On the isothermal compressible multi-component mixture flow: the local existence and maximal $L_p - L_q$ regularity of solutions

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Abstract: We consider the initial-boundary value problem for the system of equations describing the flow of compressible isothermal mixture of arbitrary large number of components. The system consists of the compressible Navier-Stokes equations and a subsystem of diffusion equations for the species. The subsystems are coupled by the form of the pressure and the strong cross-diffusion effects in the diffusion fluxes of the species. Assuming the existence of solutions to the symmetrized and linearized equations, proven in [31], we derive the estimates for the nonlinear equations and prove the local-in-time existence and maximal $L_p - L_q$ regularity of solutions.

1 Introduction

1.1 Setting of the problem

We consider the system of equations describing the motion of an isothermal mixture of compressible gases

$$\left.\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) &= 0 \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S} + \nabla p &= \boldsymbol{0} \\
\partial_t \varrho_k + \operatorname{div}(\varrho_k \boldsymbol{u}) + \operatorname{div} \boldsymbol{F}_k &= 0
\end{aligned}\right\} \quad \text{in } (0,T) \times \Omega \tag{1}$$

in the regular domain $\Omega \subset \mathbb{R}^3$, supplied with boundary conditions

$$\boldsymbol{u} = 0, \ \boldsymbol{F}_k \cdot \boldsymbol{n} = 0 \quad \text{on } (0,T) \times \partial\Omega \tag{2}$$

and initial condition

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0, \quad \varrho_k|_{t=0} = \varrho_k^0, \ k = 1 \ldots n \quad \text{in } \Omega. \tag{3}$$

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Above, in system (1), ϱ denotes the mass density of the mixture

$$\varrho = \sum_{k=1}^{n} \varrho_k, \tag{4}$$

$\boldsymbol{u}$ is the mean velocity of the mixture, and ϱ_k is the density of the k-th constituent. The remaining quantities: the stress tensor $\boldsymbol{S}$, the total internal pressure p, and the diffusion fluxes $\boldsymbol{F}_k$ are determined as functions of $(\boldsymbol{u}, \varrho, \varrho_k)$ by constitutive relations which will be specified later.

The first equation of system (1), usually called the continuity equation, describes the balance of the mass, and the second equation expresses the balance of the momentum. The last n equations describe the balances of masses of separate constituents (species). Note that the system of equations cannot be independent, as the last n equations must sum up to the continuity equation. Thus, here we meet a serious mathematical obstacle, the subsystem $(1)_4$ is degenerate parabolic in terms of ϱ_k.

The stress tensor. The viscous part of the stress tensor obeys the *Newton rheological law*

$$\boldsymbol{S}(\boldsymbol{u}) = 2\mu \mathbf{D}(\boldsymbol{u}) + \nu \operatorname{div} \boldsymbol{u} \mathbf{I}, \tag{5}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ and the nonnegative viscosity coefficients.

Internal pressure. The internal pressure of the mixture is determined through the Boyle law, when the temperature is constant it is given by

$$p(\varrho_1, \ldots, \varrho_n) = \sum_{k=1}^{n} p_k(\varrho_k) = \sum_{k=1}^{n} \frac{\varrho_k}{m_k}; \tag{6}$$

above, m_k is the molar mass of the species k, and for simplicity, we set the gaseous constant equal to 1.

Diffusion fluxes. A key element of the presented model is the structure of laws governing cross-diffusion processes in the mixture. The diffusion fluxes are given explicitly in the form

$$\boldsymbol{F}_k = -\sum_{l=1}^{n} C_{kl}\boldsymbol{d}_l, \quad k = 1, \ldots n, \tag{7}$$

where C_{kl} are multicomponent flux diffusion coefficients and $\boldsymbol{d}_k = (d_k^1, d_k^2, d_k^3)$ is the species k diffusion force

$$d_k^i = \nabla_{x_i}\left(\frac{p_k}{p}\right) + \left(\frac{p_k}{p} - \frac{\varrho_k}{\varrho}\right)\nabla_{x_i}\log p = \frac{1}{p}\left(\nabla_{x_i}p_k - \frac{\varrho_k}{\varrho}\nabla_{x_i}p\right). \tag{8}$$

Moreover, we assume that $\sum_{k=1}^{n}\boldsymbol{F}_k = \boldsymbol{0}$, pointwisely. The main properties of the flux diffusion matrix C are

$$C\mathcal{Y} = \mathcal{Y}C^T, \quad N(C) = \operatorname{lin}\{\vec{Y}\}, \quad R(C) = U^\perp, \tag{9}$$

where $Y_k = \frac{\varrho_k}{\varrho}$, $\mathcal{Y} = \operatorname{diag}(Y_1, \ldots, Y_N)$, $\vec{Y} = (Y_1, \ldots, Y_n)^t$, $N(C)$ is the nullspace of C, $R(C)$ is the range of C, $\vec{U} = (1, \ldots, 1)^T$, and $U^\perp$ is the orthogonal complement of $\operatorname{lin}\{\vec{U}\}$. The second property in (9) implies

$$\sum_{l=1}^{n}\frac{1}{p}C_{kl}\frac{\varrho_k}{\varrho}\nabla p = \frac{\nabla p}{p}\sum_{l=1}^{n}C_{kl}Y_l = 0, \quad k = 1, \ldots, n,$$

therefore (7), (8) are reduced to

$$\boldsymbol{F}_k = -\frac{1}{p}\sum_{l=1}^{n}C_{kl}\nabla p_l. \tag{10}$$

We also define

$$D_{kl} = \frac{C_{kl}}{\varrho Y_k}, \tag{11}$$

thus the properties of C (9) imply

$$D = D^T, \quad D \geq 0, \quad N(D) = \lim\{\vec{Y}\}, \quad R(D) = Y^\perp. \tag{12}$$

The first property results from $C_{kl}Y_l = C_{lk}Y_k$, the third from the fact that $\mathcal{Y}$ is diagonal. Next, $p \in R(\tilde{D}) \iff p_k = \frac{1}{Y_k} \sum_l C_{kl}q_l$ for some $q \in \mathbb{R}^n$. Finally D is positive definite over $U^\perp$.

Exemplary diffusion matrix. An example of matrix C satisfying conditions (9) that will be distinguished throughout the paper is

$$C = \begin{pmatrix} Z_1 & -Y_1 & \ldots & -Y_1 \\ -Y_2 & Z_2 & \ldots & -Y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -Y_n & -Y_n & \ldots & Z_n \end{pmatrix}, \tag{13}$$

where $Z_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} Y_i$.

Using expressions for the diffusion forces (10) and the properties of this matrix one can rewrite (7) into the following form

$$\boldsymbol{F}_k = -\frac{1}{p} \left(\nabla p_k - Y_k \nabla p \right). \tag{14}$$

Clearly for C given by (13), the matrix $D_{kl} = \frac{C_{kl}}{\varrho Y_k}$ is symmetric and positive semi-definite.

1.2 Discussion of the known results

The main result of this paper concerns the local well-posedness of system (1) in the maximal $L_p - L_q$ regularity setting. The local well-posedness as well as global well-posedness for small data for two-species variant of system (1) have been shown in authors' previous work [30]. There the so-called normal form, considered earlier e.g. in [12], allows to immediately write a parabolic equation for one of the species densities. The aim of this paper is to generalize this result to the system with arbitrary number of constituents, however still isothermal. The key difference is that in the two species case the part corresponding to diffusion flux is reduced to a single parabolic equation, while now we obtain only a symmetrized system. Nevertheless, the properties of D imply only nonnegativity of its leading order part so an important step is to show its parabolicity. Dealing with the systems of species instead of single equation also requires serious modifications in the linear theory.

In the previous results devoted to the complete mixture model, see Giovangigli and Massot [13, 14], the local smooth solutions and global smooth solutions around constant equilibrium states were considered. Their method of proof was based on normal form of equations, hyperbolic-parabolic estimates and on local strict dissipativity of linearized systems. It can be seen as an application of more abstract theory proposed for the hyperbolic-parabolic systems of conservation laws by Kawashima and Shizuta [19, 20].

When the species equations are decoupled from the fluid equations, the resulting system of PDEs is related to the Stefan-Maxwell system analyzed for example in [2, 16]. In both of these papers the isobaric isothermal systems are considered with the barycentric velocity being equal to 0. This means that, in comparison with the system of last n equations from (1), the convective term $\mathrm{div}(\varrho_k \boldsymbol{u})$ is absent and the variation of total pressure in the diffusion fluxes (14) is neglected. Essential difference between these systems is that in the present case the diffusion fluxes are explicit combination of diffusion deriving

forces, while for the Stefan-Maxwell system the flux-forces relations need to be first inverted. This can be done using the Perron-Frobenius theory as first noticed in [11]. With this at hand, the local-in-time well-posedness and maximal L_p regularity follow from classical results of Amann [1] or Prüss [32]. In the approach presented in the present paper we rather relate on the alternative approach of the second author and collaborators [9, 35, 26, 27, 33, 34] tailored to the compressible fluid systems. The main result of this paper is maximal $L_p - L_q$ regularity of solutions to (1), but it relies on the proof of existence of relevant solutions to the linearized system. The latter result is proved in our other article [31] mostly for the sake of brevity, but also as it can be of independent interest. Indeed, it applies to whole class of symmetric parabolic systems satisfying certain regularity assumptions on the coefficients, therefore it is likely to be used in other contexts.

As far as maximal $L_p - L_q$ regularity is concerned, the coupling between Stefan-Maxwell and the fluid equations, was so far considered only for the incompressible Navier-Stokes system, see [4]. It was also proven, independently in [6] and [21], that the incompressible Navier-Stokes-Stefan-Maxwell system possesses a global-in-time weak solution with arbitrary data . The approach employed by Chen and Jüngel in [6] relies on a certain symmetrization of the species subsystem with one of equations eliminated, see also [18]. They have noticed that such reformulation allows to deduce parabolicity in terms of the so-called entropic variables. See also [17] for an overview of different problems where a similar approach can be applied. The idea of our approach is similar, however the change of variables we propose is slightly different, in the spirit of normal variables from [12]. Concerning analogous results for the compressible Navier-Stokes-Stefan-Maxwell system, the existence of weak solutions is so far known either for stationary flow of species with the same molar masses [38, 15, 28, 29], or for exemplary diffusion matrix C and stress tensor $\boldsymbol{S}$ with density-dependent viscosity coefficient [39, 40, 24, 25]. There are also relevant results for multi-component systems with diffusion fluxes in the form of the Fick law [10].

2 Main result

The main result of this paper is the the local well-posedness in the maximal $L_p - L_q$ regularity setting of certain reformulation of system (1) (17). This reformulation is similar to the normal form derived in ([12], Chapter 8) for the complete system with thermal effects. In case of constant temperature derivation of the symmetrized equations can be simplified considerably, and, to make our paper self contained, we first prove the following theorem.

Proposition 2.1 *Let $(\varrho, \boldsymbol{u}, \varrho_1, \ldots, \varrho_n)$ be a regular solution to system (1-4) such that*

$$\{\varrho_1 > C, \ldots, \varrho_n > C\} \tag{15}$$

for some constant $C > 0$. Then the change of unknowns

$$(\varrho, h_1, \ldots, h_{n-1}) = \left(\sum_{i=1}^{n} \varrho_1, \log\left(\frac{\varrho_2^{\frac{1}{m_2}}}{\varrho_1^{\frac{1}{m_1}}} \right), \ldots, \log\left(\frac{\varrho_n^{\frac{1}{m_n}}}{\varrho_1^{\frac{1}{m_1}}} \right) \right) =: \Psi(\rho_1, \ldots, \rho_n). \tag{16}$$

is a diffeomorphism, and the system (1) is transformed to

$$\partial_t \varrho + \mathrm{div}(\varrho \boldsymbol{u}) = 0,$$

$$\varrho \partial_t \boldsymbol{u} + \frac{\varrho \nabla \varrho}{\Sigma_\varrho} + \sum_{l=2}^{n} \left(\varrho_l - \frac{m_l \varrho_l \varrho}{\Sigma_\varrho} \right) \nabla h_{l-1} + \varrho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \mu \Delta \boldsymbol{u} + (\mu + \nu)\nabla \mathrm{div}\boldsymbol{u},$$

$$\sum_{l=1}^{n-1} \mathcal{R}_{kl}(\partial_t h_l + \boldsymbol{u} \cdot \nabla h_l) + \left(\varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho} \right) \mathrm{div}\, \boldsymbol{u} = \mathrm{div}\left(\sum_{l=1}^{n-1} \mathcal{B}_{kl} \nabla h_l \right), \tag{17}$$

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with the boundary conditions

$$\boldsymbol{u} = 0, \quad \sum_{l=1}^{n-1} \mathcal{B}_{kl} \nabla h_l \cdot \boldsymbol{n} = 0, \quad k = 1, \ldots, n-1, \quad on \ (0, T) \times \partial\Omega, \tag{18}$$

and the initial conditions

$$(\boldsymbol{u}, \rho, \{h_k\}_{k=1,\ldots,n-1})|_{t=0} = (\boldsymbol{u}^0, \rho^0, \{h_k^0\}_{k=1,\ldots,n-1}) = \Psi(\rho_1^0(x), \ldots \rho_n^0(x)), \tag{19}$$

where

$$\Sigma_\varrho = \sum_{k=1}^{n} m_k \varrho_k \tag{20}$$

and $\mathcal{R}$ and $\mathcal{B}$ are $(n-1) \times (n-1)$ matrices given by

$$\mathcal{R}_{kl} = m_{k+1} \varrho_{k+1} \delta_{kl} - \frac{m_{k+1} m_{l+1} \varrho_{k+1} \varrho_{l+1}}{\Sigma_\varrho}, \tag{21}$$

$$\mathcal{B}_{kl} = \frac{\varrho_{k+1} \varrho_{l+1} D_{k+1,l+1}}{p}. \tag{22}$$

for $k, l = 1, \ldots, n-1$. Moreover, the matrix $\mathcal{R}$ is uniformly coercive in (x, t) and the same property holds for $\mathcal{B}$ provided that either:
Condition 1: The matrix C is of the form (13)
or
Condition 2: Ω is bounded and (12) is satisfied for $x \in \overline{\Omega}$, $t \in [0, T]$.

The local well-posedness of system (17),(18) in the maximal $L_p - L_q$ regularity setting is provided by our main result below.

Theorem 2.2 *Assume that*

- *$2 < p < \infty$, $3 < q < \infty$, $2/p + 3/q < 1$ and $L > 0$;*

- *Ω is a uniform C^3 domain in $\mathbb{R}^N$ $(N = 3)$;*

- *there exists a constant $C > 0$ such that*

$$\forall k, l \in 1, \ldots, n \quad \|\nabla D_{kl}(t, \cdot)\|_{L_q(\Omega)} \leq C \sum_{j=1}^{n} \|\nabla \rho_j(t, \cdot)\|_{L_q(\Omega)} \quad a.e. \ in \ (0, T); \tag{23}$$

- *there exist positive numbers a_1 and a_2 for which*

$$a_1 \leq \rho_k^0(x) \leq a_2 \quad \forall x \in \overline{\Omega}, \ k \in 1, \ldots, n. \tag{24}$$

Let $\rho_k^0(x), k = 1, \ldots n$, and $\boldsymbol{u}^0(x)$ be initial data for Eq. (1) and let

$$(\rho^0(x), h_1^0(x), \ldots, h_{n-1}^0(x)) = \Psi(\rho_1^0(x), \ldots \rho_n^0(x)).$$

Then, there exists a time $T > 0$ depending on a_1, a_2 and L such that if the initial data satisfy the condition:

$$\|\nabla(\rho_1^0, \ldots, \rho_n^0)\|_{L_q(\Omega)} + \|\boldsymbol{u}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|h_1^0, \ldots, h_{n-1}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq L \tag{25}$$

and the compatibility condition:

$$\boldsymbol{u}^0|_\Gamma = 0, \quad \nabla h_k^0 \cdot \boldsymbol{n}|_\Gamma = 0, \quad k = 1, \ldots, n-1, \tag{26}$$

then problem (17) *with boundary conditions* (18) *and initial conditions* (19) *admits a unique solution* $(\rho, \boldsymbol{u}, h_1, \ldots, h_{n-1})$ *with*

$$\rho - \rho^0 \in H_p^1((0,T), H_q^1(\Omega)), \quad \boldsymbol{u} \in H_p^1((0,T), L_q(\Omega)^3) \cap L_p((0,T), H_q^2(\Omega)^3),$$
$$h_1, \ldots, h_{n-1} \in H_p^1((0,T), L_q(\Omega)) \cap L_p((0,T), H_q^2(\Omega))$$

possessing the estimates:

$$\|\rho - \rho^0\|_{H_p^1((0,T), H_q^1(\Omega))} + \|\partial_t(\boldsymbol{u}, h_1, \ldots, h_{n-1})\|_{L_p((0,T), L_q(\Omega))} + \|(\boldsymbol{u}, h_1, \ldots, h_{n-1})\|_{L_p((0,T), H_q^2(\Omega))} \leq CL,$$

$$a_1 \leq \rho(x,t) \leq na_2 + a_1 \quad \text{for } (x,t) \in \Omega \times (0,T), \quad \int_0^T \|\nabla \boldsymbol{u}(\cdot, s)\|_{L_\infty(\Omega)} \leq \delta.$$

Here, C is some constant independent of L, and δ is a small positive parameter.

Remark 2.3 *Notice that due to* (11) *the requirement* (23) *is satisfied for the special form* (13) *provided $C_1 \leq |\rho_k| \leq C_2$ for some positive constants $C_1 < C_2$.*

Remark 2.4 *The parameter δ above remains small for large times. This is especially important for the existence of global-in-time solutions, not included in the present study.*

The outline of the rest of the paper is the following. In Section 3 we prove Proposition 2.1, i.e. we derive the normal form which is a symmetrization of system (1) with omitted equation for ϱ_1. We also show that the obtained system is uniformly parabolic. We show this property regardless of boundedness of Ω for the special form (13), while for general diffusion matrix we require boundedness of Ω. Then, in Sections 4–6 we prove Theorem 2.2. To this purpose we first rewrite the problem in Lagrangian coordinates in Section 4; this step is necessary to apply the maximal $L_p - L_q$ regularity theory. In Section 5 we linearize the problem around the initial condition. Section 6 is dedicated to nonlinear estimates which are used to close the fixed point argument and prove Theorem 2.2 using the existence result for linearized system from Theorem 5.2, recalled in the Appendix. The proof of Theorem 5.2 can be found in [31].

3 Proof of Proposition 2.1

3.1 Derivation of the normal form

The proof of Proposition 2.1 is split into a couple of steps. First we derive the normal form of system (1). By the change of unknowns (16) we have

$$[\nabla \varrho, \nabla h_1 \ldots \nabla h_{n-1}]^T = A[\nabla \varrho_1, \ldots \nabla \varrho_n]^T \tag{27}$$

with

$$A = \begin{pmatrix} 1 & 1_{1 \times (n-1)} \\ \left(-\frac{1}{m_1 \varrho_1}\right)_{(n-1) \times 1} & \text{diag}\left(\frac{1}{m_2 \varrho_2}, \ldots, \frac{1}{m_n \varrho_n}\right) \end{pmatrix}. \tag{28}$$

The matrix A is diagonal except the first row and first column, which also have quite simple structure. It is therefore easy to observe that its inverse reads

$$A^{-1} = \begin{pmatrix} \frac{m_1\varrho_1}{\Sigma_\varrho} & \left[\left(-\frac{m_1\varrho_1 m_k\varrho_k}{\Sigma_\varrho}\right)_{k=2\ldots n}\right]_{1\times(n-1)} \\ \left[\left(\frac{m_k\varrho_k}{\Sigma_\varrho}\right)_{k=2,\ldots,n}\right]_{(n-1)\times 1} & \mathcal{R} \end{pmatrix}, \tag{29}$$

where

$$\Sigma_\varrho = \sum_{k=1}^{n} m_k\varrho_k \tag{30}$$

and $\mathcal{R}$ is matrix of dimension $n-1$ given by

$$\mathcal{R}_{kl} = m_{k+1}\varrho_{k+1}\delta_{kl} - \frac{m_{k+1}m_{l+1}\varrho_{k+1}\varrho_{l+1}}{\Sigma_\varrho}, \quad k,l = 1,\ldots,n-1. \tag{31}$$

Therefore, from (27) we obtain

$$[\nabla\varrho_1, \ldots \nabla\varrho_n]^T = A^{-1}[\nabla\varrho, \nabla h_1 \ldots \nabla h_{n-1}]^T \tag{32}$$

and, analogously, for the time derivative

$$[\partial_t\varrho_1, \ldots \partial_t\varrho_n]^T = A^{-1}[\partial_t\varrho, \nabla h^1 \ldots \partial_t h_{n-1}]^T. \tag{33}$$

From (32), (33), and (34) we infer

$$\partial_t\varrho_{k+1} + \boldsymbol{u} \cdot \nabla\varrho_{k+1} = \frac{m_{k+1}\varrho_{k+1}}{\Sigma_\varrho}(\partial_t\varrho + \boldsymbol{u} \cdot \nabla\varrho) + \sum_{l=1}^{n-1} \mathcal{R}_{kl}(\partial_t h_l + \boldsymbol{u} \cdot \nabla h_l), \quad k = 1,\ldots,n-1. \tag{34}$$

However, from (1) we have

$$\partial_t\varrho + \boldsymbol{u} \cdot \nabla\varrho = -\varrho \operatorname{div} \boldsymbol{u}$$

as well as

$$\partial_t\varrho_k + \boldsymbol{u} \cdot \nabla\varrho_k = -\varrho_k \operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{F}_k.$$

Inserting these relations to (34) we obtain

$$\sum_{l=1}^{n-1} \mathcal{R}_{kl}(\partial_t h_l + \boldsymbol{u} \cdot \nabla h_l) + \left(\varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho}\right) \operatorname{div} \boldsymbol{u} = -\operatorname{div} \boldsymbol{F}_{k+1}. \tag{35}$$

We can further rewrite the rhs of the above equations. For this purpose we observe that

$$-\frac{\nabla p_1}{\varrho}\left(\frac{1}{\varrho_1}\sum_{l=2}^{n}\varrho_l C_{kl} + \frac{\varrho_1}{\varrho_1}C_{k1}\right) = -\frac{\nabla p_1}{\varrho_1}\sum_{l=1}^{n} Y_l C_{kl} = 0$$

due to (9). Therefore, denoting

$$\bar{m} = \frac{\varrho}{p} \tag{36}$$

we obtain from (10)

$$
\begin{aligned}
-\boldsymbol{F}_k &= \frac{1}{p}\sum_{l=1}^{n} C_{kl}\nabla p_l \\
&= \frac{\bar{m}}{\varrho}\left[\sum_{l=1}^{n} C_{kl}\nabla p_l - \nabla p_1\left(\frac{1}{\varrho_1}\sum_{l=2}^{n} C_{kl}\varrho_l + C_{k1}\right)\right] \\
&= \frac{\bar{m}}{\varrho}\sum_{l=2}^{n} C_{kl}\left(\nabla p_l - \frac{\varrho_l}{m_1}\frac{\nabla\varrho_1}{\varrho_1}\right) \\
&= \frac{\bar{m}}{\varrho}\sum_{l=2}^{n}\varrho_l C_{kl}\left(\frac{\nabla\varrho_l}{m_l\varrho_l} - \frac{\nabla\varrho_1}{m_1\varrho_1}\right) \\
&= \frac{\bar{m}}{\varrho}\sum_{l=2}^{n}\varrho_k\varrho_l D_{kl}\nabla h_{l-1}.
\end{aligned}
\tag{37}
$$

Now let us transform the pressure term, from (32) we have

$$
\begin{aligned}
\nabla p &= \sum_{k=1}^{n}\frac{\nabla\varrho_k}{m_k} \\
&= \frac{1}{m_1}\left(\frac{m_1\varrho_1}{\Sigma_\varrho}\nabla\varrho - \sum_{k=2}^{n}\frac{m_1\varrho_1 m_k\varrho_k}{\Sigma_\varrho}\nabla h_k\right) \\
&\quad + \sum_{l=2}^{n}\frac{1}{m_l}\left(\frac{m_l\varrho_l}{\Sigma_\varrho}\nabla\varrho + m_l\left(\varrho_l - \frac{m_l\varrho_l^2}{\Sigma_\varrho}\right)\nabla h_{l-1} - \sum_{\substack{k>1\\k\neq l}}\frac{m_l\varrho_l m_k\varrho_k}{\Sigma_\varrho}\nabla h_k\right) \\
&= \frac{\varrho}{\Sigma_\varrho}\nabla\varrho + \sum_{k=1}^{n-1} A_k\nabla h_k,
\end{aligned}
\tag{38}
$$

where we denoted

$$
A_k = \varrho_{k+1} - \frac{1}{\Sigma_\varrho}\left[m_{k+1}\varrho_{k+1}^2 + m_{k+1}\varrho_{k+1}\sum_{l\neq k+1}\varrho_l\right] = \varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho}.
\tag{39}
$$

From (35)-(39) we obtain the explicit form of the symmetrized system (17).

Now we have to rewrite the boundary conditions (3) for the symmetrized system (17). First note that with equation for ϱ_1 being omitted, the system (17) needs to be supplemented only with the boundary conditions for $n-1$ last species densities; due to (37) we get

$$
\boldsymbol{u} = 0, \quad \frac{\bar{m}}{\varrho}\sum_{l=2}^{n}\varrho_k\varrho_l D_{kl}\nabla h_{l-1}\cdot\boldsymbol{n} = 0, \quad k = 2,\ldots,n, \quad \text{on } (0,T)\times\partial\Omega
\tag{40}
$$

which is exactly (18) and it is a natural boundary conditions in view of the second order term in $(17)_3$.

3.2   Coercivity properties

A keynote requirement necessary to prove our main result is the coercivity of matrices $\mathcal{R}$ and $\mathcal{B}$. To prove this we need to know that fractional densities are bounded from below by a positive constant. However,

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the statement of Theorem 2.2 provides us only with bounded functions h_i given by (16). Let us first see that these conditions are in fact equivalent. The implication in one direction follows immediately from (16), for the other one we have:

Lemma 3.1 *Let h_i given by (16) be bounded and let*

$$\varrho \geq C > 0. \tag{41}$$

Then

$$\varrho_i \geq C > 0, \quad i = 1, \ldots, n. \tag{42}$$

Proof. Assume $\exists i \in \{1, \ldots, n-1\}$ and (x_0, t_0) s.t.

$$\lim_{(x,t) \longrightarrow (x_0, t_0)} \varrho_{i+1}(x, t) = 0.$$

Then

$$\lim_{(x,t) \longrightarrow (x_0, t_0)} \varrho_1(x, t) = 0 \tag{43}$$

since otherwise $h_i(x, t)$ would be unbounded from below. This in turn implies that

$$\lim_{(x,t) \longrightarrow (x_0, t_0)} \varrho_{k+1}(x, t) = 0 \quad \forall \, 1 \leq k \leq n-1 \tag{44}$$

since otherwise corresponding h_k would be unbounded from above. This means that $\sum_{k=1}^n \varrho_k(x, t) = 0$ which contradicts (41).

$\square$

We are now ready to prove the more direct coercivity of $\mathcal{R}$. Below, $\xi = (\xi_1, \ldots \xi_n)$ is a vector of complex numbers, $\overline{\xi} = (\overline{\xi_1}, \ldots \overline{\xi_n})$ is a vector of their complex conjugates, and $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{C}$.

Lemma 3.2 *Let assumptions of Lemma 3.1 be satisfied. Then there exists a constant $C_1 > 0$ independent of (x,t) such that*

$$\langle \mathcal{R}(x, t) \xi, \overline{\xi} \rangle \geq C_1 |\xi|^2. \tag{45}$$

Proof. Notice first that $\mathcal{R}_{kk} > 0$ for every $k = 1, \ldots, n-1$. We rewrite $\mathcal{R}_{kk}$ as

$$\mathcal{R}_{kk} = \frac{1}{\Sigma_\varrho} m_{k+1} \varrho_{k+1} (\Sigma_\varrho - m_{k+1} \varrho_{k+1}) = \frac{1}{\Sigma_\varrho} m_{k+1} \varrho_{k+1} \sum_{l=1, \; l \neq k+1}^{n} m_l \varrho_l.$$

Then we have due to symmetry of $\mathcal{R}$

$$
\begin{aligned}
\langle \mathcal{R} \xi, \overline{\xi} \rangle &= \sum_{k=1}^{n-1} \mathcal{R}_{kk} |\xi_k|^2 + \sum_{l=1}^{n-1} \sum_{k<l} \mathcal{R}_{kl} (\xi_k \overline{\xi_l} + \xi_l \overline{\xi_k}) \\
&\geq \sum_{k=1}^{n-1} \mathcal{R}_{kk} |\xi_k|^2 - \sum_{l=1}^{n-1} \sum_{k<l} |\mathcal{R}_{kl}| (|\xi_k|^2 + |\xi_l|^2) \\
&= \frac{m_1 \varrho_1}{\Sigma_\varrho} \sum_{k=1}^{n-1} m_{k+1} \varrho_{k+1} |\xi_k|^2 \\
&\geq \frac{m_1 \varrho_1}{\Sigma_\varrho} \min_{k \neq 1} \{ m_k \varrho_k \} |\xi|^2,
\end{aligned}
\tag{46}
$$

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which proves (45).

$\square$

Although (12) implies only semi-definitness of $D \geq 0$, the change of unknowns introduced in the previous section and resulting reduction by one row and column enables to deduce ellipticity of the resulting matrix which follows from the properties of D. The next lemma shows the coercivity of $\mathcal{B}$.

Lemma 3.3 *Assume that one of Conditions 1,2 from Proposition 2.1 hold. Then there exists a constant $C_2 > 0$ independent of (x,t) such that*

$$\langle \mathcal{B}(x,t)\xi, \overline{\xi} \rangle \geq C_2 |\xi|^2 \quad \forall\, (x,t) \in \Omega \times [0,T]. \tag{47}$$

Proof. It is convenient to rewrite the entries of $\mathcal{B}$ as

$$\mathcal{B}_{kl} = \frac{\varrho}{p} Y_{k+1} Y_{l+1} \frac{C_{k+1,l+1}}{Y_{k+1}} = \frac{\varrho}{p} Y_{l+1} C_{k+1,l+1}. \tag{48}$$

Under Condition 1 we therefore have

$$\mathcal{B} = \frac{\varrho}{p} \begin{pmatrix} Y_2 Z_2 & -Y_2 Y_3 & \ldots & -Y_2 Y_n \\ -Y_3 Y_2 & Y_3 Z_3 & \ldots & -Y_3 Y_n \\ \ldots & & & \\ -Y_n Y_2 & \ldots & & Y_n Z_n \end{pmatrix}. \tag{49}$$

In order to compute $\det \mathcal{B}$ we transform the matrix with elementary operations. First we add $n-1$ first rows to the last one. Denoting the new matrix by $\mathcal{B}^1$ we have

$$\mathcal{B}^1_{nn} = Y_n Z_n - Y_n \sum_{j=2}^{n-1} Y_j = Y_n Y_1$$

and for $k < n$ we have

$$\mathcal{B}^1_{nk} = -Y_n Y_k + Y_k Z_k - Y_k \sum_{j \neq k, j \geq 2} Y_j = Y_k Y_1,$$

therefore

$$\mathcal{B}^1 = \frac{\varrho}{p} \begin{pmatrix} Y_2 Z_2 & -Y_2 Y_3 & \ldots & -Y_2 Y_n \\ -Y_3 Y_2 & Y_3 Z_3 & \ldots & -Y_3 Y_n \\ \ldots & & & \\ Y_1 Y_2 & Y_1 Y_3 & \ldots & Y_1 Y_n \end{pmatrix}. \tag{50}$$

Notice that all entries of the last column contain Y_n and all entries of the last row contain Y_1, therefore

$$\det \mathcal{B} = \left(\frac{\varrho}{p} \right)^{n-1} Y_1 Y_n \det \underbrace{\begin{pmatrix} Y_2 Z_2 & -Y_2 Y_3 & \ldots & -Y_2 \\ -Y_2 Y_3 & Y_3 Z_3 & \ldots & -Y_3 \\ \ldots & & & \\ Y_2 & Y_3 & \ldots & 1 \end{pmatrix}}_{\mathcal{B}^2}. \tag{51}$$

Now we can easily diagonalize part of the above matrix. For this purpose we add to each k-th row, $k = 1 \ldots n-1$, the last row multiplied by Y_{k+1}. Then all the entries except the diagonal becomes zero. Namely, we have

$$\mathcal{B}^2_{k,\cdot} + Y_{k+1} \mathcal{B}^2_{n,\cdot} = Y_{k+1} \sum_{j=1}^{n} Y_j \mathbf{e}_k.$$

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Therefore (51) yields

$$\det \mathcal{B} = \left(\frac{\varrho}{p}\right)^{n-1} \prod_{k=1}^{n} Y_k \left(\sum_{k=1}^{n} Y_k\right)^{n-1} \geq C > 0, \tag{52}$$

since $Y_k(x,t) > C$ for every $k = 1,\ldots,n$ uniformly w.r.t. (x,t), due to (42). Next, denoting

$$\det \mathcal{B}_k = \begin{vmatrix} \mathcal{B}_{11} & \ldots & \mathcal{B}_{1k} \\ \vdots & \ddots & \vdots \\ \mathcal{B}_{k1} & \ldots & \mathcal{B}_{kk} \end{vmatrix} \tag{53}$$

we have $\det \mathcal{B}_k > 0$. Therefore, all the leading principal minors of matrix $\mathcal{B}$ are positive and hence we have shown

$$\mathcal{B}(x,t) > 0, \quad \det\mathcal{B}(x,t) \geq C > 0 \quad \text{uniformly in } (x,t). \tag{54}$$

Now from (54) it's easy to deduce (47). For this purpose note that the eigenvectors $\zeta_i(x,t)$ of $\mathcal{B}(x,t)$ form an orthonormal basis of $\mathbb{R}^n$ and $\mathcal{B}(x,t)$ in this basis is in a form

$$\mathcal{B}(x,t) = \text{diag}(\lambda_1(x,t),\ldots,\lambda_n(x,t)\}, \quad \lambda_i(x,t) \geq C > 0 \quad \text{uniformly in } (x,t). \tag{55}$$

Therefore, denoting $\xi = \sum_{i=1}^{n} \alpha_i \zeta_i$ we have

$$\langle B(x,t)\xi, \bar{\xi}\rangle = \sum_{i=1}^{n} \lambda_i(x,t)\alpha_i^2 \geq \min_i\{\lambda_i(x,t)\} \sum_{i=1}^{n} \alpha_i^2 \geq C|\xi|^2 \quad \text{uniformly in } (x,t).$$

Now let us consider a general form of D satisfying the assumptions (12). In this case we use the form of $\mathcal{B}$ as in (22). In particular, each entry of k-th row of $\mathcal{B}$ contains Y_{k+1}, therefore

$$\det \mathcal{B} = \left(\frac{\varrho^2}{p}\right)^{n-1} Y_2 \ldots Y_n \begin{vmatrix} Y_2 D_{22} & Y_3 D_{23} & \ldots & Y_n D_{2n} \\ Y_2 D_{32} & Y_3 D_{33} & \ldots & Y_n D_{3n} \\ \ldots & & & \\ Y_2 D_{n2} & Y_3 D_{n3} & \ldots & Y_n D_{nn} \end{vmatrix} \tag{56}$$

Similarly, since each entry of k-th column contains Y_{k+1}, we have

$$\det \mathcal{B} = \left(\frac{\varrho^2}{p}\right)^{n-1} (Y_2 \ldots Y_n)^2 \det \underbrace{\begin{pmatrix} D_{22} & D_{23} & \ldots & D_{2n} \\ D_{32} & D_{33} & \ldots & D_{3n} \\ \ldots & & & \\ D_{n2} & D_{n3} & \ldots & D_{nn} \end{pmatrix}}_{:=\bar{D}} \tag{57}$$

Due to (42) we have $Y_2 \ldots Y_n \geq C > 0$, and so, the whole coefficient in front of matrix $\bar{D}$ is positive. Notice however that we only have $D \geq 0$ in general, but $\bar{D}$ is a $(n-1) \times (n-1)$ sub-matrix of D for which we can show positive definiteness. Assume on the contrary that there is a vector $[v_2,\ldots,v_n] \neq 0$, s.t.

$$\bar{D}[v_2,\ldots,v_n] = 0.$$

Then one would also have that

$$D[0, v_2,\ldots,v_n] = 0,$$

which is in contradiction with the fact that $\text{Ker}D = \text{lin}\{\vec{Y}\}$ and all Y_k are strictly positive. Similarly we show that the minors (53) are positive, hence we conclude that

$$D(x) > 0. \tag{58}$$

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Now, as for each (x, t) fixed, $D(x, t)$ is a linear operator, we have

$$\forall (x, t) \in \Omega \; \exists c(x, t) > 0 \; s.t. \; \langle \bar{D}(x, t)\xi, \bar{\xi} \rangle \; \geq \; c(x, t)|\xi|^2, \tag{59}$$

where

$$c(x, t) = \min_{|\xi|=1} \langle \bar{D}(x, t)\xi, \bar{\xi} \rangle.$$

Finally, if Condition 2 is satisfied, we can have the function $c(x, t) > 0$ defined on a compact set $\overline{\Omega} \times [0, T]$, hence

$$\exists \kappa > 0 : \; c(x, t) \geq \kappa \quad \forall (x, t) \in \overline{\Omega} \times [0, T],$$

which completes the proof.

$\square$

Let us finish this section with a couple of remarks.

Remark 3.4 *The method which we applied for the special structure* (13) *can be to some extent repeated for a general matrix using the fact that* $\mathrm{Ker}D = \mathrm{lin}\{\vec{Y}\}$. *However, in the last step we do not obtain a diagonal sub-matrix but just a matrix with modified entries. For this matrix coercivity probably could be shown under some additional assumptions on* D *also for unbounded domain, we leave this direction for further investigation in the future.*

Remark 3.5 *Due to conditions* (47) *we can apply the inverse of* $\mathcal{B}$ *to the boundary conditions* (18) *which leads to equivalent formulation of the boundary condition in the standard form*

$$\boldsymbol{u} = 0, \quad \nabla h_k \cdot \boldsymbol{n} = 0, \quad k = 1, \ldots, n - 1, \quad on \; (0, T) \times \partial\Omega. \tag{60}$$

4 Lagrangian coordinates

We begin the proof of Theorem 2.2 by transforming the symmetrized system (17) to the Lagrangian coordinates $x = \Phi(y, t)$ related to the vector field $\boldsymbol{v}$:

$$x = y + \int_0^t \boldsymbol{v}(y, s) \, ds. \tag{61}$$

Then for any differentiable function f we have

$$\partial_t f(\Phi(t, y), t) = \partial_t f + \boldsymbol{u} \cdot \nabla_x f. \tag{62}$$

Since

$$\frac{\partial x_i}{\partial y_j} = \delta_{ij} + \int_0^t \frac{\partial v_i}{\partial y_j}(y, s) \, ds, \tag{63}$$

assuming that

$$\sup_{t \in (0, T)} \int_0^t \|\nabla \boldsymbol{v}(\cdot, s)\|_{L_\infty(\Omega)} \, ds \leq \delta \tag{64}$$

for sufficiently small positive constant δ, the matrix $\partial x / \partial y = (\partial x_i / \partial y_j)$ has the inverse

$$\left(\frac{\partial x_i}{\partial y_j} \right)^{-1} = \boldsymbol{I} + \mathbf{V}^0(\mathbf{k_v}), \quad \mathbf{k_v} = \int_0^t \nabla \boldsymbol{v}(y, s) \, ds. \tag{65}$$

Here , $\boldsymbol{I}$ is the $N \times N$ identity matrix, and $\mathbf{V}^0(\mathbf{k})$ is the $N \times N$ matrix of smooth functions with $\mathbf{V}^0(0) = 0$. We have

$$\nabla_x = (\boldsymbol{I} + \mathbf{V}^0(\mathbf{k_v}))\nabla_y, \quad \frac{\partial}{\partial x_i} = \sum_{j=1}^{N}(\delta_{ij} + V_{ij}^0(\mathbf{k_v}))\frac{\partial}{\partial y_j}. \tag{66}$$

Moreover (see for instance [36]), the map $\Phi(y,t)$ is bijection from Ω onto Ω.

We define our unknown functions in Lagrangian coordinates:

$$\boldsymbol{v}(y,t) = \boldsymbol{u}(x,t), \quad \eta(y,t) = \rho(x,t), \quad \vartheta_i(y,t) = h_i(x,t), \; i = 1,\ldots,n-1, \tag{67}$$

and we denote

$$\vec{\vartheta} := (\vartheta_1,\ldots,\vartheta_{n-1})^\top.$$

We now show that $U = (\boldsymbol{v}, \eta, \vec{\vartheta})$ satisfies the system

$$\partial_t \eta + \eta \operatorname{div} \boldsymbol{v} = R_1(U)$$

$$\eta \partial_t \boldsymbol{v} - \mu \Delta \boldsymbol{v} - (\mu + \nu)\nabla \operatorname{div} \boldsymbol{v} + \frac{\eta}{\Sigma_\varrho}\nabla \eta + \sum_{l=1}^{n-1}\left(\varrho_{l+1} - \frac{m_{l+1}\varrho_{l+1}\varrho}{\Sigma_\varrho} \right)\nabla \vartheta_l = \mathbf{R}_2(U) \tag{68}$$

$$\sum_{l=1}^{n-1}\mathcal{R}_{kl}\partial_t \vartheta_l + \left(\varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho} \right)\operatorname{div} \boldsymbol{v} - \operatorname{div}\left(\sum_{l=1}^{n-1}\mathcal{B}_{kl}\nabla \vartheta_l \right) = R_3^k(U), \quad k = 1,\ldots,n-1$$

supplemented with the boundary conditions

$$\boldsymbol{v}|_{\partial\Omega} = 0, \quad \nabla\vartheta_k \cdot \boldsymbol{n}|_{\partial\Omega} = R_4^k(U), \quad k = 1,\ldots,n-1 \tag{69}$$

where

$$(\varrho_1,\ldots,\varrho_n) = (\varrho_1,\ldots,\varrho_n)(\eta,\vec{\vartheta}) = \Psi^{-1}(\eta,\vec{\vartheta}). \tag{70}$$

Remark 4.1 *In the remainder of the paper we write simply ϱ_k keeping in mind that we have the dependence* (70) *since we work in Lagrangian coordinates.*

We now derive the precise form of terms on the right hand side of (68),(69). First of all we have

$$\operatorname{div}_x = \operatorname{div}_y + \sum_{i,j=1}^{N}V_{ij}^0(\mathbf{k_v})\frac{\partial v_i}{\partial y_j}, \tag{71}$$

therefore we easily obtain $(68)_1$ with

$$R_1(U) = -\eta \sum_{i,j=1}^{N}V_{ij}^0(\mathbf{k_v})\frac{\partial v_i}{\partial y_j}. \tag{72}$$

Now we need to transform second order operators. By (66), we have

$$\Delta_x \boldsymbol{u} = \sum_{k=1}^{N}\frac{\partial}{\partial x_k}\left(\frac{\partial \boldsymbol{u}}{\partial x_k} \right) = \sum_{k,l,m=1}^{N}\left(\delta_{kl} + V_{kl}^0(\mathbf{k_v}) \right)\frac{\partial}{\partial y_l}\left(\left(\delta_{km} + V_{km}^0(\mathbf{k_v}) \right)\frac{\partial \boldsymbol{v}}{\partial y_m} \right).$$

Therefore

$$\Delta_x \boldsymbol{u} = \Delta_y \boldsymbol{v} + A_{2\Delta}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} + A_{1\Delta}(\mathbf{k_v})\nabla_y \boldsymbol{v}$$

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with

$$A_{2\Delta}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} = 2\sum_{l,m=1}^N V_{kl}^0(\mathbf{k_v})\frac{\partial^2 \boldsymbol{v}}{\partial y_l \partial y_m} + \sum_{k,l,m=1}^N V_{kl}^0(\mathbf{k_v})V_{km}^0(\mathbf{k_v})\frac{\partial^2 \boldsymbol{v}}{\partial y_l \partial y_m}, \tag{73}$$

$$\begin{aligned}
A_{1\Delta}(\mathbf{k_v})\nabla_y \boldsymbol{v} &= \sum_{l,m=1}^N (\nabla_{\mathbf{k_v}} V_{lm}^0)(\mathbf{k_v})\int_0^t (\partial_l \nabla_y \boldsymbol{v})\,ds\frac{\partial \boldsymbol{v}}{\partial y_m} \\
&\quad + \sum_{k,l,m=1}^N V_{kl}^0(\mathbf{k_v})(\nabla_{\mathbf{k_v}} V_{km}^0)(\mathbf{k_v})\int_0^t \partial_l \nabla_y \boldsymbol{v}\,ds\frac{\partial \boldsymbol{v}}{\partial y_m},
\end{aligned} \tag{74}$$

where $(\nabla_{\mathbf{k_v}} V_{km}^0)(\mathbf{k_v})$ denotes $\left(V_{km}^0\right)'(\mathbf{k_v})$.

Similarly for $i \in \{1,\dots,N\}$ we have

$$\frac{\partial}{\partial x_j}\mathrm{div}_x \boldsymbol{u} = \sum_{k=1}^N (\delta_{jk} + V_{jk}^0(\mathbf{k_v}))\frac{\partial}{\partial y_k}\left(\mathrm{div}_y \boldsymbol{v} + \sum_{l,m=1}^N V_{lm}^0(\mathbf{k_v})\frac{\partial v_l}{\partial y_m}\right),$$

so we obtain

$$\frac{\partial}{\partial x_j}\mathrm{div}_x \boldsymbol{u} = \frac{\partial}{\partial y_j}\mathrm{div}_y \boldsymbol{v} + A_{2\mathrm{div},j}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} + A_{1\mathrm{div},j}(\mathbf{k_v})\nabla_y \boldsymbol{v},$$

where

$$A_{2\mathrm{div},j}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} = \sum_{l,m=1}^N V_{lm}^0(\mathbf{k_v})\frac{\partial^2 v_l}{\partial y_m \partial y_j} + \sum_{k=1}^N V_{jk}^0(\mathbf{k_v})\frac{\partial}{\partial y_k}\mathrm{div}_y \boldsymbol{v} + \sum_{k,l=1}^N V_{jk}^0(\mathbf{k_v})V_{lm}^0(\mathbf{k_v})\frac{\partial^2 v_l}{\partial y_k \partial y_m}, \tag{75}$$

$$\begin{aligned}
A_{1\mathrm{div},j}(\mathbf{k_v})\nabla_y \boldsymbol{v} &= \sum_{l,m=1}^N (\nabla_{\mathbf{k_v}} V_{lm}^0)(\mathbf{k_v})\int_0^t \partial_j \nabla_y \boldsymbol{v}\,ds\frac{\partial v_l}{\partial y_m} \\
&\quad + \sum_{k,l,m=1}^N V_{jk}^0(\mathbf{k_v})(\nabla_{\mathbf{k_v}} V_{lm}^0)(\mathbf{k_v})\int_0^t \partial_k \nabla_y \boldsymbol{v}\,ds\frac{\partial v_l}{\partial y_m}.
\end{aligned} \tag{76}$$

Therefore, transforming also $\nabla_x \varrho$ and $\nabla_x h_l$ we obtain $(68)_2$ with

$$\begin{aligned}
\mathbf{R}_2(U) =&\mu A_{2\Delta}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} + \mu A_{1\Delta}(\mathbf{k_v})\nabla_y \boldsymbol{v} + \nu A_{2\mathrm{div}}(\mathbf{k_v})\nabla_y^2 \boldsymbol{v} + \nu A_{1\mathrm{div}}(\mathbf{k_v})\nabla_y \boldsymbol{v} \\
&+ \frac{\eta}{\Sigma_\rho}\mathbf{V}^0(\mathbf{k_v})\nabla_y \eta + \mathbf{V}^0(\mathbf{k_v})\sum_{l=2}^n \left(\varrho_l - \frac{m_l \varrho_l \varrho}{\Sigma_\varrho}\right)\nabla_y \vartheta_{l-1},
\end{aligned} \tag{77}$$

where $A_{i\mathrm{div}}\nabla_y^i \boldsymbol{v}$, $i = 1,2$ are vectors with coordinates $A_{i\mathrm{div},j}\nabla_y^i \boldsymbol{v}$, $j = 1,\dots,N$.

Finally we transform the species balance equations. We have

$$\begin{aligned}
\mathrm{div}_x(\mathcal{B}_{kl}\nabla_x h_l) &= \mathcal{B}_{kl}(\Delta_y \vartheta_l + A_{2\Delta}(\mathbf{k_v})\nabla_y^2 \vartheta_l + A_{1\Delta}(\mathbf{k_v})\nabla_y \vartheta_l) \\
&\quad + \left(\nabla_y \mathcal{B}_{kl} + \mathbf{V}^0(\mathbf{k_v})\nabla_y \mathcal{B}_{kl}\right)\left(\nabla_y \vartheta_l + \mathbf{V}^0(\mathbf{k_v})\nabla_y \vartheta_l\right) \\
&= \mathrm{div}_y(\mathcal{B}_{kl}\nabla_y \vartheta_l) + R_3^{kl}(U),
\end{aligned}$$

where

$$\begin{aligned}
R_3^{kl}(U) =&\mathcal{B}_{kl}(A_{2\Delta}(\mathbf{k_v})\nabla_y^2 \vartheta_l + A_{1\Delta}(\mathbf{k_v})\nabla_y \vartheta_l) \\
&+ \mathbf{V}^0(\mathbf{k_v})\nabla_y \mathcal{B}_{kl}(\nabla_y \vartheta_l + \mathbf{V}^0(\mathbf{k_v})\nabla_y \vartheta_l) + (\nabla_y \mathcal{B}_{kl})\mathbf{V}^0(\mathbf{k_v})\nabla_y \vartheta_l.
\end{aligned} \tag{78}$$

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Therefore, transforming also div $\boldsymbol{u}$, we obtain $(68)_3$ with

$$R_3^k(U) = \sum_{l=1}^{n-1} R_3^{kl}(U) - \left(\varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho} \right) \sum_{j,m=1}^{N} V_{jm}^0(\mathbf{k_v}) \frac{\partial v_j}{\partial y_m}. \tag{79}$$

It remains to transform the boundary conditions. For this purpose notice that

$$\boldsymbol{n}(x) = \boldsymbol{n} \left(y + \int_0^t \boldsymbol{v}(y,s)\, ds \right) = \boldsymbol{n}(y) + \int_0^1 (\nabla \boldsymbol{n}) \left(y + \tau \int_0^t \boldsymbol{v}(y,s)\, ds \right) d\tau \int_0^t \boldsymbol{v}(y,s)\, ds,$$

and therefore we obtain (69) with

$$\begin{aligned}
R_4^k(U) =& \boldsymbol{n} \left(y + \int_0^t \boldsymbol{v}(y,s)\, ds \right) \cdot (\mathbf{V}^0(\mathbf{k_v}) \nabla_y \vartheta_k) \\
&+ \left\{ \int_0^1 (\nabla \boldsymbol{n}) \left(y + \tau \int_0^t \boldsymbol{v}(y,s)\, ds \right) d\tau \int_0^t \boldsymbol{v}(y,s)\, ds \right\} \cdot \nabla_y \vartheta_k.
\end{aligned} \tag{80}$$

5   Linearization

5.1   Formulation of linearized system

We now linearize the system in the Lagrangian coordinates (68) around the initial conditions. For this purpose we introduce small perturbations

$$\eta = \sigma + \varrho^0, \quad \varrho_l = \sigma_l + \varrho_l^0, \tag{81}$$

following the convention introduced in the previous section that ϱ_l are the functions in the Lagrangian coordinates.

Let us denote

$$\Sigma_\rho^0 = \sum_{k=1}^{n} m_k \rho_k^0, \quad p^0 = \sum_{k=1}^{n} \frac{\rho_k^0}{m_k},$$

and

$$h_k^0 = \frac{1}{m_k} \log \varrho_{k+1}^0 - \frac{1}{m_1} \log \varrho_1^0, \quad k = 1, \dots, n-1. \tag{82}$$

Observe that due to (24) we have

$$na_1 \leq \varrho^0 \leq na_2, \tag{83}$$

as well as

$$|h_k^0| \leq \frac{1}{m_{k+1}} |\log a_2| + \frac{1}{m_1} |\log a_1|.$$

The linearization of the continuity equation is straighforward, while for the momentum equation we have

$$\frac{\eta}{\Sigma_\rho} \nabla \eta = \frac{\rho^0}{\Sigma_\rho^0} \nabla \sigma + \rho^0 \nabla \sigma \left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0} \right) + \frac{\eta}{\Sigma_\rho} \nabla \rho^0 + \frac{\sigma}{\Sigma_\rho} \nabla \sigma$$

and

$$\frac{m_l \rho_l \rho}{\Sigma_\rho} = \frac{m_l \rho_l^0 \rho^0}{\Sigma_\rho^0} + m_l \rho^0 \rho_l^0 \left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0} \right) + \frac{m_l}{\Sigma_\rho} (\rho_l^0 \sigma + \rho^0 \sigma_l). \tag{84}$$

Similarly we linearize the $\mathcal{R}_{kl}$ in the species equations while for the reduced diffusion matrix we use

$$\mathcal{B}_{k-1,l-1} = \frac{\rho_k \rho_l D_{kl}}{p} = \frac{\rho_k^0 \rho_l^0 D_{kl}^0}{p^0} + \frac{\rho_k \rho_l D_{kl} - \rho_k^0 \rho_l^0 D_{kl}^0}{p} + \rho_k^0 \rho_l^0 D_{kl}^0 \left(\frac{1}{p} - \frac{1}{p^0} \right). \tag{85}$$

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Therefore we obtain the following linearized system

$$\partial_t \sigma + \rho^0 \mathrm{div}\boldsymbol{v} = f_1(U) \tag{86}$$

$$\rho^0 \partial_t \boldsymbol{v} - \mu \Delta \boldsymbol{v} - (\mu + \nu)\nabla \mathrm{div}\boldsymbol{v} + \gamma_1 \nabla \sigma + \sum_{l=1}^{n-1} \gamma_2^l \nabla \vartheta_l = \mathbf{f}_2(U) \tag{87}$$

$$\sum_{l=1}^{n-1} \mathcal{R}_{kl}^0 \partial_t \vartheta_l + \gamma_2^k \, \mathrm{div}\,\boldsymbol{v} - \mathrm{div}\left(\sum_{l=1}^{n-1} \mathcal{B}_{kl}^0 \nabla \vartheta_l\right) = f_3^k(U), \quad k = 1, \dots, n-1 \tag{88}$$

in $\Omega \times (0, T)$, supplied with the boundary conditions

$$\boldsymbol{v}|_{\partial\Omega} = 0, \quad \nabla \vartheta_k \cdot \boldsymbol{n}|_{\partial\Omega} = f_4^k(U), \quad k = 1, \dots, n-1 \tag{89}$$

and initial conditions

$$(\sigma, \boldsymbol{v}, \vec{\vartheta})|_{t=0} = (0, \boldsymbol{u}^0, \vec{h}^0), \tag{90}$$

where we denote

$$\vec{h}^0 = (h_1^0, \dots, h_{n-1}^0),$$

$$D_{kl}^0 = D_{kl}(\rho^0), \quad \mathcal{R}_{kl}^0 = m_{k+1}\rho_{k+1}^0 \delta_{kl} - \frac{m_{k+1}m_{l+1}\rho_{k+1}^0\rho_{l+1}^0}{\Sigma_\rho^0}, \quad \mathcal{B}_{kl}^0 = \frac{\rho_{l+1}^0\rho_{k+1}^0 D_{k+1,l+1}^0}{p_0},$$

$$\gamma_1 = \frac{\rho^0}{\Sigma_\rho^0}, \quad \gamma_2^k = \rho_{k+1}^0 - \frac{m_{k+1}\rho_{k+1}^0\rho^0}{\Sigma_\rho^0},$$

and the right hand side is given by

$$f_1(U) = R_1(U) - \sigma \mathrm{div}\boldsymbol{v}, \tag{91}$$

$$\begin{aligned}
\mathbf{f}_2(U) =& R_2(U) - \sigma\partial_t\boldsymbol{v} - \varrho^0\nabla\eta\left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0}\right) - \frac{\rho^0}{\Sigma_\rho}\nabla\rho^0 - \frac{\sigma}{\Sigma_\rho}\nabla\eta \\
&+ \sum_{l=1}^{n-1}\left(-\sigma_{l+1} + m_{l+1}\rho_{l+1}^0\rho^0\left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0}\right) + \frac{m_{l+1}}{\Sigma_\rho}(\rho_{l+1}\sigma + \rho^0\sigma_{l+1})\right)\nabla\vartheta_l,
\end{aligned} \tag{92}$$

$$\begin{aligned}
f_3^k(U) =& R_3^k(U) + \rho_{k+1}^0\mathrm{div}\boldsymbol{v} + \left[m_{k+1}\rho^0\rho_{k+1}^0\left(\frac{1}{\Sigma} - \frac{1}{\Sigma^0}\right) + \frac{m_{k+1}}{\Sigma_\rho}(\rho_{k+1}^0\sigma + \rho^0\sigma_{k+1})\right]\mathrm{div}\boldsymbol{v} \\
&+ \sum_{l=1}^{n-1}\left(-\delta_{kl}m_{k+1}\sigma_{k+1} + m_{k+1}m_{l+1}\left[\rho_{k+1}^0\rho_{l+1}^0\left(\frac{1}{\Sigma} - \frac{1}{\Sigma_\rho^0}\right) + \frac{\rho_{k+1}^0\sigma_{l+1} + \rho_{l+1}^0\sigma_{k+1}}{\Sigma_\rho}\right]\right)\partial_t\vartheta_l \\
&+ \mathrm{div}\left(\sum_{l=1}^{n-1}\left[\frac{\rho_{k+1}\rho_{l+1}D_{k+1,l+1} - \rho_{k+1}^0\rho_{l+1}^0 D_{k+1,l+1}^0}{p} + \rho_{k+1}^0\rho_{l+1}^0 D_{k+1,l+1}^0\left(\frac{1}{p} - \frac{1}{p_0}\right)\right]\nabla\vartheta_l\right),
\end{aligned} \tag{93}$$

$$f_4^k(U) = R_4^k(U). \tag{94}$$

5.2 Solvability of the complete linear system

5.2.1 Notation and auxiliary results

For abbreviation and clarity we introduce the following notation:

1. We will denote by $E(T)$ a continuous function of T s.t. $E(0) = 0$. Moreover, we use C to denote a generic positive constant, or we use $C(X, Y)$ to specify the dependence of parameters X and Y.

2. By $\vec{\cdot}$ we denote an $(n-1)$-vector of functions, for example $\vec{\vartheta} = (\vartheta_1, \dots, \vartheta_{n-1})^\top$, $\vec{h}^0 = (h_1, \dots, h_{n-1})^\top$.

3. We introduce the norms describing regularity of our solutions; for $T > 0$ we define:

$$
\begin{aligned}
[\boldsymbol{v}]_{T,1} &:= \|\boldsymbol{v}\|_{L_p(0,T;H_q^2(\Omega))} + \|\partial_t \boldsymbol{v}\|_{L_p(0,T;L_q(\Omega))}, \\
[\sigma]_{T,2} &:= \|\sigma\|_{H_p^1(0,T;H_q^1(\Omega))}, \\
[\sigma, \boldsymbol{v}, \vec{\vartheta}]_T &:= [\boldsymbol{v}]_{T,1} + [\sigma]_{T,2} + \sum_{k=1}^{n-1} [\vartheta_k]_{T,1}.
\end{aligned}
\tag{95}
$$

Then, for given $T, M > 0$ we define the sets in the functional spaces:

$$
\mathcal{H}_{T,M}^1 = \{\boldsymbol{v} : [\boldsymbol{v}]_{T,1} \le M\}, \qquad \mathcal{H}_{T,M}^2 = \{\sigma : [\sigma]_{T,2} \le M\}
\tag{96}
$$

and

$$
\mathcal{H}_{T,M} = \left\{ (\sigma, \boldsymbol{v}, \vec{\vartheta}) : \quad (\sigma, \boldsymbol{v}, \vartheta_k)|_{t=0} = (0, \boldsymbol{u}^0, h_k^0) \quad \text{in } \Omega, \quad [\sigma, \boldsymbol{v}, \vec{\vartheta}]_T \le M \right\}.
\tag{97}
$$

To prove existence of local-in-time strong solutions to system (86) with fixed and given $f_1, \mathbf{f}_2, f_3^k$, and f_4^k we will use some auxiliary results for two subsystems. First let us recall a relevant existence result for the fluid part (for the proof see [30], Theorem 5.1):

Theorem 5.1 *Assume* $1 < p,q < \infty$ $2/p + 1/q \ne 1$, $T > 0$ *and* Ω *is a uniformly* C^2 *domain in* $\mathbb{R}^N$ $(N \ge 2)$. *Assume moreover that* $\rho^0 \in H_q^1(\Omega)$, $\boldsymbol{u}_0 \in B_{q,p}^{2(1-1/p)}(\Omega)^n$, $f_1 \in L_p(\mathbb{R}, H_q^1(\Omega)^n)$ *and* $\mathbf{f}_2 \in L_p((0,T), L_q(\Omega)^n)$. *Then the problem*

$$
\begin{cases}
\partial_t \sigma + \varrho^0 \mathrm{div}\boldsymbol{v} = \tilde{f}_1 & \text{in } \Omega \times (0,T), \\
\rho^0 \partial_t \boldsymbol{v} - \mu\Delta\boldsymbol{v} - (\mu + \nu)\nabla\mathrm{div}\boldsymbol{v} + \gamma_1\nabla\sigma = \tilde{\mathbf{f}}_2 & \text{in } \Omega \times (0,T), \\
\boldsymbol{v}|_{\partial\Omega} = 0 & \text{on } \Gamma \times (0,T), \\
(\sigma, \boldsymbol{v})|_{t=0} = (0, \boldsymbol{u}^0) & \text{in } \Omega,
\end{cases}
\tag{98}
$$

admits a solution $(\sigma, \boldsymbol{v})$ *such that*

$$
\|\boldsymbol{v}\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t \boldsymbol{v}\|_{L_p((0,T),L_q(\Omega))} + \|\sigma\|_{H_p^1(0,T;H_q^1(\Omega))}
\tag{99}
$$

$$
\le Ce^{cT}\left(\|\rho^0\|_{H_q^1(\Omega)} + \|\boldsymbol{u}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tilde{f}_1\|_{L_p((0,T),H_q^1(\Omega))} + \|\tilde{\mathbf{f}}_2\|_{L_p((0,T),L_q(\Omega))} \right).
\tag{100}
$$

For the species subsystem we recall the following theorem which gives solvability in a maximal $L_p - L_q$ regime of a linear problem, its proof can be found in our previous work [31]. For general m species we consider $k \in \{1, \dots, m\}$ and the following set of equations

$$
\begin{cases}
\displaystyle\sum_{\ell=1}^m \mathcal{R}_{k\ell}\partial_t \vartheta_\ell - \mathrm{div}\left(\sum_{\ell=1}^m \mathcal{B}_{k\ell}\nabla\vartheta_\ell \right) = \tilde{f}_3^k & \text{in } \Omega \times (0,T), \\
\displaystyle\sum_{\ell=1}^m \mathcal{B}_{k\ell}\nabla\vartheta_\ell \cdot \boldsymbol{n} = \tilde{f}_4^k & \text{on } \Gamma \times (0,T), \\
\vartheta_k|_{t=0} = h_k^0 & \text{in } \Omega,
\end{cases}
\tag{101}
$$

where $\mathcal{B} = \mathcal{B}(x)$ and $\mathcal{R} = \mathcal{R}(x)$ are $m \times m$ matrices whose $(k, \ell)^{\text{th}}$ components are $\mathcal{B}_{k\ell}(x)$ and $\mathcal{R}_{k\ell}(x)$, respectively.

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Theorem 5.2 *Assume that*

- *there exists a number M_0 for which*

$$|\mathcal{B}_{k\ell}(x)|, |\mathcal{R}_{k\ell}(x)| \leq M_0, \quad \text{for any } x \in \Omega,$$
$$|\mathcal{B}_{k\ell}(x) - \mathcal{B}_{k\ell}(y)| \leq M_0|x-y|^\sigma, \quad |\mathcal{R}_{k\ell}(x) - \mathcal{R}_{k\ell}(y)| \leq M_0|x-y|^\sigma \quad \text{for any } x, y \in \Omega, \quad (102)$$
$$\|\nabla(\mathcal{B}_{k\ell}, \mathcal{R}_{k\ell})\|_{L_r(\Omega)} \leq M_0.$$

- *the matrices $\mathcal{B}$ and $\mathcal{R}$ are positive and symmetric and that there exist constants $m_1, m_2 > 0$ for which*

$$\langle \mathcal{B}(x)\xi, \overline{\xi} \rangle \geq m_1|\xi|^2, \quad \langle \mathcal{R}(x)\xi, \overline{\xi} \rangle \geq m_2|\xi|^2 \tag{103}$$

  for any complex m-vector ξ and $x \in \Omega$.

- $1 < p, q < \infty$ *and $T > 0$, $2/p + 1/q \neq 1$ and Ω is a uniformly C^2 domain in $\mathbb{R}^N$ ($N \geq 2$).*

- *for all $k = 1, \ldots, m$, $h_k^0 \in B_{q,p}^{2(1-1/p)}(\Omega)$, $\tilde{f}_3^k \in L_p((0,T), L_q(\Omega))$ and $\tilde{f}_4^k \in L_p(\mathbb{R}, H_q^1(\Omega)) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega))$ are given functions satisfying the compatibility conditions:*

$$\sum_{\ell=1}^m B_{k\ell}\nabla h_\ell^0 \cdot \boldsymbol{n} = \tilde{f}_4^k(\cdot, 0) \quad \text{on } \Gamma \tag{104}$$

  provided $2/p + 1/q < 1$.

Then, problem (101) admits a unique solution $\vec{\vartheta} = (\vartheta_1, \ldots, \vartheta_m)^\top$ with

$$\vec{\vartheta} \in L_p((0,T), H_q^2(\Omega)^m) \cap H_p^1((0,T), L_q(\Omega)^m) \tag{105}$$

possessing the estimate:

$$\|\vec{\vartheta}\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t\vec{\vartheta}\|_{L_p((0,T),L_q(\Omega))}$$
$$\leq Ce^{cT}(\|\vec{h}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\vec{\tilde{f}}_3\|_{L_p((0,T),L_q(\Omega))} + \|\vec{\tilde{f}}_4\|_{L_p((0,T),H_q^1(\Omega))} + \|\vec{\tilde{f}}_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}) \tag{106}$$

for some constants C and c.

5.2.2 Fixed point argument

With Theorems 5.1 and 5.2 it is easy to show solvability with appropriate estimates of complete linear system corresponding to (86)-(89):

$$\begin{cases} \partial_t\sigma + \rho^0\text{div}\boldsymbol{v} = f_1 \\[2mm] \rho^0\partial_t\boldsymbol{v} - \mu\Delta\boldsymbol{v} - (\mu + \nu)\nabla\text{div}\boldsymbol{v} + \gamma_1\nabla\sigma + \sum_{l=1}^{n-1}\gamma_2^l\nabla\vartheta_l = \mathbf{f}_2 \\[2mm] \sum_{l=1}^{n-1}\mathcal{R}_{kl}^0\partial_t\vartheta_l + \gamma_2^k\,\text{div}\,\boldsymbol{v} - \text{div}\left(\sum_{l=1}^{n-1}\mathcal{B}_{kl}^0\nabla\vartheta_l\right) = f_3^k, \quad k = 1, \ldots, n-1 \end{cases} \tag{107}$$

with given $\gamma_1, \{\gamma_2^l\}_{l=1,\ldots,n-1}$ and the boundary conditions

$$\boldsymbol{v}|_{\partial\Omega} = 0, \quad \sum_{l=1}^{n-1}\mathcal{B}_{kl}^0\nabla\vartheta_l \cdot \boldsymbol{n}|_{\partial\Omega} = f_4^k, \quad k = 1, \ldots, n-1. \tag{108}$$

and initial conditions (90).

We have the following result.

Theorem 5.3 *Assume $\mathcal{B}^0, \mathcal{R}^0$, Ω and p, q satisfy the assumptions of Theorem 5.2 with $m = n-1$. Assume moreover $\boldsymbol{u}^0, \vec{h}^0 \in B_{q,p}^{2(1-1/p)}(\Omega)$, $\rho^0 \in H_q^1(\Omega)$, $f_1 \in L_p((0,T), H_q^1(\Omega))$, $(\mathbf{f}_2, \vec{f}_3) \in L_p((0,T), L_q(\Omega))$, $\vec{f}_4 \in L_p(\mathbb{R}, H_q^1(\Omega)) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega))$. Then for any $M > 0$, if*

$$\|\boldsymbol{u}^0, \vec{h}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\rho^0\|_{H_q^1(\Omega)} + \|f_1\|_{L_p((0,T),H_q^1(\Omega))} \tag{109}$$

$$+ \|(\mathbf{f}_2, \vec{f}_3)\|_{L_p((0,T),L_q(\Omega))} + \|\vec{f}_4\|_{L_p(\mathbb{R},H_q^1(\Omega))} + \|\vec{f}_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega)^n)} \leq M, \tag{110}$$

then there exists $T > 0$ such that system (107)-(108) admits a solution $(\sigma, \boldsymbol{v}, \vec{\vartheta})$ on $(0,T)$ with

$$[\sigma, \boldsymbol{v}, \vec{\vartheta}]_T \leq \|\boldsymbol{u}^0, \vec{h}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\rho^0\|_{H_q^1(\Omega)} + \|f_1\|_{L_p((0,T),H_q^1(\Omega))} \tag{111}$$

$$+ \|(\mathbf{f}_2, \vec{f}_3)\|_{L_p((0,T),L_q(\Omega))} + \|\vec{f}_4\|_{L_p(\mathbb{R},H_q^1(\Omega))} + \|\vec{f}_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega)^n)} \tag{112}$$

Proof. We use the Banach fixed point argument. For given $\bar{\boldsymbol{v}} \in \mathcal{H}_{T,M}^1$ denote by $\vec{\vartheta}(\bar{\boldsymbol{v}})$ solution to $(107)_3$ with $\boldsymbol{v} = \bar{\boldsymbol{v}}$ and boundary condition $(108)_2$. Since $\|\boldsymbol{v}\|_{L_\infty(0,T,H_\infty^1(\Omega))} \leq CM$, (see estimate (119)) therefore by Theorem 5.2 such solution exists for arbitrary time $T > 0$, it is unique and it satisfies

$$\begin{aligned}[\vec{\vartheta}(\bar{\boldsymbol{v}})]_{T,1} \leq &C(T)\Big(\|\vec{h}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\vec{f}_3\|_{L_p(0,T;L_q(\Omega))} + E(T)\|\boldsymbol{v}\|_{L_p(0,T;H_q^1(\Omega))} \\ &+ \|\vec{f}_4\|_{L_p(\mathbb{R},H_q^1(\Omega))} + \|\vec{f}_4\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}\Big) \\ \leq &C(T,M)\left(1 + E(T)\|\bar{\boldsymbol{v}}\|_{L_p(0,T;H_q^1(\Omega))}\right).\end{aligned} \tag{113}$$

Therefore for $(\bar{\boldsymbol{v}}, \bar{\sigma}) \in \mathcal{H}_{T,M}^1 \times \mathcal{H}_{T,M}^2$ we can define $(\boldsymbol{v}, \sigma) = \mathcal{T}(\bar{\boldsymbol{v}}, \bar{\sigma})$ as a unique solution of the first two equations of system (107) with $\vec{\vartheta} = \vec{\vartheta}(\bar{\boldsymbol{v}})$ and boundary condition $(108)_1$. By Theorem 5.1 we have

$$\begin{aligned}[\sigma]_{T,2} + [\boldsymbol{v}]_{T,1} \leq &C(T)\Big(\|\rho^0\|_{H_q^1(\Omega)} + \|\boldsymbol{u}^0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &+ \|f_1\|_{L_p((0,T),H_q^1(\Omega))} + \|\mathbf{f}_2\|_{L_p((0,T),L_q(\Omega))} + \|\nabla\vec{\vartheta}(\bar{\boldsymbol{v}})\|_{L_p((0,T),L_q(\Omega))}\Big) \\ \leq &C(T,M)\left(1 + E(T)\|\bar{\boldsymbol{v}}\|_{L_p(0,T;H_q^1(\Omega))}\right).\end{aligned} \tag{114}$$

Moreover, taking different $\bar{\boldsymbol{v}}_1, \bar{\boldsymbol{v}}_2 \in \mathcal{H}_{T,M}^1$ corresponding to the same initial data $\boldsymbol{u}^0$, and then subtracting the for $\vec{\vartheta}(\bar{\boldsymbol{v}}_1)$ and $\vec{\vartheta}(\bar{\boldsymbol{v}}_2)$ we get

$$[\vec{\vartheta}(\bar{\boldsymbol{v}}_1) - \vec{\vartheta}(\bar{\boldsymbol{v}}_2)]_{T,1} \leq C(M)E(T)[\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2]_{T,1}.$$

Therefore applying Theorem 5.1 to a difference of two solutions we have

$$\begin{aligned}[\mathcal{T}(\bar{\boldsymbol{v}}_1, \bar{\sigma}_1) - \mathcal{T}(\bar{\boldsymbol{v}}_2, \bar{\sigma}_2)]_{T,1;T,2} &\leq C(M)E(T)[\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2]_{T,1} \\ &\leq C(M)E(T)[(\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2, \bar{\sigma}_1 - \bar{\sigma}_2)]_{T,1;T,2}.\end{aligned}$$

Therefore for sufficiently small T, $\mathcal{T}$ is a contraction on a set $\mathcal{H}_{T,M}^1 \times \mathcal{H}_{T,M}^2$, and applying the Banach fixed point theorem we complete the proof.

$\square$

6 Proof of Theorem 2.2

6.1 Nonlinear estimates

The aim of this section is to prove the following proposition which gives the estimate on the right hand side of linearized system in the regularity required in order to apply Theorem 5.3. We shall use notation introduced at the beginning of Section 5.2.

Proposition 6.1 *Let* $\bar{U} = (\bar{\sigma}, \bar{\boldsymbol{v}}, \bar{\vartheta}) \in \mathcal{H}_{T,M}$ *for given* $T, M > 0$, *where the initial conditions satisfy the assumptions of Theorem 2.2. Let* $f_1(U), f_2(U), f_3^k(U)$ *and* $f_4^k(U)$ *be given by (91)-(94), where* $R_1(U), R_2(U), R_3^k(U)$ *and* $R_4^k(U)$ *are defined in (72),(77),(78)-(79) and (80), respectively. Then*

$$
\begin{aligned}
&\|f_1(\bar{U})\|_{L_p(0,T;W_q^1(\Omega))} + \|f_2(\bar{U})\|_{L_p(0,T;L_q(\Omega))} + \|f_3^k\|(\bar{U})\|_{L_p(0,T;L_q(\Omega))} \\
&+ \|f_4^k(\bar{U})\|_{L_p(0,T,H_{\bar{q}}^1(\Omega))} + \|f_4^k(\bar{U})\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} \le C(M,L)E(T).
\end{aligned}
\tag{115}
$$

Let us start with recalling some auxiliary results. The first one is due to Tanabe (cf. [37] p.10):

Lemma 6.2 *Let* X *and* Y *be two Banach spaces such that* X *is a dense subset of* Y *and* $X \subset Y$ *is continuous. Then for each* $p \in (1,\infty)$

$$
H_p^1((0,\infty),Y) \cap L_p((0,\infty),X) \subset C([0,\infty),(X,Y)_{1/p,p})
$$

and for every $u \in H_p^1((0,\infty),Y) \cap L_p((0,\infty),X)$ *we have*

$$
\sup_{t \in (0,\infty)} \|u(t)\|_{(X,Y)_{1/p,p}} \le (\|u\|_{L_p((0,\infty),X)}^p + \|u\|_{H_p^1((0,\infty),Y)}^p)^{1/p}.
$$

Next two results will be needed to estimate the boundary data. For the first one see [Shibata and Shimizu [34], Lemma 2.7]:

Lemma 6.3 *Let* $1 < p < \infty$, $3 < q < \infty$ *and* $0 < T \le 1$. *Assume that* Ω *is a uniformly* C^2 *domain. Let*

$$
f \in H_\infty^1(\mathbb{R},L_q(\Omega)) \cap L_\infty(\mathbb{R},H_q^1(\Omega)), \quad g \in L_p(\mathbb{R},H_q^1(\Omega)) \cap H_p^{1/2}(\mathbb{R},L_q(\Omega)).
$$

If we assume that $f \in L_p(\mathbb{R},H_q^1(\Omega))$ *and that* f *vanishes for* $t \notin [0,2T]$ *in addition, then we have*

$$
\|fg\|_{L_p(\mathbb{R},H_q^1(\Omega))} + \|fg\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}
$$
$$
\le C(\|f\|_{L_\infty(\mathbb{R},H_q^1(\Omega))} + T^{(q-3)/(pq)}\|\partial_t f\|_{L_\infty(\mathbb{R},L_q(\Omega))}^{(1-3/(2q))}\|\partial_t f\|_{L_p((\mathbb{R},H_q^1(\Omega))}^{3/(2q)})(\|g\|_{p(\mathbb{R},H_q^1(\Omega))} + \|g\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}).
$$

Remark 6.4 (1) *The boundary of* Ω *was assumed to be bounded in [34]. However, Lemma 6.3 can be proved using Sobolev's inequality and complex interpolation theorem, and so employing the same argument as that in the proof of [34, Lemma 2.7], we can prove Lemma 6.3.*
(2) *By Sobolev's inequality,* $\|fg\|_{H_{\bar{q}}^1(\Omega)} \le C\|f\|_{H_{\bar{q}}^1(\Omega)}\|g\|_{L_q(\Omega)}$, *and so the essential part of Lemma 6.3 is the estimate of* $\|fg\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}$.

The second result has been shown in Shibata and Shimizu [35] for $\Omega = \mathbb{R}^n$ and generalized to a uniform C^2 domain in Shibata [33]:

Lemma 6.5 *Let* $1 < p, q < \infty$. *Assume that* Ω *is a uniform* C^2 *domain. Then*

$$
H_p^1(\mathbb{R},L_q(\Omega)) \cap L_p(\mathbb{R},H_q^2(\Omega)) \subset H_p^{1/2}(\mathbb{R},H_q^1(\Omega)),
$$

and

$$
\|\nabla f\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} \le C(\|f\|_{L_p(\mathbb{R},H_q^2(\Omega))} + \|\partial_t f\|_{L_p(\mathbb{R},L_q(\Omega))}).
$$

Now we show preliminary estimates for functions from the space $\mathcal{H}_{T,M}$.

Lemma 6.6 *Let* $\sigma, \boldsymbol{v}, \vartheta_1 \ldots \vartheta_{n-1} \in \mathcal{H}_{T,M}$ *and let* $\mathbf{k_v}, \mathbf{V}^0(\mathbf{k_v})$ *be defined in* (65). *Then*

$$\|\mathbf{V}^0(\mathbf{k_v}), \nabla_{\mathbf{k_v}} \mathbf{V}^0(\mathbf{k_v})\|_{L_\infty(\Omega \times (0,T))} \leq C(M,L)E(T), \tag{116}$$

$$\sup_{t \in (0,T)} \|\sigma(\cdot, t)\|_{H_q^1(\Omega)} \leq C(M,L)E(T), \tag{117}$$

$$\sup_{t \in (0,T)} \|\vec{\vartheta}(\cdot, t) - \vec{h}^0\|_{B_{q,p}^{2(1-1/p)}} + \sup_{t \in (0,T)} \|\boldsymbol{v}(\cdot, t) - \boldsymbol{u}_0\|_{B_{q,p}^{2(1-1/p)}} \leq C(M,L), \tag{118}$$

$$\|\boldsymbol{v}, \vec{\vartheta}\|_{L_\infty(0,T,H_\infty^1(\Omega))} \leq C(M), \tag{119}$$

$$\|\rho_k - \rho_k^0\|_{L_\infty(0,T;H_q^1)} \leq C(M,L) \quad \forall k = 1, \ldots, n, \tag{120}$$

$$\|\rho_k - \rho_k^0\|_{L_\infty(\Omega \times (0,T))} \leq C(M,L)E(T). \tag{121}$$

Proof. First of all, we have

$$\int_0^T \|\nabla \boldsymbol{v}(\cdot, t)\|_{L_\infty(\Omega)} \, dt \leq C \int_0^T \|\boldsymbol{v}(\cdot, t)\|_{H_q^2(\Omega)} \, dt$$
$$\leq T^{1/p'} \left(\sup_{t \in (0,T)} \int_0^T \|\boldsymbol{v}(\cdot, t)\|_{H_q^2(\Omega)}^p \, dt \right)^{1/p} \leq M T^{1/p'}, \tag{122}$$

which implies (116). Next,

$$\|\sigma(\cdot, t)\|_{H_q^1(\Omega)} \leq \int_0^t \|\partial_t \sigma)(\cdot, s)\|_{H_q^1(\Omega)} \, ds \leq T^{1/p'} \|\partial_t \sigma\|_{L_p((0,T), H_q^1(\Omega))} \leq C(M)E(T),$$

and so we have (117). In order to prove (118) we introduce extension operator

$$e_T[f](\cdot, t) = \begin{cases} 0 & t \in (-\infty, 0) \cup (2T, +\infty), \\ f(\cdot, t) & t \in (0, T), \\ f(\cdot, 2T - t) & t \in (T, 2T). \end{cases} \tag{123}$$

Obviously, $e_T[f](\cdot, t) = f(\cdot, t)$ for $t \in (0, T)$. If $f|_{t=0} = 0$, then we have

$$\partial_t e_T[f](\cdot, t) = \begin{cases} 0 & t \in (-\infty, 0) \cup (2T, +\infty), \\ (\partial_t f)(\cdot, t) & t \in (0, T), \\ -(\partial_t f)(\cdot, 2T - t) & t \in (T, 2T), \end{cases} \tag{124}$$

understood in a weak sense. Applying Lemma 6.2 with $X = H_q^2(\Omega)$, $Y = L_q(\Omega)$ and using (123) and (124) we have

$$\sup_{t \in (0,T)} \|\vartheta(\cdot, t) - h_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq \sup_{t \in (0,\infty)} \|e_T[\vartheta_k - h_k^0]\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$$
$$= (\|e_T[\vartheta_k - h_k^0]\|_{L_p((0,\infty), H_q^2(\Omega))}^p + \|e_T[\vartheta_k - h_k^0]\|_{H_p^1((0,\infty), L_q(\Omega))}^p)^{1/p}$$
$$\leq C(\|\vartheta_k - h_k^0\|_{L_p((0,\infty), H_q^2(\Omega))} + \|\partial_t \vartheta_k\|_{L_p((0,T), L_q(\Omega))}) \leq C(M,L),$$

and estimating $\|\boldsymbol{v}(\cdot, t) - \boldsymbol{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$ in the same way we obtain (118). Then (119) follows from (118) due to Sobolev imbedding theorem as $\frac{2}{p} + \frac{3}{q} < 1$. In order to prove (119) we use a fact that

$$(\rho_1, \ldots \rho_n) = \Psi^{-1}(\rho, \vartheta_1, \ldots, \vartheta_{n-1}),$$

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where Ψ is the diffeomorphism defined in (16), and therefore

$$
\sup_{t \in (0,T)} \|\rho_k(\cdot,t) - \rho_k^0(\cdot)\|_{L_q(\Omega)} \leq \int_0^T \|\partial_t(\Psi^{-1})(\vec{\theta}(\cdot,t), \rho_0(\cdot) + \sigma(\cdot,t))\|_{L_q(\Omega)}\, dt
$$
$$
\leq \int_0^T \|(\Psi^{-1})'(\vec{\theta}(\cdot,t), \rho_0(\cdot) + \sigma(\cdot,t))\|_{L_\infty(\Omega)} \|(\partial_t\vec{\theta}(\cdot,t), \partial_t\sigma(\cdot,t))\|_{L_q(\Omega)}\, dt.
$$
(125)

By (116) and (119), we have

$$
\sup_{t \in (0,T)} \|(\Psi^{-1})'(\vec{\theta}(\cdot,t), \rho_0(\cdot) + \sigma(\cdot,t))\|_{L_\infty(\Omega)} \leq C.
$$
(126)

Thus, by (125) we have

$$
\sup_{t \in (0,T)} \|\rho_k(\cdot,t) - \rho_k^0(\cdot)\|_{L_q(\Omega)} \leq C \int_0^T \|(\partial_t\vec{\theta}(\cdot,t), \partial_t\sigma(\cdot,t))\|_{L_q(\Omega)}\, dt
$$
$$
\leq C T^{1/p'} \|\partial_t(\vec{\theta}, \sigma)\|_{L_p((0,T),L_q(\Omega))} \leq C(M)E(T).
$$
(127)

Similarly,

$$
\|\nabla[\rho_k(\cdot,t) - \rho_k^0(\cdot)]\|_{L_q(\Omega)}
$$
$$
\leq \|(\Psi^{-1})'(\vec{\theta}(\cdot,t), \rho^0(\cdot) + \sigma(\cdot,t)\|_{L_\infty(\Omega)} \|(\nabla\vec{\theta}(\cdot,t), \nabla\rho^0(\cdot) + \nabla\sigma(\cdot,t))\|_{L_q(\Omega)} + \|\nabla(\vec{\rho^0})\|_{L_q(\Omega)} \leq C(L,M),
$$

which implies (120). In order to show (129) note that $W_q^{\frac{3}{q}+\epsilon}(\Omega) \subset L_\infty(\Omega)\ \forall \epsilon > 0$, therefore for $\epsilon < 1 - \frac{3}{q}$ we have

$$
\sup_{t \in (0,T)} \|\rho_k(\cdot,t) - \rho_k^0(\cdot)\|_{L_\infty(\Omega)}
$$
$$
\leq (\sup_{0 \in (0,T)} \|\rho_k(\cdot,t) - \rho_k^0(\cdot)\|_{L_q(\Omega)})^\theta
$$
$$
\times (\sup_{0 \in (0,T)} \|\rho_k(\cdot,t) - \rho_k^0(\cdot)\|_{H_q^1(\Omega)})^{1-\theta} \leq C(M,L)E(T)
$$
(128)

with $\theta = 1 - (3/q + \epsilon) \in (0,1)$. This way we obtain (129) and complete the proof.

$\square$

The next lemma gives bounds on the terms coming from the change of coordinates.

Lemma 6.7 *Let* $A_{2\Delta}(\mathbf{k_v})\nabla^2(\cdot), A_{1\Delta}(\mathbf{k_v})\nabla(\cdot), A_{2\mathrm{div}}(\mathbf{k_v})\nabla^2(\cdot), A_{1\mathrm{div}}(\mathbf{k_v})\nabla(\cdot)$ *be defined in* (73),(74),(75) *and* (76), *respectively. Then*

$$
\|A_{2\Delta}\nabla^2\boldsymbol{v}, A_{2\mathrm{div}}\nabla^2\boldsymbol{v}\|_{L_p(0,T;L_q(\Omega))} + \|A_{1\Delta}\nabla\boldsymbol{v}, A_{1\mathrm{div}}\nabla\boldsymbol{v}\|_{L_\infty(0,T;L_q(\Omega))} \leq C(M)E(T) \tag{129}
$$
$$
\|A_{2\Delta}\nabla^2\vartheta_k, A_{2\mathrm{div}}\nabla^2\vartheta_k\|_{L_p(0,T;L_q(\Omega))} + \|A_{1\Delta}\nabla\vartheta_k, A_{1\mathrm{div}}\nabla\vartheta_k\|_{L_\infty(0,T;L_q(\Omega))} \leq C(M)E(T) \tag{130}
$$

for all $k = 1, \ldots, n-1$.

Proof. By (116) and (73) we have

$$
\|A_{2\Delta}\nabla^2\boldsymbol{v}\|_{L_p(0,T;L_q(\Omega))} \leq \|\mathbf{V}^0(\mathbf{k_v})\|_{L^\infty(\Omega \times (0,T))}(1 + \|\mathbf{V}^0(\mathbf{k_v})\|_{L^\infty(\Omega \times (0,T))})\|\nabla^2\boldsymbol{v}\|_{L_p(0,T;L_q(\Omega))} \leq C(M)E(T).
$$

Next, notice that

$$
\left\|\int_0^t \nabla^2\boldsymbol{v}\right\|_{L_q(\Omega)} \leq \int_0^t \|\nabla^2\boldsymbol{v}\|_{L_q(\Omega)} \leq t^{1/p'}\|\nabla^2\boldsymbol{v}\|_{L_p(0,T;L_q(\Omega))},
$$

therefore, by (116) and (119),

$$
\left\| \nabla_{\mathbf{k_v}} V_{lm}^0(\mathbf{k_v}) \left[\int_0^t \partial_l \nabla \boldsymbol{v} ds \right] \frac{\partial \boldsymbol{v}}{\partial y_m} \right\|_{L_q(\Omega)} \le \| \nabla_{\mathbf{k_v}} V_{lm}^0(\mathbf{k_v}) \|_{L_\infty(\Omega \times (0,T))} \left\| \int_0^t \nabla^2 \boldsymbol{v} \right\|_{L_p(0,T;L_q(\Omega))} \| \nabla \boldsymbol{v} \|_{L_\infty(\Omega \times (0,T))}
$$

$$
\le C(M) E(T).
$$

The other terms in $A_{1\Delta} \nabla \boldsymbol{v}$ have a similar structure, therefore we get

$$
\| A_{1\Delta} \nabla \boldsymbol{v} \|_{L_\infty(0,T;L_q(\Omega))} \le C(M) E(T).
$$

As $A_{2\mathrm{div}}(\mathbf{k_v}) \nabla^2(\cdot)$ and $A_{1\mathrm{div}}(\mathbf{k_v}) \nabla(\cdot)$ have structure similar to $A_{2\Delta} \nabla^2(\cdot)$ and $A_{1\Delta} \nabla(\cdot)$, respectively, we conclude (129). Finally, ϑ_k have the same regularity as $\boldsymbol{v}$ so we obtain (130) in the same way. Proof of Lemma 6.7 is complete.

$\square$

With these results at hand we can proceed with the proof of Proposition 6.1.
Estimate of $f_1(U)$. Since $f_1(U)$ it is exactly the same as in the two species case, we obtain (see [30])

$$
\| f_1(U) \|_{L_p(0,T;L_q(\Omega))} \le C(M,L) E(T). \tag{131}
$$

Estimate of $f_2(U)$. Let us start with $R_2(U)$ defined in (77). By (116) we have

$$
\left\| \mathbf{V}^0(\mathbf{k_v}) \left(\varrho_l - \frac{m_l \varrho_l \varrho}{\Sigma_\varrho} \right) \nabla \vartheta_{l-1} \right\|_{L_p(0,T;L_q(\Omega))} \le C(M,L) E(T).
$$

Applying Lemma 6.7 to the remaining terms we obtain

$$
\| \mathbf{R}_2(U) \|_{L_p(0,T;L_q(\Omega))} \le C(M,L) E(T).
$$

Next, by (117)

$$
\| \sigma \partial_t \boldsymbol{v} \|_{L_p(0,T;L_q(\Omega))} \le \| \sigma \|_{L_\infty(\Omega \times (0,T))} \| \partial_t \boldsymbol{v} \|_{L_p(0,T;L_q(\Omega))} \le C(M) E(T),
$$

and similarly, using (117)-(120) we get

$$
\left\| \frac{\sigma}{\Sigma_\rho} \nabla \eta, \sigma_l \nabla \vartheta_{l-1}, \frac{m_l}{\Sigma_\rho} (\rho_l \sigma + \rho^0 \sigma_l) \nabla \vartheta_{l-1}, \frac{\rho^0}{\Sigma_\rho} \nabla \rho^0 \right\|_{L_p(0,T;L_q(\Omega))} \le C(M,L) E(T).
$$

In order to estimate the terms with $\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0}$ we write it as

$$
\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0} = \frac{\Sigma_\rho^0 - \Sigma_\rho}{\Sigma_\rho \Sigma_\rho^0}.
$$

As the denominator is bounded from below by a positive constant, using (119) we get

$$
\| \varrho^0 \nabla \eta \left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0} \right) \|_{L_p(0,T;L_q(\Omega))} \le C \sum_{k=1}^n \| \nabla \eta (\rho_k - \rho_k^0) \|_{L_p(0,T;L_q(\Omega))} \le
$$

$$
C \sum_{k=1}^n \left[\int_0^T \| \rho_k - \rho_k^0 \|_{L_\infty}^p \| \nabla \eta \|_{L_q}^p \right]^{1/p} \le \sum_{k=1}^n \| \rho_k - \rho_k^0 \|_{L_\infty(H_q^1)} \| \nabla \eta \|_{L_p(0,T;L_q(\Omega))} \le C(M,L) E(T),
$$

and similarly

$$\left\| m_l \rho^0 \rho_l^0 \left(\frac{1}{\Sigma_\rho} - \frac{1}{\Sigma_\rho^0} \right) \nabla \vartheta_{l-1} \right\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T).$$

Collecting all above estimates we get

$$\|\mathbf{f}_2(U)\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T). \tag{132}$$

Estimate of $f_3(U)$. First we estimate $R_3^k(U)$ given by (78)-(79). For this purpose we show

Lemma 6.8 *We have*

$$\|\mathcal{B}_{kl}\|_{L_\infty(\Omega \times (0,T))} \leq C(M), \tag{133}$$

$$\|\nabla \mathcal{B}_{kl}\|_{L_p(0,T;L_q(\Omega))} \leq C(M)E(T). \tag{134}$$

Proof. (133) follows directly from (120) and the form of $\mathcal{B}_{kl}$ (22). To show (134) we need a bound on ∇D_{kl}. For this purpose notice that, by (120),

$$\|\nabla \rho_k\|_{L_p(0,T;L_q(\Omega))}^p \leq \int_0^T \|(\nabla \rho_k - \nabla \rho_k^0)(t,\cdot)\|_{L_q}^p dt + \int_0^T \|\nabla \rho_k^0\|_{L_q(\Omega)} dt \leq [C(M,L)E(T)]^p.$$

Therefore, under the assumption (23) and using the fact that the fractional densities are bounded from below by a positive constant we obtain (134).

$\square$

From (130) and (133) we get

$$\|\mathcal{B}_{kl}(A_{2\Delta}(\mathbf{k_v})\nabla^2 \vartheta_l + A_{1\Delta}(\mathbf{k_v})\nabla \vartheta_l)\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T). \tag{135}$$

Next, by (119) and (134),

$$\|\nabla \mathcal{B}_{kl}\nabla \vartheta_l\|_{L_p(0,T;L_q(\Omega))} \leq \|\nabla \vartheta_l\|_{L_\infty(\Omega \times (0,T))}\|\nabla \mathcal{B}_{kl}\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T).$$

Therefore

$$\|\mathbf{V}^0(\mathbf{k_v})\nabla \mathcal{B}_{kl}\left([\nabla \vartheta_l + \nabla \vartheta_l \mathbf{V}^0(\mathbf{k_v})]\right) + (\nabla \mathcal{B}_{kl})\mathbf{V}^0(\mathbf{k_v})\nabla \vartheta_l\|_{L_p(0,T;L_q(\Omega))}$$
$$\leq C\|\|\nabla \mathcal{B}_{kl}\nabla \vartheta_l\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T). \tag{136}$$

Combining (135) and (136) we get

$$\|R_3^{kl}(U)\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T). \tag{137}$$

Finally, by (116) and (119),

$$\left\| \left(\varrho_{k+1} - \frac{m_{k+1}\varrho_{k+1}\varrho}{\Sigma_\varrho} \right) \sum_{j,m=1}^n V_{jm}^0(\mathbf{k_v})\frac{\partial v_j}{\partial y_m} \right\|_{L_p(0,T;L_q(\Omega))} \leq C \sum_{j,m=1}^n \|V_{jm}^0(\mathbf{k_v})\|_{L_\infty(\Omega \times (0,T))} \left\| \frac{\partial v_j}{\partial y_m} \right\|_{L_p(0,T;L_q(\Omega))}$$
$$\leq C(M)E(T),$$

which together with (137) yields

$$\|R_3^k(U)\|_{L_p(0,T;L_q(\Omega))} \leq C(M,L)E(T). \tag{138}$$

The remaining terms in (93) contains only components of type $\rho_k \nabla \boldsymbol{v}, \rho_k \partial_t \theta_l, \nabla \rho_k \nabla \theta_l$ and $\rho_k \nabla^2 \theta_l$, therefore we can estimate them in a similar way to $\mathbf{f}_2(U)$ using (117)-(129) obtaining

$$\|f_3^k(U)\|_{L_p(0,T;L_q(\Omega))} \le C(M, L)E(T), \quad k = 1, \ldots, n-1. \tag{139}$$

Estimate of $f_4^k(U)$. This task is more delicate since we have to find a bound on $\|f_4^k(U)\|_{H_p^{1/2}(\mathbb{R};L_q(\Omega))}$. However, the structure of boundary condition (60) is exactly the same as in the two species case, therefore we can repeat the estimate from [30]. For the sake of completeness we repeat the idea here. First we have to extend $f_4^k(U)$ to whole real line. For this purpose we apply the extension operator (123). Let us denote

$$\mathbf{J}[\boldsymbol{v}](t) = \boldsymbol{n}(x)\mathbf{V}^0(\mathbf{k}_{\boldsymbol{v}}) \left\{ \int_0^1 (\nabla \boldsymbol{n})(y + \tau \int_0^t \boldsymbol{v}(y,s)\,ds)\,d\tau \int_0^t \boldsymbol{v}(y,s)\,ds \right\}.$$

Then (80) can be rewritten as

$$R_4^k(U) = -\mathbf{J}[\boldsymbol{v}]\nabla \vartheta_k.$$

Since $\mathbf{J}[\boldsymbol{v}](0) = 0$, we can readily define

$$\tilde{\mathbf{J}}[u] = e_T(\mathbf{J}[u]) \tag{140}$$

Next, we also need to extend ϑ_k. The difference is that it does not vanish at 0, therefore first we first extend the boundary data to $\tilde{\vartheta}_k^0$ defined on $\mathbb{R}^n$ and define

$$E\vartheta_k = e_T[\vartheta_k^0 - \mathcal{T}(t)\vartheta_k^0] + \mathcal{T}(t)\vartheta_k^0, \tag{141}$$

where $\mathcal{T}(t)$ is an exponentially decaying semigroup (details can be found in Section 5 of [30]). The norms of extensions are equivalent with the norms defined on $(0, T)$, therefore we have to estimate $\|E\mathbf{J}[u]\nabla(E\vartheta_l)\|_{H_p^{1/2}(\mathbb{R}, L_q(\Omega))}$.

For this purpose we apply Lemma 6.3. As $\partial\Omega$ is uniformly C^3, we can extend the normal vector to $E\mathbf{n}$ defined on $\mathbb{R}^n$ s.t. $\|E\mathbf{n}\|_{H_\infty^2(\mathbb{R}^n)} \le C(\Omega)$. Then we obtain

$$\|EJ[\boldsymbol{v}]\|_{L_\infty(0,T;H_q^1(\Omega))} \le C(M)E(T). \tag{142}$$

and, due to (124),

$$\|\partial_t EJ[\boldsymbol{v}]\|_{L_\infty(0,T;L_q(\Omega))} + \|\partial_t EJ[\boldsymbol{v}]\|_{L_p(0,T;H_q^1(\Omega))} \le C\left[\|\boldsymbol{v}\|_{L_\infty(0,T;H_q^1(\Omega))} + \|\boldsymbol{v}\|_{L_p(0,T;H_q^2(\Omega))}\right] \le C(M). \tag{143}$$

In order to estimate $E\nabla\vartheta_k$ we apply Lemma 6.5 to obtain

$$\|E\nabla\vartheta_k\|_{H_p^{1/2}(0,T;L_q(\Omega))} + \|E\nabla\vartheta_k\|_{L_p(H_q^1(\Omega))} \le C(M, L). \tag{144}$$

Applying Lemma 6.3 with $f = E\mathbf{J}[u]$ and $g = \nabla(E\vartheta_k)$ and using (142) - (144) we obtain

$$\|R_4^k(U)\|_{L_p(\mathbb{R}, H_q^1(\Omega)) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega))} \le E(T)C(M, L). \tag{145}$$

Now, combining (131),(132),(139),(145) and (94) we obtain (115), which completes the proof of Proposition 6.1.

6.2 Fixed point argument

Theorem 5.3 allows us to define an operator $(\sigma, \boldsymbol{v}, \vartheta) = \mathcal{S}(\bar{\sigma}, \bar{\boldsymbol{v}}, \bar{\vartheta})$ as a solution to system (86) with the right hand side $f_1(\bar{U}), \mathbf{f}_2(\bar{U}), f_3^k(\bar{U}), f_4^k(\bar{U})$ where $\bar{U} = (\bar{\sigma}, \bar{\boldsymbol{v}}, \bar{\vartheta})$. From the Proposition 6.1 combined with Theorem 5.3 we easily verify that for any $M > 0$

$$\mathcal{S} : \mathcal{H}_{T,M} \to \mathcal{H}_{T,M}$$

is well defined provided $T > 0$ is sufficiently small. It remains to show that $\mathcal{S}$ is a contraction on $\mathcal{H}_{T,M}$. For this purpose we show

Proposition 6.9 *Let $\bar{U}_1 = (\bar{\sigma}_1, \bar{\boldsymbol{v}}_1, \bar{\vartheta}_1), \bar{U}_2 = (\bar{\sigma}_2, \bar{\boldsymbol{v}}_2, \bar{\vartheta}_2) \in \mathcal{H}_{T,M}$ for given $T, M > 0$, where the initial conditions satify the assumptions of Theorem 2.2. Let $f_1(U), f_2(U), f_3^k(U)$ and $f_4^k(U)$ be given by (91)-(94), where $R_1(U), R_2(U), R_3^k(U)$ and $R_4^k(U)$ are defined in (72),(77),(78)-(79) and (80), respectively. Then*

$$\|f_1(U_1) - f_1(U_2)\|_{L_p(0,T;W_q^1(\Omega))} + \|\mathbf{f}_2(U_1) - \mathbf{f}_2(U_2)\|_{L_p(0,T;L_q(\Omega))} + \|f_3^k(U_1) - f_3^k(U_2)\|_{L_p(0,T;L_q(\Omega))} +$$

$$\|f_4^k(U_1) - f_4^k(U_2)\|_{L_p(0,T,H_q^1(\Omega))} + \|f_4^k(U_1) - f_4^k(U_2)\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))} \leq E(L,M,T)[U_1 - U_2]_T. \quad (146)$$

Proof. The precise form of the terms on the left hand side of (146) is rather complicated, however what is essential is that it contains only the terms which are products of either $\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2$, $\bar{\sigma}_1 - \bar{\sigma}_2$ or $\bar{\vartheta}_1 - \bar{\vartheta}_2$ multiplied by some quantities which are small for small times. Therefore, following the lines of the proof of Proposition 6.1 we obtain (146).

$\square$

Now we can subtract systems for U_1 and U_2 to obtain a linear problem for $U_1 - U_2$ with the structure of the left hand side that same as in (68), zero initial and boundary conditions and left hand side which is estimated in (146). Therefore, combining Proposition 6.9 and Theorem 5.3 we obtain

$$[\mathcal{S}(U_1) - \mathcal{S}(U_2)]_T \leq E(T)[U_1 - U_2]_T, \quad (147)$$

which implies that for any $M > 0$, $\mathcal{S}$ is a contraction on $\mathcal{H}_{T,M}$ for sufficiently small T. Therefore, application of the Banach fixed point theorem to $\mathcal{S}$ completes the proof of Theorem 2.2.

$\square$

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