A MORE INTUITIVE PROOF OF A SHARP VERSION OF HALÁSZ'S THEOREM

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Abstract. We prove a sharp version of Halász's theorem on sums $\sum_{n\le x} f(n)$ of multiplicative functions f with $|f(n)|\le 1$. Our proof avoids the "average of averages" and "integration over α" manoeuvres that are present in many of the existing arguments. Instead, motivated by the circle method we express $\sum_{n\le x} f(n)$ as a triple Dirichlet convolution, and apply Perron's formula.

1. Introduction

Given a multiplicative function $f:\mathbb{N}\to\mathbb{C}$, for each $x\ge 2$ let its summatory function be

$$S(x):=\sum_{n\le x} f(n), \text{ and } F_x(s):=\prod_{p\le x}\left(1+\sum_{k=1}^{\infty}\frac{f(p^k)}{p^{ks}}\right)$$

denote the corresponding truncated Euler product.

In this note we shall prove the following form of Halász's theorem on mean values of multiplicative functions taking values in the unit disc.

Theorem 1. *For f, F_x and $S(\cdot)$ as above, suppose that $|f(n)|\le 1$ for all integers $n\ge 1$. Define the quantity $L(x)$ by setting*

$$L(x)^2:=\sum_{|N|\le \log^2 x+1}\frac{1}{N^2+1}\sup_{|t-N|\le 1/2}|F_x(1+it)|^2.$$

Then we have

$$|S(x)|\ll x\frac{L(x)}{\log x}\log\left(100\frac{\log x}{L(x)}\right)+x\frac{\log\log x}{\log x}.$$

Note that $|F_x(1+it)|\le \prod_{p\le x}(1-1/p)^{-1}=(e^\gamma+o(1))\log x$, from which it follows that $L(x)\le 6\log x$ for large x. Thus the quantity $100(\log x)/L(x)$ appearing in Theorem 1 is bounded away from 1.

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Theorem 1 is essentially the same as the version of Halász's theorem proved by Montgomery [6] (if we note that $|F_x(1+it)|^2 \asymp |F(1+\frac{1}{\log x}+it)|^2$, where $F(s)$ denotes the full Euler product over all primes), and is known to be quantitatively sharp (see the papers of Granville and Soundararajan [3] and Montgomery [6]). See Halász's papers [4, 5] for his original arguments which were refined by Montgomery [6], and see Chapter III.4 of Tenenbaum [7] for an elegant textbook treatment with added precision.

Our proof here is, hopefully, more intuitive and easier to motivate than the existing proofs, although it also has important features in common with several of them. We begin by expressing $S(x)$ as a triple Dirichlet convolution, and using Perron's formula to relate our triple Dirichlet convolution to the Dirichlet series $F_x(s)$ and two other Dirichlet polynomials. This is done by analogy with the circle method, as we want to use a pointwise bound for $F_x(s)$ and obtain a mean square bound for the remaining two Dirichlet polynomials. To carry everything out with little loss, we break the Dirichlet convolution into subsums which depend on the size of one of the variables p. Our proof avoids the "average of averages" step in many other treatments of Halász's theorem, and in particular it avoids the arguably slightly obscure "integration over α" device from many of the treatments.

Our longer companion paper [2] uses a similar strategy to prove various generalisations of Halász's theorem, including for multiplicative functions bounded by divisor functions, and treating sums over short intervals and arithmetic progressions. However we give here the original argument, stripped of the technicalities in the more general argument of [2] (compare, for example, the more complicated but more easily generalisable triple convolution in [2]).

2. A lemma concerning prime numbers

We will need some basic information about the integrals of Dirichlet polynomials supported on the primes. We record a suitable result here.

Lemma 1. *Uniformly for any complex numbers $(a_n)_{n=1}^{\infty}$ and any $T \geq 1$, we have*

$$\int_{-T}^{T} \Bigg| \sum_{T^2 \leq n \leq x} \frac{a_n \Lambda(n)}{n^{1+it}} \Bigg|^2 dt \ll \sum_{T^2 \leq n \leq x} \frac{|a_n|^2 \Lambda(n)}{n}.$$

Proof of Lemma 1. This follows by inserting a smooth weight $\Phi(t/T)$ into the integral, expanding out, and applying a Brun–Titchmarsh upper bound for primes in short intervals at a suitable point. See Lemma 2.6 of [2], for example, for a full proof; or Lemme 3.1 of Tenenbaum [8], who attributes such results to Gallagher. □

Mean value results and majorant principles of this kind are often used in multiplicative number theory, and proved in very similar ways (see, e.g., the Lemma in section 2 of Montgomery [6]); some would be sufficient for our purposes. However the most standard mean value theorem for Dirichlet polynomials, which implies that $\int_{-T}^{T} |\sum_{n\le x} \frac{a_n}{n^{it}}|^2 dt = \sum_{n\le x} |a_n|^2(2T+O(n))$, would *not* suffice because in Lemma 1 it would yield a multiplier $\Lambda(n)^2$, rather than $\Lambda(n)$, on the right hand side.

3. Proof of Theorem 1: the combinatorial part

We begin by expressing $S(x)$ as a triple Dirichlet convolution, up to an acceptable error. Since $\log n = \sum_{d|n} \Lambda(d)$ we have

$$\sum_{n\le x} f(n)\log n = \sum_{d\le x} \Lambda(d) \sum_{m\le x/d} f(md) = \sum_{mp\le x} f(m)f(p)\log p + O(x),$$

the error term arising from bounding trivially the contribution of prime power values of d, and the terms with $(m,d)>1$. Since

$$\sum_{n\le x} \log(x/n) = O(x),$$

we deduce that

$$S(x) = \frac{1}{\log x}\sum_{n\le x} f(n)(\log n + \log x/n) = \frac{1}{\log x}\sum_{mp\le x} f(m)f(p)\log p + O\Big(\frac{x}{\log x}\Big).$$

This is a double multiplicative convolution, since we have the two variables p and m in the sum.

We repeat the above argument to arrive at a triple convolution. For technical convenience we begin by discarding those primes p for which $p\le \log^4 x$ or $p > x/2$ from the sum, which gives rise to an acceptable error term $O(x\frac{\log\log x}{\log x})$. For primes p in the range $\log^4 x < p \le x/2$, we use the above argument to replace the sum over $m\le x/p$ by a double convolution; that is,

$$\sum_{m\le x/p} f(m) = S(x/p) = \frac{1}{\log(x/p)}\sum_{nq\le x/p} f(n)f(q)\log q + O\Big(\frac{x}{p\log x/p}\Big).$$

Therefore

$$\begin{aligned} S(x) &= \frac{1}{\log x}\sum_{\log^4 x<p\le \frac{x}{2}} f(p)\log p \sum_{m\le x/p} f(m) + O\Big(x\frac{\log\log x}{\log x}\Big) \\ &= \frac{1}{\log x}\sum_{\log^4 x<p\le \frac{x}{2}} \frac{f(p)\log p}{\log(x/p)} \sum_{nq\le x/p} f(n)f(q)\log q + O\Big(x\frac{\log\log x}{\log x}\Big), \end{aligned} \qquad (3.1)$$

since

$$\sum_{\log^4 x<p\le \frac{x}{2}} \frac{\log p}{p\log(x/p)} \ll \log\log x.$$

We have arrived at the desired triple multiplicative convolution.

The range of q in (3.1) is severely restricted when p is large, which will lead to bigger error terms, so it pays to treat summands differently depending on the size of p. We achieve this by partitioning up the range of the primes $p \in \mathcal{P} = [(\log x)^4, x/2]$ into the intervals $\mathcal{P}_k = \mathcal{P} \cap (x^{1-e^{1-k}}, x^{1-e^{-k}}]$, where k runs through the integers from 1 to $\log\log x + O(1)$. Define

$$S_k(x) = \sum_{\substack{pqn \le x\\ p \in \mathcal{P}_k}} \frac{f(p)\log p}{\log(x/p)} f(n) f(q) \log q,$$

so that (3.1) implies

$$S(x) \ll \frac{1}{\log x} \sum_{k=1}^{\log\log x + O(1)} |S_k(x)| + x\frac{\log\log x}{\log x}.$$

Since each $|f(p)f(q)f(n)| \le 1$, we may bound $S_k(x)$ trivially as follows:

$$\begin{aligned} |S_k(x)| &\le \sum_{\substack{pq \le x\\ p \in \mathcal{P}_k}} \frac{\log p}{\log(x/p)} \log q \sum_{n \le x/pq} 1 \\ &\le x \sum_{p \in \mathcal{P}_k} \frac{\log p}{p \log(x/p)} \sum_{q \le x/p} \frac{\log q}{q} \\ &\ll x \sum_{x^{1-e^{1-k}} < p \le x^{1-e^{-k}}} \frac{\log p}{p} \ll e^{-k} x \log x. \end{aligned}$$

Thus the sum of $|S_k(x)|$ over all integers $k > \log(100 \log x / L(x))$ (where $L(x)$ is as in the statement of Theorem 1) leads to a bound that is acceptable for Theorem 1.

To complete the proof of Theorem 1, it therefore suffices to show that

$$S_k(x) \ll xL(x) + x \tag{3.2}$$

for all positive integers $k \le \log(100 \log x / L(x))$.

Remark 3.1. Partitioning the range for p and applying the triangle inequality might appear to be wasteful. However, in the worst case, there is no loss in introducing absolute values, since the arguments of the values of the $f(p)$ with $p > x^{1-e^{-1}}$ could have been chosen, given the values $f(q^k)$ for $q \le x^{1-e^{-1}}$, so that $f(p)$ times the sum over $qn \le x/p$ all point in exactly the same direction. Indeed, extremal examples for Halász's theorem can behave precisely in this way, as in the introduction to Montgomery's paper [6].

Remark 3.2. If the multiplicative function $f(n)$ is supported only on numbers with all their prime factors $\le x^{0.999}$, say, (that is, the $x^{0.999}$-smooth numbers), then there will only be a bounded number of terms k in our decomposition of the p-sum. For such functions f, Halász's theorem can be improved, using (3.2), to

$$|S(x)| \ll \frac{x}{\log x}(L(x)+1),$$

after taking a little more care in handling the discarded contribution from the primes $p \le \log^4 x$. As far as we know, this has not been noted previously.

4. Proof of Theorem 1: the analytic part

It remains to prove (3.2). If pqn is a term appearing in the definition of $S_k(x)$ then note that p lies in $\mathcal{P}_k$, the prime q is constrained to $q \le x^{e^{1-k}}$ (since $pq \le x$), and n (which is less than x) is an integer with all prime factors below x. Therefore, using a truncated Perron formula (see the Lemma in Chapter 17 of [1]), we get

$$S_k(x) = \frac{1}{2\pi i}\int_{1-iT}^{1+iT} \sum_{p\in\mathcal{P}_k} \frac{f(p)\log p}{p^s \log(x/p)} \sum_{q\le x^{e^{1-k}}} \frac{f(q)}{q^s}(\log q) F_x(s)\frac{x^s}{s} ds + E, \tag{4.1}$$

where the error term E satisfies

$$E \ll \sum_{p\in\mathcal{P}_k} \sum_{q\le x^{e^{1-k}}} \sum_{p|n \overset{n}{\Longrightarrow} p\le x} \frac{\log p}{\log(x/p)}(\log q)\Big(\frac{x}{pqn}\Big)\min\Big(1, \frac{1}{T|\log(x/pqn)|}\Big).$$

We shall take $T = (\log x)^2$. First we bound E, splitting terms according to whether $1/2 \le x/(pqn) \le 2$ or not. The first type contributes (since $|\log x/(pqn)| \gg |x-pqn|/x$ here)

$$\ll \sum_{p\in\mathcal{P}_k} \sum_{q\le 2x/p} \frac{\log p}{\log(x/p)} \log q \sum_{x/(2pq)\le n\le 2x/(pq)} \min\Big(1, \frac{x}{T|x-pqn|}\Big)$$
$$\ll \sum_{p\in\mathcal{P}_k} \sum_{q\le 2x/p} \frac{\log p}{\log(x/p)} \log q\Big(1+\frac{x}{Tpq}\log T\Big) \ll x.$$

The second type contributes (since $|\log x/(pqn)| \gg 1$ here)

$$\ll \frac{1}{T}\sum_{p\in\mathcal{P}_k} \frac{\log p}{\log(x/p)} \sum_{q\le x^{e^{1-k}}} \log q \sum_{p|n \Longrightarrow p\le x} \frac{x}{pqn} \ll x.$$

Thus $E \ll x$, which is acceptable for (3.2).

Turning now to the main term in (4.1), using the triangle inequality followed by the Cauchy–Schwarz inequality, we may bound the integral there by $\ll x\sqrt{I_1 I_2}$, where

$$I_1 = \int_{1-i(\log x)^2}^{1+i(\log x)^2} \Big|\sum_{p\in\mathcal{P}_k} \frac{f(p)\log p}{p^s\log(x/p)}\Big|^2 |ds|,$$

and

$$I_2 = \int_{1-i(\log x)^2}^{1+i(\log x)^2} \Big|\sum_{q\le x^{e^{1-k}}} \frac{f(q)}{q^s}\log q\Big|^2 |F_x(s)|^2 \frac{|ds|}{|s|^2}.$$

Splitting the integral in I_2 into intervals of length 1, we may bound it by

$$I_2 \ll \sum_{|h| \leq \log^2 x + 1} \frac{1}{h^2+1} \sup_{|t-h| \leq 1/2} |F_x(1+it)|^2 \int_{1+i(h-1/2)}^{1+i(h+1/2)} \Big| \sum_{q \leq x^{e^{1-k}}} \frac{f(q)}{q^s} \log q \Big|^2 |ds|.$$

Recalling that q runs over primes, we can apply Lemma 1 with $T = 1$, $a_q = f(q)q^{-ih}$ for primes q, and $a_q = 0$ otherwise, and deduce that

$$\begin{aligned} I_2 &\ll \sum_{|h| \leq \log^2 x + 1} \frac{1}{h^2+1} \sup_{|t-h| \leq 1/2} |F_x(1+it)|^2 \sum_{q \leq x^{e^{1-k}}} \frac{\log q}{q} \\ &\ll L(x)^2 \, e^{-k} \log x. \end{aligned}$$

To bound I_1, we use Lemma 1 again (noting that $\mathcal{P}_k$ only has primes larger than $(\log x)^4$ for all k), to obtain

$$I_1 \ll \sum_{p \in \mathcal{P}_k} \frac{\log p}{p \log^2(x/p)} \ll \frac{e^{2k}}{\log^2 x} \sum_{x^{1-e^{1-k}} < p \leq x^{1-e^{-k}}} \frac{\log p}{p} \ll \frac{e^k}{\log x}.$$

Combining the foregoing estimates, we obtain (3.2) and therefore the bound claimed in Theorem 1. □

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