The D(2)-Problem for some metacyclic groups

Jason Marcus Vittis

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics

UCL

Supervisor: Professor FEA Johnson

June 2019

I, Jason Marcus Vittis confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

....

Abstract

We study problems relating to the D(2)-Problem for metacyclic groups of type $G(p, p-1)$ where p is an odd prime.

Specifically we build on Nadim's thesis [12], which showed that the $\mathbb{Z}[G(5,4)]$-module $\mathbb{Z}$ admits a diagonal resolution and a minimal representative for the third syzygy $\Omega_3(\mathbb{Z})$ is $R(2) \oplus [y-1)$. Motivated by this result, we show that the $\mathbb{Z}[G(p,p-1)]$-module $R(2) \oplus [y-1)$ is both full and straight for any odd prime p. Given Johnson's work on the D(2)-Problem [5], this leads to the conclusion that $G(5,4)$ satisfies the D(2)-property, as well as providing a sufficient condition for the D(2)-property to hold for $G(p, p-1)$, namely the condition that $R(2) \oplus [y-1)$ is a minimal representative for $\Omega_3(\mathbb{Z})$ over $\mathbb{Z}[G(p, p-1)]$, which we refer to as the condition M(p).

Following this result, we prove a theorem which simplifies the calculations required to show that the condition M(p) holds. Finally, we carry out these calculations in the case where $p = 7$ and prove that the condition M(7) holds, which is sufficient to show that $G(7,6)$ satisfies the D(2)-property.

Impact statement

This thesis studies the D(2)-Problem, specifically for metacyclic groups of type $G(p, p-1)$ where p is an odd prime. Notable strides forward are made in the study of this problem, namely the results that the D(2)-property holds for the groups $G(5,4)$ and $G(7,6)$. In the future, it is hoped that the methods which are described and used in this thesis, particularly relating to the Swan homomorphism and the condition M(p) can be utilised and/or improved upon to further the study of the D(2)-Problem.

Acknowledgements

First and foremost, I would like to thank my supervisor Professor FEA Johnson, whose mathematical interests and frequent publications have helped to guide my research. His dedication, patience and support have given me confidence in my own abilities, and the confidence to pursue my own interests, for this I will be forever grateful.

Secondly, I would like to thank the staff and students at UCL, who have helped to make the last eight years a truly enjoyable experience.

Finally, I would like to thank my family for their faith in me, and my girlfriend Jafrin, for always encouraging me to be the best that I can be.

Contents

1 Introduction

1.1 Motivation

1.1.1 The D(2)-Problem

The motivation for this thesis comes primarily from Wall's D(n)-Problem which was first formulated in [16].

The D(n)-Problem: Let X be a finite connected cell complex with geometric dimension $n+1$ and with universal cover $\tilde{X}$ such that:

$$H_{n+1}(\tilde{X};\mathbb{Z}) = 0 \text{ and } H^{n+1}(X;\mathcal{B}) = 0,$$

for all coefficient systems $\mathcal{B}$ on X. Is X homotopy equivalent to a finite complex of dimension n?

In his paper [16], Wall solved the D(n)-Problem in the affirmative for each natural number n such that $n \geq 3$. The D(1)-Problem was later solved in the affirmative by Stallings and Swan [14], [15]. This left only the D(2)-Problem, the primary focus of this thesis.

The D(2)-Problem: Let X be a finite connected cell complex with geometric dimension 3 and with universal cover $\tilde{X}$ such that:

$$H_3(\tilde{X};\mathbb{Z}) = 0 \text{ and } H^3(X;\mathcal{B}) = 0,$$

for all coefficient systems $\mathcal{B}$ on X. Is X homotopy equivalent to a finite complex of dimension 2?

The D(2)-Problem is intrinsically connected to a second problem in topology, which is known as the two-dimensional realization problem, or the R(2)-Problem.

1.1.2 The R(2)-Problem

Let $\mathcal{G} =< x_1, \ldots, x_g \mid W_1, \ldots, W_r >$ be a presentation for a group G, and let $K_{\mathcal{G}}$ be the presentation complex of $\mathcal{G}$, a two-dimensional CW complex satisfying $\pi_1(K_{\mathcal{G}}) \cong G$. Let $\tilde{K_{\mathcal{G}}}$ be the universal cover of $K_{\mathcal{G}}$, also known as the Cayley complex of $\mathcal{G}$. The cellular chain complex of $\tilde{K_{\mathcal{G}}}$ gives rise to:

$$C_*(\mathcal{G}) = (0 \to \pi_2(K_{\mathcal{G}}) \to C_2(\tilde{K_{\mathcal{G}}}) \xrightarrow{\partial_2} C_1(\tilde{K_{\mathcal{G}}}) \xrightarrow{\partial_1} C_0(\tilde{K_{\mathcal{G}}}) \xrightarrow{\partial_0} \mathbb{Z} \to 0),$$

an exact sequence of right $\mathbb{Z}[G]$-modules. Here, we have identified the module $Ker(\partial_2) = H_2(\tilde{K_{\mathcal{G}}})$ with $\pi_2(K_{\mathcal{G}})$ via the Hurewicz isomorphism and the isomorphism induced by the covering map $\tilde{K_{\mathcal{G}}} \to K_{\mathcal{G}}$.

Since each $C_n(\tilde{K_{\mathcal{G}}})$ is a free $\mathbb{Z}[G]$-module, this construction suggests that it might be informative to consider *algebraic 2-complexes* over G, which are exact sequences of $\mathbb{Z}[G]$-modules of the form

$$0 \to J \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0,$$

where each F_n is a free $\mathbb{Z}[G]$-module. By the *third syzygy* of $\mathbb{Z}$ over $\mathbb{Z}[G]$, denoted by $\Omega_3(\mathbb{Z})$, we mean the stable module $[J]$. While considering algebraic 2-complexes, an obvious question arises: whether each algebraic 2-complex over G can be written as $C_*(\mathcal{G})$ where $\mathcal{G}$ is some presentation for G.

The R(2)-Problem: Let G be a finitely presented group. Is every algebraic 2-complex over G

$$0 \to J \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0,$$

geometrically realizable; that is, homotopy equivalent to an algebraic 2-complex of the form $C_*(\mathcal{G})$, where $\mathcal{G}$ is some presentation for G?

Johnson showed in [5] that for finite groups G, the R(2)-Problem is equivalent to the D(2)-Problem, that is, if the R(2)-Problem holds true for a finite group G, then the D(2)-Problem holds true for all cell complexes X satisfying $\pi_1(X) = G$ and vice versa. This result has since been extended further by Johnson, before reaching its current form, due to Mannan [10]: the R(2)-Problem and the D(2)-Problem are equivalent for all finitely presented groups G. If the D(2)-Problem holds true for a group G, we say that G satisfies the *D(2)-property*. In this thesis, we focus on problems relating to the D(2)-Problem for metacyclic groups of type

$$G(p, p-1) = < x, y \mid x^p = 1, y^{p-1} = 1, yx = x^m y >,$$

where p is an odd prime and m is chosen so that the group isomorphism $\theta \in Aut(C_p)$ given by $\theta(x) = x^m$ satisfies $ord(\theta) = p - 1$. In Nadim's thesis [12], some work has already been completed relating to the existence of a diagonal resolution for $\mathbb{Z}$ over $\mathbb{Z}[G(5, 4)]$, we use this work as motivation to study the more general case of $G(p, p-1)$.

1.2 Statement of results

We begin in chapters 2 and 3 by taking a ring Λ with unity, and covering some preliminary information relating to the abelian group $Ext^1_\Lambda(A, B)$ and the derived module category of Λ, $\mathcal{D}er(\Lambda)$. In §3.3 we restrict to the case where G is a finite group and $\Lambda = \mathbb{Z}[G]$, in this context we consider exact sequences of Λ-lattices and homomorphisms

$$0 \to J \xrightarrow{i} S \to M \to 0,$$

such that S is stably free. We can then define an additive group homomorphism, known as the Swan homomorphism

$$\begin{aligned} S_J : Aut_{\mathcal{D}er}(J) &\to \tilde{K}_0(\Lambda), \\ \overline{f} &\mapsto [\varinjlim(f, i)], \end{aligned}$$

which plays a key role throughout the thesis.

In chapter 4, we state some results from [11] relating to Milnor squares, we then outline a method which uses Milnor squares to classify Projective modules over a ring Λ. This is followed in §4.3 by two theorems, adapted from [3]:

Theorem 4.3.1.([3], page 30, 4.1.1) Given a quasi-augmentation sequence

$$\mathcal{S} = (0 \to S_- \xrightarrow{i} \Lambda \xrightarrow{p} S_+ \to 0),$$

satisfying the condition

- $Hom_\Lambda(S_+, S_-) = 0$;

there exists a Milnor square

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & End_\Lambda(S_+) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ End_\Lambda(S_-) & \xrightarrow{j_2} & End_{\mathcal{D}er}(S_-), \end{array}$$

where i_1, j_2 and j_1 are surjective.

Theorem 4.3.2.([3], page 41, 4.5.2) With the hypotheses of the above theorem, if $f_- \in End_\Lambda(S_-)$ is such that $\overline{f}_- \in Aut_{\mathcal{D}er}(S_-)$, then

$$\varinjlim(f_-, i) \cong M(End_\Lambda(S_+), End_\Lambda(S_-), \overline{f}_-).$$

These theorems allows us to both construct Milnor squares from quasi-augmentation sequences and find the kernel and image of the Swan homomorphism in some special cases.

In chapter 5, we outline some results from [5] which allow us to formulate a sufficient condition for the D(2)-property to hold for a finite group G, namely:

Theorem 5.5.1. If a finite group G satisfies properties 1,2 and 3 below

1. G admits a balanced presentation;

2. $\Omega_3(\mathbb{Z})$ is straight;

3. the minimal module J in $\Omega_3(\mathbb{Z})$ is full;

then G satisfies the D(2)-property.

This theorem motivates the remainder of the thesis.

In chapter 6 we restrict our focus to the group ring $\Lambda = \mathbb{Z}[G(p, p-1)]$ where p is an odd prime. Motivated by the quasi-augmentation sequence

$$0 \to \mathcal{T}_{p-1}(\mathbb{Z}, p) \to \Lambda \to \mathbb{Z}[C_{p-1}] \to 0,$$

from [7] and our results in chapters 4 and 5, we work towards a deep understanding of the ring and Λ-module

$$\mathcal{T}_{p-1}(\mathbb{Z}, p) = \{(a_{i,j})_{1\leq i,j\leq p-1} \in M_{p-1}(\mathbb{Z}) \mid a_{i,j} \in p\mathbb{Z} \text{ if } i > j\}.$$

We begin by outlining results from [7], namely a group presentation

$$\lambda_* : G(p, p-1) \to \mathcal{T}_{p-1}(\mathbb{Z}, p),$$

and a ring isomorphism

$$\tilde{\lambda_*} : \mathcal{C}_{p-1}(\mathbb{Z}(\zeta), \bar{\theta}) \to \mathcal{T}_{p-1}.$$

These results are then used to endow $\mathcal{T}_{p-1}(\mathbb{Z}, p)$ with a right Λ-module structure. As a right Λ-module, $\mathcal{T}_{p-1}(\mathbb{Z}, p)$ is a direct sums of its rows, with this in mind, we denote by $R(i)$ the i^{th} row of $\mathcal{T}_{p-1}(\mathbb{Z}, p)$, and so, as right Λ-modules,

$$\mathcal{T}_{p-1}(\mathbb{Z}, p) \cong \bigoplus_{i=1}^{p-1} R(i).$$

To conclude the chapter, we provide a full description of the rings $Hom_{\Lambda}(R(i), R(j))$ for $1 \leq i, j \leq p-1$, as well as a description of the abelian group $K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p))$.

In chapter 7, we continue to work over $\Lambda = \mathbb{Z}[G(p, p-1)]$, where p is an odd prime. A proof is given that $G(p, p-1)$ admits a balanced presentation for any odd prime p, leaving only conditions 2 and 3 in **Theorem 5.5.1** to be proven true. Given that Nadim showed in [12] that over $\mathbb{Z}[G(5, 4)]$, the minimal module in $\Omega_3(\mathbb{Z})$ is $R(2) \oplus [y-1)$, we are motivated in chapter 7 to prove that conditions 2 and 3 in **Theorem 5.5.1** hold for the stable module $[R(2) \oplus [y-1)]$. Both of these conditions are proven to be true, in two of our main theorems:

Theorem 7.1.3. $[R(2) \oplus [y-1)]$ is straight over $\Lambda = \mathbb{Z}[G(p, p-1)]$ for any odd prime p.

Theorem 7.2.5. $R(2) \oplus [y-1)$ is full over $\Lambda = \mathbb{Z}[G(p, p-1)]$ for any odd prime p.

In chapter 8, we begin by defining the condition **M(p)** on $\mathbb{Z}[G(p, p-1)]$ as follows:

M(p): The third syzygy of $\mathbb{Z}$ over $\Lambda = \mathbb{Z}[G(p, p-1)]$, $\Omega_3(\mathbb{Z})$, is the stable module $[R(2) \oplus [y-1)]$.

Given our work in chapters 5 and 7, this immediately leads to the following theorem:

Theorem 8.1.1. Let $\Lambda = \mathbb{Z}[G(p, p-1)]$, if Λ satisfies $\mathbf{M(p)}$, then $G(p, p-1)$ satisfies the D(2)-property.

As previously noted, it has already been shown [12] that the condition $\mathbf{M(5)}$ is satisfied, leading to another one of our main theorems:

Theorem 8.1.2. $G(5, 4)$ satisfies the D(2)-property

The remainder of chapter 8 is dedicated to refining techniques used in [12] and using these refinements to show that the condition $\mathbf{M(7)}$ holds, which leads to our conclusion and final theorems:

Theorem 8.4.1. Over $\Lambda = \mathbb{Z}[G(7, 6)]$, $\Omega_3(\mathbb{Z}) = [R(2) \oplus [y-1)]$ i.e. the condition $M(7)$ holds.

Theorem 8.4.2. The D(2)-property holds for $G = G(7, 6)$.

2 The abelian group Ext^1_Λ

Let Λ be a ring with unity, in this chapter we briefly outline some basic definitions and properties relating to Ext^1_Λ, roughly following the scheme of sections 4.1-4.3 in [6].

2.1 The category of extensions

We denote by $\mathbf{Ext}^1_\Lambda$ the collection of exact sequences of Λ-modules and homomorphisms of the form

$$\mathbf{E} = (0 \to E_+ \xrightarrow{i} E_0 \xrightarrow{p} E_- \to 0),$$

$\mathbf{Ext}^1_\Lambda$ can be regarded as a category by taking morphims to be commutative diagrams of Λ-homomorphisms as follows:

$$\begin{array}{cccccccccccc}
\mathbf{E} & & (0 & \longrightarrow & E_+ & \xrightarrow{i} & E_0 & \xrightarrow{p} & E_- & \longrightarrow & 0) \\
\downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle h_+} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_-} & & \\
\mathbf{F} & & (0 & \longrightarrow & F_+ & \longrightarrow & F_0 & \longrightarrow & F_- & \longrightarrow & 0).
\end{array}$$

For $A, B \in \mathcal{M}od_\Lambda$ we denote by $\mathbf{Ext}^1_\Lambda(A, B)$ the full subcategory of $\mathbf{Ext}^1_\Lambda$ whose objects $\mathbf{E}$ satisfy $E_+ = B$ and $E_- = A$. If $\mathbf{E}, \mathbf{F} \in \mathbf{Ext}^1_\Lambda(A, B)$, a morphism $h : \mathbf{E} \to \mathbf{F}$ is said to be a *congruence* when it induces the identity at both ends, i.e. h takes the form:

$$\begin{array}{cccccccccccc}
\mathbf{E} & & (0 & \longrightarrow & B & \xrightarrow{i} & E_0 & \xrightarrow{p} & A & \longrightarrow & 0) \\
\downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle Id} & & \\
\mathbf{F} & & (0 & \longrightarrow & B & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow & 0).
\end{array}$$

We write '$\mathbf{E} \equiv \mathbf{F}$' when $\mathbf{E}$ and $\mathbf{F}$ are congruent. By the Five lemma, congruence is an equivalence relation on $\mathbf{Ext}^1_\Lambda(A, B)$. We denote by $Ext^1_\Lambda(A, B)$ the collection of equivalence classes in $\mathbf{Ext}^1_\Lambda(A, B)$ under '$\equiv$'. For any Λ-modules A, B, there is a distinguished extension, the *trivial extension*

$$\mathcal{T} = (0 \to B \xrightarrow{i_B} B \oplus A \xrightarrow{\pi_A} A \to 0),$$

where $i_B(b) = (b, 0)$ and $\pi_A(b, a) = a$. An extension

$$\mathcal{F} = (0 \to B \xrightarrow{j} X \xrightarrow{p} A \to 0)$$

is said to be *split* when it is congruent to the trivial extension. $\mathcal{F}$ is said to *split on the right* when there exists a Λ-homomorphism $s : A \to X$ such that $p \circ s = Id_A$. $\mathcal{F}$ is said to *split on the left* when there exists a Λ-homomorphism $r : X \to B$ such that $r \circ j = Id_B$. The splitting lemma states that for a given $\mathcal{F} \in \mathbf{Ext}^1_\Lambda(A, B)$,

$$\mathcal{F} \text{ splits} \iff \mathcal{F} \text{ splits on the left} \iff \mathcal{F} \text{ splits on the right.}$$

2.2 The group structure of $Ext^1_\Lambda(A, B)$

We now work towards describing an abelian group structure on $Ext^1_\Lambda(A, B)$ with $\mathcal{T}$ as the identity element. To define the group multiplication, we must first describe some natural constructions on $\mathbf{Ext}^1_\Lambda(A, B)$.

Pushout: Let A, B_1, B_2 be Λ-modules; if $f : B_1 \to B_2$ is a Λ-homomorphism and $\mathbf{E} = (0 \to B_1 \xrightarrow{i} E_0 \xrightarrow{\eta} A \to 0) \in \mathbf{Ext}^1_\Lambda(A, B_1)$, we define

$$f_*(\mathbf{E}) = (0 \to B_2 \xrightarrow{j} \varinjlim(f,i) \xrightarrow{\epsilon} A \to 0) \in \mathbf{Ext}^1_\Lambda(A, B_2),$$

where $\varinjlim(f,i) = \dfrac{B_2 \oplus E_0}{Im(f \times -i)}$ denotes the *colimit* and $j : B_2 \to \varinjlim(f,i)$ is the injection defined by $j(x) = [x,0]$. The correspondence $\mathbf{E} \mapsto f_*(\mathbf{E})$ determines the *pushout* mapping $f_* : \mathbf{Ext}^1_\Lambda(A, B_1) \to \mathbf{Ext}^1_\Lambda(A, B_2)$. The pushout gives rise to a commutative diagram with exact rows as follows

$$\begin{array}{ccccccccccc}
(\mathbf{E}) & \quad & (0 & \longrightarrow & B_1 & \xrightarrow{\quad i \quad} & E_0 & \xrightarrow{\quad \eta \quad} & A & \longrightarrow & 0) \\
\downarrow{\scriptstyle f_*} & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle Id} & & \\
f_*(\mathbf{E}) & & (0 & \longrightarrow & B_2 & \xrightarrow{\quad j \quad} & \varinjlim(f,i) & \xrightarrow{\quad \epsilon \quad} & A & \longrightarrow & 0),
\end{array}$$

where v(x)=[0,x]. If in addition $g : B_2 \to B_3$ is a Λ-homomorphism, it is straighforward to see that

$$(g \circ f)_*(\mathbf{E}) = g_* f_*(\mathbf{E}).$$

Pullback: Let A_1, A_2, B be Λ-modules; if $f : A_1 \to A_2$ is a Λ-homomorphism and $\mathbf{E} = (0 \to B \to E_0 \xrightarrow{\eta} A_2 \to 0) \in \mathbf{Ext}^1_\Lambda(A_2, B)$, we define

$$f^*(\mathbf{E}) = (0 \to B \to \varprojlim(\eta, f) \xrightarrow{\epsilon} A_1 \to 0) \in \mathbf{Ext}^1_\Lambda(A_1, B),$$

where $\varprojlim(\eta, f) = \{(x,y) \in E_0 \times A_1 \mid \eta(x) = f(y)\}$ and $\epsilon : \varprojlim(\eta, f) \to A_1$ is the projection $\epsilon(x,y) = y$. The correspondence $\mathbf{E} \mapsto f^*(\mathbf{E})$ defines the *pullback* mapping $f^* : \mathbf{Ext}^1_\Lambda(A_2, B) \to \mathbf{Ext}^1_\Lambda(A_1, B)$. The pullback mapping gives rise to a commutative diagram with exact rows as follows

$$\begin{array}{ccccccccccc}
f^*(\mathbf{E}) & \quad & (0 & \longrightarrow & B & \longrightarrow & \varprojlim(\eta, f) & \xrightarrow{\quad \epsilon \quad} & A_1 & \longrightarrow & 0) \\
{\scriptstyle f^*}\uparrow & & & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle f} & & \\
\mathbf{E} & & (0 & \longrightarrow & B & \longrightarrow & E_0 & \xrightarrow{\quad \eta \quad} & A_2 & \longrightarrow & 0),
\end{array}$$

where $\mu_0 : \varprojlim(\eta, f) \to E_0$ is the projection $\mu_0(x,y) = x$. If in addition $g : A_2 \to A_3$ is a Λ-homomorphism, it is straightforward to see that

$$(g \circ f)^*(\mathbf{E}) = f^* \circ g^*(\mathbf{E}).$$

Direct product: Let A_1, A_2, B_1, B_2 be Λ-modules and for $r = 1, 2$ let

$$\mathbf{E}(r) = (0 \to B_r \to E(r)_0 \to A_r \to 0) \in \mathbf{Ext}^1_\Lambda(A_r, B_r).$$

Then $\mathbf{E}(1) \times \mathbf{E}(2)$ is defined as the extension

$$\mathbf{E}(1) \times \mathbf{E}(2) = (0 \to B_1 \times B_2 \to E(1)_0 \times E(2)_0 \to A_1 \times A_2 \to 0),$$

with the obvious mappings.

Note that each of the above constructions are compatible with congruence, and so they descend to Ext^1_Λ.

Using these constructions, we can now define the group operation on $Ext^1_\Lambda(A, B)$. Note that the direct product gives a functorial pairing

$$\times : \mathbf{Ext}^1_\Lambda(A_1, B_1) \times \mathbf{Ext}^1_\Lambda(A_2, B_2) \to \mathbf{Ext}^1_\Lambda(A_1 \times A_2, B_1 \times B_2).$$

For Λ-modules A, B_1, B_2 there is a functorial pairing, the *external sum*

$$\oplus : \mathbf{Ext}^1_\Lambda(A, B_1) \times \mathbf{Ext}^1_\Lambda(A, B_2) \to \mathbf{Ext}^1_\Lambda(A, B_1 \times B_2),$$

given by $\mathbf{E} \oplus \mathbf{F} = \Delta^*(\mathbf{E} \times \mathbf{F})$, where $\Delta : A \to A \times A$ is given by $a \mapsto (a, a)$. Combining the external sum with the pushout, we obtain the Baer sum on $\mathbf{Ext}^1_\Lambda(A, B)$. Define the mapping $\alpha : B \times B \to B$ by $(b, b') \mapsto b + b'$, let $\mathbf{E}, \mathbf{F} \in \mathbf{Ext}^1_\Lambda(A, B)$, we define the *Baer sum* $\mathbf{E} + \mathbf{F}$ by

$$\mathbf{E} + \mathbf{F} = \alpha_* \Delta^*(\mathbf{E} \times \mathbf{F}).$$

It is straightforward to see that congruence is compatible with the Baer sum, and so we have a mapping

$$+ : Ext^1_\Lambda(A, B) \times Ext^1_\Lambda(A, B) \to Ext^1_\Lambda(A, B),$$

which defines the group operation on the abelian group $Ext^1_\Lambda(A, B)$.

Now that we have defined the group structure on $Ext^1_\Lambda(A, B)$, we describe some homomorphisms between these groups which arise as a result of the pushout construction. If $g : B_1 \to B_2$ is a Λ-homomorphism, the correspondence $\mathcal{E} \to g_*(\mathcal{E})$ gives a mapping $g_* : Ext^1_\Lambda(A, B_1) \to Ext^1_\Lambda(A, B_2)$ such that

$$g_*(\mathcal{E}_1 + \mathcal{E}_2) \to g_*(\mathcal{E}_1) + g_*(\mathcal{E}_2).$$

Thus g induces a group homomorphism $g_* : Ext^1_\Lambda(A, B_1) \to Ext^1_\Lambda(A, B_2)$.

We may also construct a mapping $Hom_\Lambda(A, N) \to Ext^1_\Lambda(C, N)$ using the pushout construction. Given an exact sequence of Λ-modules and homomorphisms $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$, there is a mapping $\delta : Hom_\Lambda(A, N) \to Ext^1_\Lambda(C, N)$, the *connecting mapping*, given by

$$\delta(\alpha) = \alpha_*(\mathcal{E}).$$

It is straightforward to check that δ is in fact an abelian group homomorphism. We will refer back to δ in § 3.2.

2.3 Properties of the colimit

In this section, we prove some basic properties of the colimit which will be utilised later in the thesis. All mappings in this section are Λ-module homomorphisms. A Λ-module homomorphism $g : A \oplus B \to C \oplus D$ is said to be *diagonal* if, when written in matrix form

$$g = \begin{pmatrix} g_{1,1} : A \to C & g_{1,2} : B \to C \\ g_{2,1} : A \to D & g_{2,2} : B \to D \end{pmatrix},$$

the matrix is diagonal, i.e. $g_{1,2} = 0$ and $g_{2,1} = 0$. We now prove a proposition relating to diagonal Λ-homomorphisms.

Proposition 2.3.1. *Consider a short exact sequence of Λ-modules and homomorphisms*

$$0 \to B_1 \oplus B_2 \xrightarrow{i} E(1)_0 \oplus E(2)_0 \xrightarrow{p} A_1 \oplus A_2 \to 0,$$

in which each of the mappings are diagonal, specifically $i = i_1 \oplus i_2$ where $i_r : B_r \to E(r)_0$ and $p = p_1 \oplus p_2$ where $p_r : E(r)_0 \to A_r$ for $r = 1, 2$. Let $f : B_1 \oplus B_2 \to B_1 \oplus B_2$ be diagonal, so $f = f_1 \oplus f_2$ where $f_r : B_r \to B_r$ for $r = 1, 2$. Then

$$\varinjlim(f, i) \cong \varinjlim(f_1, i_1) \oplus \varinjlim(f_2, i_2)$$

Proof. Recall that $\varinjlim(f, i) = \frac{(B_1 \oplus B_2) \oplus (E(1)_0 \oplus (E(2)_0)}{Im(f \times -i)}$,

$$\text{while } \varinjlim(f_1, i_1) = \frac{B_1 \oplus E(1)_0}{Im(f_1 \times -i_1)},$$
$$\text{and } \varinjlim(f_2, i_2) = \frac{B_2 \oplus E(2)_0}{Im(f_2 \times -i_2)}.$$

Define a Λ-homomorphism

$$(B_1 \oplus B_2) \oplus (E(1)_0 \oplus E(2)_0) \to \varinjlim(f_1, i_1) \oplus \varinjlim(f_2, i_2)$$

$$((b_1, b_2), (e_1, e_2)) \mapsto ((b_1, e_1) + Im(f_1 \times -i_1), (b_2, e_2) + Im(f_2 \times -i_2))$$

This map is clearly surjective and the mappings f and i are diagonal, therefore

$$Im(f \times -i) = Im(f_1 \times -i_1) \oplus Im(f_2 \times -i_2).$$

We deduce that the kernel of the mapping is $Im(f \times -i)$, this completes the proof. $\square$

Corollary 2.3.1.1. *Consider a short exact sequence of Λ-modules and homomorphisms*

$$0 \to B_1 \oplus B_2 \xrightarrow{i} E(1)_0 \oplus E(2)_0 \xrightarrow{p} A_1 \oplus A_2 \to 0,$$

in which each of the mappings are diagonal and $i = i_1 \oplus i_2$, as above. Let $f_2 : B_2 \to B_2$ be a Λ-homomorphism, then

$$\varinjlim(Id_{B_1} \oplus f_2, i) \cong E(1)_0 \oplus \varinjlim(f_2, i_2).$$

Proof. For any $\mathcal{E} \in Ext^1_\Lambda(A_1, B_2)$, $Id_*(\mathcal{E}) = \mathcal{E}$ and so $\varinjlim(Id_{B_1}, i_1) \cong E(1)_0$. The result now follows from the previous proposition. $\square$

We now prove a second proposition relating to pushouts.

Proposition 2.3.2. *Consider an exact sequence*

$$0 \to B_1 \oplus B_2 \xrightarrow{i} E \to A \to 0,$$

where $i = i_1 \oplus i_2$ and $i_r : B_r \to E$ for $r = 1, 2$. In this case, an isomorphism $\varinjlim(Id_{B_1} \oplus nId_{B_2}, i) \cong \varinjlim(nId_{B_2}, i_2)$ exists.

Proof. Recall that

$$\varinjlim(Id_{B_1} \oplus nId_{B_2}, i) = \frac{(B_1 \oplus B_2) \oplus E}{Im((Id_{B_1} \oplus nId_{B_2}) \times -i)},$$

$$\varinjlim(nId_{B_2}, i_2) = \frac{(B_2) \oplus E}{Im(nId_{B_2} \times -i_2)}.$$

We define a mapping

$$(B_1 \oplus B_2) \oplus E \to \frac{(B_2) \oplus E}{Im(nId_{B_2} \times -i_2)},$$
$$((b_1, b_2), e) \mapsto (b_2, e + i(b_1, 0)) + Im(nId_{B_2} \times -i_2).$$

One can check easily that the mapping is a Λ-homomorphism and by noting that $((0, b_2), e) \mapsto (b_2, e) + Im(nId_{B_2} \times -i_2)$, we see that the map is surjective. We now find the kernel of the map.

$$\begin{aligned} ((b_1, b_2), e)) \in Ker &\iff (b_2, e + i(b_1, 0)) \in Im(nId_{B_2} \times -i_2,) \\ &\iff \text{there exists } c_2 \in B_2 \text{ such that } (b_2, e + i(b_1, 0)) = (nc_2, -i(0, c_2)), \\ &\iff ((b_1, b_2), e) = ((b_1, nc_2), -i(b_1, c_2)), \\ &\iff ((b_1, b_2), e) \in Im((Id_{B_1} \oplus nId_{B_2}) \times -i). \end{aligned}$$

This completes the proof. □

We close this chapter with one final proposition relating to the colimit.

Proposition 2.3.3. *Let $i : A \oplus B \hookrightarrow M$ be injective and define $h : A \oplus B \to A$ by $(a, b) \mapsto a$. Then $\varinjlim(h, i) \cong M/i(B)$.*

Proof. By definition,

$$\varinjlim(h, i) = \frac{A \oplus M}{Im(h \times -i)}.$$

Consider the composition of the inclusion $M \hookrightarrow A \oplus M$ with the projection $A \oplus M \to \frac{A \oplus M}{Im(h \times -i)}$,

$$\varphi : M \to \frac{A \oplus M}{Im(h \times -i)},$$
$$m \mapsto (0, m) + Im(h \times -i).$$

This map is surjective:

Take a general element $(a, m) + Im(h \times -i) \in \frac{A \oplus M}{Im(h \times -i)}$, $m + i(a, 0)$ is clearly in M, and

$$\begin{aligned} \varphi(m + i(a, 0)) &= (0, m + i(a, 0)) + Im(h \times -i), \\ &= (a, m) + Im(h \times -i). \end{aligned}$$

To complete the proof, we simply note that $Ker(\varphi) \cong i(B)$.

$\square$

3 The derived module category

In this chapter we define the derived module category and discuss some of its properties. The results in the chapter are largely adapted from chapters 5 and 7 of [6] and so complete proofs are not always given, as they can be found easily in the reference. We take Λ to be a ring with unity.

3.1 Definitions

If $f : M \to N$ is in $\mathcal{M}od_\Lambda$, we say that f factors through a projective module, written '$f \approx 0$', when f can be written as a composite $f = \xi \circ \eta$ thus

$$\begin{tikzcd} & P \arrow[dr, "\xi"] & \\ M \arrow[ur, "\eta"] \arrow[rr, "f"] & & N, \end{tikzcd}$$

where $P \in \mathcal{M}od_\Lambda$ is a projective module and η, ξ are Λ-homomorphisms. We define

$$\langle M, N\rangle \ = \{f \in Hom_\Lambda(M, N) \mid f \approx 0\}.$$

By taking $\eta = 0$, we see that $0 \in \langle M, N\rangle$. If $f, g \in \langle M, N\rangle$, with their factorisations through the projective modules P, Q given by $f = \alpha \circ \beta$ and $g = \gamma \circ \delta$ respectively, then

$$f - g = \begin{pmatrix} \alpha & \gamma \end{pmatrix} \begin{pmatrix} \beta \\ -\delta \end{pmatrix}.$$

Note that $\begin{pmatrix} \alpha & \gamma \end{pmatrix} : P \oplus Q \to N$ and $\begin{pmatrix} \beta \\ -\delta \end{pmatrix} : M \to P \oplus Q$, and so $f - g \approx 0$. It follows that:

$$\langle M, N\rangle \text{ is an additive abelian subgroup of } Hom_\Lambda(M, N)$$

We extend $\approx$ to a binary relation on $Hom_\Lambda(M,N)$ via

$$f \approx g \iff f - g \approx 0.$$

By extending $\approx$ in this manner, $\approx$ becomes an equivalence relation on $Hom_\Lambda(M,N)$. The equivalence relation $\approx$ is compatible with composition:

Proposition 3.1.1. *Given Λ-homomorphisms $f, f' : M_o \to M_1$, $g, g' : M_1 \to M_2$,*

$$f \approx f' \text{ and } g \approx g' \implies g \circ f \approx g' \circ f'$$

Proof. $f \approx f'$, so $f - f' \approx 0$, therefore $g \circ (f - f') \approx 0$ and so

$$g \circ f \approx g \circ f'.$$

Similarly, $g - g' \approx 0$ and so $(g - g') \circ (f') \approx 0$, therefore

$$g \circ f' \approx g' \circ f'.$$

We deduce that $g \circ f \approx g' \circ f'$. □

We define the *derived module category* of Λ, $\mathcal{D}er(\Lambda)$ to be the category whose objects are right Λ-modules, and in which, for any two objects M, N, the set of morphisms $Hom_{\mathcal{D}er}(M,N)$ is given by

$$Hom_{\mathcal{D}er}(M,N) = Hom_\Lambda(M,N)/\langle M,N\rangle.$$

$Hom_\Lambda(M,N)$ is an abelian group, and so $Hom_{\mathcal{D}er}(M,N)$ has the natural structure of an abelian group. Throughout this thesis, we will use the notation '$\overline{}$' for homomorphisms in the derived module category, in the sense that if $f \in Hom_\Lambda(M,N)$, $\overline{f}$ is the element in the derived module category represented by f

3.2 Results

Let G be a finite group with integral group ring $\Lambda = \mathbb{Z}[G]$. In this section, we describe some results from [6] relating to $\mathcal{D}er(\Lambda)$. Each of the results will be useful during calculations later in the thesis.

A Λ-module M is said to be *coprojective* if $Ext^1_\Lambda(M, \Lambda) = 0$. We now quote a result, the 'de-stabilization lemma' from ([6], page 97, 5.17).

Lemma 3.2.1. *Let*

$$0 \to J \oplus Q_0 \xrightarrow{j} Q_1 \to M \to 0$$

be an exact sequence of Λ-modules in which Q_0, Q_1 are projective; if M is coprojective then $Q_1/j(Q_0)$ is projective.

We now work towards describing an exact sequence which can be used to describe homomorphism groups in the derived module category.

Proposition. *Consider $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0) \in Ext^1_\Lambda(C, A)$. If C is coprojective, then the connecting homomorphism*

$$\delta : Hom_\Lambda(A, N) \to Ext^1_\Lambda(C, N),$$

given by $\delta(\alpha) = \alpha_*(\mathcal{E})$ which we described in §2.2 factors through $Hom_{\mathcal{D}er}(A, N)$ according to the diagram

$$\begin{tikzcd} Hom_\Lambda(A,N) \arrow[rr, "\delta"] \arrow[dr, "\pi"'] & & Ext^1_\Lambda(C,N) \\ & Hom_{\mathcal{D}er}(A,N). \arrow[ur, "\delta_*"'] & \end{tikzcd}$$

Proof. Assume that $\alpha \in Hom_\Lambda(A, N)$ factors through a projective as follows

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\eta"'] & & N \\ & Q. \arrow[ur, "\xi"'] & \end{tikzcd}$$

Then $\alpha_*(\mathcal{E}) = \xi_* \circ \eta_*(\mathcal{E})$. Now, $\eta_* : Ext^1_\Lambda(C, A) \to Ext^1_\Lambda(C, Q)$ is a group homomorphism, and $Ext^1(C, Q) = 0$ as C is coprojective, therefore

$$\begin{aligned} \alpha_*(\mathcal{E}) &= \xi_* \circ \eta_*(\mathcal{E}), \\ &= \xi_*(0), \\ &= 0. \end{aligned}$$

In particular, δ vanishes on $\langle A, N\rangle$, as required. □

We now describe an exact sequence with connecting homomorphism δ_*.

Proposition 3.2.2. *([6], page 104, 5.28) Let $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$ be an exact sequence of Λ-modules in which C is coprojective; then for any Λ-module N we have an exact sequence of additive groups*

$$Hom_{\mathcal{D}er}(C, N) \xrightarrow{p^*} Hom_{\mathcal{D}er}(B, N) \xrightarrow{i^*} Hom_{\mathcal{D}er}(A, N) \xrightarrow{\delta_*} Ext^1_\Lambda(C, N)$$

$$\xrightarrow{p^*} Ext^1_\Lambda(B, N) \xrightarrow{i^*} Ext^1_\Lambda(A, N).$$

Here, δ_* is the homomorphism described in the previous proposition and all other mappings are the standard pullback mappings.

The above proposition has a dual proposition:

Proposition 3.2.3. *([6], page 101, 5.23) Let* $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$ *be an exact sequence of* Λ*-modules; then for any* Λ*-module* M *there is an exact sequence*

$$Hom_{\mathcal{D}er}(M, A) \xrightarrow{i_*} Hom_{\mathcal{D}er}(M, B) \xrightarrow{p_*} Hom_{\mathcal{D}er}(M, C) \xrightarrow{\partial_*} Ext^1_\Lambda(M, A)$$

$$\xrightarrow{i_*} Ext^1_\Lambda(M, B) \xrightarrow{p_*} Ext^1_\Lambda(M, C).$$

Here, $\partial_*(\overline{f}) = f^*(\mathcal{E})$ *and all other mappings are standard pushout mappings.*

The above two propositions give rise to functors which will be of use to us. Define $\mathcal{E}xact(6)$ to be the category whose objects are exact sequences

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to A_6,$$

and whose morphisms are commutative diagrams

$$\begin{array}{ccccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & \longrightarrow & A_6 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 & \longrightarrow & B_6. \end{array}$$

Let $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$ be an exact sequence of Λ-modules and homomorphisms, then **Proposition 3.2.2** defines a covariant functor

$$Hom(\mathcal{E}, -) : \mathcal{D}er(\Lambda) \to \mathcal{E}xact(6).$$

and **Proposition 3.2.3** defines a contravariant functor

$$Hom(-, \mathcal{E}) : \mathcal{D}er(\Lambda) \to \mathcal{E}xact(6).$$

To conclude this section, we will now explicitly describe a relationship between the endomorphisms in the derived module category of two modules satisfying particular conditions. Given an exact sequence

$$\mathcal{E} = (0 \to J \xrightarrow{i} P \xrightarrow{p} M \to 0),$$

of Λ-modules and homomorphisms such that P is projective, one can define a group homomorphism $\rho : End_{\mathcal{D}er}(M) \to End_{\mathcal{D}er}(J)$. Consider a homomorphism $f \in End_{\Lambda}(J)$, the universal property for projective modules implies that a lift $\tilde{f} : P \to P$ exists and so a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \xrightarrow{i} & P & \xrightarrow{p} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & J & \xrightarrow{j} & P & \xrightarrow{\rho} & M & \longrightarrow & 0,
\end{array}$$

exists. Now, the homomorphism f' need not be unique, but if we work in the derived module category we have uniqueness i.e.

$$f \approx g \implies f' \approx g'.$$

Therefore, we have a well defined mapping $\rho : End_{\mathcal{D}er}(J) \to End_{\mathcal{D}er}(M)$ One can easily check that ρ is a ring homomorphism, and in fact if M is coprojective, ρ is an isomorphism. That is

Proposition 3.2.4. *([6], page 133, 7.7) Let* $\mathcal{E} = (0 \to J \xrightarrow{i} P \xrightarrow{p} M \to 0)$ *be an exact sequence of* Λ*-modules and homomorphisms such that* P *is projective and* M *is coprojective, then* $\rho : End_{\mathcal{D}er}(J) \to End_{\mathcal{D}er}(M)$ *is an isomorphism of rings.*

3.3 The Swan homomorphism

Let G be a finite group and let $\Lambda = \mathbb{Z}[G]$ be its integral group ring. In this section, we will define the Swan homomorphism $S_J : Aut_{\mathcal{D}er}(J) \to \tilde{K}_0(\Lambda)$

and explore some of its properties. This section is adapted from chapters 5 and 7 of [6]. A Λ-module S is said to be *stably free* if there exists natural numbers n, m such that $S \oplus \Lambda^n \cong \Lambda^m$.

Let $\mathcal{E} = (0 \to J \xrightarrow{i} S \xrightarrow{p} M \to 0)$ be an exact sequence of Λ-modules and homomorphisms such that S is stably free. Define a mapping

$$s_{\mathcal{E}} : End_{\mathcal{D}er}(J) \to \mathcal{M}od_{\Lambda},$$

$$\overline{\alpha} \mapsto \varinjlim(\alpha, i).$$

Proposition 3.3.1. (**Swan′s projectivity criterion**) *If M is coprojective, then $s_{\mathcal{E}}(\overline{\alpha})$ is projective if and only if $\overline{\alpha} \in Aut_{\mathcal{D}er}(J)$.*

Proof. Assume that $\overline{\alpha} \in Aut_{\mathcal{D}er}(J)$, using the pushout construction, we know that a commutative sequence with exact rows

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "\alpha"] & S \arrow[r, "p"] \arrow[d, "v"] & M \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & J \arrow[r, "j"] & s_{\mathcal{E}}(\overline{\alpha}) \arrow[r, "\rho"] & M \arrow[r] & 0,
\end{tikzcd}$$

exists. Let N be a Λ-module, the contravariant functor described in §3.2 gives rise to the following commutative diagram with exact rows

$$\begin{tikzcd}
Hom_{\mathcal{D}er}(N, J) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & Hom_{\mathcal{D}er}(N, S) \arrow[r, "p_*"] \arrow[d, "v_*"] & Hom_{\mathcal{D}er}(N, M) \arrow[r, "\partial"] \arrow[d, "Id_*"] & Ext^1_{\Lambda}(N, J) \arrow[d, "\alpha_*"] \\
Hom_{\mathcal{D}er}(N, J) \arrow[r, "j_*"] & Hom_{\mathcal{D}er}(N, s_{\mathcal{E}}(\overline{\alpha})) \arrow[r, "\rho_*"] & Hom_{\mathcal{D}er}(N, M) \arrow[r, "\partial"] & Ext^1_{\Lambda}(N, J).
\end{tikzcd}$$

Id_* is clearly an isomorphism, and since α is an isomorphism in the derived module category, α_* is an isomorphism on both $Hom_{\mathcal{D}er}(N, J)$ and

$Ext^1_\Lambda(N, J)$. S is projective, and so $Hom_{\mathcal{D}er}(N, S) = 0$, we deduce that $Hom_{\mathcal{D}er}(N, s_{\mathcal{E}}(\overline{\alpha})) = 0$. Thus, for any Λ-module N, $Hom_{\mathcal{D}er}(N, s_{\mathcal{E}}(\overline{\alpha})) = 0$, and so $s_{\mathcal{E}}(\overline{\alpha})$ is projective.

It remains to prove the reverse implication. Assume that $s_{\mathcal{E}}(\overline{\alpha})$ is projective. By projectivity of S and $s_{\mathcal{E}}(\overline{\alpha})$, $Hom_{\mathcal{D}er}(S, N)$, $Hom_{\mathcal{D}er}(s_{\mathcal{E}}(\overline{\alpha}), N))$, $Ext^1_\Lambda(S, N)$ and $Ext^1_\Lambda(s_{\mathcal{E}}(\overline{\alpha}), N))$ are all zero for any Λ-module N. By applying **Proposition 3.2.2** to

$$0 \to J \xrightarrow{i} S \xrightarrow{p} M \to 0,$$

with $N = J$, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & End_{\mathcal{D}er}(J) & \xrightarrow{j^*} & Ext^1_\Lambda(M, J) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle Id^*} & & \\ 0 & \longrightarrow & End_{\mathcal{D}er}(J) & \xrightarrow{j^*} & Ext^1_\Lambda(M, J) & \longrightarrow & 0. \end{array}$$

By the Five lemma, α^* is bijective. Therefore, there exists a $\beta : J \to J$ such that $\alpha^*(\beta) = \beta \circ \alpha \approx Id$. Now,

$$\begin{aligned} \alpha^*(\alpha \circ \beta) &= (\alpha \circ \beta) \circ \alpha \\ &= \alpha \circ (\beta \circ \alpha) \\ &\approx \alpha \circ Id \\ &\approx Id \circ \alpha \\ &= \alpha^*(Id). \end{aligned}$$

By injectivity of α^*, $\alpha \circ \beta \approx Id$ and so $\overline{\alpha} \in Aut_{\mathcal{D}er}(J)$. This completes the proof. $\square$

We can therefore restrict $s_{\mathcal{E}}$ to $Aut_{\mathcal{D}er}(J)$ and think of it as a mapping

$$s_{\mathcal{E}} : Aut_{\mathcal{D}er}(J) \to \tilde{K}_0(\Lambda).$$

It is shown in §7 of [6] that $s_{\mathcal{E}}$ is not dependant on the choice of $\mathcal{E}$, and is in fact only dependant on J, we therefore define the mapping

$$S_J : Aut_{\mathcal{D}er}(J) \to \tilde{K}_0(\Lambda)$$

$$\overline{\alpha} \mapsto s_{\mathcal{E}}(\overline{\alpha}).$$

Furthermore, S_J is an additive group homomorphism, and so we call S_J the *Swan homomorphism*.

4 Milnor squares

Let Λ be a ring with unity. In this chapter we will briefly outline some results from [11] regarding the classification of projective modules over Λ using Milnor squares. We will then describe the Milnor square of a quasi-augmentation sequence and outline a related theorem from [3].

4.1 Projective modules over Milnor squares

Consider a commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \Lambda_2 & \xrightarrow{j_2} & \Lambda'. \end{array}$$

Consider the following conditions on the above commutative square:

Condition 1 : Given any $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$ such that $j_1(\lambda_1) = j_2(\lambda_2)$ in Λ', there is exactly one element $\lambda \in \Lambda$ such that $i_1(\lambda) = \lambda_1$ and $i_2(\lambda) = \lambda_2$.

Condition 2 : At least one of the two homomorphisms j_1 and j_2 is surjective.

We say that a commutative square of the above form satisfying **Condition 1** is a *fibre square*, and a commutative square of the above form which satisfies both **Condition 1** and **Condition 2** is a *Milnor square*. For the remainder of this section we will work only with Milnor squares. Consider projective modules P_1, P_2 over Λ_1, Λ_2 respectively and a Λ'-module isomorphism

$$h : P_1 \otimes_{\Lambda_1} \Lambda' \to P_2 \otimes_{\Lambda_2} \Lambda'.$$

Define

$$j_{i*} : P_i \to P_i \otimes_{\Lambda_i} \Lambda',$$

$$p_i \mapsto p_i \otimes 1,$$

for $i = 1, 2$. If we define $M(P_1, P_2, h)$ to be the set

$$\{(p_1, p_2) \in P_1 \times P_2 \mid h \circ j_{1*}(p_1) = j_{2*}(p_2)\},$$

we may endow $M(P_1, P_2, h)$ with a right Λ-module structure as follows:

$$(p_1, p_2) \cdot \lambda = (p_1 \cdot i_1(\lambda), p_2 \cdot i_2(\lambda)).$$

In [11], some key theorems are proven relating to the classification of projective modules using Milnor squares, which we now state without proof.

Theorem. *The module $M(P_1, P_2, h)$ is projective over Λ. Furthermore if P_1 and P_2 are finitely generated over Λ_1 and Λ_2 respectively, then M is finitely generated over Λ.*

Theorem. *Every projective Λ-module is isomorphic to $M(P_1, P_2, h)$ for some suitably chosen P_1, P_2 and h.*

4.2 Classification of projective modules

Later in this thesis, we will deal with several projective modules of type $M(P_1, P_2, h)$ where P_1, P_2 are projective modules over rings Λ_1, Λ_2 respectively which arise within Milnor squares of the form

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \Lambda_2 & \xrightarrow{j_2} & \Lambda'. \end{array}$$

Consequently, it will be useful to classify modules of this type up to isomorphism. To do this, we use results from [6].

By ([6], 3.3, page 40) we know that a 1-1 correspondence,

$$Aut_{\Lambda_1}(P_1)\backslash Iso_{\Lambda'}(P_1 \otimes \Lambda', P_2 \otimes \Lambda')/Aut_{\Lambda_2}(P_2) \leftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{modules of type} \\ M(P_1, P_2, h) \end{array} \right\},$$

exists, given by

$$[h] \mapsto M(P_1, P_2, h).$$

In particuar, when $P_1 = \Lambda_1$, $P_2 = \Lambda_2$, we have a 1-1 correspondence,

$$\Lambda_1^* \backslash \Lambda'^* / \Lambda_2^* \leftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{modules of type} \\ M(\Lambda_1, \Lambda_2, h) \end{array} \right\},$$

given by

$$[h] \mapsto M(\Lambda_1, \Lambda_2, h).$$

4.3 The Milnor square of a quasi-augmentation sequence

A Λ-module S is called *strongly Hopfian* if for each integer $n \geq 1$ any surjective homomorphism $\varphi : S^n \to S^n$ is necessarily an isomorphism. A *quasi-augmentation sequence* is then a short exact sequence

$$\mathcal{S} = (0 \to S_- \to S_0 \to S_+ \to 0)$$

over Λ such that S_0 is stably free, satisfying the following conditions:

1. $Ext^1_\Lambda(S_+, \Lambda) = 0$;

2. S_+, S_- are strongly Hopfian;

3. $Hom_\Lambda(S_-, S_+) = 0$.

We note that for $\Lambda = \mathbb{Z}[G]$ with G a finite group, in the case where S_+ and S_- are Λ-lattices, conditions 1 and 2 are automatically satisfied. For the remainder of this section, we take G to be a finite group and Λ to be the integral group ring of G, $\mathbb{Z}[G]$. We now prove a related theorem, adapted from [3]:

Theorem 4.3.1. *([3], page 30, 4.1.1) Given a quasi-augmentation sequence*

$$\mathcal{S} = (0 \to S_- \xrightarrow{i} \Lambda \xrightarrow{p} S_+ \to 0),$$

satisfying the condition

- $Hom_\Lambda(S_+, S_-) = 0$*;*

there exists a Milnor square

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & End_\Lambda(S_+) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ End_\Lambda(S_-) & \xrightarrow{j_2} & End_{\mathcal{D}er}(S_-), \end{array}$$

where i_1, j_2 and j_1 are surjective.

Proof. We first define the maps in the Milnor square. Given an element $\lambda \in \Lambda$, we define a map $f_\lambda : \Lambda \to \Lambda$ by $f(1) = \lambda$. Since $Hom_\Lambda(S_-, S_+) = 0$, f_λ defines two unique maps, f_+, f_- such that

$$\begin{array}{ccccccccccc} \mathcal{S} = (0 & \longrightarrow & S_- & \xrightarrow{i} & \Lambda & \xrightarrow{p} & S_+ & \longrightarrow & 0) \\ & & \big\downarrow{\scriptstyle f_-} & & \big\downarrow{\scriptstyle f_\lambda} & & \big\downarrow{\scriptstyle f_+} & & \\ \mathcal{S} = (0 & \longrightarrow & S_- & \longrightarrow & \Lambda & \longrightarrow & S_+ & \longrightarrow & 0), \end{array}$$

commutes. We define $i_1(\lambda) = f_+$ and $i_2(\lambda) = f_-$, by construction it is clear that these two maps are ring homomorphisms. Every map in $End_\Lambda(S_+)$ lifts to a map in $End_\Lambda(\Lambda)$ by the universal property of projective modules, and so i_1 is surjective, as claimed. We now move on to the maps j_2 and j_1. j_2 Is simply the standard projection map

$$End_\Lambda(S_-) \to End_{\mathcal{D}er}(S_-).$$

Now, utilising **Proposition 3.2.3** we see that $End_{\mathcal{D}er}(S_+) \cong Ext^1_\Lambda(S_+, S_-)$ via the pullback $\partial_* : End_{\mathcal{D}er}(S_+) \to Ext^1_\Lambda(S_+, S_-)$. By **Proposition 3.2.2** $Ext^1_\Lambda(S_+, S_-) \cong End_{\mathcal{D}er}(S_-)$ via the pushout $\delta_* : End_{\mathcal{D}er}(S_-) \to Ext^1_\Lambda(S_+, S_-)$, we can now describe j_1 as the composition of the standard projection map $End_\Lambda(S_+) \to End_{\mathcal{D}er}(S_+)$ with ∂_* and δ_*^{-1}:

$$End_\Lambda(S_+) \to End_{\mathcal{D}er}(S_+) \xrightarrow{\partial_*} Ext^1_\Lambda(S_+, S_-) \xrightarrow{\delta_*^{-1}} End_{\mathcal{D}er}(S_-).$$

Clearly j_1 and j_2 are surjective ring homomorphisms.

We will now show that $j_1 \circ i_1 = j_2 \circ i_2$. Fix a $\lambda \in \Lambda$, there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \mathcal{S} = (0 & \longrightarrow & S_- & \xrightarrow{i} & \Lambda & \xrightarrow{p} & S_+ & \longrightarrow & 0) \\ & & \big\downarrow{\scriptstyle f_-} & & \big\downarrow{\scriptstyle f_\lambda} & & \big\downarrow{\scriptstyle f_+} & & \\ \mathcal{S} = (0 & \longrightarrow & S_- & \xrightarrow{i} & \Lambda & \longrightarrow & S_+ & \xrightarrow{p} & 0), \end{array}$$

such that $f_- = i_1(\lambda)$ and $f_+ = i_2(\lambda)$. By ([8], 1.5, page 66) the above commutative diagram *factors* in the sense that a commutative diagram with exact rows

$$\begin{array}{lccccccccccc}
\mathcal{S} = & (0 & \longrightarrow & S_- & \xrightarrow{i} & \Lambda & \xrightarrow{p} & S_+ & \longrightarrow & 0) \\
& & & \downarrow{\scriptstyle f_-} & & \downarrow{\scriptstyle f^1_\lambda} & & \downarrow{\scriptstyle Id} & & \\
& 0 & \longrightarrow & S_- & \xrightarrow{i} & M & \xrightarrow{p} & S_+ & \longrightarrow & 0) \\
& & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f^2_\lambda} & & \downarrow{\scriptstyle f_+} & & \\
\mathcal{S} = & (0 & \longrightarrow & S_- & \longrightarrow & \Lambda & \longrightarrow & S_+ & \longrightarrow & 0)
\end{array}$$

exists such that $f^2_\lambda \circ f^1_\lambda = f_\lambda$. We deduce immediately that for each $\lambda \in \Lambda$, $j_1 i_1(\lambda) = j_2 i_2(\lambda)$.

It remains only to prove that given an $f_+ \in End_\Lambda(S_+), f_- \in End_\Lambda(S_-)$ such that $j_1(f_+) = j_2(f_-)$, there exists a unique $\lambda \in \Lambda$ such that $i_1(\lambda) = f_+$ and $i_2(\lambda) = f_-$. Let f_+, f_- satisfy these conditions. A commutative diagram similar to the one above shows that there exists a $\lambda \in \Lambda$ such that $i_1(\lambda) = f_+$ and $i_2(\lambda) = f_-$, the uniqueness of λ follows from the conditions $Hom_\Lambda(S_-, S_+) = 0$ and $Hom_\Lambda(S_+, S_-) = 0$ in our hypothesis. This completes the proof. $\square$

We conclude this chapter with the following theorem, which will allow us to calculate the image of the Swan map in some cases which are of particular interest in this thesis.

Theorem 4.3.2. *([3], page 41, 4.5.2) With the hypotheses of the above theorem, if $f_- \in End_\Lambda(S_-)$ is such that $\overline{f}_- \in Aut_{\mathcal{D}er}(S_-)$, then*

$$\varinjlim(f_-, i) \cong M(End_\Lambda(S_+), End_\Lambda(S_-), \overline{f}_-).$$

5 A sufficient condition for the D(2)-property

Let G be a finite group and let Λ be the integral group ring of G, $\mathbb{Z}[G]$. In this chaper, we will briefly outline some results from [5], concluding with a sufficient condition for the D(2)-property to hold for a finite group G. When we refer to Λ-lattices, it is assumed that the lattice has finite rank over $\mathbb{Z}$.

5.1 Stable modules and their associated trees

Let M be a Λ-lattice. We define the *stable module represented by* M to be the set

$$[M] = \{N \in Mod_\Lambda \mid \text{there exists } a, b \in \mathbb{N} \text{ such that } M \oplus \Lambda^a \cong N \oplus \Lambda^b \text{ and } N \ncong 0\}.$$

We can associate with $[M]$ a tree structure by drawing an upward directed arrow from N to $N \oplus \Lambda$ for each $N \in [M]$. As the $\mathbb{Z}$-rank of Λ, $rk_{\mathbb{Z}}(\Lambda) = |G|$ is finite, it is clear that the tree extends finitely downwards and infinitely upwards.

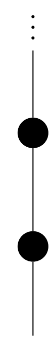
- We call $N_0 \in [M]$ a *minimal module* if it has minimal $\mathbb{Z}$-rank, i.e. $rk_{\mathbb{Z}}(N_0) = min\{rk_{\mathbb{Z}}(N) \mid N \in [M]\}$.
- We call $N_1 \in [M]$ a *root module* if there is no $N_1' \in [M]$ such that $N_1' \oplus \Lambda \cong N_1$.

Note that all minimal modules are root modules, while the converse is not true in general. Fix a minimal module $N_0 \in [M]$, let $N \in [M]$; by the definition of $[M]$, there exists an $a, b \in \mathbb{N}$ such that $N_0 \oplus \Lambda^a \cong N \oplus \Lambda^b$. As

$rk_{\mathbb{Z}}(N_0) \leq rk_{\mathbb{Z}}(N)$, $a \geq b$ and so $a - b \geq 0$, we can therefore define a *height function*

$$h : [M] \to \mathbb{N},$$
$$N \mapsto a - b.$$

Note that the height function is independent of our choice of minimal module N_0. We can now think of the minimal modules of $[M]$ simply as the elements of $[M]$ with height zero. The stable module $[M]$ is said to be *straight* if $|h^{-1}(n)| = 1$ for each natural number n, a straight stable module with root module N_0 will have the following tree structure:



N_0.

5.2 The Swan-Jacobinski Theorem

In this section, we briefly describe a special case of the Swan-Jacobinski Theorem [2] and explore some of its connotations which are useful in the context of this thesis. This treatment of the Swan-Jacobinski Theorem is adapted from section 15 in [5].

Let $\Lambda_{\mathbb{R}} = \mathbb{R}[G]$. Wedderburn's Theorem gives a decomposition of $\Lambda_{\mathbb{R}}$ into a direct sum of simple two-sided ideals

$$\Lambda_{\mathbb{R}} \cong \prod_{i=1}^{m} M_{d_i}(\mathbb{R}) \times \prod_{j=1}^{n} M_{e_j}(\mathbb{C}) \times \prod_{k=1}^{r} M_{f_k}(\mathbb{H}).$$

We say that Λ satisfies the *Eichler condition* if either $r = 0$, or $f_k \neq 1$ for all k. We say that a Λ-lattice M satisfies the *cancellation property* when, for any Λ-lattice N such that $rk_{\mathbb{Z}}(M) \leq rk_{\mathbb{Z}}(N)$,

$$N \oplus \Lambda^m \cong M \oplus \Lambda^n \implies N \cong M \oplus \Lambda^{n-m}.$$

Now, ([2], page 324, 51.28) gives the following special case of the Swan-Jacobinski Theorem:

Theorem. *Let G be a finite group such that $\Lambda = \mathbb{Z}[G]$ satisfies the Eichler condition. Let M be a Λ-lattice, then each $N \in [M]$ for which there exists an $N_0 \in \mathcal{M}od_{\Lambda}$ satisfying $N_0 \oplus \Lambda \cong N$ satisfies the cancellation property.*

We can use the Swan-Jacobinski Theorem to make deductions about the tree structure of stable modules over Λ.

Proposition 5.2.1. *Let G be a finite group such that $\Lambda = \mathbb{Z}[G]$ satisfies the Eichler condition. If M is a Λ-lattice, and $h : [M] \to \mathbb{N}$ is the height function for $[M]$, then $|h^{-1}(n)| = 1$ for each $n \geq 1$.*

Proof. Let M_n, N_n be such that $h(M_n) = h(N_n) = n$ where $n \geq 1$. We will show that $M_n \cong N_n$. Let $M_0 \in [M]$ be minimal, then there exists an r, s such that $r - s = n$ and

$$M_0 \oplus \Lambda^r \cong M_n \oplus \Lambda^s,$$

$$M_0 \oplus \Lambda^r \cong N_n \oplus \Lambda^s.$$

By the Swan-Jacobinski Theorem, $M_0 \oplus \Lambda^{r-s}$ satisfies the cancellation property and so

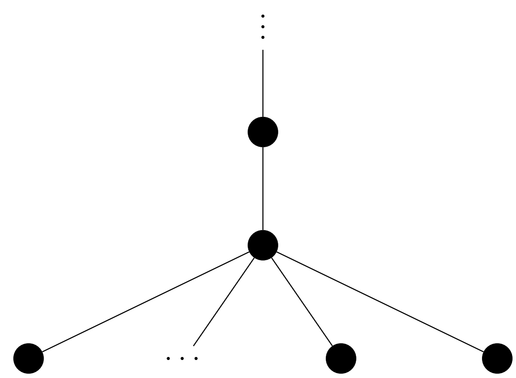
$$M_0 \oplus \Lambda^{r-s} \cong M_n,$$

$$M_0 \oplus \Lambda^{r-s} \cong N_n.$$

This completes the proof.

□

Therefore, if Λ satisfies the Eichler condition, for each Λ-lattice M, the stable module $[M]$ has the following shape:



where the minimal level of the tree has $k \geq 1$ elements. Note that for stable modules over rings satisfying the Eichler condition, minimal modules are equivalent to root modules. Recall the definition of *stably free* from §3.3. We say that Λ satisfies *stably free cancellation* (SFC) if every stably free Λ-module is free, note that the following statements are equivalent:

- Λ satisfies SFC;

- $S \oplus \Lambda^a \cong \Lambda^b$ implies that $S \cong \Lambda^{b-a}$;
- the stable module $[\Lambda]$ is straight.

A stably free Λ-lattice is said to be *non-trivial* if it is not free. It can be shown easily that any integral group ring which satisfies the Eichler condition also satisfies SFC.

Proposition. *Let G be a finite group and let $\Lambda = \mathbb{Z}[G]$. If Λ satisfies the Eichler condition, then Λ satisfies SFC.*

Proof. Let S be a stably free Λ-module. In this case, there exists integers a, b such that

$$S \oplus \Lambda^a \cong \Lambda^b.$$

Clearly $b > a$, and so Λ^{b-a} satisfies the cancellation property by the Swan-Jacobinski Theorem. We deduce that $S \cong \Lambda^{b-a}$. □

5.3 Full modules

Consider an exact sequence of Λ-modules and homomorphisms

$$\Phi = (0 \to J \xrightarrow{j} S \to M \to 0),$$

such that S is stably free. Recall the Swan homomorphism from §3.3

$$\begin{aligned} S_J : Aut_{\mathcal{D}er}(J) &\to \tilde{K_0}(\Lambda), \\ \overline{f} &\mapsto [\varinjlim(f, j)]. \end{aligned}$$

We may also define a mapping

$$v^J : Aut_\Lambda(J) \to Aut_{\mathcal{D}er}(J),$$
$$f \mapsto \overline{f}.$$

If f is in $Aut_\Lambda(J)$, we know from §2.2 that a commutative diagram

$$\begin{array}{rcccccccccc}
\Phi & = (0 & \longrightarrow & J & \longrightarrow & S & \longrightarrow & M & \longrightarrow & 0) \\
 & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Id} & & \\
f_*(\Phi) & = (0 & \longrightarrow & J & \longrightarrow & \varinjlim(f,j) & \xrightarrow{p} & M & \longrightarrow & 0),
\end{array}$$

with exact rows exists. By the Five lemma, g is an isomorphism and so $[\varinjlim(f,j)] = [S] = 0 \in \tilde{K_0}(\Lambda)$. Therefore

$$Im(v^J) \subset Ker(S_J).$$

We say that J is *full* if the reverse inclusion holds, that is, if

$$Im(v^J) = Ker(S_J).$$

5.4 Balanced presentations

Given any finitely generated Λ-module M we know that a surjective homomorphism $\Lambda^n \twoheadrightarrow M$ exists for some natural number n, therefore, an exact sequence

$$0 \to J_3 \to \Lambda^{a_2} \to \Lambda^{a_1} \to \Lambda \to \mathbb{Z} \to 0,$$

exists. We define the *third syzygy of* $\mathbb{Z}$, $\Omega_3(\mathbb{Z})$ to be the stable module $[J_3]$, the third syzygy is well defined by Schanuel's lemma. Given a group G with presentation $\mathcal{G} =< x_1, \ldots, x_g \mid W_1, \ldots, W_r >$, there exists an exact sequence (the cellular chain complex of the Cayley complex for G)

$$0 \to \pi_2(\mathcal{G}) \to \Lambda^r \to \Lambda^g \to \Lambda \to \mathbb{Z} \to 0.$$

A finite group G admits a *balanced presentation* if there exists a group presentation

$$\mathcal{G} =< x_1, \ldots, x_g \mid W_1, \ldots, W_r >,$$

for G such that $g = r$. Therefore, if G admits a balanced presentation $\mathcal{G}$, there exists an exact sequence

$$0 \to \pi_2(\mathcal{G}) \to \Lambda^g \to \Lambda^g \to \Lambda \to \mathbb{Z} \to 0.$$

We say that K in $\Omega_3(\mathbb{Z})$ is *realizable* if there exists a presentation $\mathcal{G}$ for G such that $K \cong \pi_2(\mathcal{G})$.

Now, if $\mathcal{G}$ is a balanced presentation, $rk_{\mathbb{Z}}(\pi_2(\mathcal{G})) = rk_{\mathbb{Z}}(\Lambda) - 1$, in this case, $\pi_2(\mathcal{G})$ is clearly a minimal module (and, by extension, a root module) in $\Omega_3(\mathbb{Z})$. We have shown:

Proposition. *If G admits a balanced presentation, then there exists a minimal module $J \in \Omega_3(\mathbb{Z})$ such that J is realizable.*

5.5 The sufficient condition

We now have a key theorem from [5]:

Theorem. *([5], page 216, Theorem III) If each minimal module $J \in \Omega_3(\mathbb{Z})$ is both realizable and full, then G satisfies the realization property.*

For finite groups, the realization property is equivalent to the D(2)-property [5], we can therefore conclude the chapter with the following sufficient condition for the D(2)-property to hold.

Theorem 5.5.1. *If a finite group G satisfies properties 1,2 and 3 below*

1. *G admits a balanced presentation;*

2. *$\Omega_3(\mathbb{Z})$ is straight;*

3. *the minimal module J in $\Omega_3(\mathbb{Z})$ is full;*

then G satisfies the D(2)-property.

6 The ring $\mathcal{T}_{p-1}(\mathbb{Z}, p)$

Let p be an odd prime, denote by C_p the cyclic group $C_p =< x \mid x^p = 1 >$, $Aut(C_p) \cong C_{p-1}$ and so there exists a $\theta \in Aut(C_p)$ such that $ord(\theta) = p - 1$, let m be such that $\theta(x) = x^m$. We define the metacyclic group

$$G(p, p-1) =< x, y \mid x^p = 1,\ y^{p-1} = 1,\ yx = x^m y >,$$

whose isomorphism class is independent of our choice of m. Let Λ be the integral group ring of $G(p, p-1)$, $\mathbb{Z}[G(p, p-1)]$, we will study the ring

$$\mathcal{T}_{p-1}(\mathbb{Z}, p) = \{(a_{i,j})_{1 \leq i,j \leq p-1} \in M_{p-1}(\mathbb{Z}) \mid a_{i,j} \in p\mathbb{Z} \text{ if } i > j\}.$$

For brevity, we will refer to $\mathcal{T}_{p-1}(\mathbb{Z}, p)$ as $\mathcal{T}_{p-1}$. In [7] it was shown that a surjective ring homomorphism $\Lambda \twoheadrightarrow \mathcal{T}_{p-1}$ exists, which allows us to endow $\mathcal{T}_{p-1}$ with a Λ-module structure. In this chapter we will outline several results from [7] before studying some other properties of $\mathcal{T}_{p-1}$, both as a ring and as a Λ-module.

6.1 The cyclic algebra construction

In this section, we will describe the cyclic algebra construction, as in [6]. Let S be a commutative ring and $\theta : S \to S$ be a ring automorphism with order dividing q; in particular, θ satisfies the identity $\theta^q = Id$. We define the *cyclic algebra* $\mathcal{C}_q(S, \theta)$ as the (two-sided) free S-module

$$\mathcal{C}_q(S, \theta) = S \cdot 1 + S \cdot y + \cdots + S \cdot y^{q-1},$$

of rank q with basis $\{1, y, \ldots, y^{q-1}\}$ and with multiplication determined by the relations

$$y^q = 1 \ \ ; \ \ y\xi = \theta(\xi)y \ \ (\xi \in S).$$

Two cyclic algebras are of particular interest to us. The first is constructed by taking $S = \mathbb{Z}[C_p]$, we then take $\theta \in Aut(S)$ to be given by $\theta(x) = x^m$, as in the introduction to this chapter. It is straightforward to see that

$$\Lambda \cong \mathcal{C}_{p-1}(\mathbb{Z}[C_p], \theta). \tag{1}$$

For our second cyclic algebra let $\Sigma_x = 1 + x + \cdots + x^{p-1}$ in $\mathbb{Z}[C_p]$, and let ζ be the primitive pth root of unity $e^{\frac{2\pi i}{p}}$, we can then make the identification $\mathbb{Z}(\zeta) = \mathbb{Z}[C_p]/[\Sigma_x)$, where $[\Sigma_x)$ is the right ideal generated by Σ_x. We take our commutative ring S to be $S = \mathbb{Z}(\zeta)$. As $\theta(\Sigma_x) = 0$, θ induces an automorphism $\bar{\theta} : \mathbb{Z}(\zeta) \to \mathbb{Z}(\zeta)$ given by $\bar{\theta}(\zeta) = \zeta^m$. We can now construct the cyclic algebra

$$\mathcal{C}_{p-1}(\mathbb{Z}(\zeta), \bar{\theta}).$$

From our constructions, it is obvious that a surjective ring homomorphism $C_{p-1}(\mathbb{Z}[C_p], \theta) \twoheadrightarrow C_{p-1}(\mathbb{Z}(\zeta), \bar{\theta})$ exists and so we have a surjection

$$\begin{aligned} &\Lambda \twoheadrightarrow C_{p-1}(\mathbb{Z}(\zeta), \bar{\theta}); \\ &x \mapsto \zeta; \\ &y \mapsto y. \end{aligned}$$

We can use this surjection to endow $C_{p-1}(\mathbb{Z}(\zeta), \bar{\theta})$ with a Λ-module structure. We will see later in this chapter that $C_{p-1}(\mathbb{Z}(\zeta), \bar{\theta})$ is isomorphic as a ring to $\mathcal{T}_{p-1}$, and so when we endow $\mathcal{T}_{p-1}$ with a Λ-module structure in the obvious way,

$$\mathcal{T}_{p-1}(\mathbb{Z},p) \cong_\Lambda \mathcal{C}_{p-1}(\mathbb{Z}(\zeta),\bar{\theta}). \tag{2}$$

6.2 The isomorphism $C_{p-1}(\mathbb{Z}(\zeta),\bar{\theta}) \cong \mathcal{T}_{p-1}(\mathbb{Z},p)$

We now describe a ring homomorphism $\tilde{\lambda_*} : \mathcal{C}_{p-1}(\mathbb{Z}(\zeta),\bar{\theta}) \to \mathcal{T}_{p-1}(\mathbb{Z},p)$ which was first formulated in [7]. Observe that $\{1,\zeta,\ldots,\zeta^{p-2}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}(\zeta)$. Note that $(\zeta-1)^r = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\zeta^k$ and so

$$\begin{aligned}
\zeta &= (\zeta-1)+1;\\
\zeta^2 &= (\zeta-1)^2+2(\zeta-1)+1;\\
&\ \vdots\\
\zeta^r &= (\zeta-1)^r - \sum_{k=0}^{r-1}(-1)^{r-k}\binom{r}{k}\zeta^k.
\end{aligned}$$

By making elementary basis transformations, we see that

$$\{(\zeta-1)^{p-2},(\zeta-1)^{p-3},\ldots,(\zeta-1),1\},$$

is a $\mathbb{Z}$-basis for $\mathbb{Z}(\zeta)$. Now, consider the right action of $G(p,p-1)$ on $\mathbb{Z}(\zeta)$

$$\begin{aligned}
\mathbb{Z}(\zeta)\times G(p,p-1) &\to \mathbb{Z}(\zeta)\\
z\cdot(x^r y^s) &= \theta^{-s}(z\cdot\zeta^{-r}).
\end{aligned}$$

Identifying $\mathbb{Z}(\zeta) = \mathbb{Z}^{p-1}$ with the basis $\{(\zeta-1)^{p-2-r}\}_{0\le r\le p-2}$, the above action describes a representation $\lambda : G(p,p-1)\to GL_{p-1}(\mathbb{Z})$. Observe that for $0 \leq r \leq p-3$,

$$\begin{aligned}\lambda(x^{-1})[(\zeta-1)^r] &= (\zeta-1)^r\zeta \\ &= (\zeta-1)^{r+1} + (\zeta-1)^r,\end{aligned}$$

whilst

$$\lambda(x^{-1})[(\zeta-1)^{p-2}] = (\zeta-1)^{p-1} + (\zeta-1)^{p-2}.$$

Now, it is well known ([1], page 87, 3) that $p = (\zeta-1)^{p-1}u$ for some $u \in \mathbb{Z}(\zeta)^*$, and so

$$\lambda(x^{-1})[(\zeta-1)^{p-2}] = (\zeta-1)^{p-2} + pu^{-1}.$$

Therefore, $\lambda(x^{-1})$ takes the form

$$\lambda(x^{-1}) = \begin{pmatrix} 1+pa_{p-2} & 1 & 0 & \dots & 0 & 0 \\ pa_{p-3} & 1 & 1 & \dots & 0 & 0 \\ pa_{p-4} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ pa_1 & 0 & 0 & 0 & 1 & 1 \\ pa_0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, x^{-1} generates C_p and $\lambda(x^{-1})$ lies in the unit group of $\mathcal{T}_{p-1}(\mathbb{Z},p)$, $\mathcal{U}_{p-1}(\mathbb{Z},p)$, and so

$$\lambda(C_p) \subset \mathcal{U}_{p-1}(\mathbb{Z},p).$$

This result is expanded on in [7] to show the following:

Proposition. $\lambda(G(p,p-1)) \subset \mathcal{U}_{p-1}(\mathbb{Z},p)$

We can now think of λ as a map $\lambda : G(p,p-1) \to \mathcal{U}_{p-1}(\mathbb{Z},p)$, which extends naturally to a ring homomorphism

$$\lambda_* : \Lambda \to \mathcal{T}_{p-1}.$$

It is clear from our description of λ that $\lambda(\Sigma_x) = 0$, and so λ_* induces a ring homomorphism

$$\tilde{\lambda}_* : \mathcal{C}_{p-1}(\mathbb{Z}(\zeta), \bar{\theta}) \to \mathcal{T}_{p-1}.$$

In [7], it is shown that $\tilde{\lambda}_*$ is in fact a ring isomorphism. We can therefore endow $\mathcal{T}_{p-1}$ with a right Λ-module structure which is inherited from the Λ-module structure of $\mathcal{C}_{p-1}(\mathbb{Z}(\zeta), \bar{\theta})$. As $\lambda_*(\Sigma_x) = 0$, we can now prove a proposition which we will utilise later in this thesis.

Proposition 6.2.1. *For any* $t \in \mathcal{T}_{p-1}(\mathbb{Z}, p)$, $t \cdot \Sigma_x = 0$.

Proof. $t \cdot \Sigma_x = t \cdot (\lambda_*(\Sigma_x)) = t \cdot 0 = 0$. □

We now use this result to prove that $\mathcal{T}_{p-1} \cong [x-1)$

Proposition. $Ker(\lambda_*) = Span_{\mathbb{Z}}\{\Sigma_x, \Sigma_x \cdot y, \ldots, \Sigma_x \cdot y^{p-2}\}$.

Proof. Let $S = Span_{\mathbb{Z}}\{\Sigma_x, \Sigma_x \cdot y, \ldots, \Sigma_x \cdot y^{p-2}\}$, $\lambda_*(\Sigma_x) = 0$ by **Proposition 6.2.1** and so $S \subset Ker(\lambda_*)$. We now have a short exact sequence

$$0 \to Ker(\lambda_*)/S \to \Lambda/S \to \mathcal{T}_{p-1} \to 0.$$

To prove our result, it is therefore sufficient to show that Λ/S is torsion free, but $\{x^a y^b \mid 0 \leq a, b \leq p-2\} \cup \{\Sigma_x, \Sigma_x \cdot y, \ldots, \Sigma_x \cdot y^{p-2}\}$ is a basis for Λ which includes a basis for S, and so Λ/S is torsion free. □

We now consider the map $\varphi : \Lambda \to [x-1)$ given by $\alpha \mapsto (x-1)\alpha$, we have the following proposition.

Proposition. $Ker(\varphi) = Ker(\lambda_*)$

Proof. Consider the exact sequence

$$0 \to Ker(\varphi) \to \Lambda \xrightarrow{\varphi} [x-1) \to 0.$$

Note that $Ker(\lambda_*) \subset Ker(\varphi)$, this gives rise to a second exact sequence:

$$0 \to Ker(\varphi)/Ker(\lambda_*) \to \Lambda/Ker(\lambda_*) \to [x-1) \to 0.$$

But $\Lambda/Ker(\lambda_*)$ and $[x-1)$ are both torsion free with $\mathbb{Z}$-rank $(p-1)^2$, and so any surjection $\Lambda/Ker(\lambda_*) \to [x-1)$ is necessarily an isomorphism. We deduce that $Ker(\varphi) = Ker(\lambda_*)$. $\square$

The above propositions provide us with an explicit isomorphism

$$\mathcal{T}_{p-1}(\mathbb{Z}, p) \cong [x-1.)$$

We have shown:

Corollary 6.2.1.1. $\mathcal{T}_{p-1}(\mathbb{Z}, p) \cong [x-1)$.

6.3 $Hom_\Lambda(R(i), R(j))$

When considered as a right Λ-module, one can think of $\mathcal{T}_{p-1}$ simply as a direct sum of its rows, we denote the i^{th} row of $\mathcal{T}_{p-1}$ by $R(i)$, and so as Λ-modules,

$$\mathcal{T}_{p-1}(\mathbb{Z}, p) \cong \bigoplus_{i=1}^{p-1} R(i).$$

In this section, we will find explicit descriptions of the rings $Hom_\Lambda(R(i), R(j))$, and then state the description of $Hom_{\mathcal{D}er}(R(i), R(j))$ from [7]. We begin by defining a mapping $f : \mathbb{Z}^2 \to \{0, p-1\}$ by

$$f(i,j) = \begin{cases} p-1, & \text{if } i > j, \\ 0, & \text{otherwise.} \end{cases}$$

If $\epsilon(i,j)$ is the $(p-1) \times (p-1)$ matrix described by $\epsilon(i,j)_{r,s} = \delta_{i,r}\delta_{j,s}$, then $\mathcal{T}_{p-1}$ has $\mathbb{Z}$-basis given by

$$\{t(i,j) = \epsilon(i,j)(1 + f(i,j)) \mid 1 \leq i, j \leq p-1\}.$$

We now define $p-1$ vectors in $M_{1\times(p-1)}(\mathbb{Z})$, each with a right Λ-action given via $\mathcal{T}_{p-1}$ in the obvious way.

- $a_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix}$;
- $a_2 = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \end{pmatrix}$;
  $\vdots$
- $a_{p-2} = \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \end{pmatrix}$;
- $a_{p-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

We can think of $R(i)$ as having a $\mathbb{Z}$-basis

$$\{a_j(1 + f(i,j)) = a_i t(i,j) \mid 1 \leq j \leq p-1\}.$$

We have shown:

Proposition. *$R(i)$ is generated over Λ by a_i.*

Therefore, any Λ-homomorphism $\varphi : R(i) \to R(j)$ is defined completely by $\varphi(a_i)$. Assume that

$$\varphi(a_i) = \sum_{n=1}^{p-1} x_n a_n \in R(j).$$

Where $x_n \in \mathbb{Z}$ for each n. Note that we have not placed any further restrictions on the values of x_n, and so $\varphi(a_i)$ need not be in $R(j)$ as things stand, this is done in order to simplify the following calculation, and the discrepancy is dealt with shortly. Now, $a_i \sum_{m \neq i} t(m,m) = 0$, and so

$$0 = \varphi(a_i \sum_{m \neq i} t(m,m)) = \sum_{n=1}^{p-1} x_n a_n \sum_{m \neq i} t(m,m) = \sum_{n \neq i} x_n a_n,$$

and so $x_n = 0$ for $n \neq i$. Concluding,

$$\varphi(a_i) = x_i a_i.$$

Finally,

$$\varphi(a_i t(i,k)) = \varphi(a_i) t(i,k) = x_i a_i t(i,k).$$

We have shown:

Proposition. *Let $\varphi \in Hom_\Lambda(R(i), R(j))$, then there exists an integer x_i such that if we take a general element α in $R(i)$,*

$$\alpha = \big(\, \alpha_1(1+f(i,1)),\ \alpha_2(1+f(i,2)),\ \dots\ , \alpha_{p-2}(1+f(i,p-2)),\ \alpha_{p-1}(1+f(i,p-1)) \,\big) \in R(i),$$

then the element $\varphi(\alpha)$ in $R(j)$ is given by

$$\varphi(\alpha) = \big(\, x_i\alpha_1(1+f(i,1)),\ x_i\alpha_2(1+f(i,2)),\ \dots\ , x_i\alpha_{p-2}(1+f(i,p-2)),\ x_i\alpha_{p-1}(1+f(i,p-1) \,\big) \in R(j).$$

We can therefore think of elements of $Hom_\Lambda(R(i), R(j))$ as right multiplication by some $nI_{p-1} \in \mathcal{T}_{p-1}$ for some integer n. To deal with the discrepancy mentioned above, we must ensure that right multiplication of elements of $R(i)$ by nI_{p-1} gives an element of $R(j)$. This can be ensured by placing a condition on n, we clearly have two cases:

- If $i \geq j$, $Im(nI_{p-1}) \subset R(j)$ for each $n \in \mathbb{Z}$;

- If $i < j$, $Im(nI_{p-1}) \subset R(j)$ for each $n \in p\mathbb{Z}$.

We have shown:

Proposition.

$$Hom_\Lambda(R(i), R(j)) = \begin{cases} \textit{right multiplication by } nI_{p-1},\ n \in \mathbb{Z}, & \textit{if } i \geq j, \\ \textit{right multiplication by } pnI_{p-1},\ n \in \mathbb{Z} & \textit{if } i < j. \end{cases}$$

From our description, it is clear that Λ-homomorphisms $\varphi : R(i) \to R(j)$ and $\psi : R(j) \to R(k)$ compose in the obvious manner, i.e. if $\varphi = nI_{p-1}$ and $\psi = mI_{p-1}$, then $\psi \circ \varphi = mnI_{p-1} : R(i) \to R(k)$. It is now clear that if we think of $End_\Lambda(\mathcal{T}_{p-1}(\mathbb{Z}, p))$ as a ring of matrices of the form $(m_{i,j})_{1 \leq i,j \leq p-1}$ where $m_{i,j} : R(j) \to R(i)$, then our explicit description of $Hom_\Lambda(R(i), R(j))$ shows that $End_\Lambda(\mathcal{T}_{p-1}(\mathbb{Z}, p)) \cong \mathcal{T}_{p-1}(\mathbb{Z}, p)$, as expected.

We conclude this section by stating a result from [7] which describes the rings $Hom_{\mathcal{D}er}(R(i), R(j))$.

Proposition 6.3.1.

$$Hom_{\mathcal{D}er}(R(i), R(j)) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \textit{if } i = j, \\ 0 & \textit{if } i \neq j. \end{cases}$$

6.4 Projective modules over $\mathcal{T}_{p-1}$

In [13], MI Rosen showed that the ring $\mathcal{T}_{p-1}$ is hereditary. In this section, we will use techniques from [11] to find a description for all projective modules over $\mathcal{T}_{p-1}$. This will provide a full description of all submodules of projective modules over $\mathcal{T}_{p-1}$.

Consider the Milnor square

$$\begin{array}{ccc} \mathcal{T}_{p-1}(\mathbb{Z},p) & \xhookrightarrow{i_1} & M_{p-1}(\mathbb{Z}) \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle j_1} \\ \mathcal{T}_{p-1}(\mathbb{F}_p) & \xhookrightarrow{j_2} & M_{p-1}(\mathbb{F}_p). \end{array}$$

Here, $\mathbb{F}_p$ is the finite field with p elements, and $\mathcal{T}_{p-1}(\mathbb{F}_p) = \mathcal{T}_{p-1}(\mathbb{Z},p)/pM_{p-1}(\mathbb{Z})$. The mappings are the standard inclusion and projection maps. In [11], it is shown that given such a Milnor square, there exists an exact sequence

$$K_1(\mathcal{T}_{p-1}(\mathbb{Z},p)) \to K_1(M_{p-1}(\mathbb{Z})) \oplus K_1(\mathcal{T}_{p-1}(\mathbb{F}_p)) \to K_1(M_{p-1}(\mathbb{F}_p))$$

$$\xrightarrow{\partial} K_0(\mathcal{T}_{p-1}(\mathbb{Z},p)) \to K_0(M_{p-1}(\mathbb{Z})) \oplus K_0(\mathcal{T}_{p-1}(\mathbb{F}_p)) \to K_0(M_{p-1}(\mathbb{F}_p)).$$

Here, the homomorphisms

$$K_\alpha(\mathcal{T}_{p-1}(\mathbb{Z},p)) \to K_\alpha(M_{p-1}(\mathbb{Z})) \oplus K_\alpha(\mathcal{T}_{p-1}(\mathbb{F}_p)) \to K_\alpha(M_{p-1}(\mathbb{F}_p)),$$

for $\alpha = 0, 1$ are made up of the standard induced mappings in K_α, and are defined by

$$x \mapsto (i_{1*}(x), i_{2*}(x)),$$

and

$$(y, z) \mapsto j_{1*}(y) - j_{2*}(z).$$

We will now use this exact sequence to find a practical description for $K_0(\mathcal{T}_{p-1}(\mathbb{Z},p))$, along with a set of generators. As $\mathbb{F}_p$ is a field, there is an obvious isomorphism $K_1(M_{p-1}(\mathbb{F}_p)) \cong \mathbb{F}_p^*$, where $\mathbb{F}_p^*$ is the unit group of $\mathbb{F}_p$.

Moreover, the map on unit groups $\mathcal{T}_{p-1}(\mathbb{F}_p)^* \to \mathbb{F}_p^*$ given by the determinant is surjective, and so the map

$$K_1(M_{p-1}(\mathbb{Z})) \oplus K_1(\mathcal{T}_{p-1}(\mathbb{F}_p)) \to K_1(M_{p-1}(\mathbb{F}_p)),$$

is surjective. We deduce that the mapping $\partial : K_1(M_{p-1}(\mathbb{F}_p)) \to K_0(\mathcal{T}_{p-1})$ is the zero mapping. By Morita's theorem,

$$K_0(M_{p-1}(\mathbb{Z})) \cong \mathbb{Z},$$

and

$$K_0(M_{p-1}(\mathbb{F}_p)) \cong \mathbb{Z},$$

and one can easily check that the standard projection $\mathbb{Z} \to \mathbb{F}_p$ induces an isomorphism $K_0(M_{p-1}(\mathbb{Z})) \to K_0(M_{p-1}(\mathbb{F}_p))$. Therefore, we can extract an isomorphism

$$i_{2*} : K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p)) \xrightarrow{\sim} K_0(\mathcal{T}_{p-1}(\mathbb{F}_p)),$$

from the above exact sequence. Now, to find a description and set of generators for $K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p))$, we will find a description and set of generators for $K_0(\mathcal{T}_{p-1}(\mathbb{F}_p))$ and make use of the isomorphism i_{2*}. We begin by considering some projective modules over $\mathcal{T}_{p-1}(\mathbb{F}_p)$. Let $\hat{R}(i)$ denote the i^{th} row of $\mathcal{T}_{p-1}(\mathbb{F}_p)$. As

$$\mathcal{T}_{p-1}(\mathbb{F}_p) \cong \bigoplus_{i=1}^{p-1} \hat{R}(i),$$

then $\hat{R}(i)$ is projective.

Proposition 6.4.1. *Let $\hat{R}(i)$ denote the i^{th} row of $\mathcal{T}_{p-1}(\mathbb{F}_p)$, with this notation, $K_0(\mathcal{T}_{p-1}(\mathbb{F}_p)) \cong \mathbb{Z}^{p-1}$, with free basis $\{\hat{R}(i)\}_{1\leq i\leq p-1}$.*

Proof. Define $N = \{A = (a_{i,j}) \in \mathcal{T}_{p-1}(\mathbb{F}_p) \mid a_{i,i} = 0 + p\mathbb{Z} \text{ for all } i\}$. It is easily seen that N is a radical ideal and $\mathcal{T}_{p-1}(\mathbb{F}_p)/N \cong \mathbb{F}_p^{p-1}$. By ([9], 6.37, page 183) every projective module over $\mathcal{T}_{p-1}(\mathbb{F}_p)$ can be written uniquely as a lift of projective modules over $\mathcal{T}_{p-1}(\mathbb{F}_p)/N$. As noted, each $\hat{R}(i)$ is projective, and the modules $\hat{R}(i)/N$, $i = 1, \ldots, p-1$ provide a complete list of generators for $K_0(\mathcal{T}_{p-1}(\mathbb{F}_p)/N)$, this leads to our result. $\square$

We can now describe $K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p))$ explicitly.

Proposition 6.4.2. *$K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p)) \cong \mathbb{Z}_{p-1}$, with basis $\{R(i)\}_{1 \le i \le p-1}$.*

Proof. Clearly $R(i)$ maps to $\hat{R}(i)$ for $i = 1, 2, \ldots, p-1$ under the isomorphism

$$i_{2*} : K_0(\mathcal{T}_{p-1}(\mathbb{Z}, p)) \to K_0(\mathcal{T}_{p-1}(\mathbb{F}_p)),$$

the result then follows from **Proposition 6.4.1**. $\square$

7 The module $R(2) \oplus [y-1)$

Recall the sufficient condition, discussed in §5, for a finite group to satisfy the $D(2)$-property.

Theorem 5.5.1. If a finite group G satisfies properties 1,2 and 3 below

1. G admits a balanced presentation;
2. $\Omega_3(\mathbb{Z})$ is straight;
3. The minimal module J in $\Omega_3(\mathbb{Z})$ is full;

then G satisfies the D(2)-property.

Let p be an odd prime and let $\Lambda = \mathbb{Z}[G(p, p-1)]$. In [12], it was shown that $R(2) \oplus [y-1)$ is minimal in $\Omega_3^{G(5,4)}(\mathbb{Z})$, and we will see in §8 that $R(2) \oplus [y-1)$ is minimal in $\Omega_3^{G(7,6)}(\mathbb{Z})$. If this property were to hold for the group $G(p, p-1)$ where p is any odd prime, that is, $R(2) \oplus [y-1)$ is minimal in $\Omega_3^{G(p,p-1)}(\mathbb{Z})$ for each odd prime p, then we can rewrite properties 2 and 3 as follows:

2′. $[R(2) \oplus [y-1)]$ is straight over $\mathbb{Z}[G(p, p-1)]$;

3′. $R(2) \oplus [y-1)$ is full over $\mathbb{Z}[G(p, p-1)]$.

In our specific case, namely the group $G(p, p-1)$ where p is an odd prime, it was shown by JW Wamsley [17] that $G(p, p-1)$ admits a balanced presentation. We will now give an alternative proof of this result.

Proposition. *Let p be an odd prime and let*

$$G(p, p-1) = < x, y \mid x^p = 1, y^{p-1} = 1, yxy^{-1} = x^m >,$$

as defined in §6, then

$$G(p, p-1) \cong < x, y \mid x^p = y^{p-1}, yx^iy^{-1} = x^{i+1} >,$$

where i is any integer such that $i \equiv (m-1)^{-1} (modp)$.

Proof. Firstly, we note that in $< x, y \mid x^p = 1, y^{p-1} = 1, yxy^{-1} = x^m >$, the equality $yx^iy^{-1} = x^{i+1}$ holds true, as $yxy^{-1} = x^m$ implies that

$$\begin{aligned} yx^iy^{-1} &= x^{mi} \\ &= x^{i+1}. \end{aligned}$$

Secondly, we see that in $< x, y \mid x^p = y^{p-1}, yx^iy^{-1} = x^{i+1} >$, $x^p = 1$ and so $y^{p-1} = 1$:

$$\begin{aligned} x^p &= x^{(i+1)p-ip} \\ &= yx^{ip}y^{-1}x^{-ip} \\ &= yx^{ip-ip}y^{-1} \\ &= 1, \end{aligned}$$

since $x^p = y^{p-1}$ and so x^{-ip} commutes with y^{-1}. It remains to show that in $< x, y \mid x^p = y^{p-1}, yx^iy^{-1} = x^{i+1} >$, $yxy^{-1} = x^m$. By our hypothesis,

$i(m-1) \equiv 1(mod p)$, and we have already seen that $x^p = 1$, therefore

$$\begin{aligned} yxy^{-1} &= yx^{i(m-1)}y^{-1} \\ &= (yx^iy^{-1})^{m-1} \\ &= x^{(i+1)(m-1)} \\ &= x^{i(m-1)}x^{m-1} \\ &= x^m. \end{aligned}$$

□

We have shown:

Theorem 7.0.1. *Let $\Lambda = \mathbb{Z}[G(p, p-1)]$ where p is an odd prime. If the third syzygy of $\mathbb{Z}$, $\Omega_3(\mathbb{Z}) = [R(2) \oplus [y-1)]$ and the properties $2'$, $3'$ are satisfied, then $G(p, p-1)$ satisfies the $D(2)$-property.*

In §7.1 we will show that $2'$ holds for any odd prime p. In §7.2 we will show that $3'$ holds for any odd prime p.

7.1 Straightness of $R(2) \oplus [y-1)$

7.1.1 An exact sequence

By the Swan-Jacobinski Theorem, discussed in §5.2, in order to show that $[R(2) \oplus [y-1)]$ is straight over $\Lambda = \mathbb{Z}[G(p, p-1)]$, it is sufficient to show that if S is a Λ-module satisfying

$$\Lambda \oplus S \cong R(2) \oplus [y-1) \oplus \Lambda,$$

then $S \cong R(2) \oplus [y-1)$. To do this, we begin by showing that S can be expressed as an extension of $I(C_{p-1})$ by $R(2) \oplus \bigoplus_{i \neq 1} R(i)$ where $I(C_{p-1})$ will be defined shortly. For brevity, we denote $\bigoplus_{i \neq k} R(i)$ by $R(\hat{k})$. Assume that S is a Λ-module such that

$$\Lambda \oplus S \cong R(2) \oplus [y-1) \oplus \Lambda.$$

Let $I(C_{p-1})$ be the kernel of the augmentation mapping $\epsilon : \mathbb{Z}[C_{p-1}] \to \mathbb{Z}$ when considered as a Λ-homomorphism. Consider the short exact sequence

$$0 \to R(\hat{1}) \to [y-1) \to I(C_{p-1}) \to 0,$$

from [7]. We can now easily construct a second exact sequence

$$\mathcal{E} = (0 \to R(2) \oplus R(\hat{1}) \to R(2) \oplus [y-1) \to I(C_{p-1}) \to 0), \tag{3}$$

in the obvious manner. It is also shown in [7] that another exact sequence

$$0 \to \bigoplus_{j=1}^{p-1} R(j) \to \Lambda \to \mathbb{Z}[C_{p-1}] \to 0,$$

exists. By taking the direct sum of this exact sequence with $\mathcal{E}$, we can construct the following exact sequence:

$$0 \to R(2) \oplus R(\hat{1}) \oplus \bigoplus_{j=1}^{p-1} R(j) \to R(2) \oplus [y-1) \oplus \Lambda \to I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}] \to 0.$$

By assumption, $S \oplus \Lambda \cong R(2) \oplus [y-1) \oplus \Lambda$, and so we also have a short exact sequence

$$0 \to R(2) \oplus R(\hat{1}) \oplus \bigoplus_{j=1}^{p-1} R(j) \to S \oplus \Lambda \xrightarrow{p} I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}] \to 0. \tag{4}$$

We will now manipulate (4) to extract an exact sequence Φ of the following form:

$$\Phi = (0 \to R(2) \oplus R(\hat{1}) \to S \to I(C_{p-1}) \to 0). \tag{5}$$

To complete the manipulation we will require some propositions.

Proposition. $Hom_{\Lambda}(\mathcal{T}_{p-1}, \mathbb{Z}[C_{p-1}] \oplus I(C_{p-1})) = 0$

Proof. $I(C_{p-1})$ is a Λ-submodule of $\mathbb{Z}[C_{p-1}]$, so it suffices to show that $Hom_{\Lambda}(\mathcal{T}_{p-1}, \mathbb{Z}[C_{p-1}]) = 0$. We know from **Proposition 6.2.1** that for any $t \in \mathcal{T}_{p-1}$, $t \cdot (1 + x + \cdots + x^{p-1}) = 0$. Now, as x acts trivially on $\mathbb{Z}[C_{p-1}]$ and $\mathbb{Z}[C_{p-1}]$ is a Λ-lattice, we may deduce the result. $\square$

Proposition. *For the surjection $p : S \oplus \Lambda \to I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]$ defined by the short exact sequence (4), $p(S) \cap p(\Lambda) = 0$.*

Proof. For a Λ-lattice M, we set $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, which we think of as a $\Lambda_{\mathbb{Q}}$-module. As $S \oplus \Lambda \cong R(2) \oplus [y-1) \oplus \Lambda$, $S_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} \cong R(2)_{\mathbb{Q}} \oplus [y-1)_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}$. By Wedderburn's Theorem, it follows that $S_{\mathbb{Q}} \cong R(2)_{\mathbb{Q}} \oplus [y-1)_{\mathbb{Q}}$. By the Wedderburn-Maschke Theorem, we can use the exact sequence (3) to form a split exact sequence

$$0 \to (R(2) \oplus R(\hat{1}))_{\mathbb{Q}} \to S_{\mathbb{Q}} \to I(C_{p-1})_{\mathbb{Q}} \to 0,$$

and so

$$S_{\mathbb{Q}} \cong I(C_{p-1})_{\mathbb{Q}} \oplus (R(2) \oplus R(\hat{1}))_{\mathbb{Q}}.$$

Now, $Hom_{\Lambda_{\mathbb{Q}}}((\mathcal{T}_{p-1})_{\mathbb{Q}}, (I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}])_{\mathbb{Q}}) = 0$, therefore

$$Hom_{\Lambda_{\mathbb{Q}}}(S_{\mathbb{Q}}, (I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}])_{\mathbb{Q}}) \cong Hom_{\Lambda_{\mathbb{Q}}}(I(C_{p-1})_{\mathbb{Q}}, I(C_{p-1})_{\mathbb{Q}} \oplus \mathbb{Z}[C_{p-1}]_{\mathbb{Q}}).$$

We deduce that the image of Λ-homomorphisms $f : S \to I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]$ can have rank of at most $p-2$. By utilising the split exact sequence

$$0 \to \mathcal{T}_{p-1_{\mathbb{Q}}} \to \Lambda_{\mathbb{Q}} \to \mathbb{Z}[C_{p-1}]_{\mathbb{Q}} \to 0,$$

in a similar manner, we see that the maximal rank of the image of a Λ-homomorphism $g : \Lambda \to I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]$ is $p-1$. As p is surjective and $rk_{\mathbb{Z}}(I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]) = (p-2)+(p-1)$ we can now deduce the result. $\square$

Let $K_1 = S \cap ker(p)$ and $K_2 = \Lambda \cap ker(p)$. We can now think of (4) as a diagonal short exact sequence of the following form.

$$0 \to K_1 \oplus K_2 \to S \oplus \Lambda \to p(S) \oplus p(\Lambda) \to 0.$$

Given our work in §6.4, we know that the modules K_1 and K_2 must be direct sums of $R(i)$ modules, namely $K_1 = \bigoplus_{i=1}^{m} R(a_i)$ and $K_2 = \bigoplus_{j=1}^{n} R(b_j)$ for some a_i, b_j such that $m+n = 2(p-1)$. Recall the exact sequence

$$0 \to \bigoplus_{i=1}^{p-1} R(i) \to \Lambda \to \mathbb{Z}[C_{p-1}] \to 0,$$

from [7]. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^{p-1} R(i) \arrow[r] & \Lambda \arrow[r] \arrow[d, "Id"] & \mathbb{Z}[C_{p-1}] \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{j=1}^{n} R(b_j) \arrow[r] & \Lambda \arrow[r] & p(\Lambda) \arrow[r] & 0.
\end{tikzcd}$$

Now, as $Hom_{\Lambda}(\mathcal{T}_{p-1}, p(\Lambda)) = 0$ and $Hom_{\Lambda}(\mathcal{T}_{p-1}, \mathbb{Z}[C_{p-1}]) = 0$, $Id : \Lambda \to \Lambda$ induces and restricts to isomorphisms

$$\mathbb{Z}[C_{p-1}] \cong p(\Lambda),$$

and

$$\bigoplus_{i=1}^{p-1} R(i) \cong \bigoplus_{j=1}^{n} R(b_j),$$

respectively. Clearly $n = p-1$ and so $m = p-1$. We now wish to show that $p(S) \cong I(C_{p-1})$, to do this we require a proposition.

Proposition. *Let J be a $\mathbb{Z}[C_{p-1}]$-module such that*

$$J \oplus \mathbb{Z}[C_{p-1}] \cong I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}],$$

as $\mathbb{Z}[C_{p-1}]$-modules. Then $J \cong \mathbb{Z}[C_{p-1}]$.

Proof. Let J satisfy the above hypothesis. By stabilising the augmentation sequence on C_{p-1}, we construct the following exact sequence:

$$0 \to I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}] \to \mathbb{Z}[C_{p-1}]^{(2)} \to \mathbb{Z} \to 0.$$

By substituting $J \oplus \mathbb{Z}[C_{p-1}]$ for $I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]$, we construct the exact sequence

$$0 \to J \oplus \mathbb{Z}[C_{p-1}] \xrightarrow{j} \mathbb{Z}[C_{p-1}]^{(2)} \to \mathbb{Z} \to 0.$$

Let $T = \mathbb{Z}[C_{p-1}]^{(2)}/Im(j|_{\mathbb{Z}[C_{p-1}]})$, taking quotients, we can now construct the exact sequence

$$0 \to J \to T \to \mathbb{Z} \to 0.$$

By **Lemma 3.2.1**, T is projective and so the exact sequence

$$0 \to \mathbb{Z}[C_{p-1}] \xrightarrow{j} \mathbb{Z}[C_{p-1}]^{(2)} \to T \to 0,$$

splits, therefore $T \oplus \mathbb{Z}[C_{p-1}] \cong \mathbb{Z}[C_{p-1}]^{(2)}$. Now, $\mathbb{Z}[C_{p-1}]$ satisfies the Eichler condition and so by the Swan-Jacobinski theorem discussed in §5.2, $\mathbb{Z}[C_{p-1}]$

has stably free cancellation. Therefore, $T \cong \mathbb{Z}[C_{p-1}]$ and we have a short exact sequence

$$0 \to J \to \mathbb{Z}[C_{p-1}] \to \mathbb{Z} \to 0.$$

Up to sign, the augmentation mapping $\epsilon : \mathbb{Z}[C_{p-1}] \to \mathbb{Z}$ is the only surjective homomorphism $\mathbb{Z}[C_{p-1}] \to \mathbb{Z}$ and so $J \cong I(C_{p-1})$ as claimed. $\square$

Using this proposition, we see that $p(S) \cong I(C_{p-1})$ as a $\mathbb{Z}[C_{p-1}]$-module. Given that $p(S) \oplus p(\Lambda) \cong I(C_{p-1}) \oplus \mathbb{Z}[C_{p-1}]$ as Λ-modules, $\mathbb{Z}[C_p]$ clearly acts trivially on $p(S)$, and so $p(S) \cong I(C_{p-1})$ as a Λ-module. We now have an exact sequence

$$0 \to \bigoplus_{i=1}^{p-1} R(a_i) \to S \xrightarrow{p|_S} I(C_{p-1}) \to 0.$$

Now, $K_2 \cong \bigoplus_{i=1}^{p-1} R(i)$ and so

$$R(2) \oplus R(\hat{1}) \oplus \bigoplus_{i=1}^{p-1} R(i) \cong K_1 \oplus K_2 \cong \bigoplus_{j=1}^{p-1} R(a_j) \oplus \bigoplus_{i=1}^{p-1} R(i).$$

We can now use **Proposition 6.3.1.** to show that

$$R(2) \oplus R(\hat{1}) \cong \bigoplus_{j=1}^{p-1} R(a_j).$$

We have shown that we can express S as an extension Φ of $I(C_{p-1})$ by $R(2) \oplus R(\hat{1})$:

$$\Phi = (0 \to R(2) \oplus R(\hat{1}) \to S \to I(C_{p-1}) \to 0). \tag{5}$$

Which we wish to compare to

$$\mathcal{E} = (0 \to R(2) \oplus R(\hat{1}) \to R(2) \oplus [y-1] \to I(C_{p-1}) \to 0), \tag{3}$$

in order to show that $S \cong R(2) \oplus [y-1)$.

7.1.2 $Ext^1_\Lambda(I(C_{p-1}), R(2) \oplus R(\hat{1}))$

Let $\Lambda = \mathbb{Z}[G(p, p-1)]$ where p is an odd prime. We wish to find a practical description for $Ext^1_\Lambda(I(C_{p-1}), R(2) \oplus R(\hat{1}))$, in order to compare the extensions (3) and (5). We will use the exact sequence from **Proposition 3.2.2** to describe $Ext^1_\Lambda(I(C_{p-1}), R(2) \oplus R(\hat{1}))$ as a quotient of the additive abelian group $End_{\mathcal{D}er}(R(2) \oplus R(\hat{1}))$. Recall the short exact sequence

$$\mathcal{E} = (0 \to R(2) \oplus R(\hat{1}) \to R(2) \oplus [y-1) \to I(C_{p-1}) \to 0), \qquad (3)$$

Applying **Proposition 3.2.2** to $\mathcal{E}$, we find an exact sequence

$$Hom_{\mathcal{D}er}(I_{p-1}, R(2) \oplus R(\hat{1})) \to Hom_{\mathcal{D}er}(R(2) \oplus [y-1), R(2) \oplus R(\hat{1})) \quad (6)$$

$$\to End_{\mathcal{D}er}(R(2) \oplus R(\hat{1})) \to Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1}))$$

$$\to Ext^1_\Lambda(R(2) \oplus [y-1), R(2) \oplus R(\hat{1})) \to \dots$$

where, for brevity, we denote $I(C_{p-1})$ by I_{p-1}. We now find practical descriptions for some of the groups in this exact sequence, beginning with $Hom_{\mathcal{D}er}(I_{p-1}, R(2) \oplus R(\hat{1}))$. Let $t \in \mathcal{T}_{p-1}$, then by **Proposition 6.2.1** $t \cdot (1 + x + \dots + x^{p-1}) = 0$, and for $i \in I_{p-1}$ $i \cdot (1 + x + \dots + x^{p-1}) = p \cdot i$. Now, I_{p-1} is a Λ-lattice, and so $Hom_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1})) = 0$, therefore $Hom_{\mathcal{D}er}(I_{p-1}, R(2) \oplus R(\hat{1})) = 0$.

We now state without proof a proposition which will allow us to simplify the exact sequence (6).

Proposition. *([7], page 47, 5.8)*

$$Ext^1_\Lambda(I_{p-1}, R(k)) = \begin{cases} 0, & \text{if } k = 1 \\ \mathbb{Z}/p\mathbb{Z}, & \text{otherwise} \end{cases}$$

To calculate the second term, we require a lemma.

Lemma 7.1.1. $Hom_{\mathcal{D}er}([y-1), \mathcal{T}_{p-1}) = 0$

Proof. Recall that $[y-1)$ occurs in an exact sequence

$$0 \to R(\hat{1}) \to [y-1) \to I_{p-1} \to 0.$$

Applying this exact sequence to **Proposition 3.2.2** with $N = \mathcal{T}_{p-1}$, we construct the exact sequence

$$Hom_{\mathcal{D}er}(I_{p-1}, \mathcal{T}_{p-1}) \to Hom_{\mathcal{D}er}([y-1), \mathcal{T}_{p-1}) \to Hom_{\mathcal{D}er}(R(\hat{1}), \mathcal{T}_{p-1})$$

$$\to Ext^1_\Lambda(I_{p-1}, \mathcal{T}_{p-1}) \to Ext^1_\Lambda([y-1), \mathcal{T}_{p-1}) \to Ext^1_\Lambda(R(\hat{1}), \mathcal{T}_{p-1}).$$

By the above proposition, $Ext^1_\Lambda(I_{p-1}, \mathcal{T}_{p-1}) = (\mathbb{Z}/p\mathbb{Z})^{p-2}$, using this result in conjunction with **Proposition 6.3.1** the above exact sequence becomes

$$0 \to Hom_{\mathcal{D}er}([y-1), \mathcal{T}_{p-1}) \to (\mathbb{Z}/p\mathbb{Z})^{p-2} \to (\mathbb{Z}/p\mathbb{Z})^{p-2}$$

$$\to Ext^1_\Lambda([y-1), \mathcal{T}_{p-1}) \to \dots$$

Therefore, if we can show that $Ext^1_\Lambda([y-1), \mathcal{T}_{p-1}) = 0$, our proof of the lemma will be complete. Let $j : \mathbb{Z}[C_p] \to \Lambda$ be the standard inclusion map and let $[x-1)'$ be the right $\mathbb{Z}[C_p]$-module generated by $x-1$, then

$$\mathcal{T}_{p-1} \cong [x-1) \cong j_*([x-1)'),$$

by **Corollary 6.2.1.1**. We also note that $j^*([y-1]) \cong \mathbb{Z}[C_p]^{p-2}$, therefore, by the Eckmann-Shapiro lemma,

$$\begin{aligned} Ext^1_\Lambda([y-1], \mathcal{T}_{p-1}) &\cong Ext^1_\Lambda([y-1], j_*([x-1]')) \\ &\cong Ext^1_{\mathbb{Z}[C_p]}(\mathbb{Z}[C_p]^{p-2}, [x-1)') \\ &= 0. \end{aligned}$$

This completes the proof. $\square$

Collecting our results, we can rewrite the exact sequence (6) as

$$0 \to (\mathbb{Z}/p\mathbb{Z})^2 \to (\mathbb{Z}/p\mathbb{Z})^{p+1} \to (\mathbb{Z}/p\mathbb{Z})^{p-1} \xrightarrow{p^*} Ext^1_\Lambda(R(2) \oplus [y-1), R(2) \oplus R(\hat{1})) \to \dots$$

We deduce that $p^* = 0$, rewriting (6) a final time, we find the exact sequence

$$0 \to Hom_{\mathcal{D}er}(R(2) \oplus [y-1), R(2) \oplus R(\hat{1})) \xrightarrow{i^*} End_{\mathcal{D}er}(R(2) \oplus R(\hat{1}))$$

$$\xrightarrow{\delta} Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1})) \to 0.$$

We will now calculate $Im(i^*)$, and use this information to give a practical description for $Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1}))$. When considered as a matrix, elements f in $End_\Lambda(R(2) \oplus R(\hat{1}))$ take the form

$$f = \left(\begin{array}{c|c} f_1 : R(2) \oplus R(2) \to R(2) \oplus R(2) & f_2 : \bigoplus_{i \neq 1,2} R(i) \to R(2) \oplus R(2) \\ \hline f_3 : R(2) \oplus R(2) \to \bigoplus_{i \neq 1,2} R(i) & f_4 : \bigoplus_{i \neq 1,2} R(i) \to \bigoplus_{i \neq 1,2} R(i) \end{array} \right).$$

By **Proposition 6.3.1**, the element represented by f in the derived module category, $\overline{f}$ takes the form

$$\overline{f} = \left(\begin{array}{c|c} \overline{f_1} & 0 \\ \hline 0 & \overline{f_4} \end{array}\right) = \left(\begin{array}{cc|cccc} a_{1,1}+p\mathbb{Z} & a_{1,2}+p\mathbb{Z} & & & & \\ a_{2,1}+p\mathbb{Z} & a_{2,2}+p\mathbb{Z} & & & 0 & \\ \hline & & a_{3,3}+p\mathbb{Z} & & & \\ & & & a_{4,4}+p\mathbb{Z} & & \\ & 0 & & & \ddots & \\ & & & & & a_{p-1,p-1}+p\mathbb{Z} \end{array}\right),$$

Here, elements in the matrix are zero if not otherwise specified. The elements $a_{n,n}+p\mathbb{Z}$ are in $End_{\mathcal{D}er}(R(n))$ for $3 \leq n \leq p-1$. To find $Im(i^*)$, take a general element $f' \in Hom_\Lambda(R(2) \oplus [y-1], R(2) \oplus R(\hat{1}))$ which, when considered as a matrix takes the form

$$f' = \left(\begin{array}{c|c} f'_1 : R(2) \to R(2) \oplus R(2) & f'_2 : [y-1] \to R(2) \oplus R(2) \\ \hline f'_3 : R(2) \to \bigoplus_{i \neq 1,2} R(i) & f'_4 : [y-1] \to \bigoplus_{i \neq 1,2} R(i) \end{array}\right).$$

We proved earlier in this section that $Hom_{\mathcal{D}er}([y-1], \mathcal{T}_{p-1}) = 0$, and so the element represented by f' in the derived module category, $\overline{f}'$ is given by

$$\overline{f}' = \left(\begin{array}{c|c} \overline{f_1}' : R(2) \to R(2) \oplus R(2) & \overline{0} : [y-1] \to R(2) \oplus R(2) \\ \hline \overline{0} : R(2) \to \bigoplus_{i \neq 1,2} R(i) & \overline{0} : [y-1] \to \bigoplus_{i \neq 1,2} R(i) \end{array}\right) = \left(\begin{array}{c|c} \overline{f_1}' & 0 \\ \hline 0 & 0 \end{array}\right).$$

Now, $i^*(\overline{f'}) = \overline{f' \circ i} = \overline{f'} \circ \overline{i}$ and so

$$Im(i^*) = \{ \left(\begin{array}{cc|c} a_{1,1}+p\mathbb{Z} & 0+p\mathbb{Z} & \\ a_{2,1}+p\mathbb{Z} & 0+p\mathbb{Z} & 0 \\ \hline & 0 & 0 \end{array}\right) \in End_{\mathcal{D}er}(R(2) \oplus R(\hat{1})) \mid a_{1,1}, a_{2,1} \in \mathbb{Z} \}.$$

We conclude that given an α in $End_\Lambda(R(2) \oplus R(\hat{1}))$, which takes the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,p-2} & a_{1,p-1} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,p-2} & a_{2,p-1} \\ pa_{3,1} & pa_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,p-2} & a_{3,p-1} \\ pa_{4,1} & pa_{4,2} & pa_{4,3} & a_{4,4} & \cdots & a_{4,p-2} & a_{4,p-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ pa_{p-1,1} & pa_{p-1,2} & pa_{p-1,3} & pa_{p-1,4} & \cdots & pa_{p-1,p-2} & a_{p-1,p-1} \end{pmatrix},$$

the element in $End_{\mathcal{D}er}(R(2) \oplus \bigoplus_{i \neq 1} R(i))/Im(i^*)$ represented by α, which we denote by $[\alpha]$ is given by

$$[\alpha] = \left(\begin{array}{cc|cccc} 0 & a_{1,2} + p\mathbb{Z} & & & 0 & \\ 0 & a_{2,2} + p\mathbb{Z} & & & & \\ \hline & & a_{3,3} + p\mathbb{Z} & & & \\ & & & a_{4,4} + p\mathbb{Z} & & \\ & 0 & & & \ddots & \\ & & & & & a_{p-1,p-1} + p\mathbb{Z} \end{array}\right).$$

Using our rewritten version of (6), we know that

$$End_{\mathcal{D}er}(R(2) \oplus R(\hat{1})/Im(i^*) \cong Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1}))$$

via the mapping $[\alpha] \mapsto \alpha_*(\mathcal{E})$, we have shown:

Theorem 7.1.2. *Every element of* $Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1}))$ *can be written* *uniquely as* $\alpha_*(\mathcal{E})$*, where* $\alpha \in End_\Lambda(R(2) \oplus R(\hat{1}))$ *is given by a matrix*

$$\alpha = \left(\begin{array}{cc|cccc} 0 & a_{1,2} & & & & \\ 0 & a_{2,2} & & & 0 & \\ \hline & & a_{3,3} & & & \\ & & & a_{4,4} & & \\ & 0 & & & \ddots & \\ & & & & & a_{p-1,p-1} \end{array}\right),$$

where $a_{i,j} \in \mathbb{Z}$ *and* $0 \leq a_{i,j} \leq p-1$ *for each* i, j*. Elements of the matrix are zero if not otherwise specified.*

Recall Φ, the short exact sequence (5), we deduce that $\Phi = \alpha_*(\mathcal{E})$ for some $\alpha \in End_\Lambda(R(2) \oplus R(\hat{1}))$ of the above form. We now prove some lemmas which allow us to place restrictions on the form of α.

Lemma. *Assume that* $\alpha_*(\mathcal{E}) = \Phi$*, then it can not be true that* $a_{1,2} = a_{2,2} = 0$*.*

Proof. Recall that Φ takes the form

$$\Phi = (0 \to R(2) \oplus \bigoplus_{i \neq 1} R(i) \to S \to I(C_{p-1}) \to 0).$$

Assume that $a_{1,2} = a_{2,2} = 0$, then by **Proposition 2.3.1** S must take the form

$$S \cong R(2) \oplus R(2) \oplus X,$$

for some Λ-module X. We know that $S \oplus \Lambda \cong R(2) \oplus [y-1) \oplus \Lambda$, and so

$$Hom_{\mathcal{D}er}(R(2) \oplus R(2) \oplus X \oplus \Lambda, R(2)) \cong Hom_{\mathcal{D}er}(R(2) \oplus [y-1) \oplus \Lambda, R(2)),$$

but

$$Hom_{\mathcal{D}er}(R(2) \oplus R(2) \oplus X \oplus \Lambda, R(2)) \cong (\mathbb{Z}/p\mathbb{Z})^2 \oplus Hom_{\mathcal{D}er}(X, R(2)),$$

while

$$Hom_{\mathcal{D}er}(R(2) \oplus [y-1) \oplus \Lambda, R(2)) \cong \mathbb{Z}/p\mathbb{Z}.$$

This gives a contradiction, completing the proof. □

Similarly, we can prove the following:

Lemma. *Assuming that $\alpha_*(\mathcal{E}) = \Phi$, it can not be true that $a_{i,i} = 0$ for any i satisfying $3 \leq i \leq p-1$.*

7.1.3 Realising the matrices as automorphisms

In this section, we will prove that there must exist an automorphism

$$\alpha : R(2) \oplus R(\hat{1}) \to R(2) \oplus R(\hat{1}),$$

such that $\Phi = \alpha_*(\mathcal{E})$, by the Five lemma, this results leads to the conclusion that $R(2) \oplus [y-1) \cong S$. Let α be a general element of $End_\Lambda(R(2) \oplus R(\hat{1}))$, when considered as a matrix, α takes the form

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,p-2} & a_{1,p-1} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,p-2} & a_{2,p-1} \\
pa_{3,1} & pa_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,p-2} & a_{3,p-1} \\
pa_{4,1} & pa_{4,2} & pa_{4,3} & a_{4,4} & \cdots & a_{4,p-2} & a_{4,p-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
pa_{p-1,1} & pa_{p-1,2} & pa_{p-1,3} & pa_{p-1,4} & \cdots & pa_{p-1,p-2} & a_{p-1,p-1}
\end{pmatrix}.$$

Define a map

$$k : End_\Lambda(R(2) \oplus \bigoplus_{i \neq 1} R(i)) \to (\mathbb{Z}/p\mathbb{Z})^{p-1},$$

by

$$\alpha \mapsto (a_{1,2} + p\mathbb{Z}, a_{2,2} + p\mathbb{Z}, a_{3,3} + p\mathbb{Z}, a_{4,4} + p\mathbb{Z}, \ldots, a_{p-1,p-1} + p\mathbb{Z}).$$

By §7.1.2, $End_{\mathcal{D}er}(R(2) \oplus R(\hat{1}))/Im(i^*) \cong Ext^1_\Lambda(I_{p-1}, R(2) \oplus R(\hat{1}))$ via the pushout map δ, and so if $\alpha, \beta \in End_\Lambda(R(2) \oplus R(\hat{1}))$, then $[\alpha] = [\beta]$ if and only if $k(\alpha) = k(\beta)$. Therefore $\delta(\alpha) = \delta(\beta)$ if and only if $k(\alpha) = k(\beta)$. We noted in the lemmas at the end of the previous section that $\Phi = \alpha_*(\mathcal{E})$ for some α such that $k(\alpha)$ can have at most one zero entry, and $a_{i,i} + p\mathbb{Z} \neq 0 + p\mathbb{Z}$ for $3 \leq i \leq p-1$, therefore, if we can find an automorphism α for each of these cases, it must be true that $S \cong R(2) \oplus [y-1)$ by the Five lemma.

Define the set of units

$$U_{(a_{1,2}, a_{2,2}, \ldots, a_{p-1,p-1})} = \{\alpha \in Aut_\Lambda(R(2) \oplus R(\hat{1})) \mid k(\alpha) = (a_{1,2} + p\mathbb{Z}, a_{2,2} + p\mathbb{Z}, a_{3,3} + p\mathbb{Z}, \ldots, a_{p-1,p-1} + p\mathbb{Z})\}.$$

To prove that $S \cong R(2) \oplus [y-1)$, we will show that $U_{(a_{1,2}, a_{2,2}, \ldots, a_{p-1,p-1})}$ is non-empty whenever $a_{1,2}, a_{2,2}$ are not both zero $(modp)$ and $a_{i,i} \not\equiv 0 (modp)$ for $3 \leq i \leq p-1$. We begin by noting some generating elements:

$$f_{(n,1,1,\ldots,1)} = \left(\begin{array}{cc|c} 1 & n & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{p-3} \end{array}\right) \in U_{(n,1,1,\ldots,1)},$$

$$f_{(1,0,1,\dots,1)} = \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{0} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{p-3} \end{array}\right) \in U_{(1,0,1,\dots,1)},$$

$$f_{(0,2,1,1,\dots,1)} = \left(\begin{array}{cc|cccc} \frac{p+1}{2} & 0 & 1 & 0 & \dots & 0 \\ -1 & 2 & 0 & 0 & \dots & 0 \\ \hline 0 & p & 1 & & & \\ 0 & 0 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & 1 \end{array}\right) \in U_{(0,2,1,\dots,1)}.$$

Note that

$$U_{(a_{1,2},a_{2,2},\dots,a_{p-1,p-1})} \cdot f_{(0,2,1,\dots,1)} \subset U_{(2a_{1,2},2a_{2,2},a_{3,3},\dots a_{p-1,p-1})}.$$

Therefore, by considering automorphisms of the form

$$f_{(n,1,1,\dots,1)} \cdot f^a_{(0,2,1,1,\dots,1)} \text{ and } f_{(1,0,1,\dots,1)} \cdot f^a_{(0,2,1,1,\dots,1)},$$

one sees easily that $U_{(a_{1,2},a_{2,2},1,1\dots,1)}$ is non-empty whenever $a_{1,2}, a_{2,2}$ are not both zero.

We now note two more generating elements:

$$f_{(0,1,2,1,\dots,1)} = \left(\begin{array}{cc|cccc} \frac{p+1}{2} & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \hline p & 0 & 2 & & & \\ 0 & 0 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & 1 \end{array}\right) \in U_{(0,1,2,1,\dots,1)},$$

$$f_{(0,1,\frac{p+1}{2},2,1,\ldots,1)} = \left(\begin{array}{cc|cccc} 1 & 0 & & & & \\ 0 & 1 & & & 0 & \\ \hline & & \frac{p+1}{2} & 1 & & \\ & & p & 2 & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{array}\right) \in U_{(0,1,\frac{p+1}{2},2,1,\ldots,1)}.$$

By considering automorphisms of the form $f^a_{(0,1,2,1,\ldots,1)}$, we see that

$$U_{(0,1,a_{3,3},1,\ldots,1)} \text{ is non-empty whenever } a_{3,3} \not\equiv 0(modp).$$

Now, note that $f_{(0,1,2,1,\ldots,1)} \cdot f_{(0,1,\frac{p+1}{2},2,1,\ldots 1)} \in U_{(0,1,1,2,1,\ldots,1)}$. If we define the automorphism $f_{(0,1,1,2,1,\ldots,1)} = f_{(0,1,2,1,\ldots,1)} \cdot f_{(0,1,\frac{p+1}{2},2,1,\ldots 1)}$, then by considering automorphisms of the form $f^a_{(0,1,1,2,1,\ldots,1)}$ we can see similarly that

$$U_{(0,1,1,a_{4,4},\ldots,1)} \text{ is non-empty whenever } a_{4,4} \not\equiv 0(modp).$$

Repeating this process, we see that $U_{(0,1,1,\ldots a_{i,i},\ldots,1)}$ is non-empty whenever $a_{i,i} \not\equiv 0(modp)$ for $3 \leq i \leq p-1$. Now, note that

$$U_{(0,1,a_{3,3},1,\ldots,1)} \cdot U_{(0,1,1,a_{4,4},\ldots,1)} \cdot \cdots \cdot U_{(0,1,1,\ldots,a_{p-1,p-1})} \subset U_{(0,1,a_{3,3},a_{4,4},\ldots,a_{p-1,p-1})}.$$

We deduce that

$$U_{(0,1,a_{3,3},a_{4,4},\ldots,a_{p-1,p-1})} \text{ is non-empty whenever } a_{i,i} \not\equiv 0(modp)\ 3 \leq i \leq p-1.$$

Finally, we note that

$$f_{(a_{1,2},a_{2,2},1,\ldots,1)} \cdot U_{(0,1,a_{3,3},a_{4,4},\ldots,a_{p-1,p-1})} \subset U_{(a_{1,2},a_{2,2},a_{3,3},a_{4,4},\ldots,a_{p-1,p-1})}.$$

We have shown:

Proposition. *$U_{(a_{1,2},a_{2,2},\ldots,a_{p-1,p-1})}$ is non-empty whenever $a_{1,2}, a_{2,2}$ are not both zero $(modp)$ and $a_{i,i} \not\equiv 0 (modp)$ for $3 \leq i \leq p-1$*

Therefore, there exists an automorphism $\alpha \in Aut_\Lambda(R(2) \oplus \bigoplus_{i\neq 1} R(i))$ such that $\Phi = \alpha_*(\mathcal{E})$. In conclusion:

Theorem 7.1.3. *$S \cong R(2) \oplus [y-1)$ and so $[R(2) \oplus [y-1)]$ is straight over $\Lambda = \mathbb{Z}[G(p,p-1)]$ for any odd prime p.*

7.2 $R(2) \oplus [y-1)$ is full

In this section, we will show that for any odd prime p, $R(2) \oplus [y-1)$ is full over $\Lambda = \mathbb{Z}[G(p,p-1)]$. We will begin by showing that $R(2)$ and $[y-1)$ are both full Λ-modules, before building upon these results to show that $R(2) \oplus [y-1)$ is full.

To show that $R(2)$ is full, we must find the kernel of the Swan homomorphism discussed in §3.3, $S_{R(2)} : Aut_{\mathcal{D}er}(R(2)) \to \tilde{K}_0(\Lambda)$. Recall that $End_\Lambda(R(2)) = \{nId_{R(2)} \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, and $End_{\mathcal{D}er}(R(2)) \cong \mathbb{Z}/p\mathbb{Z}$, so $Aut_{\mathcal{D}er}(R(2)) = \{\overline{nId_{R(2)}} \mid 1 \leq n < p\} \cong (\mathbb{Z}/p\mathbb{Z})^*$, the units of $\mathbb{Z}/p\mathbb{Z}$. Recall the exact sequence

$$0 \to \bigoplus_{i=1}^{p-1} R(i) \xrightarrow{\iota} \Lambda \to \mathbb{Z}[C_{p-1}] \to 0,$$

which is in fact a quasi-augmentation sequence which satisfies the condition that $Hom_\Lambda(\mathbb{Z}[C_{p-1}], \bigoplus_{i=1}^{p-1} R(i)) = 0$. Using this exact sequence, we can form an exact sequence

$$0 \to R(2) \xrightarrow{\iota|_{R(2)}} \Lambda \to \Lambda/\iota(R(2)) \to 0.$$

In order to find $Ker(S_{R(2)})$, we must answer the question: 'for which $\overline{n} \in (\mathbb{Z}/p\mathbb{Z})^*$ is $\varinjlim(\overline{nId_{R(2)}}, \iota|_{R(2)})$ stably free?' As Λ satisfies the Eichler condition, and so by the Swan-Jacobinski Theorem satisfies SFC, we simply need to find when $\varinjlim(\overline{nId_{R(2)}}, \iota|_{R(2)}) \cong \Lambda$. Using the Five lemma, it is immediately obvious that for $\overline{n} = \overline{1}$, $\overline{n} = \overline{p} - \overline{1}$, $\varinjlim(\overline{nId_{R(2)}}, \iota|_{R(2)}) \cong \Lambda$. We will show that these are the only two choices for $\overline{n} \in (\mathbb{Z}/p\mathbb{Z})^*$ such that this isomorphism holds.

Let the mapping

$$\widetilde{nId_{R(2)}} : \bigoplus_{i=1}^{p-1} R(i) \to \bigoplus_{i=1}^{p-1} R(i),$$

be defined by

$$(r_1, r_2, r_3, \ldots, r_{p-1}) \mapsto (r_1, nr_2, r_3, \ldots r_{p-1}),$$

by **Proposition 2.3.2**, $\varinjlim(\widetilde{nId_{R(2)}}, \iota) \cong \varinjlim(nId_{R(2)}, \iota|_{R(2)})$. We can therefore use the Milnor square of the above quasi-augmentation sequence which also satisfies the extra condition $Hom_{\Lambda}(\mathbb{Z}[C_{p-1}], \bigoplus_{i=1}^{p-1}) = 0$ to calculate $Ker(S_{R(2)})$. The Milnor square takes the form

$$\begin{tikzcd}
\Lambda \arrow[r, "i_1"] \arrow[d, "i_2"'] & End_{\Lambda}(\mathbb{Z}[C_{p-1}]) \arrow[d, "j_1"] \\
End_{\Lambda}(\bigoplus_{i=1}^{p-1} R(i)) \arrow[r, "j_2"] & End_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i)).
\end{tikzcd}$$

Recall that in matrix form, $End_{\Lambda}(\bigoplus_{i=1}^{p-1} R(i)) \cong \mathcal{T}_{p-1}(\mathbb{Z}, p)$ and $End_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i))$ is the ring of $(p-1)\times(p-1)$ matrices with elements of $\mathbb{Z}/p\mathbb{Z}$ on the diagonal, and zeroes elsewhere. The map $j_1 : End_{\Lambda}(\bigoplus_{i=1}^{p-1} R(i)) \to End_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i))$, when considered in matrix form is given by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,p-1} \\ pa_{2,1} & a_{2,2} & \dots & a_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ pa_{p-1,1} & pa_{p-1,2} & \dots & a_{p-1,p-1} \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1}+p\mathbb{Z} & & & \\ & a_{2,2}+p\mathbb{Z} & & \\ & & \ddots & \\ & & & a_{p-1,p-1}+p\mathbb{Z} \end{pmatrix}.$$

By **Theorem 4.3.2**, $\varinjlim(\widetilde{nId_{R(2)}}, \iota) \cong M(End_\Lambda(\mathbb{Z}[C_{p-1}]), \mathcal{T}_{p-1}, \widetilde{nId_{R(2)}})$. As a result of our work in §4.2, projective modules of type $M(End_\Lambda(\mathbb{Z}[C_{p-1}]), \mathcal{T}_{p-1}, h)$ are in $1-1$ correspondence with the quotient set

$$j_2(End_\Lambda(\mathbb{Z}[C_{p-1}])^*)\backslash Aut_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i))/j_1(\mathcal{T}_{p-1}^*).$$

To show that $\varinjlim(\widetilde{nId_{R(2)}}, \iota) \ncong \Lambda$ when $n \not\equiv \pm 1 (mod p)$ it is therefore sufficient to show that the class of $\widetilde{Id_{R(2)}}$ is different to that of $\widetilde{nId_{R(2)}}$ in the above quotient set. In order to do this, we begin by simplifying our description of the quotient set.

Lemma.

$$j_2(End_\Lambda(\mathbb{Z}[C_{p-1}])^*)\backslash Aut_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i))/j_1(\mathcal{T}_{p-1}^*) \cong Aut_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i))/j_1(\mathcal{T}_{p-1}^*)$$

Proof. Take a unit $f \in End_\Lambda(\mathbb{Z}[C_{p-1}])$, f clearly lifts to a unit $f_\lambda \in End_\Lambda(\Lambda)$, which in turn lifts to an $\tilde{f} \in End_\Lambda(\bigoplus_{i=1}^{p-1} R(i))$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^{p-1} R(i) & \longrightarrow & \Lambda & \longrightarrow & \mathbb{Z}[C_{p-1}] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \bigoplus_{i=1}^{p-1} R(i) & \longrightarrow & \Lambda & \longrightarrow & \mathbb{Z}[C_{p-1}] & \longrightarrow & 0, \end{array}$$

commutes. By the Five lemma, $\tilde{f}$ is an isomorphism, and by our construction of the Milnor square, $j_1(\tilde{f}) = j_2(f)$. Therefore, there exists an $\tilde{f} \in \mathcal{T}^*_{p-1}$ such that $j_1(\tilde{f}) = j_2(f)$, completing the proof. $\square$

To show that $\varinjlim(\widetilde{nId_{R(2)}}, \iota) \ncong \Lambda$ for $n \not\equiv \pm 1(modp)$, it is now sufficient to show that no unit in $\mathcal{T}^*_{p-1}$ has $(1(modp), n(modp), 1(modp), \ldots, 1(modp))$ on the diagonal. Basic considerations relating to the determinant show that any such matrix would have determinant $n(modp)$, and so if $n \not\equiv \pm 1(modp)$, $\varinjlim(\widetilde{nId_{R(2)}}, \iota) \ncong \Lambda$. We have shown:

Proposition. *Over* $\Lambda = \mathbb{Z}[G(p, p-1)]$, $Ker(S_{R(2)}) = \{\pm Id\}$.

Now, $\{\pm Id\} \subset Im(v^{R(2)})$ and so $Im(v^{R(2)}) = Ker(S_{R(2)})$, concluding:

Proposition. *Over* $\Lambda = \mathbb{Z}[G(p, p-1)]$, $R(2)$ *is full.*

We will now show that $[y-1)$ is full, to do this, we will first consider the problem over $\mathbb{Z}[C_{p-1}]$. Let $[y-1)'$ be the right $\mathbb{Z}[C_{p-1}]$-module generated by $(y-1)$.

Proposition 7.2.1. *Over* $\mathbb{Z}[C_{p-1}]$, $[y-1)'$ *is full and*

$$S_{[y-1)'} : Aut_{\mathcal{D}er}([y-1)') \to \tilde{K}_0(\mathbb{Z}[C_{p-1}])$$

is the zero map.

Proof. Consider the standard augmentation sequence

$$0 \to [y-1)' \to \mathbb{Z}[C_{p-1}] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $\epsilon(y) = 1$. By **Proposition 3.2.4**, $Aut_{\mathcal{D}er}([y-1)') \cong Aut_{\mathcal{D}er}(\mathbb{Z})$, and so $Aut_{\mathcal{D}er}([y-1)') = \{\overline{nId}_{[y-1)'} \mid gcd(n, p-1) = 1; 1 \leq n \leq p-2\}$. We will find a representative for each element of $Aut_{\mathcal{D}er}([y-1)')$ which is also an element of $Aut_{\Lambda}([y-1)')$, and our result will follow by the Five lemma.

Fix an r such that $1 \leq r \leq p-2$ and $gcd(r, p-1) = 1$. Let

$$(1 + y + \cdots + y^{r-1}) : [y-1)' \to [y-1)'$$

be the $\mathbb{Z}[C_{p-1}]$-homomorphism given by multiplication by $(1+y+\cdots+y^{r-1})$.

Firstly, we will show that $(1 + y + \cdots + y^{r-1}) : [y-1)' \to [y-1)'$ is an automorphism by showing that $y-1$ lies in its image. By assumption, r is coprime to $p-1$, so there exists an s such that $rs \equiv 1(mod(p-1))$. Clearly $y^r - 1, y^{2r} - y^r, \ldots, y^{sr} - y^{(s-1)r}$ are elements of $Im(1 + y + \cdots + y^{r-1})$, and their sum is $y-1$. Therefore $(1 + y + \cdots + y^{r-1}) \in Aut_{\Lambda}([y-1)')$.

To complete the proof, it remains only to show that

$$\overline{(1 + y + \cdots + y^{r-1})} = \overline{rId} \in End_{\mathcal{D}er}([y-1)').$$

To prove this, it is sufficient to show that

$$-r + (1 + y + \cdots + y^{r-1}) \in [y-1)'.$$

But

$$\begin{aligned} -r + (1 + y + \cdots + y^{r-1}) &= -(r-1) + y + y^2 + \cdots + y^{r-1}, \\ &= (y-1) + (y^2 - 1) + \cdots + (y^{r-1} - 1) \in [y-1)'. \end{aligned}$$

Therefore, each $\overline{f} \in Aut_{\mathcal{D}er}([y-1)')$ lifts to an automorphism over Λ and $S_{[y-1)'} = 0$, as required. □

Before we can extend this proposition to Λ-modules, we first need a description for $End_{\mathcal{D}er}([y-1))$.

Proposition 7.2.2. *Over* $\Lambda = \mathbb{Z}[G(p, p-1)]$,

$$End_{\mathcal{D}er}([y-1)) = \{\overline{nId}_{[y-1)} \mid 0 \leq n \leq p-2\} \cong \mathbb{Z}/(p-1)\mathbb{Z}.$$

Proof. Consider the exact sequence of Λ-modules and homomorphisms

$$0 \to [y-1) \to \Lambda \to [\Sigma_y) \to 0,$$

where $\Sigma_y = 1 + y + \cdots + y^{p-2}$. By **Proposition 3.2.2**,

$$End_{\mathcal{D}er}([y-1)) \cong Ext^1_{\Lambda}([\Sigma_y), [y-1)).$$

Let $j : \mathbb{Z}[C_{p-1}] \hookrightarrow \Lambda$ be the standard inclusion, and let $[y-1)'$ be the right $\mathbb{Z}[C_{p-1}]$-module generated by $(y-1)$. By the Eckmann-Shapiro lemma,

$$Ext^1_{\Lambda}([\Sigma_y), [y-1)) \cong Ext^1_{\mathbb{Z}[C_{p-1}]}(j^*([\Sigma_y)), [y-1)').$$

We claim that $j^*([\Sigma_y)) \cong \mathbb{Z} \oplus \mathbb{Z}[C_{p-1}]$. Fix the basis $\{\Sigma_y, \Sigma_y \cdot x, \ldots, \Sigma_y \cdot x^{p-1}\}$ for $[\Sigma_y)$. The action of y on the basis gives rise to a representation of $j^*([\Sigma_y))$ of the form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array}\right) \in GL_p(\mathbb{Z}),$$

where X is a $(p-1) \times (p-1)$ matrix with exactly one non-zero element in each row and column, and every non-zero element is 1. Clearly X represents a $\mathbb{Z}[C_{p-1}]$-module which is isomorphic to $\mathbb{Z}[C_{p-1}]$, so $j^*([\Sigma_y)) \cong \mathbb{Z} \oplus \mathbb{Z}[C_{p-1}]$

as claimed. Therefore

$$\begin{aligned} End_{\mathcal{D}er}([y-1)) &\cong Ext^1_{\Lambda}([\Sigma_y), [y-1)) \\ &\cong Ext^1_{\mathbb{Z}[C_{p-1}]}(j^*[\Sigma_y), [y-1)') \\ &\cong Ext^1_{\mathbb{Z}[C_{p-1}]}(\mathbb{Z} \oplus \mathbb{Z}[C_{p-1}], [y-1)') \\ &\cong Ext^1_{\mathbb{Z}[C_{p-1}]}(\mathbb{Z}, [y-1)') \\ &\cong \mathbb{Z}/(p-1)\mathbb{Z}. \end{aligned}$$

Here, the final isomorphism follows from **Proposition 3.2.4** and the existence of the exact sequence

$$0 \to [y-1)' \to \mathbb{Z}[C_{p-1}] \to \mathbb{Z} \to 0.$$

$\square$

We can now extend **Proposition 7.2.1** to an analogous result over Λ.

Proposition 7.2.3. *Over* Λ, $[y-1)$ *is full and*

$$S_{[y-1)} : Aut_{\mathcal{D}er}([y-1)) \to \tilde{K}_0(\mathbb{Z}[C_{p-1}])$$

is the zero map.

Proof. Let $j : \mathbb{Z}[C_{p-1}] \to \Lambda$ be the standard inclusion. Fix an r such that $1 \leq r \leq p-2$ and $gcd(r, p-1) = 1$. By **Proposition 7.2.1** a commutative diagram of $\mathbb{Z}[C_{p-1}]$ modules and homomorphisms with exacts rows of the form

$$\begin{tikzcd} 0 \arrow[r] & {[y-1)'} \arrow[r] \arrow[d, "(1+y+\cdots+y^{r-1})"] & \mathbb{Z}[C_{p-1}] \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & {[y-1)'} \arrow[r] & \mathbb{Z}[C_{p-1}] \arrow[r] & \mathbb{Z} \arrow[r] & 0, \end{tikzcd}$$

exists. By applying j_* to the above diagram, we construct a second commutative diagram with exact rows, this time made up of Λ-modules and homomorphisms,

$$\begin{tikzcd}
0 \arrow[r] & {[y-1)} \arrow[r] \arrow[d, "(1+y+\cdots+y^{r-1})"] & \Lambda \arrow[r] \arrow[d, "\wr"] & {[\Sigma_y)} \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & {[y-1)} \arrow[r] & \Lambda \arrow[r] & {[\Sigma_y)} \arrow[r] & 0,
\end{tikzcd}$$

where $\Sigma_y = 1 + y + \cdots + y^{p-2}$. By **Proposition 7.2.2**,

$$Aut_{\mathcal{D}er}([y-1)) = \{\overline{nId}_{[y-1)} \mid gcd(n, p-1) = 1, 1 \leq n \leq p-2\},$$

and $\overline{(1 + y + \cdots + y^{r-1})} = \overline{r} \in End_{\mathcal{D}er}([y-1))$ by the same argument used over $\mathbb{Z}[C_{p-1}]$. This leads to our result. $\square$

We have shown that both $R(2)$ and $[y-1)$ are full Λ-modules. In order to extend these results to $R(2) \oplus [y-1)$, we require a lemma.

Lemma 7.2.4. *Let A, B be full Λ-modules. If A, B satisfy the following properties:*

- $Hom_{\mathcal{D}er}(A, B) = 0$*;*
- $Hom_{\mathcal{D}er}(B, A) = 0$*;*
- $S_B : Aut_{\mathcal{D}er}(B) \to \tilde{K}_0(\Lambda)$ *is the zero mapping.*

Then $A \oplus B$ is full.

Proof. Let $\overline{f}$ be an element of $Ker(S_{A\oplus B})$, then

$$\overline{f} = \begin{pmatrix} \overline{f_1} : A \to A & \overline{f_2} : B \to A \\ \overline{f_3} : A \to B & \overline{f_4} : B \to B \end{pmatrix} = \begin{pmatrix} \overline{f_1} & \overline{0} \\ \overline{0} & \overline{f_4} \end{pmatrix} \in Aut_{\mathcal{D}er}(A \oplus B),$$

So $\overline{f_1} \in Aut_{\mathcal{D}er}(A)$ and $\overline{f_4} \in Aut_{\mathcal{D}er}(B)$. Now, $S_{A\oplus B}(\overline{f}) = S_A(\overline{f_1}) + S_B(\overline{f_4})$ by **Proposition 2.3.1**, by our hypothesis, $S_B = 0$ and so $S_{A\oplus B}(\overline{f}) = S_A(\overline{f_1})$. We assumed that $\overline{f} \in Ker(S_{A\oplus B})$ and so $S_A(\overline{f_1}) = 0$, but A is full and so $\overline{f_1} \in Im(v^A) = Ker(S_A)$. Finally, $\overline{f_4} \in Aut_{\mathcal{D}er}(B)$, but $S_B = 0$ and B is full, therefore $\overline{f}_4 \in Ker(S_B) = Im(v^B)$. Concluding, because $\overline{f_1} \in Im(v^A)$ and $\overline{f}_4 \in Im(v^B)$, $\overline{f} \in Im(v^{A\oplus B})$, completing the proof. □

By **Lemma 7.1.1** and **Proposition 7.2.3**, $Hom_{\mathcal{D}er}([y-1), R(2)) = 0$ and $S_{[y-1)} = 0$. To utilise the above lemma to show that $R(2) \oplus [y-1)$ is full, it remains only to show that $Hom_{\mathcal{D}er}(R(2), [y-1)) = 0$, we instead prove the following, stronger result:

Proposition. $Hom_{\mathcal{D}er}(\mathcal{T}_{p-1}, [y-1)) = 0$.

Proof. Recall the exact sequence

$$0 \to \mathcal{T}_{p-1} \to \Lambda \to \mathbb{Z}[C_{p-1}] \to 0,$$

from [7]. By applying **Proposition 3.2.2** to the above exact sequence with $N = [y-1)$, we see that

$$Hom_{\mathcal{D}er}(\mathcal{T}_{p-1}, [y-1)) \cong Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], [y-1)).$$

Let $j : \mathbb{Z}[C_{p-1}] \hookrightarrow \Lambda$ be the standard inclusion mapping. If $[y-1)'$ is the right $\mathbb{Z}[C_{p-1}]$-module generated by $(y-1)$, then $j_*([y-1)') = [y-1)$, and $j^*(\mathbb{Z}[C_{p-1}]) = \mathbb{Z}[C_{p-1}]$. Therefore, by the Eckmann-Shapiro lemma

$$
\begin{aligned}
Hom_{\mathcal{D}er}(\mathcal{T}_{p-1}, [y-1)) &\cong Ext^1_{\Lambda}(\mathbb{Z}[C_{p-1}], [y-1)) \\
&\cong Ext^1_{\mathbb{Z}[C_{p-1}]}(\mathbb{Z}[C_{p-1}], [y-1)') \\
&= 0.
\end{aligned}
$$

□

Collecting our results:

Theorem 7.2.5. *$R(2) \oplus [y-1)$ is full over $\Lambda = \mathbb{Z}[G(p, p-1)]$.*

8 The condition M(7)

In this chapter, we will begin by defining a condition $M(p)$ over $\mathbb{Z}[G(p,p-1)]$ where p is an odd prime, which we will use to give a practical sufficient condition for the $D(2)$-property to hold over $\mathbb{Z}[G(p,p-1)]$. We will then prove a theorem which significantly shortens the calculations necessary to show that the condition $M(p)$ holds. We will close the chapter by proving that the condition $M(7)$ holds, which in turn leads to one of our main results, namely that D(2)-property holds for the group $G(7,6)$.

8.1 The condition M(p)

Let $\Lambda = \mathbb{Z}[G(p,p-1)]$ where p is an odd prime, we define the condition **M(p)** on Λ as follows:

M(p) : The third syzygy of $\mathbb{Z}$ over $\Lambda = \mathbb{Z}[G(p,p-1)]$ is the stable module $[R(2) \oplus [y-1)]$.

Basic considerations relating to the rank of $R(2) \oplus [y-1)$ as a $\mathbb{Z}$-module show that if **M(p)** is satisfied, then $R(2) \oplus [y-1)$ is in fact a minimal representative for $\Omega_3^{G(p,p-1)}(\mathbb{Z})$. Therefore, if **M(p)** is satisfied, our results relating to $R(2) \oplus [y-1)$ in §7 mean that all three conditions in **Theorem 5.5.1** are satisfied, we have shown:

Theorem 8.1.1. *Let $\Lambda = \mathbb{Z}[G(p,p-1)]$, if Λ satisfies **M(p)**, then $G(p,p-1)$ satisfies the D(2)-property.*

It has already been shown [12] that **M(5)** is satisfied, this leads to one of our main theorems.

Theorem 8.1.2. $G(5,4)$ *satisfies the D(2)-property*

The remainder of this thesis will be devoted to showing that **M(7)** is satisfied.

8.2 A theorem relating to the condition $M(p)$

In this section, we take p to be an odd prime and let $\Lambda = \mathbb{Z}[G(p, p-1)]$. We wish to study $Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], R(\hat{n}))$ by expressing its elements as pushouts of elements of $Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], \bigoplus_{i=1}^{p-1} R(i))$. Recall the exact sequence,

$$\Psi = (0 \to \bigoplus_{i=1}^{p-1} R(i) \xrightarrow{j} \Lambda \to \mathbb{Z}[C_{p-1}] \to 0),$$

from [7]. Applying **Proposition 3.2.2** to Ψ with $N = R(\hat{n})$, we find that

$$Hom_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n})) \cong Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], R(\hat{n})),$$

via the map

$$\delta_* : Hom_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n})) \to Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], R(\hat{n})),$$

$$\overline{f} \mapsto f_*(\Psi).$$

Given our discussion in §6.3, we can explicitly describe the additive groups $Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ and $Hom_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ in matrix form:

Let $f \in Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$, we can think of f as a $(p-2) \times (p-1)$

matrix such that:

$$(f)_{i,j} \in \begin{cases} Hom_\Lambda(R(j), R(i)), & \text{if } i \leq n-1 \\ Hom_\Lambda(R(j), R(i+1)), & \text{if } i \geq n. \end{cases}$$

This is equivalent to

$$(f)_{i,j} \in \begin{cases} \mathbb{Z}, & \text{if } i \leq n-1 \text{ and } j \geq i \\ p\mathbb{Z}, & \text{if } i \leq n-1 \text{ and } i \geq j+1 \\ \mathbb{Z}, & \text{if } i \geq n \text{ and } j \geq i+1 \\ p\mathbb{Z}, & \text{if } i \geq n \text{ and } i+1 \geq j+1. \end{cases}$$

The representative of $f \in Hom_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$, $\overline{f}$ can then be expressed as a $(p-2) \times (p-1)$ matrix such that

$$(\overline{f})_{i,j} \in \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } i = j \text{ and } i \leq n-1 \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } i+1 = j \text{ and } i \geq n \\ \{0\}, & \text{otherwise.} \end{cases}$$

Note that the standard projection map

$$Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n})) \to Hom_{\mathcal{D}er}(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$$

takes the obvious form when considered in matrix form.

Consider an $f \in Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ in matrix form, we define an additive group homomorphism

$$k : Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n})) \to (\mathbb{Z}, p\mathbb{Z})^{p-2},$$

by

$$f \mapsto (f_{1,1} + p\mathbb{Z}, f_{2,2} + p\mathbb{Z}, \dots, f_{n-1,n-1} + p\mathbb{Z}, f_{n,n+1} + p\mathbb{Z}, \dots, f_{p-2,p-1} + p\mathbb{Z}).$$

Note that k descends to an isomorphism in the derived module category i.e. $k(f) = k(f')$ if and only if $\overline{f} = \overline{f'}$. Therefore $k(f) = k(f')$ if and only if $\delta_*(f) = \delta_*(f')$. It is now clear that for each $\Phi \in Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], R(\hat{n}))$, there exists an $f \in Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ such that $\Phi = \delta_*(f)$, moreover $\delta_*(f) = \delta_*(f')$ if and only if $k(f) = k(f')$, we therefore classify $Ext^1_\Lambda(\mathbb{Z}[C_{p-1}], R(\hat{k}))$ by the *k-invariants* of f, which we define to be

$$f_{1,1} + p\mathbb{Z}, f_{2,2} + p\mathbb{Z}, \dots, f_{n-1,n-1} + p\mathbb{Z}, f_{n,n+1} + p\mathbb{Z}, \dots, f_{p-2,p-1} + p\mathbb{Z}.$$

We aim to show that if there exists a short exact sequence

$$\mathcal{S} = (0 \to R(\hat{n}) \to K \to \mathbb{Z}[C_{p-1}] \to 0),$$

with all non-zero k-invariants, then there exists an imbedding $i : R(n) \hookrightarrow \Lambda$ such that $K \cong \Lambda / i(R(n))$.

We begin by defining the set $U_{(x_1, x_2, \dots, x_{n-1}, x_{n+1}, x_{n+2}, \dots x_{p-1})}$ as follows:

$$\{u \in \mathcal{T}_{p-1}(\mathbb{Z}, p)^* \mid u \text{ has } (y_1, y_2, \dots, y_{n-1}, \star, y_{n+1}, \dots, y_{p-1}) \text{ on the diagonal}$$
$$\text{and } y_i \equiv x_i (mod p) \text{ for all } i\},$$

where $\star$ can represent any integer.

Lemma. *For any $(p-2)$-tuple of integers $(c_1, \dots c_{n-1}, c_{n+1}, \dots, c_{p-1})$ which are not in $p\mathbb{Z}$, the set $U_{(c_1, \dots c_{n-1}, c_{n+1}, \dots, c_{p-1})}$ is non-empty.*

Proof. By Bézout's lemma, $U_{(c_1,1,1,\dots)}, U_{(1,c_2,1,\dots,1)}, \dots U_{(1,1,\dots,1,c_{p-1})}$ are all non-empty. By taking the product of an element from each of these sets, we construct an element of $U_{(c_1,\dots c_{k-1},c_{k+1},\dots,c_{p-1})}$, completing the proof. $\square$

Fix a $(p-2)$-tuple of integers $(d_1, d_2, \dots, d_{n-1}, d_{n+1}, \dots, d_{p-1})$ which are not in $p\mathbb{Z}$. We will find an $f \in Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ such that

$$k(f) = (d_1 + p\mathbb{Z}, d_2 + p\mathbb{Z}, \dots, d_{n-1} + p\mathbb{Z}, d_{n+1} + p\mathbb{Z}, \dots, d_{p-1} + p\mathbb{Z}),$$

and $\delta_*(f)$ takes the form

$$\delta_*(f) = (0 \to R(\hat{n}) \to \Lambda/R(n) \to \mathbb{Z}[C_{p-1}] \to 0.$$

By the lemma, $U_{(d_1,d_2,\dots,d_{n-1},d_{n+1},\dots,d_{p-1})}$ is non-empty, and so contains an element u, we can therefore push out Ψ along u to give the following.

$$\begin{tikzcd}
\mathcal{E} = (0 \arrow[r] & \bigoplus_{i=1}^{p-1} R(i) \arrow[r, "j"] \arrow[d, "u"] & \Lambda \arrow[r] \arrow[d] & \mathbb{Z}[C_{p-1}] \arrow[r] \arrow[d, "Id"] & 0) \\
u_*(\mathcal{E}) = (0 \arrow[r] & \bigoplus_{i=1}^{p-1} R(i) \arrow[r, "i"] & \Lambda \arrow[r] & \mathbb{Z}[C_{p-1}] \arrow[r] & 0),
\end{tikzcd}$$

where the central mapping is an isomorphism by the Five lemma. Define $h \in Hom_\Lambda(\bigoplus_{i=1}^{p-1} R(i), R(\hat{n}))$ to be the $(p-2) \times (p-1)$ matrix described by

$$(h)_{i,j} = \begin{cases} 1, & i = j \text{ and } i \leq n-1 \\ 1, & i+1 = j \text{ and } n \geq k \\ 0, & \text{otherwise.} \end{cases}$$

We can simply think of h as the mapping

$$h : R(\hat{n}) \oplus R(n) \to R(\hat{n})$$
$$(\hat{n}, n) \mapsto \hat{n}.$$

By **Proposition 2.3.3**, if $i : \bigoplus_{i=1}^{p-1} R(i) \hookrightarrow \Lambda$ is an injection, then an isomorphism $\varinjlim(h,i) \cong \Lambda/i(R(n))$ exists. This leads to the conclusion that $\varinjlim(h \circ u, j) \cong \Lambda/i(R(n))$. But

$$k(h \circ u) = (d_1 + p\mathbb{Z}, d_2 + p\mathbb{Z}, \ldots, d_{n-1} + p\mathbb{Z}, d_{n+1} + p\mathbb{Z}, \ldots, d_{p-1} + p\mathbb{Z}).$$

Therefore, if we define $f = h \circ u$, then

$$k(f) = (d_1 + p\mathbb{Z}, d_2 + p\mathbb{Z}, \ldots, d_{n-1} + p\mathbb{Z}, d_{n+1} + p\mathbb{Z}, \ldots, d_{p-1} + p\mathbb{Z}),$$

and

$$\delta_*(f) = 0 \to R(\hat{k}) \to \Lambda/i(R(n)) \to \mathbb{Z}[C_{p-1}] \to 0.$$

Concluding:

Theorem 8.2.1. *For every extension*

$$O \to R(\hat{n}) \to K \to \mathbb{Z}[C_{p-1}] \to 0,$$

with all non-zero k-invariants, there exists an imbedding $i : R(n) \to \Lambda$ such that $K \cong \Lambda/i(R(n))$.

This theorem is useful in showing that the condition $M(p)$ holds, as if we can show that a surjective homomorphism $\pi : \Lambda \to R(1)$ exists with kernel K, and that K is an extension of $\mathbb{Z}[C_{p-1}]$ by $R(\hat{2})$ with all non-zero k-invariants, then $K \cong \Lambda/R(2)$. Given that the Kernel of the augmentation map $\epsilon : \Lambda \to \mathbb{Z}$ is isomorphic to $R(1) \oplus [y-1)$ [7], the isomorphism $K \cong \Lambda/R(2)$ then leads to the conclusion that $\Omega_3^{G(p,p-1)}(\mathbb{Z}) = [R(2) \oplus [y-1)]$. This is the scheme of proof which was used in [12] to show that the condition $M(5)$ holds. We will use this scheme later in this chapter to show that the condition $M(7)$ holds.

8.3 $\mathcal{T}_6(\mathbb{Z},7)$

For the remainder of this thesis, we will work towards a proof that the condition $M(7)$ holds. Fix the presentation

$$G(7,6) = < x, y \mid y^6 = 1, x^7 = 1, yx = x^3 y >,$$

for the group $G(7,6)$. Let Λ be the integral group ring of $G(7,6)$. In this section, by studying the representation $\lambda : G(7,6) \to \mathcal{T}_6(\mathbb{Z},7)^*$ described in §6.2, we will find characteristic equations for each of the rows of $\mathcal{T}_6(\mathbb{Z},7)$. We will then use these characteristic equations to find representations for each of the rows of $\mathcal{T}_6(\mathbb{Z},7)$. For brevity, we abbreviate $\mathcal{T}_6(\mathbb{Z},7)$ to $\mathcal{T}_6$.

As in §6.2, we take ζ to be a primitive 7^{th} root of unity, we define the mapping $\theta : \mathbb{Z}(\zeta) \to \mathbb{Z}(\zeta)$ to be the obvious extension from the map defined by $\zeta \mapsto \zeta^3$. We then define a right action of $G(7,6)$ on $\mathbb{Z}(\zeta)$ by,

$$\mathbb{Z}(\zeta) \times G(7,6) \to \mathbb{Z}(\zeta),$$
$$z \cdot (x^r y^s) = \theta^{-s}(z \cdot \zeta^{-r}).$$

Now, as $\mathbb{Z}(\zeta) \cong_{\mathbb{Z}} \mathbb{Z}^6$ with $\mathbb{Z}$-basis given by

$$\{(\zeta-1)^5, (\zeta-1)^4, (\zeta-1)^3, (\zeta-1)^2, (\zeta-1)^1, 1\},$$

the above action defines a group representation $\lambda : G(7,6) \to GL_6(\mathbb{Z})$. In fact, as discussed in §6.2, λ is a group homomorphism $\lambda : G(7,6) \to \mathcal{T}_6$ which extends to a ring isomorphism $\tilde{\lambda_*} : \mathcal{C}_6(\mathbb{Z}(\zeta),\theta) \xrightarrow{\sim} \mathcal{T}_6(\mathbb{Z},7)$. One can calculate $\lambda(x^{-1})$ and $\lambda(y^{-1})$ by hand, giving the following result:

$$\lambda(x^{-1}) = \begin{pmatrix} -6 & 1 & 0 & 0 & 0 & 0 \\ -21 & 1 & 1 & 0 & 0 & 0 \\ -35 & 0 & 1 & 1 & 0 & 0 \\ -35 & 0 & 0 & 1 & 1 & 0 \\ -21 & 0 & 0 & 0 & 1 & 1 \\ -7 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\lambda(y^{-1}) = \begin{pmatrix} 5 & -5 & 3 & -1 & 0 & 0 \\ 35 & -31 & 18 & -6 & 0 & 0 \\ 105 & -84 & 48 & -17 & 1 & 0 \\ 175 & -126 & 70 & -26 & 3 & 0 \\ 161 & -105 & 56 & -21 & 3 & 0 \\ 70 & -42 & 21 & -7 & 0 & 1 \end{pmatrix}.$$

The Λ-action on $\mathcal{T}_6$ is given by

$$t \cdot x = t \cdot \lambda(x^{-1}),$$

and

$$t \cdot y = t \cdot \lambda(y^{-1}).$$

Let $v_1 = (20,\ -10,\ 4,\ -1,\ 0,\ 0) \in \mathcal{T}_6$. One can easily check that $v_1 \cdot y = -v_1 \cdot (1 + x)$. Now, $[v_1)$ is a right Λ-module with Λ-action given by the representation λ. Also, since $v_1 \in R(1), [v_1) \subset R(1)$, but $R(1)$ is generated by $(1,\ 0,\ 0,\ 0,\ 0,\ 0)$ and

$$(20,\ -10,\ 4,\ -1,\ 0,\ 0)\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -21 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (1,\ 0,\ 0,\ 0,\ 0,\ 0),$$

therefore $[v_1) = R(1)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(1)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;
- $M(1)$: M has a generator v_1 such that $v_1 \cdot y = -v_1 \cdot (1 + x)$.

Similarly, if we define $v_2 = (7,\ -1,\ 0,\ 0,\ 0,\ 0) \in \mathcal{T}_6$, then it is easily checked that $v_2 \cdot y = v_2 \cdot (1 + x)^2$ and $[v_2) = R(2)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(2)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;
- $M(2)$: M has a generator v_2 such that $v_2 \cdot y = v_2 \cdot (1 + x)^2$.

Similarly, if we define $v_3 = (21,\ -7,\ 1,\ 1,\ -1,\ 0)$, then $v_3 \cdot y = -v_3$ and $[v_3) = R(3)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(3)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;

- $M(3)$: M has a generator v_3 such that $v_3 \cdot y = -v_3$.

Similarly, if we define $v_4 = (0,\ 0,\ 0,\ 1,\ -2,\ 2)$, then $v_4 \cdot y = v_4 \cdot (1+x)$ and $[v_4) = R(4)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(4)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;
- $M(4)$: M has a generator v_4 such that $v_4 \cdot y = v_4 \cdot (1+x)$.

Similarly, if we define $v_5 = (77,\ -49,\ 28,\ -14,\ 6,\ -2)$, then one can easily check that $v_5 \cdot y = -v_5 \cdot (1+x)^2$ and $[v_5) = R(5)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(5)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;
- $M(5)$: M has a generator v_5 such that $v_5 \cdot y = -v_5 \cdot (1+x)^2$.

Finally, if we define $v_6 = (7,\ -7,\ 7,\ -7,\ 7,\ -6)$, then $v_6 \cdot y = v_6$ and $[v_6) = R(6)$. We deduce that a six-dimensional Λ-lattice M is isomorphic to $R(6)$ if and only if the following conditions are satisfied:

- $M(\Sigma)$: $m \cdot (1 + x + x^2 + \cdots + x^6) = 0$ for each $m \in M$;
- $M(6)$: M has a generator v_6 such that $v_6 \cdot y = v_6$.

Using the above characteristic equations, we can find representations $\theta_k : \mathbb{Z}[G(7,6)] \to \mathcal{T}_6$ for each row $R(k)$ of $\mathcal{T}_6$. We give these explicitly now:

$$\theta_1(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_1(y^{-1}) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\theta_2(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_2(y^{-1}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 2 & -2 & 0 & 1 & 0 & -1 \\ 1 & -2 & 0 & 2 & -1 & -1 \\ 0 & -2 & 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix},$$

$$\theta_3(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_3(y^{-1}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\theta_4(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_4(y^{-1}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\theta_5(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_5(y^{-1}) = \begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix},$$

$$\theta_6(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ \theta_6(y^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

8.4 The mapping $\pi_* : \Lambda \to [\pi)$

In this section, we define an element $\pi \in \Lambda$ and show that $[\pi) \cong R(1)$ using the characteristic equations from the §8.3. We then go on to find the kernel of the map $\pi_* : \Lambda \to [\pi)$ defined by $\alpha \mapsto \pi \cdot \alpha$.

We begin by defining

$$\pi = (x-1)((2+x^2+x^5)y + (-1+x^2+2x^3+2x^4+x^5)y^2 + y^3)(1-y^3).$$

Let $v = \pi(1+x^2)$, Clearly $[v) \subset [\pi)$, but

$$\begin{aligned} v(1+x)(1+x^4) &= \pi(1+x^2)(1+x)(1+x^4) \\ &= \pi(1+x+x^2+\cdots+x^7) \\ &= \pi \end{aligned}$$

and so $[v) = [\pi)$. One can check easily that $v \cdot y = -v \cdot (1+x)$, and so since $[\pi)$ is clearly a 6-dimensional Λ-lattice, we have shown the following:

Proposition. *The right Λ-module $[\pi)$ is isomorphic to $R(1)$.*

We therefore have an explicit description of a surjection $\pi_* : \Lambda \to R(1)$, we define $K = \mathrm{Ker}(\pi_*)$. We now proceed to find a $\mathbb{Z}$-basis for K which includes a $\mathbb{Z}$-basis for $R(1,3,4,5,6)$. This will be followed by a proof that $K/R(1,3,4,5,6) \cong \mathbb{Z}[C_6]$, and so K is described by an extension of the type mentioned in §8.2. Utilising the identity $(1-y^3)(1+y^3) = 0$, we see that $[y^3+1) \cap [x-1) \subset K$. We now define a $\mathbb{Z}$-basis $\{e(i)\}_{1 \leq i \leq 18}$ for $[y^3+1) \cap [x-1)$:

- $e(i) = (y^3 + 1)(x - 1)x^{i-1}$, $1 \leq i \leq 6$;
- $e(i) = (y^4 + y)(x - 1)x^{i-7}$, $7 \leq i \leq 12$;
- $e(i) = (y^5 + y^2)(x - 1)x^{i-13}$, $13 \leq i \leq 18$.

If we define the representation $L' : \Lambda \to GL_6(\mathbb{Z})$ by

$$L'(x^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}; \ L'(y^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

standard calculations on the basis $\{e(i)\}_{1 \leq i \leq 18}$ produce the representation $L : \Lambda \to GL_{18}(\mathbb{Z})$ for $[y^3 + 1) \cap [x - 1)$ given by

$$L(x^{-1}) = \begin{pmatrix} L'(x^{-1}) & 0 & 0 \\ 0 & L'(x^{-1}) & 0 \\ 0 & 0 & L'(x^{-1}) \end{pmatrix};$$

and

$$L(y^{-1}) = \begin{pmatrix} 0 & 0 & L'(y^{-1}) \\ L'(y^{-1}) & 0 & 0 \\ 0 & L'(y^{-1}) & 0 \end{pmatrix}.$$

In order to show that $[y^3 + 1) \cap [x - 1) \cong R(1, 3, 5)$, we begin by defining the following representation $\theta_{1,3,5} : \Lambda \to GL_{18}(\mathbb{Z})$ for $R(1, 3, 5)$ using the representations found in §8.3:

$$\theta_{1,3,5}(x^{-1}) = \begin{pmatrix} \theta_1(x^{-1}) & 0 & 0 \\ 0 & \theta_3(x^{-1}) & 0 \\ 0 & 0 & \theta_5(x^{-1}) \end{pmatrix};$$

and

$$\theta_{1,3,5}(y^{-1}) = \begin{pmatrix} \theta_1(y^{-1}) & 0 & 0 \\ 0 & \theta_3(y^{-1}) & 0 \\ 0 & 0 & \theta_5(y^{-1}) \end{pmatrix}.$$

Now, if we define

$$h = \begin{pmatrix}
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1
\end{pmatrix}$$

Then h has determinant 1 and satisfies the following:

- $h \cdot \theta_{1,3,5}(x^{-1}) \cdot h^{-1} = L(x^{-1})$;
- $h \cdot \theta_{1,3,5}(y^{-1}) \cdot h^{-1} = L(y^{-1})$.

We deduce that $[y^3+1) \cap [x-1) = [\eta(1)) \dotplus [\eta(3)) \dotplus [\eta(5))$ where $[\eta(i)) \cong R(i)$ and the elements $\eta(i)$ are defined as follows:

- $\eta(1) = h(v_1) = (y^3+1)(1+y+y^2)(x^5-x^4)$;
- $\eta(3) = h(v_3) = (y^3+1)(-(x^4-x^3)+y(x^6-x)+y^2(x^5-x^2))$;
- $\eta(5) = h(v_5) = (y^3+1)(-(x^6-x^5)+y((x^6-x^5)+(x^4-x^3)+(x-1))+y^2(-(x^4-x^3)-(x-1))$.

Now, as $\{e(i)\}_{1\leq i\leq 18} \cup \{y^i \cdot x^j \mid 0 \leq i \leq 2,\ 0 \leq j \leq 6\} \cup \{y^3, y^4, y^5\}$ is a basis for Λ, and $\{e(i)\}_{1\leq i\leq 18}$ is a basis for $[y^3+1) \cap [x-1)$, $\Lambda/[y^3+1) \cap [x-1)$ is torsion free. From the exact sequence

$$0 \to K/[y^3+1) \cap [x-1) \to \Lambda/[y^3+1) \cap [x-1) \to R(1) \to 0,$$

we can now deduce that $K/[y^3+1) \cap [x-1)$ is torsion free. We therefore have the following short exact sequence of Λ-lattices:

$$0 \to [y^3+1) \cap [x-1) \to K \to K/[y^3+1) \cap [x-1) \to 0.$$

Using this exact sequence, we can form a basis for K using bases for $[y^3+1) \cap [x-1)$ and $K/[y^3+1) \cap [x-1)$. As $[y^3+1) \cap [x-1) \cong R(1,3,5)$, in order to find a basis for K which contains a basis for $R(1,3,4,5,6)$, we will find elements $\eta(4), \eta(6) \in K$ such that there exists a basis for $K/[y^3+1) \cap [x-1)$ which contains the set $\{\eta(i)X^j \mid i = 4, 6;\ j = 0, 1, \ldots, 5\}$ and $[\eta(i)) \cong R(i)$ for $i = 4, 6$. Here, we use capitalisation to represent the image of x in $K/[y^3+1) \cap [x-1)$. We define $\eta(4), \eta(6)$ as follows,

- $\eta(4) = (x-1)((1+x^5) + (-x+x^4+x^5)y + (x^2+x^3+x^4+x^5)y^2 + (x^3 + x^5)y^3 + (-1-x-2x^2-x^3-x^4)y^4 + (-x^2-x^3-x^4)y^5)$,
- $\eta(6) = (x-1)(1+x^5)(1+y+y^2+y^3+y^4+y^5)$.

Through tedious calculations one can check that $\pi \cdot \eta(4) = 0$ and $\pi \cdot \eta(6) = 0$ and so $\eta(4), \eta(6) \in K$; it is immediately clear that $\eta(6) \cdot y = \eta(6)$, and through another tedious calculation one can check that $\eta(4) \cdot y = \eta(4) \cdot (1+x)$, therefore $[\eta(i)) \cong R(i)$ for $i = 4, 6$. We will now find a basis for $K/[y^3+1) \cap [x-1)$, before transforming the basis to find a basis which contains $\{\eta(i)X^j \mid i = 4, 6 \; j = 0, 1, \ldots, 5\}$. We begin by expressing the mapping

$$\overline{\pi_*} : \Lambda/[y^3+1) \cap [x-1) \to [\pi), \quad \alpha + [y^3+1) \cap [x-1) \mapsto \pi \cdot \alpha$$

in matrix form, where we take $\Lambda/[y^3+1) \cap [x-1)$ to have basis

$$\{1, X, X^2, X^3, X^4, X^5, X^6, Y, YX, YX^2, YX^3, YX^4, YX^5, YX^6$$
$$Y^2, Y^2X, Y^2X^2, Y^2X^3, Y^2X^4, Y^2X^5, Y^2X^6, Y^3, Y^4, Y^5\}.$$

Here, capitalisation is used to represent the image in $\Lambda/[y^3+1) \cap [x-1)$. We take $[\pi)$ to have basis

$$\{\pi, \pi \cdot x, \pi \cdot x^2, \pi \cdot x^3, \pi \cdot x^4, \pi \cdot x^5\}.$$

Now, $\pi(1+x^2)y = -\pi(1+x^2)(1+x)$ and so

$$\begin{aligned}\pi y &= \pi(1+x^2)(1+x)(1+x^4)y \\ &= \pi(1+x^2)y(1+x^5)(1+x^6) \\ &= -\pi \cdot (1+x^2)(1+x)(1+x^5)(1+x^6), \\ &= \pi(-1+x^2+2x^3+2x^4+x^5)\end{aligned}$$

Using this equality, we can form the following matrix for $\overline{\pi_*}$:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 2 & -1 & 1 & 0 & -1 & 1 & -2 & -1 & 1 & -2 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & -2 & -1 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & -3 & -2 & 0 & 2 & 3 & 1 & -1 & 2 & 0 & 0 & 0 & -2 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 2 & 0 & -2 & -3 & -1 & 1 & 3 & 0 & 0 & 0 & 2 & -1 & 1 & -2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 1 & -1 & -2 & -2 & 0 & 2 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 2 & -2 & 0 & -1 & -1
\end{pmatrix}.$$

By utilising elementary linear algebra we can now easily find a $\mathbb{Z}$-basis for $K/[y^3+1) \cap [x-1)$. By expressing $\{\eta(i)X^j \mid i = 4, 6\ , j = 0, 1, \ldots, 5\}$ in terms of the basis for $\mathrm{Ker}(\overline{\pi_*})$ and utilising the Smith Normal Form, we construct the following basis for $K/[y^3+1) \cap [x-1)$:

$$\begin{aligned}
&\{\,\eta(4)X^i \mid 0 \le i \le 6\} \cup \{\eta(6)X^i \mid 0 \le i \le 6\} \cup \{1 + Y^3, -2 - X^2 - X^5 + Y^2, \\
&\ 2 + X^2 + X^5 + Y^5, -1 + X^2 + 2X^3 + 2X^4 + X^5 + Y^4, 1 - X^2 - 2X^3 - 2X^4 - X^5 + Y, \\
&\ 1 + X + X^2 + X^3 + X^4 + X^5 + X^5\}.
\end{aligned}$$

We therefore have a basis for K which includes bases for $[\eta(i))$ for $i = 1, 3, 4, 5, 6$. Using basic linear algebra one can calculate the representation $\rho : \Lambda \to GL_6(\mathbb{Z})$ for $K/R(1, 3, 4, 5, 6)$. The result it as follows.

$$\rho(x^{-1}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} ; \ \rho(y^{-1}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 4 & 5 & -5 & 7 \\
0 & -3 & 3 & 3 & -3 & 5
\end{pmatrix}.$$

Now, if we let

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & -2 & -1 & 1 \end{pmatrix}.$$

Then $f^{-1}\rho(g)f = \sigma(g)$ for each $g \in \Lambda$, where $\sigma(g)$ is the regular representation of the Λ-module $\mathbb{Z}[C_6]$. We deduce that K lies in a short exact sequence of the form

$$0 \to R(1,3,4,5,6) \to K \to \mathbb{Z}[C_6] \to 0.$$

By **Theorem 8.2.1**, to prove that $K \cong \Lambda/i(R(2))$ for some injective Λ-homomorphism $i : R(2) \to \Lambda$, it is sufficient to show that the above exact sequence has all non-zero k-invariants. The representation $\varphi : \Lambda \to GL_{36}(\mathbb{Z})$ of K given by the above exact sequence after the change of basis defined by f takes the following form:

$$\varphi(x^{-1}) = \begin{pmatrix} \theta_1(x^{-1}) & 0 & 0 & 0 & 0 & C(1) \\ 0 & \theta_3(x^{-1}) & 0 & 0 & 0 & C(3) \\ 0 & 0 & \theta_4(x^{-1}) & 0 & 0 & C(4) \\ 0 & 0 & 0 & \theta_5(x^{-1}) & 0 & C(5) \\ 0 & 0 & 0 & 0 & \theta_6(x^{-1}) & C(6) \\ 0 & 0 & 0 & 0 & 0 & I_6 \end{pmatrix};$$

$$\varphi(y^{-1}) = \begin{pmatrix} \theta_1(y^{-1}) & 0 & 0 & 0 & 0 & D(1) \\ 0 & \theta_3(y^{-1}) & 0 & 0 & 0 & D(3) \\ 0 & 0 & \theta_4(y^{-1}) & 0 & 0 & D(4) \\ 0 & 0 & 0 & \theta_5(y^{-1}) & 0 & D(5) \\ 0 & 0 & 0 & 0 & \theta_6(y^{-1}) & D(6) \\ 0 & 0 & 0 & 0 & 0 & \sigma(y^{-1}) \end{pmatrix}.$$

Now, to check that the k-invariant corresponding to $Ext^1_\Lambda(\mathbb{Z}[C_6], R(i))$ for $i = 1, 3, 4, 5, 6$ is non-zero, we must show that the extension defined by

$$\varphi_i(x^{-1}) = \begin{pmatrix} \theta_i(x^{-1}) & C(i) \\ 0 & I_6 \end{pmatrix};\ \varphi_i(y^{-1}) = \begin{pmatrix} \theta_i(y^{-1}) & D(i) \\ 0 & \sigma(y^{-1}) \end{pmatrix},$$

is not congruent to the trivial extension in $Ext^1_\Lambda(R(i), \mathbb{Z}[C_6])$ for any i. This is equivalent to showing that there is no matrix $\psi_i \in GL_{12}(\mathbb{Z})$ of the form

$$\psi_i = \begin{pmatrix} I_6 & X_i \\ 0 & I_6 \end{pmatrix}$$

such that

$$\begin{pmatrix} I_6 & X_i \\ 0 & I_6 \end{pmatrix} \begin{pmatrix} \theta_i(x^{-1}) & C(i) \\ 0 & I_6 \end{pmatrix} = \begin{pmatrix} \theta_i(x^{-1}) & 0 \\ 0 & I_6 \end{pmatrix} \begin{pmatrix} I_6 & X_i \\ 0 & I_6 \end{pmatrix},$$

and

$$\begin{pmatrix} I_6 & X_i \\ 0 & I_6 \end{pmatrix} \begin{pmatrix} \theta_i(y^{-1}) & D(i) \\ 0 & \sigma(y^{-1}) \end{pmatrix} = \begin{pmatrix} \theta_i(y^{-1}) & 0 \\ 0 & \sigma(y^{-1}) \end{pmatrix} \begin{pmatrix} I_6 & X_i \\ 0 & I_6 \end{pmatrix}.$$

This is equivalent to showing that there is no $X_i \in M_{6\times 6}(\mathbb{Z})$ such that

$$C(i) = (\theta_i(x^{-1}) - I_6)X_i \tag{7}$$

and

$$D(i) + X_i\sigma(y^{-1}) = \theta_i(y^{-1})X_i \tag{8}$$

In our case, we calculate the $C(i)$ and $D(i)$ to be:

$$C(1) = \begin{pmatrix} -5 & -2 & 2 & 5 & 2 & -2 \\ -4 & 1 & 2 & 4 & -2 & -1 \\ -5 & -3 & 2 & 4 & 3 & -1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & -3 & -1 & 0 & 2 & 2 \end{pmatrix};\ C(3) = \begin{pmatrix} 7 & 3 & -2 & -8 & -1 & 1 \\ 5 & -1 & -2 & -6 & 3 & 1 \\ 14 & 4 & -6 & -14 & -4 & 6 \\ 7 & -5 & -4 & -8 & 7 & 3 \\ 8 & 4 & -3 & -7 & -3 & 1 \\ 3 & 3 & 0 & -4 & -1 & -1 \end{pmatrix};$$

$$C(4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 2 & 1 \\ 2 & 0 & -1 & -2 & 0 & 1 \\ 1 & -2 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};\ C(5) = \begin{pmatrix} -7 & 0 & 4 & 6 & 1 & -4 \\ -4 & 3 & 3 & 3 & -3 & -2 \\ 0 & -2 & 0 & -2 & 2 & 2 \\ 7 & -3 & -4 & -8 & 3 & 5 \\ 10 & -2 & -5 & -11 & 3 & 5 \\ 5 & -2 & -3 & -5 & 2 & 3 \end{pmatrix};$$

$$C(6) = \begin{pmatrix} 2 & 0 & -1 & -2 & 0 & 1 \\ 2 & 0 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$D(1) = \begin{pmatrix} 1 & -1 & 2 & -1 & 1 & -2 \\ 1 & 0 & 5 & -1 & 0 & -5 \\ 2 & 2 & 6 & -2 & -2 & -6 \\ 2 & 1 & 5 & -2 & -1 & -5 \\ 0 & -1 & 3 & 0 & 1 & -3 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{pmatrix};\ D(3) = \begin{pmatrix} -7 & 3 & 4 & 7 & -3 & -4 \\ -2 & -2 & 0 & 2 & 2 & 0 \\ -7 & -2 & -3 & 7 & 2 & 3 \\ -6 & -1 & -7 & 6 & 1 & 7 \\ -4 & -3 & -7 & 4 & 3 & 7 \\ -9 & -2 & -1 & 9 & 2 & 1 \end{pmatrix};$$

$$D(4) = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}; \ D(5) = \begin{pmatrix} 3 & 2 & 7 & -3 & -2 & -7 \\ 4 & 3 & 13 & -4 & -3 & -13 \\ 3 & 4 & 13 & -3 & -4 & -13 \\ 4 & 2 & 6 & -4 & -2 & -6 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -3 & -4 & -1 & 3 & 4 \end{pmatrix};$$

$$D(6) = \begin{pmatrix} -1 & 0 & -1 & 1 & 0 & 1 \\ -2 & 0 & -2 & 2 & 0 & 2 \\ 0 & -1 & -3 & 0 & 1 & 3 \\ -3 & 0 & -1 & 3 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -2 & 1 & 1 & 2 & -1 & -1 \end{pmatrix}.$$

One can check easily that there are no solutions $X_i \in M_{6\times 6}(\mathbb{Z})$ for equation (7) for $i = 1, 3, 4, 5, 6$. We have shown that the kernel K of the map π_*, lies in an extension of the form

$$0 \to R(1,3,4,5,6) \to K \to \mathbb{Z}[C_{p-1}] \to 0$$

with all non-zero k-invariants. Therefore, by **Theorem 8.2.1**, and our discussion at the end of §8.2, we have shown:

Theorem 8.4.1. *Over* $\Lambda = \mathbb{Z}[G(7,6)]$, $\Omega_3(\mathbb{Z}) = [R(2) \oplus [y-1)]$ *i.e. the condition* $M(7)$ *holds.*

Therefore, by **Theorem 8.1.1**

Theorem 8.4.2. *The D(2)-property holds for* $G = G(7,6)$.

References

[1] BJ Birch: *Cyclotomic fields and Kummer extensions; page 85-93 of 'Algebraic Number Theory' JWS Cassels and A Fröhlich (Editors). Academic Press 1965*

[2] CW Curtis & I Reiner: *Methods of Representation Theory, Volume II, John Wiley and Sons, 1987*

[3] T Edwards: *Algebraic 2-complexes over certain infinite abelian groups, Doctoral Thesis, University College London, 2006*

[4] FEA Johnson: *Explicit homotopy equivalences in dimension two, Mathematical Proceedings of the Cambridge Philosophical Society, Volume 133, Issue 3, 2002.*

[5] FEA Johnson: *Stable Modules and the D(2)-Problem, Cambridge University Press, 2003.*

[6] FEA Johnson: *Syzygyies and Homotopy Theory, Springer, 2011.*

[7] FEA Johnson, JJ Remez: *Diagonal resolutions for metacyclic groups, Journal of Algebra, Vol 474, 329-360, 2017.*

[8] S Maclane: *Homology, Springer-Verlag, 1963*

[9] BA Magurn: *An Algebraic Introduction to K-Theory, Cambridge University Press, 2002*

[10] WH Mannan: *Realizing algebraic 2-complexes by cell complexes, Math. Proc. Camb, Phil. Soc., 146, 2009, 671-673*

[11] J Milnor: *Introduction to algebraic K-Theory, Princeton University Press and University of Tokyo Press, 1971*

[12] AJ Nadim: *A periodic monogenic resolution, Doctoral Thesis, University College London, 2015*

[13] MI Rosen: *Representations of Twisted Group Rings, Doctoral Thesis, Princeton University, 1963*

[14] JR Stallings: *On torsion free groups with infinitely many ends, Ann. of Math., 84 (1962) 387-399*

[15] RG Swan: *Groups of cohomological dimension one, J. of algebra, 12 (1969) 585-610*

[16] CTC Wall: *Finiteness conditions for CW-complexes. Ann. of Math. 81 (1965) 56-69.*

[17] JW Wamsley: *The Deficiency of Metacyclic Groups, Proceedings of the American Mathematical Society Vol. 24, No. 4 (Apr. 1970), pp. 724-726 (3 pages)*