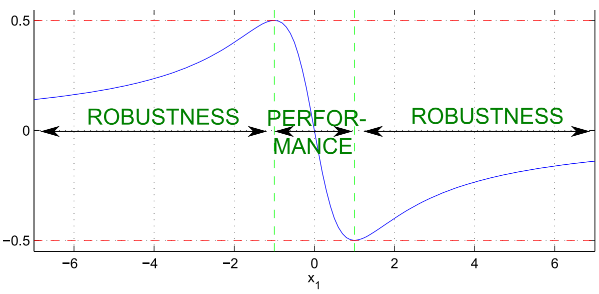


Fig. 1. Structurally bounded control term $\tilde{u}(x_1)=-k_1\frac{x_1}{k_2+x_1^2}$. $k_1=1$; $k_2=1$.

$$\dot{W}=\frac{\partial W}{\partial x}\dot{x}=L_fW+(L_gW)\,u \tag{7a}$$

$$=L_fW-k_1\frac{\|L_gW\|^2}{k_2+\|L_gW\|^2}\leq 0 \tag{7b}$$

In (7b) the first term $L_fW\leq 0$ due to passivity property; the second term is made up of only positive quantities and the minus sign ensures it is ≤ 0 for all $||L_gW||$. (This second term in the above relation has similar shape to one of the curves represented later on, in Fig. 4(a).) Now, by making sure that $L_gW\neq 0$, the stronger property of GAS is ensured.

Next, to prove the structural boundedness of (6), let us notice that $\lim_{|y|\to\infty}u(y)=0$. By imposing $\dot{u}=0$, the upper and lower bounds for control law (6) can be calculated: this gives $y=\sqrt{\pm k_2}$. Thus, the bounds of the control law:

$$u(\sqrt{\pm k_2})=\mp\frac{k_1}{2\sqrt{k_2}}\leq u_{Max} \tag{8}$$

Parameters k_1 and k_2 can therefore easily be selected, such that (8) should hold.

As guideline for the interested reader, we mention that relation (6), can be extended to form a more general class of bounded control laws, $u(y)=-k_1\frac{y}{(k_2+y^2)^{\theta}}$, with $\theta\in\mathbb{Z}_{>1}$. Other feasible alternatives can be constructed in the same way.

2) Practice: **Application to seismically base-isolated structures.** The ability to efficiently protect buildings against unpredictable earthquakes is an open problem. The concept of *base-isolated structures* consists of placing passive, semi-active or active devices, at the floor (or base) level, such that, energy transferred to the structure by hazardous earthquakes effects, can be dissipated (see [18]).

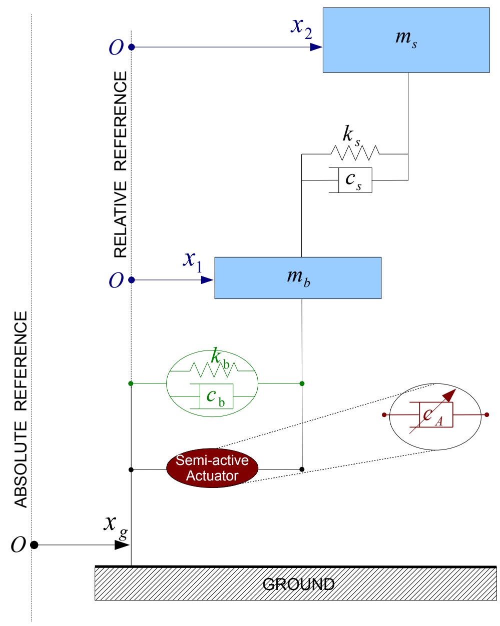


Fig. 2. Lumped-mass model of a 2DOF seismically isolated structure.

In this paper, we deal exclusively with *semi-active control* problem of a *base-isolated structure*, modeled as a two degree-of-freedom (2DOF) stable system, with lumped-mass idealization:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m_b}\begin{bmatrix} -(c_b+c_s)x_2 + c_s x_4 \\ -(k_b+k_s)x_1 + k_s x_3 \end{bmatrix} - \ddot{x}_g - \dfrac{x_2}{m_b}c_A \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{1}{m_s}\left[c_s(x_2-x_4) + k_s(x_1-x_3)\right] - \ddot{x}_g \end{cases} \tag{9}$$

In Fig. 2 and system (9), m_s stands for mass of structure, while m_b is mass of the base. The *control device* is an ideal damper with variable damping constant $c_A \in [0, c_{Max}]$, with $c_{Max} \in \mathbb{R}_{>0}$; x_1, x_2 are relative position and velocity of m_b, with respect to a fixed point attached to structure's base; x_3, x_4 are relative position and velocity of m_s, with respect to ground; k_s, c_s are linear-spring stiffness and dumping constants of m_s with respect to m_b; k_b, c_b are linear-spring stiffness and dumping constants of m_b with respect to ground.

The problem is to design a stationary *feedback law* $c_A(x)$, structurally bounded by physical constraints $c_A \in [0, c_{Max}]$, in spite of any *a priori* unknown, unpredictable, non-stationary earthquake signal, modeled as base-level relative acceleration $\ddot{x}_g$.

Solution: To start with, the control solution that we propose, consists, in a first phase, to relax a little bit the structural constraint on control law, into $|c_A| \le c_{Max}$. Then, we apply straightforwardly Prop. 1 on the bilinear nominal system (9). By doing so, we will notice that $c_A(t) \in [0, c_{Max}]$, which is what we want.

In the absence of perturbation $\ddot{x}_g(t)$, nominal plant model (9) is naturally GAS at the origin $x = 0$ and consequently *passive* with storage function chosen to be the Hamiltonian of the system. To recall, the Hamiltonian $H(x)$ of Euler-Lagrange models like system (9) is the sum of kinetic and elastic energies:

$$H(x) = \frac{1}{2}\begin{bmatrix}\dot{x}_1 & \dot{x}_3\end{bmatrix} M \begin{bmatrix}\dot{x}_1 \\ \dot{x}_3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}x_1 & x_3\end{bmatrix} K \begin{bmatrix}x_1 \\ x_3\end{bmatrix} \ge 0 \tag{10}$$

where matrices

$$M = \begin{bmatrix} m_b & 0 \\ 0 & m_s \end{bmatrix}, K = \begin{bmatrix} k_b + k_s & -k_s \\ -k_s & k_s \end{bmatrix}, C = \begin{bmatrix} c_b + c_s & -c_s \\ -c_s & c_s \end{bmatrix}$$

are symmetric positive-definite matrices by construction.

One may notice that

$$\dot{H}(x) = -\begin{bmatrix}\dot{x}_1 & \dot{x}_3\end{bmatrix} C \begin{bmatrix}\dot{x}_1 \\ \dot{x}_3\end{bmatrix} \le 0$$

emphasizing energy-loss mechanism is ensured naturally, due to existing damping. Additive damping c_A is intended to reinforce this energy-loss mechanism, in the presence of hazardous earthquake accelerations $\ddot{x}_g$.

All requirements of Prop. 1 are easy to be verified, with

$$y(x) = L_g V(x) = -\dot{x}_1^2 = -x_2^2$$

and relation (6) leads to bounded control law

$$c_A(y) = -k_1 \frac{y}{k_2 + y^2} = k_1 \frac{x_2^2}{k_2 + x_2^4} \ge 0 \tag{11}$$

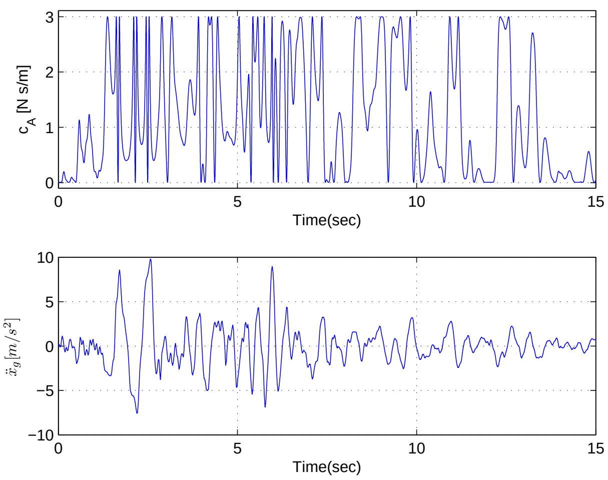


Fig. 3. (a) Semi-active controller results, with c_A from (11); (b) ground-level, total acceleration, seismic excitation signal $\ddot{x}_g(t)$. $k_1 = 0.6$; $k_2 = 0.01$

with parameters $k_1 \in \mathbb{R}_{>0}$ and $k_2 \in \mathbb{R}_{>0}$, adjusted such that (8) holds. Simulation results of Fig. 3 show that the time-evolution of control law during earthquake excitation is within desired constraints $c_A \in [0, c_{Max}]$.

C. Summarize

In Part 1 of this paper we have presented one way to achieve a bounded control on a nonlinear system. The theoretical result was shown to have practical value on an earthquake engineering problem.

In order to make the transition towards Part 2 of this paper, it is important to make a few observations:

- the key to obtaining *bounded control* was the user-defined choice for the bounded term in the storage function *derivative* (7b)
- the so-called storage functions used in passivity-based control are actually a generalization of Lyapunov functions
- plant model (9) can be written in lower-triangular form, which is typical for *backstepping*

These observations encouraged us to attempt to extend the same methodology towards bounded backstepping.

III. Part 2. Bounded Backstepping-Based Control

A. Preliminaries: recall

Definition 3: (CLF in [14]; [3, Def. 3.41]; [5, Def. 2.6]; [10]; [2, Def. 12.1, pp. 313]) Again, we consider *autonomous*, *affine in control*, nonlinear dynamical systems:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{12}$$

with $x \in \mathbb{R}^n$ state-space vector, $n \in \mathbb{Z}_{>0}$; $u \in \mathbb{R}^m$ is input vector, $m \in \mathbb{Z}_{>0}$; $y \in \mathbb{R}^p$ output vector, $p \in \mathbb{Z}_{>0}$; the continuous vector fields f and h, together with the columns of g, are locally Lipschitz in x; and a differentiable (i.e. of class $\mathscr{C}^1$), positive definite and radially unbounded function $V:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we call V a *control Lyapunov function* (CLF) for (12), provided that any of the relations hold:

(i) for all $|x| \neq 0$ there exists u, such that

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x) \cdot [f(x) + g(x)u] < 0 \tag{13a}$$

$$\text{i.e., } \dot{V}(x) = L_f V(x) + L_g V(x) u < 0$$

(ii) for all $|x| \neq 0$,

$$L_g V(x) = 0_{1\times m} \quad \Rightarrow \quad L_f V(x) < 0 \tag{13b}$$

Relation (13) translates the *global asymptotic stabilization* (GAS) condition at the origin, for controlled *nominal system* (12). □

B. Motivation

Backstepping based-control (BBC) (see [7], [5], [4, Ch. 5], [3, Ch. 6.1], [16, Ch. 14.3], [2, Ch. 12.5]) consists of an iterated step-by-step procedure, constructing at each ith step, $(i = 2,..,n)$, an augmented Lyapunov function

$$V_i = V_{i-1} + \tfrac{1}{2}\varepsilon_i^2$$

ending at the nth step with a control law and a global CLF. At each step, a condition of the type (13), translating into $\dot{V}_i < 0$ is necessary to calculate *virtual controls* and *tracking error variables* ε_i. This leads to the main question, motivating this work, namely: Next we will study various shapes of $\dot{V}_i$ terms

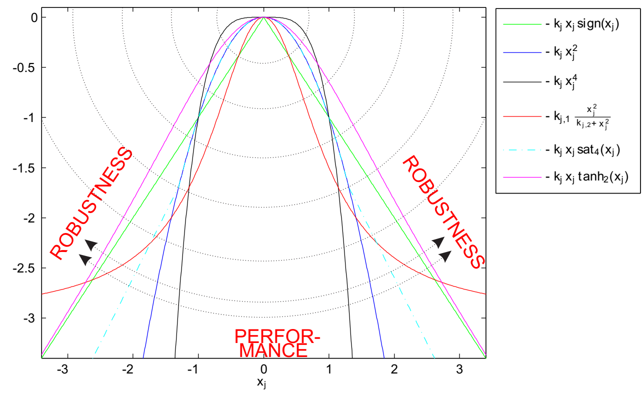
How to choose, geometrically, the shape of $\dot{V}_i$ in order to obtain bounded control ?

and then, we will use them in order to check feedback loop properties on a simple one-dimensional nonlinear system. Although the aim of this paper is indeed on *bounded control*, here the discussion will be broader (e.g. robustness and performance).

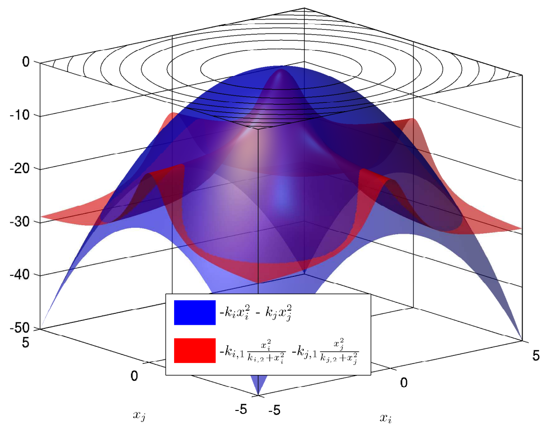
C. Geometrically shaping Lyapunov function derivatives

In the sequel, Lyapunov function derivatives (LFD) concept is revisited. With present topic, we hope to give an insight, easy to follow and, to some extent, originally structured, for the 'right' choice of LFD. LFD, let $\dot{V}(x_1, x_2, \ldots, x_n)$ be, are n-dimensional manifolds (i.e. hypersurfaces) in a $(n+1)$-dimensional space, with $n \in \mathbb{Z}_{\geq 1}$ being the dimension of state-space. Next, we intend to show that, from a geometrical perspective, by manipulating these shapes, one can efficiently *tune* behavior of closed-loop system. How many possibilities do we have? Theoretically, an infinity, provided that (13) holds. This paper discusses only a few.

Throughout this section, curves in Fig. 4(a) and surfaces in Fig. 4(b), will serve as simple illustrative, visual guideline. Some curves correspond to classical choice of LFD terms of known control techniques, others are not. We will analyze



(a) in $\mathbb{R}^1$ state-space. $j = 1,..,n$; $k_j = 1$; $k_{j,1} = 3$; $k_{j,2} = 1$; $\varepsilon = 1.3$ for $\text{sat}_4(\cdot)$ and $\tanh_2(\cdot)$.



(b) in $\mathbb{R}^2$ state-space. $i, j = 1,..,n$; $i \neq j$; $k_i = 1$; $k_j = 1$; $k_{i,1} = 15$; $k_{i,2} = 1$; $k_{j,1} = 15$; $k_{j,2} = 1$.

Fig. 4. A catalog of feasible choices of LFD terms.

which are the advantages/disadvantages of choosing one or another, with respect to *control system* properties (e.g., trade-off between robustness and performance; boundedness of control law).

The following functions were used to draw the curves in Fig. 4(a):

$$\text{sign}(\zeta) \triangleq \begin{cases} +1 & \text{if } \zeta > 0 \\ \ \ 0 & \text{if } \zeta = 0 \\ -1 & \text{if } \zeta < 0 \end{cases}$$

$$\text{sat}_4(\zeta) \triangleq \begin{cases} \zeta & \text{if } |\zeta| \leq \varepsilon \\ \varepsilon\ \text{sign}(\zeta) & \text{elsewhere} \end{cases}$$

$$\tanh_2(\zeta) \triangleq \tanh(\zeta/\varepsilon)$$

with $\tanh(\cdot)$ hyperbolic tangent function; parameter $\varepsilon \in \mathbb{R}_{>0}$. Instead of $\tanh(\cdot)$, one may use any sigmoid-type functions, like the error function $\text{Erf}(\zeta) \triangleq \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-s^2} \mathrm{d}s$.

In Fig.4(b), the following two LFD terms, part of $\dot{V}$, were represented in blue and red, respectively:

- $-\begin{bmatrix} x_i & x_j \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} k_i & 0 \\ 0 & k_j \end{bmatrix}}_{\stackrel{\text{not.}}{=} Q} \begin{bmatrix} x_i \\ x_j \end{bmatrix}$
- $-\begin{bmatrix} \frac{x_i}{\sqrt{k_{i,2}+x_i^2}} & \frac{x_j}{\sqrt{k_{j,2}+x_j^2}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} k_{i,1} & 0 \\ 0 & k_{j,1} \end{bmatrix} \begin{bmatrix} \frac{x_i}{\sqrt{k_{i,2}+x_i^2}} \\ \frac{x_j}{\sqrt{k_{j,2}+x_j^2}} \end{bmatrix}$

with k_i, k_j, $k_{i,1}$ and $k_{j,2} \in \mathbb{R}_{>0}$.

Next, the focus is on a simple one-dimensional ($n = 1$) case study. Hopefully, ideas are easier to be shared this way, thus facilitating possible reflection on extensions to higher order systems defined in $\mathbb{R}^n$ space, with $n \geq 2$.

D. A case study

Given the nominal, nonlinear, dynamical system

$$\dot{x}_1 = \frac{x_1^2}{1+x_1^2} + u \tag{14}$$

we seek stationary control law u, such that GAS is ensured at the origin $x_1 = 0$.

Solution. Let us choose the *candidate Lyapunov function*

$$V_1(x) = \tfrac{1}{2}x_1^2 \tag{15}$$

To construct a CLF from V_1, (13) should hold. In Fig. 4(a), we give some feasible choices for $\dot{V}_1$, leading to robust vs. performance and bounded control.

For BBC, the classical, most common choice is this $\mathscr{C}^2$ class function, $\dot{V}_1 = -k_1 x_1^2 < 0$, with $k_1 \in \mathbb{R}_{>0}$; it follows that

$$\underset{\text{BBC}}{u} = -\tfrac{x_1^2}{1+x_1^2} - k_1 x_1 \tag{16}$$

which is unbounded by construction, as

$$\lim_{|x_1| \to \infty} |u(x_1)| = +\infty$$

On the other hand, the classical, most common choice for sliding mode control is this $\mathscr{C}^0$ class function, $\dot{V}_1 = -k_1|x_1| = -k_1 x_1 \operatorname{sign}(x_1) < 0$, with $k_1 \in \mathbb{R}_{>0}$; it leads to

$$\underset{\text{SMC}}{u} = -\tfrac{x_1^2}{1+x_1^2} - k_1 \operatorname{sign}(x_1) \tag{17}$$

which is bounded for this particular system (although this does not necessarily hold in general); major drawbacks consist of inherent *chattering* problems (see [16]) and uniqueness loss of system trajectories. To overcome them, one can use regularization techniques, e.g., by choosing (see Fig. 4(a))

(i) $\dot{V}_1 = -k_1 x_1 \tanh_2(x_1)$ will lead to (17) with $\tanh_2(\cdot)$ instead of $\operatorname{sign}(\cdot)$
(ii) $\dot{V}_1 = -k_1 x_1 \operatorname{sat}_4(x_1)$, resulting in (17) with $sat_4(\cdot)$ instead of $\operatorname{sign}(\cdot)$

Discussion on feedback properties. For parameter choice indicated in Fig. 4(a), the former control law (i) is more robust then (ii) for $x_1(t_0) \in \mathbb{R}$, while both are more robust then (17) for $|x_1(t_0)| < 1$. When $|x_1(t_0)| > 1$, (ii) achieves better performance then any of the other two. All this information can easily be read on Fig. 4(a): details are provided hereafter.

Performance. Roughly speaking, performance increases as the curve $\dot{V}_1(x_1)$, assimilated to any LFD term illustrated in Fig. 4(a), approaches the axis of ordinates. The higher the values of $\dot{V}_1(x_1)$, for any fixed x_1 (this is equivalent to increasing the value of adjustment parameter k_1), the better nominal system behavior gets, in terms of lower *settling time* values, while system trajectories converge faster towards the origin. It also leads to higher peak values for control law u, which might be a practical drawback; sensitivity to noise increases. In order to reduce actuator wear and stress, it might be better avoided. If the curve $\dot{V}_1(x_1)$ is chosen such that, it is situated too close to axis of ordinates, one gets into situations:

- where singularities appear inside control low: e.g. when choosing $\dot{V}_1(x_1) = -k_1|x_1|^\gamma$, with $\gamma \in (0,1)$, a singularity will appear at the origin $x_1 = 0$;
- where multiple (i.e. non-unique) feedback trajectories are possible: e.g. when choosing $\dot{V}_1(x_1) = -k_1 x_1^\gamma$, with $\gamma = 4/3$, there are two feasible solutions: $x_1(t) \equiv 0$ and $x_1(t) = \left(-\frac{2t}{3}\right)^{3/2}$, for $k_1 = 1$ and initial condition $x_1(0) = 0$ (see, for details, [16, pp. 88])

In $\mathbb{R}^2$ space, ensuring performance is equivalent to saying that LFD surface should draw near z-axis. See Fig. 4(b).

Robustness. Robustness should be pursued:

- to achieve 'moderate' variations of controller output
- to reduce system sensibility and reactiveness to fast noise variations; noise might be due, e.g., to (faulty) measurement equipment

Let us come back to Fig. 4(a). Improved robustness properties of feedback system is achieved, i.e., overall robustness increases, as $\dot{V}_1(x_1)$ curve furthers the axis of ordinates: for any fixed x_1 on the axis of abscissas, lowering $\dot{V}_1(x_1)$.

In Fig. 4(b), robustness increases, as the surface associated to LFD terms, steps away from z-axis. This figure shows that, the blue surface is capable to ensure more feedback system robustness than the red one, around system origin $(x_1, x_2) = (0,0)$. While for sufficiently large $|x_1|$, $|x_2|$ values, the contrary holds.

Bounded control. Let us come back to system (14); let (15) be, and choose $\dot{V}_1 = -k_{1,1}\frac{x_1^2}{k_{1,2}+x_1^2} < 0$, with $k_{1,1} \in \mathbb{R}_{>0}$ and $k_{1,2} \in \mathbb{R}_{>0}$ (see Fig. 4(a)); the control law is calculated

$$\underset{\text{bounded}}{u} = -\tfrac{x_1^2}{1+x_1^2} - k_{1,1}\tfrac{x_1}{k_{1,2}+x_1^2} \tag{18}$$

(see Fig. 5). It is structurally bounded, since, as one may notice, $\sup_{x_1} |u(x_1)|$ is finite; also it is worth to notice that, $\lim_{|x_1| \to \infty} u(x_1) = -1$, meaning that controller is not running to the saturation limit, for large $|x_1|$ values, as it would have been the case, if we explicitly saturated, using (1), any linear-type or nonlinear controller, like (16). In Fig. 1, we illustrated graphically only the second term of (18).

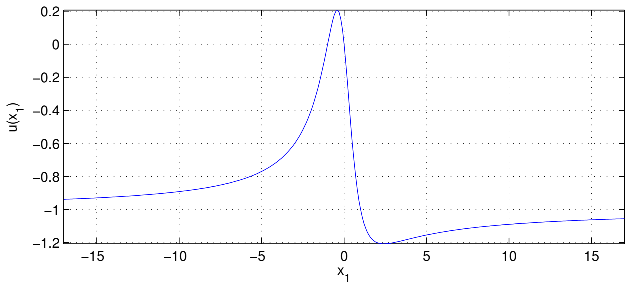


Fig. 5. Structurally bounded control (18), ensuring GAS. $k_{1,1} = 1$; $k_{1,2} = 1$.

E. Summarize

Bounded backstepping has been achieved on a simple one-dimensional nonlinear system. However, attempts to generalize this methodology towards n-dimensional systems have not been successful so far.

F. Further work

This section is intended to be an open discussion. It would be interesting to know if this methodology consisting on shaping Lyapunov function derivative terms could lead to *bounded control* on other nonlinear control designs. We have two in mind, briefly introduced hereafter.

Forwarding (see [8], [3, Ch. 6.2], [2, Ch. 12.6]) is another example of recursive design. At the last step one has built $V(x)$ which ensures that the non-controlled system $\dot{x} = f(x)$ is $\mathscr{C}^1$ dissipative. Consequently, by calculating $\dot{V}(x)$ on the controlled system (12) and properly imposing its shape, the control law $u(x)$ is extracted.

In *sliding mode control* (SMC) (see [16]), one needs to build a stable sliding surface, let us call it $\sigma = \sigma(x)$, and let $V(\sigma(x)) = \frac{1}{2}\sigma(x)^2$ be. Once more, a condition of the type (13) is necessary, i.e. $\dot{V}(\sigma) = \sigma\dot{\sigma} < 0$ (except at the origin, on the zero-error manifold $\sigma(x) = 0$), prior to calculating the control law. Again, the interest would be here on how to choose $\dot{V}$ in order to obtain bounded control?

IV. Conclusions

In the first part of this paper we have presented a general result on bounded passivity-based control, that can be applied to n-dimensional systems. In particular, it was applied to a common problem arising in earthquake engineering ($n = 2$). In the second part of the paper we have presented preliminary results on bounded backstepping. They were applied on a one-dimensional system. Hopefully this proposed methodology will be extended in the future to higher dimensional systems and/or other nonlinear control techniques.

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