HOLOMORPHIC CURVES IN COMPACT SHIMURA VARIETIES.

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Abstract. We prove a hyperbolic analogue of the Bloch-Ochiai theorem about the Zariski closure of holomorphic curves in abelian varieties.

Résumé. On démontre un analogue hyperbolique du théorème de Bloch-Ochiai sur l'adhérence de Zariski d'une courbe holomorphe dans une variété abélienne.

Contents

1. Introduction.

The following theorem of Bloch-Ochiai (see Chapter 9, theorem 3.9.19 of [3]) is classically proved using Nevanlinna theory.

Theorem 1.1 (Bloch-Ochiai). *Let A be an abelian variety and $f\colon \mathbb{C} \longrightarrow A$ be a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.*

In this paper we formulate and prove an analogue of this theorem for a certain type of locally symmetric varieties, namely the compact Shimura varieties.

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For notations and facts about Shimura varieties and weakly special subvarieties, we refer to [9] and references therein. Recall that any hermitian symmetric domain X, admits a realisation $X \subset \mathbb{C}^n$ (with $n = \dim(X)$) as a bounded symmetric domain. See [5], Chapter 4 for details.

Recall also that given an arithmetic lattice $\Gamma \subset \mathrm{Aut}(X)^+$, such that the quotient $\Gamma\backslash X$ is compact, there exists a fundamental domain $\mathcal{F}$ for the action of Γ on X which is an open subset of X such that $\overline{\mathcal{F}}$ is compact. For a bounded hermitian symmetric domain $X \subset \mathbb{C}^n$, we denote by ∂X the boundary of X, i.e. $\partial X = \overline{X}\backslash X$ where $\overline{X}$ denotes the topological closure of X in $\mathbb{C}^n$.

For notions of Shimura data, Shimura varieties and their weakly special subvarieties we refer to [2], [9] and references contained therein. We just recall that weakly special subvarieties are defined in terms of Shimura subdata, but as shown in [6], they are exactly the totally geodesic subvarieties of $\Gamma\backslash X$ and terms 'weakly special' and 'totally geodesic' are used in the literature interchangeably.

Let (G, X) be a Shimura datum with G anisotropic over $\mathbb{Q}$, let X^+ be a connected component of X and K a compact open subgroup of $G(\mathbb{A}_f)$. As above, $X^+ \subset \mathbb{C}^n$ is a bounded symmetric domain.

We let Γ be the intersection of K with the stabiliser of X^+ in $G(\mathbb{Q})$. Then Γ is an arithmetic congruence group acting on X^+.

Then $S = \Gamma\backslash X^+$ is compact. Let $\pi\colon X^+ \longrightarrow \Gamma\backslash X^+$ be the quotient map.

Theorem 1.2. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}^n$ be a holomorphic map such that $C = f(\mathbb{C}) \cap X^+$ is non-empty. The folowing holds:*

1. *Let C' be an analytic irreducible component of $f(\mathbb{C}) \cap X^+$. Then the Zariski closure $Zar(\pi(C'))$ is weakly special.*
2. *The components of the Zariski closure $Zar(\pi(C))$ of $\pi(C)$ are weakly special subvarieties of S.*

An easy analytic continuation argument shows that if Y is an irreducible analytic subset of X^+, then $Zar(\pi(Y))$ is (algebraic) irreducible. It follows that in the statement (1) of 1.2, one does not need to consider irreducible components of $Zar(\pi(C'))$.

Another comment is that (2) does not follow directly from (1) since $f(\mathbb{C}) \cap X^+$ can have infinitely analytic irreducible components. Additional arguments are necessary. They are given in section (2).

This result is partly inspired by the so-called hyperbolic Ax-Lindemann theorem whose slightly different but equivalent formulation is proven in [9], Théorème 1.3 in the co-compact case and in [4] for all Shimura varietes. The statement of the hyperbolic Ax-Lindemann theorem is as follows.

Theorem 1.3. *Keep the notations of theorem 1.2.*

Let Y be an irreducible algebraic subset of X^+. Then the Zariski closure of $\pi(Y)$ is a weakly special subvariety of S.

The proof of theorem 1.2 relies on the theory of o-minimality and the Pila-Wilkie counting theorem and is inspired by the proof of the hyperbolic Ax-Lindemann theorem in the co-compact case as in [9]. The proof also uses in an essential way the hyperbolic Ax-Lindemann theorem itself and the results of [8].

An analogous question in the context of abelian varieties has been investigated in [10]. In that paper we have not been able to re-prove Bloch-Ochiai theorem using o-minimal techniques. We however obtained a result analogous to 1.2 in the abelian context for certain sets definable in the usual o-minimal structures. Our result in [10] is in some ways more general than the Bloch-Ochiai theorem. It is surprising and interesting that the obstructions to prove the Bloch-Ochiai theorem using o-minimality do not occur in the hyperbolic case we consider here, however additional serious difficulties arise which we overcome in section 3.

The strategy of the proof is as follows. We start by decomposing $f^{-1}(f(\mathbb{C}) \cap X^+)$ as a union of connected components $U_i \subset \mathbb{C}$. For a given i we prove that for some $R_i > 0$, it is in fact enough to prove the conclusion for $\pi \circ f(U_i \cap B(0, R_i))$ where $B(0, R_i)$ is the open ball centered at the origin of radius R_i. This is done in section 2.

We now set $C_i = f(U_i \cap B(0, R_i))$. Section 3 is the technical heart of the proof. The analytic curve C_i in X^+ is definable in the o-minimal structure $\mathbb{R}_{an}$ (here $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$). For o-minimality, related notions and results we refer to [11]. We fix a fundamental domain $\mathcal{F}$ for the action of Γ on X^+.

We let V_i be the Zariski closure of $\pi(C_i)$ and $\widetilde{V_i}$ be $\pi^{-1}(V_i) \cap \mathcal{F}$. We associate to C_i a certain definable (in $\mathbb{R}_{an}$) set $\Sigma \subset G(\mathbb{R})$ and show that $\Sigma \cdot C_i \subset \widetilde{V_i}$. The main technical work is to prove that Σ contains *a lot* of points of $G(\mathbb{Q})$ of height up to T. Pila-Wilkie theorem then allows us to conclude that Σ contains a positive dimensional semi-algebraic subset W and the hyperbolic Ax-Lindemann theorem allows us to conclude that V_i contains a Zariski dense set of weakly special subvarieties. Using results from [8] and some additional arguments, we conclude the proof of theorem 1.2.

We do not know whether conclusions of theorem 1.2 remain true without the assumption that S is compact. The main difficulty in removing the compactness assumption lies in proving that C_i (with the above notations) intersects "many" translates of a fundamental set. We in fact do not know whether it is possible for C_i to be contained in a union of finitely many such translates.

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2. Preliminaries.

Keep notations as in Theorem 1.2. For simplicity of notation, we write X for X^+. Let

$$f^{-1}(f(\mathbb{C}) \cap X) = \coprod_{i \in I} U_i$$

be the decomposition of $f^{-1}(f(\mathbb{C}) \cap X)$ into connected components. By definition of the U_i, for each i, we have

$$\overline{f(U_i)} \cap \partial X \neq \emptyset.$$

For $R_i > 0$ large enough, we have

$$\overline{f(B(0, R_i) \cap U_i)} \cap \partial X \neq \emptyset$$

(where $B(0, R_i)$ is the open ball of radius R_i centered at the origin). For each i, we fix an R_i with this property.

Proposition 2.1. *We have*

$$Zar(\pi \circ f(U_i)) = Zar(\pi \circ f(B(0, R_i) \cap U_i).$$

Proof. One inclusion is obvious. Write $Zar(\pi \circ f(B(0, R_i) \cap U_i)) \subset \mathbb{P}^m$ for some m and let $s \in H^0(\mathbb{P}^m, \mathcal{O}(l))$ for $l \geq 1$ such that s is zero on $\pi \circ f(B(0, R_i) \cap U_i)$. Then the function $s \circ f \circ \pi \colon U_i \longrightarrow \mathbb{C}$ is zero on $B(0, R_i) \cap U_i)$. Since U_i is connected, by analytic continuation, the function $s \circ f \circ \pi$ is zero on U_i. It follows that s is zero on $\pi \circ f(U_i)$. This proves the other inclusion. □

In this paper we will prove the following:

Theorem 2.2. *The Zariski closure of $\pi \circ f(B(0, R_i) \cap U_i)$ contains a Zariski dense subset of weakly special subvarieties.*

Let $V = V_i$ be the Zariski closure of $\pi \circ f(B(0, R_i) \cap U_i)$. The theorem 2.2 will be deduced from the following:

Theorem 2.3. *There exists a positive dimensional semialgebraic set W in $G(\mathbb{R})$ such that*

$$W \cdot f(B(0,R_i)\cap U_i) \subset \pi^{-1}(V).$$

To deduce 2.2 from 2.3, let $P \in f(B(0,R_i)\cap U_i)$. For the notion of algebraic subset of X, we refer to Appendix B of [4]. There exists a complex algebraic subset $Y_P \subset \pi^{-1}(V)$ such that $W\cdot P \subset Y_P$ (see [4], Lemma B.3). By Ax-Lindemann theorem 1.3, the Zariski closure of $\pi(Y_P) \subset V$ is weakly special. Therefore, through each point of $\pi f(B(0,R_i)\cap U_i)$ there passes a weakly special subvariety and hence V contains a dense set of weakly special subvarieties.

We will now prove that theorem 1.2 follows from theorem 2.2. Let V be a component of the Zariski closure of $\pi(f(\mathbb{C})\cap X)$. By theorem 2.2, V contains a Zariski dense set of weakly special subvarieties.

If V is a weakly special subvariety, then we are done. Assume that V is not weakly special. By [8], Corollary 1.4, there exists a special subvariety $S' \subset S$ containing V and such that $S' = S_1 \times S_2$ (S_i special and positive dimensional) and such that

$$V = S_1 \times V'$$

where V' is a subvariety of S_2.

There exists a sub-Shimura datum (G', X') of (G, X) and a decomposition

$$(G'^{ad}, X'^{ad}) = (G_1, X_1) \times (G_2, X_2)$$

such that $S_1 = \Gamma_1\backslash X_1$ and $S_2 = \Gamma_2\backslash X_2$ (where as usual we omit the superscript $+$) and Γ_1 and Γ_2 are suitable arithmetic lattices in $G_1(\mathbb{Q})^+$ and $G_2(\mathbb{Q})^+$.

Let $\mathfrak{p}_1 \cong \mathbb{C}^{r_1}$ and $\mathfrak{p}_2 \cong \mathbb{C}^{r_2}$ be the holomorphic tangent spaces to X_1 and X_2. Then $\mathfrak{p}_1 \times \mathfrak{p}_2$ is a subspace of the holomorphic tangent space $\mathfrak{p} \cong \mathbb{C}^n$ to X. Let

$$f^{-1}(f(\mathbb{C})\cap X) = \coprod_{i\in I} U_i.$$

be as before, the connected component decomposition. There exists a U_i such that the restriction, $f\colon U_i \longrightarrow \mathbb{C}^n$ factors through $\mathbb{C}^{r_1}\times\mathbb{C}^{r_2}$. By analytic continuation $f\colon \mathbb{C} \longrightarrow \mathbb{C}^n$ factors through $\mathbb{C}^{r_1}\times\mathbb{C}^{r_2}$.

Let f_1 and f_2 be the holomorphic functions from $\mathbb{C}$ to $\mathbb{C}^{r_1}$ and $\mathbb{C}^{r_2}$ respectively such that $f = (f_1, f_2)$.

Similarly, write

$$f_2^{-1}(f_2(\mathbb{C})\cap X_2) = \coprod_{j\in J} V_j.$$

the connected component decomposition.

From the definition of U_is and V_js, it follows that any $i \in I$, there exists an $j \in J$ such that $U_i \subset V_j$. It follows that for any $i \in I$, there exists $j \in J$ such that:

$$Zar(\pi_2 \circ f_2(U_i)) = Zar(\pi_2 \circ f_2(V_j)).$$

Note that V' is the Zariski closure of the union of the $\pi_2 \circ f_2(U_i)$. Therefore V' is the Zariski closure of

$$\bigcup_{i \in I} Zar(\pi_2 \circ f_2(U_i)).$$

By theorem 2.2, $Zar(\pi_2 \circ f_2(U_i)) = Zar(\pi_2 \circ f_2(V_j))$ contains a Zariski dense set of weakly special subvarieties.

It follows that V' contains a Zariski dense set of weakly special subvarieties of S_2. An inductive argument finishes the proof of theorem 1.2 assuming theorem 2.2.

3. Counting lattice elements.

In this section we show that $f(U_i)$ (as in the previous section) in X intersects "exponentially many" (in a suitable sense) Γ-translates of a fixed fundamental domain. This section constitutes the technical heart of the paper.

Recall the following notations from [9]. Let X be a connected Hermitian symmetric domain (as usual we omit the superscript $+$), realised as a bounded symmetric domain in some $\mathbb{C}^n$. We let C to be $f(B(0, R_i) \cap U_i)$ with R_i and U_i as in the previous section.

Let Γ be a cocompact arithmetic lattice in the group G of holomorphic isometries of X. For a point $x_0 \in X$, we let $\mathcal{F}$ be a fundamental domain for the action of Γ on X such that $x_0 \in \mathcal{F}$. We assume that $\mathcal{F}$ is an open connected set such that $\overline{\mathcal{F}}$ is compact. The set

$$\mathcal{S}_{\mathcal{F}} = \{\gamma \in \Gamma : \gamma\overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}$$

is finite and generates Γ.

The "word metric" $l \colon \Gamma \longrightarrow \mathbb{N}$ with respect to $\mathcal{S}_{\mathcal{F}}$ is defined as follows $l(1) = 0$ and for $\gamma \neq 1$, $l(\gamma)$ is the minimal length of a word in the elements of $S_{\mathcal{F}}$ representing γ.

We also let $K(Z, W)$ be the Bergmann kernel on X and we let

$$\omega = \sqrt{-1}\partial\overline{\partial}K(Z, Z)$$

be the associated Kähler form. We refer to [5] 4.1 for details on this.

We define the following functions:

$$N_C(n) = |\{\gamma \in \Gamma : \dim(\gamma\mathcal{F} \cap C) = 1, l(\gamma) \leq n\}|$$

and

$$N'_C(n) = |\{\gamma \in \Gamma : \dim(\gamma\mathcal{F} \cap C) = 1, l(\gamma) = n\}|.$$

The main result of this section is the following theorem:

Theorem 3.1. *There is a positive constant c such that for all $n \gg 0$, we have*

$$N_C(n) \geq e^{cn}.$$

Let b be a point of the boundary of $\overline{C} \cap \partial X$ and a neighbourhood V_b of b such that $\overline{C} \cap \partial X \cap V_b$ is a real analytic curve.

We parametrise $\overline{C} \cap \partial X \cap V_b$ as follows. For $0 < \alpha, \beta < 2\pi$, let $\Delta_{\alpha,\beta}$ be the sector of the unit disc Δ defined as follows:

$$\Delta_{\alpha,\beta} = \{z = re^{i\theta} : 0 \leq r < 1, \alpha \leq \theta \leq \beta\}.$$

Let $C_{\alpha,\beta}$ be the subset of $\partial\Delta_{\alpha,\beta}$ defined as

$$C_{\alpha,\beta} = \{z = e^{i\theta} : \alpha \leq \theta \leq \beta\}.$$

We can find α, β and a real analytic map ψ from a neignbourhood of $\Delta_{\alpha,\beta}$ to $\mathbb{C}^n$ such that $\psi(\Delta_{\alpha,\beta}) \subset \overline{C} \cap X$ and $\psi(C_{\alpha,\beta}) \subset \overline{C} \cap \partial X$.

Let Δ be the open unit disk. We let ω_Δ be the usual Poincaré $(1,1)$-form on Δ ($\omega_\Delta = \sqrt{-1}\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$). By lemma 2.8 of [9], there exists a smooth $(1,1)$-form η on $\Delta_{\alpha,\beta}$ such that

$$\psi^*\omega = s\omega_\Delta + \eta$$

for some integer $s > 0$.

Let $\gamma \in \Gamma$ be such that $\dim(\gamma\mathcal{F} \cap C) = 1$ and $\gamma\mathcal{F} \cap C \subset \psi(\Delta_{\alpha,\beta})$, then

$$\int_{\gamma\mathcal{F}\cap C} \omega = s\int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \omega_\Delta + \int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \eta \qquad (1)$$

Proposition 3.2. *There exists a constant B such that for any $\gamma \in \Gamma$ such that* $\dim(\gamma\mathcal{F} \cap C) = 1$, *we have*

$$\int_{\gamma\mathcal{F}\cap C} \omega \leq B.$$

Proof. We consider the compact dual X_c of X which is a closed algebraic subvariety of some projective $\mathbb{P}^m$. Let $\mathcal{L}$ be the dual of the canonical line bundle endowed with the Fubini-Study metric $||.||_{FS}$. We let ω_{FS} the associated $(1,1)$-form: $\omega_{FS} = c_1(\mathcal{L}, ||.||_{FS})$.

By Harish-Chandra embedding theorem, there is a biholomorphism λ from $\mathfrak{p} \cong \mathbb{C}^n$ to an open dense subset of X_c. For details, see Theorem 1, section 5.2 of [5].

Let $\gamma \in \Gamma$ be such that $\gamma\mathcal{F} \cap C \neq \emptyset$. Since ω is Γ-invariant, we have

$$\int_{\gamma\mathcal{F}\cap C} \omega = \int_{\gamma^{-1}(\gamma\mathcal{F}\cap C)} \omega.$$

On the compact set $\overline{\mathcal{F}}$, the two forms ω and $\lambda^*(\omega_{FS})$ are positive holomorphic forms, therefore there is a constant B_1 such that on $\mathcal{F}$, we have

$$\omega \leq B_1 \lambda^*(\omega_{FS}).$$

We have

$$\int_{\gamma\mathcal{F}\cap C} \omega \leq B_1 \int_{\gamma^{-1}(\gamma\mathcal{F}\cap C)} \lambda^*(\omega_{FS}).$$

Furthermore,

$$\int_{\gamma^{-1}(\gamma\mathcal{F}\cap C)} \lambda^*(\omega_{FS}) \leq \int_{\gamma^{-1}\lambda(C)} \omega_{FS}.$$

The conclusion of proposition 3.2 follows from the following lemma that will be proven in the following section. :

Lemma 3.3. *There is a constant B_2 such that for all $\gamma \in \Gamma$, we have*

$$\int_{\gamma^{-1}\lambda(C)} \omega_{FS} \leq B_2.$$

□

3.1. **Proof of lemma 3.3.** The volume of the analytic curve $\lambda(C)$ is defined as

$$\mathrm{Vol}(\lambda(C)) = \int_{\lambda(C)} \omega_{FS}.$$

Let $\mathbb{P}^{m\vee}$ be the dual projective space, the set of hyperplanes in $\mathbb{P}^m$. Let dJ be the invariant volume element on $\mathbb{P}^{m\vee}$ normalised to have total mass one.

By Generalised Crofton's formula (see [1] and references therein), we have

$$\mathrm{Vol}(\gamma^{-1}\lambda(C)) = \alpha \int_{\mathbb{P}^{m\vee}} n_{\gamma^{-1}\lambda(C)}(J) dJ.$$

where α is a uniformisation constant and $n_{\gamma^{-1}\lambda(C)}(J)$ is the number of points (counted with multiplicity) of the intersection of $\gamma^{-1}\lambda(C)$ with J. Note that the function $n_{\gamma^{-1}\lambda(C)}(J)$ is a function defined on the open subset of $\mathbb{P}^{m\vee}$ consisting of hyperplanes J such that $\gamma^{-1}\lambda(C)$ is not contained J. The complement of this open set is of measure zero, therefore, by lemma 3.5, the integral is well defined.

Lemma 3.4. *Let J be a hyperplane in $\mathbb{P}^m$ and $\gamma \in \Gamma$. There exists a hyperplane J' such that*

$$n_{\gamma^{-1}\lambda(C)}(J) = n_{\lambda(C)}(J').$$

Proof. Recall that $\mathcal{L}$ is Γ-invariant and $\mathcal{L}$ is very ample i.e. $\mathcal{L} = \mathcal{O}(1)|_{X_c}$. Write s the section of $\mathcal{L}$ such that $J \cap X_c = \operatorname{div}(s)$. Let $s' = \gamma^* s$. Then s' is a restriction of a section of $\mathcal{O}(1)$ corresponding to some hyperplane J' and we thus have

$$\gamma(J \cap X_c) = J' \cap X_c.$$

Therefore

$$\lambda(C) \cap \gamma(J \cap X_c) = \lambda(C) \cap (J' \cap X_c).$$

We also have

$$\lambda(C) \cap \gamma(J \cap X_c) = \gamma(\gamma^{-1}\lambda(C) \cap J).$$

We conclude using the fact that

$$|\gamma(\gamma^{-1}\lambda(C) \cap J)| = n_{\gamma^{-1}\lambda(C)}(J).$$

□

We finish by proving a general lemma:

Lemma 3.5. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{P}^m(\mathbb{C})$ be a holomorphic map. Let $R > 0$. There exists a constant $\Theta = \Theta(R, f)$ such that for any hyperplane H of $\mathbb{P}^m(\mathbb{C})$ not containing $f(\mathbb{C})$, one has*

$$|\{f(B(0,R)) \cap H\}| \leq \Theta.$$

Proof. A reference for notions of Nevanlinna-Cartan theory is [3], Chapter 3, Section B. We use notations from this reference.

Let $N(R, f, H)$ be the counting function associated to f, R and H. Let $\alpha_1, \dots, \alpha_t \in B(0,R)$ be the complex numbers such that $f(\alpha_i) \in H$.

Let $\nu(f, \alpha_i, H)$ be the multiplicity of f in H at α_i. We have

$$N(R, f, H) = \sum_{i=1}^{t} \nu(f, \alpha_i, H) \log(\frac{R}{|\alpha_i|}).$$

Therefore

$$N(2R, f, H) \geq \sum_{i=1}^{t} \nu(f, \alpha_i, H) \log(\frac{2R}{|\alpha_i|}).$$

We have $\log(\frac{2R}{|\alpha_i|}) \geq \log(2)$, therefore a bound on $N(2R, f, H)$ implies a bound on $\sum_{i=1}^{t} \nu(f, \alpha_i, H) = |\{f(B(0,R)) \cap H\}|$. It is hence enough to bound $N(2R, f, H)$.

The first main theorem of Cartan-Nevanlinna theory ([3], 3.B.16), we have

$$N(2R, f, H) \leq T(2R, f) + c$$

where c is a uniform constant and $T(2R, f)$ is the order function defined in [3], 3.B.2.

Since $T(2R, f)$ does not depend on H, this concludes the proof. □

3.2. **End of proof of theorem 3.1.** As η is smooth on $\Delta_{\alpha,\beta}$, the integral $\int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \eta$ is bounded independently of γ. Equation 1 and lemma 3.2 imply that $\int_{\psi^{-1}(\gamma\mathcal{F}\cap C)} \omega_\Delta$ is bounded by a constant B', independent of γ.

Recall the following lemma (Lemma 2.1) from [9]. Note that this lemma is proved in [9] for C algebraic but the algebraicity assumption is not used, the statement and proof remain the same in our situation. In fact the proof is a combination of some general facts about hermitian symmetric domains and word metrics.

Lemma 3.6. *There exist positive constants λ_1 and λ_2 and D such that for all $z \in \Delta_{\alpha,\beta}$ with $z \in \psi^{-1}(\gamma\mathcal{F} \cap C)$,*

$$\lambda_1 l(\gamma) \leq -\log(1 - z\overline{z}) \leq \lambda_2 l(\gamma) + D.$$

We now follow the end of section 2 of [9].

For $n > 0$, let

$$I_n = \{z \in \Delta_{\alpha,\beta}, e^{-(n+1)} \leq 1 - |z|^2 \leq e^{-n}\}.$$

The hyperbolic volume of I_n satisfies

$$\mathrm{Vol}(I_n) \geq \delta_1 e^n$$

where δ_1 is a positive constant.

The set I_n is covered by the $\psi^{-1}(\gamma\mathcal{F} \cap C)$. For each n large enough and for all $z \in I_n$, by lemma 3.6, there exists a γ such that $\psi(z) \in \gamma\mathcal{F}$ with γ satisfying

$$c_1 n \leq l(\gamma) \leq c_2 n$$

with uniform (i.e independent of n) constants c_1 and c_2.

On the other hand, for all $z \in \Delta_{\alpha,\beta}$, such that $\psi(z) \in \gamma\mathcal{F}$ for some $\gamma \in \Gamma$,

$$\mathrm{Vol}(\psi^{-1}(\gamma\mathcal{F} \cap C)) \leq B'.$$

Therefore, by the computation of $\mathrm{Vol}(I_n)$ above, there exists a $\delta_1 > 0$ such that

$$\sum_{c_1 n \leq k \leq c_2 n} N'_C(k) \geq \delta_1 e^n.$$

This finishes the proof of theorem 3.1.

4. A definable set and application of Pila-Wilkie theorem.

In this section we prove theorem 2.3 and hence our main theorem. We follow section 5 of [9] with appropriate modifications.

Let U be as before a connected component of $f^{-1}(f(\mathbb{C}) \cap X)$ and R such that $f(U \cap B(0,R)) \cap \partial X \neq \emptyset$. Note that $C = f(B(0,R) \cap U)$ is definable in $\mathbb{R}_{an}$. Let $\mathcal{F}$ be as in the previous section. Recall (see [9], Proposition 4.2) that π restricted to $\mathcal{F}$ is definable in $\mathbb{R}_{an}$.

Consider

$$\Sigma(C) = \{g \in G(\mathbb{R}) : \dim(gC \cap \pi^{-1}(V) \cap \mathcal{F}) = 1\}.$$

The set $\Sigma(C)$ is definable in $\mathbb{R}_{an}$.

We prove the following.

Lemma 4.1. (1) *For all $g \in \Sigma(C)$, $gC \subset \pi^{-1}(V)$.*

(2) *Define*

$$\Sigma'(C) = \{g \in G(\mathbb{R}) : g^{-1}\mathcal{F} \cap C \neq \emptyset\}.$$

Then

$$\Sigma(C) \cap \Gamma = \Sigma'(C) \cap \Gamma.$$

Proof. Let $g \in \Sigma(C)$, then

$$gC \cap \mathcal{F} \subset \pi^{-1}(V).$$

By analytic continuation, this implies that $gC \subset \pi^{-1}(V)$.

The proof of (2) is exactly identical to the proof of [9], Lemma 5.2 and relies on the fact that $\pi^{-1}(V)$ is Γ-invariant. □

From previous lemma and theorem 3.1, we obtain the following.

Lemma 4.2. *Let*

$$N_{\Sigma(C)}(n) = |\{\gamma \in \Gamma \cap \Sigma(C) : l(\gamma) \leq n\}|.$$

For all n large enough,

$$N_{\Sigma(C)}(n) \geq e^{cn}.$$

The height $H(\gamma)$ of an element γ of Γ is defined by viewing Γ as a subgroup of some $\mathrm{GL}_m(\mathbb{Z})$ and taking the maximum of the absolute values of the entries. If $l(\gamma) \leq n$, then $H(\gamma) \leq (mA)^n$ where A is the maximum of heights of elements of $\mathcal{S}_{\mathcal{F}}$.

Let now

$$\Theta(\Sigma(C), T) = \{g \in G(\mathbb{Q}) \cap \Sigma(C) : H(g) \leq T\}$$

and

$$N(\Sigma(C), T) = |\Theta(\Sigma(C), T)|$$

Lemma 4.3.

$$N(\Sigma(C), T) \geq T^{c_1}.$$

We now appeal to the Pila-Wilkie theorem (see [7], Theorem 1.8).

For a definable (in some o-minimal structure) subset $\Theta \subset \mathbb{R}^n$, we define Θ^{alg} to be the union of all positive dimensional semi-algebraic subsets contained in Θ. We define Θ^{tr} to be $\Theta \backslash \Theta^{alg}$.

Theorem 4.4 (Pila-Wilkie)**.** *Let Θ be a subset of $\mathbb{R}^n$ definable in an o-minimal structure. Let $\epsilon > 0$. There exists a constant $c = c(\Theta, \epsilon)$ such that for any $T \geq 0$,*

$$|\{x \in \Theta^{tr} \cap \mathbb{Q}^n : H(x) \leq T\}| \leq cT^{\epsilon}.$$

In view of lemma 4.3, by Pila-Wilkie theorem, there exists a positive dimensional semi-algebraic subset $W \subset \Sigma(C)$ and by (1) of lemma 4.1, we have $W \cdot C \subset \pi^{-1}(V)$. This finishes the proof of theorem 2.3.

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