HIGHER ORDER DEFORMATIONS OF HYPERBOLIC SPECTRA

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In memory of Erik Balslev

Abstract. This is an expanded writeup of a talk given by the second author at Erik Balslev's 75th birthday conference on October 1-2, 2010 at Aarhus University. We summarize our work on Fermi's golden rule and higher order phenomena for hyperbolic manifolds, a topic which occupied the last part of Erik Balslev's research.

1. Introduction

In 1911 Herman Weyl [Wey11] proved, that the number $N(\lambda)$ of eigenvalues λ_n less than λ of the Dirichlet Laplacian on a bounded domain $X \subseteq \mathbb{R}^n$ with sufficiently nice boundary has the following asymptotic behaviour:

$$N(\lambda) \sim \frac{\omega_d \mathrm{vol}(X)}{(2\pi)^d}\lambda^{d/2}, \text{ as } \lambda \to \infty. \tag{1.1}$$

Here ω_d is the volume of the unit ball in $\mathbb{R}^n$, and $\mathrm{vol}(X)$ is the volume of X. For us it is useful to know that Weyl's law holds for compact Riemannian manifolds, see [MP49, Bus92]. Weyl's law has been generalized and extended to many other cases, see e.g. [Ivr16].

In a seemingly unrelated direction Erich Hecke [Hec36] showed that the zeta function ζ_K of an *imaginary* quadratic field K is related in a simple manner to a certain modular form through Mellin transform. A modular form of weight k is a holomorphic function on the upper half-plane $\mathbb{H}$ such that the differential $f(z)(dz)^{k/2}$ is invariant under the action of certain subgroups of the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Hans Maass [Maa49] investigated whether an analogous relation were true for *real* quadratic fields. This led him to consider eigenfunctions of the hyperbolic Laplacian

$$-\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

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on square integrable functions f on the upper half-plane $\mathbb{H}$ transforming as

$$f(\gamma z) = \chi(\gamma) f(z), \text{ for } \in \mathbb{H}, \gamma \in \Gamma.$$

Here Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}$ by linear fractional transformations, and $\chi : \Gamma \to S^1$ is a unitary character. We denote the induced automorphic Laplacian by L. Such eigenfunctions have since been called Maass forms. Maass managed to show that – at least for certain Hecke congruence groups $\Gamma_0(N)$ and Dirichlet characters $\chi \mod N$ – such forms exist. Moreover he showed that these forms are related to zeta functions of real quadratic fields in a way similar to how zeta functions of imaginary quadratic fields are related to modular forms. At that time it was not clear whether a single Maass form existed for $\mathrm{SL}_2(\mathbb{Z})$, $\chi = 1$. On the other hand, Maass constructed non-holomorphic Eisenstein series, i.e. generalized eigenfunctions of L, but these are not square integrable.

Roelcke [Roe53, Roe66] and Selberg [Sel56, Sel89] gave a detailed description of the spectrum of L when the hyperbolic volume $\mathrm{vol}(\Gamma\backslash\mathbb{H})$ is finite. It consists of two parts:

(i) A discrete set of eigenvalues

$$0 \leq \lambda_0 \leq \lambda_1 \leq \dots \lambda_n \leq \dots$$

This part may be finite or infinite, and does not have accumulation points.

(ii) Furthermore, if $\Gamma\backslash\mathbb{H}$ is not compact, then the spectrum also contains a continuous part $[1/4, \infty[$ with multiplicity equal to the number of inequivalent open cusps for (Γ, χ).

The continuous spectrum associated with the cusp $\mathfrak{a}$ is provided by Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ for $s = 1/2 + it$, provided the cusp $\mathfrak{a}$ is open, i.e. for its stabilizer $\Gamma_{\mathfrak{a}}$ in Γ we have $\chi(\Gamma_{\mathfrak{a}}) = 1$.

We denote by $N_d(\lambda) = \#\{\lambda_n \leq \lambda\}$ the counting function for the *discrete* part. Using his newly developed trace formula, Selberg proved the following groundbreaking result: if $\Gamma\backslash\mathbb{H}$ is not compact, but Γ is a congruence group and χ a Dirichlet character, then the set of discrete eigenvalues satisfies Weyl's law, i.e.

$$N_d(\lambda) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda, \text{ as } \lambda \to \infty. \tag{1.2}$$

Roelcke and Selberg independently speculated about the behaviour of $N_d(\lambda)$ for general cofinite groups, i.e. Γ with $\mathrm{vol}(\Gamma\backslash\mathbb{H}) < \infty$, and general characters χ. The belief that $N_d(\lambda) \to \infty$ as $\lambda \to \infty$ for such groups has been called the *Roelcke–Selberg conjecture* by several authors, even if it is unclear to what extent Roelcke and Selberg formally stated it as a conjecture. Even if there may be an infinite number of Maass forms, they may or may not satisfy Weyl's law.

The difficulty of this conjecture lies in trying to count the discrete eigenvalues embedded in the continuous spectrum $[1/4, \infty[$. The general belief in the conjecture weakened after the work of Phillips and Sarnak on the stability of eigenvalues, see [PS85b, PS85a, Sar90, PS92] and our description below. A similar stability phenomenon occurs in the study of the Schrödinger Hamiltonian for the helium atom, see [Sim73]. In the physics literature embedded eigenvalues tend to be unstable and turn into scattering poles or resonances under perturbation. Resonances are poles of the analytic continuation of the resolvent in a second sheet (as opposed to the physical plane). The same phenomenon is true for the hyperbolic Laplacian. With the parametrization $\lambda = s(1-s)$ the second sheet corresponds to the left half-plane $\operatorname{Re} s < 1/2$. The instability of embedded eigenvalues for the Schrödinger operator is described by Fermi's Golden Rule, proved rigorously by Simon in [Sim73] following the work on analytic dilations of Balslev and Combes [BC71]. Phillips and Sarnak turned their attention to the analogous situation for hyperbolic surfaces with cusps.

Motivated by Selberg's trace formula, Phillips and Sarnak defined the singular set. For a given eigenvalue λ_j we consider the two values s_j counting multiplicity satisfying $\lambda_j = s_j(1-s_j)$. *The singular set* is then defined as follows: It is the multiset consisting of

(i) s_j counted with the multiplicity of the corresponding eigenvalue.
(ii) ρ_j the poles of the scattering determinant $\varphi(s)$ counted with multiplicity the order of the pole.
(iii) $1/2$ with multiplicity $(n+\operatorname{tr}(\Phi(1/2)))/2$ where n is the number of open cusps of (Γ, χ) and Φ is the scattering matrix related to (Γ, χ).

We refer to [Sel89] for the definition of scattering matrix etc. Note that the above definition differs form [PS92] by a rotation by i followed by a shift of $1/2$. Using the Lax–Phillips scattering theory [LP76] as applied to automorphic functions, Phillips and Sarnak showed that the singular set is better behaved under deformations than the discrete spectrum. We consider the following three types of deformations of (Γ, χ):

(i) Character deformations defined by
$$\begin{aligned} \chi_\varepsilon : &\Gamma \to S^1 \\ &\gamma \mapsto \exp\left(2\pi i \varepsilon \int_{z_0}^{\gamma z_0} \alpha\right), \end{aligned}$$
where $\alpha = \operatorname{Re}(f(z)dz)$ is a real Γ-invariant holomorphic 1-form, and ε is a real parameter.
(ii) Real analytic deformations in Teichmüller space generated by f a holomorphic cusp form of weight 4, see [PS85a] for details.

(iii) Real analytic compact deformations in the set of admissible surfaces, i.e. Riemannian surfaces of finite area with hyperbolic ends, see [Mül92] for details.

Phillips and Sarnak proved that in the cases (i), (ii) the singular set has at most algebraic singularities. It follows in particular that, if $s(0)$ has multiplicity one, then $s(\varepsilon)$ is analytic for small ε. Müller [Mül92] extended this to case (iii), and Balslev [Bal97] gave a different proof.

In all three cases described above the Laplacian $L(\varepsilon)$ admits a real analytic expansion

$$L(\varepsilon) = L(0) + \varepsilon L^{(1)} + \varepsilon^2 \frac{L^{(2)}}{2} + \cdots$$

after possibly making a suitable conjugation, and adjustments of the corresponding metric high in the cusps. See Section 2.2 for additional details.

Phillips and Sarnak identified a condition that will ensure that an embedded eigenvalue $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2 > 1/4$ will dissolve into a resonance when $\varepsilon \neq 0$. For simplicity we restrict ourselves to the case of only one open cusp. Let $E(z, 1/2+it)$ be the generalized eigenfunction for the continuous spectrum at $1/4+t^2$ (see Section 2.1). Let furthermore $\hat{s}_j(\varepsilon)$ be the weighted mean of the branches of the singular points generated by splitting the eigenvalue $s_j(0) = s_j$ of multiplicity m under perturbation, i.e.

$$\hat{s}_j(\varepsilon) = \frac{1}{m}\sum_{k=1}^{m} s_{j,k}(\varepsilon).$$

Let $u_{j,1}, \dots u_{j,m}$ be an orthonormal basis of the eigenspace of λ_j.

Theorem 1.1 (Fermi's Golden Rule)**.** *If $\left\langle L^{(1)}u_{j,k}, E(\cdot, 1/2+it_j)\right\rangle \neq 0$ for some k, then some of the eigenvalues with eigenvalue λ_j turn into resonances under the perturbation. More precisely:*

$$\operatorname{Re}\hat{s}_j^{(2)}(0) = -\frac{1}{4t_j^2}\sum_{k=1}^{m}\left|\left\langle L^{(1)}u_{j,k}, E(\cdot, 1/2+it_j)\right\rangle\right|^2.$$

For $m=1$ this is Eq. (5.29) in [PS92]. For $m>1$ this is discussed in [Pet94a].

Since the singular spectrum cannot move to the right under perturbation we always have $\operatorname{Re}\hat{s}_j^{(1)}(0) = 0$, so $\operatorname{Re}\hat{s}_j^{(2)}(0)$ determines if $\hat{s}_j^{(2)}(0)$ moves to the left up to second order.

It turns out that for Γ a Hecke congruence group and perturbations of type (i) and (ii) the dissolving condition

$$\left\langle L^{(1)}u_{j,k}, E(\cdot, 1/2+it_j)\right\rangle \neq 0, \text{ for some } k = 1, \dots, l. \tag{1.3}$$

is equivalent to the nonvanishing of a special value of a Rankin–Selberg L-function. This allows one to use techniques from the analytic theory of L-functions to investigate how many eigenvalues are dissolved under perturbations. Luo [Luo01] succeeded in proving that a positive proportion of these special values of Rankin–Selberg L-functions are indeed non-zero. Assuming that multiplicities of the eigenvalues for the Laplacian of a fixed Hecke congruence groups are all bounded by the same common bound – which is indeed expected to hold – this allowed him to prove that a small deformation of the Hecke congruence groups does *not* satisfy Weyl's law, i.e. (1.2) does *not* hold.

Phillips and Sarnak [DIPS85] conjectured something much stronger: the generic cofinite hyperbolic surface should only have finitely many discrete eigenvalues.

1.1. **Erik Balslev's interest in spectral deformations in the context of hyperbolic surfaces.** Throughout his career Erik Balslev was interested in various properties of spectra of Schrödinger operators. One of his major contributions to this field was the use of analytic dilation techniques in the setting of quantum mechanical many–body systems; see [BC71].

Balslev knew Ralph Phillips from his time in the United States in the 1960s and 1970s, and from Phillips numerous long-term visits to Denmark. Balslev was employed at Aarhus University, Denmark during most of his career. In the autumn of 1991 Balslev was visiting Stanford and found himself in Phillips' office. Simultaneously Petridis went to the same office to explain to Phillips his work on the genericity of the L^2-eigenvalue $1/4$ and half-bound states, i.e. $E(z, 1/2)$, which are also known as nullvectors [Pet94b]. This work was complementing that of Phillips and Sarnak. Soon after this encounter Balslev realized that the analytic dilation techniques, which he had used so effectively in the context of Schrödinger operators, could be used also in the context of deformations of hyperbolic surfaces. This realization led him to write [Bal97], where he reproved much of the theory of Phillips and Sarnak, including Fermi's Golden rule, using analytic dilation techniques. It is well-known that the Eisenstein series $E(z, 1/2 + it)$, which provide the continuous spectrum, have zero Fourier coefficient non-vanishing for $t \in \mathbb{R}^*$. However, the eigenfunctions with eigenvalue embedded in the continuous spectrum have vanishing zero Fourier coefficient. Balslev introduced a family of operators $U(\lambda)$ acting only on the zero Fourier coefficient, corresponding to dilations in the hyperbolic distance for λ real. For λ complex the continuous spectrum $[1/4, \infty[$ of Δ is rotated by an angle $-2 \arg \lambda$ to provide the continuous spectrum of the conjugated operator $U(\lambda)\Delta U(\lambda)^{-1}$. The embedded eigenvalues do not change location, so they become isolated. Resonances of Δ also turn into discrete eigenvalues (for appropriate choice of angle). This allowed him to use analytic perturbation theory and to reprove Fermi's Golden rule.

In the early 1990s Balslev met Alexei B. Venkov. This became the beginnning of a fruitful collaboration and close friendship which would last for the rest of Balslev's life. At this point Venkov had already been thinking about the Roelcke–Selberg conjecture and the Phillips–Sarnak conjecture for a long time [Ven79, Ven90a, Ven90b, Ven92], and together they started discussing the implications of Balslev's work [Bal97]. Venkov started to visit Balslev at Aarhus University regularly, and in 2001 he joined their faculty. Together they worked on how to refine, use, and extend the deformation ideas of Phillips and Sarnak. This lead to several joint results on Weyl's law [BV98, BV01, BV07] as well as on other related topics [BV00, BV05].

1.2. **Higher order deformation.** The current work was inspired by the following question posed by Erik Balslev to the authors: *If the Phillips-Sarnak condition* (1.3) *is* not *satisfied, can one give simple conditions that ensures that an eigenvalue is dissolved?* We will report on our work in this direction. We refer to [PR13] for full details.

Understanding higher order deformations seems daunting at first. If one considers general expressions for the perturbation series of eigenvalues under analytic deformations one finds e.g. a 15-term expression for $\hat{\lambda}^{(4)}(0)$, see [Kat76, p. 80]. We managed to find simpler expressions assuming that the lower order terms vanish, see Theorem 2.1 below.

Our motivation to understand what happens when the Phillips–Sarnak condition (1.3) is not satisfied came from the numerical investigation by Farmer and Lemurell [FL05] and Avelin [Ave07]. For a given cusp form Farmer and Lemurell found curves (branches) in Teichmüller space where a cusp form for $\varepsilon = 0$ remains cusp form, i.e. is not destroyed to *any order*. For specific even cusp forms Avelin identified an analytic curve in Teichmüller space such that the movement of the poles of the scattering matrix gives a fourth order contact to the line $\mathrm{Re}(s) = 1/2$. Our work aims to explain such phenomena theoretically.

2. Stability of eigenvalues under character deformations

We start by recalling a basic few properties of Eisenstein series. We refer to [Iwa02] for additional details.

2.1. **Standard non-hololorphic Eisenstein series.** For simplicity of exposition we assume that Γ has precisely one cusp, and that it is located at infinity. Placing the cusp at infinity can always be achieved by conjugation. Assume further that the stabiliser of the cusp is generated by $\gamma_\infty : z \mapsto z + 1$, and that $\chi(\gamma_\infty) = 1$. Recall that the standard non-holomorphic Eisenstein series is defined by

$$E(z,s,\chi)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\chi}(\gamma)\,\mathrm{Im}(\gamma z)^s,\ \text{ when }\ \mathrm{Re}(s)>1,$$

and that it admits meromorphic continuation to $s\in\mathbb{C}$. Recall also that $E(z,1/2+it,\chi)$ is a generalized eigenfunction of Δ with eigenvalue $1/4+t^2$. These numbers, with $t\geq 0$ span the continuous spectrum. Clearly $E(\gamma z,s,\chi)=\chi(\gamma)E(z,s,\chi)$. The zero Fourier coefficient of $E(z,s,\chi)$ has the form

$$\int_0^1 E(z,s,\chi)dx=y^s+\varphi(s,\chi)y^{1-s}.$$

This defines the scattering matrix $\varphi(s,\chi)$, which in the one cusp case is just a function. We recall that $\varphi(s,\chi)$ satisfies the functional equation

$$\varphi(s,\chi)\varphi(1-s,\chi)=1, \tag{2.1}$$

and that it is unitary on the line $\mathrm{Re}(s)=1/2$. Furthermore

$$E(z,s,\chi)=\varphi(s,\chi)E(z,1-s,\chi). \tag{2.2}$$

We define

$$M(T)=-\frac{1}{4\pi}\int_{-T}^{T}\frac{\varphi'}{\varphi}(1/2+it,\chi)dt. \tag{2.3}$$

Selberg proved that for *all* cofinite groups Γ we have

$$N_d(\lambda)+M(\sqrt{\lambda-1/4})\sim\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda.$$

Hence N_d satisfies Weyl's law precisely if

$$M(T)=o(T^2). \tag{2.4}$$

Selberg proved that (2.4) holds for Γ a congruence group and χ a congruence character. For such a group the scattering determinant φ can be computed explicitly in terms of completed L-functions, and the bound (2.4) follows from classical bounds on these L-functions.

2.2. **Character deformations.** For the rest of the paper we consider for simplicity the case of character deformations. The cases of real analytic Teichmüller deformations and real analytic compact deformations within the set of admissible surfaces can be dealt with in a similar way. We refer to [PR13] for additional details.

Let f be a cusp form of weight two, i.e. $f:\mathbb{H}\to\mathbb{C}$ is a holomorphic function that satisfies

$$f(\gamma z)=(cz+d)^2f(z)\ \text{for all}\ \gamma\in\Gamma,$$

and admits a Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}a_ne^{2\pi inz}. \tag{2.5}$$

Let $\omega = \mathrm{Re}(f(z)dz)$ be the corresponding real invariant 1-form, and let α be a compactly supported 1-form in the same cohomology class as ω. We now define the modular symbol to be

$$\langle \gamma, \alpha \rangle := -2\pi i \int_{z_0}^{\gamma z_0} \alpha.$$

Here z_0 is any point in $\mathbb{H} \cup \{i\infty\}$. The modular symbol is independent of the choice of z_0, the choice of compact form α in the same cohomology class, as well as choice of path between z_0 and γz_0. The modular symbol mapping is an additive homomorphism, i.e. $\langle \gamma_1\gamma_2, \alpha \rangle = \langle \gamma_1, \alpha \rangle + \langle \gamma_2, \alpha \rangle$, and, moreover, vanishes at parabolic elements: $\langle \gamma_\infty, \alpha \rangle = 0$.

With the help of modular symbols we create a one-parameter family of unitary characters. Consider now the unitary characters

$$\chi_\varepsilon(\gamma) = \exp(\varepsilon \langle \gamma, \alpha \rangle)$$

and the space

$$L^2(\Gamma\backslash\mathbb{H}, \overline{\chi_\varepsilon})) = \left\{ f : \mathbb{H} \to \mathbb{C} : f(\gamma z) = \overline{\chi_\varepsilon}(\gamma) f(z), \int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 \, d\mu(z) < \infty \right\}.$$

Here $d\mu(z) = y^{-2} dx dy$ is the $\mathrm{PSL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$. We denote the induced automorphic Laplacian by $\tilde{L}(\varepsilon)$. We now conjugate this family of operators to the fixed space $L^2(\Gamma\backslash\mathbb{H})$ by using unitary operators

$$\begin{aligned} U(\varepsilon) : L^2(\Gamma\backslash\mathbb{H}) \to & L^2(\Gamma\backslash\mathbb{H}, \overline{\chi_\varepsilon})) \\ & f(z) \mapsto \exp\left(2\pi i \varepsilon \int_{z_0}^{z} \alpha\right) f(z), \end{aligned}$$

and let $L(\varepsilon) = U^{-1}(\varepsilon) \tilde{L}(\varepsilon) U(\varepsilon)$. This new family of operators has the advantage of being defined on a fixed space. It is now a straightforward computation to show that on smooth functions h

$$L(\varepsilon) h = \Delta h + \varepsilon L^{(1)} h + \frac{\varepsilon^2}{2} L^{(2)} h,$$

where

$$\begin{aligned} L^{(1)} h &= 4\pi i \langle dh, \alpha \rangle - 2\pi i \delta(\alpha) h, \\ L^{(2)} h &= -8\pi^2 \langle \alpha, \alpha \rangle h. \end{aligned} \tag{2.6}$$

Here

$$\begin{aligned} \langle f_1 dz + f_2 d\overline{z}, g_1 dz + g_2 d\overline{z} \rangle = 2y^2 \left(f_1 \overline{g}_1 + f_2 \overline{g}_2 \right), \\ \delta(p dx + q dy) = -y^2 (p_z + q_y). \end{aligned}$$

We want to investigate whether embedded eigenvalues are destroyed under this perturbation, i.e. they turn into resonances.

2.3. **Goldfeld Eisenstein series.** Goldfeld [Gol99b, Gol99a] introduced in the late 1990s a generalization of the standard Eisenstein series $E(z,s)$, which has since been studied by several people. It turns out that the stability of eigenvalues under perturbations as described above can be analyzed using such series.

The Goldfeld Eisenstein series, also known as Eisenstein series twisted by modular symbols, is defined by

$$E^n(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle\gamma,\alpha\rangle^n \operatorname{Im}(\gamma z)^s, \text{ when } \operatorname{Re}(s)>1.$$

It is well establised [PR04, JO08] that $E^n(z,s)$ admits meromorphic continuation to $s\in\mathbb{C}$.

For $n>0$ the function $E^n(z,s)$ is not invariant but satisfies an nth order automorphy relation, i.e. $E^n(z,s)\in A^n_\Gamma$ where A^n_Γ is defined recursively as follows: the set A^0_Γ is simply the set of Γ-invariant functions on $\mathbb{H}$, and A^n_Γ consists on functions f on $\mathbb{H}$ satisfying $f(\gamma z)-f(z)\in A^{n-1}_\Gamma$ for all $\gamma\in\Gamma$. For details on higher order Maass forms see e.g. [BD12].

As $\operatorname{Im}(\gamma z)^s$ is formally an eigenfunction of Δ, we have furthermore

$$(\Delta + s(1-s))E^n(z,s) = 0.$$

There is a related *invariant* function constructed by automorphizing $\left(-2\pi i\int_{z_0}^z\alpha\right)^n \operatorname{Im}(z)^s$ as follows:

$$D^n(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(-2\pi i\int_{z_0}^{\gamma z}\alpha\right)^n \operatorname{Im}(\gamma z)^s, \text{ when } \operatorname{Re}(s)>1.$$

Similarly to $E^n(z,s)$ the function $D^n(z,s)$ also admits meromorphic continuation to $s\in\mathbb{C}$. Indeed there is a simple way to relate the two:

$$D^n(z,s) = \sum_{j=0}^n \binom{n}{j}\left(-2\pi i\int_{z_0}^z\alpha\right)^{n-j} E^j(z,s).$$

The function $D^n(z,s)$ is not an eigenfunction of Δ but satisfies

(2.7)
$$(\Delta+s(1-s))D^n(z,s) = -\binom{n}{1}L^{(1)}D^{n-1}(z,s) - \binom{n}{2}L^{(2)}D^{n-2}(z,s),$$

where $L^{(1)}$ and $L^{(2)}$ are as in (2.6). Here we interpret $D^n(z,s)=0$, if n is negative.

2.4. **Higher order Fermi's Golden Rules.** We are now ready to formulate our main theorem, which answers the question of Balslev mentioned in the introduction:

Theorem 2.1 ([PR13]). *Let s_j be a cuspidal eigenvalue for $L(0)$ and let $\hat{s}_j(\varepsilon)$ be the weighted mean of the branches of the singular points for $L(\varepsilon)$ generated by s_j. Assume that*

$$\hat{s}_j^{(l)}(0) = 0, \qquad \textit{for } l \leq 2(n-1).$$

Then

(i) $\hat{s}_j^{(2n-1)}(0) = 0$,
(ii) $D^n(z,s)$ *has at most a first order pole at s_j, and*
(iii) $\operatorname{Re} \hat{s}_j^{(2n)}(0) = -\dfrac{1}{2n}\dbinom{2n}{n}\left\| \operatorname*{res}_{s=s_j} D^n(z,s)\right\|^2$.

We note that if

$$\operatorname*{res}_{s=s_j} D^n(z,s) \neq 0, \tag{2.8}$$

then at least one eigenvalue s_j will be become a resonance under the deformation, so we may interpret this as a *higher order vanishing condition.*

We also note that when $n = 1$ this reduces to Theorem 1.1. We note that the Phillips–Sarnak vanishing condition can be formulated as

$$\operatorname*{res}_{s=s_j} D^1(z,s) \neq 0, \tag{2.9}$$

since we have

$$\operatorname*{res}_{s=s_j} D^1(z,s) = \sum_{k=1}^{m} \frac{c}{\pi^{s_j}} L(u_{j,k} \otimes f, s_j + 1/2)\Gamma(s_j - 1/2)u_{j,k}(z),$$

see [Pet02, Eq. (1.13)] and [PS85a].

3. Relation to special values of Dirichlet series

We first need to setup some additional notation. An eigenfunction u_j with eigenvalue $s_j(1-s_j) > 1/4$ has a Fourier expansion

$$u_j(z) = \sum_{n\neq 0} b_n \sqrt{y} K_{s_j - 1/2}(2\pi |n| y) e^{2\pi i n x},$$

where $K_s(t)$ is the McDonald–Bessel function. We assume that u_j has been normalized to have L^2-norm equal to 1.

The Phillips–Sarnak condition for dissolving a cuspidal eigenvalue (1.3) can be expressed as the non-vanishing of a special value of a Rankin–Selberg L-function, see [Sar90, PS85a]. This is defined as

$$L(u_j \otimes f, s) = \sum_{n=1}^{\infty} \frac{a_n b_{-n}}{n^{s+1/2}} \text{ for } \operatorname{Re}(s) > 1.$$

We will now explain that something similar happens for the higher order dissolving conditions in Theorem 2.1.

We consider the antiderivative of the cusp form f inducing χ_ε, i.e.

$$F(z) = \int_{i\infty}^{z} f(w)dw = \sum_{n=1}^{\infty} \frac{a_n}{2\pi i n} e^{2\pi i n z}.$$

Then for $\mathrm{Re}(s) > 1$ we define the convergent Dirichlet series

$$L(u_j \otimes F^2, s) = \sum_{n=1}^{\infty} \sum_{k_1+k_2=n} \frac{a_{k_1}}{k_1} \frac{a_{k_2}}{k_2} \frac{b_{-n}}{n^{s-1/2}}. \tag{3.1}$$

One can show that $L(u_j \otimes F^2, s)$ admits meromorphic continuation to $s \in \mathbb{C}$ and satisfies a functional equation relating its value at s and $1-s$. The possible poles of $L(u_j \otimes F^2, s)$ are at the singular points.

A holomorphic form of weight 2 as in Section 2.2 gives rise to two character deformations, namely those induced from $\omega_1 = \mathrm{Re}(f(z)dz)$ and $\omega_2 = \mathrm{Re}(if(z)dz)$.

Theorem 3.1. *Assume that the Phillips–Sarnak condition* (2.9) *at a cuspidal eigenvalue s_j is* not *satisfied for either of ω_1, ω_2. Then $L(u_j \otimes F^2, s)$ has a removable singularity at s_j.*

Assume further that $L(u_j \otimes F^2, s_j) \neq 0$. Then in all directions ω in the real span of ω_1, ω_2 with at most two exceptions we have

$$\mathrm{Re}\, \hat{s}_j^{(4)}(0) \neq 0.$$

In particular there exists a cusp form with eigenvalue $s_j(1-s_j)$ that is dissolved in this direction.

We refer to [PR13, Sec. 4.3] for proofs of this theorem. The function $L(u_j \otimes F^2, s)$ is not as well studied as the Rankin–Selberg L-function, and, although it does share many of its properties (continuation to $s \in \mathbb{C}$, functional equation, bounds on vertical lines), there are important differences. Most importantly $L(u_j \otimes F^2, s)$ does not admit an Euler product.

4. Idea of proof

We now indicate the main steps of Theorem 2.1, and refer to [PR13] for details.

For the fixed group Γ and the family of characters χ_ε we consider the scattering matrix $\varphi(s, \varepsilon)$. Besides properties that we have already stated one can show that

$$\varphi(s, \varepsilon) = \overline{\varphi}(\overline{s}, \varepsilon), \tag{4.1}$$

see [Hej83, page 218, Remark 61].

We track the movement of the singular set close to an embedded eigenvalue $s_j(1-s_j) > 1/4$ i.e. the embedded eigenvalue/resonance in the half-plane left of $\mathrm{Re}(s) = 1/2$ using complex analysis, in particular, a simple variation of the argument principle. Define Λ as the half circle $\gamma_1(t) = ue^{it} + s_j$, $\pi/2 \leq t \leq 3\pi/2$ followed by the line $\gamma_2(t) = s_j + it$,

$-u \le t \le u$. Here u is chosen small enough, so that the only singular point for $\varepsilon = 0$ inside the ball $B(s_j, u)$ is s_j with multiplicity $m = m(s_j)$. This contour is traversed counterclockwise. For ε sufficiently small the total multiplicities of the singular points $s_j(\varepsilon)$ inside $B(s_j, u)$ is $m(s_j)$.

We have

$$m(\hat{s}(\varepsilon) - s_j) = -\frac{1}{2\pi i}\int_\Lambda (s - s_j)\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\,ds + \sum_{j\in C}(s_j(\varepsilon) - s_j), \tag{4.2}$$

where C is indexing the cusp forms eigenbranches inside $B(s_j, u)$, i.e. the cusp forms that remain cusp forms. Let the last sum be denoted by $p(\varepsilon)$. The reason for using Λ and not the whole $\partial B(s_j, u)$ is that on the right half-disc $\varphi(s,\varepsilon)$ has zeros, which we do not want to count. Note that by well-known properties of $\varphi(s,\varepsilon)$ [Iwa02, Chapter 6] it has no zeroes in Λ. Notice that $\overline{\int_\gamma f(s)\,ds} = \int_{\bar\gamma} \bar{f}(\bar{s})\,ds$ and, therefore we find by using (4.1) that

$$\begin{aligned} m(\overline{\hat{s}(\varepsilon) - s_j}) &= \frac{1}{2\pi i}\int_{\bar\Lambda} (s - \bar{s}_j)\overline{\left(\frac{\varphi'(\bar{s},\varepsilon)}{\varphi(\bar{s},\varepsilon)}\right)}\,ds + \overline{p(\varepsilon)} \\ &= \frac{1}{2\pi i}\int_{\bar\Lambda} (s - \bar{s}_j)\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\,ds + \overline{p(\varepsilon)}. \end{aligned}$$

Denoting by $-\gamma$ the contour γ traversed in the opposite direction, we get

$$\begin{aligned} m(\overline{\hat{s}(\varepsilon) - s_j}) &= -\frac{1}{2\pi i}\int_{-\bar\Lambda} (s - \bar{s}_j)\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\,ds + \overline{p(\varepsilon)} \\ &= -\frac{1}{2\pi i}\int_{T^{-1}(-\bar\Lambda)} (1 - w - \bar{s}_j)\frac{\varphi'(1-w,\varepsilon)}{\varphi(1-w,\varepsilon)}\,(-dw) + \overline{p(\varepsilon)}, \end{aligned}$$

where $s = T(w) = 1 - w$ is a conformal map. By (2.2) we get

$$\varphi'(s,\varepsilon)\varphi(s,\varepsilon) - \varphi(s,\varepsilon)\varphi'(1-s,\varepsilon) = 0,$$

giving that

$$\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)} = \frac{\varphi'(1-s,\varepsilon)}{\varphi(1-s,\varepsilon)}.$$

We plug this into the expression for $m(\overline{\hat{s}(\varepsilon) - s_j})$ to get

$$m(\overline{\hat{s}(\varepsilon) - s_j}) = -\frac{1}{2\pi i}\int_{T^{-1}(-\bar\Lambda)} (w - s_j)\frac{\varphi'(w,\varepsilon)}{\varphi(w,\varepsilon)}\,dw + \overline{p(\varepsilon)}. \tag{4.3}$$

We sum (4.2) and (4.3) and notice that the cuspidal branch contributions cancel, since the function $s_{j,l}(\varepsilon) - s_j$ is purely imaginary for a cuspidal branch $s_{j,l}(\varepsilon)$. We therefore conclude that

$$
\begin{aligned}
2m\operatorname{Re}(\hat{s}(\varepsilon)-s_j) &= -\frac{1}{2\pi i}\int_{\Lambda+T^{-1}(-\bar{\Lambda})}(s-s_j)\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\,ds \\
&= -\frac{1}{2\pi i}\int_{\partial B(s_j,u)}(s-s_j)\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\,ds, \qquad (4.4)
\end{aligned}
$$

since the contribution from the line segment on $\operatorname{Re}(s)=1/2$ from Λ and $T^{-1}(-\bar{\Lambda})$ cancel. By uniform convergence we can differentiate the last formula in ε. We get

(4.5)

$$
\begin{aligned}
2m\frac{d^{2n}}{d\varepsilon^{2n}}\operatorname{Re}(\hat{s}(\varepsilon))\Big|_{\varepsilon=0} &= -\frac{1}{2\pi i}\int_{\partial B(s_j,u)}(s-s_j)\frac{d^{2n}}{d\varepsilon^{2n}}\left(\frac{\varphi'(s,\varepsilon)}{\varphi(s,\varepsilon)}\right)\Big|_{\varepsilon=0}ds \\
= -\frac{1}{2\pi i}\int_{\partial B(s_j,u)}(s-s_j)\sum_{k=0}^{2n}\binom{2n}{k}\,&\frac{d^k\varphi'(s,\varepsilon)}{d\varepsilon^k}\Big|_{\varepsilon=0}\,\frac{d^{2n-k}(\varphi(s,\varepsilon)^{-1})}{d\varepsilon^{2n-k}}\Big|_{\varepsilon=0}ds.
\end{aligned}
$$

We can interchange the order of differentiation:

$$\frac{d^k}{d\varepsilon^k}\varphi'(s,\varepsilon)=\frac{d}{ds}\varphi^{(k)}(s).$$

Note that the prime denotes derivative in s, whereas the $\varphi^{(k)}(s)$ denotes the kth derivative in ε evaluated at $\varepsilon=0$. By differentiating m times $\varphi(s,\varepsilon)^{-1}\varphi(s,\varepsilon)=1$ we find

$$\sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{d\varepsilon^k}\varphi(s,\varepsilon)^{-1}\Big|_{\varepsilon=0}\varphi^{(m-k)}(s,0)=0.$$

These observations combined with (4.5) show that in order to compute $2m\frac{d^{2n}}{d\varepsilon^{2n}}\operatorname{Re}(\hat{s}(\varepsilon))\Big|_{\varepsilon=0}$ it suffices to understand the analytic behaviour of $\varphi^{(k)}(s)$ at $s=s_j$.

The general functional equation (2.2) implies that

$$D^n(z,s)=\sum_{k=0}^{n}\binom{k}{n}\varphi^{(k)}(s)D^{n-k}(z,1-s).$$

Combining this with (2.7) and properties of the resolvent kernel $R(s)=(\Delta+s(1-s))^{-1}$ we can show that

(4.6)

$$\varphi^{(n)}(s)=\frac{1}{2s-1}\int_{\Gamma\backslash\mathbb{H}}E(z,s)\left(\binom{n}{1}L^{(1)}D^{n-1}(z,s)+\binom{n}{2}L^{(2)}D^{n-2}(z,s)\right)d\mu(z).$$

After some computations this and (4.5) lead to

$$2m\frac{d^{2n}}{d\varepsilon^{2n}}\operatorname{Re}(\hat{s}(\varepsilon))\Big|_{\varepsilon=0}=\frac{\operatorname{res}_{s=s_j}\varphi^{(2n)}(s,0)}{\varphi(s_j,0)}.$$

Combining this with the following result gives Theorem 2.1.

Theorem 4.1. *Assume that* $D^i(z,s)$ *is regular at* $s_j = 1/2 + ir_j$ *for* $i = 0, \dots, n-1$. *Then*

(i) *the function* $\varphi^{(l)}(s)$ *is regular at* s_j *for* $l = 0, 1, \dots, 2n-1$.

(ii) *the function* $\varphi^{(2n)}(s)$ *has at most a simple pole at* s_j. *Furthermore the residue at* s_j *is given by*

$$\operatorname*{res}_{s=s_j} \varphi^{(2n)}(s) = -\varphi(s_j)\binom{2n}{n}\left\|\operatorname*{res}_{s=s_j} D^n(z,s)\right\|^2.$$

This theorem is proved by investigating further (4.6) and (2.7). We refer to [PR13, Thm 3.3] for details.

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