ON THE SPECTRUM OF AN "EVEN" PERIODIC SCHRÖDINGER OPERATOR WITH A RATIONAL MAGNETIC FLUX

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Abstract. We study the Schrödinger operator on $L_2(\mathbb{R}^3)$ with periodic variable metric, and periodic electric and magnetic fields. It is assumed that the operator is reflection symmetric and the (appropriately defined) flux of the magnetic field is rational. Under these assumptions it is shown that the spectrum of the operator is absolutely continuous. Previously known results on absolute continuity for periodic operators were obtained for the zero magnetic flux.

1. Introduction and results

In the last two decades a good deal of attention was focused on the absolute continuity of self-adjoint periodic differential elliptic operators of second order in dimension $d \geqslant 2$, i.e. of the operators of the form

$$H = \sum_{j,l=1}^{d} (D_j - A_j) g_{jl} (D_l - A_l) + V, \quad D_j = -i\partial_j, \tag{1.1}$$

with a periodic symmetric positive-definite matrix $\{g_{jl}\} = \mathbf{G}$, and coefficients $\mathbf{A} = \{A_l\}$, V which we interpret as magnetic and electric potentials respectively. If all the coefficients in (1.1) are periodic and satisfy suitable integrability and/or smoothness conditions, then the operator H is known to be absolutely continuous for $d = 2$. If $\mathbf{G}(\mathbf{x}) = g(\mathbf{x})\mathbf{I}$ with a positive function g then this conclusion extends to arbitrary $d \geqslant 2$. We do not provide a thorough bibliographical account and refer e.g. to [3], [11] and [14] for more detailed references.

The case of general variable $\mathbf{G}$ in dimensions $d \geqslant 3$ remains unassailable, but there are some partial results. First, if the matrix $\mathbf{G}$ is

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not smooth then the spectrum of H may not be absolutely continuous, see [4]. Second, in L. Friedlander's paper [7] the absolute continuity was obtained for smooth variable matrix $\mathbf{G}$ and smooth $\mathbf{A}$, V for all dimensions $d \geqslant 2$ under the condition that the operator H is reflection symmetric. Later the smoothness assumptions were relaxed by N. Filonov, M. Tikhomirov in [6].

In this note we address another open question of the theory: is the spectrum absolutely continuous if instead of the magnetic potential $\mathbf{A}$ we assume that the magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{A}$ is periodic? The traditional methods used to study the spectra of periodic operators are not directly applicable. However, under the additional condition of the reflection symmetry one can still use the ideas of [7] and [6]. We concentrate on the physically relevant case $d = 3$. Note that the case $d = 2$ is also of interest but the requirement of the reflection symmetry automatically implies that the constant component of the magnetic field is zero, i.e. the magnetic potential itself becomes periodic. Thus for $d = 2$ the Friedlander's method would give no new information. At this point we should note that in general (i.e. without reflection symmetry), the two-dimensional case is dramatically different from the three-dimensional one. It suffices to observe that in the absence of electric field for $d = 3$ a constant magnetic field induces absolutely continuous spectrum, whereas for $d = 2$ the spectrum consists of equidistant eigenvalues of infinite multiplicity, called Landau levels, see [12]. Thus for $d = 2$ mechanisms responsible for the possible formation of the absolute continuous spectrum (e.g. with non-trivial periodic V) are very different. For this case absolute continuity was proved in [9] for constant magnetic field with a rational flux and a generic periodic potential V.

Let us proceed to the precise formulations. The operator H is defined via the quadratic form

$$h[u] = \int\limits_{\mathbb{R}^3} \langle \mathbf{G}(\mathbf{x}) \ (-i\nabla - \mathbf{A}(\mathbf{x}))\, u(\mathbf{x}), (-i\nabla - \mathbf{A}(\mathbf{x}))\, u(\mathbf{x})\rangle d\mathbf{x}$$

$$+ \int\limits_{\mathbb{R}^3} V(\mathbf{x}) |u(\mathbf{x})|^2 d\mathbf{x}, \tag{1.2}$$

with the domain $D[h] = C_0^\infty(\mathbb{R}^3)$ in the Hilbert space $L_2(\mathbb{R}^3)$. The coefficient $\mathbf{G} = \{g_{jl}(\mathbf{x})\}, j, l = 1, 2, 3,$ is a symmetric matrix-valued function with real-valued entries $g_{jl}(\mathbf{x})$ which satisfies the conditions

$$c|\boldsymbol{\xi}|^2 \leqslant \langle \mathbf{G}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \leqslant C|\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3, \text{ a.e. } \mathbf{x} \in \mathbb{R}^3, \tag{1.3}$$

$$\mathbf{G} \in \mathrm{Lip}(\mathbb{R}^3). \tag{1.4}$$

Here and everywhere below by C and c with or without indices we denote various positive constants whose precise value is unimportant. The vector-field $\mathbf{A}$ and the function V satisfy the conditions

$$\mathbf{A} \in L_{p,\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3), \qquad V \in L_{3/2,\mathrm{loc}}(\mathbb{R}^3), \tag{1.5}$$

with $p = 3$. Under the assumptions (1.3), (1.5) with $p = 3$, and that V is periodic, the form (1.2) is semibounded from below and closable (see e.g. [13, §2]). We denote by H the self-adjoint operator in $L_2(\mathbb{R}^3)$ which corresponds to the closure of the form h. We write it formally as

$$H = \langle (-i\nabla - \mathbf{A}), \mathbf{G}(-i\nabla - \mathbf{A})\rangle + V. \tag{1.6}$$

Since we assume that the magnetic field $\mathbf{B}(\mathbf{x}) = \mathrm{curl}\, \mathbf{A}(\mathbf{x})$ is periodic, the magnetic potential can be represented in the form

$$\mathbf{A}(\mathbf{x}) = \mathbf{a}_0(\mathbf{x}) + \mathbf{a}(\mathbf{x}),$$

where $\mathbf{a}_0$ is a linear magnetic potential associated with the constant component $\mathbf{B}_0 = \mathrm{curl}\, \mathbf{a}_0(x)$ of the magnetic fields, and $\mathbf{a}$ is a periodic vector-potential. We align $\mathbf{B}_0$ with the positive direction of the x_3-axis, and choose for $\mathbf{a}_0(\mathbf{x})$ the gauge $(-bx_2, 0, 0), b = |\mathbf{B}_0| \geqslant 0$, so that $\mathbf{B}_0 = (0, 0, b)$ and

$$\mathbf{A}(\mathbf{x}) = (-bx_2, 0, 0) + \mathbf{a}(\mathbf{x}). \tag{1.7}$$

We assume that with this choice of coordinates the matrix-valued function $\mathbf{G}$, the potentials V and $\mathbf{a}$ are $(2\pi\mathbb{Z})^3$-periodic:

$$\mathbf{G}(\mathbf{x} + 2\pi\mathbf{n}) = \mathbf{G}(\mathbf{x}), \ V(\mathbf{x} + 2\pi\mathbf{n}) = V(\mathbf{x}),$$

$$\mathbf{a}(\mathbf{x} + 2\pi\mathbf{n}) = \mathbf{a}(\mathbf{x}), \quad \forall \mathbf{n} \in \mathbb{Z}^3. \tag{1.8}$$

Furthermore, to ensure that the operator (1.6) is partially diagonalizable via the Floquet-Bloch-Gelfand decomposition, we assume that the flux of the constant component $\mathbf{B}_0$ is integer, i.e.

$$\frac{1}{2\pi} \int\limits_{(-\pi,\pi)^2} |\mathbf{B}_0| dx_1 dx_2 = 2\pi b \in \mathbb{Z}_+ = \{0, 1, \dots\}. \tag{1.9}$$

To describe the symmetry of the operator H introduce the reflection map $\mathbf{R}:\mathbb{R}^3\to\mathbb{R}^3$:

$$\mathbf{R}(x_1,x_2,x_3)=(x_1,x_2,-x_3),$$

and the associated operation on $u\in L_2(\mathbb{R}^3)$:

$$(Ju)(\mathbf{x})=u(\mathbf{R}\mathbf{x}). \tag{1.10}$$

It is straightforward to check that H commutes with J if $\mathbf{G}$, $\mathbf{a}$ and V satisfy the conditions

$$\begin{gathered}\mathbf{G}(\mathbf{R}\mathbf{x})=\mathbf{R}\mathbf{G}(\mathbf{x})\mathbf{R},\ \ \mathbf{A}(\mathbf{R}\mathbf{x})=\mathbf{R}\mathbf{A}(\mathbf{x}),\\ V(\mathbf{R}\mathbf{x})=V(\mathbf{x}),\qquad \text{a.e. } \ \mathbf{x}\in\mathbb{R}^3.\end{gathered} \tag{1.11}$$

Obviously the symmetry condition for $\mathbf{A}$ is equivalent to that for $\mathbf{a}$.

The next theorem constitutes the main result of the paper.

Theorem 1.1. *Assume that the matrix $\mathbf{G}$, the potentials $\mathbf{A}, V$ satisfy the conditions* (1.3), (1.4) *and* (1.5) *with $p>3$. Assume also that* (1.7), (1.8), (1.9) *and* (1.11) *are satisfied. Then the spectrum of the operator* (1.6) *is absolutely continuous.*

Throughout the paper we always assume the periodicity (1.8). As a special case this allows a constant magnetic field i.e. $\mathbf{a}=0$. With regard to the regularity, we normally need only (1.3) and (1.5) with $p=3$. The assumptions (1.4) and $p>3$ are required only once when employing the unique continuation argument, see Lemma 4.4. Recall that if (1.4) is not satisfied, the spectrum may not be absolutely continuous, see [4].

Note that the condition (1.9) can be replaced by $2\pi b\in\mathbb{Q}$. This case reduces to that of an integer flux by taking an appropriate sublattice of $\mathbb{Z}^3$ and rescaling. If the flux is irrational we cannot say anything about the nature of the spectrum.

As mentioned earlier, one can state a theorem similar to Theorem 1.1 in the two-dimensional case as well. However, in this case the reflection symmetry would imply that $b=0$, see (1.7), and hence such a theorem would not say anything new compared to the known results.

Theorem 1.1 can be conceivably generalized to arbitrary dimensions $d\geqslant 3$ with the standard changes to the conditions (1.5). We have chosen not to clutter the presentation with these details but to focus on the lowest dimension where the reflection symmetry leads to a non-trivial effect.

The proof of absolute continuity amounts to showing that the operators $H(\mathbf{k})$ in the Bloch decomposition of H have no eigenvalues which are constant as functions of the quasi-momentum $\mathbf{k}$. The operator-function $H(\mathbf{k})$ is a quadratic pencil, and in general the study of its spectrum is a non-trivial problem. As mentioned earlier, L. Friedlander (see [7]) was the first to understand how the reflection symmetry can be used to establish the absolute continuity of H. Friedlander's approach is based on a reduction to a pair of Dirichlet-to-Neumann maps for the operator H on the fundamental domain of the lattice $\mathbb{Z}^3$, see Section 4 for a short discussion. The crucial observation is that in the presence of reflection symmetry the quadratic operator pencil $H(\mathbf{k})$ reduces to a linear one involving the mentioned Dirichlet-to-Neumann operators. This dramatically simplifies the proof of the fact that the eigenvalues of $H(\mathbf{k})$ are not constant in $\mathbf{k}$.

In the paper [6] the Friedlander's argument was translated into the language of quadratic forms which allowed the authors to prove the absolute continuity with "minimal" regularity assumptions. Although in the current paper we follow the paper [6] our proof is simpler and somewhat shorter, and we consider it worthy of dissemination.

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2. Floquet-Bloch-Gelfand transformation

Denote by Ω the interior of the standard fundamental domain of the lattice $\Gamma = \mathbb{Z}^3$: $\Omega = (-\pi, \pi)^3$. We also need separate notation for the top and bottom faces of this cube:

$$\Lambda_\pm = \{\mathbf{x} \in \mathbb{R}^3 : \hat{\mathbf{x}} \in (-\pi,\pi)^2,\ x_3 = \pm\pi\},\ \hat{\mathbf{x}} = (x_1, x_2).$$

The interior of the fundamental domain of the dual lattice is denoted $\Omega^\dagger = (0,1)^3$.

The Floquet-Bloch-Gelfand transform is defined as the operator

$$(Uf)(\mathbf{x},\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} e^{-i\mathbf{k}\cdot(\mathbf{x}+2\pi\mathbf{n})} e^{i2\pi b n_2 x_1} f(\mathbf{x}+2\pi\mathbf{n}),\ \mathbf{x}\in\Omega, \mathbf{k}\in\Omega^\dagger,$$

for functions $f \in C_0^\infty(\mathbb{R}^3)$. It is clear that $Uf \in C^\infty(\overline{\Omega}\times\overline{\Omega^\dagger})$. Moreover, the function $v(\ \cdot\ ) = Uf(\ \cdot\ ; \mathbf{k})$ is periodic in x_1 (due to the condition (1.9)), and in x_3:

$$v(-\pi, x_2, x_3) = v(\pi, x_2, x_3), \tag{2.1}$$

$$v(x_1, x_2, -\pi) = v(x_1, x_2, \pi). \tag{2.2}$$

It is quasiperiodic in x_2:

$$v(x_1, \pi, x_3) = e^{-i2\pi b x_1} v(x_1, -\pi, x_3). \tag{2.3}$$

A direct calculation shows that the transform U can be extended to $L_2(\mathbb{R}^3)$ as a unitary operator

$$U : L_2(\mathbb{R}^3) \to \int_{\Omega^\dagger}^{\oplus} L_2(\Omega) d\mathbf{k}.$$

For each $\mathbf{z} \in \mathbb{C}^3$ introduce the quadratic form

$$\begin{aligned} h(\mathbf{z})[v] = & \int_\Omega \big\langle \mathbf{G}(\mathbf{x})(-i\nabla + |\mathbf{z} - \mathbf{A}(x))v(\mathbf{x}), (-i\nabla + \overline{\mathbf{z}} - \mathbf{A}(\mathbf{x}))v(\mathbf{x})\big\rangle d\mathbf{x} \\ & + \int_\Omega V(\mathbf{x})|v(\mathbf{x})|^2 d\mathbf{x}. \end{aligned} \tag{2.4}$$

Under the conditions (1.3), (1.5) with $p = 3$ the potentials $\mathbf{A}$ and V induce on $C^\infty(\overline{\Omega})$ a perturbation which is infinitesimally bounded by the standard Dirichlet form, and hence

$$C_0^{-1}\|v\|^2_{H^1(\Omega)} \leqslant |h(\mathbf{z})[v]| + C\|v\|^2_{L_2(\Omega)} \leqslant C_0\|v\|^2_{H^1(\Omega)}, \tag{2.5}$$

with some positive constants $C = C(\mathbf{z})$ and $C_0 = C_0(\mathbf{z}) > 1$ uniformly in $\mathbf{z}$ on a compact subset of $\mathbb{C}^3$. Thus (2.4) naturally extends to all $v \in H^1(\Omega)$ as a closed form. In order to relate this form to the form (1.2) we consider (2.4) on a smaller domain. It is convenient to introduce a special notation for the function spaces with the conditions (2.1) and (2.3):

$$W^1 = \{u \in H^1(\Omega) : \ u \text{ satisfies (2.1) and (2.3)}\}. \tag{2.6}$$

Now we consider the form (2.4) on the domain

$$D[h(\mathbf{z})] = D[h(\mathbf{0})] = \{v \in W^1 : v \text{ satisfies (2.2)}\}.$$

Clearly the form (2.4) is closed on $D[h(\mathbf{0})]$ and analytic (quadratic) in $\mathbf{z} \in \mathbb{C}^3$. One checks directly that

$$h[v] = \int_{\Omega^\dagger} h(\mathbf{k})[(Uv)(\ \cdot\ , \mathbf{k})] d\mathbf{k}, \tag{2.7}$$

for any $v \in D[h(\mathbf{0})]$. The form $h(\mathbf{z})[v]$ is sectorial, i.e. for a suitable number $\gamma = \gamma(\mathbf{z}) \in \mathbb{R}$,

$$\operatorname{Re} h(\mathbf{z})[v] \geqslant -\gamma\|v\|^2,\ |\operatorname{Im} h(\mathbf{z})[v]| \leqslant C\big(\operatorname{Re} h(\mathbf{z})[v]+\gamma\|v\|^2\big),\ v \in D[h(\mathbf{0})],$$

with some positive constant $C = C(\mathbf{z})$ uniformly in $\mathbf{z}$ on a compact subset of $\mathbb{C}^3$. Hence it defines a sectorial operator (m-sectorial in T. Kato's terminology, see [8]) which we denote by $H(\mathbf{z})$. As the form $h(\mathbf{z})$ is compact in $H^1(\Omega)$ the resolvent of $H(\mathbf{z})$ is compact whenever it exists. For the values $\mathbf{k} \in \mathbb{R}^3$ the operator $H(\mathbf{k})$ is self-adjoint: $H(\mathbf{k}) = H(\mathbf{k})^*$. In view of (2.7) the following unitary equivalence

$$UHU^* = \int_{\Omega^\dagger}^{\oplus} H(\mathbf{k})\, d\mathbf{k} \tag{2.8}$$

holds. Although this formula requires only the values $\mathbf{k} \in \Omega^\dagger$ it is important for us to have the operator $H(\mathbf{z})$ defined for $\mathbf{z} \in \mathbb{C}^3$. Sometimes we use the notation $\mathbf{z} = (\hat{\mathbf{z}}, z_3)$, with $\hat{\mathbf{z}} = (z_1, z_2)$.

In order to prove Theorem 1.1 it is sufficient to show that H has no eigenvalues, see [5]. The proof of this fact reduces to the analysis of the following boundary value problem for a function $u \in W^1$ with the form $h_0 = h(\mathbf{0})$ and a number $\zeta \in \mathbb{C}$:

$$u \in W^1, \quad u|_{\Lambda_+} = \zeta\, u|_{\Lambda_-}\,, \tag{2.9}$$

$$h_0[u, w] = 0, \qquad \forall w \in W^1, \ \text{ s.t. } \ \overline{\zeta}\, w|_{\Lambda_+} = w|_{\Lambda_-}\,. \tag{2.10}$$

Theorem 1.1 is derived from the next theorem:

Theorem 2.1. *Suppose that* $\mathbf{G}$, $\mathbf{A}$ *and* V *satisfy the conditions of Theorem 1.1. Let* $X \subset \mathbb{C}$ *be the subset of the complex plane consisting of the points* ζ *such that*

(1) $\operatorname{Im}\zeta \neq 0$, $|\zeta| \neq 1$,
(2) *there exists at least one function* $u \in W^1, u \not\equiv 0$ *satisfying* (2.9) *and* (2.10).

Then X *is at most finite.*

Derivation of Theorem 1.1 from Theorem 2.1. By virtue of [5] it suffices to show that any arbitrarily chosen number $\lambda \in \mathbb{R}$ is not an eigenvalue of H. Replacing the potential V with $V - \lambda$ we may assume that $\lambda = 0$.

Suppose on the contrary that $\lambda = 0$ is an eigenvalue of H, so that there exists at least one $\hat{\mathbf{k}} \in [0, 1]^2$ such that the measure of the set

$$\{k \in (0, 1) : \lambda \in \sigma(H(\hat{\mathbf{k}}, k))\}$$

is positive. Since $H(\mathbf{z})$ has compact resolvent, by the analytic Fredholm alternative (see [8], Theorems VII.1.10, VII.1.9), this implies that the point $\lambda = 0$ is an eigenvalue of $H(\hat{\mathbf{k}}, k)$ for all $k \in \mathbb{C}$. Replacing the magnetic potential $\mathbf{A}$ with $\mathbf{A} - (\hat{\mathbf{k}}, 0)$ we may assume that $\hat{\mathbf{k}} = \mathbf{0}$, so that for any $k \in \mathbb{C}$ there exists a function $v \in D[h_0]$, $v \not\equiv 0$, such that

$$h(\hat{\mathbf{0}}, k)[v, \eta] = 0, \quad \forall \eta \in D[h_0].$$

Introducing

$$u(\mathbf{x}) = e^{ikx_3} v(\mathbf{x}), \quad w(\mathbf{x}) = e^{i\bar{k}x_3} \eta(\mathbf{x}),$$

we reduce the above equation to equation (2.10) for the function $u \not\equiv 0$ satisfying (2.9) with $\zeta = e^{i2\pi k}, k \in \mathbb{C}$. Thus the set X defined in Theorem 2.1 has positive (in fact, infinite) measure. This contradicts the conclusion of Theorem 2.1, and hence $\lambda = 0$ cannot be an eigenvalue of H. As explained at the beginning of the proof, this implies that the spectrum of H is absolutely continuous, as claimed. ∎

3. Associated boundary-value problem

We begin the analysis of the system (2.9), (2.10) with introducing the subspaces

$$W^{1,0} = \{v \in W^1 : v|_{\Lambda_+} = v|_{\Lambda_-} = 0\},$$

$$W^1_+ = \{u \in W^1 : u|_{\Lambda_+} = 0\},$$

with the standard H^1-inner product. Now define the subspaces

$$N = \left\{v \in W^{1,0} : h_0[v, w] = 0, \ \forall w \in W^{1,0}\right\},$$

$$M = \{u \in W^1_+ : h_0[u, v] = 0, \ \forall v \in N\},$$

and

$$Z = \{u \in W^1_+ : h_0[u, w] = 0, \quad \forall w \in W^{1,0}, \quad u \perp N\}.$$

The subspace Z consists of weak solutions $u \in W^1_+$ of the equation $Hu = 0$ which are orthogonal to N. By definition of M we automatically have $Z, W^{1,0} \subset M$.

First of all consider the following boundary-value problem.

Lemma 3.1. *Let the conditions* (1.3) *and* (1.5) *with* $p = 3$ *be satisfied. Then for any function* $u \in M$ *the system*

$$\begin{cases} h_0[\phi, w] = 0, \quad \forall w \in W^{1,0}, \\ \phi - u \in W^{1,0}, \end{cases} \tag{3.1}$$

is solvable for the function $\phi \in W^1_+$. *The solution is unique under the condition* $\phi \perp N$. *Moreover,* $\dim N < \infty$.

Proof. The system is studied in the standard way. Namely, the function $\psi = \phi - u \in W^{1,0}$ satisfies

$$h_0[\psi, w] = -h_0[u, w], \quad \forall w \in W^{1,0}. \tag{3.2}$$

Referring to (2.5) introduce on $W^{1,0}$ the inner product

$$(f, g)_1 = h_0[f, g] + \gamma(f, g)$$

choosing $\gamma \geqslant 0$ in such a way that the induced norm $\|f\|_1$ is equivalent to the standard H^1-norm. The L_2-inner product is an example of a symmetric compact form in H^1, and hence there is a compact self-adjoint operator $T : W^{1,0} \to W^{1,0}$ such that $(f, g) = (Tf, g)_1, f, g \in W^{1,0}$. As a result, the left-hand side of (3.2) rewrites as $((I-\gamma T)\psi, w)_1$. The right-hand side of (3.2) is a continuous linear functional of $w \in W^{1,0}$ so there is a function $q \in W^{1,0}$ such that $-h_0[u, w] = (q, w)_1$, $\|q\|_1 \leqslant C\|u\|_1$. Thus (3.2) takes the form

$$\psi - \gamma T\psi = q. \tag{3.3}$$

Now it follows from the classical Fredholm Theory that (3.3) has a solution $\psi \in W^{1,0}$ if and only if $(q, v)_1 = 0$ for all $v \in \ker(I - \gamma T)$. Under this condition there is a unique solution ψ_0 satisfying the property $(\psi_0, v)_1 = 0$ for all $v \in \ker(I - \gamma T)$, and this solution satisfies the bound $\|\psi_0\|_1 \leqslant C\|q\|_1$. Note that $N = \ker(I - \gamma T)$, so by definition of q the equality $(q, v)_1 = 0$, $\forall v \in \ker(I - \gamma T)$ follows from the condition $u \in M$. Thus (3.2) is solvable and hence so is (3.1). As T is compact, it immediately follows that $\dim N < \infty$.

Denote $\phi_0 = \psi_0 + u \in W^1_+$. Any other solution of (3.1) has the form $\phi = \phi_0 + w$ with a suitable $w \in N$. If one demands that $\phi \perp N$ then $w = -\mathcal{P}\phi_0$ where $\mathcal{P}$ is the projection in $L_2(\Omega)$ on the finite-dimensional subspace N. Therefore such a solution $\phi \in W^1_+$ is uniquely defined, as required. ∎

The following elementary lemma is crucial for us.

Lemma 3.2. *Let the conditions* (1.3), (1.5) *with* $p = 3$ *be satisfied. Let the subspaces* M, Z *be as defined above. Then the subspace* Z *is non-trivial, and*

$$M = Z \dot{+} W^{1,0}.$$

In other words, any function $u \in M$ *is uniquely represented as the sum* $\phi + w$ *with some* $\phi \in Z$ *and* $w \in W^{1,0}$.

Proof. By Lemma 3.1 for any function $u \in M$ there is a solution ϕ of (3.1) orthogonal to N. Furthermore, ϕ is uniquely defined and $w = \phi - u \in W^{1,0}$, so $M = Z \dotplus W^{1,0}$, as claimed. Recall that codim $M < \infty$ in W^1_+ whereas codim $W^{1,0} = \infty$, so $M \neq W^{1,0}$. This implies that Z is non-trivial. ∎

4. The Dirichlet-to-Neumann forms

4.1. **General facts.** On the subspace Z considered as a Hilbert space with the H^1-inner product introduce the forms

$$t_0[u,v] = h_0[u,v], \quad t_1[u,v] = h_0[u,Jv], \; u,v \in Z,$$

where J is defined in (1.10). We call t_0 and t_1 the Dirichlet-to-Neumann forms. Their properties are listed in the following lemma.

Lemma 4.1. *Let the conditions* (1.3), (1.5) *with* $p = 3$ *be satisfied. Let* t_0, t_1 *be as defined above. Then*

(1) *Both forms* t_0 *and* t_1 *are bounded on* Z*:*

$$|t_0[\phi,\psi]| + |t_1[\phi,\psi]| \leqslant C\|\phi\|_{H^1} \, \|\psi\|_{H^1}. \tag{4.1}$$

(2) *The form* t_0 *is Hermitian. If the condition* (1.11) *is satisfied then* t_1 *is also Hermitian.*

(3) *Let* $\mathcal{L} \subset Z$ *be a linear set such that* $t_0[\phi] \leqslant 0$ *for all* $\phi \in \mathcal{L}$. *Then*

$$\sup_{\mathcal{L}} \dim \mathcal{L} < \infty. \tag{4.2}$$

Proof. The bound (4.1) immediately follows from (2.5).

The form t_0 is clearly Hermitian. If (1.11) is satisfied, then

$$t_1[u,v] = h_0[u,Jv] = h_0[Ju,v] = \overline{t_1[v,u]}, \quad \forall u,v \in Z,$$

i.e. t_1 is Hermitian.

Consider the form h_0 on $H^1(\Omega)$, and recall that by (2.5) with $\mathbf{z} = \mathbf{0}$ it is closed and semibounded from below. Moreover, $H^1(\Omega)$ embeds into $L_2(\Omega)$ compactly, and hence the associated self-adjoint operator has discrete spectrum accumulating at $+\infty$. The number of eigenvalues $n(\lambda)$ which are less than or equal to an arbitrary number $\lambda \in \mathbb{R}$ can be found in terms of the form h_0 in the standard way. Precisely, let $\mathcal{L}_\lambda \subset H^1(\Omega)$ be a linear set such that $h_0[u] \leqslant \lambda\|u\|^2$ for all $u \in \mathcal{L}_\lambda$. Then

$$n(\lambda) = \max_{\mathcal{L}_\lambda} \dim \mathcal{L}_\lambda < \infty,$$

see [1], Ch. 10, Theorem 2.3. The form t_0 is the restriction of h_0 to the subspace Z, and hence (4.2) is a direct consequence of the above bound with $\lambda = 0$. ∎

Instead of the solution space Z we could have considered the spaces of traces on the faces Λ_-, Λ_+. Then the forms t_0 and t_1 would correspond to two Dirichlet-to-Neumann operators T_0 and T_1 which map the trace $\phi|_{\Lambda_-}$, $\phi \in Z$, into the normal derivative of ϕ on the faces Λ_- and Λ_+ respectively. This approach was adopted in the paper [7]. We do not make explicit use of the Dirichlet-to-Neumann operators but it seems appropriate to use this terminology for the forms t_0, t_1.

4.2. **Reflection symmetry.** From now we assume that $\mathbf{G}$, $\mathbf{A}, V$ satisfy the symmetry condition (1.11). Thus using the operator J defined in (1.10) we get

$$h_0[Ju, Jv] = h_0[u, v], \quad \forall u, v \in H^1(\Omega).$$

Another consequence of the symmetry is that $JN = N$.

The next property is crucial for our argument.

Theorem 4.2. *Let the conditions* (1.3), (1.4) *and* (1.5) *with* $p > 3$ *be satisfied. Denote*

$$\ker t_1 \equiv \{u \in Z : t_1[u, v] = 0, \ \forall v \in Z\}. \tag{4.3}$$

If (1.11) *is satisfied then* $\ker t_1 = \{0\}$.

For the proof of this fact we need two lemmas.

Lemma 4.3. *Let $\mathfrak{H}$ be a Hilbert space, and ℓ, $\ell_1, \dots, \ell_n$, $n < \infty$, be bounded linear functionals on $\mathfrak{H}$. If*

$$\bigcap_{k=1}^{n} \ker \ell_k \subset \ker \ell, \tag{4.4}$$

then the functional ℓ is a linear combination of the others: $\ell = \sum_{k=1}^n \alpha_k \ell_k$ with some coefficients α_k, $k = 1, 2, \dots, n$.

Although this fact is elementary we provide a proof for the sake of completeness.

Proof. Let z, $z_1, \dots, z_n \in \mathfrak{H}$ be the uniquely defined vectors such that

$$\ell(x) = (x, z), \quad \ell_k(x) = (x, z_k), \quad k = 1, \dots, n, \quad \forall x \in \mathfrak{H}.$$

The condition (4.4) is equivalent to the following implication: if $x \perp \mathfrak{L} = \mathrm{span}\{z_1, \dots, z_n\}$, then $x \perp z$. This means that $z \in \mathfrak{L}$, i.e. $z = \sum_{k=1}^n \alpha_k z_k$ with suitable coefficients α_k, $k = 1, 2, \dots, n$.■

Lemma 4.4. *Let* $\mathbf{G}$, $\mathbf{A}$ *and* V *satisfy* (1.3), (1.4) *and* (1.5) *with* $p > 3$. *Let a function* $w \in H^1(\Omega)$ *be such that* $w|_{\Lambda_+} = 0$ *and*

$$h_0[w, Jv] = 0, \quad \forall v \in W_+^1.$$

Then $w = 0$.

Proof. We extend the function w by zero into the parallelepiped $\Xi = \Lambda_- \times (-\pi, 4)$:

$$\tilde{w}(\mathbf{x}) = \begin{cases} w(\mathbf{x}), \ \text{when } x_3 \leqslant \pi, \\ 0, \ \text{when } x_3 > \pi. \end{cases}$$

Clearly, $\tilde{w} \in H^1(\Xi)$, and

$$\int_\Xi \left(\langle \mathbf{G}(-i\nabla\tilde{w} - \mathbf{A}\tilde{w}), -i\nabla v - \mathbf{A}v \rangle + V\tilde{w}\overline{v} \right) d\mathbf{x} = 0, \quad \forall v \in \mathring{H}^1(\Xi).$$

Therefore $\tilde{w}$ is a weak solution of the equation $H\tilde{w} = 0$ in Ξ. Now, the unique continuation principle for elliptic equations, see [10], Theorem 1, implies that $\tilde{w} \equiv 0$ in Ξ. ■

Remark 4.5. We need the conditions $\mathbf{G} \in \mathrm{Lip}$ and $A \in L_{p,\mathrm{loc}}$, $p > 3$, instead of the "sharp" condition $A \in L_{3,\mathrm{loc}}$ for the unique continuation principle only.

Proof of Theorem 4.2. By definition (4.3), for $u \in \ker t_1$ we have

$$h_0[u, J\phi] = 0 \quad \forall \phi \in Z. \tag{4.5}$$

By Lemma 3.1 the subspace N is finite-dimensional. Let $\{u_k\}, k = 1, 2, \dots, n$, be a basis in N. Consider on the Hilbert space W_+^1 linear functionals

$$\ell(\psi) = \overline{h_0[u, J\psi]}, \quad \ell_k(\psi) = \overline{h_0[u_k, J\psi]}, \ \psi \in W_+^1.$$

Since $JN = N$, by definition of M we have $\cap_k \ker \ell_k = M$. On the other hand, if $\psi \in M$ then by Lemma 3.2 $\psi = \phi + w$ with $\phi \in Z, w \in W^{1,0}$, so

$$h_0[u, J\psi] = h_0[u, J\phi] + h_0[u, Jw] = 0,$$

where we have used (4.5) and the fact that $u \in Z$. Thus $M \subset \ker \ell$. By virtue of Lemma 4.3 there exists a function $u_0 \in N$ such that

$$\ell(\psi) = \overline{h_0[u_0, J\psi]}, \quad \forall \psi \in W_+^1.$$

Therefore,

$$h_0[u - u_0, J\psi] = 0, \quad \forall \psi \in W_+^1.$$

Putting $v = u - u_0$ we have $h_0[v, J\psi] = 0$ for all $\psi \in W_+^1$. By Lemma 4.4 $v = 0$, so that $u = u_0 \in W^{1,0} \cap Z$. By Lemma 3.2 $u = 0$ as claimed. ∎

5. Proof of the main result

Recall that the operator $H(\mathbf{k})$ depends on the quasi-momentum $\mathbf{k}$ quadratically, i.e. it is a quadratic operator pencil. The decisive observation due to L. Friedlander [7] is that the reflection symmetry allows one to reduce the analysis of $H(\mathbf{k})$ to a linear operator pencil.

5.1. **An abstract lemma on linear operator pencils.** We will need the following abstract result. Let $\mathfrak{H}$ be a Hilbert space, and let $\mathfrak{t}$ be a bounded sesquilinear form defined on $\mathfrak{H}$. Similarly to (4.3) we introduce the notation

$$\ker \mathfrak{t} = \{\phi \in \mathfrak{H} : \mathfrak{t}[\phi, \psi] = 0, \ \forall \psi \in \mathfrak{H}\}.$$

The set $\ker \mathfrak{t}$ is a (closed) subspace.

Lemma 5.1. *Let $\mathfrak{H}$ be a Hilbert space, and let $\mathfrak{t}_0$, $\mathfrak{t}_1$ be two bounded Hermitian sesquilinear forms on $\mathfrak{H}$. Let $\mathfrak{L} \subset \mathfrak{H}$ be a linear set such that $\mathfrak{t}_0[\phi] \leqslant 0$ for any $\phi \in \mathfrak{L}$. Suppose that*

$$m = \sup_{\mathfrak{L}} \dim \mathfrak{L} < \infty. \tag{5.1}$$

Assume that $\ker \mathfrak{t}_1 = \{0\}$. Then

$$\# \{z \in \mathbb{C} \setminus \mathbb{R} : \ker(\mathfrak{t}_0 + z\mathfrak{t}_1) \neq \{0\}\} \leqslant 2m.$$

Clearly this Lemma can be generalised to unbounded forms with appropriate restrictions on $\mathfrak{t}_0, \mathfrak{t}_1$ but it is unnecessary for our purposes.

Proof. Let

$$F = \{z_1, z_2, \dots, z_n\}, \ \operatorname{Im} z_j > 0, j = 1, 2, \dots, n,$$

be a finite set of distinct points in the complex plane such that

$$\mathfrak{G}_j = \ker(\mathfrak{t}_0 + z_j \mathfrak{t}_1) \neq \{0\}, j = 1, 2, \dots, n.$$

Let us show that the subspaces $\mathfrak{G}_j$ are linearly independent. We proceed by induction. If $n = 1$ then there is nothing to proof.

Let $1 \leqslant p \leqslant n-1$. Suppose that any p-tuple of non-zero vectors $\phi_k \in \mathfrak{G}_k, k = 1, 2, \dots, p$ are linearly-independent. Suppose also that $\phi_{p+1} \in \mathfrak{G}_{p+1}$ is a vector such that

$$\phi_{p+1} = \sum_{k=1}^{p} \alpha_k \phi_k, \tag{5.2}$$

and at least one coefficient α_k is non-zero. By definition of $\mathfrak{H}_k$,

$$\mathfrak{t}_0[\phi_k, w] + z_k \mathfrak{t}_1[\phi_k, w] = 0, \quad \forall w \in \mathfrak{H},$$

for all $k = 1, 2, \dots, p+1$. Therefore

$$\sum_{k=1}^{p} \alpha_k \mathfrak{t}_0[\phi_k, w] + \sum_{k=1}^{p} \alpha_k z_k \mathfrak{t}_1[\phi_k, w] = 0,$$

and

$$\sum_{k=1}^{p} \alpha_k \mathfrak{t}_0[\phi_k, w] + \sum_{k=1}^{p} \alpha_k z_{p+1} \mathfrak{t}_1[\phi_k, w] = 0,$$

for all $w \in \mathfrak{H}$, where we have used (5.2). Subtracting one equation from the other we get

$$\mathfrak{t}_1 \left[\sum_{k=1}^{p} \alpha_k (z_k - z_{p+1}) \phi_k, w \right] = 0, \quad \forall\ w \in \mathfrak{H}.$$

Recalling again that $\ker \mathfrak{t}_1 = \{0\}$, we conclude that

$$\sum_{k=1}^{p} \alpha_k (z_k - z_{p+1}) \phi_k = 0,$$

which means that the set $\{\phi_1, \phi_2, \dots, \phi_p\}$ is linearly dependent. This gives a contradiction, and hence the $(p+1)$-tuple containing also ϕ_{p+1} are linearly independent as well. By induction all kernels $\mathfrak{G}_j, j = 1, 2, \dots, n$ are linearly-independent, and as a consequence, $\#F \leqslant \dim \mathfrak{G}$ where

$$\mathfrak{G} = \bigoplus_{j=1}^{n} \mathfrak{G}_j.$$

Now, for any $\phi_j \in \mathfrak{G}_j, \phi_k \in \mathfrak{G}_k$, we have

$$\begin{cases} \mathfrak{t}_0[\phi_j, \phi_k] + z_j \mathfrak{t}_1[\phi_j, \phi_k] = 0, \\ \mathfrak{t}_0[\phi_j, \phi_k] + \overline{z}_k \mathfrak{t}_1[\phi_j, \phi_k] = 0, \end{cases}$$

where we have used that $\mathfrak{t}_0, \mathfrak{t}_1$ are Hermitian. Since $\operatorname{Im} z_j, \operatorname{Im} z_k > 0$, we conclude that $\mathfrak{t}_0[\phi_j, \phi_k] = \mathfrak{t}_1[\phi_j, \phi_k] = 0$. As a consequence,

$$\mathfrak{t}_0[\phi, \psi] = \mathfrak{t}_1[\phi, \psi] = 0, \quad \forall \phi, \psi \in \mathfrak{G}.$$

In particular, $\mathfrak{t}_0[\phi] = 0$, so that $\dim \mathfrak{G} \leqslant m$, and hence, $\#F \leqslant m$, i.e.

$$\#\{z \in \mathbb{C},\ \operatorname{Im} z > 0 : \ker(\mathfrak{t}_0 + z\mathfrak{t}_1) \neq \{0\}\} \leqslant m.$$

In the same way one proves that the number of such points in the lower half-plane is also bounded by m. This completes the proof. ∎

Note in passing that if any of the forms $\mathfrak{t}_0$ or $\mathfrak{t}_1$ is positive-definite then the set

$$\{z \in \mathbb{C} \setminus \mathbb{R} : \ker(\mathfrak{t}_0 + z\mathfrak{t}_1) \neq \{0\}\} \tag{5.3}$$

is trivially empty. Indeed, assume for example that $\mathfrak{t}_1$ is positive-definite. Let T_0, T_1 be the operators associated with the forms $\mathfrak{t}_0, \mathfrak{t}_1$ respectively. Thus $\ker(\mathfrak{t}_0 + z\mathfrak{t}_1) \neq \{0\}$ iff the number z belongs to the spectrum of the self-adjoint operator $-T_1^{-\frac{1}{2}} T_0 T_1^{-\frac{1}{2}}$. Thus $z \in \mathbb{R}$, which implies that the set (5.3) is empty, as claimed.

5.2. **Proof of Theorem 2.1.** We begin the study of the problem (2.9), (2.10) with the analysis of the following system for a function $u \in W^1$:

$$\begin{cases} h_0[u, v] = 0, & \forall v \in W^{1,0}, \\ u|_{\Lambda_+} = \zeta\, u|_{\Lambda_-}. \end{cases} \tag{5.4}$$

Lemma 5.2. *Suppose that the conditions* (1.3), (1.5) *with* $p = 3$, *and* (1.11) *are satisfied. Let* $\zeta \neq \pm 1$. *Then any solution of* (5.4) *has the form*

$$u = \phi + \zeta J\phi + \omega, \quad \textit{where} \quad \phi \in Z,\ \omega \in N. \tag{5.5}$$

Proof. Let u be a solution to (5.4). Then the function $\psi = (1 - \zeta^2)^{-1}(u - \zeta Ju)$ belongs to W^1_+ and solves the equation $h_0[\psi, v] = 0$, $\forall v \in W^{1,0}$, and hence $\psi \in M$. By Lemma 3.2, $\psi = \phi + w$ where $\phi \in Z$ and $w \in W^{1,0}$. Consequently $w, Jw \in N$. By inspection,

$$u = \psi + \zeta J\psi, \tag{5.6}$$

so that the representation (5.5) holds with $\omega = w + \zeta Jw \in N$. ∎

Proof of Theorem 2.1. Let $\zeta \in X$, and let $u \in W^1$ be a non-trivial solution of the system (2.9), (2.10). By virtue of Lemma 5.2, $u = \phi + \zeta J\phi + \omega$, with some $\phi \in Z$ and $\omega \in N$.

First, consider the case $\phi = 0$. Then $u = \omega \in N$. Let us use (2.10) with the function $w = \overline{\zeta} f + Jf$ where $f \in W^1_+$ is an arbitrary function. Thus

$$h_0[\zeta Ju + u, Jf] = h_0[u, \overline{\zeta} f + Jf] = 0.$$

Lemma 4.4 yields $Ju = -\zeta^{-1}u$. On the other hand, the spectrum of the operator J consists of two numbers 1 and -1 only, but $|\zeta| \neq 1$, so $u = 0$, which gives a contradiction.

Now, assume that $\phi \neq 0$. For a function $\psi \in Z$ substitute $w = J\psi + \overline{\zeta}\psi$ into (2.10), and obtain

$$\begin{aligned} 0 = h_0[u, w] &= h_0\left[\phi + \zeta J\phi + \omega, J\psi + \overline{\zeta}\psi\right] \\ &= \left(1 + \zeta^2\right) t_1[\phi, \psi] + 2\zeta t_0[\phi, \psi], \end{aligned}$$

where we have used the fact that $JW^{1,0} = W^{1,0}$. Therefore,

$$t_0[\phi, \psi] + z t_1[\phi, \psi] = 0, \quad z = \frac{1 + \zeta^2}{2\zeta}.$$

In view of the conditions $\operatorname{Im} \zeta \neq 0$, $|\zeta| \neq 1$ we have $\operatorname{Im} z \neq 0$. The forms t_0, t_1 satisfy all the conditions of Lemma 5.1. Indeed, both forms are bounded on Z, $\ker t_1 = \{0\}$ by Theorem 4.2, and the condition (5.1) is satisfied by virtue of (4.2). Therefore Lemma 5.1 yields that $\#X \leqslant 2m < \infty$. This completes the proof. ∎

As explained earlier, Theorem 2.1 implies Theorem 1.1 stating the absolute continuity of the operator H.

References

[1] M. Sh. Birman, M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Reidel, 1987.

[2] M. Sh. Birman, M. Z. Solomyak, *Schrödinger operator. Estimates for number of bound states as function-theoretical problem* (Russian), Spectral theory of operators (Novgorod, 1989), p. 1–54. Transl. in Amer. Math. Soc. Transl. Ser. 2, 150, AMS, Providence, RI, 1992.

[3] M. Sh. Birman, T.A. Suslina, *Periodic magnetic Hamiltonian with a variable metric. The problem of absolute continuity* (Russian), Algebra i Analiz **11**, no. 2 (1999), 1-40. English translation in: St. Petersburg Math. J. **11**, no. 2 (2000), 203-232.

[4] N.Filonov, *Second-order elliptic equation of divergence form having a compactly supported solution*, Problemy Mat. Analiza, **22** (2001), 246–257 (Russian); Engl. transl. Journal of Mathematical Sciences, Vol. **106**, No. 3, 2001, 3078-3086.

[5] N. Filonov, A. V. Sobolev, *Absence of the singular continuous component in the spectrum of analytic direct integrals*, Zap. Nauch. Sem. POMI **318** (2004), p. 298–307.

[6] N. Filonov, M. Tikhomirov, *Absolute Continuity of an Even Periodic Schrödinger Operator with Nonsmooth Coefficients* (Russian), Algebra i Analiz **16** (2004), no. 3, p. 201–210; transl. in St. Petersburg Math. J. **16** (2005), 583-589.

[7] L. Friedlander, *On the spectrum of a class of second order periodic elliptic differential operators*, Comm. Math. Phys. **229** (2002), no. 1, p. 49–55.

[8] T. Kato, *Perturbation theory for liner operators*, Springer, New-York 1966.

[9] F. Klopp, *Absolute continuity of the spectrum of a Landau Hamiltonian perturbed by a generic periodic potential*, Math. Ann. **347** (2010), no. 3, 675–687.

[10] H. Koch, D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math. **54** (2001), no. 3, p. 339-360.

[11] P.Kuchment, S.Levendorskii, *On the structure of spectra of periodic elliptic operators*, Transactions AMS, **354** (2001), 2, 537–569.

[12] L. D. Landau, E. M. Lifshitz, *Course of Theoretical Physics*, vol. 3, Butterworth-Heinemann, 1981.

[13] M. Melgaard, G. V. Rozenblum, *Spectral estimates for magnetic operators*, Math. Scand. **79** (1996), p. 237–254.

[14] E. Shargorodsky, A.V.Sobolev, *Quasiconformal mappings and periodic spectral problems in dimension two*, J. Anal. Math. **91** (2003), 67–103.

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