

Itô's formula for finite variation Lévy processes: The case of non-smooth functions

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Abstract

Extending Itô's formula to non-smooth functions is important both in theory and applications. One of the fairly general extensions of the formula, known as Meyer-Itô, applies to one dimensional semimartingales and convex functions. There are also satisfactory generalizations of Itô's formula for diffusion processes where the Meyer-Itô assumptions are weakened even further. We study a version of Itô's formula for multi-dimensional finite variation Lévy processes assuming that the underlying function is continuous and admits weak derivatives. We also discuss some applications of this extension, particularly in finance.

Keywords: Itô's formula, Finite variation Lévy process, Weak derivative, PIDE

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1 Introduction

In order to motivate our study, we consider the following Partial Integro-Differential Equation (PIDE):

$$\begin{aligned} \frac{\partial P}{\partial t}(t, x) + rx \frac{\partial P}{\partial x}(t, x) + \frac{\sigma^2 x^2}{2} \frac{\partial^2 P}{\partial x^2}(t, x) - rP(t, x) \\ + \int v(dy) \left(P(t, xe^y) - P(t, x) - x(e^y - 1) \frac{\partial P}{\partial x}(t, x) \right) = 0, \\ P(T, x) = (x - K)^+, \text{ for all } x \in (0, D), \\ P(t, x) = 0, \text{ for all } x \geq D, \text{ and all } t \in [0, T], \end{aligned} \quad (1)$$

where $D > K > 0$, $r > 0$, $T > 0$, are constants, and v is the Lévy measure of a Lévy process X with characteristic triplet (σ^2, v, γ) . Furthermore, it is assumed that $(e^{X_t})_{0 \leq t \leq T}$ is a martingale with respect to the natural filtration generated by X and a risk-neutral probability measure.

Finding the solution of this PIDE (or similar ones) is of particular interest in different applied fields. For instance, under some circumstances the solution of PIDE (1) can be identified as the price of a financial security. As it follows, Itô's formula is a key element in this procedure.

More precisely, assume that the risk-neutral evolution of an asset is modeled by $S_t = S_0 e^{rt + X_t}$, where r and X are the same as above such that $(e^{-rt} S_t)_{0 \leq t \leq T}$ is a martingale under the risk-neutral probability measure. Suppose that we are interested in pricing a barrier option with maturity T , strike price K , barrier $D > K$, and the payoff $\max(S_T - K, 0) 1_{\{\max_{0 \leq s \leq T} S_s < D\}}$. If $\sigma > 0$, then using Itô's formula one can show that there is a $C^{1,2}$ solution of PIDE (1) which is in fact the price of this barrier option given by

$$P(t, x) = e^{-r(T-t)} \mathbb{E}[H(S_{T \wedge \tau_D}) | S_t = x], \quad (2)$$

where \mathbb{E} is the expectation under the risk-neutral measure, $H(x) := (x - K)^+ 1_{\{x < D\}}$, and $\tau_D := \inf\{s \geq t; X_s \geq D\}$, see Proposition 12.2 of Cont and Tankov (2004).

Equation (2) is in fact the Feynman-Kac representation of the solution of PIDE (1) which can be numerically calculated through simulation techniques. Note that the condition $\sigma > 0$ is crucial for this argument to work which guarantees that the purposed solution (2) is smooth and hence Itô's formula is applicable. However, in the case of pure jump Lévy processes, i.e. when $\sigma = 0$, the smoothness is not obvious and it can fail. The situation is more complicated for American options where the smoothness of the purposed solution is not known even in the presence of a non-zero volatility, see Chapter 12 of Cont and Tankov (2004) for more detail. For example, Theorem 7.2 of Boyarchenko and Levendorskiĭ (2002) shows that the smoothness of

the proposed solution in the case of American option fails for tempered stable Lévy processes with finite variation.

One purpose of this work is to fix this kind of problems for models using finite variation Lévy processes. For this class of processes, under some conditions, we obtain an Itô formula that works well with non-smooth continuous functions. In particular, this can provide a solution to PIDE (1) when $\sigma = 0$ and X is a finite variation Lévy process. This problem is investigated at the end of this paper. We continue with some literature review.

A version of Itô's formula is obtained in Aebi (1992) where the underlying process is a continuous semimartingale with a special structure. In this paper, the first and second order derivatives of the function are defined in the sense of distributions and they satisfy some local integrability conditions. Föllmer et al. (1995) discuss an extension of the formula to a one-dimensional standard Brownian motion and an absolutely continuous function with a locally square integrable derivative. This result was further extended by Föllmer and Protter (2000) to a multi-dimensional Brownian motion.

Following the idea of Föllmer et al. (1995), an extension of Itô's formula is proved in Bardina and Jolis (1997) for a one-dimensional diffusion process such that its law has a density satisfying certain integrability conditions. In their work, it is assumed that the underlying function $f = f(t, x)$ is absolutely continuous in x with a locally square integrable derivative satisfying a mild form of continuity in time t .

In all the above works, the sample paths of the underlying processes are continuous. Concerning discontinuous processes, Theorem 70, Chapter IV of Protter (2004) (known as Meyer-Itô's formula) provides a fairly general extension of Itô's formula to semimartingales and one dimensional convex functions.

Comparing to Theorem 70, Chapter IV of Protter (2004), our extension applies to finite variation Lévy processes and continuous functions that admit weak derivatives. Therefore this generalizes Meyer-Itô's formula for finite variation Lévy processes. In addition, it is assumed that the function is multi-dimensional and time-dependent. Beside the motivation provided at the beginning and theoretical interests to extend Itô's formula for these processes, it is also argued in Geman (2002) that the evolution of asset prices are better modeled by finite variation processes with infinite activity¹.

The structure of the paper is as follows. The theoretical backgrounds, in particular some fundamental results in real and functional analysis are reviewed in Section 2. Section 3 concentrates on hypotheses and key tools. The main result is proved in Section 4. Finally, the

¹A Lévy process X in \mathbb{R}^d is of infinite activity, if there are infinite number of jumps on any finite time interval, i.e. $\nu(\mathbb{R}^d) = \infty$, where ν is the Lévy measure of X .

paper ends with some examples and conclusions.

2 Preliminaries and Definitions

In this section, we recall a few results from real and functional analysis (basically Distribution theory) that will be used later. We begin with some definitions. In what follows, \mathbb{R} is the set of real numbers; $U \subset \mathbb{R}^d$ is a nonempty open set, $d \geq 1$; $|\cdot|$ and $\|\cdot\|_d$ are respectively the one-dimensional and d -dimensional Euclidean norms; and m is the Lebesgue measure. For simplicity, regardless of the dimension of the space, the Lebesgue measure is always denoted by m .

Definition 2.1. A point $x \in U \subset \mathbb{R}^d$ is a Lebesgue point of a function $f : U \mapsto \mathbb{R}$ if

$$\lim_{r \rightarrow 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0,$$

where $B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}$ and the limit is taken for those r small enough to guarantee that $B_r(x)$ is a subset of U .

Definition 2.2. The set of all Lebesgue points of $f : U \mapsto \mathbb{R}$ is denoted by L_f and it is called the Lebesgue set.

Definition 2.3. A family $\{E_r\}_{r>0}$ of Borel subsets of U is said to shrink nicely to $x \in U$ if the following two conditions hold

- $E_r \subset B_r(x) \subset U$ for each r ,
- there is a constant $\alpha > 0$, independent of r , such that $m(E_r) > \alpha m(B_r(x))$.

Theorem 2.1. The Lebesgue Differentiation Theorem. Suppose that $f \in L^1_{loc}(U)$ and $\text{supp}(f) \subset U$. Then we have

- $m(U - L_f) = 0$,
- for every x in the Lebesgue set of f , in particular for almost every x in U , we have

$$\lim_{r \rightarrow 0^+} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$$

where $\{E_r\}_{r>0}$ is a family of Borel subsets of $U \subset \mathbb{R}^d$ that shrinks nicely to x .

For a proof of this theorem in the case of $U = \mathbb{R}^d$, see Theorem 3.21 of Folland (1999). The generalization to an open set $U \subset \mathbb{R}^d$ is straightforward. Note that following the Lebesgue Differentiation Theorem we have $\lim_{r \rightarrow 0^+} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$, where f and E_r are the same as in the above theorem. Therefore, this can be thought of as a generalization of the fundamental theorem of calculus. In general, determining the Lebesgue points of a function is not an easy task. The next lemma gives a partial answer to this challenge; the proof is simple and hence omitted.

Lemma 2.1. *If $f \in L^1_{loc}(U)$, $U \subset \mathbb{R}^d$, and f is continuous at $x \in U$, then $x \in L_f$.*

Definition 2.4. *If $g : \mathbb{R}^d \mapsto \mathbb{R}$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$, are measurable functions, then the convolution $g * f : \mathbb{R}^d \mapsto \mathbb{R}$ is defined by $(g * f)(x) := \int_{\mathbb{R}^d} g(x - y)f(y) dy$, provided that for every x in \mathbb{R}^d , the integral is well defined.*

Some basic properties of convolution can be found in standard text books such as Folland (1999) or Brezis (2011). The next lemma provides a simple and sufficient condition for the existence of convolution.

Lemma 2.2. *Let $f \in L^p_{loc}(U)$, $p \geq 1$, and $\text{supp}(f) \subset U$. Suppose that $g : \mathbb{R}^d \mapsto \mathbb{R}$ is bounded and compactly supported. Then $g * f 1_U$, $f 1_U(x) = \begin{cases} f(x), & x \in U; \\ 0, & x \notin U, \end{cases}$ is well defined on \mathbb{R}^d , i.e. the integral $\int_U g(x - y)f(y) dy$ is finite for all x in \mathbb{R}^d .*

Let η be any function in $C_c^\infty(\mathbb{R}^d)$ such that it satisfies the following conditions

$$\eta \geq 0, \quad \int_{\mathbb{R}^d} \eta(x) dx = 1, \quad \text{supp}(\eta) = \overline{B_1(0)}.$$

For any $\epsilon > 0$, define $\eta^\epsilon(x) = \frac{1}{\epsilon^d} \eta\left(\frac{x}{\epsilon}\right)$ then clearly we have

$$\eta^\epsilon \in C_c^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \eta^\epsilon(x) dx = 1, \quad \text{supp}(\eta^\epsilon) = \overline{B_\epsilon(0)}.$$

The next definition provides an example of such a function.

Definition 2.5. *Let*

$$\eta(x) = \begin{cases} ce^{-\frac{1}{1-\|x\|_d^2}}, & \|x\|_d < 1; \\ 0, & \|x\|_d \geq 1, \end{cases}$$

and take c such that $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Then η^ϵ is called the standard mollifier.

Our discussion does not depend on a specific choice of η^ϵ . However, if necessary, the reader can always consider the standard mollifier. Suppose that $f \in L^p_{loc}(U)$, $p \geq 1$, and for every $\epsilon > 0$, let $f^\epsilon : \mathbb{R}^d \mapsto \mathbb{R}$ be defined by

$$f^\epsilon(x) := (\eta^\epsilon * f1_U)(x) = \int_U \eta^\epsilon(x-y)f(y) dy.$$

For a fixed x and ϵ small enough (that depends on x), $\overline{B_\epsilon(x)} \subset U$ and so $f^\epsilon(x)$ exists. However, if $\text{supp}(f) \subset U$ and since $f \in L^p_{loc}(U)$, $p \geq 1$, by Lemma 2.2, f^ϵ is well defined on \mathbb{R}^d for all $\epsilon > 0$. The following theorem is a classical well-known result in the theory of distributions. Parts (1) and (2) can be found in Section 4.4 of Brezis (2011), and part (3) is a conclusion of Theorem 2.1.

Theorem 2.2. *Assume that $f \in L^p_{loc}(U)$, $p \geq 1$, $\text{supp}(f) \subset U$, and $\epsilon > 0$. Then*

1. $f^\epsilon \in C^\infty(\mathbb{R}^d)$ and $\partial^\alpha f^\epsilon = \partial^\alpha \eta^\epsilon * f1_U$,
2. $f^\epsilon \rightarrow f1_U$ in $L^p_{loc}(\mathbb{R}^d)$ as $\epsilon \rightarrow 0^+$,
3. $f^\epsilon \rightarrow f1_U$ pointwise on L_{f1_U} as $\epsilon \rightarrow 0^+$, hence $f^\epsilon \rightarrow f$ pointwise on L_f as $\epsilon \rightarrow 0^+$.

Note that part (3) of Theorem 2.2 implies that $f^\epsilon \rightarrow f$, Lebesgue almost every where on U . Let \mathbb{N}_0 be the set of non-negative integers and $\mathbb{N}_0^d = \{(\alpha_1, \alpha_2, \dots, \alpha_d) : \alpha_i \in \mathbb{N}_0, i = 1, 2, \dots, d\}$. An element of the set \mathbb{N}_0^d is called a multi-index. In our extended version of Itô's formula instead of classical strong differentiability, we apply weak differentiability which is defined below.

Definition 2.6. *Suppose that $\alpha \in \mathbb{N}_0^d$ is a multi-index. We say that a function $f \in L^1_{loc}(U)$, $U \subset \mathbb{R}^d$, is weakly differentiable; and also its α -th-weak derivative denoted by $\partial^\alpha f \in L^1_{loc}(U)$, if*

$$\int_U (\partial^\alpha f(x))\phi(x) dx = (-1)^{|\alpha|} \int_U f(x)(\partial^\alpha \phi(x)) dx, \text{ for all } \phi \in C_c^\infty(U),$$

where $|\alpha| = \sum_{i=1}^d \alpha_i$, and the functions $\phi \in C_c^\infty(U)$ are called test functions.

By applying Theorem 2.2 and simple properties of weak derivatives, we can get the following theorem.

Theorem 2.3. *Let $f \in L^1_{loc}(U)$ and $\text{supp}(f) \subset U$. We further assume that f admits the weak derivative $\partial^\alpha f \in L^1_{loc}(U)$, then:*

1. $f^\epsilon \in C^\infty(\mathbb{R}^d)$, and $\partial^\alpha(f^\epsilon) = \eta^\epsilon * (\partial^\alpha f)$ on U ,
2. $\partial^\alpha(f^\epsilon) \rightarrow \partial^\alpha f$ in $L^1_{loc}(U)$ as $\epsilon \rightarrow 0^+$,

3. $\partial^\alpha(f^\epsilon) \rightarrow \partial^\alpha f$ pointwise on $L_{\partial^\alpha f}$ as $\epsilon \rightarrow 0^+$.

Remark 2.1. Note that part(1) of Theorems 2.2 and 2.3 still holds if we replace η^ϵ by a test function.

Though it is very simple, the next lemma is a key point in our discussion.

Lemma 2.3. Assume that $f \in L_{loc}^1(U)$ has the weak derivative $\partial^\alpha f \in L_{loc}^1(U)$. Suppose that $\phi \in C_c^\infty(\mathbb{R}^d)$ is a test function with support of K such that $\phi(x) \geq 0$, for all $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} \phi(x) dx = 1$. Then for every $x \in \mathbb{R}^d$ we have

$$|\partial^\alpha(f * \phi)(x)| \leq \sup_{z \in U \cap \Lambda(x)} |\partial^\alpha f(z)|,$$

where $\Lambda(x) = \{y \in \mathbb{R}^d : x - y \in K\}$.

Proof. By using Remark 2.1 we get

$$\partial^\alpha(f * \phi)(x) = (\phi * 1_U \partial^\alpha f)(x) = \int_U \phi(x - y) \partial^\alpha f(y) dy = \int_{U \cap \Lambda(x)} \phi(x - y) \partial^\alpha f(y) dy.$$

Using this equation and the following inequalities, we get the result

$$\begin{aligned} |\partial^\alpha(f * \phi)(x)| &\leq \sup_{z \in U \cap \Lambda(x)} |\partial^\alpha f(z)| \int_{U \cap \Lambda(x)} \phi(x - y) dy \\ &\leq \sup_{z \in U \cap \Lambda(x)} |\partial^\alpha f(z)| \int_{\mathbb{R}^d} \phi(x) dx = \sup_{z \in U \cap \Lambda(x)} |\partial^\alpha f(z)|. \end{aligned}$$

□

Remark 2.2. Note that the value of the right-hand side of the inequality in Lemma 2.3 can be infinity.

3 Discussion of Assumptions and Key Tools

In applying classical Itô's formula on smooth functions $f : [0, \infty) \times U \mapsto \mathbb{R}$, $U \subset \mathbb{R}^d$, the differentiability at $t = 0$ is understood by being the right-hand side derivative. Note that since the Lebesgue measure of $\{0\} \times U$ is zero, the weak derivatives of f can be defined similar to Definition 2.6.

Assume that $f : [0, \infty) \times U \mapsto \mathbb{R}$ is a Lebesgue measurable function. In accordance with Definition 2.6, we say that $f \in L_{loc}^1([0, \infty) \times U)$ has weak derivatives $\partial^\alpha f \in L_{loc}^1([0, \infty) \times U)$ if

$$\int_{[0, \infty) \times U} (\partial^\alpha f(x)) \phi(x) dx = (-1)^{|\alpha|} \int_{[0, \infty) \times U} f(x) (\partial^\alpha \phi(x)) dx, \text{ for all } \phi \in C_c^\infty([0, \infty) \times U). \quad (3)$$

Note that since a test function ϕ is smooth, its derivatives at the origin are understood as the right-hand side ones. The results of Section 2 are stated for open subsets of \mathbb{R}^d . However, $[0, \infty) \times U$ is not an open set. So in our first step we fix this problem by introducing an extended version of f .

Suppose that the function $f : [0, \infty) \times U \mapsto \mathbb{R}$ is continuous on $[0, \infty) \times U$. This function can be continuously extended to a new function $\tilde{f} : \mathbb{R} \times U \mapsto \mathbb{R}$:

$$\tilde{f}(t, x) = \begin{cases} f(t, x), & (t, x) \in [0, \infty) \times U; \\ f(-t, x), & (t, x) \in (-\infty, 0) \times U. \end{cases} \quad (4)$$

Now in addition assume that $f \in L^1_{loc}([0, \infty) \times U)$ and it is weakly differentiable in the sense of equation (3). Then one can easily show that $\tilde{f} \in L^1_{loc}(\mathbb{R} \times U)$ and it is weakly differentiable on the open set $\mathbb{R} \times U$ in the sense of Definition 2.6. The weak derivatives of \tilde{f} can be stated explicitly based on weak derivatives of f . For instance in the case of $d = 1$, one can easily check that

$$\frac{\partial \tilde{f}}{\partial t}(t, x) = \begin{cases} \frac{\partial f}{\partial t}(t, x), & (t, x) \in [0, \infty) \times U; \\ -\frac{\partial f}{\partial t}(-t, x), & (t, x) \in (-\infty, 0) \times U, \end{cases}$$

and

$$\frac{\partial \tilde{f}}{\partial x}(t, x) = \begin{cases} \frac{\partial f}{\partial x}(t, x), & (t, x) \in [0, \infty) \times U; \\ \frac{\partial f}{\partial x}(-t, x), & (t, x) \in (-\infty, 0) \times U, \end{cases}$$

where $\frac{\partial f}{\partial t}(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are weak derivatives of f in the sense of equation (3).

Assume that $(\Omega, \mathfrak{F}, \mathbb{P})$ is a complete probability space. Let $X = (X_t)_{t \geq 0}$, $X_t : \Omega \mapsto U$, $U \subset \mathbb{R}^d$, be a càdlàg stochastic process that is defined on this space. In any extension of Itô's formula, it is important to somehow measure the amount of time that the process spends in some certain regions of the domain. In particular, this is crucial for those points for which the function is not smooth. For instance, in the case of Meyer-Itô formula (see Theorem 70, Chapter IV of Protter (2004)), this is done through local times. In the next proposition, we discuss a similar tool which is a key result in our extension. The proposition is provided for a certain class of processes explained below.

Assumption 3.1. *Suppose that $X : [0, \infty) \times \Omega \mapsto U$ is a càdlàg stochastic process defined on the complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, that satisfies the following condition: If $A \subset U$ is a Borel measurable set such that $m(A) = 0$, where m is the Lebesgue measure, then for all $s \in \mathbb{R}^+$, $\mathbb{P}(X_s \in A) = 0$. In other words, for all $s \in \mathbb{R}^+$, the measure μ_s on U defined by $\mu_s(A) = \mathbb{P}(X_s \in A)$, is absolutely continuous with respect to the Lebesgue measure.*

Proposition 3.1. *Assume that the process X satisfies Assumption 3.1. Let $A \subset [0, \infty) \times U$ be any Lebesgue measurable set such that $m(A) = 0$, then for all $t \geq 0$ we have*

$$\mathbb{P}\{\omega \in \Omega : m(\{s \in [0, t] : (s, X_s(\omega)) \in A\}) = 0\} = 1.$$

In particular, this implicitly implies that for almost all $\omega \in \Omega$, the set $\{s \in [0, t] : (s, X_s(\omega)) \in A\}$ is Lebesgue measurable for all $t \geq 0$.

Proof. First assume that A is a Borel measurable set. Define the process $Y : [0, \infty) \times \Omega \mapsto [0, \infty) \times U$ by $Y(s, \omega) = (s, X_s(\omega))$. The process Y is càdlàg and by Proposition 1.21 of Jacod and Shiryaev (1987), Y is $\mathfrak{B}_{[0, \infty)} \times \mathfrak{F}$ measurable, where $\mathfrak{B}_{[0, \infty)}$ is the Borel σ -algebra on $[0, \infty)$ and \mathfrak{F} is the σ -algebra on Ω . Hence $Y^{-1}(A)$ belongs to $\mathfrak{B}_{[0, \infty)} \times \mathfrak{F}$ and so $\llbracket 0, t \rrbracket \cap Y^{-1}(A)$ is in $\mathfrak{B}_{[0, \infty)} \times \mathfrak{F} \subset \mathcal{L} \times \mathfrak{F}$, where $\llbracket 0, t \rrbracket = [0, t] \times \Omega$, and \mathcal{L} is Lebesgue σ -algebra on $[0, \infty)$. Therefore the function $f : [0, \infty) \times \Omega \mapsto \mathbb{R}$ defined by $f := 1_{\llbracket 0, t \rrbracket \cap Y^{-1}(A)}$ belongs to $L^1(m \times \mathbb{P})$.

From Fubini-Tonelli Theorem, see Theorem 2.37 of Folland (1999), it follows that f_ω defined by $f_\omega := (\cdot, \omega)$ is in $L^1(m)$ for almost all ω . So for a fixed ω , $\llbracket 0, t \rrbracket \cap Y^{-1}(A)$ is Lebesgue measurable, and $m\{s \in [0, t] : (s, X_s(\omega)) \in A\}$ is well defined for almost all $\omega \in \Omega$.

Moreover, let $Z(\omega) := \int f_\omega dm = m\{s \in [0, t] : (s, X_s(\omega)) \in A\}$, then again by Fubini-Tonelli Theorem Z is a random variable and $Z \in L^1(\mathbb{P})$, furthermore, we can calculate its expectation

$$\begin{aligned} \mathbb{E}[Z] &= \int \int f_\omega dm d\mathbb{P} = \int_0^t \int f_s d\mathbb{P} ds \\ &= \int_0^t \mathbb{E}[1_{\{(s, X_s) \in A\}}] ds. \end{aligned}$$

Note that for a fixed s , $1_{\{(s, X_s) \in A\}} = 1_{\{X_s \in A_s\}}$, where $A_s = \{y \in \mathbb{R}^d : (s, y) \in A\}$ is Borel measurable, hence we obtain

$$\mathbb{E}[Z] = \int_0^t \mathbb{P}(X_s \in A_s) ds. \quad (5)$$

The set A is Borel measurable and hence Lebesgue measurable as well. By Theorem 2.36 of Folland (1999) the function $s \mapsto m(A_s)$ is Lebesgue measurable and $m(A) = \int_{[0, \infty)} m(A_s) ds$. By the proposition's assumption $m(A) = 0$ which concludes that $m(A_s) = 0$ for Lebesgue almost all $s \geq 0$, i.e. there exists a set $N \subset [0, \infty)$ such that $m(N) = 0$ and if $s \notin N$ then $m(A_s) = 0$. From equation (5) and Assumption 3.1, we get

$$\mathbb{E}[Z] = \int_{[0, t] \cap N^c} \mathbb{P}(X_s \in A_s) ds = \int_{[0, t] \cap \{s : m(A_s) = 0\}} \mathbb{P}(X_s \in A_s) ds = 0.$$

The random variable Z is non-negative and $\mathbb{E}[Z] = 0$, hence $Z = 0$, \mathbb{P} -almost surely which means that for almost all $\omega \in \Omega$, $m(\{s \in [0, t] : (s, X_s(\omega)) \in A\}) = 0$. This completes the proof when A is Borel measurable.

Next, suppose that A is a Lebesgue measurable set, then $A = A' \cup A''$, $A'' \subset B$, where A' and B are Borel measurable and $m(B) = 0$. Now the result follows from the previous part, and the facts that $m(A) = 0$ and the probability space is complete. \square

Note that if $A = [0, t] \times B$, where $B \subset \mathbb{R}^d$ a Borel set, then $\{s \in [0, t] : (s, X_s) \in A\}$ is the amount of time that the process X spends in Borel set B . So under Assumption 3.1, Proposition 3.1 concludes that almost surely the Lebesgue measure of this amount of time is zero for any zero Borel measurable set.

We would like to point out that this measure can be quite different than local times. For instance, let X be a standard Brownian motion, then by Proposition 3.1, $m\{s \in [0, t] : X_s = a\} = 0$, \mathbb{P} -almost surely for all real numbers a whereas the local time of a Brownian motion at the level a is not zero. This is also because of the fact that as a measure the local time of a Brownian motion is singular with respect to the Lebesgue measure.

4 The Main Result

In this section, we state and prove our main result. First, we mention that the result holds for a finite variation Lévy process that satisfies Assumption 3.1. This assumption is not valid for a compound Poisson process X as $\mathbb{P}[X_t = 0] > 0$, for $t > 0$, and therefore, the measure μ_t defined in Assumption 3.1 is not absolutely continuous with respect to the Lebesgue measure, see Remark 27.3 of Sato (1999). However, based on Theorem 27.7 of Sato (1999), Assumption 3.1 is always satisfied for a finite variation Lévy process with infinite activity, if its Lévy measure is absolutely continuous with respect to the Lebesgue measure.

For simplicity we present the theorem for the case of $d = 1$, however there is no restriction on extending the result to a general d .

Theorem 4.1. *Assume that $f : [0, \infty) \times U \mapsto \mathbb{R}$ is a continuous function on $[0, \infty) \times U$ such that $f \in L^1_{loc}([0, \infty) \times U)$, $\text{supp}(f) \subset [0, \infty) \times U$, and U is an open set of \mathbb{R} . Let the weak derivatives $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial x} \in L^1_{loc}([0, \infty) \times U)$ be locally bounded and defined by equation (3). Suppose that X is a finite variation Lévy process satisfying Assumption 3.1 such that for all $t \geq 0$, X_t and X_{t-} are in U . Then*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \gamma \int_0^t \frac{\partial f}{\partial x}(s, X_s) ds \\ &\quad + \iint_{[0, t] \times \mathbb{R}} (f(s, X_{s-} + x) - f(s, X_{s-})) J_X(ds \times dx), \end{aligned}$$

where J_X and γ are respectively the Poisson random measure and the drift coefficient of the process X admitting the following representation: $X_t = \gamma t + \int_{[0, t] \times \mathbb{R}} x J_X(ds \times dx)$.

Proof. Assume that \tilde{f} is an extension of the function f to $\mathbb{R} \times U$ given by equation (4), note that $\text{supp}(\tilde{f}) \subset \mathbb{R} \times U$. Let $\phi_n = \eta_n^{\frac{1}{n}}$ and $f_n(t, x) := (\phi_n * \tilde{f} \mathbf{1}_{\mathbb{R} \times U})(t, x)$, where $(t, x) \in \mathbb{R}^2$, $n \geq 1$,

and $\eta_n^{\frac{1}{n}}$ is defined in Section 2. Since $\tilde{f} \in L_{loc}^1(\mathbb{R} \times U)$, by Theorem 2.3, $f_n \in C^\infty(\mathbb{R} \times \mathbb{R})$ for all $n \geq 1$. Hence from Itô's formula, see Theorem 4.2 of Kyprianou (2006), we have

$$\begin{aligned} f_n(t, X_t) &= f_n(0, X_0) + \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) ds + \gamma \int_0^t \frac{\partial f_n}{\partial x} ds \\ &\quad + \iint_{[0,t] \times \mathbb{R}} (f_n(s, X_{s^-} + x) - f_n(s, X_{s^-})) J_X(ds \times dx). \end{aligned}$$

The rest of the proof is divided into five steps:

Step 1. Since \tilde{f} is a continuous function, by Lemma 2.1 $L_{\tilde{f}} = \mathbb{R} \times U$. On the other hand for all $t \geq 0$, X_t is in U and so by Theorem 2.2, $f_n(t, X_t) \rightarrow \tilde{f}(t, X_t)$, for all $\omega \in \Omega$ and $t \in \mathbb{R}$. Especially $f_n(0, X_0) \rightarrow \tilde{f}(0, X_0)$. Also note that for $t \geq 0$, $\tilde{f}(t, X_t) = f(t, X_t)$ by the definition of \tilde{f} .

Step 2. From Theorem 2.3, if $(s, X_s) \in L_{\frac{\partial \tilde{f}}{\partial s}}$, then we have

$$\frac{\partial f_n}{\partial s}(s, X_s) \rightarrow \frac{\partial \tilde{f}}{\partial s}(s, X_s).$$

Let $L_1 = \mathbb{R} \times U - L_{\frac{\partial \tilde{f}}{\partial s}}$, then

$$\int_0^t \frac{\partial f_n}{\partial s}(s, X_s) ds = \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) 1_{\{(s, X_s) \notin L_1\}} ds + \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) 1_{\{(s, X_s) \in L_1\}} ds.$$

By Theorem 2.1, $m(L_1) = 0$, therefore by Proposition 3.1, $m\{s \in [0, t] : (s, X_s) \in L_1\} = 0$, \mathbb{P} -almost surely. Hence because of the properties of Lebesgue integral, for each fixed t , the integral

$$\int_0^t \frac{\partial f_n}{\partial s}(s, X_s) 1_{\{(s, X_s) \in L_1\}} ds = \int_{[0,t] \cap \{s : (s, X_s) \in L_1\}} \frac{\partial f_n}{\partial s}(s, X_s) ds,$$

is zero \mathbb{P} -almost surely. Therefore for a fixed t ,

$$\int_0^t \frac{\partial f_n}{\partial s}(s, X_s) ds = \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) 1_{\{(s, X_s) \notin L_1\}} ds, \quad \mathbb{P} - \text{almost surely.}$$

By Lemma 2.3, for all $(s, x) \in \mathbb{R}^2$, $|\frac{\partial f_n}{\partial s}(s, x)| \leq \sup_{z \in (\mathbb{R} \times U) \cap \Lambda(s, x)} |\frac{\partial \tilde{f}}{\partial s}(z)| \leq \sup_{z \in \Lambda(s, x)} |\frac{\partial \tilde{f}}{\partial s}(z)|$, where $\Lambda(s, x) = \{y \in \mathbb{R}^2 : (s, x) - y \in K\}$, and $K = \sup \phi_n = \overline{B_{\frac{1}{n}}(0)} \subset \overline{B_1(0)}$ which results

$$\left| \frac{\partial f_n}{\partial s}(s, X_s) \right| \leq \sup_{z \in \Lambda(s, X_s)} \left| \frac{\partial \tilde{f}}{\partial s}(z) \right|, \quad 0 \leq s \leq t.$$

For a fixed $\omega \in \Omega$, $\Lambda(s, X_s)$ is bounded, because X is bounded on $[0, t]$ (due to being a càdlàg process). Therefore for a fixed $\omega \in \Omega$ and $s \in [0, t]$, one can find an upper bound for $|\frac{\partial f_n}{\partial s}(s, X_s)|$ that depends only on ω, t , and the minimum, maximum of $\frac{\partial \tilde{f}}{\partial s}(s, X_s)$ on $[0, t]$. This

upper bound is finite because the weak derivatives of f are locally bounded by the assumption of the theorem and so the weak derivatives of \tilde{f} must be locally bounded too. Therefore, one can apply Lebesgue Dominated Convergence theorem and we obtain:

$$\lim_{n \rightarrow \infty} \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) ds = \int_0^t \lim_{n \rightarrow \infty} \frac{\partial f_n}{\partial s}(s, X_s) 1_{\{(s, X_s) \notin L_1\}} ds, \mathbb{P} - \text{almost surely.}$$

By Theorem 2.3, this is \mathbb{P} -almost surely equal to $\int_0^t \frac{\partial \tilde{f}}{\partial s}(s, X_s) 1_{\{(s, X_s) \notin L_1\}} ds$. Since \mathbb{P} -almost surely, $m\{s \in [0, t] : (s, X_s) \in L_1\} = 0$, and for each $s \in [0, t]$, $\frac{\partial \tilde{f}}{\partial s}(s, X_s) = \frac{\partial f}{\partial s}(s, X_s)$, we have

$$\lim_{n \rightarrow \infty} \int_0^t \frac{\partial f_n}{\partial s}(s, X_s) ds = \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds, \mathbb{P} - \text{almost surely.}$$

Step 3. Similar to Step 2, one can prove that

$$\lim_{n \rightarrow \infty} \int_0^t \frac{\partial f_n}{\partial x}(s, X_s) ds = \int_0^t \frac{\partial f}{\partial x}(s, X_s) ds, \mathbb{P} - \text{almost surely.}$$

Step 4. Let $I_n = \iint_{[0, t] \times \mathbb{R}} (f_n(s, X_{s^-} + x) - f_n(s, X_{s^-})) J_X(ds \times dx)$, by using mean-value theorem we have $|f_n(s, X_{s^-} + x) - f_n(s, X_{s^-})| = |\frac{\partial f_n}{\partial x}(s, C)| |x|$, where C is a random variable between X_{s^-} and $X_{s^-} + x$. By applying Lemma 2.3 and the same procedure as Step 2, we can show that $|f_n(s, X_{s^-} + x) - f_n(s, X_{s^-})| \leq C' |x|$, where C' is a finite random variable, free from s, x, n . On the other hand, since X is a finite variation Lévy process, we also have that $\int_{[0, t] \times \mathbb{R}} |x| J_X(ds \times dx) < \infty$, \mathbb{P} -almost surely.

Therefore by applying Lebesgue Dominated Convergence theorem, one can interchange the limit and the integral in expression I_n as n goes to infinity. Since $L_{\tilde{f}} = \mathbb{R} \times U \supseteq [0, t] \times U$, and for all $s \geq 0$, X_s and X_{s^-} are in U , by part three of Theorem 2.2, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} I_n &= \iint_{[0, t] \times \mathbb{R}} (\tilde{f}(s, X_{s^-} + x) - \tilde{f}(s, X_{s^-})) J_X(ds \times dx) \\ &= \iint_{[0, t] \times \mathbb{R}} (f(s, X_{s^-} + x) - f(s, X_{s^-})) J_X(ds \times dx). \end{aligned}$$

Step 5. From Steps 1, 2, 3, 4, for a fixed $t \geq 0$, we have \mathbb{P} -almost surely the following identity

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \gamma \int_0^t \frac{\partial f}{\partial x} ds \\ &\quad + \iint_{[0, t] \times \mathbb{R}} (f(s, X_{s^-} + x) - f(s, X_{s^-})) J_X(ds \times dx). \end{aligned} \quad (6)$$

The process X is càdlàg, so the left-hand side and the right-hand side of the above equality are well defined processes. Therefore so far we have shown that the two sides of the above equation (when considered as processes) are in fact modifications of each other. Now we prove that as processes the left-hand side and the right-hand side are indeed indistinguishable.

1. First note that since f is continuous on $[0, \infty) \times U$, then $(f(t, X_t))_{t \geq 0}$ is càdlàg.
2. The function $\frac{\partial f}{\partial s}$ is Borel measurable and for a fixed $\omega \in \Omega$, $(X_s)_{0 \leq s \leq t}$ is also Borel measurable. Hence for a fixed ω , $\frac{\partial f}{\partial s}(s, X_s)$ is Borel measurable. So it is also Lebesgue measurable and by Fundamental theorem of Lebesgue integral calculus $t \mapsto \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds$ is uniformly continuous in t . Note that in Step 2, we actually showed that $\frac{\partial \tilde{f}}{\partial s}(s, X_s)$ is Lebesgue integrable and on $[0, t]$, $\frac{\partial \tilde{f}}{\partial s}(s, X_s) = \frac{\partial f}{\partial s}(s, X_s)$.
3. Similarly to the previous case, $t \mapsto \int_0^t \frac{\partial f}{\partial x}(s, X_s) ds$ is also continuous in t .
4. Let $Z_t := \iint_{[0, t] \times \mathbb{R}} (f(s, X_{s^-} + x) - f(s, X_{s^-})) J_X(ds \times dx)$. For all $s \geq 0$, X_s and X_{s^-} are in U , therefore $Z_t = \sum_{0 \leq s \leq t} (f(s, X_s) - f(s, X_{s^-}))$. If the function f is $C^{1,1}$, then obviously the process $Z = (Z_t)_{t \geq 0}$ is right continuous. However, since here f is not necessarily smooth, to show the right continuity of Z , we do as follows:

$$\begin{aligned}
\lim_{h \rightarrow 0^+} |Z_{t+h} - Z_t| &= \lim_{h \rightarrow 0^+} \left| \sum_{t < s \leq t+h} (f(s, X_s) - f(s, X_{s^-})) \right| \\
&\leq \lim_{h \rightarrow 0^+} \sum_{t < s \leq t+h} |f(s, X_s) - f(s, X_{s^-})| \\
&= \lim_{h \rightarrow 0^+} \sum_{t < s \leq t+h} \left| \lim_{n \rightarrow \infty} (f_n(s, X_s) - f_n(s, X_{s^-})) \right| \\
&\leq \lim_{h \rightarrow 0^+} \sum_{t < s \leq t+h} C'' |\Delta X_s|,
\end{aligned}$$

where similar to Step 4, one can show that C'' is a finite random variable free from s, h, n , so we obtain

$$\lim_{h \rightarrow 0^+} |Z_{t+h} - Z_t| \leq C'' \lim_{h \rightarrow 0^+} \sum_{t < s \leq t+h} \Delta X_s = 0, \mathbb{P} - \text{ almost surely.}$$

This shows that the process Z is right continuous.

Thus the left-hand side and the right-hand side of equation (6), when considered as processes, are right continuous, and we already know that they are also modification of each other. By Theorem 4, Chapter I of Protter (2004), we conclude that the left-hand side and the right-hand side of this equation define two processes that are indistinguishable.

This proves our theorem.

□

The next example shows that even in one dimensional cases, there are simple functions for which Meyer-Itô formula is not applicable but Theorem 4.1 can be used.

Example 4.1. Assume that $X : [0, \infty) \times \Omega \mapsto \mathbb{R}$ is a finite variation Lévy process that satisfies Assumption 3.1. Let the function $f : \mathbb{R} \mapsto \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

This function is continuous, but its derivative is not continuous at origin. So the classical Itô's formula cannot be applied. Moreover, one can show that f cannot be written as the difference of two convex functions, and hence Meyer-Itô's formula (Theorem 70, Chapter IV of Protter (2004)) is not applicable as well. However, f is weakly differentiable, its weak derivative is locally bounded, and therefore Theorem 4.1 is in force.

Example 4.2. Let the function f and the process X be the same as Theorem 4.1. In addition, we equip the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with the natural filtration $\mathbb{F}^X = \{\mathcal{F}_t; t \geq 0\}$ generated by the history of X , i.e. for each $t \geq 0$, \mathcal{F}_t is the sigma algebra generated by $\{X_s; s \leq t\}$ and all the null sets of \mathfrak{F} . Since X is a finite variation Lévy process, similar to Step 4 of Theorem 4.1, one can show that for every $t \geq 0$, $\iint_{[0,t] \times \mathbb{R}} |f(s, X_{s^-} + x) - f(s, X_{s^-})| ds \times v(dx) < C \int_{\mathbb{R}} x v(dx) < \infty$, \mathbb{P} -almost surely, where C is a random variable free from s and x . Then we have the following decomposition: $f(t, X_t) = f(0, X_0) + M_t + \int_0^t \mathcal{A}f(s, X_s) ds$, where M is a local martingale with respect to \mathbb{F}^X given by $M_t = \iint_{[0,t] \times \mathbb{R}} (f(s, X_{s^-} + x) - f(s, X_{s^-})) \tilde{J}_X(ds \times dx)$, $\tilde{J}_X(ds \times dx) = J_X(ds \times dx) - ds \times v(dx)$, and

$$\mathcal{A}f(s, X_s) = \frac{\partial f}{\partial s}(s, X_s) + \gamma \frac{\partial f}{\partial x}(s, X_s) + \int_{\mathbb{R}} (f(s, X_{s^-} + x) - f(s, X_{s^-})) v(dx).$$

In other words, this shows that the process $(f(t, X_t))_{t \geq 0}$ is a special semimartingale.

In the next lemma, we get back to the motivation provided in the introduction. This lemma also highlights applications of Theorem 4.1 in Feynman-Kac representations. Comparing to similar results, for instance Rong (1997), this representation is valid in the absence of diffusions terms. In addition, there are less restrictive assumptions on the underlying function.

Lemma 4.1. Suppose that X is a finite variation Lévy process that satisfies Assumption 3.1 for $U = \mathbb{R}$. Let the function $P = P(t, x)$, defined by equation (2), admit $L_{loc}^1([0, T] \times (0, \infty))$ -weak derivatives which are locally bounded. Then using Theorem 4.1 and following the same

procedure as Proposition 12.2 of Cont and Tankov (2004) (or Rong (1997)), one can show that $P = P(t, x)$ is the solution of PIDE (1).

5 Conclusions

A version of Itô's formula is studied under multi-dimensional finite variation Lévy processes that is time-dependent and requires weak differentiability. The formula can be particularly useful for functions that are continuous and piecewise smooth. The possible formula's applications were motivated by a financial example.

The two main assumptions are that the process is finite variation and the weak derivatives of the functions are locally bounded. The extension of the formula to pure jump semimartingales, using the theory of distributions (in functional analysis), is interesting for future work.

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