Spekkens' toy model and contextuality as a resource in quantum computation

L. Catani and D. E. Browne

University College London, Physics and Astronomy department
Gower St, London WC1E 6BT, UK

N. de Silva

University College London, Computer Science department
Gower St, London WC1E 6BT, UK

Summary. — Spekkens' toy model (SM) is a non-contextual hidden-variable model made to support the epistemic view of quantum theory, where quantum states are states of partial knowledge about a deeper underlying reality. Despite being a classical model, it has reproduced many features of quantum theory (entanglement, teleportation, ...): (almost) everything but contextuality, which therefore seems to be the inherent quantum feature. In addition to the importance in foundation of quantum theory, the notion of contextuality seems to be a crucial resource for quantum computation. In particular it has been proven that, in the case of odd prime discrete dimensional systems, contextuality is necessary for universal quantum computation in state-injection schemes of computation based on stabilizer quantum mechanics (SQM). The latter is a subtheory of quantum mechanics which is very popular in the field of quantum computation and quantum error correction. State-injection schemes consist of a classically-simulable part (like SQM) and a resource state that boosts the computation to a quantum improvement. In the odd-dimensional case, SM is operationally equivalent to SQM. In the even-dimensional case, the equivalence only holds in terms of structure, not in terms of statistical predictions. This because qubit-SQM shows contextuality, while qudit(odd dimensions)-SQM does not. We believe that SM can be a valid tool to study contextuality as a resource in the field of quantum computation. Restricted versions of SM compatible with quantum mechanics (QM) can be used as the non-contextual classically-simulable part of state-injection schemes thus opening other scenarios where studying if contextuality is necessary for quantum computational speed-up.

1. – Spekkens' toy model

In the last decades many attempts to better understand quantum theory through hidden-variable models have been developed [1-3]. Nowadays a big question in quantum foundations is whether to interpret the quantum state according to the ontic view, *i.e.* where it completely describes reality, or to the epistemic view, where it is a state of incomplete knowledge of a deeper reality which can be described by the hidden variables. In 2005 R. Spekkens [4] constructed a non-contextual hidden-variable model to support the epistemic view of quantum mechanics. The aim of the model was to replace quantum mechanics by a hidden-variable theory with the addition of an epistemic restriction (*i.e.* a restriction on what an observer can know about reality). The first version of the model [4] refers only to two-dimensional systems (inspired by the quantum bits) and, despite its simplicity, it has obtained many results that were thought to belong only to quantum mechanics (*e.g.* the no-cloning theorem and teleportation). A later version of the model [5], with a more rigorous mathematical formulation, has extended the theory to all discrete prime and continuous degrees of freedom. The latter has been shown to be operationally equivalent, except for the two-dimensional case, to sub-theories of quantum mechanics, so-called *quadrature quantum mechanics*, which in the discrete case correspond to SQM. Almost all the features of quantum mechanics are reproduced there, approximately everything but contextuality (and the related Bell non-locality), which therefore arises as the signature of quantumness. Spekkens' theory has influenced much research over the years (*e.g.*, [16-20]) and it also addresses many key issues in quantum foundations: whether the quantum state describes reality or not, finding a derivation of quantum theory from intuitive physical principles and classifying the inherent non-classical features.

In this section we describe Spekkens' theory for discrete dimensional systems by defining, in a physically motivated way, what are the states, the measurement observables and the outcomes. The updating rules for the state of a system after a measurement can be found in [6].

We denote with $\Omega \equiv (\mathbb{Z}_d)^{2n}$ the discrete phase space of n d-dimensional systems([1]). Let us consider a fiducial set of quadrature variables in the phase space (capital letters), X_j and P_j (like position and momentum in the standard classical mechanics), on each system j, where $j \in 0, \dots, n-1$, taking values in $0, \dots, d-1$. These variables allow us to define the *ontic states* (the reality) associated to the systems. Each ontic state represents a set of values for the fiducial observables X_j and P_j, and so an ontic state is denoted by a point in the phase space $\lambda \in \Omega$. We call X_j and P_j observables because they correspond to proper measurable quantities that uniquely define the ontic state. We can refer to Ω as a vector space where the ontic states are vectors (bold characters) whose components (small letters) are the values of the fiducial variables:

$$\boldsymbol{\lambda} = (x_0, p_0, x_1, p_1, \dots, x_{n-1}, p_{n-1}). \tag{1}$$

([1]) The dimension d is any positive number, odd or even, prime or non-prime, unless differently specified.

Spekkens' theory imposes a restriction on what an observer can know about the ontic state of a system. This means that what an observer can know about the system is described by an *epistemic state* which is a probability distribution $p(\lambda)$ over Ω.

A generic observable, denoted by O, is defined by any linear combination of fiducial variables:

$$O = \sum_m (a_m X_m + b_m P_m), \tag{2}$$

where $a_m, b_m \in \mathbb{Z}_d$ and $m \in 0, \ldots, n-1$. The observables live in the dual space Ω^*, which is isomorphic to Ω itself. Therefore we can define them as vectors, in analogy with ontic states, $\mathbf{O} = (a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1})$. The formalism provides a simple way of *evaluating* the outcome σ of any observable measurement O given the ontic state λ, *i.e.* by computing their *inner product*:

$$\sigma = O^T \lambda = \sum_j (a_j x_j + b_j p_j), \tag{3}$$

where all the arithmetic is over $\mathbb{Z}_d$.

The epistemic restriction of ST is called *classical complementarity principle* and it states that two observables can be simultaneously measured when their Poisson bracket is zero, and in this case we will say that they *commute*. This can be simply rephrased in terms of the *symplectic inner product*:

$$\langle \mathbf{O_1}, \mathbf{O_2} \rangle \equiv \mathbf{O_1}^T J \mathbf{O_2} = 0, \tag{4}$$

where $J = \bigoplus_{j=1}^n \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]_j$ is the symplectic matrix.

A subspace of commuting observables, thus satisfying the classical complementarity principle, is called *isotropic*. This means that the subspace of the variables jointly known by the observer is an isotropic subspace. We denote the subspace of known variables as $V = span\{O_1, \ldots, O_n\} \subseteq \Omega$, where O_i denotes one of the generators (commuting observables) of V. Taking into account this definition we can define an epistemic state by the set of known variables V, and also the values $\sigma_1, \ldots, \sigma_n$ that these variables take. This means that $\mathbf{O}_j^T \cdot \mathbf{w} = \sigma_j$, where $w \in V$ is the ontic state that evaluates the known observables and it is called the *shift vector*. The set of ontic states consistent with the epistemic state described by $(V, \mathbf{w})$ is $V^\perp + \mathbf{w}$, where the perpendicular complement of V is, by definition, $V^\perp = \{a \in \Omega \mid \mathbf{a}^T \mathbf{b} = 0 \ \forall \ b \in V\}$. The proof of this result can be found in [6]. By assumption the probability distribution associated to the epistemic state $(V, \mathbf{w})$ is uniform (indeed we expect all possible ontic states to be equiprobable), so the probability distribution of one of the possible ontic states in the epistemic state $(V, \mathbf{w})$ is

$$P_{(V,\mathbf{w})}(\boldsymbol{\lambda}) = \frac{1}{d^n} \delta_{V^\perp + \mathbf{w}}(\lambda), \tag{5}$$

where the delta is equal to one only if $\boldsymbol{\lambda} \in V^{\perp} + \mathbf{w}$ (note this means that the theory is a *possibilistic* theory).

The aim of Spekkens is to show that epistemic states in his theory are the analogue of quantum states in quantum theory (fig. 1). The analogue of unitary evolutions in quantum theory here corresponds to a subset of all the possible permutations (symplectic affine transformations). The elements of a sharp measurement are here represented as an epistemic state, by virtue of the duality between states and measurements. In the odd-dimensional case the operational equivalence between SM and SQM is proven by using Gross' Wigner functions [6, 15] Figures 1(a) and 1(b) picture the notions defined so far in the two-dimensional case. The notion of entanglement is depicted in fig. 1(c).

2. – Contextuality

In 1967 Kochen-Specker showed that no non-contextual hidden-variable models can ever reproduce all the results of quantum mechanics [3]. The theorem, that can be seen as a complement to Bell's theorem [2], highlights the new concept of contextuality. Probably the most intuitive and popular way to express this concept is through the so-called Mermin square [9]:

$X \otimes \mathbb{I}$	$\mathbb{I} \otimes X$	$X \otimes X$
$\mathbb{I} \otimes Z$	$Z \otimes \mathbb{I}$	$Z \otimes Z$
$X \otimes Z$	$Z \otimes X$	$Y \otimes Y$

The square is composed by nine Pauli observables on a two-qubit system. Each row and each column is composed by commuting (simultaneously measurable) observables. With the assumption that the functional relation between observables is preserved in terms of their outcomes (*e.g.* if an observable C is the product of two observables A, B, also its outcome c is the product of the outcomes a, b of A, B), and that the outcome of each observable does not depend on which other commuting observables are performed with it (non contextuality), the square shows that it is impossible to predict the outcome of each observable among all the rows and columns without falling into contradiction. For example, if we start by assigning values, say ± 1, to the observables starting from the first (top left) row on, the contradiction can be easily seen when we arrive at the last column and last row (red circles), that bring different results to the same observable $Y \otimes Y$, as witnessed by the following simple calculation, $(X \otimes Z) \cdot (Z \otimes X) = Y \otimes Y$, and $(X \otimes X) \cdot (Z \otimes X) = -(Y \otimes Y)$. Measurement contextuality refers to the fact that the outcome of a measurement does depend on the other compatible measurements that we perform with it (*i.e.* on the contexts). More recent versions of contextuality do not only consider sharp measurements, but also preparations and transformations [7]. Non-contextual inequalities have been developed to quantify contextuality [11-13] and in 2014 Howard *et al.* [14] used Cabello-Severini-Winter inequality to prove that contextuality is

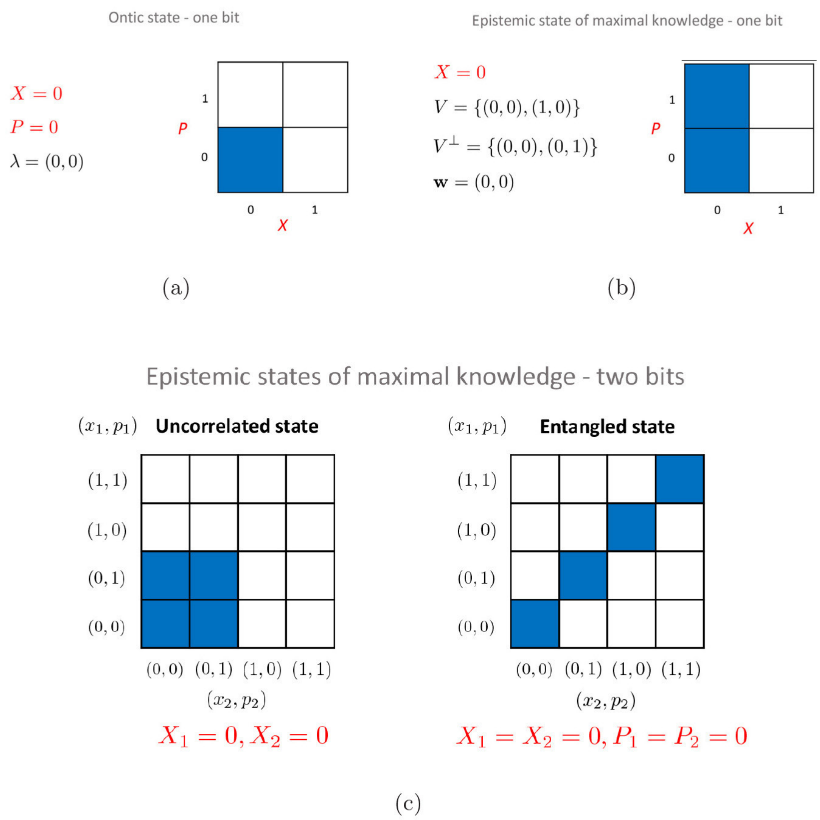

Fig. 1. – Spekkens' toy states of one and two bits. Panels (a) and (b) show the elementary system of Spekkens' theory in two dimension: the bit. One possible ontic state of one bit is shown in (a), where the observer both knows $X = 0$ and $P = 0$, so $\boldsymbol{\lambda} = (0, 0)$. The epistemic restriction — classical complementarity principle — in this case corresponds to saying that at maximum the observer has "half" of the knowledge about the ontic state. For example a possible epistemic state is shown in (b), where the observer only knows the variable $X = 0$, so $V = span\{(1, 0)\}$ and $\mathbf{w} = (0, 0)$. In this case the epistemic state $X = 0$ of one bit can be seen as the analogue of the quantum state $|0\rangle$ of one qubit. Panel (c) shows two kinds of two-bits epistemic states of maximal knowledge. The state on the left is a non-correlated state ($X_1 = 0 = X_2$), indeed we have the knowledge of the states of the individual subsystems, while the state on the right ($X_1 = X_2$ *and* $P_1 = P_2$) is perfectly correlated (*i.e.* entangled), indeed it would be impossible to know the states of the individual subsysytems, but we know exactly the correlation between them (in the case above we know that they have the same ontic states). This trade-off in choosing if knowing the correlation or the states of the individual subsystems is something which is not present in any classical theory.

necessary for universal quantum computation in a state-injection scheme of computation based on stabilizer quantum mechanics. The latter is a subtheory of quantum mechanics where only common eigenstates of tensors of Pauli operators are considered and only unitaries belonging to the Clifford group and Pauli measurements are allowed. The previous state-injection scheme consists of two parts (fig. 2): the "classical" part of the

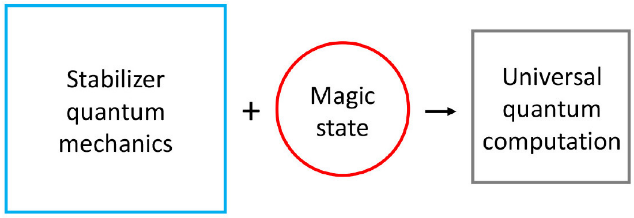

Fig. 2. – State-injection scheme of computation. Quantum computation by state injection consists of a non-contextual and efficiently classically simulable part (like a stabilizer circuit) and a "magic" resource state (given by a non-Clifford gate) that boosts it to universal quantum computation. Howard *et al.* [14] have found that if the resource state can be distilled into a magic one then it must show contextuality. We want to study a similar state-injection scheme for qubits based on other sub-theories of SM compatible with QM instead of SQM.

computation which is composed by stabilizer circuits, which are known to be efficiently classically simulable [8], and the resource state. The classical part of the computation is boosted to quantum universality by injecting a particular resource state, called *magic* state. The result states that if the resource state can be distilled into a magic one then it shows a violation of CSW-inequality, *i.e.* contextuality. The result holds only for qudits (odd dimensions). Note that the contextuality considered in the above scenario is *state-dependent* (indeed it is injected by the magic state), while the one presented in the Mermin square scenario is state-independent. We conclude this section on contextuality by highlighting which are the philosophical assumption behind the notion of contextuality: counterfactual realism and counterfactual compatibility. The former says that results of unperformed tests have the same degree of reality of the results of performed tests. In this sense contextuality consists of a logical contradiction between an *actual* outcome that happens and a *potential* outcome that does not. Counterfactual compatibility roughly says that the common observable (*e.g.* $Y \otimes Y$) considered in the two incompatible contexts (*e.g.* last row and last column of the Mermin square) is exactly the same in the two scenarios. In other words the result of an unperformed test does not depend on the choice of compatible observables that can be performed with it [10].

3. – Restricting SM as a subtheory of QM

Spekkens' model is a hidden-variable model that is not operationally equivalent to quantum theory, indeed it does not show contextuality. Nevertheless we could consider sub-theories of SM that are compatible (thus showing the same statistics of measurements) with QM. A simple example of a sub-theory of SM which is compatible with QM is the following. Let us consider one qubit system where the allowed measurements([2]) are the Pauli X, Z and the allowed gates are H, X, Y, Z, where H is the Hadamard gate

([2]) Note that setting the allowed measurements also sets the allowed states. If the allowed measurements are the Pauli X, Z, the allowed states are the eigenstates of $\pm X, Z$.

and X, Y, Z are the Pauli unitary transformations. All the measurements/states have a faithful representation in SM and the unitaries are symplectic affine transformations. However if we add the Pauli Y measurement and the S gate, where $S = \left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]$ in the computational basis, we obtain a theory which is not faithfully represented in SM. The gate S is not symplectic and it would imply a different action in SM and QM. Our aim is to understand the mathematical prerequisites for sub-theories of SM to coincide with QM (even in the case of many qubits with entangling gates) and develop frameworks for that. An interesting idea is to treat these restricted versions of SM compatible with QM as the non-contextual classically simulable part of state-injection schemes. Inspired by Howard's result [14], we can then analyse the role of contextuality in qubit quantum computation by injecting resources that boost valid Spekkens' sub-theories of QM to universal quantum computation. We think that the above construction [21] is just one possible application of Spekkens' toy model in quantum computation. Its correspondence with SQM could suggest other possible applications, like the ones related to SQM in non-prime dimensions [6].

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REFERENCES

[1] Einstein A., Podolsky B. and Rosen N., *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?*, *Phys. Rev.*, **47** (1935) 777.
[2] Bell J. S., *On the problem of hidden variables in quantum mechanics*, *Rev. Mod. Phys.*, **38** (1966) 447.
[3] Kochen S. and Specker E. P., *The problem of hidden variables in quantum mechanics*, *J. Math. Mech.*, **17** (1967) 59.
[4] Spekkens R. W., *Evidence for the epistemic view of quantum states: A toy theory*, *Phys. Rev. A*, **75** (2007) 032110.
[5] Spekkens R. W., *Quasi-quantization: classical statistical theories with an epistemic restriction*, arXiv:1409.5041 (2014).
[6] Catani L. and Browne D. E., *Spekkens' toy model in all dimensions and its relationship with stabiliser quantum mechanics*, *New J. Phys.*, **19** (2017) 073035.
[7] Spekkens R. W., *Contextuality for preparations, transformations, and unsharp measurements*, *Phys. Rev. A*, **71** (2005) 052108.
[8] Gottesman D., *The Heisenberg representation of quantum computers*, arXiv:quant-ph/9807006 (1998).
[9] Mermin N. D., *Simple unified form for the major no-hidden-variables theorems*, *Phys. Rev. Lett.*, **65** (1990) 3373.
[10] Peres A., *Quantum theory: concepts and methods* (Kluwer Academic Publishers) 1995.
[11] Cabello A., Severini S. and Winter A., *Graph-Theoretic Approach to Quantum Correlations*, *Phys. Rev. Lett.*, **112** (2014) 040401.
[12] Abramsky S. and Brandenburger A., *The sheaf-theoretic structure of non-locality and contextuality*, *New. J. Phys.*, **13** (2011) 113036.
[13] de Silva N., *Graph-theoretic strengths of contextuality*, *Phys. Rev. A*, **95** (2017) 032108.

[14] Howard M., Wallman J., Veitch V. and Emerson J., *Contextuality supplies the magic for quantum computation*, *Nature*, **510** (2014) 351.

[15] Gross D., *Hudson's theorem for finite-dimensional quantum systems*, *J. Math. Phys.*, **47** (2006) 122107.

[16] Pusey M. F., *Stabilizer notation for Spekkens' toy theory*, *Found. Phys.*, **42** (2012) 688.

[17] Coecke B., Edwards B. and Spekkens R. W., *Phase Groups and the Origin of Non-locality for Qubits*, *Electron. Notes Theor. Comput.*, **270** (2011) 29.

[18] Disilvestro L. and Markham D., *Quantum Protocols within Spekkens' Toy Model*, *Phys. Rev. A*, **95** (2017) 052324.

[19] Blasiak P., *Quantum cube: A toy model of a qubit*, *Phys. Lett. A*, **377** (2013) 847.

[20] Larsson J., *A contextual extension of Spekkens' toy model*, *AIP Conf. Proc.*, **1424** (2012) 211.

[21] Catani L., de Silva N. and Browne D. E., *On the computational power of Spekkens' toy model and contextuality as a resource*, work in progress.