Optimal Auction Design under Non-Commitment *

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Abstract

We characterize revenue-maximizing mechanisms in Myerson's (1981) environment when the seller behaves sequentially rationally, in the sense that she cannot commit not to propose a new mechanism if the previously chosen one fails to allocate the object. We show that the seller-optimal mechanism takes the same form as in the case when there is commitment: The seller maximizes revenue by assigning, at $t = 1$, the good to the buyer with the highest virtual valuation if it is above a buyer-specific reserve price. If no trade takes place at $t = 1$, at $t = 2$, the seller assigns the object to the buyer with the highest posterior virtual valuation, provided that it is above the seller's value.

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The classic works on auctions (Myerson (1981); Riley and Samuelson (1981)) characterize the revenue-maximizing allocation mechanism for a risk-neutral seller who owns one object and faces a fixed number of buyers whose valuations are private information. An important assumption in these papers is that the seller commits to withdraw the item from the market if it is not sold. This commitment assumption is far-fetched and often not met in reality. Christie's in Chicago auctions bottles of wine that failed to sell in earlier auctions. The U.S. government re-auctions properties that fail to sell: Lumber tracts, oil tracts, and real estate are put up for a new auction if no bidder bids above the reserve price.[1] As Porter (1995) reports, 46.8 percent of the oil and gas tracts with rejected high bids were put up for a new auction. In March 2010, the FCC announced that in 2011, it would re-auction part of the 700 MHz wireless spectrum that failed to sell in 2009. The key issue is not only that commitment is often unrealistic, but, more importantly, that an auction that is desirable with commitment may lead to poor outcomes if there is limited commitment.

The durable-good monopolist literature was the first to study the effects of a seller's inability to commit to a given institution if it fails to realize all gains of trade (Bulow (1982); Gul, Sonnenschein, and Wilson (1986); Stokey (1981)). McAfee and Vincent (1997) study an auction setup in which the seller behaves sequentially rationally. The papers cited above restrict the procedures that the seller can employ (the seller chooses prices in the durable-goods papers and reservation prices in McAfee and Vincent (1997) and in Liu, Mierendorff, and Shi (2015))[2] and show that the seller's inability to commit reduces monopoly profits. Here, we maintain the assumption that the seller behaves sequentially rationally, but we allow the seller to choose any selling procedure (mechanism). Our goal is to determine which procedure maximizes revenue and to investigate the extent to which allowing for general mechanisms enables the seller to mitigate revenue loss due to the lack of commitment.

We consider the following scenario: There is a risk-neutral seller who owns a single object and faces I risk-neutral buyers. Valuations are private, independently distributed across buyers, and constant over time. The buyers and the seller interact for two periods and discount the future with the same discount factor. At the beginning of each period, the seller proposes a mechanism to sell the object. If the object is sold, the game ends; otherwise, the seller returns in the next period and offers a new mechanism. The game ends after two periods even if the object remains unsold.[3] We show that the optimal mechanism takes the same form as in the case when there is commitment: First- (or second-) price auctions with optimally chosen reserve prices are revenue-maximizing when buyers are ex-ante

[1]These examples are also mentioned in McAfee and Vincent (1997).

[2]Other papers that study reserve price dynamics without commitment are Burguet and Sákovics (1996), which examines cases of costly bidding, and Caillaud and Mezzetti (2004), which looks at sequential auctions of many identical units.

[3]The analysis of the case of $T = 2$ contains the most essential insights and can be carried out with less-burdensome notation. Section 6 presents an overview of the analysis of the case in which $2 < T < \infty$.

identical. When buyers' valuations are drawn from different distributions, the seller maximizes revenue by assigning the good to the buyer with the highest virtual valuation if it is above a buyer-specific reserve price. Reserve prices drop over time. How much the optimal reserve prices drop depends on the discount factor. Inability to commit is costly for the seller. The revenue loss is highest for intermediate values of the discount factor and when the number of buyers is small.

In the U.S., the FDIC runs a large number of auctions of distressed assets (real estate, in particular). Properties are auctioned off with reserve prices, and in a number of cases, the initial reserve price is too high and the property is sold later with a lower reserve.[4] In both the U.S. and Europe, fiscal crises have led to a surge in distressed assets for sale.[5] The amounts that financial institutions recover from these sales is very important for their future solvency and the health of the financial sector. Given that unsold assets are placed back on the market, there is a lack of commitment to the initial reserve price. Our analysis establishes that auctions with reserve prices are revenue-maximizing and that revenue loss due to the lack of commitment is small when the seller is either very patient or very impatient and when the number of buyers is relatively large. This last finding says that when demand exceeds supply significantly, the impact of lack of commitment is very modest, but the finding also stresses that the auction design becomes more important when supply surges relative to demand, as happens during a crisis.

Methodologically, this is the first paper that solves for the optimal mechanism under limited commitment in a multi-agent environment allowing for a continuum of valuations, and for the possibility that the seller controls what agents observe–the *transparency* of mechanisms.[6] Mechanism design under non-commitment is notoriously difficult even in single-agent environments because, as the literature on the ratchet effect (Freixas, Guesnerie, and Tirole, 1985; Laffont and Tirole, 1988) first observed, one cannot use the revelation principle,[7] which asserts that the choice of mechanism(s) is final. This implies that the designer can never change the rules in the future, even though it might become obvious that better ones exist. When this commitment assumption fails, there is no generally applicable canonical class of mechanisms.

Kumar (1985), who formulated what he refers to as the "noisy revelation principle," provides a first step towards providing a canonical class of mechanisms when the principal behaves sequentially ratio-

[4]See http://www.fdic.gov/buying/historical/index.html and McAfee, Quan, and Vincent (2002), who thoroughly document this phenomenon.

[5]See Stovall and Tor (2011) or "Troubled European Assets Come to Market," in *The Wall Street Journal*, Feb. 5 2013.

[6]Bester and Strausz (2007) allow a mediator to execute the principal's mechanism and then release a noisy signal of the agent's choice. In other words, the mediator in Bester and Strausz (2007) controls what the principal observes. In our paper, the transparency of mechanisms controls what *agents* observe. This dimension of transparency is not relevant in Bester and Strausz (2007)'s model because there is only one agent.

[7]See Laffont and Tirole (1988), Salanie (1997), or, for a more recent treatment, Skreta (2006b).

nally.[8] Despite this result, finding the optimal mechanism is not simple, as showing which posteriors are optimal is challenging. Bester and Strausz (2001) establish that for single-agent and finite-type models, it is without loss of generality for the principal to restrict attention to mechanisms with message spaces that have the same cardinality as the type space, and in which the agent reports his true type with strictly positive probability. Again, finding the optimal set of types that each type of the agent is randomizing over (and with what probabilities) is not trivial. In addition, Bester and Strausz (2000) show that this result fails in a two-agent example.[9]

This paper employs the method developed in Skreta (2006b), which relies on characterizing equilibrium outcomes and is introduced in Section 1. Section 2.1 describes necessary conditions for an outcome to be PBE-feasible, and, using these conditions as constraints, in Section 2.2, we formulate the seller's search for the revenue-maximizing sequentially rational auctions as a constrained maximization problem–Program NC. We solve Program NC in Section 3. In Subsection 3.3 we find an assessment that implements the solution of Program NC and is a PBE. These are the two main steps of the proof of the main result, Theorem 1. The value of commitment is discussed in Section 4, which also illustrates our characterization in a simple example.

To solve Program NC, we analyze how the second-period optimal mechanism, which is a vector of functions, varies as a function of arbitrarily complex posteriors. Lemma 1 formalizes the result that in the absence of commitment, eliciting information sequentially is costly because the buyers at $t = 1$ anticipate that the seller will be exploiting this information at $t = 2$ and, hence, require that they be rewarded for it in advance. Based on Lemma 1, Proposition 2 shows that the best action for the seller is to pool all valuations below a cutoff until the second period of the game, when she has commitment power, since trying to separate them at $t = 1$ is too costly. Establishing Proposition 2 is the central and most challenging part of the analysis since we have to show that pooling the low end of valuations is better than any other conceivable alternative in which buyers mix over several reports, use reporting rules with non-convex supports, and so forth. This step is harder here, where there are many agents compared to the single-agent environment in Skreta (2006b); this is the case because when there is only one agent, the optimal second-period mechanism is a posted price, whereas, here, it is an auction and performing comparisons of how it varies as a function of various posteriors is more complex.

Given Proposition 2, Lemma 2 establishes that, at an optimum, valuations below the cutoff do not get the good and do not pay anything at $t = 1$. Lemma 3 shows that this cutoff is higher than the

[8] Kumar's principle states that without any loss, we can restrict attention to mechanisms where the agents report a probability distribution about their types. Kumar (1985)'s result captures the essential difference between mechanism design with and without commitment–namely, that without commitment, in all but the first period, the principal's posterior beliefs are endogenous and depend on her previous mechanism choice.

[9] Evans and Reiche (2008) show that the revelation principle does extend in the Bester and Strausz (2000) example if one allows ex-ante payments, but it fails even with ex-ante payments if more than one agent has private information.

revenue-maximizing reserve price with commitment (the valuation where a buyer's *virtual* valuation is equal to the seller's value). Then, a solution of Program NC separates valuations into two groups: the-no-trade-at-$t = 1$-region, where all valuations below a cutoff are pooled together with the lowest possible valuation and never get the good or pay anything at $t = 1$; and the-trade-at $t = 1$-region, where the seller assigns the object to a buyer with the highest virtual valuation.

Apart from the challenge arising from the lack of a "revelation principle" result that is common to single- and multi-agent environments, there are two conceptual issues specific to multi-agent environments: The first is related to the fact that what buyers (or agents, more generally) observe at each stage–the transparency of mechanisms–critically affects their beliefs about each other, which may, in turn, affect their future behavior. For example, competing in a sealed-bid, versus an open-outcry, auction has different implications for what buyers learn about each other. The second difference is that with multiple agents, the mechanism designer may endogenously become privately informed over time.[10] This is possible because the designer (the seller) observes more than the agents (the buyers) observe about the behavior of their competitors; think, for instance, of sealed-bid auctions. These two issues do not arise when there is a single buyer and compound the difficulties arising from the lack of an appropriate "revelation principle."

In Section 5 we address the issue of transparency by introducing an alternative definition of mechanisms that determines not only who gets the object and the payments as a function of buyers' behavior, but also what buyers know after the mechanism is played. Formally, this is done by modeling first-period mechanisms as game forms endowed with an information-disclosure policy. Then, by applying the information-disclosure irrelevance result from Skreta (2011), we show that the set of sequentially optimal mechanisms at $t = 2$ is independent of the disclosure policy used at $t = 1$. We conclude that without any loss, we can model period-one mechanisms as game forms and assume that all buyers observe all actions chosen, which, in turn, implies that the seller does not become privately informed.

The first paper to incorporate disclosure policies as part of the design of the optimal mechanism is Pavan and Calzolari (2006), who consider a sequential common agency problem. There, disclosure determines how much information is communicated by the stage-one principal to the stage-two principal. Calzolari and Pavan (2006) apply this approach in a model of monopoly with resale. In their paper, the disclosure rule discloses to each buyer an offer/ask price recommendation for the resale game. As in this paper, the disclosure policy in Calzolari and Pavan (2006) is meant to capture what agents learn about the other agents' behavior in the previous stage. The only difference is that, in this paper, in the second period, agents participate in another mechanism offered by the original principal, whereas in Calzolari and Pavan (2006), in a resale market, that does not involve the original principal.

[10]For a brief account of the literature on informed-principal problems, see Skreta (2011).

The ideas and techniques developed in the present paper have a large set of potential applications. One area in which the designer chooses mechanisms sequentially is that of government procurement and, in particular, defense procurement:[11] There are typically multiple stages until the final winner is determined; in each of these stages, sellers submit bids, and based on the bids, a subset of them advances to the next stage. If a bidder signals too much about his private information early on, his rents may be reduced at a later stage. Also, how bidders compete at each stage may depend on the information they obtain about their competitors in earlier stages, so the issue of transparency arises here, too.

Our techniques could be applicable to situations in which other issues relating to transparency–privacy,[12] for example–are important.[13] Nowadays, sellers not only keep track of their buyers but also design their pricing schemes (coupons, loyalty programs) based on the information that they have obtained thus far.[14] When sellers track their interactions with various buyers, the issues of ratcheting and transparency are central.

Other related literature: "No sale" is not the only form of inefficiency of the classical optimal auction. Sometimes, it allocates the object to a buyer other than the one with the highest valuation, thus leaving resale opportunities open for the new owner. Zheng (2002) studies optimal auctions allowing for resale, assuming that the original seller controls the bargaining power of the new owner in the resale market. With an impressive construction, that paper derives conditions under which Myerson (1981)'s optimal allocation can also be attained by a seller who cannot prevent resale. In Zheng (2002), there is no discounting. Calzolari and Pavan (2006) also study revenue-maximizing auctions with resale. Compared to Zheng (2002), they drop the assumption that the initial owner controls the bargaining power in the resale market. They also allow the seller to disclose information to the reseller and find that it is impossible to maximize revenue with a deterministic selling procedure. Here, we look at the complementary problem of characterizing mechanisms for a seller who cannot prevent herself from re-auctioning the good, and we allow for discounting.

Other works assume that the seller has even less commitment than the seller in this paper; in particular, the seller cannot even commit to carry out the rules of the current auction. In McAdams and Schwarz (2007), the seller, after observing the bids, cannot commit not to ask for another round of bids. The current paper, in contrast, assumes that the seller chooses revenue-maximizing procedures

[11] Bower (1993) outlines some of the commitment issues that arise in defense procurement.

[12] The pioneering papers on the economics of privacy are Hirshleifer (1980); Posner (1981); Stigler (1980). For a recent survey, see Hui and Png (2006). For a paper contributing to the policy debate on privacy issues, see Varian (1996).

[13] The recent work of Izmalkov, Lepinski, and Micali (2005, 2008) addresses these issues and marries mechanism design theory with cryptography.

[14] See, for instance, Acquisti and Varian (2005) and Fudenberg and Villas-Boas (2007) for an excellent and comprehensive survey of the work in this area.

and commits at each stage to carry out the mechanism for that stage. Vartiainen (2011), too, examines a symmetric model in which buyers' types are finite and there is no discounting.[15] He allows both forms of no commitment (no commitment to current or to future mechanisms), but he assumes that all actions are publicly observable and asks what mechanism leads to a sustainable outcome given complete lack of commitment. He shows that, essentially, only the English auction achieves sustainability. Again, this paper is different from Vartiainen (2011) in that our interest is in designing optimal mechanisms and there is discounting. We also allow for a continuum of valuations and for non-transparent mechanisms.

1 The Model

A risk-neutral seller, indexed by S, owns a unit of an indivisible object, and faces I risk-neutral buyers.[16] The set of all players–the buyers and the seller–is denoted by $\bar{I} = \{S, 1, .., I\}$. The seller's valuation is known and normalized to $v_S \equiv 0$, whereas v_i–the valuation of buyer i–is private information and is distributed according to F_i on $V_i = [0, b_i]$, with $0 \leq b_i < \infty$ which has a *continuous* and *strictly positive* density f_i.[17] Buyers' valuations are distributed independently of one another and remain constant over time. Let $F(v) = \times_{i \in I} F_i(v_i)$, where $v \in V = \times_{i \in I} V_i$ and $F_{-i}(v_{-i}) = \times_{\substack{j \in I \\ j \neq i}} F_j(v_j)$. Time is discrete and finite, and the game lasts $2 \leq T < \infty$ periods. The seller and the buyers have the same discount factor $\delta \in [0, 1]$. All elements of the game, apart from the realization of the buyers' valuations, are common knowledge. The seller aims to maximize expected discounted revenue, whereas buyers aim to maximize expected discounted surplus. Since most of the intricacies of the problem appear even when $T = 2$, we initially focus on this case and defer the analysis of $T > 2$ to Section 6.

The set of buyer i's actions (or reports) is the Borel set A_i, which contains all valuations $V_i \subset A_i$, and can be a strict superset of V_i. Let $A = \times_{i \in I} A_i$. A *mechanism* $M_t = (q^t, z^t) : A \to [0, 1]^{\bar{I}} \times \mathbb{R}^{\bar{I}}$ specifies for each vector of actions (or reports) $a \equiv (a_i, a_{-i})$, the probability i is assigned the good at t, $q_i^t(a)$, and i's expected payment at t, $z_i^t(a)$. Each mechanism has the same set of actions for buyer i–namely, A_i–so the set of actions is not an object of choice. This is without loss of generality[18] and

[15] In contrast to the problem in Vartiainen (2011), without discounting, our problem is trivial: In that case, the seller would wait until the last period of the game and offer the mechanism described in Myerson (1981), obtaining the highest possible revenue given the presence of asymmetric information.

[16] We use the female pronoun for the seller and male pronoun for the buyers.

[17] The fact that we take each buyer's lowest valuation to be zero is not important for the results. Nothing in the analysis hinges on that.

[18] To see why this is without loss of generality, consider a case in which the seller offers mechanisms where at $t = 1$ the action set is A_i^1 and at $t = 2$ it is A_i^2. This can be equivalently represented by the seller offering a mechanism in both periods with action set $A_i = A_i^1 \cup A_i^2$, and by amending the mechanisms as follows: At $t = 1$, for all vectors of actions where $a_i \in A_i^2 \backslash A_i^1$ set $q_i^1(a_i, a_{-i}) = q_i^1(\hat{a}_i, a_{-i})$ and $z_i^1(a_i, a_{-i}) = z_i^1(\hat{a}_i, a_{-i})$, for some $\hat{a}_i \in A_i^1$. The $t = 2$-mechanism can be amended analogously. Doing so does not add any new outcomes (assignment probabilities and payments) at period 1. Moreover, it does not alter the sequentially rational outcomes at $t = 2$ since–as we later establish–in equilibrium, actions associated with identical $t = 1$-assignment probabilities must be followed by identical $t = 2$ assignment, and the proof of Proposition 2 shows that this $t = 2$-assignment is identical to the one that would be sequentially rational if we would merge all actions with identical $t = 1$-assignment probabilities into one action. Also, the formulation allows the action

avoids the complication of having the space of all mechanisms contain the set of all sets, which doesn't exist in the usual naive set theory. Note that the period-t mechanism specifies only what happens at period t, so it is analogous to a spot contract. Also, because the seller collects buyers' payments and keeps the good if no buyer gets it, we have $z^t_S(a) = -\Sigma_{i \in I} z^t_i(a)$ and $q^t_S(a) = 1 - \Sigma_{i \in I} q^t_i(a)$.

Timing and Strategies:

t=1: At the beginning of $t = 1$, nature determines the buyers' valuations; the seller chooses a mechanism and commits to it; buyers observe it and choose their actions. If, given the agents' actions, the mechanism prescribes trade, then the game ends; otherwise, we move to $t = 2$. All actions are publicly observable. This assumption is relaxed in Section 5.

t=2: At $t = 2$, given the vector of actions chosen at $t = 1$, a, the seller chooses a mechanism; buyers observe it and choose their actions. The game ends at $T = 2$, irrespective of whether or not trade takes place.

The fact that the seller commits to the period-1 mechanism implies that the period-1 allocation of the good, payments and disclosure are determined by the period-1 mechanism along with the actions chosen by the agents. Given that mechanisms and all actions are publicly observable, the seller's information sets I_S coincide with public histories, whereas buyer i's information sets, I_i, contain v_i in addition to the public histories. A public history at $t = 1$ contains M_1, while at $t = 2$, it contains M_1, the vector of period-1 reports a, and M_2. The seller's strategy specifies a mechanism for each element in I_S. We restrict attention to pure strategies for the seller.

Buyers' Strategies: At each information set, buyer i's *behavioral strategy* specifies a family of reporting rules m^t_i, where for each $v_i \in [0, b_i]$, m^t_i is a probability distribution over A_i: $\int_{A_i} m^t_i(a_i|v_i)da_i = 1$. The reporting rule $m^t_i(.|v_i)$ is a density that can be a degenerate function with one or more mass points. When it has a unique mass point, it is the Dirac function, which is not absolutely continuous with respect to the Lebesgue measure.[19] The proofs of the results do not require the reporting rule $m^t_i(.|v_i)$ to be absolutely continuous with respect to the Lebesgue measure.

Reporting Functions: Every reporting rule $m^t_i(.|v_i)$ is associated with a *reporting function* $m^t_i(a_i|.)$, which is a measurable function of v_i, and (roughly) represents the relatively likelihood that different valuations employ a given message a_i. If buyers employ pure strategies, then $m^t_i(a_i|.)$ is equal to one

set to be larger than the type space. If the seller wants to use a mechanism with fewer actions then, for all "superfluous" actions $\tilde{a}_i$, set $q^t_i(\tilde{a}_i, a_{-i}) = q^t_i(a_i, a_{-i})$ and $z^t_i(\tilde{a}_i, a_{-i}) = z^t_i(a_i, a_{-i})$, where a_i is an action in the restricted set that the seller wants to employ.

[19] When the density has mass points, the integral expression $\int_{A_i} m^t_i(a_i|v_i)da_i = 1$ abuses notation, but this is somewhat standard.

for the valuations that report a_i and zero otherwise. The support of the reporting function $m_i^t(a_i|.)$–denoted by $V_i(a_i)$–is the set of v_i's for which message a_i is in the support of the reporting rule $m_i^t(.|v_i)$. Let $\bar{V}_i(a_i) \equiv [\underline{v}_i(a_i), \bar{v}_i(a_i)]$ denote the closure of the convex hull of $V_i(a_i)$. Reporting functions are used to obtain Bayesian beliefs as follows:

Bayesian Beliefs: The seller's posterior belief at $t = 2$ about i's valuation after observing a_i at $t = 1$ is denoted by $f_i(.\,|a_i)$ and satisfies

$$f_i(v_i\,|a_i) \int_{V_i} m_i^1(a_i|t_i) f_i(t_i) dt_i = m_i^1(a_i|v_i) f_i(v_i). \tag{1}$$

Because buyers behave non-cooperatively, they choose their actions at $t = 1$ independently from one another. Then, upon observing the vector of actions a, the seller's posterior about the buyers' valuations satisfies

$$f(v\,|a) \equiv \times_{i\in I} f_i(v_i\,|a_i).^{20} \tag{2}$$

Our goal is to find a strategy profile that is PBE of the game and is associated with the highest expected discounted revenue for the seller among all PBE's.

Preliminary Observations: To simplify matters and to avoid introducing excessive notation, we make a few preliminary observations. First, what matters for the seller's mechanism-choice at $t = 2$ is her posterior beliefs about the buyers' valuations, which (for a given strategy profile) are determined by the vector a of period-1 reports. Thus, we index $t = 2$-mechanisms by a. Second, since $t = 2$ is the last period of the game, the seller commits to the outcome of the $t = 2$-mechanism, and the revelation principle applies as usual. We can then, at $t = 2$, restrict attention to mechanisms where it is a best-response for all buyers to participate and to report their true valuations. Thus, buyers' $t = 2$-reporting function is deterministic, and all reports in A_i other than those in V_i are superfluous.

Key Definitions: Fix a strategy profile in which buyers report the truth in the last period of the game, and i employs a reporting rule m_i at $t = 1$.[21] Then, when valuations are v, and $t = 1$-reports are a, the expected discounted probability that buyer i gets the good, and his expected discounted

[20] The way we formalized buyers' reporting rules is identical to the way signalling rules are formalized in Crawford and Sobel (1982). As in Crawford and Sobel (1982), the existence of regular distributions can be established as follows: Define a buyer's reporting rule to be a probability distribution $\hat{\mu}$ on the Borel subsets of $[0, b_i] \times [A]$ for which $\hat{\mu}(\tilde{V} \times [A]) = \int_{\tilde{V}} f$, for all measurable sets $\tilde{V}$. Loève (1955), pp.137-138, shows that in this setting there exist regular distributions $m(.|v)$ and $f(.|a)$ for $(v, a) \in V \times A$. Milgrom and Weber (1985), who introduced this distributional approach, show that it is equivalent to the mixed strategies used in the text. In this paper's setup, this formulation guarantees that $m(.|v)$ and $f(.|a)$ are measurable functions of v and a, respectively, and the integral expressions in the text are well-defined.

[21] To simplify notation, we omit the $t = 1$-superscript and write, for instance, $m_i(a_i|v_i)$ instead of $m_i^1(a_i|v_i)$, q_i instead of q_i^1 and so forth, while we keep the $t = 2$-superscripts, which are necessary for clarity.

payment are, respectively, given by:

$$\bar{q}_i(v,a) \equiv q_i(a) + q_S(a)\delta q_i^{2(a)}(v) \text{ and } \bar{z}_i(v,a) \equiv z_i(a) + q_S(a)\delta z_i^{2(a)}(v), \tag{3}$$

whereas his expected discounted payoff at (v,a) is:

$$u_i(v,a) \equiv \bar{q}_i(v,a)v_i - \bar{z}_i(v,a). \tag{4}$$

From (3) and (4), we also define: $U_i(v_i) \equiv E_{V_{-i}}E_A[u_i(v,a)]$ and $U_i^{a_i}(v_i) \equiv E_{V_{-i}}E_{A_{-i}}[u_i(v,a)]$, where $U_i^{a_i}(v_i)$ is i's expected payoff when his valuation is v_i; he reports a_i at $t=1$, and play proceeds according to the specified strategy profile. These definitions imply that

$$U_i(v_i) = E_{A_i}[U_i^{a_i}(v_i)], \tag{5}$$

where the expectation is taken with respect to the density $m_i(.|v_i)$ that describes the reporting rule.

Analogously, we define

$$P_i(v_i) \equiv E_{V_{-i}}E_A[\bar{q}_i(v,a)] \text{ and } P_i^{a_i}(v_i) \equiv E_{V_{-i}}E_{A_{-i}}[\bar{q}_i(v,a)] \tag{6}$$

and $X_i(v_i) \equiv E_{V_{-i}}E_A[\bar{z}_i(v,a)]$ and $X_i^{a_i}(v_i) \equiv E_{A_{-i}}E_{V_{-i}}[\bar{z}_i(v,a)]$. Note that the above definitions imply $U_i(v_i) = P_i(v_i)v_i - X_i(v_i)$. We also use:

$$P_i^{1(a_i)} \equiv E_{V_{-i}}E_{A_{-i}}[q_i(a)] \tag{7}$$

and

$$P_i^{2(a_i)}(v_i) \equiv E_{V_{-i}}E_{A_{-i}}[q_S(a)q_i^{2(a)}(v)], \tag{8}$$

which denote the interim expected probability that i gets the good at $t=1$ and at $t=2$, respectively, given a $t=1$-report a_i. Analogously, we define $X_i^{1(a_i)}$ and $X_i^{2(a_i)}(v_i)$.

In the above definitions, the expectation over A is taken with respect to

$$m(a|v) \equiv \times_{i\in I} m_i(a_i\,|v_i\,), \tag{9}$$

while the expectation over A_{-i} with respect to $m(a_{-i}|v_{-i}) \equiv \times_{\substack{j\in I\\ j\neq i}} m_j(a_j\,|v_j\,)$.

Outcomes: The *outcome* of a strategy profile (not necessarily an equilibrium) is an *allocation rule* p and a *payment rule* x obtained as follows:

$$p_i(v) \equiv E_A[\bar{q}_i(v,a)] \text{ and } x_i(v) \equiv E_A[\bar{z}_i(v,a)]. \tag{10}$$

The allocation rule specifies for all $i \in I$ the expected, discounted probability that player i obtains the object, and the payment rule specifies the corresponding expected, discounted payment when the realized vector of buyers' valuations is v. Analogously, we can define the outcomes of continuation games. Note that (6) and (10) imply that

$$P_i(v_i) = E_{V_{-i}}[p_i(v)]. \tag{11}$$

If (p, x) is an outcome of a strategy profile that is a PBE, it is called PBE-feasible.

Methodology: Our goal is to find a strategy profile that is PBE of the game and is associated with the highest expected discounted revenue for the seller. For this, it would suffice to find all PBEs of our game. However, as explained in the introduction, there is no "revelation principle"type of result for the first-stage mechanism, given that the seller behaves sequentially rationally and cannot commit not to re-optimize her choice at $t = 2$. In order not to impose any ad hoc restrictions, the seller's choice set at $t = 1$ is the set of all mechanisms. The resulting game, however, is too complex to establish that a strategy profile is a PBE: The difficulty arises because the seller may deviate at $t = 1$ to mechanisms where no continuation equilibrium exists. Because it is impossible to evaluate such deviations, it is impossible to establish that a proposed strategy profile is an equilibrium.[22] To circumvent this difficulty, we introduce a *canonical game* that has the same outcomes as the original game but is simpler in that the seller is not a player: In the canonical game, the seller chooses a *dynamic mechanism* once and for all:[23]

Definition 1 A *dynamic mechanism* $\mathcal{M}$ specifies a period-1 mechanism $M_1 = (q^1, z^1)$, and for all $t = 1$-vectors of actions a for which the seller keeps the good at $t = 1$, a period-2 mechanism $M_{2(a)} = (q^{2(a)}, z^{2(a)})$.

A dynamic mechanism is *PBE-feasible* if, given the proposed mechanisms there exist buyers' strategies such that: For all i, buyer i's strategy specifies: (i) for each $M_{2(a)}$ a continuation reporting rule that is a best response given $M_{2(a)}$ and his posterior beliefs consistent with Bayes' rule along the path, and, (ii) a $t = 1$-reporting rule that is a best response given M_1 and continuation strategies. Moreover, given these buyers' strategies, for all a, mechanism $M_{2(a)} = (q^{2(a)}, z^{2(a)})$ is optimal for the seller at $t = 2$ given the buyers' $t = 2$-continuation reporting rules and the seller's posterior beliefs derived from the buyers' $t = 1$-reporting rules using Bayes' rule (given by (2)).

[22] Existence of an off-path equilibrium outcome, no matter the mechanism the principal deviates to (even deviation to indirect mechanisms), is assumed in Myerson (1983) but, there, agents' type and strategy spaces are finite. In this paper, we assume a continuum of valuations and the possibility of uncountable action spaces.

[23] I am grateful to the editors of this journal for suggesting this insightful way to deal to the equilibrium-existence issue.

In the canonical game, once-and-for-all, the seller chooses a mechanism as in standard mechanism-design with commitment. However, she chooses a dynamic mechanism, and sequential rationality is introduced by requiring the dynamic mechanism to be *PBE-feasible*. It is immediate to see that a PBE-feasible outcome of the original game is a PBE-feasible outcome of the canonical game. Also, arguments similar to those establishing the Renegotiation-Proofness Principle motivate restricting attention to dynamic mechanisms that are not reneged at $t = 2$: For any dynamic mechanism and associated PBE, such that the seller deviates at $t = 2$ to another mechanism, there exists another dynamic mechanism and associated PBE in which the seller does not deviate at $t = 2$ that implements the same outcomes.

Definition 2 A dynamic PBE-feasible mechanism $\mathcal{M}$ is *optimal* if the seller's expected revenue, given $\mathcal{M}$ is higher compared to any other PBE-feasible dynamic mechanism.

Still, given the generality of the stage mechanisms, it is not obvious how to find an optimal dynamic mechanism. To sidestep this difficulty, we focus on *equilibrium-feasible outcomes* rather than on strategies. This methodology, introduced in Skreta (2006b), relies on the idea that the seller's equilibrium revenue is determined by the equilibrium outcome (described by the allocation and payment rule, defined in (10)).

PBE-feasible Outcomes: An allocation rule p and a payment rule x are *PBE-feasible* if they are the outcome of a dynamic PBE-feasible mechanism.

Then, the seller seeks an allocation rule and a payment rule that maximize expected discounted revenue among all PBE-feasible allocation and payment rules:

$$\max_{(p,x)} \int_V \Sigma_{i\in I} x_i(v) dF(v), \tag{12}$$

subject to (p, x) being PBE-feasible.

To find an optimal dynamic mechanism, we proceed as follows: We start by deriving *necessary* conditions for allocation and payment rules to be PBE-feasible. Using these conditions as constraints, we then formulate a maximization problem in Section 2.2 and choose the revenue-maximizing PBE-feasible outcome in Section 3.2. Finally, in Section 3.3, we find a PBE-feasible dynamic mechanism that implements this outcome.

2 Mechanism Selection under Limited Commitment

2.1 Necessary Conditions

We start by deriving *necessary* conditions for allocation and payment rules to be PBE-feasible.

Resource Constraints: The allocation rule p has to satisfy *resource constraints,* which, since there is only one object, translate to:

$$\textbf{RES} : 0 \leq p_i(v) \text{ and } \Sigma_{i \in I} p_i(v) \leq 1 \text{ for all } v \in \times_{i \in I} V \text{ and } i \in I.$$

Participation Constraints: Following Myerson (1981), we assume that the seller employs mechanisms that guarantee that buyers' interim payoff at the beginning of the game is higher than their outside options, which we assume to be equal to zero. We call these *participation constraints:*

$$\textbf{PC}_i : U_i(v_i) \geq 0 \text{ for all } i \in I, v_i \in V_i.$$

It turns out that, despite imposing this weak version of participation constraints, the optimal mechanisms satisfy ex-post participation in each period.

Best-Response Constraints: Consider a buyer i whose strategy is part of a PBE. At a PBE, buyer i's strategy must be a best response at each information set. This implies that there is no type of buyer i that can strictly benefit by mimicking the reporting rule of another type, which, in turn, implies that:

$$\textbf{IC}_i\text{: } P_i(v_i)v_i - X_i(v_i) \geq P_i(v_i')v_i - X_i(v_i'), \text{ for all } i \in I, v_i, v_i' \in V_i.$$

We call these the *anti-mimic (or incentive) constraints.* Also, if a_i' and a_i'' are in the support of m_i, both are optimal and, hence, maximize $U_i^{a_i}(v_i)$, implying that i is indifferent between these two reports: $U_i^{a_i'}(v_i) = U_i^{a_i''}(v_i)$.

Sequential Rationality Constraints: At a PBE-feasible dynamic mechanism, for all vectors $a \in A$, such that $q_S(a) > 0$, mechanism $(q^{2(a)}, z^{2(a)})$ must be optimal given (2). Since $T = 2$ is the last period of the game, the seller's problem is equivalent to finding the revenue-maximizing auction in a static setup, with the only difference that beliefs are endogenous and depend on the vector of $t = 1$ reports a.

Let $J_i(v_i\,|a_i)$ denote buyer i's posterior virtual valuation at $t = 2$ when he reports a_i at $t = 1$.[24] Myerson (1981) shows that a revenue-maximizing mechanism assigns the object with probability one to the buyer with maximal posterior virtual valuation,[25] provided that his reported valuation is above

[24] Myerson (1981) solves this problem assuming that all buyers' distributions of valuations have strictly positive densities. In general, distributions of valuations may fail to have strictly positive densities. In such cases, we cannot straightforwardly express the buyers' virtual valuations. One can, however, use the method of Skreta (2007) or of Monteiro and Svaiter (2010), or can appropriately approximate the problem as we explain in Appendix E, so as to get meaningful expressions of virtual valuations. If the posterior is well-behaved, then it is written as usual: $J_i(v_i|a_i) \equiv v_i - \frac{(1-F_i(v_i|a_i))}{f_i(v_i|a_i)}$.

[25] If $J_i(v_i\,|a_i)$ is increasing in v_i the problem is regular, meaning that the point-wise optimum is incentive-compatible. If not, we replace them with their "ironed" versions according to a procedure described in Myerson (1981), Skreta (2007) or Monteiro and Svaiter (2010). In what follows, in order to avoid extra notation, when we write J_i, we mean its *ironed* version.

a buyer-specific reserve price. Skreta (2006a) establishes that the optimal reserve price is given by[26]

$$r_i^2(a_i) \equiv \inf\left\{v_i \in V_i \text{ s.t. } \int_{v_i}^{\tilde{v}} \left[s f_i(s|a_i) - \int_s^{b_i} f_i(t|a_i)dt\right] ds \geq 0, \text{ for all } \tilde{v} \in [v_i, b_i]\right\}. \tag{13}$$

In words, the optimal reserve price is the smallest valuation v_i such that the average virtual valuation to the right of v_i is positive. The reserve price $r_i^2(a_i)$ depends on the seller's posterior (1), which is determined by i's equilibrium reporting-$t = 1$ strategy m_i and the realized report a_i. Note that (13) implies that

$$r_i^2(a_i) \geq \underline{v}_i(a_i) \tag{14}$$

since $\underline{v}_i(a_i)$ is the smallest valuation in the support of $f_i(.|a_i)$.

Let $v_j(v_i, a)$, denote j's valuation that satisfies

$$J_i(v_i|a_i) - J_j(v_j|a_j) = 0. \tag{15}$$

If $v_j < v_j(v_i, a)$, i's posterior virtual valuation is higher than j's, and the reverse if $v_j > v_j(v_i, a)$. Let $v_{-i}(v_i, a)$, be the vector consisting of all $v_j(v_i, a)$ with j different from i. An optimal allocation at $t = 2$, $q_i^{2(a)}$, is determined by the boundaries $r_i^2(a_i)$ and $v_{-i}(v_i, a)$ as follows:

$$\begin{aligned} q_i^{2(a)}(v) &= \begin{cases} 1 \text{ if } v_i \geq r_i^2(a_i) \text{ and } v_{-i} \leq v_{-i}(v_i, a), \\ 0 \text{ otherwise} \end{cases} \\ z_i^{2(a)}(v) &= q_i^{2(a)}(v)v_i - \int_0^{v_i} q_i^{2(a)}(t_i, v_{-i})dt_i, \end{aligned} \tag{16}$$

where $r_i^2(a_i)$ satisfies (13), and $v_{-i}(v_i, a)$ is the vector of v_j, with j different from i that satisfy (15).[27] We call the requirement that $q^{2(a)}$ and $z^{2(a)}$ satisfy (16) the *sequential-rationality constraint*, and we denote them by $\mathbf{SRC}(a)$.

Observation: T=2-reserve prices matter: Note that i obtains the good at the reserve price $r_i^2(a_i)$ with positive probability. If that were not true, then for *all* $t = 1$-reports a_{-i}, and realizations of v_{-i}, the posterior virtual valuations of all buyers other than i are strictly higher than the seller's value, which is zero in our case. This is impossible since posterior virtual valuations are below the seller's value at least in a neighborhood around zero.

[26]This expression shows how to obtain the optimal reserve price when i's distribution of valuations does not necessarily have a density or does not satisfy the monotone hazard rate property. So, in the one-buyer case solved in Skreta (2006a), one does not need to rely on techniques of Skreta (2007) or of Monteiro and Svaiter (2010), or on the approximation technique described in Appendix E.

[27] In case of ties, $q_i^{2(a)}(v) = 1$ for the lowest-index buyer among those buyers with maximal posterior virtual valuations.

2.2 The Seller's Problem

We start by expressing the seller's revenue as a function of the allocation rule. Lemma 2 in Myerson (1981), establishes that the constraints $\mathbf{IC_i}$, $\mathbf{PC_i}$ and **RES** are equivalent to: (i) $P_i(v_i)$ is increasing in v_i; (ii) $U_i(v_i) = \int_0^{v_i} P_i(t_i)dt_i + U_i(0)$; $U_i(0) \geq 0$; and, finally, (iii) $\Sigma_{i \in I} p_i(v) \leq 1$, $p_i(v) \geq 0$ for all i and $v \in V$. Myerson (1981) established that if (p, x) satisfy (i)-(iii), then the seller's expected revenue can be expressed as $\int_V \Sigma_{i \in I} x_i(v)dF(v) = \int_V \Sigma_{i \in I} p_i(v)J_i(v_i)f(v)dv - \Sigma_{i \in I} U_i(0)$, where $J_i(v_i) \equiv v_i - \frac{(1-F_i(v_i))}{f_i(v_i)}$ denotes buyer i's (prior) virtual valuation. Given that the necessary conditions for PBE-feasibility include $\mathbf{IC_i}$, $\mathbf{PC_i}$ and **RES**, if (p, x) is PBE-feasible, it satisfies (i)-(iii) and revenue can be expressed exactly as in Myerson (1981). As is standard, we assume that $J_i(v_i)$ is strictly increasing in v_i for all i.[28]

We now formulate a constrained maximization problem, which we call **Program NC**:

$$\sup_{\{m_i, q_i(a)\}_{a \in A, i \in \bar{I}}} \int_V \Sigma_{i \in I} p_i(v)J_i(v_i)f(v)dv - \Sigma_{i \in I} U_i(0),$$

where $p_i(v)$ is given by (10), and subject to:

$m_i(.|v_i)$ is a density function for all $v_i \in V_i$ and $i \in I$;

$m_i(a_i|.)$ is a measurable function of v_i for all $a_i \in A_i$ and $i \in I$;

$IC_i : P_i(v_i)$ increasing in v_i, for all $v_i \in V_i$ and $i \in I$;

$IC_i(a_i)$: if a_i and $\hat{a}_i$ are in the support of $m_i(.|v_i)$, $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$;

$PC_i : U_i(0) \geq 0$, for all i;

RES: for all $v \in V$, $0 \leq p_i(v) \leq 1$, $\Sigma_{i \in I} p_i(v) \leq 1$, and $i \in I$
and $\sum_{i=S}^{\bar{I}} q_i(a) = 1$, $q_i(a) \geq 0$, for all $i \in \bar{I}$;

$SRC(a)$: for all a s.t. $q_S(a) > 0$, $q^{2(a)}, z^{2(a)}$ are given by (16) for beliefs (2);

$Beliefs$: posterior beliefs are given by (2) .

To solve the problem, we have to optimally specify the $t = 1$-assignment probabilities and payments $\{q_i(a), z_i(a); i \in \bar{I}\}_{a \in A}$; and the reporting rules–the m_i's–which, in turn, determine the $t = 2$ sequentially rational mechanism through (16).

The following Proposition establishes that if a solution of Program NC exists and turns out to be PBE-feasible,[29] then the dynamic mechanism that implements it is optimal.

[28] This assumption allows us to avoid the complications that result from not having well-defined virtual valuations or from having to iron them and to focus on the ones that arise from the sequential rationality constraints.

[29] To see that it is possible that (p, x) satisfy all the constraints of Program NC but are *not* PBE-feasible, note that constraint IC_i requires that P_i is increasing in v_i, whereas constraint $IC_i(a_i)$ requires that $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$ if a_i and $\hat{a}_i$ are in the support of $m_i(.|v_i)$. Even if these two constraints are satisfied (as well as the rest of the constraints of Program NC), this does not exclude the possibility that there is an $\tilde{a}_i$ that i can choose where he gets the good with probability 1 and he pays nothing, but that $\tilde{a}_i$ is *not* in the support of $m_i(.|v_i)$ for any v_i. Obviously if such an action

Proposition 1 *If a solution of Program NC exists and is PBE-feasible, then the dynamic mechanism that implements it is optimal.*

Proof. Let (p^*, x^*) denote the allocation and payment rule arising at a solution of Program NC. Suppose that it is PBE-feasible, and denote the corresponding seller's expected revenue by $R(p^*, x^*)$. We argue by contradiction. Let $(\tilde{p}, \tilde{x})$ be a revenue-maximizing PBE-feasible outcome given some dynamic mechanism, and suppose that $R(\tilde{p}, \tilde{x}) > R(p^*, x^*)$. Since $(\tilde{p}, \tilde{x})$ is PBE-feasible it satisfies all the constraints of Program NC and, hence, is feasible (since the constraints are a few *necessary* conditions for PBE-feasibility). However, since it generates strictly higher revenue than (p^*, x^*), it contradicts the fact that (p^*, x^*) is a solution of Program NC. ■

In what follows, we obtain a solution of Program NC and construct a dynamic mechanism that is PBE-feasible and that implements it. Proposition 1 then guarantees that this is an optimal dynamic mechanism.

3 Revenue-Maximizing Sequentially Rational Auctions

The main result of this paper is the characterization of revenue-maximizing sequentially rational auctions. We first state the result and its implications, and then, establish it.

Theorem 1 *The optimal dynamic mechanism allocates, at $t = 1$, the good to the buyer with the highest virtual valuation if it is above a buyer-specific threshold. If no trade takes place at $t = 1$, at $t = 2$, it allocates the object to the buyer with the highest posterior virtual valuation if it is above the seller's value–which is equivalent to the actual valuation being above a buyer-specific threshold. If all buyers are ex-ante symmetric, the $t = 1$- and $t = 2$-thresholds are the same across buyers.*

An interesting and practically relevant implication of Theorem 1 is the following Corollary:

Corollary 1 *When buyers are ex-ante symmetric, the symmetric equilibrium of the game in which the seller runs in each period a second-price (SPA) or a first-price (FPA) auction with optimally chosen reserve prices and in which buyers not bidding above the reserve do not participate (report the same message), generates maximal revenue for the seller.*

Proof. At a sequential SPA (or FPA) with reserve prices, conditional on at least one buyer bidding above the reserve in period 1, trade takes place at $t = 1$ and the game ends. Following standard arguments, it is then easy to see that buyers at $t = 1$ bid their true valuation in an SPA (or

exists, i must choose it in equilibrium, so it must be in the support of $m_i(.|v_i)$ for all v_i. This shows that there can exist allocation and payment rules satisfying all the constraints of Program NC that are not PBE-feasible.

use a strictly increasing bidding function at a symmetric equilibrium of a FPA) if their value (or the prescribed bid in the case of FPA) is above a threshold; otherwise, they do not participate and wait until period $t = 2$.

Conditional on at least one buyer bidding above the reserve in period 1, trade takes place at $t = 1$ and the game ends, so it does not matter what the seller learns (or infers) about a buyer's true valuation. Only when *all* buyers choose not to participate at $t = 1$ does the game continue to $t = 2$. In that case, the only thing that the seller learns is that all buyers' valuations are below a threshold, exactly as is prescribed by the optimal mechanism. Then, at a symmetric equilibrium of an SPA with a reserve price, at $t = 1$, the object is assigned to the buyer with the highest valuation–who, due to symmetry, is the buyer with the highest virtual valuation–among all buyers who submit a bid above the $t = 1$-reserve price. If no one bids above the reserve at $t = 1$, the game proceeds to $t = 2$. Given ex-ante symmetric buyers, at a symmetric equilibrium, the buyers are symmetric at $t = 2$, as well. At $t = 2$, an SPA assigns the object to the buyer with the highest valuation–who, due to symmetry, is also the buyer with the highest posterior virtual valuation–if his valuation is above the reservation price posted at $t = 2$.[30] Similar arguments hold for symmetric equilibria of sequential first-price auctions, FPA with reserve prices. ■

We establish Theorem 1 in Subsections 3.2-3.4 as follows: We solve Program NC and then find a PBE-feasible dynamic mechanism that implements its solution. Then, Proposition 1 allows us to conclude that this is an optimal dynamic mechanism.

3.1 Benchmark: The Solution Ignoring Sequential Rationality Constraints

To better understand the solution of Program NC, it is helpful to consider the solution of a relaxed program–called Program C–that ignores the sequential rationality constraints. This gives us the revenue-maximizing outcome when the seller has commitment.

Myerson (1981) solves this program and shows that, at the optimum, the object goes to the buyer with the highest virtual valuation, provided that his valuation is above a buyer-specific reserve price r_i^* that satisfies (13) for the prior. The optimal $t = 1$-assignment is:

$$q_i(v) \quad = \quad \begin{cases} 1 \text{ if } v_i \geq r_i^* \text{ and } J_i(v_i) \geq J_j(v_j), \text{ for all } j \\ 0 \text{ otherwise} \end{cases} \tag{17}$$

with payment $z_i(v) = q_i(v)v_i - \int_0^{v_i} q_i(t_i, v_{-i})dt_i$, while the optimal $t = 2$-mechanism is the same as the $t = 1$ one: $q_i^{2(a)}(v) = q_i(v)$ and $z_i^{2(a)}(v) = z_i(v)$ for all v and all i (the seller commits not to change the

[30]If ties occur–a probability zero event given increasing virtual valuations–the lowest-index buyer among the ones who tie is assigned the good with probability one.

$t = 1$-allocation at $t = 2$). The optimal assignment with commitment is characterized by boundaries dividing the set of valuations into regions in which either the seller or a buyer is assigned the object, as is depicted in Figure 1. The region in which the seller is assigned the good is $\times_{i \in I}[0, r_i^*]$, whereas the

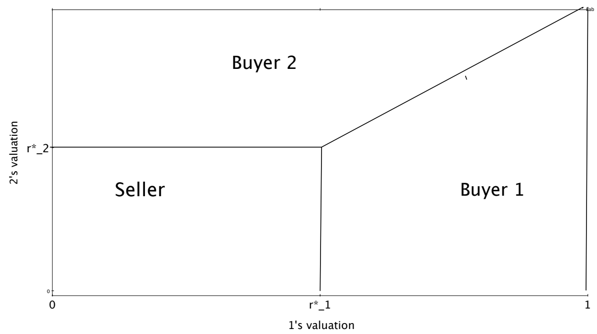

Figure 1: The revenue-maximizing allocation ignoring sequential rationality constraints when $I = 2$

region in which i gets it is characterized by r_i^* and the boundary that determines when i's valuation is the highest. When buyers are ex-ante symmetric, this boundary is given by the locus where all buyers' valuations are equal, which is simply the 45-degree line between any two buyers.

The commitment solution trivially satisfies the sequential rationality constraints if the good is always assigned to a buyer at $t = 1$. This occurs when $r_i^* = 0$ for at least one i. Otherwise, the seller keeps the object at $t = 1$ when all realized valuations are in $\times_{i \in I}[0, r_i^*]$. Commitment means that this outcome remains unchanged at $t = 2$, and it imposes that $q_i^{2(a)}(v) = 0$ for all i when $v \in \times_{i \in I}[0, r_i^*]$. This assignment violates sequential rationality, however, since at $t = 2$, the seller infers that all realized valuations lie in $\times_{i \in I}[0, r_i^*]$. It is immediate to see that i's posterior virtual valuation at r_i^* is equal to $r_i^* \geq 0$, so the seller would, at the very least, prefer to give the good to i when his valuation is r_i^*, rather than keep it.

3.2 The Solution of Program NC

When thinking about what is optimal, taking the sequential rationality constraints into account, it helps to examine the features of the solution when there is commitment. First, note that the reason the commitment solution pools the lower end of each buyer's valuations is that the cost of separating them (which is the higher rents that must be left to types above the excluded ones) outweighs the benefit, so the net benefit of separation (as captured by those types' virtual valuation) is negative. An optimal reporting rule given commitment is, then, as follows: All i's valuations below r_i^* report 0, while all valuations above r_i^* report the truth.

When we introduce the sequential rationality constraints, the seller may want to expand or shrink the region of valuations that pool with valuation 0. She may actually want to use a first-period mechanism that supports complex reporting rules as best-responses: for instance, mixed strategies that lead to posteriors with non-convex supports, etc. A key question is whether such reporting rules make it sequentially rational for the seller to select $t = 2$-optimal mechanisms that induce $t = 2$-allocations closer to the ex-ante optimal one, compared to $t = 2$-allocations that are optimal when buyers use pure reporting rules. We start our characterization by investigating which reporting rules are optimal given the presence of sequential rationality. In order to do this, we define what it means for a report to be relevant for sequential rationality constraints:

SRC-Relevant Reports: A report a_i is relevant for sequential rationality constraints if $q_S(a_i, a_{-i}) > 0$ for at least one vector a_{-i} with $\int_{V_{-i}} m_{-i}(a_{-i}|v_{-i})f_{-i}(v_{-i})dv_{-i} > 0$.

We focus on the case in which sequential rationality constraints bind. Sequential rationality constraints bind when for all i, there is a SRC-relevant message in the support of the reporting rule of type 0–namely, the function $m_i(.|0)$.[31] If the support of $m_i(.|0)$ is a singleton–that is, if type 0 of i employs a pure strategy–we call this message employed by type 0 message 0. If the support $m_i(.|0)$ is not a singleton, then we call message 0 the message a_i associated with the smallest $P_i^{1(a_i)}$ (defined in (7)) among all messages a_i in the support of $m_i(.|0)$.

Proposition 2 below establishes that at a solution of Program NC, the optimal reporting rule is pure and that all valuations below some threshold $\bar{v}_i$ report 0. The reporting function $m_i(0|.)$ associated

[31] To see this, recall that prior virtual valuations are increasing (without this assumption, we would have to employ the "ironed "virtual valuations, which are, by definition, increasing), hence, when a buyer's realized valuation is zero, the virtual valuation is negative and at its lowest level. Now, suppose that at a solution of Program NC, trade takes place with probability 1 when at least one buyer's valuation is zero, no matter the realization of the other buyers' valuations (that is, for some i, $q_S(a_i, a_{-i}) = 0$ for all a_{-i} and for all a_i in the support of $m_i(.|0)$). This implies that trade takes place at $t = 1$ also after a vector of reports a_{-i} chosen by $-i$ buyers when all their valuations are zero. It is immediate to see that the seller would find it optimal to assign the object at $t = 1$ given reports that are chosen by higher valuations. Then sequential rationality constraints do not bind since trade takes place with probability 1 at $t = 1$.

with such a reporting rule is

$$m_i(0|v_i) = \begin{cases} 1, \text{ for } v_i \in [0, \bar{v}_i] \\ 0, \text{ otherwise} \end{cases} . \tag{18}$$

The support of the reporting rule in (18)–namely, the set $V_i(0)$–is the interval $[0, \bar{v}_i]$. With the help of this reporting function, we can obtain the seller's posterior at $t = 2$ after i reports message 0 at $t = 1$, which is:

$$f_i(v_i|\bar{v}_i) = \begin{cases} \frac{f_i(v_i)}{F_i(\bar{v}_i)}, \text{ for } v_i \in [0, \bar{v}_i] \\ 0, \text{ otherwise} \end{cases} . \tag{19}$$

Proposition 2 (Reporting rules) *At a solution of Program NC: (i) there is a unique vector of reports that is associated with no trade at $t = 1$, which we denote by $\mathbf{0} \equiv 0, ..., 0$; and (ii) buyers use a degenerate (pure) reporting rule where all valuations below some cutoff $\bar{v}_i$ report 0. When buyers are ex-ante symmetric, the cutoff is equal across buyers: $\bar{v}_i(0) = \bar{v}$.*

The proof of Proposition 2 proceeds in three Steps and relies on Lemma 1, which follows. This Lemma establishes that buyers' best-response constraints imply that they must be rewarded at $t = 1$ for any eventual reduction to their surplus due to the seller's re-optimizing behavior at $t = 2$. Hence, sequential rationality constraints increase the cost of separating valuations.

Lemma 1 *For any type v_i for which there are two actions a_i and $\hat{a}_i$ such that v_i belongs to $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$, it holds that $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$.*

Consider a valuation in $v_i \in V_i(a_i)$ for which a_i and $\hat{a}_i$ are in the support of $m_i(.|v_i)$. Then, best-response constraints imply that $U_i^{a_i}(v_i) = U_i^{\hat{a}_i}(v_i)$. Lemma 1 establishes that $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$ for all valuations in $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$, even for v_i's for which a_i is not in the support of their reporting rule $m_i(.|v_i)$. This, in turn, implies that $\frac{dU_i^{a_i}(v_i)}{dv_i} = \frac{dU_i^{\hat{a}_i}(v_i)}{dv_i}$ a.e. on $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$, which is equivalent to

$$P_i^{a_i}(v_i) = P_i^{\hat{a}_i}(v_i), \text{ a.e. on } \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i). \tag{20}$$

Equality (20) tells us that (even) if mixing can reduce the probability that i gets the good at $t = 2$ (which may be desirable from the ex-ante perspective), this reduction must be balanced with an increase of the expected probability of trade at $t = 1$. This implies that the sequential rationality constraints increase the $t = 1$-incentive rents since, now, the rewards of separating a valuation at $t = 1$ must include the anticipated reduction in the rents that occurs at $t = 2$ because the seller exploits the information obtained at $t = 1$. Equality (20) is the basis of the proof of Proposition 2, which can be found in Appendix B.

We now proceed to specify the optimal $t = 1$-assignment at $a = \mathbf{0}$–the vector when all buyers report 0.

Lemma 2 (First-Period Assignment) *At an optimal first-period assignment, either $q_S(\mathbf{0}) = 1$ or $q_S(\mathbf{0}) = 0$. Moreover, if $q_S(\mathbf{0}) = 1$, then $q_i(0, a_{-i}) = 0$ for all a_{-i}.*

This Lemma follows almost immediately from the following observation: Given the optimal reporting rule established in Proposition 2, the seller's expected revenue can be written as:

$$R = \int_{\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I} \left[q_i(\mathbf{0}) + \delta q_S(\mathbf{0}) q_i^{2(\mathbf{0})}(v)\right] J_i(v_i) f(v) dv + \int_{V\setminus\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I} p_i(v) J_i(v_i) f(v) dv, \quad (21)$$

which is linear in $q_i(\mathbf{0})$, for $i \in \bar{I}$.

Lemma 2 establishes that at a solution of Program NC, the seller keeps the good when all buyers' valuations lie below some cutoff. At the "commitment solution,"the cutoff is r_i^*–the valuation where i's virtual valuation is zero (the seller's valuation). We now show that with sequential rationality constraints, the cutoff $\bar{v}_i$ must be (weakly) greater than r_i^*:

Lemma 3 (Optimal Cutoffs) *At a solution of Program NC, $\bar{v}_i \geq r_i^*$: that is, $\times_{i\in I}[0, \bar{v}_i]$ contains the region where all virtual valuations are below the seller's value.*

Lemma 3 shows that at a solution of Program NC, the seller keeps the good at $t = 1$ when all virtual valuations are less than a threshold that is larger than the seller's value. The intuition for why this threshold is higher compared to the one with commitment is that the seller anticipates at $t = 1$ her temptation at $t = 2$ to over-assign the object compared to what is revenue-maximizing from the $t = 1$ perspective.[32] The fact that with limited commitment, the seller may want to induce a larger set of high-value buyers to postpone trade, so as to sustain a higher price in the continuation game, has been already noted in the single-buyer environment studied in Skreta (2006b) and in the continuum of buyers sequential screening problem studied in Deb and Said (2015). The optimal level of $\bar{v}_i$ depends on the discount factor δ, which, as we illustrate in examples in Section 4, determines the revenue loss due to sequential rationality.

[32]The reason for this over-assignment is that the posterior virtual valuation is higher compared to the prior. In particular, when the posterior is (19), i's posterior virtual valuation is equivalent to $v_i - \frac{[F_i(\bar{v}_i) - F_i(v_i)]}{f_i(v_i)}$, which is equal to the prior virtual valuation $v_i - \frac{[1-F_i(v_i)]}{f_i(v_i)}$ *plus* the term $\frac{[1-F_i(\bar{v}_i)]}{f_i(v_i)}$. Note, however, that this intuition oversimplifies matters. The cutoffs (the $\bar{v}_i$'s) also determine the ranking of the buyers' posterior virtual valuations. It is then conceivable that the ranking that is supported by some vector of cutoffs with $\bar{v}_i < r_i$ for some i is optimal because it reduces the costs of sequential rationality constraints. We show (in Appendix D) that this cannot happen because, roughly, the effect of $\bar{v}_i$'s on the ranking of posterior virtual valuations across buyers is secondary.

The fact that at a solution of Program NC, it holds that $\bar{v}_i \geq r_i^*$ implies that in the remaining area of vectors of valuations–that is, in $V \backslash \times_{i\in I}[0, \bar{v}_i]$–at least one buyer has a virtual valuation that is higher than the seller's value. For $V \backslash \times_{i\in I}[0, \bar{v}_i]$, we choose the first-period assignment that maximizes (21) pointwise (for each v), subject to the resource constraint. Clearly, this point-wise optimum of a relaxed program is the best the seller can achieve. The above considerations lead to the following:

Proposed Solution: At a solution of Program NC, the optimal reporting rule $m_i(.|v_i)$ is a degenerate function with a mass point at 0 for all $v_i \in [0, \bar{v}_i]$, and a degenerate function with a mass point at v_i for all $v_i > \bar{v}_i$. The optimal $t = 1$ assignment is: $q_S(a_i, a_{-i}) = 1$ if for all i $a_i = 0$; otherwise, $q_S = 0$; $q_i(0, a_{-i}) = 0$ for all a_{-i}; and, finally, $q_i(v_i, v_{-i}) = 1$ if i has the maximum virtual valuation, zero otherwise, while q^2 is given by (16), for beliefs given by (19). The resulting allocation and payment rules are:[33]

$$\begin{aligned} \text{for } v &\in V\backslash \times_{i\in I}[0,\bar{v}_i]: \; p_i^*(v_i, v_{-i}) = \begin{cases} 1 \text{ if } i \in \arg\max_{i\in I^1(v)} J_i(v_i) \\ 0 \text{ otherwise} \end{cases}, \\ \text{for } v &\in \times_{i\in I}[0,\bar{v}_i]: \; p_i^*(v_i, v_{-i}) = \delta q_i^2(v), \end{aligned} \tag{22}$$

where $I^1(v) = \{i \in I : v_i \in V_i \text{ and } v_i > \bar{v}_i\}$ and q^2 is given by (16), for beliefs given by (19);

$$x_i^*(v) = p_i^*(v)v_i - \int_0^{v_i} p_i^*(t_i, v_{-i})dt_i. \tag{23}$$

We now verify that, indeed, this proposed solution is feasible for Program NC:

Lemma 4 (Feasibility) *The allocation rule in (22) is feasible for Program NC.*

Finally, we note that the revenue-maximizing assignment under limited commitment can be fully characterized by choosing the cutoffs $\bar{v}_i$ optimally. This can be achieved by solving the following simplified program: After substituting (22) in (21), Program NC reduces to the problem of finding $\bar{\mathbf{v}} = (\bar{v}_1, ..., \bar{v}_I)$, with $\bar{v}_i \in [0, b_i]$, which maximizes:

$$R(\bar{\mathbf{v}}) \equiv \delta \int_{\times_{i\in I}[0,\bar{v}_i]} \Sigma_{i\in I} q_i^2(v) J_i(v_i) f(v) dv + \int_{V\backslash \times_{i\in I}[0,\bar{v}_i]} J^{\max}(v) f(v) dv, \tag{24}$$

where $J^{\max}(v) = \max_{i\in I}\{J_1(v_1), ..., J_I(v_I)\}$, and q^2 satisfies (16) for beliefs given by (19).

This program is tremendously simpler than Program NC: Instead of maximizing over an infinite dimensional space, the seller now chooses a vector $\bar{v}$ out of a compact set $\times_{i\in I}[0, b_i]$. Moreover, it is routine to verify that (24) is continuous in $\bar{v}$ since, as we establish in Appendix D, q^2–more

[33] Because virtual valuations are increasing, so are posterior virtual valuations; see Lemma 8 in Appendix D. Ties, then, occur with probability zero and can be broken arbitrarily.

precisely the boundaries that describe q^2–$r_i^2(\bar{v}_i)$ and $v_j(v_i, \bar{v}_i, \bar{v}_j)$, vary continuously with $\bar{v}$. Hence, this maximization problem is well-defined and a solution exists. We illustrate how to obtain this solution in Section 4.

3.3 Implementation

We have, thus far, obtained a solution of Program NC described in (22) and (23) for $\bar{\mathbf{v}}$ that solves (24). Since this solution satisfies only necessary conditions of being PBE-feasible, to complete our characterization, we provide a PBE-feasible dynamic mechanism that implements it:

Dynamic Mechanism: The assignment rule of the $t = 1$-mechanism is:

$$q_i(v_i, v_{-i}) = \begin{cases} 1 \text{ if } i \in \arg\max_{i \in I^1(v)} J_i(v_i) \\ 0 \text{ otherwise,} \end{cases}, \tag{25}$$

where $I^1(v) = \{i \in I : v_i \in V_i \text{ and } v_i > \bar{v}_i\}$, and where $\bar{\mathbf{v}} = (\bar{v}_i, \bar{v}_{-i})$ solves (24). The payment rule of the $t = 1$-mechanism is:

$$\begin{aligned} z_i(0, a_{-i}) &= 0, \text{ for all } a_{-i} \in A_{-i} \\ z_i(v_i, v_{-i}) &= \begin{cases} r_i^1(v_{-i}) \text{ if } i \in \arg\max_{i \in I_1(v)} J_i(v_i) \text{ and } I^1(v) \neq i \\ \bar{r}_i^1 \text{ if } I^1(v) = i \\ 0 \text{ otherwise} \end{cases}. \end{aligned} \tag{26}$$

This payment rule (26) is derived using (23): Buyer i pays only when he wins the object. If i wins while facing some competition at $t = 1$–that is, when $I^1(v) \neq i$–he pays the lowest possible valuation that would still allow him to win given v_{-i}–that is, $r_i^1(v_{-i}) \equiv \inf\{v_i \text{ such that } q_i(v) = 1\}$.[34] If i does not face any competition at $t = 1$–that is, if $I^1(v) = \{i\}$–i pays a buyer-specific reserve price $\bar{r}_i^1$. Full derivation details can be found in Appendix F.

The $t = 2$-mechanism is a direct mechanism described in (16) that is optimal given (19).

Buyers' Strategies: At period $t = 1$, buyer i employs a reporting rule $m_i(.|v_i)$ that is a degenerate function with a mass point at 0 for all $v_i \in [0, \bar{v}_i]$, and a degenerate function with a mass point at v_i for all $v_i > \bar{v}_i$. At $t = 2$, buyers report their valuations truthfully, so the $t = 2$ reporting rule is a degenerate function with a mass point at v_i for all v_i.

Beliefs: Given the buyers' behavior described by their reporting rules, when trade does not occur at $t = 1$, the seller's posterior beliefs along the path are given by (19). When the seller observes an off-path vector of actions that leads to no trade at $t = 1$, her posterior belief at $t = 2$ remains equal to the prior (passive beliefs).

[34] The payment rule in this region is analogous to the one described by the optimal mechanism in Myerson (1981). The difference is that, here, not all buyers report a valuation above the cutoff at $t = 1$.

It is immediate to see that this strategy profile implements (22) and (23). Now, we establish that this dynamic mechanism is PBE-feasible. At $t = 2$, the dynamic mechanism specifies a direct revelation mechanism with an allocation rule described in (16), which is revenue-maximizing given the seller's posterior beliefs derived from the buyers' reporting rules. Since $\bar{\mathbf{v}}$ is optimally chosen, the seller cannot do any better by changing the regions where trade takes place at $t = 1$, versus $t = 2$. The fact that buyers' strategies are best responses at $t = 2$ follows immediately from the incentive compatibility of the direct mechanism that the seller employs at $t = 2$. Hence, the only requirement that remains to be verified is that buyers' strategies are best responses at $t = 1$. This follows from the incentive compatibility of (22) established in Proposition 4 and from the fact that payments are constructed from (23). Putting all the pieces together, we have shown that:

Proposition 3 (PBE-implementability) *The allocation rule described in (22) can be implemented by a PBE-feasible dynamic mechanism.*

3.4 Proof of Theorem 1

In Proposition 1, we argued that if a solution of Program NC is PBE-feasible, then the dynamic mechanism that implements it is optimal. We then showed that the allocation rule in (22) with optimally chosen $\bar{\mathbf{v}}$ solves Program NC. Finally, Proposition 3 states that this solution can be implemented by a PBE-feasible dynamic mechanism. Hence, this dynamic mechanism is optimal.

Discussion:[35] Put differently, this paper proves the existence of a dynamic mechanism $\mathcal{M}$–in which $A_i = V_i$, for both $t = 1$ and $t = 2$, and such that the following three conditions are satisfied:

(a) Given $\mathcal{M}$, there exists a PBE in the game among the buyers *only*, in which: (i) at $t = 1$, each buyer reports 0 for $v_i < \bar{v}_i$ and reports truthfully for $v_i > \bar{v}_i$; and (ii) if the good remains unassigned at $t = 1$, then all v_i report truthfully at $t = 2$.

(b) Given buyers' strategies, it is optimal for the seller not to change the mechanism at $t = 2$ if the good is unassigned at $t = 1$; that is, given the seller's $t = 2$-posteriors, deviation to any $t = 2$-feasible direct mechanism yields a lower payoff.

(c) For any other dynamic mechanism $\mathcal{M}'$ and any PBE of $\mathcal{M}'$ (in the game among the buyers), one of the following holds:

   (i) either the mechanism is not sequentially rational, in the sense that the seller has a profitable deviation to a different feasible mechanism $t = 2$; or

[35] I am grateful to the editors for providing this insightful discussion.

(ii) the seller's expected payoff under the proposed PBE associated to M' is lower than under the original PBE of M.

Observation: While we could not a priori argue that it is wlog to restrict attention to dynamic mechanisms in which the t-1 message spaces coincide with the type spaces, the results of this paper de facto imply that it is without loss of optimality for the seller to restrict attention to such mechanisms: Setting $A_i = V_i$ suffices to get the revenue-maximizing equilibrium feasible outcome of the general game, as we established in Section 3.2. Put differently, the findings of this paper extend the Bester and Strausz (2001) result to the IPV auction setup considered in this paper.

4 Illustration and the Value of Commitment

Here, we provide an illustration of the result of Theorem 1 in a simple example: Suppose that there are I buyers whose valuations are distributed uniformly on $[0, 1]$, that the seller's valuation is zero, and that $T = 2$. For this example, the commitment benchmark–that is, a revenue-maximizing auction without the sequential rationality constraints–is a second-price auction with a reserve price of $r_i^* = 0.5$ for all $i \in I$. When $I = 2$, the seller's expected revenue is 0.4166.

With sequential rationality constraints, at $t = 1$, buyer i gets the object if $v_i \geq v_j$, for all $i \neq j$, and $v_i \geq \bar{v}_i$. Given a vector of first-period cutoffs $\bar{\mathbf{v}} = (\bar{v}_1, ..., \bar{v}_I)$, the posterior is $f_i(v_i|\bar{v}_i) = \frac{1}{\bar{v}_i}$, and i's posterior virtual valuation is $J_i(v_i\,|\bar{v}_i) = 2v_i - \bar{v}_i$. It is easy to see that the optimal $t = 2$-reserve price is $r_i^2(\bar{v}_i) = \frac{\bar{v}_i}{2}$. Then, at a revenue-maximizing mechanism at $t = 2$, buyer i obtains the object if $v_i \geq v_j - \frac{(\bar{v}_j - \bar{v}_i)}{2}$ and $v_i \geq \frac{\bar{v}_i}{2}$ for all $j \neq i$.

Since, in this example, buyers are ex-ante identical, at the optimum, $\bar{v}_i = \bar{v}_j \equiv \bar{v}$ and (24) reduces to:

$$I\delta \int_{\frac{\bar{v}}{2}}^{\bar{v}} (2v - 1)v^{I-1}dv + I \int_{\bar{v}}^{1} (2v - 1)v^{I-1}dv. \tag{27}$$

It is straightforward to verify that the revenue-maximizing first-period cutoff is:

$$\bar{v} = \frac{\delta - 2^I\delta + 2^I}{\delta - 2^{I+1}\delta + 2^{I+1}}. \tag{28}$$

Comparative Statics: We first observe that the first-period cutoff, described in (28), is increasing in δ. Differentiating (28) with respect to δ, we get $\frac{\partial \bar{v}}{\partial \delta} = \frac{2^I + 2^{I+1}}{(\delta - 2^{I+1}\delta + 2^{I+1})^2} > 0$. Differentiating (28) with respect to I, we get that $\frac{\partial \bar{v}}{\partial I} = \frac{\frac{1}{2}2^I I\delta(\delta-1)}{(\delta - 2^{I+1}\delta + 2^{I+1})^2} \leq 0$. The first-period cutoff decreases, and it converges to $\frac{1}{2}$ as I gets large. The left panel of Figure 2 depicts how the first-period cutoff varies with the discount factor: Each curve corresponds to a different number of buyers: the dotted line to $I = 2$, the dashed line to $I = 3$ and the solid line to $I = 10$. The right panel of the same figure depicts how

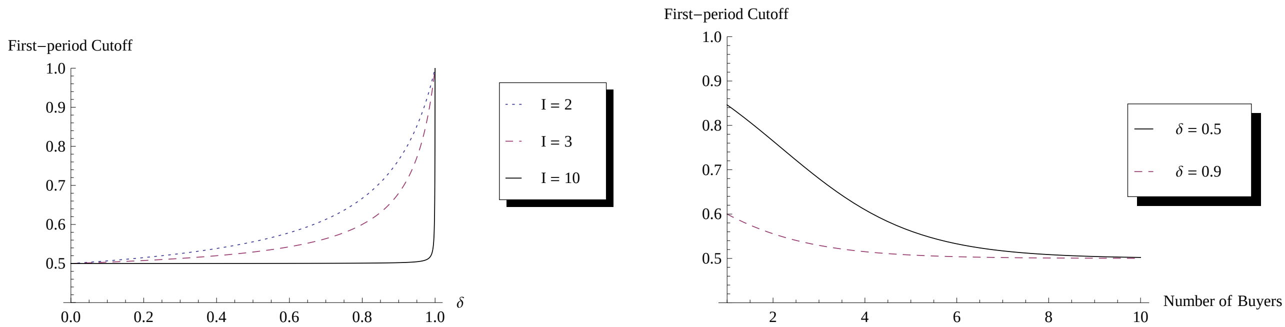

Figure 2: First-Period Cutoff

the first-period cutoff varies with the number of buyers. The dotted curve corresponds to a discount factor of $\delta = 0.5$, and the solid line corresponds to a discount factor of $\delta = 0.9$.

Expected revenue as a function of the discount factor when $I = 2$ and $T = 2$ is given by $\frac{8}{3(7\delta-8)^3}(53\delta^3 - 183\delta^2 + 210\delta - 80) - \frac{1}{12}\delta\frac{(3\delta-4)^2}{(7\delta-8)^3}(21\delta - 16)$ and is depicted by the solid line in Figure 3 below. The dotted line on the same graph depicts the expected revenue in the commitment benchmark. The revenue loss is zero for extreme values of the discount factor and highest for δ close to 0.8. We can also look at at the first-period reserve price. For the example under consideration and when $I = 2$, the first-period reserve price derived in (47) becomes $\bar{r}_i^1 = \bar{v}(1 - 0.375\delta)$, with $\bar{v} = \frac{3\delta-4}{7\delta-8}$. Interestingly, the first-period reserve price varies non-monotonically with the discount factor.

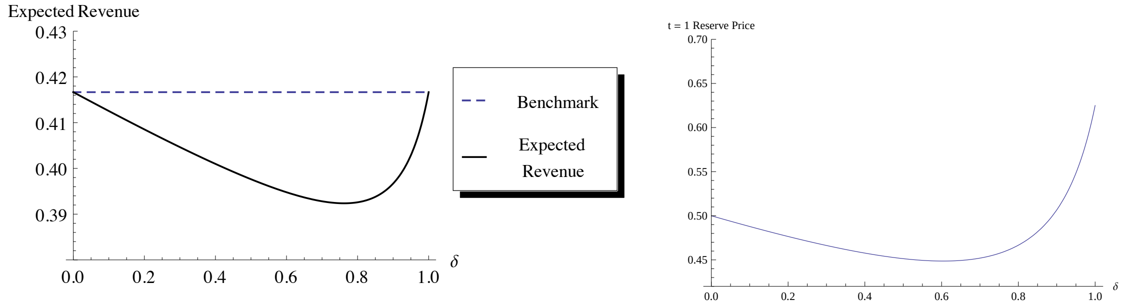

Figure 3: Left Panel: Revenue; Right Panel: First-period reserve price

How Much is Commitment Worth? As Figure 3 illustrates, for extreme discount factors–namely, for $\delta = 0$ and for $\delta = 1$–the revenue loss due to lack of commitment is zero: When $\delta = 0$, the future does not matter at all, so the sequential rationality constraints disappear, and the optimal vector of cutoffs is given by the vectors of valuations where all buyers' virtual valuations are equal to the seller's valuation; that is, $\bar{\mathbf{v}}(\delta=0)=(r_1^*,...,r_I^*)$. When $\delta=1$, waiting is costless, so the seller can wait until the last period of the game and offer an optimal mechanism without sequential-rationality constraints. This can be achieved by selecting $\bar{\mathbf{v}}$ to be equal to the vector of the highest possible valuations of all buyers: $\bar{\mathbf{v}}(\delta=1)=(b_1,...,b_I)$. This can also be seen in Figure 2 above. For intermediate discount factors, an optimal vector of cutoffs is somewhere between $\bar{\mathbf{v}}(0)$ and $\bar{\mathbf{v}}(1)$.

Since the seller can always run an efficient auction at $t=1$ (which allocates the good with probability one, and, hence, limited commitment has no bite), an upper bound on the value of commitment in terms of the seller's expected revenue is equal to the difference between the revenue at the optimal Myerson benchmark and the revenue generated by an efficient auction. The difference between these two benchmarks depends on the number of buyers, the discount factor, and the distribution of valuations. If the seller faces many buyers, then the probability that trade occurs at period one is very high, and her lack of commitment becomes less important: The design matters the most when she faces a small number of buyers.

Longer Games: Following an analogous procedure, we can obtain the optimal cutoffs for the case in which $I=2$ as a function of the discount factor for longer games.[36] In Figure 4, the number of buyers is held at two, and we depict the first-period cutoff as a function of the discount factor for games of various lengths. The dashed line corresponds to $t=2$, the dotted line to $t=3$ and the solid

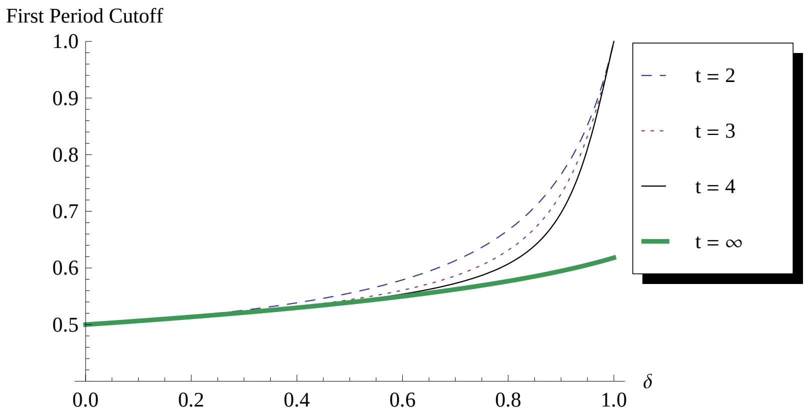

Figure 4: First-Period Cutoff and Length of T

line to $t=4$. The last thick line is the first-period cutoff of a stationary equilibrium of the $t=\infty$ game of McAfee and Vincent (1997). This cutoff is implicitly given by a solution of $2\bar{v}-1=\delta\bar{v}^3$.[37]

McAfee and Vincent (1997) study reserve-price dynamics in an infinite-horizon model in which the seller uses standard auctions; buyers are symmetric; and the lowest possible valuation is strictly

[36] The analysis of the derivation of a revenue-maximizing PBE for the case $T>2$ is overviewed in Section 6.

[37] More specifically, McAfee and Vincent (1997) consider a uniform $[0,1]$, infinite-horizon, no-gap example and establish that it has a symmetric linear stationary equilibrium.

higher than the seller's value–"the gap case."[38] We show that in a finite-horizon version of their model, standard auctions together with sequentially rational reserve price paths–analogous to the one characterized in Theorem 1 in McAfee and Vincent (1997) for the infinite-horizon case–indeed constitute the revenue-maximizing equilibrium when the seller can choose any mechanism in each period.

5 Mechanisms with Variable Transparency

We have, thus far, assumed that all players observe the entire vector of actions chosen at period t; that is, we have assumed that the seller employs fully transparent mechanisms. This can be restrictive since, in dynamic settings where we require players to behave sequentially rationally, what buyers observe determines their beliefs about their competitors, which may affect their future behavior and, hence, the set of continuation equilibrium outcomes. Here, we allow for the seller to control how much buyers observe in each period: The seller privately observes all the actions chosen by the buyers at period t, and the *transparency* of the mechanism determines how much information about these actions is disclosed to buyers at the end of period t.

The degree of transparency of the period-t mechanism is modeled as an *information-disclosure rule*, that is a mapping from the vector of actions chosen by the buyers (the seller's information), to a vector of messages, one for each buyer: $c^t : A \to \Delta(\Lambda)$, where $\Lambda := \times_{i \in I} \Lambda_i$, and Λ_i is the set of messages that the seller can send to buyer i. Let $\lambda \equiv (\lambda_i, \lambda_{-i}) \in \Lambda$ denote a vector of messages released by the disclosure rule. Buyer i observes only λ_i.

Definition 3 A *variable transparency mechanism* $M_t = (q^t, z^t, c^t) : \times_{i \in I} A_i \to [0,1]^{\bar{I}} \times \mathbb{R}^{\bar{I}} \times \Delta(\Lambda)$ specifies for a vector of actions a, the probability i is assigned the good at t, $q_i^t(a)$; i's expected payment at t, $z_i^t(a)$; and a probability distribution of vectors of messages that buyers observe $c^t(\lambda|a)$.

Note that the disclosure rule is part of the characteristics of the period-t mechanism; hence, the seller commits within period t to the disclosure rule. This is realistic since when, in practice, a seller chooses to use a sealed bid instead of an open-outcry auction, the choice of the selling procedure itself captures its transparency level.[39] It is also easy to see that this formulation encompasses full disclosure (then, c^t is deterministic and maps a vector of actions a to itself); no disclosure (then c^t maps every vector of actions a to the same message; hence, it is completely uninformative); as well as intermediate cases. When the seller employs a variable transparency mechanism, she not only determines how a

[38] In "the gap case,"there is always some period in which trade takes place with probability one.

[39] It is worth noting that Proposition 4 below holds even without such commitment, as its proof does not rely on it, and it remains true even if the seller chooses the disclosure rule *after* observing the period-t actions and before proposing the $t + 1$ mechanism.

vector of actions is mapped to assignment probabilities and transfers, but also specifies what buyers observe based on a vector of actions.[40]

As in the baseline model, we assume that *all* players observe the dynamic mechanism (which now consists of a period-1 mechanism that includes the information-disclosure rule) and whether or not trade takes place. Given that the seller observes the entire vector of actions and what is disclosed to all buyers, whereas each buyer observes only the message he receives, the seller has superior information vis-a-vis the buyers, and she becomes *an informed principal*.

Proposition 4 *The* full-transparency *optimal mechanism–the mechanism that is optimal when a is common knowledge– is optimal for the seller at $t = 2$ irrespective, of the disclosure rule associated with the $t = 1$ mechanism.*

Proposition 4 relies on the following observations: First, the *full-transparency* optimum is a feasible choice for every type of the seller, regardless of the buyers' posterior beliefs, because it is dominant-strategy incentive-compatible. This implies that no matter the seller's $t = 2$ information, the expected revenue from the $t = 2$-mechanism must be (weakly) higher than the revenue from the *full-transparency* optimum. Second, because there is a *common prior*, from the ex-ante point of view, the expectation of i's $t = 2$-payment from his and from the seller's perspective coincide. This is despite the fact that this expectation can differ at the interim, given that the seller's and buyer i's beliefs about the types of $-i$ buyers may differ. Skreta (2011) employs these two observations to establish that, at an equilibrium of the informed-seller game, this *interim* expectation must also coincide, regardless of the disclosure policy. Using this finding, that paper obtains the usual expression of the seller's revenue in terms of virtual valuations, as in the standard model with common beliefs.[41] It proceeds to establish[42] that the *full-transparency* optimal mechanism is optimal at $t = 2$ (and, more generally, at the last period of the game), regardless of the disclosure rule associated with the $t = 1$ mechanism.

Proposition 4 implies that optimal $t = 2$-mechanisms are independent of the disclosure policy of the $t = 1$-mechanism. Since disclosure policies can affect revenue only through changing the revenue-maximizing choices at $t = 2$, we can conclude the following:

[40] In the current formulation, a disclosure rule maps a vector of actions to a probability distribution over signals. This is assumed for simplicity. Another possibility would have been that the disclosure rule is a mapping from actions *and* allocations to a probability distribution over signals. Given that allocations are themselves functions of a, we conjecture that the reduced-form formulation is without loss since we can define a new composite disclosure rule that is simply a function of a and is equivalent to the original one.

[41] Some more details are provided in online Appendix D.

[42] Full details are available from the author upon request.

Corollary 2 *Without loss of generality for the seller, we can assume that all buyers observe the entire vector of actions chosen in the first period.*

6 Analysis of the Problem when $2 < T < \infty$: An Overview

We have shown that if $T = 2$, the seller maximizes expected revenue by employing a "Myerson" auction in each period. Here, we sketch out how we can extend this result by induction for any T finite.

We describe the induction step for $T = 3$: First, we establish the analog of Lemma 1, the proof of which remains essentially unchanged, with the only modification that the allocations at $t = 2$ are now continuation allocations of a two-period game that starts at $t = 2$. Given Lemma 1, we can establish Proposition 2 using arguments parallel to the ones used in the $T = 2$ case: When two different actions of a buyer are followed by identical $t = 2$ continuation allocations, we can merge the actions, and that does not change the sequentially-rational continuation allocation at $t = 2$. Actions followed by different continuation allocations increase the cost of sequential rationality constraints. Hence, pooling the lower end of valuations at $t = 1$ is optimal. Given Proposition 2, the analog of Lemma 2 for the $T = 3$ game follows, using arguments identical to those in the $T = 2$ case. Proposition 2 implies that at the optimum, posteriors after all buyers choose the vector $\mathbf{0}$ are truncations of the priors–given by (19). Lemma 8 (in Appendix D) then establishes that the corresponding posterior virtual valuations are strictly increasing. Then, we can apply our $T = 2$ result to obtain the revenue-maximizing sequentially-rational allocation rule at the continuation game that starts at $t = 2$ after the vector $\mathbf{0}$ is chosen at $t = 1$, which is dominant-strategy incentive-compatible (Remark 1 in Appendix E).

To establish Lemma 3, we need to show that the $t = 1$ boundary $\bar{v}_i$ is greater or equal to r_i^* (the commitment reserve price) for all $i \in I$ when the problem lasts three periods. We already know from Lemma 2 of Skreta (2006b) that the $T = 3$-boundaries $r_i^3(\bar{v}_i^2(\bar{v}_i))$ are increasing in $\bar{v}_i^2(\bar{v}_i)$ (or, more generally, $r_i^T(\bar{v}_i^{T-1}(\bar{v}_i))$ are increasing in $\bar{v}_i^{T-1}(\bar{v}_i)$.) The proof of the analog of Lemma 3 mimics that for the two-period version of the game, after we establish that the boundaries $\bar{v}_i^2(\bar{v}_i)$ are increasing in $\bar{v}_i$. The $t = 2$-boundaries $\bar{v}_i^2(\bar{v}_i)$ give the valuation of i below which the seller keeps the object at $t = 2$ when the posteriors are given by equation (19). When $T = 3$, posterior virtual valuations at the continuation game that starts at $t = 2$ are equivalent to $J_i(v_i|\bar{v}_i) \equiv v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)}$. Notice that $J_i(v_i|\bar{v}_i)$ is decreasing in $\bar{v}_i$ since F_i is increasing in $\bar{v}_i$. Then, because posterior virtual valuations fall when $\bar{v}_i$ increases, the region in which the seller keeps the good at $t = 2$ increases; that is, $\bar{v}_i^2(\bar{v}_i)$ is (weakly) increasing in $\bar{v}_i$. With this simple observation in hand, we can straightforwardly establish

Lemma 3 for the case in which the game lasts for three periods. Lemma 4 and Proposition 3 are, then, routine to generalize. Finally, given that the two-period sequentially optimal allocation rule is dominant-strategy incentive-compatible (recall Remark 1), we can easily establish[43] the analog of Corollary 2 for $T = 3$ and for longer finite games. And we conclude that we can, without loss, assume that at $t = 1$, all buyers observe the entire vector of actions chosen.

Given the result for $T = 3$, we can continue to get the result for $T = 4$, since all steps followed in the case of $T = 3$ immediately generalize to the case of $T = 4$. The general induction is as follows: once we have the characterization for a game that lasts t periods, we can use this to establish the result for a game that lasts $t + 1$, periods and so forth.

7 Robustness and Concluding Remarks

We have shown that when sellers cannot resist the temptation to place unsold items back on the market simple selling procedures are optimal: When buyers are ex-ante symmetric, first- or second-price auctions with optimally chosen reservation prices are revenue-maximizing. Lack of commitment is costly for the seller, especially when demand is thin and when the seller is moderately patient.

Governments sell important assets, such as oil tracts, timber tracts, spectrum and treasury bills, through auctions. Optimal design is especially important for revenue generation when the number of buyers who participate in the auction is very small and there is little competition. This is usually the case for auctions of very valuable assets. This observation, together with the fact that a large fraction of items that remain unsold are placed back on the market, makes the characterization obtained in this paper a relevant extension of the optimal-auction literature.

We now offer a few remarks on the generality of the solution. With respect to the definition of "mechanisms," we have been very general: We have assumed that a mechanism consists of some abstract game form endowed with an information-disclosure policy as a way of capturing different scenarios of what buyers observe during the play of an auction. Regarding the generality of buyers' strategies, we allow for mixed strategies, and for non-convex sets of types to be making the same report. Finally, regarding what the seller observes during play, we have assumed that she observes the vector of actions that buyers choose at each stage. This assumption makes the non-commitment constraints quite strong, and intentionally so: The point of our analysis is to find what is best for the seller, given that she cannot commit. If we had assumed that the seller observes nothing over time then, she would not have any reason to update her beliefs, and, trivially, the commitment solution would be sequentially-rational.

[43] Full details are available from the author upon request.

A limitation of this work is that we analyze a finite-horizon problem. This goes somewhat against the spirit of our analysis since the seller *commits* in the last period. However, in many situations in practice, financial or political constraints impose a hard deadline by which an asset must be sold. For example, financial institutions have a certain period of time to sell distressed confiscated property. Technically, an infinite-horizon mechanism-design problem in which the designer behaves sequentially rationally is complex, and its analysis is beyond the scope of the present paper. In such a problem, continuation allocation rules need not be revenue-maximizing, so Lemma 1 may not hold. In addition, it seems impossible to express the problem in a recursive way, thus precluding the use of dynamic programming techniques. It is very likely that the solution of an infinite-horizon problem would be based on an argument that relies on the properties of the revenue-maximizing mechanisms in the finite horizon established in this paper, making the current analysis a key stepping stone towards such a characterization.

In recent years, motivated by the large number and the importance of applications,[44] there has been substantial work on dynamic mechanism design.[45] A key assumption in that literature is that the principal can fully commit ex-ante to mechanisms that are ex-post suboptimal. This can be sometimes unrealistic: Contracting parties often renegotiate or change a contract if it becomes clear that there exist others that dominate it. Thus, designing multi-period incentives schemes under various assumptions of commitment is an important and active area of research: Hörner and Samuelson (2011) study a revenue-management problem in the absence of commitment when the seller is posting prices. Deb and Said (2015) characterize the revenue-maximizing sequential screening contract in a two-period model in which the monopolist cannot commit to the second-period price. Finally, Evans and Reiche (2015) characterize implementable outcomes for an uninformed principal and an informed agent if, having observed the agent's contract choice, the principal can offer a new menu of contracts. Despite the progress, general dynamic mechanism design in the absence of commitment is largely an understudied area. We hope and expect that the ideas and tools developed in this paper will be useful for further work.

Appendices

A Proof of Lemma 1

Proof. Our goal is to establish that $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$ for all $v_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$. First, consider a valuation in $V_i(a_i)$ that, in addition to a_i, reports $\hat{a}_i$, that is, a_i and $\hat{a}_i$ are in the support of $m_i(.|v_i)$. Then, best-response constraints imply that at valuation v_i, buyer i is indifferent between a_i and $\hat{a}_i$,

[44] For example, the classical airline revenue-management problem, the allocation of advertising inventory, the design of incentive schemes that take into account inter-temporal considerations of managers, etc.

[45] See, for example, the survey of Bergemann and Said (2011), and the references therein.

that is, $U_i^{\hat{a}_i}(v_i) = U_i^{a_i}(v_i)$. Valuations in $\bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i) \backslash V_i(a_i)$ do not report a_i at $t = 1$; still, we now show that *all* $v_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ must be indifferent between a_i and $\hat{a}_i$. We argue by contradiction: Consider a $\tilde{v}_i \in \bar{V}_i(a_i) \cap \bar{V}_i(\hat{a}_i)$ that strictly prefers $\hat{a}_i$ to a_i, implying that:

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(v_i)\tilde{v}_i - X_i^{a_i}(v_i), \tag{29}$$

for all options in $\{P_i^{a_i}(v_i), X_i^{a_i}(v_i)\}_{v_i \in V_i(a_i)}$. Given the linearity and the single-crossing property of payoffs, (29) implies that there is an open set of valuations neighboring $\tilde{v}_i$ that strictly prefer $\hat{a}_i$ to a_i at $t = 1$. Let $(\tilde{v}_i^L, \tilde{v}_i^H)$ be the largest such neighborhood. Then, by definition, a_i is not in the support of $m_i(.|v_i)$ for $v_i \in (\tilde{v}_i^L, \tilde{v}_i^H)$, while $\tilde{v}_i^L$ and $\tilde{v}_i^H$ (at least weakly) prefer a_i. Then, there is a gap of $(\tilde{v}_i^L, \tilde{v}_i^H)$ in the support of $f_i(.|a_i)$–the seller's posterior at $t = 2$ after he observes a_i at $t = 1$–and, because the seller employs an optimal mechanism at $t = 2$, it holds:[46]

$$P_i^{a_i}(\tilde{v}_i^H)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^H) = P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^L). \tag{30}$$

Note that (29) trivially[47] implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i - X_i^{a_i}(\tilde{v}_i^L). \tag{31}$$

Now, observe that it is not possible that $P_i^{\hat{a}_i}(\tilde{v}_i) = P_i^{a_i}(\tilde{v}_i^L)$ because, then, (31) implies that $X_i^{\hat{a}_i}(\tilde{v}_i) < X_i^{a_i}(\tilde{v}_i^L)$, and, in that case, both $\tilde{v}_i$ and $\tilde{v}_i^L$ strictly prefer to choose $\hat{a}_i$ at $t = 1$, which contradicts the definition $\tilde{v}_i^L$. If $P_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)$, then because $\tilde{v}_i^H > \tilde{v}_i$, (31) implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i^H - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^H - X_i^{a_i}(\tilde{v}_i^L),$$

which, together with (30), implies that type $\tilde{v}_i^H$ strictly prefers $\hat{a}_i$ to a_i. Contradiction.

If $P_i^{\hat{a}_i}(\tilde{v}_i) < P_i^{a_i}(\tilde{v}_i^L)$, then, because $\tilde{v}_i^L < \tilde{v}_i$, (31) implies that

$$P_i^{\hat{a}_i}(\tilde{v}_i)\tilde{v}_i^L - X_i^{\hat{a}_i}(\tilde{v}_i) > P_i^{a_i}(\tilde{v}_i^L)\tilde{v}_i^L - X_i^{a_i}(\tilde{v}_i^L),$$

which implies that type v_i^L strictly prefers $\hat{a}_i$ to a_i. Contradiction. ∎

B Proof of Proposition 2

The goal is to establish that at a solution of Program NC, (i) there is a unique vector of reports that is associated with no trade at $t = 1$, which we denote by $\mathbf{0} \equiv 0, ..., 0$; and (ii) buyers use a degenerate (pure) reporting rule where all valuations below some threshold $\bar{v}_i$ report 0. When buyers are ex-ante symmetric, the cutoff is equal across buyers: $\bar{v}_i(0) = \bar{v}$. The proof is broken down into three main steps.

Step 1: Reserve price at $t = 2$. We show that among all posteriors associated with reporting functions $m_i(0|.)$ with $\bar{V}_i(0) = [0, \bar{v}_i(0)]$, for some cutoff $\bar{v}_i(0) \in [0, b_i]$, the reporting function (and associated posterior) that leads to the $t = 2$-reserve price closest to r_i^* is the partitional one given by (18) for $\bar{v}_i = \bar{v}_i(0)$. We assume that $\bar{v}_i(0) > 0$; otherwise, messages 0 and a_i are in the support of just

[46] This is easy to see: At an optimal mechanism at $t = 2$, after a vector of actions a_i, a_{-i} was chosen at $t = 1$, the incentive constraint for v_i^H not to mimic v_i^L is tight for all a_{-i}, because, otherwise, the seller could generate strictly more revenue at $t = 2$ by increasing the payments for all valuations greater or equal to v_i^H. This result is analogous to the one in the optimal-pricing problem with two types, where the price is chosen to be just big enough to make the high type indifferent about mimicking the low type.

[47] If a_i is in the support of $m_i(.|\tilde{v}_i^L)$, this is immediate; if not, denote by a slight abuse of notation $P_i^{a_i}(\tilde{v}_i^L)$ and $X_i^{a_i}(\tilde{v}_i^L)$ the best option for $\tilde{v}_i^L$ out of $\{P_i^{a_i}(v_i), X_i^{a_i}(v_i)\}_{v_i \in V_i(a_i)}$.

one type, namely 0, and the result trivially follows. A $t = 2$-reserve price that is as close as possible to r_i^* is preferred from the $t = 1$-perspective since it reduces the region in which the seller assigns the good to negative (prior) virtual valuations (namely, valuations below r_i^*). The result is established with the help of the following Lemmas:

Lemma 5 *Consider first-period reporting rules and mechanisms that satisfy the constraints of Program NC. If a_i is in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$ and $r_i^2(a_i) < \bar{v}_i(0)$, then, $r_i^2(a_i) = r_i^2(0)$.*

Lemma 5[48] tells us that if a report a_i is in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$ and $r_i^2(a_i) < \bar{v}_i(0)$, then $r_i^2(a_i) = r_i^2(0)$. Consider, now, the following modification of the reporting rule: Valuations using a $t = 1$-report a_i associated with a $t = 2$-reserve equal to $r_i^2(0)$ "pool" and simply report 0 instead of a_i. For the modified reporting rule, it holds that $\tilde{m}_i(0|v_i) = m_i(0|v_i) + m_i(a_i|v_i) + ...$, where we add over all reports associated with $t = 2$-reserve price equal to $r_i^2(0)$. We denote the optimal reserve price given the resulting posterior as $\tilde{r}_i^2(0)$. Step 1 of the proof of Proposition 8 in Skreta (2006b)) establishes that $\tilde{r}_i^2(0) = r_i^2(0)$.[49] Now, we move on to compare $\tilde{r}_i^2(0)$ with $r_i^2(\bar{v}_i(0))$—the optimal reserve price given posterior beliefs (19) for $\bar{v}_i = \bar{v}_i(0)$. Note that Lemma 2 in Skreta (2006b) establishes that $r_i^2(\bar{v}_i)$ is increasing in $\bar{v}_i$, which immediately implies that $r_i^2(\bar{v}_i) \leq r_i^2(b_i) \equiv r_i^*$. This last equality tells us that in order to sustain r_i^* as a $t = 2$-optimal reserve price, all $v_i \leq b_i$ must pool with 0 at $t = 1$.

Lemma 6 *Sequential rationality implies that $\tilde{r}_i^2(0) \leq r_i^2(\bar{v}_i(0))$.*

Proof. We argue by contradiction. Suppose that $\tilde{r}_i^2(0) > r_i^2(\bar{v}_i(0))$. First, observe that all reports made by valuations below $r_i^2(0)$ (which, as we have argued, is equal to $\tilde{r}_i^2(0)$) lead to the same reserve price (Lemma 5), implying that $\tilde{m}_i(0|v_i) = m_i(0|v_i) + m_i(a_i|v_i) + ... = 1$ for $v_i \leq \tilde{r}_i^2(0)$, so the seller's posterior after observing 0, given the modified reporting rule $\tilde{m}_i$, is

$$f_i(v_i|0) = \begin{cases} \frac{f_i(v_i)}{F_i(\tilde{r}_i^2(0)) + \int_{\tilde{r}_i^2(0)}^{\bar{v}_i} \tilde{m}_i(0|s) f_i(s) ds}, & v_i \in [0, \tilde{r}_i^2(0)) \\ \frac{\tilde{m}_i(0|v_i) f_i(v_i)}{F_i(\tilde{r}_i^2(0)) + \int_{\tilde{r}_i^2(0)}^{\bar{v}_i} \tilde{m}_i(0|s) f_i(s) ds}, & v_i \in [\tilde{r}_i^2(0), \bar{v}_i(0)] \end{cases}. \tag{32}$$

Define $A(s) \equiv s f_i(s) - F_i(\bar{v}_i) - F_i(s) = s f_i(s) - \int_s^{\bar{v}_i(0)} f_i(t) dt$ and $B(s) \equiv s f_i(s) - \int_s^{\bar{v}_i(0)} \tilde{m}_i(0|t) f_i(t) dt$. Note, that since $\tilde{m}_i(0|t) \leq 1$ for all t, we have that for all s, $A(s) \leq B(s)$. Recalling (13) and the definition of $r_i^2(\bar{v}_i(0))$, it follows that $\int_{r_i^2(\bar{v}_i(0))}^{\tilde{v}} A(s) ds \geq 0$ for all $\tilde{v} \geq r_i^2(\bar{v}_i(0))$, which, because $A(s) \leq B(s)$, immediately implies that

$$\int_{r_i^2(\bar{v}_i(0))}^{\tilde{v}} B(s) ds \geq 0 \text{ for all } \tilde{v} \geq r_i^2(\bar{v}_i(0)),$$

contradicting the definition of $\tilde{r}_i^2(0)$ according to which it satisfies (13) for beliefs (32). ∎

A rough intuition for Lemma 6 is that (32) puts less weight on the higher valuations (above $\tilde{r}_i^2(0)$) compared to (19), so the optimal reserve price given posterior (32) is lower compared to the optimal reserve price given a truncation of the prior (19). Step 1 follows from Lemmas 5 and 6. ∎

We now proceed to establish that the proposed solution minimizes the distortions arising from sequential rationality constraints when it comes to inter-buyer boundaries. First, we consider the case in which buyers are ex-ante symmetric.

[48] See the online appendix for its proof.

[49] This results also follows from the proof of Step 3, Case 1 below, where we also provide the intuition.

Step 2: Inter-buyer boundaries: Symmetric Buyers: When buyers are ex-ante symmetric, at the commitment solution, the revenue-maximizing boundary dividing the regions where i and j get the object is the forty-five degree line. We now establish that the posteriors associated with the reporting rules described in the statement of the Proposition imply that the revenue-maximizing boundary at $t = 2$ dividing the regions where i and j get the object is the forty-five degree line and, hence, are optimal:

When, at period one, the reporting rule is pure and all i report 0 for $v_i \in [0, \bar{v}_i(0)]$, with $\bar{v}_i(0) = \bar{v}_j(0) = \bar{v}$ for all $i, j \in I$, after the seller observes $\mathbf{0}$, her posterior is given by (19) with $\bar{v}_i(0) = \bar{v}_j(0) = \bar{v}$ for all $i, j \in I$. Buyers are, then, still symmetric in the eyes of the seller at $t = 2$, so the optimal boundary dividing the regions where i and j get the object remains the forty-five degree line.

Step 3: Inter-buyer boundaries: Asymmetric Buyers: When buyers are asymmetric, the shape of the ex-ante optimal boundary can be very complex and depends on the particular distributions.

Case 1–Same t=2 boundaries: Suppose that for all v_i in $[0, \bar{v}_i(0)]$, if a report a_i is in the support of their reporting rule $m_i(.|v_i)$, then it holds that: $r_i^2(a_i) = r_i^2(0)$ and $v_{-i}(v_i, 0, a_{-i}) = v_{-i}(v_i, a_i, a_{-i})$ for all a_{-i} and all v_i. If this is the case, the second-period allocation is identical for all a_{-i}, regardless of whether buyer i reported 0 or a_i at $t = 1$. We argue that if we replace buyer i's reporting rule by one, where i pools and reports 0, instead of reporting any other message associated with the same $t = 2$-boundaries (which fully describe the $t = 2$-mechanism), the $t = 2$-mechanism remains optimal and, hence, is sequentially-rational if all valuations in $\bar{V}_i(0)$ pool and report 0. Full details of this proof can be found in the online Appendix. The intuition is as follows: given that the seller's optimal $t = 2$-mechanism does not differ, regardless of whether or not she can condition on having observed message 0 or a_i, nothing changes in the seller's response when she conditions on the pooled information that i reported *either* 0 *or* a_i. Hence, a reporting rule that pools the information does not change anything in terms of the seller's choice at $t = 2$, and, hence, it is without loss and we are done.

We proceed to establish that at a solution of Program NC, reporting rules must be such that we are in Case 1–hence, the case that follows cannot arise at a solution of Program NC:

Case 2–Different t=2 boundaries: We establish that this case cannot arise at a solution of Program NC. We start by establishing an auxiliary Lemma:

Lemma 7 *At a solution of Program NC, if a_i is in the support of $m_i(.|v_i)$ for some $v_i \in [0, \bar{v}_i(0)]$, with $\bar{v}_i(0) > 0$, then $r_i^2(a_i) < \bar{v}_i(0)$.*

Lemma 7[50] implies that valuations in $[0, \bar{v}_i]$ report either 0 or messages a_i associated with $r_i^2(a_i) < \bar{v}_i(0)$. Then, together with Lemma 5, this implies that $r_i^2(a_i) = r_i^2(0)$. Given that the optimal $t = 2$ reserve price must be in the support of $m_i(a_i|.)$, it follows that $\underline{v}_i(a_i) \leq r_i^2(a_i) = r_i^2(0)$.

Observe now that at a solution of Program NC, it holds for all a_{-i} that $q_S(0, a_{-i}) \geq q_S(a_i, a_{-i})$. This observation states that the probability that the seller keeps the good at $t = 1$ is weakly higher when i reports 0 compared to any other message. This is almost immediate and follows from the fact that 0 is chosen by the lower-end of valuations, which are also associated with the lowest virtual valuations–the region in which seller aims to minimize the extent to which the object is allocated to one of the buyers. We proceed with the help of this observation and of Lemma 7:

Recalling (3), (6), (7) and (8), (20) can be more explicitly rewritten as:

[50]See the online Appendix for its proof.

$$P_i^{1(0)} + \delta P_i^{2(0)}(v_i) = P_i^{1(a_1)} + \delta P_i^{2(a_1)}(v_i). \tag{33}$$

If $\underline{v}_i(a_i) < r_i^2(a_i)$, then for $v_i \in [0, r_i^2(a_i))$, (33) reduces to $P_i^{1(0)} = P_i^{1(a_i)}$, which implies that

$$P_i^{1(0)} - P_i^{1(a_i)} = \delta \left[P_i^{2(a_i)}(v_i) - P_i^{2(0)}(v_i) \right] = 0 \tag{34}$$

for v_i a.e. in $\bar{V}_i(0) \cap \bar{V}_i(a_i)$. If $\underline{v}_i(a_i) = r_i^2(a_i)$, at $v_i = r_i^2(a_i) = r_i^2(0)$, $r_i^2(0)$ enjoys zero surplus at $t = 2$. Then, Lemma 1 implies that the first-period surplus after reporting 0 and a_i must be the same, so

$$P_i^{1(0)} r_i^2(a_i) - X_i^{1(0)} - [P_i^{1(a_i)} r_i^2(a_i) - X_i^{1(a_i)}] = 0.$$

In this case, if $P_i^{1(0)} = P_i^{1(a_i)}$ fails because, say, $P_i^{1(0)} > P_i^{1(a_i)}$, then valuations arbitrarily close to $r_i^2(a_i)$ would strictly prefer to report 0 rather than a_i, and this would contradict the optimality of $r_i^2(a_i)$. Hence, (34) holds in this case, as well.

Note, now, that at $v_i = r_i^2(a_i) = r_i^2(0)$, i's posterior virtual valuation is zero, so he gets the object at $t = 2$ when all $-i$'s realized valuations are below the reserve, namely when $v_{-i} < r_{-i}(a_{-i})$, where $r_{-i}(a_{-i})$ consists of the vector of the reserve prices for $-i$ buyers. Then, recalling (8), the RHS of (34) can be rewritten as:

$$\delta \int_{A_{-i}} \left[[q_S(a_i, a_{-i}) - q_S(0, a_{-i})] \int_{\underline{v}_{-i}(a_{-i})}^{r_{-i}(a_{-i})} f_{-i}(v_{-i}) m_i(a_{-i}|v_{-i}) dv_{-i} \right] da_{-i}. \tag{35}$$

Combining (34) with (35) we get that at $v_i = r_i^2(a_i) = r_i^2(0)$:

$$P_i^{1(0)} - P_i^{1(a_i)} = \delta \int_{A_{-i}} \left[[q_S(a_i, a_{-i}) - q_S(0, a_{-i})] \int_{\underline{v}_{-i}(a_{-i})}^{r_{-i}(a_{-i})} f_{-i}(v_{-i}) m_i(a_{-i}|v_{-i}) dv_{-i} \right] da_{-i} = 0. \tag{36}$$

Since we have argued that at a solution of NC, it must be the case that $q_S(0, a_{-i}) \geq q_S(a_i, a_{-i})$, for all a_{-i}, the only possibility for the equality in (36) to hold is that

$$q_S(0, a_{-i}) = q_S(a_i, a_{-i}), \tag{37}$$

for all a_{-i}.

From (16), we know that after reporting a_i at $t = 1$, i gets the good at $t = 2$ when $v_{-i} \leq v_{-i}(v_i, a)$. Also, since the RHS of (34) equals the LHS of (34) for all v_i, (34) can be rewritten as:

$$0 = \delta \int_{A_{-i}} \left[\int_{\underline{v}_{-i}(a_{-i})}^{v_{-i}(v_i, a)} q_S(a) f_{-i}(v_{-i}) m_{-i}(a_{-i}|v_{-i}) dv_{-i} - \int_{\underline{v}_{-i}(a_{-i})}^{v_{-i}(v_i, 0, a_{-i})} q_S(0, a_{-i}) f_{-i}(v_{-i}) m_{-i}(a_{-i}|v_{-i}) dv_{-i} \right] da_{-i}. \tag{38}$$

Suppose that $v_{-i}(v_i, a) > v_{-i}(v_i, 0, a_{-i})$ at some v_i. Note that if $v_{-i}(v_i, a) > v_{-i}(v_i, 0, a_{-i})$ for some a_{-i}, this means that the posterior virtual valuation at v_i after message a_i is higher than after 0,[51] so it holds that $v_{-i}(v_i, a_i, \tilde{a}_{-i}) > v_{-i}(v_i, 0, \tilde{a}_{-i})$ for all $\tilde{a}_{-i}$. With the help of this observation, (38)

[51] We can see this because when $v_{-i}(v_i, a_i, a_{-i}) > v_{-i}(v_i, 0, a_{-i})$ the area of v_{-i} where i gets the good at $t = 2$ when he chooses a_i at $t = 1$ contains the area of v_{-i} where i gets the good at $t = 2$ when he chooses 0 at $t = 1$.

can be rewritten as:

$$0=\delta\int_{A_{-i}}\int_{v_{-i}(v_i,0,a_{-i})}^{v_{-i}(v_i,a)} q_S(a)f_{-i}(v_{-i})m_{-i}(a_{-i}|v_{-i})dv_{-i}da_{-i}+$$

$$\delta\int_{A_{-i}}\int_{\underline{v}_{-i}(a_{-i})}^{v_{-i}(v_i,0,a_{-i})}[q_S(a)-q_S(0,a_{-i})]f_{-i}(v_{-i})m_{-i}(a_{-i}|v_{-i})dv_{-i}da_{-i}.$$

Since (37) implies that the second term is zero, the only possibility for this equality to hold is that $v_{-i}(v_i,a)=v_{-i}(v_i,0,a_{-i})$ for all a_{-i}. Contradiction. Hence, at a solution of Program NC, we are in Case 1, where we have shown that replacing i's reporting rule with a partitional one where all valuations in $[0,\bar{v}_i(0)]$ report 0 does not affect the optimal $t=2$-mechanism. This establishes that the seller does not benefit by mixed or other kinds of complex reporting rules. ■

C Proof of Lemma 2

Linearity of (21) implies that if we choose $q_S(\mathbf{0})$ so as to maximize it, ignoring all but the resource constraint of Program NC, at such a relaxed solution, either $q_S(\mathbf{0})=1$ or $q_S(\mathbf{0})=0$.

When $q_S(\mathbf{0})=1$, $q_i(\mathbf{0})=0$ for all i, which says that no buyer obtains the good at $t=1$ for vectors of valuations in $\times_{i\in I}[0,\bar{v}_i]$. Also, when $q_S(\mathbf{0})=1$, at an optimum $q_i(0,a_{-i})=0$, for all a_{-i} different from $\mathbf{0_{-i}}$. This is because buyers $j\in I, j\neq i$ report a_{-i} different from $\mathbf{0_{-i}}$ when $v_j\geq\bar{v}_j$ for all $j\in I, j\neq i$. Then, if i does not get the good when $v_j\leq\bar{v}_j$ for all $j\in I, j\neq i$, he should not be getting it when competing buyers have higher realized valuations–and, hence, higher virtual valuations. ■

D Proof of Lemma 3

In order to establish Lemma 3, we prove an intermediate Lemma:

Lemma 8 *If $J_i(v_i)=v_i-\frac{[1-F_i(v_i)]}{f_i(v_i)}$ is increasing in v_i, then so is $J_i(v_i\,|\bar{v}_i\,)=v_i-\frac{F_i(\bar{v}_i)-F_i(v_i)}{f_i(v_i)}$.*

Proof. In order for $J_i(v_i\,|\bar{v}_i\,)$ to be increasing in v_i, the following inequality must hold:

$$f_i'(v_i)[F_i(\bar{v})-F_i(v_i)]\geq -2f_i^2(v_i). \tag{39}$$

Now, if $J_i(v_i)$ is increasing in v_i, we have that:

$$f_i'(v_i)[1-F_i(v_i)]\geq -2f_i^2(v_i). \tag{40}$$

If $f_i'\geq 0$, (39) is automatically satisfied. If $f_i'<0$, then we have that

$$f_i'(v_i)\,[F_i(\bar{v})-F_i(v_i)]\geq f_i'(v_i)\,[1-F_i(v_i)]\geq -2f_i^2(v_i).$$

■

From Lemma 2, it follows that buyer i never gets the object at $t=2$ when his valuation is below the optimal second-period reserve price denoted by $r_i^2(\bar{v}_i)$, which satisfies (13) for beliefs given by (19). With the help of this observation, and after substituting (22) in (21), the seller's expected revenue can be rewritten as:

$$\Sigma_{i\in I}\left[\int_{r_i^2(\bar{v}_i)}^{\bar{v}_i}\delta P_i^2(v_i)J_i(v_i)f_i(v_i)dv_i+\int_{\bar{v}_i}^{b}P_i(v_i)J_i(v_i)f_i(v_i)dv_i\right]. \tag{41}$$

We employ (41) to show that at a solution of Program NC, $\bar{v}_i \geq r_i^*$, where r_i^* is the optimal reserve price when we ignore the sequential rationality constraints (given by (13) for the prior). In order to do so, we evaluate the impact of a marginal increase in $\bar{v}_i$ on the seller's revenue and show that it is strictly positive whenever $\bar{v}_i < r_i^*$, implying that at a revenue-maximizing assignment, it must be the case that $\bar{v}_i \geq r_i^*$.

As a preliminary step, we now establish that r_i^2 is a continuous function of $\bar{v}_i$, and, hence, it is differentiable almost everywhere: From Lemma 8, if $J_i(v_i)$ is strictly increasing, so is $J_i(v_i \,|\bar{v}_i)$. Hence, r_i^2 is unique. Moreover, since f_i is continuous, so is F_i, which ensures that r_i^2 is a continuous function of $\bar{v}_i$, and, hence, it is differentiable almost everywhere.[52]

Now, when we increase $\bar{v}_i$ for some buyer, this has a direct and an indirect effect on (41). The direct effect is a change in the range of integration for i. The indirect effect is a change on q_j^2, for all $j \in I$, and it results because an increase in $\bar{v}_i$ changes the ranking of the posterior virtual valuations: Recall that buyer i wins the object at $t = 2$ if his posterior virtual valuation is the highest and is above the seller's value-that is, we must have

$$v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)} \geq v_j - \frac{F_j(\bar{v}_j) - F_j(v_j)}{f_j(v_j)} \tag{42}$$

and $v_i - \frac{F_i(\bar{v}_i) - F_i(v_i)}{f_i(v_i)} \geq 0$. The valuation v_j that satisfies (42) with equality given $v_i, \bar{v}_i, \bar{v}_j$–namely, the boundary $v_j(v_i, \bar{v}_i, \bar{v}_j)$–varies continuously with $\bar{v}_i$ and $\bar{v}_j$. We now examine each of these effects separately. The direct effect of increasing $\bar{v}_i$ is:

$$\begin{aligned} & \delta P_i^2(\bar{v}_i) J_i(\bar{v}_i) f_i(\bar{v}_i) - \delta P_i^2(r_i^2(\bar{v}_i)) J_i(r_i^2(\bar{v}_i)) f_i(r_i^2(\bar{v}_i)) \frac{\partial r_i^2(\bar{v}_i)}{\partial \bar{v}_i} - P_i(\bar{v}_i) J_i(\bar{v}_i) f_i(\bar{v}_i) \qquad (43) \\ = \; & -(P_i(\bar{v}_i) - \delta P_i^2(\bar{v}_i)) J_i(\bar{v}_i) f_i(\bar{v}_i) - \delta P_i^2(r_i^2(\bar{v}_i)) J_i(r_i^2(\bar{v}_i)) f_i(r_i^2(\bar{v}_i)) \frac{\partial r_i^2(\bar{v}_i)}{\partial \bar{v}_i} > 0. \end{aligned}$$

This inequality results from the following observations: At a solution of Program NC, P_i must be increasing, implying that $P_i(\bar{v}_i) \geq \delta P_i^2(\bar{v}_i)$. Also, from Lemma 2 in Skreta (2006b), we have that $r_i^2(\bar{v}_i)$ is increasing in $\bar{v}_i$, from which we obtain that $\frac{\partial r_i^2(\bar{v}_i)}{\partial \bar{v}_i} \geq 0$. From the last two observations, it follows that this partial effect (43) is strictly positive for $\bar{v}_i < r_i^*$ since, if this is the case, we have $J_i(\bar{v}_i) < 0$ and, hence, $J_i(r_i^2(\bar{v}_i)) < 0$.

We now move on to establish that the indirect effect of $\bar{v}_i$ on expected revenue is zero:

$$\Sigma_{i \in I} \int_{r_i^2(\bar{v}_i)}^{\bar{v}_i} \delta \frac{\partial P_i^2(v_i)}{\partial \bar{v}_i} J_i(v_i) f_i(v_i) dv_i = 0.$$

To see this, let $q^2(v)$ denote the allocation at $t = 2$ given $\bar{v}_i$, and let $\hat{q}^2(v)$ denote the allocation rule at $t = 2$ given cutoff $\bar{v}_i + \varepsilon$ when the realized vectors of valuations are v. Observe that $q_i^2(v) = \hat{q}_i^2(v)$ for all $v \in V_{-i} \times [0, \bar{v}_i]$, except at the vectors of valuations where the ranking of virtual valuations changes. This happens along the boundaries where posterior virtual valuations are equal, which is a set of measure zero.

Therefore, the direct effect of increasing $\bar{v}_i$, as captured in (43), is equal to the total. We can, then, conclude that at a revenue-maximizing PBE, $\bar{v}_i \geq r_i^*$. ■

[52] This cutoff can be alternatively obtained as the solution of the optimal price by a monopolist who is facing a downward-sloping demand $[F_i(\bar{v}_i) - F_i(r_i^2)]$. The monopolist problem is $[F_i(\bar{v}_i) - F_i(r_i^2)]r_i^2$. The first-order necessary conditions for a maximum (which are also sufficient given MHR), are $[F_i(\bar{v}_i) - F_i(r_i^2)] - f_i(r_i^2) r_i^2 = 0$ or, since $f_i(r_i^2) > 0$, $v_i - \frac{F_i(\bar{v}_i) - F_i(r_i^2)}{f_i(r_i^2)} = 0$. Then, the continuity of r_i^2 follows by the continuity of the seller's objective function and the Theorem of the Maximum.

E Proof of Lemma 4

First, observe that p^* in (22) satisfies the resource constraints. Moreover, it satisfies the sequential rationality constraints since q^2 in (22) is given by (16) for beliefs given by (19), and $\mathbf{0}$ is the only vector of actions that leads to no trade at $t = 1$ (hence, it is the only vector of actions relevant for sequential rationality constraints). We now show that P_i is increasing in v_i. We actually establish a stronger result-namely, that $p_i^*(v)$ is increasing in v_i for each v_{-i}: From standard arguments, it is easy to see that $p_i^*(v)$ is increasing in v_i for $v_i \in [0, b_i] \backslash \{\bar{v}_i\}$ and for all v_{-i}. Hence, it remains to show that it does not drop at $\bar{v}_i$: When at least one $v_j > \bar{v}_j$, some buyer among those with valuations above the cutoff gets the good at $t = 1$, so we have that $p_i^*(\bar{v}_i - \varepsilon, v_{-i}) = 0$ (where $\varepsilon > 0$), whereas $p_i^*(\bar{v}_i + \varepsilon, v_{-i})$ is either 0 or 1. In both cases, it is increasing in v_i. Now, when for all $j \neq i$, $v_j < \bar{v}_j$, we have that $p_i^*(\bar{v}_i - \varepsilon, v_{-i})$ is equal to either 0 or δ, whereas $p_i^*(\bar{v}_i + \varepsilon, v_{-i}) = 1$ for that region of v_{-i}; hence, again, p_i^* is increasing. Finally, it is routine to verify that given the payments specified in (23), the participation constraints are satisfied. ■

Remark 1 From Lemma 4, we can conclude that (22) is dominant-strategy incentive-compatible since p_i^* is increasing for each realization of v_{-i}.

F Derivation of the Payment Rule at the Optimal Mechanism

We rewrite (23) as

$$x_i(v) = q_i(a(v))v_i - \int_0^{v_i} q_i(t_i, v_{-i})dt_i + \delta \left[q_S(v)q_i^2(v)v_i - \int_0^{v_i} q_S(t_i, v_{-i})q_i^2(t_i, v_{-i})dt_i \right]. \quad (44)$$

Recalling the optimal first-period assignment described in Lemma 2, for $v \in \times_{i \in I}[0, \bar{v}_i]$, (44) reduces to $x_i(v) = \delta \left[q_i^2(v)v_i - \int_0^{v_i} q_i^2(t_i, v_{-i})dt_i \right] = \delta z_i^2(v)$, where q^2 and z^2 are given by (16). This is because for $v \in \times_{i \in I}[0, \bar{v}_i]$, $q_S = 1$, and the first two terms of (44) are zero. Now, for $v \in V \backslash \times_{i \in I} [0, \bar{v}_i]$, (44) reduces to:

$$x_i(v_i, v_{-i}) = q_i(v)v_i - \int_{\bar{v}_i}^{v_i} q_i(t_i, v_{-i})dt_i - \delta \int_{\max\{r_i^2(\bar{v}_i), r_i^2(v_{-i})\}}^{\bar{v}_i} q_i^2(t_i, v_{-i})dt_i, \quad (45)$$

where $r_i^2(\bar{v}_i)$ is given by (13) given beliefs (19) and $r_i^2(v_{-i}) = \inf\{v_i \in [0, \bar{v}_i] \text{ such that } q_i^2(v) = 1\}$.

There are two cases to consider, depending on whether or not buyer i faces some competition at $t = 1$: When buyer i faces some competition at $t = 1$–that is, when $I^1(v) \neq \{i\}$–i pays only when he wins, and his payment is equal to $r_i^1(v_{-i})$. When $v_{-i} \in \times_{j \neq i}[0, \bar{v}_j]$, i does not face competition at $t = 1$, and using q_i from (25) and q^2 from (16), (45) can be further simplified to

$$x_i(v) = (1 - \delta)\bar{v}_i + \delta \max\left\{r_i^2(\bar{v}_i), r_i^2(v_{-i})\right\}. \quad (46)$$

However, it is not possible to implement (46) as is because it varies with v_{-i}, whereas all buyers $-i$ with $v_{-i} \times_{j \neq i} [0, \bar{v}_j]$, pool at $t = 1$ and report 0. But given that in the case under consideration, buyer i faces no competition at $t = 1$, we can use (46) to determine a personalized reserve price, $\bar{r}_i^1$, that satisfies

$$\bar{r}_i^1 = \frac{1}{F_{-i}(\bar{v}_{-i})} E_{v_{-i} \in \times_{j \neq i}[0, \bar{v}_j]} \left[(1 - \delta)\bar{v}_i + \delta \max\left\{r_i^2(\bar{v}_i), r_i^2(v_{-i})\right\} \right]. \quad (47)$$

Since buyer i does not know v_{-i}, from his perspective, he is indifferent between paying $\bar{r}_i^1$ whenever all other buyers report 0 (which occurs with probability $F_{-i}(\bar{v}_{-i})$) and incurring payments according to (46). Moreover, these two different payment methods are equivalent from the seller's perspective because they are associated with the same allocation rule.

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Online Appendix: Omitted Proofs

Optimal Auction Design under Non-Commitment

Vasiliki Skreta

A Proof of Lemma 5

Lemma 5 *Consider first-period reporting rules and mechanisms that satisfy the constraints of Program NC. If a_i is in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$ and $r_i^2(a_i) < \bar{v}_i(0)$, then, $r_i^2(a_i) = r_i^2(0)$.*

Proof. We argue by contradiction. Consider a report a_i that satisfies the conditions in the Lemma. There are a number of cases to consider:

Case A: $\underline{v}_i(a_i) \leq r_i^2(0)$.

Case A1: $\underline{v}_i(a_i) < r_i^2(a_i)$.

For $v_i \in [\underline{v}_i(a_i), r_i^2(0))$, i's valuation is below the $t = 2$-reserve $r_i^2(0)$, so he does not obtain the good at $t = 2$ after reporting 0 at $t = 1$. Hence, for $v_i \in [0, r_i^2(0))$,

$$P_i^0(v_i) = P_i^{1(0)}. \tag{48}$$

Similarly, for $v_i \in [\underline{v}_i(a_i), r_i^2(a_i))$, we have

$$P_i^{a_i}(v_i) = P_i^{1(a_i)}. \tag{49}$$

In addition, if $r_i^2(a_i) < r_i^2(0)$, then for $v_i \in [r_i^2(a_i), r_i^2(0)]$, we have that

$$P_i^{a_i}(v_i) \geq P_i^{1(a_i)} + \delta P_i^{2(a_i)}(r_i^2(a_i)) > P_i^{1(a_i)}, \tag{50}$$

which follows because $P_i^{2(a_i)}(r_i^2(a_i)) > 0$–an implication of the fact that reserve prices matter. An analogous expression can be obtained if $r_i^2(a_i) > r_i^2(0)$. Lemma 1 implies that for $v_i \in [\underline{v}_i(a_i), r_i^2(0))$, we have

$$P_i^{a_i}(v_i) = P_i^0(v_i). \tag{51}$$

Case A1.1 $r_i^2(a_i) < r_i^2(0)$. Note that for $v_i \in [\underline{v}_i(a_i), r_i^2(a_i))$, (51) implies:

$$P_i^{1(a_i)} = P_i^{1(0)}, \tag{52}$$

while with the help of (50), (51) implies that for $v_i \in [r_i^2(a_i), r_i^2(0)]$,

$$P_i^0(v_i) = P_i^{1(0)} = P_i^{a_i}(v_i) > P_i^{1(a_i)}. \tag{53}$$

Contradiction.

Case A1.2 $r_i^2(a_i) > r_i^2(0)$. The fact that this case is also impossible follows from arguments parallel to the ones employed in Case A1.1.

Case A2: $\underline{v}_i(a_i) = r_i^2(a_i) < r_i^2(0)$.

When $r_i^2(a_i) = \underline{v}_i(a_i)$, the posterior virtual valuation $J_i(.|a_i)$ is positive at $\underline{v}_i(a_i)$, implying that i obtains the good with positive probability at $t = 2$. Also, because the seller chooses an optimal mechanism at $t = 2$, it must be the case that the surplus for valuation $\underline{v}_i(a_i) = r_i^2(a_i)$ at $t = 2$ must be zero since it is the smallest possible valuation in the support of $m_i(a_i|.)$. This implies that

$$P_i^{a_i}(\underline{v}_i(a_i))\underline{v}_i(a_i) - X_i^{a_i}(\underline{v}_i(a_i)) \quad = \quad P_i^{1(a_i)}\underline{v}_i(a_i) - X_i^{1(a_i)}. \tag{54}$$

Now, (54) implies that for $v_i < \underline{v}_i(a_i)$, it must be the case that

$$P_i^0(v_i) \leq P_i^{1(a_i)}. \tag{55}$$

To see this, note that if $P_i^0(v_i) > P_i^{1(a_i)}$ for $v_i < \underline{v}_i(a_i)$, then best-responding implies that

$$P_i^0(v_i)v_i - X_i^0(v_i) \geq P_i^{1(a_i)}v_i - X_i^{1(a_i)},$$

but, then, since $v_i < \underline{v}_i(a_i)$ and $P_i^0(v_i) > P_i^{1(a_i)}$, we have that

$$P_i^0(v_i)\underline{v}_i(a_i) - X_i^0(v_i) > P_i^{1(a_i)}\underline{v}_i(a_i) - X_i^{1(a_i)},$$

implying that $\underline{v}_i(a_i)$ strictly prefers to report 0 rather than a_i, contradicting the fact that $\underline{v}_i(a_i)$ is at the closure of the support of $m_i(a_i|.)$

Combining (55) with (50) and recalling (48) we obtain that $P_i^{a_i}(v_i) > P_i^0(v_i)$, which contradicts (51). This concludes all subcases of case A.

Case B: $\underline{v}_i(a_i) > r_i^2(0)$.

We establish that this case is impossible. First, note that in this case it immediately follows that $r_i^2(0) < \underline{v}_i(a_i) \leq r_i^2(a_i)$.

Given the earlier observations, it is easy to see that for $v_i \in [r_i^2(0), r_i^2(a_i)]$,

$$P_i^0(v_i) = P_i^{1(0)} + \delta P_i^{2(0)}(v_i) \leq P_i^{1(a_i)}. \tag{56}$$

Also, at $r_i^2(a_i) + \varepsilon$, with $\varepsilon > 0$ it holds that:

$$P_i^0(r_i^2(a_i) + \varepsilon) = P_i^{a_i}(r_i^2(a_i) + \varepsilon) = P_i^{1(a_i)} + \delta P_i^{2(a_i)}(r_i^2(a_i) + \varepsilon), \tag{57}$$

which, together with (56), implies that for some $\varepsilon > 0$ small enough:

$$P_i^0(r_i^2(a_i) + \varepsilon) - P_i^0(r_i^2(a_i) - \varepsilon) \geq \delta P_i^{2(a_i)}(r_i^2(a_i) + \varepsilon)) > 0, \tag{58}$$

where $P_i^{2(a_i)}(r_i^2(a_i) + \varepsilon)) > 0$, since at $t = 2$ trade takes place with positive probability at the reserve.

Observe that (58) implies that $P_i^{2(0)}$ must jump at $r_i^2(a_i)$. We now establish that this is impossible. For transparency of the arguments, we suppose (without loss) that, other than 0, a_i is the only other report in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$. The definition of $r_i^2(a_i)$ in (13) implies that

$$r_i^2(a_i)m_i(a_i|r_i^2(a_i))f_i(r_i^2(a_i)) - \int_{r_i^2(a_i)}^{b_i} m_i(a_i|t_i)f_i(t_i)dt_i > 0, \tag{59}$$

while

$$0 > (r_i^2(a_i) - \varepsilon)m_i(a_i|r_i^2(a_i) - \varepsilon)f_i(r_i^2(a_i) - \varepsilon) - \int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(a_i|t_i)f_i(t_i)dt_i, \tag{60}$$

where $\varepsilon > 0$ arbitrarily small. Note that (59) can be equivalently written as $r_i^2(a_i) - \frac{\int_{r_i^2(a_i)}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i}{m_i(a_i|r_i^2(a_i)) f_i(r_i^2(a_i))} > 0$ or

$$r_i^2(a_i) > \frac{\int_{r_i^2(a_i)}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i}{m_i(a_i|r_i^2(a_i)) f_i(r_i^2(a_i))}, \tag{61}$$

while (60) can be equivalently written as $0 > (r_i^2(a_i) - \varepsilon) - \frac{\int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i}{m_i(a_i|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon)}$ or

$$\frac{\int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i}{m_i(a_i|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon)} > (r_i^2(a_i) - \varepsilon). \tag{62}$$

Note that since both (61) and (62) are strict, and hold for $\varepsilon > 0$ but arbitrarily small, combining (61) and (62), we get:

$$m_i(a_i|r_i^2(a_i)) f_i(r_i^2(a_i)) \int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i > m_i(a_i|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon) \int_{r_i^2(a_i)}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i. \tag{63}$$

Now, in order for $P_i^{2(0)}$ to have a discrete jump at $r_i^2(a_i)$, it must be the case that the posterior virtual valuation jumps at that point; that is,

$$J_i(r_i^2(a_i)|0) - J_i(r_i^2(a_i) - \varepsilon|0) > \tilde{J} > 0, \tag{64}$$

where $\varepsilon > 0$ arbitrarily small and $\tilde{J} > 0$. Note that (64) can be rewritten as:

$$r_i^2(a_i) - \frac{\int_{r_i^2(a_i)}^{b_i} m_i(0|t_i) f_i(t_i) dt_i}{m_i(0|r_i^2(a_i)) f_i(r_i^2(a_i))} - \left[(r_i^2(a_i) - \varepsilon) - \frac{\int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(0|t_i) f_i(t_i) dt_i}{m_i(0|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon)} \right] > \tilde{J}, \tag{65}$$

which, since ε is arbitrarily small, implies:

$$\begin{aligned} m_i(0|r_i^2(a_i)) f_i(r_i^2(a_i)) \int_{r_i^2(a_i)-\varepsilon}^{b_i} m_i(0|t_i) f_i(t_i) dt_i > m_i(0|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon) \int_{r_i^2(a_i)}^{b_i} m_i(0|t_i) f_i(t_i) dt_i \\ + \tilde{J} m_i(0|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon). \end{aligned} \tag{66}$$

Since a_i is the only other report in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$, for all $v_i \in [0, \bar{v}_i(0)]$, it holds that

$$m_i(0|v_i) + m_i(a_i|v_i) = 1. \tag{67}$$

By adding (63) and (66), by letting $\bar{J} \equiv \tilde{J} m_i(0|r_i^2(a_i)-\varepsilon) f_i(r_i^2(a_i)-\varepsilon) > 0$, and by using (67), we obtain

$$f_i(r_i^2(a_i)) \int_{r_i^2(a_i)-\varepsilon}^{b_i} f_i(t_i) dt_i > f_i(r_i^2(a_i)-\varepsilon) \int_{r_i^2(a_i)}^{b_i} f_i(t_i) dt_i + \bar{J}, \tag{68}$$

which is a *strict* inequality for all $\varepsilon > 0$, arbitrarily small. This is impossible since f_i is continuous.

Recall that for transparency of the arguments, we have assumed that, other than 0, a_i is the only other report in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$. If that is not the case, the posterior virtual valuation must jump at $r_i^2(a_i)$ for all $\tilde{a}_i$ in the support of $m_i(.|v_i)$ for $v_i \in [0, \bar{v}_i(0)]$, and we obtain the same result by adding up inequalities analogous to (65) for all such $\tilde{a}_i$'s. ■

B Omitted Details Proof of Proposition 2

Details of Step 3, Case 1:

We formalize the intuition discussed in the text below following a few long and rather tedious steps. For transparency of the arguments, suppose, without loss of generality, that other than 0, a_i is the only report employed by valuations in $V_i(0)$; then, we have that $m_i(0|v_i) + m_i(a_i|v_i) = 1$ for all $V_i(0)$. Because $r_i^2(a_i) = r_i^2(0)$ and $v_{-i}(v_i, 0, a_{-i}) = v_{-i}(v_i, a_i, a_{-i})$ for all a_{-i} and v_i, we have that the allocation rule $q_i^2(v, a_i, a_{-i}) = q_i^2(v, 0, a_{-i})$ for all $i \in I$, v and all vectors of actions $a_{-i} \in A_{-i}$ that are chosen with strictly positive probability. From now on, we call this commonly optimal allocation rule q^2 (it also depends on the vector of the other buyers' reports a_{-i}, which is held fixed below). Given that q^2, z^2 is revenue-maximizing at $t = 2$ given 0, for all a_{-i}, it satisfies that[53]

$$\begin{aligned} & \int_V \Sigma_{i\in I} q_i^2(v_i, v_{-i}) \left[v_i f_i(v_i|0) - (1 - F_i(v_i|0))\right] f_{-i}(v_{-i}|a_{-i}) dv \\ \geq & \int_V \Sigma_{i\in I} \tilde{q}_i^2(v_i, v_{-i}) \left[v_i f_i(v_i|0) - (1 - F_i(v_i|0))\right] f_{-i}(v_{-i}|a_{-i}) dv \end{aligned} \tag{69}$$

for all $\tilde{q}^2$ feasible. By substituting the exact expressions of $f_i(v_i\,|0)$ and of $F_i(v_i|0)$, and by multiplying through with the constant $\int_{V_i} m_i(0|t_i) f_i(t_i) dt_i$, one can easily see that (69) is equivalent to

$$\begin{aligned} & \int_V q_i^2(v_i, v_{-i}) \left[v_i m_i(0|v_i) f_i(v_i) - \int_{v_i}^{b_i} m_i(0|t_i) f_i(t_i) dt_i\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I\\ j\neq i}} q_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(0|v_i) f_i(v_i) dv \\ \geq & \int_V \tilde{q}_i^2(v_i, v_{-i}) \left[v_i m_i(0|v_i) f_i(v_i) - \int_{v_i}^{b_i} m_i(0|t_i) f_i(t_i) dt_i\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I\\ j\neq i}} \tilde{q}_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(0|v_i) f_i(v_i) dv, \end{aligned} \tag{70}$$

for all $\tilde{q}^2$.

Similarly, given that q^2, z^2 is revenue-maximizing at $t = 2$ given a_i, a_{-i}, it satisfies that

$$\begin{aligned} & \int_V q_i^2(v_i, v_{-i}) \left[v_i m_i(a_i|v_i) f_i(v_i) - \int_{v_i}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I\\ j\neq i}} q_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(a_i|v_i) f_i(v_i) dv \\ \geq & \int_V \tilde{q}_i^2(v_i, v_{-i}) \left[v_i m_i(a_i|v_i) f_i(v_i) - \int_{v_i}^{b_i} m_i(a_i|t_i) f_i(t_i) dt_i\right] f_{-i}(v_{-i}|a_{-i}) dv \\ & + \int_V \Sigma_{\substack{j\in I\\ j\neq i}} \tilde{q}_j^2(v_j, v_{-j}) \left[v_j f_j(v_j|a_j) - (1 - F_j(v_j|a_j))\right] f_{-i-j}(v_{-i-j}|a_{-i-j}) m_i(a_i|v_i) f_i(v_i) dv, \end{aligned} \tag{71}$$

for all $\tilde{q}^2$.

Recall that for transparency of the arguments, we have supposed that a_i is the only action other than 0 that is chosen by valuations in $V_i(0)$ (otherwise, we would have to write and add more inequalities like the ones above). Then, it holds that $m_i(0|v_i) + m_i(a_i|v_i) = 1$ for all $v_i \in \bar{V}_i(a_i)$, and by adding

[53]Note that it is without loss to take $q_i^2(., a_i, a_{-i})$ (resp. $q_i^2(., 0, a_{-i})$) to be defined on V since for $v_i \in V_i \backslash V_i(0)$, $m_i(0|v_i) = 0$.

up (70) and (71), we get that

$$\begin{aligned}
&\int_V q_i^2(v_i,v_{-i})\left[v_if_i(v_i)-\int_{v_i}^{b_i}f_i(t_i)dt_i\right]f_{-i}(v_{-i}|a_{-i})dv\\
&+\int_V \Sigma_{\substack{j\in I\\ j\neq i}}q_j^2(v_j,v_{-j})\left[v_jf_j(v_j|a_j)-(1-F_j(v_j|a_j))\right]f_{-i-j}(v_{-i-j}|a_{-i-j})f_i(v_i)dv\\
\geq\ &\int_V \tilde{q}_i^2(v_i,v_{-i})\left[v_if_i(v_i)-\int_{v_i}^{b_i}f_i(t_i)dt_i\right]f_{-i}(v_{-i}|a_{-i})dv\\
&+\int_V \Sigma_{\substack{j\in I\\ j\neq i}}\tilde{q}_j^2(v_j,v_{-j})\left[v_jf_j(v_j|a_j)-(1-F_j(v_j|a_j))\right]f_{-i-j}(v_{-i-j}|a_{-i-j})f_i(v_i)dv,
\end{aligned}$$

for all $\tilde{q}^2$, which is equivalent to saying that the allocation rule q^2 is revenue-maximizing at $t=2$ when all valuations in $\bar{V}_i(0)$ choose 0 with probability one, in which case i where is using a partitional strategy. This implies that if we are in this case, we can without any loss assume that all valuations in $\bar{V}_i(0)=[0,\bar{v}_i]$ choose action 0 with probability 1 and the result follows. ■

C Proof of Lemma 7

Lemma 7. *At a solution of Program NC, if a_i is in the support of $m_i(.|v_i)$ for some $v_i\in[0,\bar{v}_i(0)]$, with $\bar{v}_i(0)>0$, then $r_i^2(a_i)<\bar{v}_i(0)$.*

Proof. We need to establish that at a solution of Program NC, if a_i is in the support of $m_i(.|v_i)$ for some $v_i\in[0,\bar{v}_i(0)]$, it cannot be the case that $r_i^2(a_i)\geq\bar{v}_i(0)$. We argue by contradiction. If this were the case, then Lemma 1 implies that $P_i^0(v_i)=P_i^{1a_i}$ for $v_i\in[\underline{v}_i(a_i),\bar{v}_i(0)]$, which implies, in turn, that the probability that i obtains the good at $t=2$ after reporting 0 for that region of valuations is constant.

This is only possible when there almost surely is no buyer $j\neq i$ and no report a_j and v_j with $J_j(v_j|a_j)\in[J_i(\underline{v}_i(a_i)|0),J_i(\bar{v}_i(0)|0)]$. In other words, for all $j\neq i$, v_j and a_j, the posterior virtual valuations satisfy either $J_j(v_j|a_j)\leq J_i(\underline{v}_i(a_i)|0)$ or $J_j(v_j|a_j)>J_i(\bar{v}_i(0)|0)$. Observe, that $J_i(\bar{v}_i(0)|0)=\bar{v}_i(0)>0$, so having $J_j(v_j|a_j)>J_i(\bar{v}_i(0)|0)>0$ for *all* j, v_j and reports a_j is impossible: The inequality is violated when, for instance, all realized v_j's are around a neighborhood of zero.

Now, consider the case in which $J_j(v_j|a_j)\leq J_i(\underline{v}_i(a_i)|0)$ for all $j\neq i$ and a_j; then, slightly increasing $m_i(0|v_i)$ in the region $[\underline{v}_i(a_i),\bar{v}_i(0)]$ increases the $t=2$-reserve price. If the increase is small, the reduction in $J_i(v_i|0)$ does not affect the $t=2$ inter-buyer allocation–i's virtual valuation is still higher. This modification increases the $t=2$-reserve price $r_i^2(0)$ without any further effects, and, hence, increases revenue from the $t=1$-perspective. ■

D Details for Section 5

Skreta (2011) solves the informed-seller problem discussed in Section 5 as follows: It first observes that the seller's revenue at $t=2$ after having observed a at $t=1$ and given a vector of disclosed messages λ can be rewritten as

$$\int_V \Sigma_{i\in I}q_i^{2(a)}(v,\lambda)J_i(v_i|a_i)f(v\,|a)dv-\Sigma_{i\in I}\underline{U}_i(\underline{v}_i(a_i),a,\lambda)+\int_V\Sigma_{i\in I}\tau_i(v,a,\lambda)f(v\,|a)dv, \tag{72}$$

where

$$-\tau_i(v,a,\lambda) \equiv u_i(v,a,\lambda) - \int_{\underline{v}_i(a_i)}^{v_i} p_i(t_i,v_{-i},a,\lambda)dt_i - u_i(\underline{v}_i(s_i),v_{-i},a,\lambda) \quad (73)$$

and where $\underline{U}_i(\underline{v}_i(a_i),a,\lambda) \equiv \int_V [q_i^{2(a)}{}_i(\underline{v}_i(a_i),v_{-i},\lambda)\underline{v}_i(a_i) - z_i^{2(a)}{}_i(\underline{v}_i(a_i),v_{-i},\lambda)]f(v\,|a)dv$ denotes i's expected payoff at valuation $\underline{v}_i(a_i)$, from $q^{2(a)}, z^{2(a)}$, where the expectation is taken with respect to the *seller's* beliefs. Buyer i's interim payoff at a truth-telling equilibrium is

$$U_i(v_i,a_i,\lambda_i) = \int_{\underline{v}_i(a_i)}^{v_i} P_i(t_i,a_i,\lambda_i)dt_i + U_i(\underline{v}_i(s_i),a_i,\lambda_i), \text{ for } v_i \in V_i(s_i), \quad (74)$$

where $P_i(v_i,a_i,\lambda_i) = E_{v_{-i},a_{-i},\lambda_{-i}}[p_i(v,a,\lambda)\,|a_i,\lambda_i]$ is the interim expected probability of trade. This expectation is taken with respect to i's beliefs given a_i,λ_i. From (73) and (74), one can immediately see that the interim expectation of $\tau_i(v,a,\lambda)$ from i's perspective is zero, implying that it is also zero from the ex-ante perspective.

Skreta (2011) uses this observation, together with the fact that the *full-transparency* optimum is dominant-strategy incentive-compatible—which implies that it is a feasible choice for the seller, regardless of the buyers' posterior beliefs (which depend on the disclosure policy)–to establish that at an equilibrium of the informed-seller game $\tau_i(v,a,\lambda)$ is zero at each v,a,λ. Then, for a given stage-1 disclosure rule (which determines the stage-2 types and beliefs), at stage 2, the seller chooses, for *each* of her types, (a,λ), a mechanism that maximizes revenue (given by (72) using that $\int_V \Sigma_{i\in I}\tau_i(v,a,\lambda)f(v\,|a)dv = 0$) subject to Bayesian incentive and participation constraints for the buyers *ignoring* the incentive and participation constraints for the seller. That paper establishes in Proposition 5 that the solution of this program, regardless of the disclosure rule, coincides with the mechanism that is optimal when a is common knowledge–the *full-transparency* optimum–and it satisfies the seller's incentive and participation constraints.

E Optimal last-period mechanism for complex posteriors.

Given that at the last period, the seller de facto commits to the outcome of the mechanism we can appeal to the revelation principle and restrict attention to incentive-compatible direct revelation mechanisms, consisting of an assignment rule $q^2 : V(a) \longrightarrow \Delta(\bar{I})$ and a payment rule $z^2 : V(a) \longrightarrow \mathbb{R}^{\bar{I}}$. When posterior densities are zero, the sets of valuations $V_i(a_i)$ and $V(a) = \times_{i\in I}V_i(a_i)$ are not necessarily convex, and two difficulties arise: First, we cannot express expected revenue only as a function of the allocation rule, and second, the formula of posterior virtual valuation is not well-defined for valuations where the posterior density is zero. To address the first difficulty, we establish that without loss we can define the mechanism and impose the feasibility constraints on the closure of the convex hull of valuations; to address the second, we approximate the problem with one in which virtual valuations are well-defined and show that the solution of the approximate problem is arbitrarily close to the one of the problem we are interested in.

The Convexified Problem: We consider an artificial problem that has the same objective function as the problem of interest (i.e., the seller's expected revenue), but a different feasible set because we impose incentive and participation constraints on the closures of the convex hulls of $V_i(a_i)$ and $V(a)$. Proposition 1 in Skreta (2006) shows that these problems are equivalent in the following sense: One can obtain a solution of the program of interest by solving the convexified problem and by restricting the solution to the actual set of valuations. Conversely, any solution of the program of interest can be extended appropriately to the convex hull of valuations, and it is a solution of the convexified problem. Hence, without loss of generality, we can consider the convexified problem. The intuition for this result is that "adding" types that occur with probability zero does not change the value of the program. For the convexified problem, we have, as usual, a "revenue equivalence theorem," and we use standard arguments to express the seller's problem as follows:

$$\max_{q^2,z^2} \int_{\bar{V}(a)} \Sigma_{i\in I} q_i^{2(a)}(v) \left[v_i f_i(v_i|a_i) - (1 - F_i(v_i|a_i))\right] f_{-i}(v_{-i}|a_{-i}) dv - \Sigma_{i\in I} U_i^{2(a)}(q^2, z^2, \underline{v}_i(a_i)), \quad (75)$$

subject to: $P_i^{2(a)}(v_i) \equiv E_{v_{-i}}[q_i^{2(a)}(v)]$ increasing in v_i on $\bar{V}_i(a_i)$; $0 \leq q_i^{2(a)}(v) \leq 1$ and $\Sigma_{i\in I} q_i^{2(a)}(v) \leq 1$ for all $v \in \bar{V}(a)$.

Unfortunately, we still have to deal with the fact that a buyer's virtual valuation is not necessarily well-defined because the density f_i can be zero for some valuations on $\bar{V}_i(a)$. The inability to divide by $f_i(v_i|a_i)$ to obtain the virtual valuation creates difficulties, as one cannot compare the benefit from assigning the good to one buyer versus another.[54] We address this issue by considering another artificial program that is "close" to the convexified problem:

The Approximate Problem: We approximate the objective function with one in which the density of each i is replaced by $f_i^\varepsilon(v_i|a_i) = \begin{cases} f_i(v_i|, a_i) \text{ if } f_i(v_i|a_i) > 0 \\ \varepsilon \text{ otherwise} \end{cases}$, where $\varepsilon > 0$, arbitrarily small. The resulting problem has the *same* constrained set as the program in (75) because, in (75), incentive and participation constraints are imposed on $\bar{V}(a)$, which includes vectors of valuations that occur with probability zero. Moreover, the objective function is arbitrarily close to the one in (75): It is routine to check that the objective function is continuous in ε, and the feasible set is sequentially compact in the topology of point-wise convergence (for similar arguments, see the Technical Appendix of Skreta (2006b)). Then, the Theorem of the Maximum implies that the value and the solution of the approximate problem are arbitrarily close to the ones of (75).

[54] In (75), each buyer is assigned a different weight in the objective function: For example, buyer i's weight is given by $f_{-i}(v_{-i}|a_{-i})$.