Monopoles in $\mathbb{R}^3$

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Abstract

The aim of this thesis is to investigate the moduli space of framed monopoles with structure group $SU(N)$ over the radial compactification of $\mathbb{R}^3$. The moduli space of monopoles is provided with the differential structure of a smooth manifold in a similar way as it is done for instantons, that is, via a slice theorem. Later, the dimension of this manifold is computed using an index formula developed by C. Kottke. This result agrees with the one coming from the bijection with rational maps obtained by S. Jarvis.

This thesis was completed under the supervision of Professor Michael Singer.

Impact Statement

This thesis might be useful to researchers, as well as other people interested in widening their knowledge of Gauge Theory and Mathematical Physics. In particular, it explores magnetic monopoles in $\mathbb{R}^3$ with arbitrary symmetry breaking at infinity. It shows some applications from the analysis on manifolds with boundary, b-Pseudodifferential operators, and index formulas.

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To my parents, for their love and caring.

Contents

Chapter 1

Introduction.

Monopoles are solitons in $\mathbb{R}^3$ that generalise the concept of magnetic $U(1)$-monopoles introduced by Dirac in 1931. This generalisation comes from considering a principal $SU(N)$-bundle over $\mathbb{R}^3$ and looking for solutions (A, Φ) to the Bogomolny equations

$$* F_A = d_A \Phi. \tag{1.1}$$

Here F_A is the curvature of a connection A, and Φ is the Higgs field, i.e. a section of the adjoint bundle $\operatorname{ad} P$. These equations are supplemented with some boundary conditions and solutions are taken modulo gauge transformations that tend to the identity at infinity.

The purpose of this thesis is to provide the space of monopoles with a manifold structure and to compute the dimension of this manifold.

The manifold structure is proved following the lines of irreducible anti-self-dual Yang-Mills connections over a compact four-manifold. In particular, a Coulomb gauge fixing condition is shown to exist. This provides slices on which the pre-image of 0 by the Bogomolny map gives a local model for the moduli space of these framed monopoles. The dimension of this manifold is then computed via a Callias-type index formula developed by C. Kottke.

Chapter outline

In Chapter 2 we give some background on the theory of monopoles. We revise some concepts of particle physics, recall the well-known monopoles with structure group $SU(2)$ and define the mass and charge of a monopole.

In Chapter 3 we formulate the definition of a monopole in terms of the radial compactification of $\mathbb{R}^3$. Then using weighted b and sc Sobolev spaces introduced by R. Melrose, hybrid Sobolev spaces, similar to those defined in [32], are presented. With the aid of these spaces, a configuration space is introduced and a Coulomb gauge fixing condition is shown to exist. This allows us to have local models on the space of solutions to the Bogomolny equations in the slices produced by the Coulomb gauge fixing condition. Then it is shown that these hybrid Sobolev spaces are just an artefact and that in fact, these monopoles are gauge equivalent to the usual ones. Finally, it is shown that there is a natural hyperkähler metric on these moduli spaces.

In Chapter 4 we compute the dimension of the moduli space of framed monopoles, obtaining four times the sum of the topological and holomorphic charges. This is done using an index formula for asymptotically conic three-manifolds by C. Kottke.

Chapter 2

Review on Monopole theory.

2.1 Electromagnetism, motivation for monopoles.

Before delving into the development and theory of monopoles, some features of classical electromagnetism need to be discussed. Then we will see why monopoles were introduced by Dirac and how in some sense it is natural to consider the generalisation to non-abelian gauge groups.

The magnetic field $\vec{B}$, unlike the electric field $\vec{E}$ (we use the over arrow to distinguish it from a vector bundle and to make more apparent the relation with the physical theory of electromagnetism) belongs to what traditionally was called 'axial vectors'. The components of these types of vectors do not change sign in the same way as the canonical basis in $\mathbb{R}^3$ does under a parity transformation, i.e. under a transformation of $O(3) \setminus SO(3)$ the magnetic field changes the sign with respect to the transformed basis. To make this more intuitive think about a loop in the $\{z = 0\}$-plane carrying some electric current, with an associated magnetic field pointing in the positive z-direction. We now consider the reflection of this system by a plane mirror outside the loop intersecting orthogonally the plane $\{z = 0\}$. With a suitable choice of axis, this would be just the change $x \mapsto -x$. In the new system the magnetic field will point in the negative z-direction instead of its mirror image (the positive z-direction) as a 'true' vector would have done. After this behaviour with orientations, it seems natural then to consider $\vec{B}$ as a 2-form and $\vec{E}$ as a 1-form over $\mathbb{R}^3$. This corresponds to the traditional point of view where both are considered vector fields on $\mathbb{R}^3$ after the application of the Hodge star operator on the magnetic field.

Maxwell's equations for the magneto-static field are $\mathrm{div}\vec{B} = 0$ ($d\vec{B} = 0$) and $\mathrm{curl}\vec{B} = 0$ ($\delta\vec{B} = 0$), that is, the magnetic field is a harmonic 2-form. If we add a time dimension to our space, and consider then $\mathbb{R} \times \mathbb{R}^3$ as our space-time, we can combine both fields into the *electromagnetic field*:

$$F = \vec{B} + \vec{E} \wedge dt. \tag{2.1}$$

To be more precise, if we choose a space-time decomposition $M = \mathbb{R} \times \mathbb{R}^3$, a 2-form F in M corresponds to a pair of vector fields $(\vec{E}, \vec{B})$ in $\mathbb{R}^3$ via,

$$\wedge^2 \left(\mathbb{R} \oplus \mathbb{R}^3\right) = \left(\wedge^1\mathbb{R} \otimes \wedge^1\mathbb{R}^3\right) \oplus \left(\wedge^2\mathbb{R}^3\right) \cong \mathbb{R}^3 \oplus \mathbb{R}^3. \tag{2.2}$$

Using equation (2.1), Maxwell's equations can now be rewritten as

$$dF = 0, \tag{2.3}$$

$$d^*F = *d*F = J. \tag{2.4}$$

Where the Hodge star is with respect to the Lorentzian metric, i.e. signature $(-,+,+,+)$, and volume element $dt \wedge dx^1 \wedge dx^2 \wedge dx^3$.

It can be observed that we recover Maxwell's equation for vacuum when the electric four-current $J = (\rho, \vec{j})$ vanishes, and in this case the electromagnetic field is a harmonic 2-form.

Using the first equation and the fact that $H^2(M, \mathbb{Z}) = 0$, we can consider F being the curvature of an Hermitian connection A on a trivial complex line bundle over M. Dealing with complex bundles is not a problem to obtain reasonable physical solutions, since due to the linearity of Maxwell's equations the real part and complex part of a solution are still solutions. It can be observed that for a fixed time t_0, the pull-back of the curvature under the inclusion $\iota(p) = (t_0, p)$ of $\mathbb{R}^3$ into M is just the magnetic field. However after performing a Lorentzian isometry in M the pullback of the curvature will also have an electric term, showing that these two fields are intrinsically intertwined, and depend on how we chose the splitting of space and time in our manifold.

A crucial difference between the electric field and the magnetic, is that the source for the latter has never been isolated in contrast to the case of an electron. The fundamental objects found to produce a magnetic field (in particular a magnetic dipole) are elementary charged particles with non-zero spin. Nevertheless, the existence of magnetic monopoles is predicted by all Grand Unified Theories [52].

By analogy with electrostatics, if a point-source of magnetic field existed, then assuming it is situated at the origin of $\mathbb{R}^3$, it would have a magnetic charge density $\rho_m = k\delta_0$, where k is a real number representing the magnetic charge. This would produce a harmonic potential of the form:

$$\Phi^D = -i\frac{k}{2r} + \phi, \tag{2.5}$$

where ϕ is a constant. The factor i is introduced so that we can think of Φ^D as a section of a $\mathfrak{u}(1)$-vector bundle over $\mathbb{R}^3 \setminus \{0\}$.

The force due to such a potential is $ik\frac{dr}{2r^2}$, whose Hodge dual is the magnetic field,

$$\vec{B} = i\frac{k}{2}\sin\theta d\theta \wedge d\varphi. \tag{2.6}$$

This is a multiple of the volume form in S^2, which is homotopic to $\mathbb{R}^3 \setminus \{0\}$, and in particular, if the magnetic charge k is an integer, $\frac{i}{2\pi}\vec{B}$ is an integral closed 2-form in S^2. Therefore the magnetic charge parametrises $H^2(S^2, \mathbb{Z})$ and by the isomorphism with $H^1(S^2, \mathcal{C}^*)$ (the sheaf cohomology with coefficients in non-vanishing C^∞ functions) given by the Chern class, it parametrizes complex line bundles over the sphere.

It can be observed, that in the present case the magnetic potential determines the magnetic field uniquely. More precisely, by Kostant's theorem, up to gauge transformations, there exists a unique $U(1)$-bundle over S^2 with connection A, such that its curvature is $\vec{B} = *d\Phi^D$. [1]

From the expression of the curvature, we see that the inclusion of the factor $\frac{1}{2}$ in the magnetic potential is so that the magnetic field is the curvature of a $U(1)$-bundle isomorphic to $\mathcal{O}(-k)$.

The change in phase in the wave function of an electron travelling around a closed loop surounding an infinitely long solenoid, which represents the magnetix flux of a monopole along a line, is proportional to the electric charge of the electron e, times the integral along the path of the magnetic potential, i.e. the connection. In order to have a well-defined wave function, this phase should be a multiple of $2\pi n$, leading to Dirac's quantisation condition [6],

$$ek = 2\pi n, \text{ where } n \in \mathbb{Z}. \tag{2.7}$$

Moreover, it is believed that any physical theory explaining the quantisation of electric charge would predict the existence of monopoles [46].

We have seen that geometrically it is natural for the magnetic charge k to take values in a discrete set, such as the integers, therefore the quantisation condition explains the existence of the smallest electric charge.

Another relation between the magnetic and electric charges can be seen by first studying a symmetry that appears in Maxwell's equations when the electric four-current vanishes. This symmetry is given by the interchange of F and $*F$. It is called a *duality transformation* and in terms of the three-dimensional fields is just:

$$\vec{E} \mapsto -\vec{B}, \tag{2.8}$$

$$\vec{B} \mapsto \vec{E}. \tag{2.9}$$

In the case of existence of monopoles in space-time, there would be a magnetic four-current $J_m = (\rho_m, \vec{j}_m)$ that would have to be added to Maxwell's equations, the extension to this case of the duality transformation would be,

$$\begin{cases} \vec{E} \mapsto -\vec{B}, \; \vec{B} \mapsto \vec{E}, \\ q \leftrightarrow k. \end{cases} \tag{2.10}$$

[1] for non-abelian structure groups this is not the case and a monopole will consist of a pair given by a connection and Higgs field.

This would lead to the very desirable condition that there is a symmetry interchanging the electric and magnetic charge. An immediate consequence of this symmetry together with Dirac's quantisation condition is that in perturbative expansions the coupling constant can be exchanged with its inverse.

More concretely, this symmetry in terms of the $U(1)$-curvature tensor would imply to modify Maxwell's first equation $dF = 0$ by changing the curvature by its Hodge dual and adding the magnetic four-current:

$$* \, dF = J_m. \tag{2.11}$$

Unfortunately, Bianchi identity prevents this equation from holding unless the magnetic current is zero. We will see how this is can be fixed by upgrading the structure group of the bundle, but first it is desirable to revise some more physics from a geometric point of view.

2.2 Physics of Monopoles.

2.2.1 Particle Physics.

General references for the material in this section include [16], [37], Chapter 11 of [45] and [27].

To describe a physical system, in general it is preferable to work with the action S (or Lagrangian L, whose integral over time gives the action) instead of the equations of motion. One of the reasons is that the action for a composite system is obtained by adding the initial actions and the interaction terms, where for the equations of motion the procedure is not that trivial. Moreover, the equations of motion can be obtained by the critical points of the action, i.e. the Euler-Lagrange equations. In particular forces between particles will emerge from these equations and there is no need of adding by hand extra equations like the Coulomb force in Electromagnetism.

In classical mechanics the Lagrangian is given by $L = \mathcal{K} - \mathcal{P}$ (kinetic energy − potential energy). For a free particle the potential is 0 and the Lagrangian has only the free term: $L = \frac{1}{2}m\dot{r}^2$ which gives, via the Euler-Lagrange equations, Newton's law $m\ddot{r} = 0$.

For a relativistic field theory, we can start with the well known relation of the 4-momentum $p_\mu = (E, \vec{p})$ in special relativity,

$$p^\mu p_\mu = m^2. \tag{2.12}$$

Using the quantum correspondence $\vec{p} \mapsto -\nabla$ and $E \mapsto \frac{\partial}{\partial t}$, and substituting in the last equation we obtain the *Klein-Gordon equation* for a scalar field of mass m (by this we mean that once it is quantised, it corresponds to a particle with mass m in natural units where $\hbar = 1 = c$),

$$\partial^\mu \partial_\mu \phi - m^2 \phi = 0. \tag{2.13}$$

This is the Euler-Lagrange equation for the Lagrangian (spatial)-density

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right). \tag{2.14}$$

It can be observed that the quadratic term in the field has a coefficient of the form $-\alpha^2$, with α a positive constant representing the mass. The sign here is important, since there might happen to exist a potential term in the Lagrangian density with a coefficient of the form $+\alpha^2$, and in this case it does not represent the mass, we shall give a more general definition of the mass term in the Lagrangian density in the next subsection.

Until now a field was just a function on $\mathbb{R}^n$, but in a general space-time manifold M, a field is a section of a complex vector bundle $P\times_\rho\mathbb{V}$ associated with a principal G-bundle $\pi : P \to M$ and a representation ρ of its structure group. We will see that in the case of monopoles for example, the Higgs field is a section of $\operatorname{ad} P$.

The Standard Model deals with a vector bundle over Minkowski space which is the tensor product of two vector bundles. The first one is associated to a principal bundle whose structure group is the universal cover of the identity component of the Poincaré group, $R^4 \rtimes \mathrm{Spin}(3,1)$, and the second to a principal bundle with the structure group (known as the internal symmetry group),

$$G_{SM} = [SU(3)\times SU(2)\times U(1)]/\mathbb{Z}_6, \tag{2.15}$$

where $\mathbb{Z}_6$ is certain subgroup of the centre of $SU(3)\times SU(2)\times U(1)$ which acts trivially on all known particles in the Standard Model [4].

As we are not going to deal with dynamics of particles, such as cross-sections, scattering, decays... (these topics can be found in [51] and [12]) we will not need Quantum Field Theory and the infinite dimensional unitary representations of the Poincaré group ($\mathrm{Spin}(3,1)$ is isomorphic to the non-compact group $SL(2,\mathbb{C})$). Instead, we will consider *elementary fermions* to be sections of the bundle

$$S\otimes E, \tag{2.16}$$

over Minkowski space, where S is either S^L, the complex linear representation $(1/2,0)$ of $SL(2,\mathbb{C})$, i.e. its natural action on $\mathbb{C}^2$, or is its complex linear conjugate $(0,1/2) = S^R$, (see Chapter 2 of [31]) and E is a tensor product of vector bundles associated with irreducible representations of the factors in G_{SM}. Table 1 in [4] shows the precise correspondence between elementary fermions and some of the irreducible representations of the structure group $\mathrm{Spin}(3,1)\times G_{SM}$.

The usual derivatives make no sense in the Lagrangian now, and the rule for constructing a valid Lagrangian is to substitute these derivatives by covariant derivatives (so we actually need a connection $\omega\in\Omega^1(P,\mathfrak{g})^G$ in our original principal bundle). The connection on the principal bundle gives a covariant derivative via the operator $d+\rho_*(\omega)$ acting on $C^\infty(P,\mathbb{V})^G$, that in a local trivialisation of P given by a local section s, corresponds to the covariant derivative $d + A$ acting on sections of the associated bundle (here $A = s^*(\omega)$). The connection is a new field introduced in the theory called a *gauge field* and it is responsible for the transmission of an interaction. For example, the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0, \tag{2.17}$$

where ψ is a *Dirac spinor* (a section of the spin bundle $S = S^L \oplus S^R$), comes from the Lagrangian $\mathcal{L} = i\overline{\psi}\gamma^\mu \partial\psi - m\overline{\psi}\psi$, with the γ^μ matrices representing the Clifford action of the standard basis of Minkowski space. After twisting this bundle with a vector bundle E, with structure group G, the spinor is promoted to a section of the twisted bundle $S \otimes E$, where there is a tensor product connection coming from the Levi-Civita connection on S and a connection A on E. In this way, substituting the usual derivatives by covariant derivatives $d_A = \partial_\mu + iqA_\mu$ (it is standard in physics to write it in this way, so in the case of $G = SU(n)$, the matrix A is self-adjoint, so it corresponds to an observable, and the real number q represents the 'charge' of the particle to which the gauge field is coupled to) and expanding out the expression we obtain the *Dirac Lagrangian density*,

$$\mathcal{L}_D = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu A_\mu \psi. \tag{2.18}$$

The quantisation of the gauge fields gives *vector particles*, which are represented by A_μ, which are also called *gauge bosons*. They are the force carriers that mediate a fundamental interaction. Mathematically speaking, they come from the complexified adjoint representation of G, so their number equals the dimension of G. For example, for $G = U(1)$ there is only one gauge boson and corresponds to the photon.

Having introduced a new physical variable in the Lagrangian we should also add to it the free term, which by extension of the electromagnetic theory it will be just $|F_A|^2$. The Euler-Lagrange equations of the new Lagrangian with respect to the gauge field yields the generalised Maxwell's equations

$$d_A^* F = *d_A * F_A = J, \tag{2.19}$$

where J is the current produced by the Dirac particles and it is given by

$$J^\mu = q\overline{\psi}\gamma^\mu \psi. \tag{2.20}$$

Bianchi identity implies $d_A^* J = 0$. In the case of electromagnetism, Stoke's theorem gives the well-known conservation of charge $\int J^0$ over time as long as J^i decays sufficiently fast as $|x| \mapsto \infty$.

2.2.2 Mass in the Standard Model.

To determine the masses of particles appearing in a Lagrangian density, the first step is to identify the potential term $\mathcal{P}(\phi^1, ..., \phi^n)$ and find its minima. The *ground state* or *vacuum* is a section that achieves a minimum of this potential. This is a solution to the Euler-Lagrange equations when the kinetic term of the Lagrangian vanishes, i.e. it is a *static solution*. Then the Lagrangian density is re-written in terms of the fields arising from the Taylor expansion around the ground state.

The second step is to compute the Hessian of the Lagrangian density with respect to non-derivative fields in this expansion (the field derivatives are treated as an independent dynamical entity but we do not differentiate with respect to them) and evaluate at the ground state. In this way we obtain the *mass matrix*,

$$m^{ab} = -\left(\frac{\partial \mathcal{L}}{\partial \varphi_a \partial \varphi_b}\right)_{\phi=\phi_0}, \tag{2.21}$$

where the φ are the fields in the expanded Lagrangian and ϕ_0 is the ground state. The physical fields are the ones that diagonalise this matrix (see chapter 11 of [16]), and the eigenvalues of the mass matrix are the masses of the corresponding particles. It can be observed that what has been denoted as m in the Klein-Gordon Lagrangian corresponds to this mass matrix.

In conclusion, the quadratic terms in the Lagrangian carries the information for the masses of the particles.

This procedure is problematic when the gauge fields introduced are experimentally known to be massive, as the Lagrangian with covariant derivatives incorporated together with the term introduced $|F_A|^2$, lack the necessary quadratic terms to provide these bosons with mass. In fact, in the standard model, the weak interaction is modelled by a $SU(2)$-principal bundle and the three particles carrying the interaction (Z and $W^{\pm}$) are experimentally known to have mass. The naive idea of introducing a term of the form $-\alpha^2 A^2$ in the Lagrangian will not work since this term is not invariant under $SU(2)$ transformations.[2] The solution for generating these masses while preserving the symmetry was given in 1964 in the series papers [9], [18], [21], and lead to a Nobel Prize in Physics in 2013, this procedure is called *the Higgs mechanism*.

We will explain how this mechanism works to provide masses to the gauge bosons in the electroweak model of Glashow-Weinberg-Salam, where

[2] also the mass term for fermions cannot be introduced by hand in the Lagrangian, e.g, the combination of Weyl spinors $\overline{\psi^R}\psi^L$ is not invariant under $SU(2)_L$ (the meaning of the subscript is that it only acts on the left spinors, i.e. sections of S^L). On the other hand for a scalar field ϕ in the fundamental representation of the structure group, the term $\phi\overline{\phi}$ is invariant, i.e. the Lagrangian in the Standard Model can have terms like the Klein-Gordon Lagrangian but not like Dirac's Lagrangian where the mass term is included by hand.

the internal structure group is $SU(2) \times U(1)$, and later we will see how the theory of monopoles uses the same ideas.

Given a Lagrangian describing the dynamics of Hermitian scalar fields and without gauge bosons, *Goldstone's Theorem* [14] states that if the Lagrangian is invariant under the internal structure group $G \subset O(N)$ but the stabiliser of the vacuum is a proper subgroup $H \subset G$ then there are $\dim G - \dim H$ zero eigenstates of the mass matrix, which are massless and spinless bosons known as *Goldstone bosons.*

In the above scenario, i.e. when the Lagrangian restricted to the fluctuations around a ground state is invariant only under the Lie algebra of H instead of the full Lie algebra of G, it is said that *symmetry breaking* occurs.

If gauge fields are now introduced in the Lagrangian, then the *Higgs mechanism* is the process by which a Goldstone boson combines with a gauge boson to give rise to a massive vector particle. In physics jargon 'the gauge boson eats the Goldstone boson and becomes heavy'.

To be more precise, in the original Lagrangian the gauge bosons do not have mass but expanding around a ground state (ϕ_0, A_0), some of the fluctuations of the gauge field obtain mass, that is, evaluating at the ground state the expansion of $|d_A\phi|^2$, there is a quadratic term in the expansion of the connection $A = A_0 + a$ proportional to

$$\langle \rho_*(a)\phi_0, \rho_*(a)\phi_0 \rangle. \tag{2.22}$$

It should be recalled that a constant term in the Lagrangian does not affect the equation of motion and therefore it can be omitted. In particular, this applies to the term $\langle \rho_*(A_0)\phi_0, \rho_*(A_0)\phi_0 \rangle$. There will be $\dim G$ gauge bosons, and if we denote by $\{e_i\}$ a basis of the Lie algebra $\mathfrak{g}$, the mass matrix for the gauge bosons is

$$m^{ij} = \langle \rho_*(e_i)\phi_0, \rho_*(e_j)\phi_0 \rangle. \tag{2.23}$$

This matrix has precisely $\dim\, H$ zero eigenvalues because $\mathrm{Stab}_G\phi_0 = H$. The acquisition of mass for each of these $\dim G - \dim H$ bosons implies that there will be one (longitudinal) degree of freedom extra for each of these bosons, as there are two degrees of freedom for massless bosons and three for massive bosons. This extra degree of freedom comes precisely from the disappearance of a massless Goldstone boson.

We are ready to see how the gauge bosons intermediating the electroweak interaction obtain their masses via the Higgs mechanism. The process for which the elementary fermions obtain their mass is analogous and can be found in chapter 6 of [24].

A term of the form

$$L_H = \|d_A\phi\|^2 - V(\phi), \tag{2.24}$$

needs to be added to the Lagrangian of the electroweak model. This term corresponds to a new particle called the *Higgs boson* that together with the gauge bosons form the *elementary bosons* of the Standard Model. The Higgs boson for the electroweak theory is a section of the vector bundle of complex rank 2, associated with the standard representation of $U(2)$ and the trivial representation $(0,0)$ of $SL(2,\mathbb{C})$. This is coherent with the observation that in the electroweak theory, as in the full Standard Model, there are three massed gauge bosons, and therefore the Higgs boson must be a section of a vector bundle of rank at least four (to provide these three bosons with mass and to leave the photon massless). The connection A is the tensor product connection $W \otimes B$ where the first factor is a connection in a $SU(2)$-bundle and the second of a $U(1)$-bundle.

The potential term in the Higgs Lagrangian L_H, has the form

$$V(\phi) = -\mu_1\|\phi\|^2 + \mu_2\|\phi\|^4, \tag{2.25}$$

with μ_1, μ_2 positive constants that physicists determine from experiments. For obvious reasons this type of potential is known as a 'Mexican-hat type potential'. The electroweak Lagrangian together with the added Higgs Lagrangian, are invariant under the structure group $SU(2) \times U(1)$ and the subgroup H preserving the ground state ϕ_0 with $\|\phi_0\|^2 = \mu_1/2\mu_2$ is isomorphic to $U(1)$. Its embedding on $SU(2) \times U(1)$ depends on the precise choice of vacuum but it is not the copy $\{1\} \times U(1)$ in $SU(2) \times U(1)$ (see chapter 8 in [19] for the explicit form of the action).

The Lie algebra of H, the stabiliser of the Higgs boson at vacuum, is the one dimensional Lie algebra generated by the *electric charge* Q, which is given by the Gell-Mann–Nishijima formula,

$$Q = I_3 + Y/2, \tag{2.26}$$

where Y –the (weak) *hypercharge*– is a complex number corresponding to the generator of the complexified Lie algebra of the $U(1)$ factor and I_3 –the (weak) *isospin*– is the generator in the Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ corresponding to the $SU(2)$ factor.

The gauge boson arising from fluctuations around the vacuum in the direction of the electric charge is the *photon*, and the above formula shows how it is a linear combination of the gauge bosons coming from the connection on the $SU(2) \times U(1)$-bundle. Intuitively, the reason for being a massless particle is due to the fact that the potential energy is not changing in the direction of Q and thus the photon corresponds to a 0-eigenvector of the Hessian of the Lagrangian, i.e., a massless particle.

The other 3 generators of $\mathfrak{u}(2)$ acquire mass by the Higgs mechanism and become the Z and $W^{\pm}$ gauge bosons. In the above picture they correspond to fluctuations in the orthogonal direction to Q and they must have mass since they are subject to a quadratic potential, in other words, they have non-zero eigenvalues for the Hessian of the Lagrangian.

The physical reason why the electric charge is this linear combination and cannot simply be the generator of $\{1\} \times U(1)$, is that particles whose electric charge is zero –like a neutrino– would have an electromagnetic coupling coming from the B factor of the connection appearing in their covariant derivative in the Lagrangian.

The spontaneous symmetry breaking $SU(2) \times U(1) \to U(1)$ is mostly manifest at low energies; at high energies the kinetic term in the Lagrangian dominates the potential energy and the vacuum has less effect.

2.3 Yang-Mills-Higgs Lagrangian.

From now on, G will be a compact, simple and simply connected Lie group. Given a principal bundle over Minkowski space-time with structure group G, let A be a smooth connection and Φ a *Higgs field*, that is, a smooth section of the associated adjoint bundle. The Yang-Mills-Higgs Lagrangian density is given by,

$$\mathcal{L}_{YMH} = \frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\Phi|^2 + V(\Phi), \tag{2.27}$$

with the field strength components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and where we have set the coupling constant equal to 1.

As mentioned previously, the potential should be a Mexican hat type potential (so that around the minima only a subgroup of the symmetry group of the Lagrangian is preserved, and hence spontaneous symmetry breaking occurs), which will be taken to be:

$$V(\Phi) = \frac{\lambda}{2}(|\Phi|^2 - m^2)^2. \tag{2.28}$$

The critical points for the functional (2.27) together with the Bianchi identity are:

$$d_A F_A = 0, \tag{2.29}$$

$$d_A * F_A = - * [\Phi, d_A\Phi], \tag{2.30}$$

$$*d_A * d_A\Phi = 2\lambda\Phi(|\Phi|^2 - m^2). \tag{2.31}$$

The right hand side of the second equation is $*J$ with

$$J = -[\Phi, d_A\Phi], \tag{2.32}$$

the Yang-Mills-Higgs current. These extend Maxwell's equations together with the Coulomb force, encoded in a harmonic electric potential. Precisely in this simplest case where $\lambda = 0$, Prasad and Sommerfield [47] were the first ones to find a solution by giving an explicit solution to the Bogomolny equations (which we will be introduce later). This is why the case where $\lambda = 0$ is known as the *BPS limit*.

The non-abelian magnetic field is $B = *_3 F_A^3$ –we will use the label 3 referring to the restriction to Euclidean $\mathbb{R}^3$, i.e. $\{t = \text{constant}\}$, only when there is danger of confusion with the four dimensional object– and the non-abelian electric field the temporal components of the curvature, in coordinates:

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \quad i,j,k = 1,2,3. \tag{2.33}$$

$$E_i = F_{0i}. \tag{2.34}$$

The Lagrangian comes from the difference of the kinetic energy $\mathcal{K}$ and the potential energy $\mathcal{P}$. Using that $\langle F_A, F_A\rangle = -E_iE_i + \langle B_i, B_i\rangle$[3], they can be written as

$$\mathcal{K} = \frac{1}{2}\int_{\mathbb{R}^3}\left[-\vec{E}^2 - (\nabla_0\Phi, \nabla_0\Phi)\right]d^3x, \tag{2.35}$$

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}^3}\left[-\vec{B}^2 - (\nabla_i\Phi, \nabla_i\Phi) - V(\Phi)\right]d^3x. \tag{2.36}$$

From here one sees that $\mathcal{L}_{YMH}$ generalises the classical electromagnetic Lagrangian by adding the Higgs field terms.

The kinetic energy at a point (A, Φ) comes from considering the L_2-norm of the change in the connection and Higgs field due to a small change in time in the direction orthogonal to the gauge orbit through (A, Φ), that is, discarding the perturbations due to gauge transformations. These projected velocities on the slice determined by the Coulomb gauge condition are $(E, \nabla_0\Phi)$.

In the static case the kinetic energy is zero, and the Lagrangian is just minus the potential energy. That is, the energy density of the static configuration is given by:

$$\frac{1}{2}\left(|F_A^3|^2 + |\nabla_i\Phi|^2 + V(\Phi)\right), \tag{2.37}$$

where the induced metric from $\mathbb{R}^3$ is being used on forms and the multiple $-\frac{1}{2}\operatorname{Tr}(AB)$ of the killing form $\langle A, B\rangle$ used for the $\mathfrak{g}$-valued part. We will restrict attention to the static case and therefore consider our connection and Higgs field relative to a bundle over $\mathbb{R}^3$.

This energy density in the BPS limit, is the Lagrangian density that will be considered, that is,

$$\mathcal{E}(A, \Phi) = \frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\Phi|^2. \tag{2.38}$$

[3] $\langle F_A, F_A\rangle = \frac{1}{2}\sum_{\mu,\nu}F_{\mu\nu}F^{\mu\nu} = -F_{0i}F_{0i} + \frac{1}{2}\sum_{i,j}F_{ij}F_{ij} = -E_iE_i + \langle F_A^3, F_A^3\rangle = -E_iE_i + \langle B_i, B_i\rangle$ where in the last equality we used that the Hodge star is an isomorphism, or equivalently, the component expression (2.33) together with $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

It can be observed that an equipartition theorem (or a scaling argument known as Derrick's theorem [37]) shows that a pure Yang-Mills gauge theory in dimension less than 4 has no non-trivial smooth, with finite energy critical points, and that in dimension 4, solutions to Yang-Mills-Higgs are gauge equivalent to pure Yang-Mills (Corollary 2.3 in [27]). It is then natural to include a Higgs field when studying gauge theories over a based manifold of dimension 3.

In fact, the critical points of this Langrangian density are formally the same as those of (2.27) in the BPS limit, where now the curvature is the restriction of the full curvature to a 3-dimensional Euclidean space (as mentioned above we have suppressed the index '3' in the curvature). The restriction to the 3-hypersurface is implemented by considering fields independent of time together with connections with vanishing time-component.

2.4 Bogomolny equations.

Following the discussion from the last section, in order to study the static case of the Yang-Mills-Higgs theory in the BPS limit, we should consider a principal G-bundle P over $\mathbb{R}^3$ and the Lagrangian (2.38).

A solution to the *Bogomolny equations*

$$* F_A = \pm d_A \Phi, \tag{2.39}$$

consists of a pair (A, Φ), where A is a smooth connection on P and Φ, the Higgs field, is a smooth section of the associated adjoint bundle

$$\mathrm{ad}\, P = P \times_{Ad} \mathfrak{g}. \tag{2.40}$$

We will see in section 2.5.1 that solutions to the Bogomolny equations are critical points of (2.38).

Solutions to the Bogomolny equations with the $+$ $(-)$ sign can be obtained by *dimensional reduction* of the self-dual (anti self-dual) Yang-Mills equations in Euclidean space $\mathbb{R}^4$ [22]. This basically consists of assuming that the connection does not depend on one of the variables and renaming one of its components as the Higgs field. To be more precise, let $\pi : E \to \mathbb{R}^4$ be a vector bundle which is invariant under translations on the last coordinate of $\mathbb{R}^4$, that is, if p is the projection $(x^1, x^2, x^3, x^4) \mapsto (x^1, x^2, x^3, 0)$, the bundle $p^*(E\mid_{\mathbb{R}^3\times\{0\}})$ is isomorphism to E via

$$\left((x^1, x^2, x^3, x^4), e\right) \mapsto \left((x^1, x^2, x^3, x^4), v\right), \tag{2.41}$$

where e in $E\mid_{\mathbb{R}^3\times\{0\}}$ is given by $((x^1, x^2, x^3, 0), v)$. Let ∇ be an invariant connection under these translations satisfying the selfdual equations $F_\nabla = *F_\nabla$. Fix a trivialisation so that $\nabla = d + A$ with,

$$A = \sum_{i=1}^{4} A_i(x^1, x^2, x^3) dx^i. \tag{2.42}$$

Then on the bundle $p^*(E\mid_{\mathbb{R}^3\times\{0\}})$ there are two natural connections: the pull-back connection $p^*\nabla^{(3)}$, where $\nabla^{(3)}$ is the connection induced by ∇ over $E\mid_{\mathbb{R}^3\times\{0\}}$, and the connection $\nabla^{(4)}$ coming from the isomorphism with E.

The pull-back connection has $(p^*A)_4 = 0$ and the other three components do not depend on x^4, therefore the connection $\nabla^{(4)}$ differs from the pull-back connection in a term of the form ψdx^4, where $\psi = \psi(x^1, x^2, x^3)$ takes values on the Lie algebra $\mathfrak{g}$ and as it does not depend on the fourth coordinate we can write it as the pull-back of a Higgs field. Therefore on the pull-back bundle $p^*(E\mid_{\mathbb{R}^3\times\{0\}})$ this connection has the form,

$$\nabla^{(4)} = p^*(\nabla^{(3)}) + p^*(\Phi)dx^4, \tag{2.43}$$

and its curvature is,

$$F_{\nabla^{(4)}} = F_{p^*\nabla^{(3)}} + p^*(\nabla^{(3)}\Phi) \wedge dx^4. \tag{2.44}$$

With the usual orientations, we see that this curvature on the pull-back bundle[4] is self-dual if and only if the Bogomolny equations hold,

$$F_{\nabla^{(4)}} = *F_{\nabla^{(4)}} \Leftrightarrow *_{(3)}F_{\nabla^{(3)}} = \nabla^{(3)}\Phi. \tag{2.45}$$

Conversely, the Bogomolny equations provide a solution to the self-dual Yang-Mills equations with translational invariance. Given a connection on a principal bundle over $\mathbb{R}^3$, consider the associated vector bundle E and associated connection $\nabla^{(3)}$. In the pull-back bundle p^*E we consider the connection (2.43) which is clearly invariant under x^4-translations, and again its curvature is self-dual if and only if the Bogomolny equations hold.

It can be observed that because of the translational invariance, the instantons coming from these self-dual equations will not have finite energy and according to our next definition cannot be considered as monopoles.

2.5 Definition of monopoles in $\mathbb{R}^3$.

In order to talk about monopoles we need to introduce the *gauge group* $\mathscr{G}$ of a G-bundle $P \to M$. This is the group of bundle automorphisms that lift the identity on M, in other words, G-equivariant [5] diffeomorphisms of P lifting the identity map on M.

It should be recalled (see section 2 in [1]) that there is a one-to-one correspondence between gauge transformations and smooth sections of the bundle of groups $AdP = P \times_{Ad} G$ over M (which can be identified with smooth maps $f : P \to G$ satisfying the equivariant condition $f(p\cdot h) = h^{-1}f(p)h$ with $h \in G$).

[4] we can consider the self-dual equations in this pull-back bundle instead of E since the equations are invariant under gauge transformations, i.e. bundle isomorphisms.

[5] $F(p\cdot h) = F(p)\cdot h$ for every $p \in P$ and $h \in G$.

Since the Lie group G is 2-connected, that is, its first two homotopy groups vanish identically, then its classifying space BG is 3-connected. The isomorphism classes of G-bundles over M are classified by $[M : BG]$, the homotopy classes of maps from M to BG, but if M is a n-dimensional CW-complex and Y is n-connected then $[M : Y] = 0$.

Therefore we conclude that if G is a simply connected Lie group (hence 2-connected), a G-bundle over a three dimensional manifold X is trivialisable, see Theorem 13.1 in [25]. In this case, the gauge group is just

$$\mathscr{G} = C^{\infty}(X, G) . \tag{2.46}$$

Gauge transformations act naturally on connections and sections of associated bundles. In particular on sections of $\operatorname{ad} P$ it acts via the adjoint representation that for matrix groups is given by,

$$g \cdot \Phi = g \Phi g^{-1}. \tag{2.47}$$

Similarly, the connection induces a covariant derivative in an associated bundle E and the action of the gauge group seen as an automorphism of E is given by,

$$g \cdot d_A = d_{g \cdot A} = g \circ d_A \circ g^{-1} = d_A - (d_A g) g^{-1}, \tag{2.48}$$

where the last 'd_A' is the covariant derivative induced in the bundle $\operatorname{End} E$.

It is easy to check that the Bogomolny equations are gauge invariant and therefore the gauge group acts on the space of solutions.

On the other hand, the transformation $(A, \Phi) \leftrightarrow (A, -\Phi)$, or a change of orientation on the base manifold, transforms a solution to the Bogomolny equations using the '$+$' sign to other using the '$-$' sign and vice-versa. Thus it is enough to consider only one of the signs in the equations.

Definition 2.5.1. Given an $SU(N)$-bundle over $\mathbb{R}^3$, a *monopole* in $\mathbb{R}^3$ is an equivalence class of solutions to the Bogomolny equations

$$* F_A = d_A \Phi \tag{2.49}$$

under gauge transformations, subject to the following conditions:

As it is required for solitons, we assume that they have finite energy:

$$\int_{\mathbb{R}^3} \mathcal{E}(A, \Phi) d^3 x < \infty, \tag{2.50}$$

where $\mathcal{E}(A, \Phi)$ is the energy density from equation (2.38). We also require that in the limit where r, the distance from the origin, tends to infinity both the Higgs field and the connection have limits depending smoothly on the angular variables. These limits will be denoted as ϕ and A_0 respectively,

$$\Phi \to \phi, \, A \to A_0 \quad \text{as} \quad r \to \infty. \tag{2.51}$$

Moreover, there is a trivialisation such that the Higgs field has a uniform asymptotic expansion given by

$$\Phi = \phi - \frac{1}{2r}\gamma_m + O(r^{-(1+\epsilon)}), \tag{2.52}$$

for some positive real number ϵ. It is required that the *mass section* ϕ is nowhere-vanishing and that it commutes pointwise with the *magnetic charge section* γ_m.

It is also required that in the same trivialisation where the Higgs field has the above asymptotic expansion, for large enough r:

$$d_A\Phi = \frac{1}{2r^2}\gamma_m dr + O(r^{-(2+\epsilon)}). \tag{2.53}$$

Some comments about these conditions are in order:

1. Similar conditions are imposed in [29] and [42], where the finite energy condition is supplemented with the following conditions: along each ray out of the origin, there is a gauge and an $\epsilon > 0$ such that for sufficiently large r:

$$\Phi = \mu - \frac{1}{2r}k + O(r^{-(1+\epsilon)}), \tag{2.54}$$
$$d_A\Phi = \frac{1}{2r^2}kdr + O(r^{-(2+\epsilon)}), \tag{2.55}$$

   where $\mu \neq 0$ and $[\mu, k] = 0$. In Section III of [42] it is shown that the smooth pair (ϕ, γ_m) is determined by (μ, k) and therefore these conditions are equivalent to ours.

2. The commutativity

$$[\phi, \gamma_m] = 0, \tag{2.56}$$

   is necessary as it is implied when the Bogomolny equations hold. Other implications that the fulfilment of the Bogomolny equations impose are

$$d_{A_0}\phi = 0, \tag{2.57}$$
$$d_{A_0}\gamma_m = 0, \tag{2.58}$$
$$-\frac{1}{2}\gamma_m = *_{S^2_\infty}F_{A_0}. \tag{2.59}$$

   The notation S^2_∞ comes from performing the radial compactification of $\mathbb{R}^3$ and will be explicitly defined in the next chapter. These equations imply a reduction of the bundle at infinity to a principal bundle with structure group the stabiliser of the mass section.

3. It is clear that the above conditions are independent of the choice of origin. The origin could have been substituted by any other point $\vec{r_0}$ in $\mathbb{R}^3$, as the well-known asymptotic expansion of the 'Newtonian potential' (in terms of the Legendre polynomials) shows:

$$\frac{1}{|\vec{r}-\vec{r_0}|} = \sum_{l=0}^{\infty} \frac{r_0^l}{r^{l+1}} P_l(\cos\theta) = \frac{1}{r} + O(r^{-2}). \tag{2.60}$$

4. In [53] Taubes showed via a gluing construction that if all of the topological charges (see Definition 2.7.2 below) are non-negative and at least one of them is positive, then there are an infinite number of distinct gauge inequivalent solutions. Later, in [42] it was shown that for the asymptotic conditions in the first remark, there is always a conjugate element to k by the centraliser of μ such that the topological charges are non-negative. In the case of maximal symmetry breaking, as we will see in Section 2.7.1, the topological charges –if we assume for simplicity that the purely imaginary eigenvalues $\{-i\phi_j\}_{j=1}^{N-1}$ of ϕ are ordered in such a way that $\phi_i < \phi_{i+1}$– are given by $n_a := \sum_{j=1}^{a} \gamma_j$ where $a = 1, ..., N-1$ and $\{i\gamma_j\}_{j=1}$ are the eigenvalues of γ_m.

5. In the case of $SU(2)$-monopoles, Jaffe and Taubes showed [27] that the finite action condition together with the Bogomolny equations, imply the above asymptotic expansions.

The following notation will be used frequently in this thesis: by condition (2.52) there exists an R such that for $|x| > R$ the Higgs field does not have any zeroes. In general, U will denote the complement of the closed ball of radius R. The dependence of U on R will not be written explicitly as they are all homotopic equivalent (to S^2_∞) and these sets will be mainly considered when dealing with topological characterisations of monopoles at infinity.

In the simpler case of a $U(1)$-bundle, the Bogomolny equations become:

$$d\Phi = *F_A. \tag{2.61}$$

In this way one recovers *Dirac's monopole* on $\mathbb{R}^3 \setminus \{0\}$ if Φ and F_A are defined as in equations (2.5) and (2.6) respectively. It can be observed that $U(1)$-monopoles over the whole base manifold $\mathbb{R}^3$, are not interesting since the above equation together with Bianchi identity imply that Φ is harmonic over $\mathbb{R}^3$. Therefore if Φ is bounded then it must be constant by Liouville's theorem (see page 30 in [10]).

Definition 2.5.2. A *framing* is a choice of a smooth mass section ϕ and limiting connection A_0. The space of *framed monopoles* are the solutions to the Bogomolny equations satisfying the above boundary conditions with a fixed framing, determining the asymptotic mass and magnetic charge section, modulo the *reduced gauge transformations*,

$$\mathscr{G}_0 := \{g \in \mathscr{G} : \lim_{r\to\infty} g = 1\}. \tag{2.62}$$

These framed monopoles are equivalent to those defined fixing the values of the mass section and connection in certain direction from the origin, modulo the gauge transformations that tend to the identity along the chosen direction in $\mathbb{R}^3$ (see Section III.A in [42]). The space of framed monopoles is not empty, it is shown in [28] that there are large families of framed monopoles.

Remark 2.5.3. When the group acting among solutions to the Bogomolny equations is the reduced gauge group, the framing is not equivalent to a choice of mass and magnetic sections at S^2_∞. This can be understood by splitting the the trivial bundle over U into line bundles, where the curvature F_{A_0} induces a curvature on each one of them. From the fibration

$$\mathcal{J}_Y \to \mathcal{A}/\mathscr{G} \to c_L, \tag{2.63}$$

where $\mathcal{J}_Y = \frac{H^1(Y;\mathbb{R})}{H^1(Y;\mathbb{Z})}$ is the Jacobian torus of Y and c_L is the set of closed 2-forms representing the Chern class of a line bundle L over Y (which in our case $Y = S^2_\infty$), one can deduce that the curvature in each line bundle determines a unique connection up to gauge equivalence over S^2_∞, but the gauge transformations here are set to be the identity.

When the value of the mass section ϕ at a point in the sphere at infinity is a regular element, i.e. its centraliser is a maximal torus in G, it is said to be the case of *maximal symmetry breaking*. This definition is independent of the point at the sphere at infinity chosen: seeing the mass section as an equivariant function $P|_{S^2_\infty} \to \mathfrak{g}$, the condition of being covariant constant (2.57), implies that its value along horizontal paths is constant, since elements in a fibre over a point are conjugated the statement follows.

The space of framed monopoles comes in topological families indexed by integer numbers. Before explaining how this works, we need to have a closer look at the boundary conditions.

Equation (2.57) implies that $|\phi|$ is a constant m, that for reasons that will be explained later, it is known as the *mass of the monopole*. This mass can be set to 1 at the expense of a dilation of the metric on $\mathbb{R}^3$. Specifically, for m a positive real number, consider the new metric

$$\tilde{g} = m^2 g, \tag{2.64}$$

then the new volume form is $\widetilde{dvol} = m^3 dvol$ and the product of two p-forms with respect to the new metric is m^{-2p} times the original Euclidean metric. In summary,

$$\tilde{*} = m^{3-2p} * \tag{2.65}$$

and therefore (A, Φ) is a monopole using the metric $\tilde{g}$ if and only if $(A, m\Phi)$ is a monopole with the metric g.

The positiveness of the mass is guaranteed by our assumption of nowhere-vanishing mass section, and this requirement is justified by the following well-known lemma.

Lemma 2.5.4. *A solution to the Bogomolny equations cannot have zero mass unless it is a trivial solution.*

Proof. Taking the Laplacian[6] δd of the function $|\Phi|^2 = -\frac{1}{2}\operatorname{Tr}(\Phi^2)$, and making use of the Bogomolny equations and the Bianchi identity we obtain,

$$\begin{aligned}\triangle|\Phi|^2 &= -\delta\operatorname{Tr}(d_A\Phi\Phi) = *d\operatorname{Tr}(*d_A\Phi\Phi) = *(\operatorname{Tr}(d_AF_A\Phi) + \operatorname{Tr}(F_A \wedge d_A\Phi)) \\ &= *\operatorname{Tr}(F_A \wedge *F_A) = -2|F_A|^2 \leq 0, \end{aligned} \tag{2.66}$$

therefore $|\Phi|^2$ is a subharmonic function, and by the maximum principle, $|\Phi|^2$ cannot have interior maxima unless it is constant (intuitively it is clear as its Hessian cannot be negative definite). So if $\Phi \to 0$ when $r \to \infty$, we must have a null Higgs field everywhere. This together with a flat connection in $\mathbb{R}^3$ constitutes the trivial monopole. □

Because of the above bijection, monopoles with mass 1 can be considered without any loss of generality.

2.5.1 Bogomolny Energy Bound.

In this subsection we show how the second order PDEs coming from the critical points of the Yang-Mills-Higgs functional, can be reduced to a first order equation when one considers only minimisers of the energy density functional. Interestingly, Taubes [55] showed, using Morse theory, that there are smooth critical points with finite action of the $SU(2)$ Yang-Mills-Higgs equations in the BPS limit which does not satisfy the first-order Bogomolny equations. More generally, for structure group $SU(n)$, Ioannidou and Sutcliffe in [26], using harmonic maps into $\mathbb{C}P^{n-1}$, showed that there are spherically symmetric critical points which are not solutions to the Bogomolny equations.

We start with the observation that solutions to the Bogomolny equations are critical points of the energy density (2.38) in the BPS limit:

$$F_A = \pm * d_A\Phi \implies \begin{cases} d_A * F_A = \pm d_A d_A \Phi = [F_A, \pm\Phi] = [\pm * d_A\Phi, \pm\Phi] \\ \qquad\qquad = - * [\Phi, d_A\Phi] \\ d_A^* d_A \Phi = 0 \text{ (from Bianchi identity).} \end{cases}$$

[6] the sign convention for the Laplacian is such that it is non-negative definite, so in $\mathbb{R}^n$ is given by $-\sum \partial_i^2$.

In fact the following proposition shows that solutions to the Bogomolny equations are actually minimisers among the pairs (A, Φ) having the same limiting projection of the curvature into the Higgs field:

$$\lim_{r\to\infty} \int_{S_r^2} \langle \Phi, F_A \rangle d^2x, \tag{2.67}$$

where S_r^2 stands for the two-sphere of radius r. Later we will see that this term has a topological nature.

Proposition 2.5.5. *If the domain of the energy functional $\mathcal{E}(A, \Phi)$ in the BPS limit is the smooth pairs (A, Φ) with finite energy and that satisfy the boundary conditions (2.52) and (2.53), (the same conditions imposed on monopoles except for being solutions to the Bogomolny equations) with a fixed value of the integral (2.67), then solutions to the Bogomolny equations are minimisers.*

Proof. It can be observed that

$$\int_{\mathbb{R}^3} \left(\mathcal{E}(A, \Phi) - \frac{1}{2} |F_A \mp *d_A\Phi|^2 \right) d^3x = \pm \int_{\mathbb{R}^3} d\langle \Phi, F_A \rangle. \tag{2.68}$$

This result comes from expanding

$$(F_A \mp *d_A\Phi, F_A \mp *d_A\Phi) = (F_A, F_A) + (d_A\Phi, d_A\Phi) \mp 2(F_A, *d_A\Phi), \tag{2.69}$$

and using the Bianchi identity to write

$$\int_{\mathbb{R}^3} \langle F_A, *d_A\Phi \rangle d^3x = \int_{\mathbb{R}^3} F_A \wedge d_A\Phi = \int_{\mathbb{R}^3} d_A \langle F_A, \Phi \rangle = \int_{\mathbb{R}^3} d\langle F_A, \Phi \rangle, \tag{2.70}$$

where in the last equality it was used that $\langle F_A, \Phi \rangle$ is a 2-form (using the Killing form, the product on the $\operatorname{ad} P$ part is merely being evaluated). By Stokes' theorem this last integral coincides with the value of the integral that has been fixed. □

In summary, the energy for monopoles in the domain of the previous proposition is precisely the fixed value of the above integral,

$$E(A, \Phi) = \int_{\mathbb{R}^3} \mathcal{E}(A, \Phi) = \lim_{r\to\infty} \int_{S_r^2} \langle \Phi, F_A \rangle d^2x. \tag{2.71}$$

This energy, which is the rest mass of the monopole, for the case of $SU(2)$-monopoles is a multiple of $|\phi|$ and the charge, while for $SU(N)$-monopoles the value of this integral intertwines the eigenvalues of the Higgs field and curvature at 'infinity' (2.113).

We will discuss a bit further the concept of charge of a monopole but first it is convenient to start with the simplest case of structure group $SU(2)$ and then later discuss the analogue for $SU(N)$-monopoles.

2.6 $SU(2)$-Monopoles.

The singularity of the Dirac monopole can be overcome if we consider 'tHooft and Polyakov type monopoles with structure group $SU(N)$. The first non-abelian monopoles studied used the structure group $SU(2)$. These behave like Dirac monopoles at large distances from the origin, where they are centred, and contrary to Dirac's monopole, they are smooth in the whole $\mathbb{R}^3$ having finite energy.

Before we explore a particular solution we expose some general principles of these monopoles.

As mentioned above, the charge $k \in \mathbb{N}_0$ of an $SU(2)$-monopole can be defined via the integral (2.67):

$$\lim_{r\to\infty} \int_{S^2_r} \langle \Phi, F_A \rangle d^2x = 2\pi m k. \tag{2.72}$$

In fact, it can be assumed that the charge is non-zero, as otherwise the monopole would be trivial, i.e. consisting of a flat connection and a covariantly constant Higgs field. We will next give a geometrical meaning of the charge and see why it must be a natural number.

The asymptotic condition (2.57) implies that the eigenvalues of ϕ are constant, and with the invariant norm on the Lie algebra

$$|\phi|^2 = -\frac{1}{2} \operatorname{Tr} \phi^2, \tag{2.73}$$

the eigenvalues $\{im, -im\}$ are determined by the value of the mass. Moreover, the eigenvectors of ϕ are covariant constant and the eigenvectors of the Higgs field in the region U consisting of those points whose distance to the origin is greater than a certain fixed r_0, split the associated vector bundle $E = P \times_{Id} \mathbb{C}^2$, i.e.,

$$E|_U = L \oplus L^{-1}, \tag{2.74}$$

where L is the line bundle corresponding to the im-eigenvector of Φ along U. The covariant derivative d_A induced in this associated bundle, splits as the direct sum of the covariant derivatives induced in these line bundles, plus terms of order $O(r^{-(2+\epsilon)})$ (see (3.33) where a similar argument is done). Writing $d_A = d + A$, this translates to

$$A = \begin{pmatrix} A_L & 0 \\ 0 & -A_L \end{pmatrix} + O(r^{-(2+\epsilon)}), \tag{2.75}$$

where A_L is the $\mathfrak{u}(1)$-valued 1-form representing the connection on L. Hence, the limiting curvature has the form $\operatorname{diag}(F_{A_L}, -F_{A_L})$, where F_{A_L} is the curvature of L. Then we see that the integral (2.67) is just a multiple of the first Chern number of L:

$$\lim_{r\to\infty} \int_{|x|=r} \langle \Phi, F_A \rangle =: \int_{S^2_\infty} \langle \Phi, F_A \rangle = -im \int_{S^2_\infty} F_{A_L} = -2\pi m c_1(L). \tag{2.76}$$

Along the set U we will denote the normalised Higgs field as

$$\hat{\Phi} := \frac{\Phi}{|\Phi|}. \tag{2.77}$$

Since the mass does not take the value 0 for large enough distances from the origin, we have a map

$$\hat{\Phi} : S^2_\infty \to \mathfrak{su}(2) \setminus \{0\} \approx S^2, \tag{2.78}$$

given by $\lim_{|x|\to\infty} \frac{\Phi}{|\Phi|}(x)$. The homotopy type of this map is called the *topological charge* of the monopole, and by the Poincaré-Hopf theorem, is the number of zeroes counted with multiplicity of the Higgs field on $\mathbb{R}^3$ (see Figure 2.3 for an explicit example).

On the other hand, when the Lie algebra of the structure group is $\mathfrak{su}(2)$, the commutation relation (2.56) implies that

$$\gamma_m = \hat{k}\phi, \tag{2.79}$$

in principle $\hat{k}$ is a function depending on the angular variables, but condition (2.58) implies that it must be constant. Using (2.59) to evaluate (2.67) we deduce,

$$\hat{k} = -\frac{k}{m}. \tag{2.80}$$

The following result shows that the function k is minus the degree of the map $\hat{\Phi}$ and therefore an integer.

Proposition 2.6.1. *The topological charge N of the monopole is the Chern number of the line bundle of the im-eigenspace of Φ over U, i.e.,*

$$N = -\frac{1}{2\pi} \lim_{r\to\infty} \int_{|x|=r} \langle \hat{\Phi}, F_A \rangle. \tag{2.81}$$

Proof. The Hurewicz homomorphism $\mathcal{H} : \pi_k(X, x) \to H_k(X)$, is given by associating to the homotopy class $\{f : S^k \to X\}$ the push-forward, by any of the representatives in this class, of the fundamental class of the sphere: $f_*[S^k]$. When the space X is $(k-1)$-connected the Hurewicz homomorphism is in fact an isomorphism (see Theorem 4.32 in [20]). Since the two-sphere is 1-connected there is an isomorphism $\pi_2(S^2) \cong H_2(S^2)$. The homotopy class $\{\hat{\Phi}\}$ corresponds to $\hat{\Phi}_*[S^2_\infty] \in H_2(\mathfrak{su}(2) \setminus \{0\})$. We will see that $\langle \hat{\Phi}_*[S^2_\infty], \omega \rangle$, where ω is the volume form of the unit 2-sphere in $\mathfrak{su}(2) \setminus \{0\}$ coincides with the integral of the Lie algebra projection of the curvature into the Higgs field.

In a global trivialisation of the bundle, where the monopole asymptotic conditions hold, the relation $\gamma_m = \hat{k}\phi$ yields,

$$|\Phi|^2 = m^2 \left(1 - \frac{\hat{k}}{2r}\right)^2 + \mathcal{O}(r^{-1-\epsilon}). \tag{2.82}$$

Using this equation and the isomorphism of Lie algebras $(\mathfrak{su}(2), [\cdot,\cdot]) \cong (\mathbb{R}^3, \times)$,[7]

$$[\Phi, [d\Phi, \Phi]] = d\Phi|\Phi|^2 - \Phi\langle\Phi, d\Phi\rangle = d\Phi|\Phi|^2 - \Phi\frac{m^2\hat{k}}{2r^2}dr + \mathcal{O}(r^{-2-\epsilon}). \quad (2.83)$$

With this observation we deduce that any connection such that the boundary condition (2.53) holds, for large distances from the origin where $|\Phi|$ does not vanish must be of the form:

$$A = \frac{1}{|\Phi|^2}[d\Phi, \Phi] + a \otimes \Phi + \mathcal{O}(r^{-2-\epsilon}), \quad (2.84)$$

where a is an arbitrary 1-form. We should compute now the curvature of this connection $F_A = dA + \frac{1}{2}[A, A]$, the second term is:

$$\frac{1}{2}[A, A] = \frac{1}{|\Phi|^2}[[d\Phi, \Phi], a\Phi] = a \wedge d\Phi - \frac{1}{|\Phi|^2}a \wedge \Phi\langle\Phi, d\Phi\rangle, \quad (2.85)$$

and the first

$$dA = -\frac{1}{|\Phi|^2}[d\Phi, d\Phi] + \Phi da + d\Phi \wedge a + d(\frac{1}{|\Phi|^2})[d\Phi, \Phi]. \quad (2.86)$$

The product of the normalised Higgs field $\hat{\Phi}$ with the curvature of this connection and in particular with that of a monopole is therefore

$$\begin{aligned}\langle\frac{\Phi}{|\Phi|}, F_A\rangle &= -\frac{1}{|\Phi|^3}\langle\Phi, [d\Phi, d\Phi]\rangle + |\Phi|da - \frac{1}{|\Phi|}a \wedge \langle\Phi, d\Phi\rangle \\ &+ d(\frac{1}{|\Phi|^2})\langle\frac{\Phi}{|\Phi|}, [d\Phi, \Phi]\rangle + \mathcal{O}(r^{-2-\epsilon}) \\ &= -\frac{1}{|\Phi|^3}\langle\Phi, [d\Phi, d\Phi]\rangle + d(|\Phi|a) + \mathcal{O}(r^{-2-\epsilon}), \quad (2.87)\end{aligned}$$

where we have used that $\Phi \perp [d\Phi, \Phi]$ and that $d(|\Phi|a) = |\Phi|da + \langle\frac{\Phi}{|\Phi|}, d\Phi\rangle \wedge a$.

$$\begin{aligned}2\pi k = \lim_{r\to\infty}\int_{S_r^2}\langle\frac{\Phi}{|\Phi|}, F_A\rangle &= \lim_{r\to\infty}\int_{S_r^2} -\frac{1}{|\Phi|^3}\langle\Phi, [d\Phi, d\Phi]\rangle \quad (2.88) \\ &= \lim_{r\to\infty}\int_{S_r^2} -\langle\hat{\Phi}, [d\hat{\Phi}, d\hat{\Phi}]\rangle, \quad (2.89)\end{aligned}$$

where again the perpendicularity property $\Phi \perp [\Phi, \cdot]$ has been used.

On the other hand,[8]

$$N := \langle\hat{\Phi}_*[S^2_\infty], \omega\rangle = \frac{1}{2\pi}\int_{S^2_\infty}\langle\hat{\Phi}, [d\hat{\Phi}, d\hat{\Phi}]\rangle, \quad (2.90)$$

Combining both equations we obtain that $N = -k$. □

[7] that is, in coordinates: $[d\Phi, \Phi] = \partial_\mu\Phi^a\Phi^b[\tau_a, \tau_b] \otimes dx^\mu = \partial_\mu\Phi^a\Phi^b\epsilon_{abc}\tau_c \otimes dx^\mu$.

[8] cf. equation (5.7) on page 44 in [27], where $de \wedge de = \frac{1}{2}[de, de]$ and the inner product on the Lie algebra is 4 times our inner product.

Over the region U, this proposition allows us to interpret the Lie algebra projection of the curvature into the Higgs field as the electromagnetic tensor, and in this way generalises the abelian case of Maxwell's curvature tensor.

The general principles that have been stated can be checked with a concrete example of a monopole. The first explicit monopole solution was found by Prasad and Sommerfield [47] and corresponds to the case where the structure group is $SU(2)$. The Prasad-Sommerfield solution is spherically symmetric and has charge one. It was found using the Ansatz of a 'spherically symmetric' monopole, that is, after the action of the group $SO(3)$ the solution to the Bogomolny equations changes by a gauge transformation and hence the monopole is invariant under this action. They realised that this can be accomplished if when the fields are written in terms of the basis $T = \{\tau^a\}$ of the Lie algebra, Φ is a radial function times $\hat{r} \cdot T$ and A is a radial function times $\hat{r} \times T$. In this way, a rotation of $\hat{r}$ should have the same effect as the conjugation on T due to the infinitesimal gauge action. Specifically, they looked at:

$$\Phi^a = \hat{r}^a h(r)/r, \tag{2.91}$$

$$A_i^a = \epsilon_{aij} \frac{\hat{r}^j}{r}(1 - \alpha(r)). \tag{2.92}$$

Finiteness of energy implies $h(r)/r \to 1$ and $\alpha(r) \to 0$ as $r \to \infty$. Moreover, writing down the Bogomolny equations for these fields shows that in order to avoid singularities at the origin it is necessary to have $h(0) = 0$ and $\alpha(0) = 1$. With these conditions, the explicit solution found by Prasad-Sommerfield [47] is

$$h(r) = Cr \coth(Cr) - 1, \tag{2.93}$$

$$\alpha(r) = \frac{Cr}{\sinh(Cr)}, \tag{2.94}$$

where C a physical constant that by a re-scaling of units we can take it to be 1.

This solution is important because it exemplifies the properties of a monopole, and it was used in the gluing constructions by Taubes which allowed him to show that the moduli space of framed monopoles of charge k is a hyperkähler manifold of dimension $4k$ where k is the charge.

Some pictures of this spherically symmetric solution can be drawn, see the figures below.

2.7 Topological and Holomorphic charges.

In this section we will generalise the charge concept seen in the previous section for $SU(2)$-monopoles, to the case where the structure group is $SU(N)$.

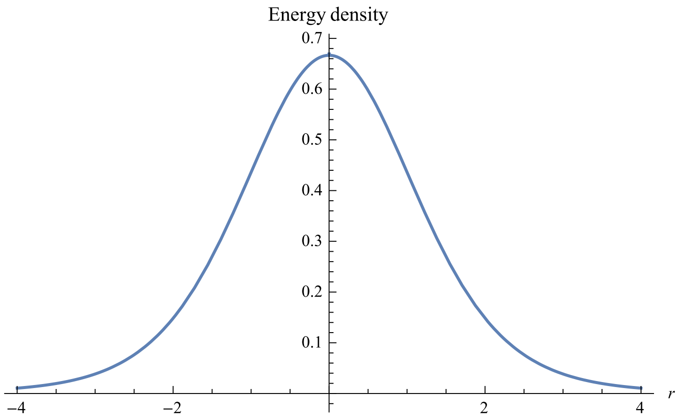

Figure 2.1: Energy density of the $k = 1$ Prasad and Sommerfield monopole.

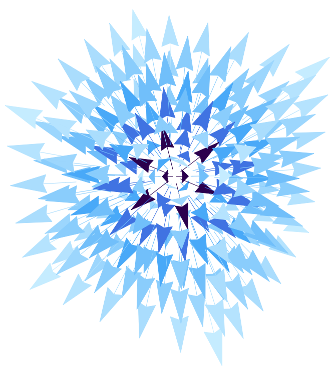

Figure 2.2: Hedgehog shape of the Higgs field.

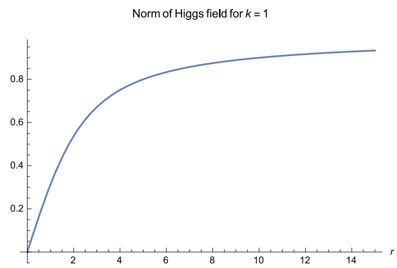

Figure 2.3: Higgs field's norm.

The main difference here is that we are not necessarily in a case of maximal symmetry breaking.

We will use the notation of $\mathfrak{g}$ for the Lie algebra of $G = SU(N)$ and of $\mathfrak{s}$ for the Lie algebra of $S := \mathrm{Stab}(h)$, the stabiliser of the value of Higgs field at a point in the sphere at infinity, which up to conjugation is independent of the chosen point. The rank $N - 1$ of G will be denoted by r.

If $|\Phi|$ is uniformly bounded and for some $\epsilon > 0$, $|d_A\Phi|$ decays faster than $\frac{1}{|x|^{1+\epsilon}}$, then for each monopole there is a well-defined element in $\pi_2(G/S)$ given by the homotopy class of ϕ. More precisely, in Theorem 3.1, Chapter II of [27] it is proved:

Theorem 2.7.1. *If A is a continuous connection in $P = \mathbb{R}^3 \times G$, and the*

Higgs field is continuously differentiable and such that,

$$\lim_{R\to\infty}\sup_{|x|=R}|1-|\Phi(x)||=0, \tag{2.95}$$

$$\textit{for some } \epsilon>0, \quad |x|^{1+\epsilon}|d_A\Phi|\leq \textit{ constant.} \tag{2.96}$$

Then there is a gauge such that the limiting value of the Higgs field at infinity ϕ, is continuous and the configuration (A,Φ) defines a homotopy class in $\pi_2(G/S)$. This class is invariant under gauge transformations and under C^1 perturbations (a,φ) of (A,Φ) satisfying $\lim_{R\to\infty}\sup_{|x|=R}|\varphi(x)|=0=\lim_{R\to\infty}\sup_{|x|=R}|x||a(x)|$.

The class of (A,Φ) in $\pi_2(G/S)$ is represented by ϕ mapping S^2_∞ into its orbit space in $\mathfrak{g}$ which by (2.57) is diffeomorphic to G/S. For the reasons mentioned in the paragraph above equation (2.46) a principal G-bundle over S^2_∞ is trivialisable and since $\pi_2(G)=0$ the gauge transformations are homotopic to the identity. This makes the homotopy class of $\phi: S^2_\infty\to\mathfrak{g}$ well-defined, independent of gauge transformations.

By the exact homotopy sequence

$$\cdots\to\pi_2(G)\to\pi_2(G/S)\to\pi_1(S)\to\pi_1(G)\to\cdots \tag{2.97}$$

this theorem allows us to classify monopoles according to their homotopy class in

$$\pi_2(G/S)\cong\pi_1(S). \tag{2.98}$$

The Lie algebra of the stabiliser of ϕ at a point in S^2_∞ is a direct sum of an abelian Lie algebra of rank l and a semisimple Lie algebra whose rank is $r-l$. By the exact homotopy sequence for the corresponding Lie groups we deduce that

$$\pi_1(S)=\mathbb{Z}^l. \tag{2.99}$$

Definition 2.7.2. The *topological charge* of a monopole (A,Φ) is the string of integers $(n_1,...,n_l)$ that by the group isomorphism (2.98) give the homotopy class of ϕ.

To explore the relation of the topological charges with the magnetic charge section γ_m we need to recall some definitions from Lie algebras.

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, the idealiser of a subalgebra $\mathfrak{h}$ is the largest subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}$ is an ideal:

$$I(\mathfrak{h})=\{x\in\mathfrak{g}:[x,y]\in\mathfrak{h}\quad\forall y\in\mathfrak{h}\}. \tag{2.100}$$

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* (CSA) if it equals its idealiser and it is nilpotent, that is, in the sequence defined recursively as $\mathfrak{h}^1=\mathfrak{h}$, $\mathfrak{h}^{n+1}=[\mathfrak{h}^n,\mathfrak{h}]$, there is an m such that $\mathfrak{h}^m=0$.

Every complex finite dimensional Lie algebra has a CSA and any two CSA are conjugated.

The generalised λ-eigenspace of $\operatorname{ad} x$ is

$$L_x^\lambda := \{y \in \mathfrak{g} : (\operatorname{ad} x - \lambda 1)^j y = 0 \quad \text{for some } j \in \mathbb{N}\}. \tag{2.101}$$

An element $x \in \mathfrak{g}$ is *regular* if the dimension of L_x^0 is as small as possible, i.e. the function $x \mapsto \dim L_x^0$ takes its minimal value on x (in terms of Lie groups, this coincides with our previous definition of a regular element as one whose stabiliser is a maximal torus, see Chapter VIII in [50]). In particular, a diagonalisable matrix $x \in \mathfrak{sl}(n, \mathbb{C})$ is regular if and only if its eigenvalues are pairwise different, and a non-diagonalisable matrix cannot be regular.

When x is regular L_x^0 is a CSA, and (as any two CSA are conjugated) every CSA is the generalised 0-eigenspace of $\operatorname{ad} x$ for some x.

When the Lie algebra $\mathfrak{g}$ is semisimple, a CSA turns out to be a maximal abelian subalgebra consisting of semisimple elements (those y such that $\operatorname{ad} y$ is diagonalisable).

A CSA determines a set of roots Λ (generalised eigenvalues of the ad action of the CSA on $\mathfrak{g}$). In the semisimple case, Λ spans the dual of the CSA, and the Killing form $\langle \cdot, \cdot \rangle$, which by Cartan's second criterion is non-degenerate, when restricted to the CSA is still non-degenerate. In this way, if μ is an element in the CSA, its metric dual ${}^\flat\mu$ can be defined as the element in the dual of the CSA that satisfies,

$${}^\flat\mu(h) = \langle \mu, h \rangle \quad \forall h \in \text{CSA}. \tag{2.102}$$

If we denote by $\mathfrak{g}_\alpha$ the CSA-invariant vector subspaces of $\mathfrak{g}$ associated with the root $\alpha \in \Lambda$, then

$$\mathfrak{sl}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \tag{2.103}$$

is a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Moreover, if we define the *co-root* H_α associated with α as

$$H_\alpha := 2\frac{\alpha^\sharp}{\langle \alpha^\sharp, \alpha^\sharp \rangle} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \tag{2.104}$$

with $\alpha^\sharp$ the metric dual to α, and take $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, then $H_\alpha, X_\alpha, X_{-\alpha}$ can be associated with the natural basis of $\mathfrak{sl}(2, \mathbb{C})$.

The span over the real numbers of the co-roots is a real vector space $\text{CSA}_\mathbb{R}$ where the killing form restricts to an inner Euclidean product, that we denote by $(\cdot, \cdot)$. As a consequence of this, $\text{CSA}_\mathbb{R}$ is a real form of the CSA. From the definition of co-root we see that $\langle \alpha^\sharp, \alpha^\sharp \rangle = 4/\langle H_\alpha, H_\alpha \rangle$ hence it is real and by the same equation we conclude that $\alpha^\sharp \in \text{CSA}_\mathbb{R}$. The roots form a basis for the CSA and this implies that the metric duals with respect to $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ coincide.

A regular element μ in $\mathrm{CSA}_{\mathbb{R}}$ allows us to define in a natural way a set of simple roots: the positive roots Λ^+ are the roots in the half-space where ${}^\flat\mu$ belongs i.e. α is in Λ^+ if $({}^\flat\mu,\alpha) > 0$. Then the simple roots $\{\alpha_1, ..., \alpha_r\}$ are the set

$$\{\alpha \in \Lambda^+ : \alpha \text{ is not the sum of two elements in } \Lambda^+\}. \tag{2.105}$$

The simple roots are a basis for $\mathrm{CSA}^*_{\mathbb{R}}$ and every other root can be written as an integer linear combination of the simple roots where all the non-zero coefficients are either all positive or all negative.[9]

The *Weyl chambers* are the connected components of $\mathrm{CSA}^*_{\mathbb{R}} \setminus \cup_{\alpha\in\Lambda} P_\alpha$ where P_α is the hyperplane orthogonal to α. If two vectors in $\mathrm{CSA}_{\mathbb{R}}$ belong to the same Weyl chamber then they are in the same side of every hyperplane P_α and therefore define the same set of simple roots. As the next proposition shows, the metric dual of a regular element belongs to the closure of exactly one Weyl chamber and therefore there is a 1-1 correspondence between Weyl chambers and bases of simple roots.

Proposition 2.7.3. *Let $\mathfrak{t}$ denote a Cartan subalgebra containing μ. Then μ is non-regular if and only if its metric dual ${}^\flat\mu$ is orthogonal to a root in $\mathfrak{t}^*$.*

Proof. Assume ${}^\flat\mu$ is orthogonal to the root $\alpha \in \mathfrak{t}^*$, then μ commutes with $\mathfrak{sl}_\alpha$, the $\mathfrak{sl}(2,\mathbb{C})$-algebra generated by α. This is a simply consequence of the fact that if $E_\alpha \in \mathfrak{g}_\alpha$ then

$$[\mu, E_\alpha] = \alpha(\mu)E_\alpha = \langle\alpha^\sharp, \mu\rangle E_\alpha = \langle\alpha, {}^\flat\mu\rangle E_\alpha = 0. \tag{2.106}$$

Similarly it commutes with $E_{-\alpha}$ and obviously with the element H_α in the Cartan subalgebra. This implies that L^0_μ strictly contains $\mathfrak{t}$ and therefore μ is not regular.

Conversely, if μ is non-regular, then there is an element $e \in \mathfrak{g}$ which is not in the CSA and such that $[\mu, e] = 0$. If in the decomposition of the Lie algebra into root spaces we have that $e \in \mathfrak{g}_\gamma$, then $k({}^\flat\mu, \gamma) = 0$ since

$$\langle\mu, \gamma^\sharp\rangle e = \gamma(\mu)e = [\mu, e] = 0. \tag{2.107}$$

□

[9]The proof of this fact is not hard: assume this property does not hold. Among the roots for which it does not hold, take the root $\alpha \in \Lambda^+$ with the smallest inner product with ${}^\flat\mu$. As α cannot be a simple root, then there exists $\alpha_1, \alpha_2 \in \Lambda^+$ such that $\alpha = \alpha_1 + \alpha_2$, and the product of one of them with ${}^\flat\mu$ must be strictly smaller than $(\alpha, {}^\flat\mu)$ contradicting the assumption of minimality. This shows that the simple roots are the 'closest' the roots can be to the hyperplane orthogonal to ${}^\flat\mu$.

The boundary conditions for the Higgs field imply that its asymptotic value in a fixed direction $\mu \in \mathfrak{su}(n)$ belongs to a real form of a CSA of $\mathfrak{sl}(n,\mathbb{C})$. Given this CSA, that will be denoted by $\mathfrak{t}$, if we denote by β the value of γ in the chosen direction, then in the case of maximal symmetry breaking, i.e. μ is a regular element, because of the commutation relation (2.56), $\beta \in L^0_\mu$ and therefore it is in $\mathfrak{t}$. The same reasoning applies when β is regular: both μ and β are in the same CSA.

So the only case where it is not necessarily true that a CSA contains both μ and β is when they are both non-regular elements. In this case we can only guarantee that a conjugate of β belongs to a CSA containing μ. More precisely, given a CSA containing μ, and a set of simple roots $\{\alpha_1, ..., \alpha_l, \alpha_{l+1}, ..., \alpha_r\}$ such that $\{\alpha_i\}_{i=1}^l$ are positive i.e.

$$\langle \alpha_i, {}^\flat \mu\rangle > 0 \quad i = 1, ..., l \tag{2.108}$$

and $\{\alpha_i\}_{i=l+1}^r$ orthogonal to ${}^\flat\mu$, it is shown in Proposition 4.1 in [42] that the magnetic orbit of β under the stabiliser of μ intersects this CSA in a full orbit of the subgroup of the Weyl group that fixes μ. Moreover there exists a unique point $\tilde{\beta}$ in this intersection that satisfies $\alpha(\tilde{\beta}) \leq 0$ for every root α orthogonal to ${}^\flat\mu$, and when $\tilde{\beta}$ is expressed as

$$\tilde{\beta} = \sum_{a=1}^{r} n_a H_a, \tag{2.109}$$

where H_a are the co-roots associated with the simple roots α_a, then the coefficients n_a are non-negative integers. These integers are called the *magnetic weights* and are divided into *holomorphic charges* which are the ones associated with the simple roots orthogonal to ${}^\flat\mu$ and the *topological charges*, associated with the positive simple roots determined by μ. Taubes showed that this definition of topological charges is the same as the one in Definition 2.7.2. Specifically, if we let the *generalised magnetic charges* be,

$$q_k := \lim_{R\to\infty} \int_{S_R} \langle \Phi^k, F_A\rangle \quad k = 1, ..., l. \tag{2.110}$$

The invariance of the Killing form under the adjoint action implies that they are gauge invariant, and moreover the following theorem shows that the topological charges are given in terms of these generalised magnetic charges, promoting in this way Proposition 2.6.1.

Theorem 2.7.4. *[54] Assume (A, Φ) satisfy the conditions on the previous theorem and that F_A and $d_A\Phi$ are sections in $L^2(\mathbb{R}^3)$. There are constants b^k_a, $a, k = 1, ..., l$, depending only on the conjugacy class of $Ad_G\mu$, such that,*

$$n_a = \sum_{k=1}^{l} b^k_a q_k \tag{2.111}$$

are integers. Moreover the set $\{n_a\}_{a=1}^{l}$ *are the topological charges, i.e. they determine the homotopy class of the monopole in* $\pi_1(S)$*.*

Dirac's quantisation condition has also an analogue here, the fact that the magnetic weights are integers implies that the magnetic charge β lies in the co-root lattice generated by the co-roots. The set $\{2\pi H_a\}$ spans the kernel of the group homomorphism given by the exponential map from the CSA containing μ, to a maximal torus in $SU(N)$.

Therefore we have the 'quantisation condition':

$$e^{2\pi i\beta} = 1. \tag{2.112}$$

The Bogomolny energy bound for a monopole (A, Φ) takes the form $E(A, \Phi) \geq |q|$ with

$$q = \lim_{R\to\infty} \int_{S_R} \langle \Phi, F_A \rangle = 4\pi(\mu, \beta) = 4\pi \sum_{a=1}^{r} n_a \mu_b (H_a, H_b), \tag{2.113}$$

where μ_b are the coordinates of μ with respect to the co-root basis and we have used that because of (2.57) and (2.58), $\langle \phi, \gamma_m \rangle$ is constant along S^2_∞.

When a root α_a is orthogonal to ${}^\flat\mu$ then the corresponding summand with n_a in the above expression does not contribute. If $r - l$ denotes the number of simple roots orthogonal to μ, only l of the magnetic charges contribute to the above sum bounding the energy. Therefore we can think of the energy bound as coming from l fundamental monopoles. These are $SU(N)$-monopoles coming from embedding via $\mathfrak{sl}_{\alpha_i}$, the PS solution for every simple root $\{\alpha_i\}_{i=1}^{l}$. These also define a class in $\mathbb{Z}^l$ corresponding to the topological charge of the monopole.

Some examples will clarify the previous situations.

2.7.1 Examples.

Let λ be a diagonal matrix in $\mathfrak{sl}(n, \mathbb{C})$ with entries $\{\lambda_i\}$. The roots $\{\alpha_{ij}\}$ are defined by $\alpha_{kl}(\lambda) = -i(\lambda_k - \lambda_l)$.

Let the structure group be $SU(3)$ and suppose that $\mu = \frac{1}{\sqrt{3}}\mathrm{diag}(i, i, -2i)$ then,

$$({}^\flat\mu, \alpha_{12}) = \alpha_{12}(\mu) = 0. \tag{2.114}$$

In the Euclidean plane spanned by the roots, see Figures 2.4 and 2.5, if we move ${}^\flat\mu$ a small amount in a clockwise direction the simple roots defined by ${}^\flat\mu$ (the closest positive roots to the hyperplane orthogonal to ${}^\flat\mu$) are $\{\alpha_{12}, \alpha_{23}\}$ but if instead, we move a bit ${}^\flat\mu$ in an anti-clockwise direction, the simple roots are $\{\alpha_{13}, -\alpha_{12}\}$.

In the first case the co-roots are

$$H_{12} = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{23} = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.115}$$

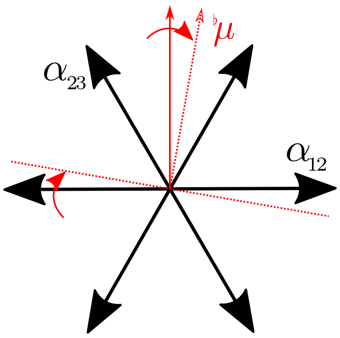

Figure 2.4: Clockwise rotation.

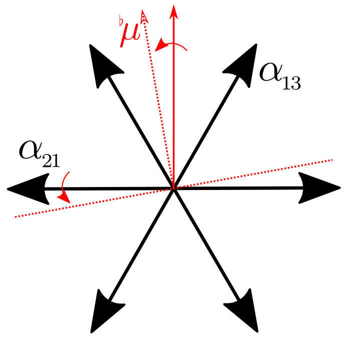

Figure 2.5: Anti-Clockwise rotation.

and corresponding fundamental weights $\{\omega_{12}, \omega_{23}\}$, with $\omega_{12}(\lambda) = -i\lambda_1$, $\omega_{23}(\lambda) = -i(\lambda_1 + \lambda_2)$. In the second case the co-roots are

$$H_{13} = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_{21} = i \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.116}$$

with corresponding fundamental weights $\{\omega_{13}, \omega_{21}\}$, where $\omega_{13}(\lambda) = -i(\lambda_1 + \lambda_2)$, $\omega_{21}(\lambda) = -i\lambda_2$.

If the magnetic charge is $\beta = i\text{diag}(k_1, k_2, k_3)$, with $k_i \in \mathbb{R}$, we see that with the first choice of simple roots, the magnetic weights are k_1 and $k_1 + k_2$, and with the second choice of simple roots the magnetic weights are $k_1 + k_2$ and k_2. From this we deduce that what is invariant is $k_1 + k_2 = -k_3$, that is, the topological charge is given by k_3.

The conjugated element $\mu' = \frac{1}{\sqrt{3}}\text{diag}(-2i, i, i)$ satisfies $^\flat\mu' \perp \alpha_{23}$ and the corresponding natural choices of simple roots are $\{\alpha_{12}, \alpha_{23}\}$ or $\{\alpha_{13}, -\alpha_{23}\}$. In this case, if β is not conjugated, $k_2 + k_3$ is invariant and the topological charge is now k_1.

In the case where μ is a regular element, hence in the maximal symmetry scenario, we can write

$$\mu = -i \text{ diag}(\phi_1, \phi_2, ..., \phi_N), \tag{2.117}$$

with the components ϕ_i being real and in increasing order $\phi_i < \phi_{i+1}$. In this case the simple roots defined by μ are $\{\alpha_{i,i+1}\}_{i=1}^{N-1}$ since they satisfy,

$$(^\flat\mu, \alpha_{i,i+1}) = \alpha_{i,i+1}(\mu) > 0. \tag{2.118}$$

The fundamental weights are given by $\omega_{j,j+1}(\lambda) = -i(\lambda_1 + \cdots + \lambda_j)$ so if $\beta = i\text{diag}(k_1, k_2, ..., k_N)$ then the magnetic weights, which in this maximal symmetry case are the same as the topological charges, are given by

$$n_p = \sum_{j=1}^{p} k_j, \tag{2.119}$$

for $1 \leq p \leq N-1$, this is in agrement with the convention on page 340 in [37].

2.7.2 Monopole particle spectrum.

Here the procedure described in Subsection 2.2.2 is used to study the particles arising from Yang-Mills-Higgs theory. We start with the particles arising from the linearisation around a static solution in the case of structure group $SU(2)$. This static solution will be a minimum of minus the potential energy, and in this case, since there are no boundary/asymptotic conditions, from (2.36) it is clear that the connection must be flat, and the Higgs field constant and minimising $V(\Phi)$. Therefore, the vacuum must be of the form $v + (d_1 g)g^{-1}$ where g is a gauge transformation, d_1 is the linearised gauge action, (see defining equation (3.15)), and

$$v = (\Phi_v, A_v) = (m\frac{i}{2}\sigma_3, 0). \tag{2.120}$$

The particle spectrum comes from perturbations around the vacuum and the particles that arise (see section 5.1.2 in [48]) are

$$\mathcal{A}_\mu, W_\mu^\pm, \varphi, \tag{2.121}$$

which come from the corresponding quanta of the fields:

$$A_\mu^3,\ \frac{A_\mu^1 \pm iA_\mu^2}{\sqrt{2}},\ \Phi^3. \tag{2.122}$$

The first particle is the photon, and corresponds to the Lie algebra of the stabiliser of v. Specifically, if we write the connection as $A_\mu = \sum A_\mu^a \tau^a$, with $\{\tau^a\}$ a basis of $\mathfrak{sl}(2,\mathbb{C})$ the quadratic term (2.22) coming from the Lagrangian that determines the mass matrix of the gauge boson near the ground state is proportional to

$$\mathrm{tr}([\Phi_v, \tau^a][\tau^b, \Phi_v]), \tag{2.123}$$

from where it can be deduced that $A_\mu^3 \tau^3$ is in the null space, and corresponds to a massless boson. The two charged boson particles $W_\mu^\pm$ come from perturbations in the other two components of the connection and their mass $e \cdot m$ comes from the Higgs mechanism. These are the $\pm e$-eigenvectors of the electric charge operator, an element in the Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ corresponding to a generator of the Lie algebra of the stabiliser group of the vacuum. The last particle is the (monopole) Higgs boson, originating from a perturbation φ of the Higgs field around the vacuum. Its mass is $\sqrt{2\lambda}m$, which can be seen from the Hessian of the potential (2.28) evaluated at the vacuum state. It can be observed, that in the BPS limit, where $\lambda = 0$, this particle becomes massless, and therefore the field is long-ranged decaying like $\frac{1}{r}$.

In the BPS limit, where the Higgs potential vanishes, the approximation of the dynamics is given by taking a ground state that satisfies boundary conditions which replace the minima condition of the potential, that is, we take a Higgs field such that $|\Phi| \to m$ at spatial infinity and whose covariant derivative by the induced connection on S^2_∞ vanishes, which is called a *Higgs vacuum*.

In contrast with the masses of elementary particles which arise from perturbations around a ground state, based on Einstein's equation the rest mass of a soliton is given by its energy, which is deduced from the Lagrangian. In our case, the rest mass of a monopole comes from the integration of the energy density (2.38), and the particle spectrum will then be the particles (2.121) coming from linearising around this monopole solution.

What we have said above, extrapolates easily to $SU(N)$-monopoles with maximal symmetry breaking, i.e. the stabiliser of the Higgs field at infinity is $U(1)^{N-1}$. In this case there are $N-1$ 'electrodynamics', meaning there are $(N-1)$ photons, $(N-1)$ Higgs bosons and $\frac{N(N-1)}{2}$ charged and massive W bosons. In the case of non-maximal symmetry breaking, there are additional massless gauge bosons coming from having a larger stabiliser group of ϕ. In this case the massless bosons merge together into a 'monopole cloud' that behaves like a massless monopole at large distances, see Chapter 6 in [58].

Chapter 3

Monopole moduli space.

In this chapter we will introduce the analytic set up to show that the space of framed monopoles is a smooth manifold. We follow the same approach as with instantons [40], [8] together with the aid of the analytic tools developed in [38], [32] to provide with charts, via a slice theorem, the moduli space of framed monopoles.

We start by compactifying $\mathbb{R}^3$ and making sense of the sphere at infinity S^2_∞. Then we extend the definition of monopoles in $\mathbb{R}^3$ to this compactified setting and introduce the hybrid Sobolev spaces that will be used in the analysis required to construct charts in the moduli space of framed monopoles. Then we show that the monopoles which originally are in these hybrid Sobolev spaces are in fact (gauge equivalent to) monopoles which are polyhomogeneous conormal to the boundary. Finally we show that as in the $SU(2)$-case, these framed moduli spaces have a natural hyperkähler metric.

3.1 Monopoles in the radial compactification of $\mathbb{R}^3$.

Unless otherwise stated, the radial compactification of $\mathbb{R}^3$ will be denoted by X. This space can be defined as

$$X := \mathbf{R}^3 \sqcup_I \left([0,\infty)_x \times S^2\right), \tag{3.1}$$

where $I : \mathbf{R}^3 \setminus \{0\} \to (0,\infty)_x \times S^2$ is the homeomorphism that inverts the radial coordinate. In other words, X is the quotient space of $\mathbf{R}^3 \sqcup \left([0,\infty)_x \times S^2\right)$ by identifying each $p \in \mathbf{R}^3 \setminus \{0\}$ with its image $I(p)$.

Topologically this space is just $X = \overline{B^3}$, that is, $\mathbb{R}^3$ with a two-sphere S^2_∞ attached at infinity. This space clearly has a C^∞-structure making it a smooth manifold with boundary. The radial function $1/r$ on $\mathbb{R}^3 \setminus \{0\}$ extends smoothly to a collar neighbourhood of the boundary of X and

$$x := 1/r, \tag{3.2}$$

defines the canonical boundary defining function (a non-negative smooth function on a collar neighbourhood of the boundary such that the boundary is given by the set of points where x vanishes and dx has no zeroes on ∂X) for this space.

With this boundary defining function, X has a natural scattering metric, i.e. a tensor field of type (2,0) with respect to the bundle ${}^{sc}TX$, which is symmetric and positive definite at each point of X. This scattering metric can be written on $X \setminus \{0\}$ as

$$g_{sc} = \frac{dx^2}{x^4} + \frac{h}{x^2}, \tag{3.3}$$

with h a smooth symmetric co-tensor on X which restricts to a metric on the boundary, in this case h is the standard round metric of the two-sphere. In a general sc-metric of the form (3.3) it is still true that h can be written as $h(x, y, dy)$ [30], where y are coordinates in ∂X.

On the interior of X, under the isomorphism ${}^{sc}TX \cong TX$, the standard Euclidean metric coincides with (3.3). In this sense the scattering metric 'extends' the Euclidean metric to the boundary.

The manifold with boundary X also has a conformally related b-metric

$$g_b := x^2 g_{sc} = \frac{dx^2}{x^2} + h. \tag{3.4}$$

This is a metric for the b-tangent bundle ${}^{b}TX$, which contrary to the scattering bundle is defined independently of the chosen boundary defining function.

These metrics via the volume form, induce trivialisations of the b/sc-density bundles. The associated b/sc-measures are related by

$$x^3 \mu_{sc} = \mu_b, \tag{3.5}$$

and in particular we have the following relation of half densities,

$$\Omega^{1/2}_{sc}(X) \cong x^{-3/2}\Omega^{1/2}_{b}(X). \tag{3.6}$$

Unless otherwise stated we shall be working with the measure induced by the scattering metric. The results on Fredholm properties for b-operators are usually stated with the more natural (in that scenario) b-measure, which is encoded in the b-half-density bundles e.g. as in [38]. The above relation of half-densities shows that an operator $D \in \mathrm{Diff}^k(X; E \otimes \Omega^{1/2}_{sc})$ has the same local expression as the associated b-operator $D_b \in \mathrm{Diff}^k(X; E \otimes \Omega^{1/2}_{b})$ given by conjugation by the factor $x^{3/2}$,

$$D = x^{3/2} D_b x^{-3/2}. \tag{3.7}$$

3.1.1 Analytic set-up

Let P be a smooth principal $SU(N)$-bundle over X, where $N \geq 2$ and X is the radial compactification of $\mathbb{R}^3$. As X is contractible, or by the arguments at the beginning of Section 2.5, the bundle is necessarily trivialisable. Using the scattering metric, the Bogomolny equations

$$F_A = *d_A\Phi, \tag{3.8}$$

extend to this setting, and monopoles are defined similarly as in $\mathbb{R}^3$, however, some comments about the regularity of (A, Φ) are in order. It is assumed that the connection and Higgs field are polyhomogeneous conormal to the boundary (phgc). The reader is referred to the Appendix for the definition and some properties of phgc sections. In particular they are smooth over the boundary S^2_∞. The requirement of phgc is the natural extension criteria to the smoothness property over the interior, since roughly the only extra assumption is an asymptotic expansion whose existence is in the properties to be satisfied by a monopole.

In the same way the gauge group consists now of the phgc sections of the bundle of groups Ad P.

Definition 3.1.1. A *monopole* on the bundle P is a pair (A, Φ), where A is a phgc connection and the Higgs field Φ is a phgc section of ad P, which are solutions to the Bogomolny equations modulo $\mathscr{G}$, the phgc group of gauge transformations. They are also required to satisfy the following conditions:

1. They have finite energy,

$$\int_X \left(|F_A|^2 + |d_A\Phi|^2\right) \mu_{sc} < \infty. \tag{3.9}$$

2. There exists a trivialisation over a collar neighbourhood of the boundary U such that for some $\epsilon > 0$,

$$\Phi = \phi - \frac{1}{2r}\gamma_m + O(x^{1+\epsilon}). \tag{3.10}$$

3. In the above trivialisation,

$$d_A\Phi = \frac{1}{2r^2}\gamma_m dr + O(x^{2+\epsilon}). \tag{3.11}$$

4. The mass section ϕ is a nowhere-vanishing smooth section over S^2_∞ and commutes with the smooth magnetic charge section γ_m.

The remarks following our definition of monopoles in $\mathbb{R}^3$ still hold here, see for example Propositions 2.5 and 2.8 in [34] for the second remark.

As mentioned in the previous chapter, it will be assumed without loss of generality that the mass function $|\phi|$ is 1.

Definition 3.1.2. A *framing* is a choice of a smooth mass section ϕ and connection A_0 over S^2_∞. The space of *framed monopoles* are the solutions to the Bogomolny equations satisfying the above boundary conditions with a fixed framing modulo the *reduced gauge transformations*,

$$\mathscr{G}_0 := \{g \in \mathscr{G} : g|_{S^2_\infty} = 1\} \lhd \mathscr{G}. \tag{3.12}$$

Definition 3.1.3. We will fix a pair $(\underline{A}, \underline{\Phi})$ consisting of a phgc connection and Higgs field that satisfy the above boundary conditions, but which is not necessarily a solution to the Bogomolny equations. The existence of monopoles [29], guarantees that such a pair exists.

Remark 3.1.4. The following conventions and notation will be used:

As in the previous chapter, U will denote a collar neighbourhood of ∂X where the above asymptotic expansions hold and π the canonical projection of U onto ∂X.

The bundle of k-forms in the scattering cotangent bundle $\wedge^k({}^{sc}T^*X)$ will be denoted as $\wedge^k$.

The domain and codomain of the next maps will be clear in context, here we just give the expression that they have:

- The gauge equivariant Bogomolny map

$$\mathcal{B}(A, \Phi) = *F_A - d_A\Phi. \tag{3.13}$$

- If (A, Φ) is a solution to the Bogomolny equations, the differential of the Bogomolny map at (A, Φ) is

$$d_2(a, \varphi) = *d_A a - d_A\varphi + [\Phi, a]. \tag{3.14}$$

- The linearisation of the gauge action around a pair (A, Φ) will be denoted as d_1. For γ a section of the bundle $\operatorname{ad} P$:

$$d_1\gamma = (-d_A\gamma, -[\Phi, \gamma]). \tag{3.15}$$

- The formal L_2-adjoint of the infinitesimal gauge transformations when either the gauge transformation comes from the reduced gauge group or (a, φ) vanish at infinity is,

$$d_1^*(a, \varphi) = -d_A^* a + [\Phi, \varphi]. \tag{3.16}$$

- The gauge action on a configuration point $m = (A, \Phi)$ given by $g \cdot m = (A - d_A g g^{-1}, g\Phi g^{-1})$ can be written concisely abusing the notation for d_1 as

$$g \cdot m = m + (d_1 g)g^{-1}. \tag{3.17}$$

It should be recalled that a function is smooth up to the boundary if it is smooth on some manifold in which X embeds. They can be characterised by Seeley's theorem as those functions having all derivatives bounded on bounded sets of $\mathring{X}$.

Analogously, a connection is smooth up to the boundary, when it comes from a smooth connection of a bundle extending P. This implies in particular that if A is the pulled-back connection on X by a trivialisation of the $SU(N)$-bundle, the 1-form component comes from the composite map

$$C^\infty(X; T^*X \otimes adP) \to C^\infty(X; {}^bT^*X \otimes adP) \to C^\infty(X; {}^{sc}T^*X \otimes adP) \tag{3.18}$$

and therefore in local coordinates near the boundary of X it satisfies

$$A\left([x^2\frac{\partial}{\partial x}]\right) = x^2 A_x, \quad A\left([x\frac{\partial}{\partial y_i}]\right) = x A_{y_i}, \tag{3.19}$$

where A_x, $A_{y_i} \in C^\infty(X; \operatorname{ad} P)$ are the local components of the connection when considered as a smooth section of $T^*X \otimes \operatorname{ad} P$. Similarly, a lift of a b-connection comes from the image of the second map in (3.18). In order for a connection to restrict to the boundary it must be the lift of a b-connection. In particular, for a scattering vector field $V = xV_b$, with V_b a b vector field, a connection ∇ which is a lift of a b-connection satisfies,

$$\nabla_V = x\nabla_{V_b}. \tag{3.20}$$

Proposition 3.1.5. *There exists a smooth up to the boundary section s of $\operatorname{ad} P$, such that, on the trivialisation over U being used to define the boundary condition (3.10),*

$$\underline{\Phi} - s = -\frac{1}{2r}\gamma_m + O(r^{-(1+\epsilon)}). \tag{3.21}$$

Proof. Let π be the radial projection of U onto ∂X, then as U deformation retracts onto ∂X, the bundles $\pi^*P|_{\partial X}$ and $P|_U$ are isomorphic. Let

$$\psi : \pi^* \operatorname{ad} P|_{\partial X} \to \operatorname{ad} P|_U, \tag{3.22}$$

denote an isomorphism of the associated adjoint bundles, whose restriction to the fibres over ∂X is the identity, then over U the smooth section

$$\psi\left((m, (\phi \circ \pi)(m)\right), \tag{3.23}$$

is in fact smooth up to the boundary by Seeley's extension theorem (see section 1.14 in [39]).

We shall fix an isomorphism ψ_0 so that in the trivialisation where the boundary conditions (3.10), (3.11), in the definition of a monopole hold, the section over U defined by

$$(\pi^*\phi)(m) := \psi_0\left((m, (\phi \circ \pi)(m))\right), \tag{3.24}$$

satisfies,

$$\underline{\Phi} - \pi^*\phi = -\frac{1}{2r}\gamma_m + O(r^{-(1+\epsilon)}). \tag{3.25}$$

The existence of such an isomorphism follows by considering the bundle isomorphism:

$$\begin{aligned} \psi_0^{-1} : \operatorname{ad} P|_U &\to \pi^* \operatorname{ad} P|_{\partial X} \\ \xi_m &\mapsto (m, \mathcal{P}_{\underline{A}}(\xi_m)), \end{aligned} \tag{3.26}$$

where $\mathcal{P}_{\underline{A}}$ denotes the parallel transport with respect to the connection $\underline{A}$, and here we are parallel transporting the fibre over m along the straight radial path connecting m with the boundary point $\pi(m)$. The monopole boundary conditions satisfied by $\underline{\Phi}$ imply that over U (3.25) holds, and therefore it is enough to take the bundle isomorphism $\psi_0 : (m, v_{\pi(m)}) \mapsto \mathcal{P}_{\underline{A}}^{-1}(v_{\pi(m)})$.

The dependence on the particular choice of the isomorphism such that equation (3.25) holds, is removed after quoting out by the reduced gauge group, as any two such isomorphisms ψ_1, ψ_2 define a reduced-gauge transformation by extending $\psi_1\psi_2^{-1}$ smoothly over X.

It is then enough to take a section s extending $\pi^*\phi$ to X. From the definitions, it is clear that any other section s' satisfying the same asymptotic condition, over U differs from s in a term of the form $O(r^{-(1+\epsilon)})$. □

Denote by $V = V_0 \oplus V_1$ the splitting of $\operatorname{ad} P$ over U, corresponding to the fibrewise centraliser of $\pi^*\phi$ and its orthogonal complement with respect to the normalised Killing form (2.73). It can be observed that since the eigenvalues of ϕ are constant, the spaces V_i are eigenbundles. The analysis leading to prove a slice theorem would be simpler if the rank of the eigenspaces of $\underline{\Phi}$ stayed constant, which is discussed in the appendix.

As mentioned above, we shall be working with the scattering measure on X. Define in a similar way as in [32] the hybrid Sobolev spaces,

$$\mathcal{H}_{\underline{\Phi}}^{\alpha,\beta,k}(X;V) = \begin{cases} x^\alpha H_b^k(X;V_0) \oplus x^\beta H_{sc}^k(X;V_1) \text{ over } U \\ H^k(\mathring{X};V) \text{ on the interior of } X. \end{cases} \tag{3.27}$$

More precisely, take χ a smooth cut-off function on X with values in $[0,1]$, supported on U and identically 1 in a neighbourhood $\tilde{U}$ of ∂X contained in U. If we let π_i be the orthogonal projection of V onto the sub-bundles V_i over U, then a distribution $s \in C^{-\infty}(X;V)$ is in $\mathcal{H}_{\underline{\Phi}}^{\alpha,\beta,k}(X;V)$ if the sum

$$\|x^{-\alpha}\pi_0(\chi s)\|_{H_b^k} + \|x^{-\beta}\pi_1(\chi s)\|_{H_{sc}^k} + \|(1-\chi)s\|_{L_k^2} \tag{3.28}$$

is finite. When s is in $\mathcal{H}_{\underline{\Phi}}^{\alpha,\beta,k}(X;V)$, we define $\|s\|_{\mathcal{H}_{\underline{\Phi}}^{\alpha,\beta,k}(X;V)}$ to be the value of this sum.

Remark 3.1.6. The definition of the hybrid Sobolev spaces extends in a natural way to vector bundles of the form $W \otimes V$, where W is a Hermitian vector bundle with a connection. If E is a vector bundle over X, the Sobolev spaces $H_b^k(X;E)$ and $H_{sc}^k(X;E)$ are the completions of $C_c^\infty(\mathring{X};E)$ with respect to the corresponding b/sc-norms. In particular over the complement of $\tilde{U}$ they are equivalent to $L_k^2(X \setminus \tilde{U};E)$. In this way we obtain a family of Hilbert spaces, that although they depend on U and χ, the important point is that for a given $\underline{\Phi}$, they are defined over the same space of distributions and their norms are all equivalent.

We shall fix a weight α, so that $\alpha > -1/2$ (as we shall see later, $\alpha - 1 \geq -3/2$ is needed in order to consider the subgroup of reduced gauge transformations, as this condition guarantees that the gauge transformations restrict to the identity element at infinity), and such that $\alpha - 1$ is not an indicial root of the b-operator $x^{-2}d_{\underline{A}}^* d_{\underline{A}} - \frac{1}{4}(\operatorname{ad}\gamma_m)^2$ restricted to V_0. This last condition is needed in order to have a slice theorem and can always be achieved since the set of these indicial roots is a discrete set, and although not necessary, these indicial roots are calculated in Proposition 5.3.2.

We shall also fix a real number l to be in the interval,

$$3/2 < l \leq 3. \tag{3.29}$$

Definition 3.1.7. The charge $[\phi]$ *configuration space* $\mathcal{C}^{\alpha,l}$ consists of pairs (A,Φ) with

$$(A,\Phi) = (\underline{A},\underline{\Phi}) + (a,\varphi) \text{ for some } (a,\varphi) \in \mathcal{H}_{\underline{\Phi}}^{\alpha,\alpha+l,l}(X;(\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P), \tag{3.30}$$

such that, the boundary conditions imposed to a monopole (3.9), (3.10), (3.11) are satisfied.

In this way, $\mathcal{C}^{\alpha,l}$ is the $\mathcal{H}_{\underline{\Phi}}^{\alpha,\alpha+l,l}$-completion of the affine space $(\underline{A},\underline{\Phi}) + C_c^\infty(\mathring{X};(\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$, such that, the boundary conditions imposed to a monopole are satisfied. The chosen weight and regularity, will allow us to provide with charts the moduli space of framed monopoles.

Remark 3.1.8. The choice of the weight $\beta = \alpha + l$ for the hybrid Sobolev spaces appearing in the definition of the configuration space, together with the Sobolev embedding (5.11), imply that $\mathcal{H}^{\alpha-k,\alpha+l,l+k}(X;V)$ embeds into $x^{\alpha-k}H_b^{l+k}(X;V)$.

Remark 3.1.9. For $l > 3/2$ (by a Sobolev embedding theorem) the space of sections in $x^\omega L_l^2(X)$ embeds into the space of continuous sections, and in this case, for a weight $\omega \geq -3/2$, a section in $x^\omega L_l^2(X)$ is forced to vanish at S_∞^2. Therefore, each section in $\mathcal{C}^{\alpha,l}$ has a continuous representative, and the elements in our configuration space will have the same limiting value (A_0,ϕ) over S_∞^2 as $(\underline{A},\underline{\Phi})$. In particular, all the connections in the configuration space $\mathcal{C}^{\alpha,l}$ define the same covariant derivative at the boundary

S^2_∞. Moreover, as we have fixed $\alpha > -1/2$ one has $(a, \varphi) = O(x^{1+\epsilon})$ for some ϵ, thus the boundary conditions (3.10) and (3.11), hold automatically.

3.2 Manifold structure.

In this section we prove a slice theorem using the existence of Coulomb gauge representatives. These slices will provide the moduli space of framed monopoles with charts, giving it in this way, the structure of a smooth manifold.

Let (A, Φ) be a point in the configuration space $\mathcal{C}^{\alpha,l}$. On the associated vector bundle E obtained from P via the standard representation of $SU(n)$ on $\mathbb{C}^n$, consider an orthonormal set of smooth sections $\{e_1, ..., e_n\}$ over U, such that, except for a finite number of points in U, where some of the e_i may vanish, they are eigenvectors of the Higgs field. Taking the covariant derivative in the equation

$$\Phi e_i = \lambda_i e_i, \tag{3.31}$$

we obtain,

$$(d_A\Phi - d\lambda_i)e_i = -\Phi d_A e_i + \lambda_i d_A e_i. \tag{3.32}$$

In the trivialisation where we have expressed the asymptotic conditions for the Higgs field (3.10), (3.11), the coefficient of the left-hand-side is of order $O(r^{-(2+\epsilon)})$. Therefore if in the last equation we take the Hermitian-inner product with e_j, using the skew-symmetry of the Higgs field and that therefore its eigenvalues are pure imaginary we conclude that

$$\langle d_A e_i, e_j\rangle(\lambda_i - \lambda_j) = O(r^{-(2+\epsilon)}). \tag{3.33}$$

From this we deduce that when $\lambda_i \neq \lambda_j$ the connection matrix has components A_{ij} in $O(r^{-(2+\epsilon)})$. That is, in this gauge and with respect to the eigenvectors $\{e_1, ..., e_n\}$ the connection matrix is block-diagonal up to order $O(r^{-(2+\epsilon)})$, and the dimension of each of these blocks is given by the dimension of the eigenspace of the corresponding eigenvalue of the Higgs field. We can cover U with a finite number of charts in which the above holds, and in particular

$$[A_0, \phi] = 0. \tag{3.34}$$

To summarise, up to terms in $O(r^{-(2+\epsilon)})$, the component of the connection matrix on the associated bundle $\operatorname{ad} P$ is in C_Φ, the centraliser of Φ in $\operatorname{ad} P$. This in turn implies that with respect to the splitting $\operatorname{ad} P = C_\Phi \oplus C_\Phi^\perp$ the connection has the form

$$\begin{pmatrix} \nabla_{00} & \nabla_{01} \\ \nabla_{10} & \nabla_{11} \end{pmatrix} = \begin{pmatrix} \nabla_{00} & 0 \\ 0 & \nabla_{11} \end{pmatrix} + \begin{pmatrix} 0 & \nabla_{01} \\ \nabla_{10} & 0 \end{pmatrix}. \tag{3.35}$$

The first term is a direct sum of connections and the second term an element in $x^{2+\epsilon}\Omega^1_{sc}(\operatorname{ad} P)$. Similarly, for the components of $\operatorname{ad}\Phi$: Φ_{00} and Φ_{10} vanish by definition, while Φ_{01} vanish since for any $v_1 \in C_\Phi^\perp, v_0 \in C_\Phi$, the product $([\Phi, v_1], v_0)$ is 0, and therefore the only non-vanishing component of $\operatorname{ad}\Phi$ in this decomposition is Φ_{11}.

Equation (3.33) can be used to give an estimate on the off-diagonal terms of the matrix connection with respect to the decomposition of $\operatorname{ad} P$ induced by the centraliser of $\pi^*\phi$, proving in this way the following lemma.

Lemma 3.2.1. *Let (A, Φ) be a point in the configuration space $\mathcal{C}^{\alpha,l}$. Over the open set U, in the trivialisation where the asymptotic conditions (3.10), (3.11) hold, the off-diagonal components of the connection matrix with respect to the splitting $V = V_0 \oplus V_1$ are of order $O(x^{2+\epsilon})$.*

The next theorem, whose proof is similar to that of Theorem 2.4 in [32], will be crucial to prove the slice theorem in the next subsection.

Theorem 3.2.2. *Let (A, Φ) be a point in the configuration space $\mathcal{C}^{\alpha,l}$. The linear map*

$$\begin{aligned} d_1^* d_1 : \mathcal{H}_{\underline{\Phi}}^{\alpha-1,\beta,l+1}(X; \operatorname{ad} P) &\to \mathcal{H}_{\underline{\Phi}}^{\alpha+1,\beta,l-1}(X; \operatorname{ad} P) \\ \gamma &\mapsto d_A^* d_A \gamma - [\Phi, [\Phi, \gamma]], \end{aligned} \tag{3.36}$$

is a continuous Fredholm operator for $3 \geq \beta - \alpha \geq l - 1$ and $\alpha - 1$ not an indicial root of the b-operator $x^{-2} d_{\underline{A}}^* d_{\underline{A}} - \frac{1}{4}(\operatorname{ad}\gamma_m)^2$ restricted to V_0.

Proof. The same notation as the one used in the definition of the hybrid Sobolev spaces will be used along this proof. We shall first show that the operator is bounded.

The operator $d_1^* d_1$ is clearly bounded on the complement of U, where the Sobolev spaces are the usual ones and the difference in regularity between its domain and codomain is 2, which is the degree of the operator. The same reasoning applies to the complement of $\tilde{U}$ since over the interior, the weighted b/sc-Sobolev spaces with k-weak derivatives are equivalent to the usual Sobolev spaces with k-weak derivatives. To prove that it is bounded over U we will see that it is bounded on V_0 and V_1. It is then convenient to obtain expressions for $d_1^* d_1|_{V_0}$ and $d_1^* d_1|_{V_1}$. Observe that by definition $\Phi - \underline{\Phi} = O(x^{1+\tilde{\epsilon}})$ for some $\tilde{\epsilon} > 0$, and this implies that over U,

$$\Phi - \pi^*\phi = -\frac{1}{2r}\gamma_m + O(x^{1+\xi}), \tag{3.37}$$

where $\xi = \min\{\epsilon, \tilde{\epsilon}\}$. Using the Jacobi identity and the commutativity of ϕ with γ_m we obtain,

$$d_1^* d_1|_{V_0} = \left[d_A^* d_A - \frac{x^2}{4}[\gamma_m, [\gamma_m, \cdot]] + O(x^{2+\xi}) \right]_{V_0}. \tag{3.38}$$

and,

$$d_1^* d_1|_{V_1} = \left[d_A^* d_A - [\phi, [\phi, \cdot]] + O(x) \right]_{V_1}. \tag{3.39}$$

It is clear that the map

$$D_0 := \pi_0 \circ d_1^* d_1|_{V_0} : H_b^{\alpha-1,l+1}(V_0) \to H_b^{\alpha+1,l-1}(V_0), \tag{3.40}$$

is a continuous operator between b-Sobolev spaces, where $H_b^{\alpha-1,l+1}(V_0)$ is short hand for $x^{\alpha-1}H_b^{l+1}(X;V_0)$. Similarly,

$$D_1 := \pi_1 \circ d_1^* d_1|_{V_1} : H_{sc}^{\beta,l+1}(V_1) \to H_{sc}^{\beta,l-1}(V_1) \tag{3.41}$$

is a bounded map between sc-Sobolev spaces.

We will see now that the boundedness — and the more demanding compactness needed for the Fredholmness property — of the other components: $D_{10} := \pi_1 \circ d_1^* d_1|_{V_0}$ and $D_{01} := \pi_0 \circ d_1^* d_1|_{V_1}$, imposes a constraint on the weights. From the above expansion of $d_1^* d_1|_{V_0}$, Lemma 3.2.1 and the fact that the connection is a lift of a b-connection,

$$D_{10} : H_b^{\alpha-1,l+1}(V_0) \to H_b^{\alpha+3+\epsilon,l}(V_1) \subset H_{sc}^{\beta,l-1}(V_1). \tag{3.42}$$

where the inclusion is compact if

$$\alpha + 3 \geq \beta. \tag{3.43}$$

On the other hand, for any $v_1 \in V_1$, $v_0 \in V_0$

$$([\phi,[\gamma_m,v_1]],v_0) = -([\gamma_m,v_1],[\phi,v_0]) = 0, \tag{3.44}$$

thus the order x term of the potential $\pi_0 \circ [\Phi,[\Phi,\cdot]]_{V_1}$ vanishes and using Lemma 3.2.1 to estimate the off-diagonal components of the connection matrix we obtain,

$$D_{01} : H_{sc}^{\beta,l+1}(V_1) \to H_{sc}^{\beta+2+\xi,l}(V_0) \subset H_b^{\alpha+1,l-1}(V_0). \tag{3.45}$$

The inclusion $H_{sc}^{\beta+2+\xi,l}(V_0) \subset H_b^{\alpha+1+\xi,l}(V_0)$ holds if $\beta+2+\xi \geq \alpha+1+\xi+l$, therefore D_{01} is compact if

$$\beta - \alpha \geq l - 1. \tag{3.46}$$

In summary, for these two operators to be compact it is required that

$$3 \geq \beta - \alpha \geq l - 1. \tag{3.47}$$

To prove the Fredholmness it can be observed from the above expression that D_1 is a fully elliptic sc-operator, hence Fredholm, and $x^{-2}D_0$ is an elliptic b-operator, i.e. it is Fredholm for $\alpha - 1$ not an indicial root of $[x^{-2}d_A^* d_A - \frac{1}{4}(\operatorname{ad}\gamma_m)^2]|_{V_0}$. This allows us to construct a parametrix, i.e. a distribution P such that $d_1^* d_1 P - 1$ and $P d_1^* d_1 - 1$ are compact operators. P will be a pseudodifferential operator of order -2 on the interior of X such that on U it decomposes as a b-pseudodifferential operator P_0 on V_0 and as a sc-pseudodifferential operator P_1 on V_1. More precisely, on the blown-up space $X_b^2 = [X^2; \partial X \times \partial X]$ and in terms of Schwartz kernels, we consider a distribution K_P conormal to the lifted diagonal of degree -2 such that its restriction to U_b^2 decomposes as $K_{x^2 P_0} + K_{P_1}$. Here P_0 is a b-pseudodifferential operator in $\Psi_b^{-2,\mathcal{E}}(U, V_0 \otimes \Omega_{sc}^{1/2})$ with principal symbol $\sigma_{b,-2}(P_0) = (1-\varphi_b)\sigma_{b,2}(D_0)^{-1}$, with φ_b a bump-function based at 0, and indicial operator the inverse of $I(D_0)$ [1]. Similarly, P_1 is a sc-pseudodifferential operator in $\Psi_{sc}^{-2,0}(U, V_1 \otimes \Omega_{sc}^{1/2})$ with principal symbol $\sigma_{sc,-2}(P_1) = (1-\varphi_{sc})\sigma_{sc,2}(D_1)^{-1}$ and normal operator $\hat{N}_{sc}(P_1) = [\hat{N}_{sc}(D_1 + \Phi_{11})]^{-1}$.

Fixing a smooth cut-off function χ with support on U and letting $\mathring{P}$ be a pseudodifferential operator in $\Psi_{cl}^{-2}(\mathring{X}, V \otimes \Omega_{sc}^{1/2})$ with principal symbol $\sigma_{-2}(\mathring{P}) = (1-\varphi)[\sigma_2(d_1^* d_1)]^{-1}$ we define

$$Ps := x^2 P_0(\pi_0 \chi s) + P_1(\pi_1 \chi s) + \mathring{P}(1-\chi)s, \quad s \in \mathcal{H}^{\alpha+1,\beta,l-1}(X; \mathrm{ad}\, P). \tag{3.48}$$

Each summand is implicitly extended by 0 outside its domain of definition and in order to have a well-defined distribution, the sub-leading terms in the asymptotic expansion of $\mathring{P}$ are required to agree with that of $x^2 P_0 + P_1$ over $\mathring{U}$.

To show the compactness of $d_1^* d_1 P - Id$ it is enough to see that

$$R := d_1^* d_1 P \chi - Id \tag{3.49}$$

is compact, as the operator $d_1^* d_1 P(\chi - 1) - Id$ is compact by the classical theory of pseudodifferential operators. With respect to the splitting of ad P, the diagonal terms of R are compact by the b/sc theory of pseudodifferential operators. The off-diagonal terms are compact since they are a composition of a bounded map with a compact operator, which are given by (3.42) and (3.45). The compactness of $Pd_1^* d_1 - Id$ follows from an analogous argument.

Finally, as already noted, the indicial roots coming from the connection A coincide with those from $\underline{A}$, as they both restrict to A_0 at S^2_∞. □

It can be observed that the theorem still holds for any other choice of set-up, i.e. choosing another background pair $(\underline{A}', \underline{\Phi}')$ and a different bundle automorphism will give an $\pi^*\phi'$ that differs from $\pi^*\phi$ in a term of order $x^{1+\epsilon}$.

From now on, we shall drop the lower index $\underline{\Phi}$ from the notation of the hybrid Sobolev spaces, and on the configuration space $\mathcal{C}^{\alpha,l}$ unless otherwise stated we shall consider the distance induced by the norm $\|\cdot\|_{\mathcal{H}^{\alpha,\alpha+l,l}}$.

[1] the Schwartz kernel of the distribution having indicial operator $I(D_0)^{-1}$, does not have to vanish to all orders in the boundary faces (lb, rb) (see section 5.13 of [38]). It forces us to use the full-calculus and yields the index family $\mathcal{E}$, which is given by $(0, E_{lb}, E_{rb})$ corresponding to the boundary hypersurfaces (ff, lb, rb) of X_b^2 and with $E_{lb} \cup (-E_{rb}) = \mathrm{Spec}_b\, I(D_0)$.

Definition 3.2.3. The group of gauge transformations $\mathscr{G}_{\alpha,l+1}$, consists of the sections in the space $H_b^{l+1}(X; \mathrm{Ad}P)$ which are the exponential of sections in $\mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \mathrm{ad}\, P)$.

It can be observed that in order for Theorem 5.1.2 to hold, the minimum regularity that is required is $l+1 > 3/2$, in this way the charts coming from the pointwise exponential have smooth inverse, making the space of gauge transformation a Hilbert Lie group (see Section 4.2 in [8]). Also, as $\alpha > -1/2$, these continuous gauge transformations restrict to the identity element at S^2_∞, as it is required for the subgroup of reduced gauge transformations.

Proposition 3.2.4. *The gauge group action on the configuration space is smooth, i.e.,* $\mathscr{G}_{\alpha,l+1} \times \mathcal{C}^{\alpha,l} \to \mathcal{C}^{\alpha,l}$ *is a smooth map on Banach manifolds.*

Proof. Writing $g = e^\gamma$ with $\gamma \in \mathcal{H}^{\alpha-1,\alpha+l,l+1}(\mathrm{ad}\, P)$ we see that $d_A g g^{-1}$ is an element in $\mathcal{H}^{\alpha,\alpha+l,l}(\mathrm{ad}\, P)$. On the other hand, it can be observed that conjugation by an element in the gauge group preserves the decay rate. Moreover, using Theorem 5.1.4 in the Appendix, conjugation by g is a bounded operator on $\mathcal{C}^{\alpha,l}$. Therefore

$$(g, (A, \Phi))) \mapsto (A - d_A g g^{-1}, g\Phi g^{-1}) \tag{3.50}$$

is a smooth map into $\mathcal{C}^{\alpha,l}$. □

Slice theorem. The idea of a slice theorem is as follows: assume there is a compact Lie group G (we require compactness in order to have a G-invariant metric and a manifold structure on the orbits) acting on a manifold M, then there is an orthogonal decomposition of the tangent space at $p \in M$,

$$T_p M = T_p\mathcal{O} \oplus (T_p\mathcal{O})^\perp, \tag{3.51}$$

where $\mathcal{O}$ is the G-orbit of p, and $\perp$ denotes the orthogonal complement with respect to a G-invariant metric on M. The image of the exponential map restricted to an $d(\mathrm{Stab}(p))$-stable neighbourhood in $(T_p\mathcal{O})^\perp$ will be denoted by S_p and is called a *slice*. Under the action $h \cdot (y, g) = (y \cdot h, h^{-1} \cdot g)$ of $\mathrm{Stab}(p)$ on $S_p \times G$ the multiplication map $S_p \times G \to M$ descends to $(S_p \times G)/\mathrm{Stab}(p)$ and gives a diffeomorphism from a small ball around $[(p, 1)]$ onto a neighbourhood of its image. This construction allows us to define a manifold structure to the space M/G when G acts freely on M.

The existence of a slice for a compact Lie group acting on a manifold was shown in [41] extending the paper [13], and was later generalised in [43] to locally compact Lie groups acting on G-spaces with compact isotropy groups. In our case the existence of a slice for the gauge group acting on the configuration space is based on the existence of Coulomb gauge representatives. The proof of this existence is similar to Proposition 2.3.4 in [8] or Theorem 2.5 in [56].

Theorem 3.2.5. *Let $m = (A, \Phi)$ be a point in the configuration space $\mathcal{C}^{\alpha,l}$. There exists a constant $c(m)$ such that for any other $\tilde{m} \in \mathcal{C}^{\alpha,l}$ at distance less than $c(m)$ from m, there exists a gauge transformation $g_{\tilde{m}} \in \mathscr{G}_{\alpha,l+1}$ such that $g_{\tilde{m}} \cdot \tilde{m}$ is in Coulomb gauge relative to m, i.e.*

$$m - g_{\tilde{m}} \cdot \tilde{m} \in \ker d_1^*. \tag{3.52}$$

Proof. This is an application of the implicit function theorem in Banach spaces. The gauge action on a configuration point $(A+a, \Phi+\varphi)$ is given by

$$g \cdot (A+a, \Phi+\varphi) = (A + a - (d_{A+a}g)g^{-1}, g(\Phi+\varphi)g^{-1}). \tag{3.53}$$

We need to prove the existence of a gauge transformation g such that

$$\begin{aligned} d_1^*(m - g \cdot (A+a, \Phi+\varphi)) &= d_1^*((d_A g)g^{-1} - gag^{-1}, [\Phi, g]g^{-1} - g\varphi g^{-1}) \\ &= 0. \end{aligned} \tag{3.54}$$

We write $g = e^\gamma$ with $\gamma \in \mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \operatorname{ad} P)$, and express this last equation via the map $G : \mathcal{H}^{\alpha-1,\alpha+l,l+1}(\operatorname{ad} P) \times \mathcal{H}^{\alpha,\alpha+l,l}((\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P) \to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(\operatorname{ad} P)$ as

$$G(\gamma, (a, \varphi)) = 0, \tag{3.55}$$

with $(a, \varphi) \in \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$. Note that the composition of a smooth map with an element in L^2_k with conformal weight $w(k,p) = k - n/p$ (in this case $n = 3$, $p = 2$) positive is again an element in L^2_k. A corresponding statement for weighted Sobolev spaces holds, the regularity is preserved but the weights are not. In particular e^γ has the same regularity as γ.

Also note that as $w(l+1, 2)$ is positive, sections in $\mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \operatorname{ad} P)$ are continuous by the Sobolev embedding theorem. This guaranties that the map G which can be considered an extension from the operator acting on smooth sections to the above Sobolev spaces, is smooth. This follows from the fact that once the multiplication is continuous it is automatically smooth (the image though gains some powers of the weight function x) see Theorem 5.1.4 of Appendix. The differential of G at $(0, (0, 0))$ is given by the composition of the Lie algebra action on $\mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$ and d_1^*:

$$\begin{aligned} G_* : \mathcal{H}^{\alpha-1,\alpha+l,l+1}(\operatorname{ad} P) \times \mathcal{H}^{\alpha,\alpha+l,l}((\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P) &\to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(\operatorname{ad} P) \\ (\xi, \tilde{a}, \tilde{\varphi}) &\mapsto d_1^*(d_A \xi - \tilde{a}, [\Phi, \xi] - \tilde{\varphi}). \end{aligned} \tag{3.56}$$

The partial derivative with respect to γ is

$$\partial_\gamma G_{(0,(0,0))} = -d_A^* d_A + [\Phi, [\Phi, \cdot]] = -d_1^* d_1. \tag{3.57}$$

The implicit function theorem states that we have γ as a function of (a, φ) and solving equation (3.55) in a small neighbourhood of (a, φ) if $-d_1^* d_1$ is an isomorphism, where

$$\begin{aligned} d_1^* d_1 : \mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \operatorname{ad} P) &\to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X; \operatorname{ad} P) \\ \gamma &\mapsto d_A^* d_A \gamma - [\Phi, [\Phi, \gamma]]. \end{aligned} \tag{3.58}$$

It should be recalled that the gain in weight in the V_0 factor is due to the fact that we can extract a factor of x in the connection A to obtain a b-connection.

To see that the map is injective, consider γ_0 in the kernel of $-d_1^* d_1$, then $d_A^* d_A \gamma_0 = [\Phi, [\Phi, \gamma_0]]$, pairing this equation with γ_0 we see that the left hand side is non-negative and the right hand side is non-positive, therefore γ_0 is a continuous section which must be covariantly constant and taking values in $\operatorname{ad} P_0$. As we have by hypothesis $\alpha - 1 \geq -3/2$, our Sobolev spaces extend the infinitesimal reduced gauge action where the gauge transformations are required to be the identity at the boundary and therefore $\gamma_0 = 0$ at the boundary. This together with the fact that it is a parallel section implies $\gamma_0 = 0$.

To prove the surjectivity we have to show that there is a solution to the equation $d_1^* d_1 \xi = \eta$ for any section η in the image of d_1^* (see the definition of G in equation (3.54)).

The previous Theorem 3.2.2 implies that the image of $d_1^* d_1$ is closed, which allows us to have a *Fredholm alternative*: the space $\ker(d_1^* d_1)^*$ is the L_2-orthogonal subspace to $\operatorname{Im}(d_1^* d_1)$, and as this space is closed we have that,

$$\eta \text{ is } L_2\text{-orthogonal to} \ker(d_1^* d_1)^* \text{ if and only if } \eta \in \operatorname{Im}(d_1^* d_1). \tag{3.59}$$

Surjectivity now follows immediately from the Fredholm alternative writing $\eta = d_1^* \eta_0$ and pairing it with $\tilde{\gamma_0}$ in the kernel of $(d_1^* d_1)^*$,

$$(d_1^* \eta_0, \tilde{\gamma_0}) = (\eta_0, d_1 \tilde{\gamma_0}) = 0. \tag{3.60}$$

□

Let ϵ be a positive real number, the *ϵ-slice based at $m \in \mathcal{C}^{\alpha,l}$* is defined to be,

$$\begin{aligned} S_{m,\epsilon} := \{ m + (a, \varphi) : (a, \varphi) \in \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P), \\ d_1^*(a, \varphi) = 0, \|(a, \varphi)\|_{\mathcal{H}^{\alpha,\alpha+l,l}} < \epsilon \}. \end{aligned} \tag{3.61}$$

By the previous theorem, for sufficiently small enough ϵ, the ϵ-slice is not empty. As in the finite dimensional case the first thing to check is the following short lemma.

Lemma 3.2.6. *The ϵ-slice based at m is stable under the action of the stabiliser of m.*

Proof. As the gauge transformations have two weak derivatives in L^2, the multiplication rule for Sobolev spaces (see the Appendix) implies that the regularity of (a, φ) is preserved under gauge transformations, moreover the weights do not change as the action is given by

$$g\cdot(m + (a, \varphi)) = m+(gag^{-1}, g\varphi g^{-1}) \in \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1\oplus\wedge^0)\otimes \mathrm{ad}\, P). \quad (3.62)$$

As the action is an isometry and $d_1^* g = 0$, the other two conditions also hold:

$$d_1^*(g \cdot (a, \varphi)) = 0, \quad \|g \cdot (a, \varphi)\|_{\mathcal{H}^{\alpha,\alpha+l,l}} < \epsilon. \quad (3.63)$$

□

By Proposition 3.2.4, there is a natural smooth multiplication map given by the action of the gauge group on the ϵ-slice based at m:

$$\begin{aligned} M : S_{m,\epsilon} \times \mathscr{G}_{\alpha,l+1} &\to \mathcal{C}^{\alpha,l} \\ (s, g) &\mapsto g \cdot s. \end{aligned} \quad (3.64)$$

If again we denote $g = e^\gamma$, the differential of this map at $(m, 1)$ is given by

$$\begin{aligned} M_* : \ker d_1^* \times \mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \mathrm{ad}\, P) &\to T_m\mathcal{C}^{\alpha,l} \\ ((a, \varphi), \gamma) &\mapsto d_1\gamma + (a, \varphi). \end{aligned} \quad (3.65)$$

One of the advantages of the completion from the spaces of smooth sections to the Banach spaces of Sobolev sections is that there is an implicit function theorem that we can use.

On the other hand, a crucial difference with the finite dimensional case is that there is no guarantee that $T_x\mathcal{O}$ is closed and therefore there might not be a decomposition of the tangent space as above. This is solved by the following lemma.

Lemma 3.2.7. *The operator*

$$\begin{aligned} d_1 : \mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \mathrm{ad}\, P) &\to \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \mathrm{ad}\, P) \\ \gamma &\mapsto (-d_A\gamma, -[\Phi, \gamma])\,. \end{aligned} \quad (3.66)$$

has closed range.

Proof. As γ must vanish at infinity, the kernel of d_1 is 0. In fact, there is a Poincaré type inequality i.e. there is constant $c > 0$ such that,

$$\|d_1\gamma\|_{\mathcal{H}^{\alpha-1,\alpha+l,l+1}(X;\mathrm{ad}\, P)} \geq c\|\gamma\|_{\mathcal{H}^{\alpha,\alpha+l,l}(X;(\wedge^1\oplus\wedge^0)\otimes\mathrm{ad}\, P)}. \quad (3.67)$$

This follows from the standard elliptic estimates on the interior and integration by parts on a collar neighbourhood of the boundary. In particular, as γ vanishes at the boundary, the norm of the radial derivative of $|\gamma|$ bounds $\|\gamma\|$. Then from Kato's inequality $|d|\gamma|| \leq |d_A\gamma|$ it follows that $\|d_A\gamma\| \geq c\|\gamma\|$, which implies the above inequality.

With the Poincaré inequality: if $\{d_1\gamma_k\} \to \xi$ in $\mathcal{H}^{\alpha,\alpha+l,l}(X;(\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$, then $\{\gamma_k\}$ is Cauchy and therefore $\xi \in \operatorname{Im} d_1$. □

As $\operatorname{Im} d_1$ is closed in $\mathcal{H}^{\alpha,\alpha+l,l}(X;(\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$, it is also closed in $L^2(X;(\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P)$. With this fact we arrive at the conclusion that there is an isomorphism,

$$T_m\mathcal{C}^{\alpha,l} \cong \operatorname{Im} d_1 \oplus (\operatorname{Im} d_1)^{\perp} = \operatorname{Im} d_1 \oplus \ker d_1^*. \tag{3.68}$$

This allows us to write the differential of the multiplication operator (3.65) as $d_1 \oplus Id$ (with the obvious re-ordering of the factors). This maps onto $\operatorname{Im} d_1 \oplus \ker d_1^*$, so in order to apply the implicit function theorem we must show that d_1 is injective.

The stabiliser of the configuration point $m \in \mathcal{C}^{\alpha,l}$, will be denoted by Γ_m, and is defined as

$$\Gamma_m := \{g \in \mathscr{G}_{\alpha,l+1} : g \cdot m = m\}. \tag{3.69}$$

It consists of the covariant constant gauge transformations (with respect to the connection of the configuration point m) that commute with the Higgs field. As the elements in the stabiliser are covariantly constant they are determined by their value at a fibre and therefore we can think of Γ_m as a subset of the structure group. In fact, it is not hard to see that it is a closed subgroup and therefore Γ_m is an embedded Lie subgroup of $SU(N)$. In the case of interest, where the action is given by the reduced gauge group $\mathscr{G}_0$, the stabiliser of any configuration point is the identity, as that is the value that these gauge transformation must take on the fibres over S^2_∞. The elements in the kernel of d_1 are the sections in the Lie algebra of Γ_m, and therefore for the action of reduced gauge transformations,

$$\ker d_1 = 0. \tag{3.70}$$

In this way we obtain a generalisation of the finite dimensional case

$$\begin{aligned} M : (S_{m,\epsilon} \times \mathscr{G}_{\alpha,l+1})/\Gamma_m &\to \mathcal{C}^{\alpha,l} \\ [(s,g)] &\mapsto g \cdot s. \end{aligned} \tag{3.71}$$

The differential of this map is now an isomorphism of Hilbert spaces and this proves the following theorem.

Theorem 3.2.8. *Given a configuration point $m \in \mathcal{C}^{\alpha,l}$, there exists a positive real number δ such that, a δ-ball based at $(m,1) \in S_m \times \mathscr{G}_{\alpha,l+1}$ is diffeomorphic to a small neighbourhood of m in the configuration space $\mathcal{C}^{\alpha,l}$.*

To have a picture in mind of this theorem it is useful to recall the construction of a fibre bundle F with fibre f out of a principal G-bundle P. The structure group G acts on f and we have the fibre bundle $F = P \times_G f$ over P/G. With this in mind, the statement that a δ-ball based at $[(m,1)] \in (S_m \times \mathscr{G}_{\alpha,l+1})/\Gamma_m$ is diffeomorphic to a small neighbourhood of m in the configuration space $\mathcal{C}^{\alpha,l}$ (if two configuration points are in the δ-slice based at m are gauge related by a transformation close to the identity, then in fact they are gauge equivalent by an element in the stabiliser of m), translates into having a small ball around m in the configuration space which is diffeomorphic to a ball in the total space of the fibre bundle

$$S_{m,\epsilon} \times_{\Gamma_m} \mathscr{G} \to S_{m,\epsilon}/\Gamma_m. \tag{3.72}$$

For a more concrete example, assume that the configuration space is finite dimensional, say $\mathbb{R}^3$, and that the gauge group is isomorphic to $U(1) \times \mathbb{R}$. The action of the factor $U(1)$ corresponding to rotations around the z-axis and the action of the $\mathbb{R}$-factor to translations in the z-direction. Then with the obvious notations, at a point $p \in \mathbb{R}^3$ the adjoint of the infinitesimal gauge action is given by $d_1^*(y) = -p_2y_1 + p_1y_2 + y_3$. The stabiliser of 0 is $U(1) \times \{0\}$ and therefore $S_0/U(1)$ can be taken to be the non-negative y-axis. In this case the theorem says that a small cylinder centred at 0, corresponding to the fibration $S_{0,\epsilon} \times_{U(1)} \mathscr{G}$, is diffeomorphic to a small ball around the origin in the configuration space.

The next thing to do is to generalise the above theorem by removing the restriction to the elements of the gauge group of being close to the identity. In the above picture this corresponds to have that the fibre bundle over $S_{m,\epsilon}/\Gamma_m$ with fibre the whole gauge group, is diffeomorphic to a neighbourhood of the orbit of m (in the space of orbits).

A pathological situation that might happen is illustrated by the following scenario: again consider the configuration space to be $\mathbb{R}^3$ with a gauge group such that the orbit of $m = 0$ is the z-axis and there is a sequence of orbits with a $\subset$-shape having its vertex approaching a point p in the z-axis and as they are closer to p they turn in a fixed direction so that they do not intersect each other and they do not cross the slice that we take to be the $\langle x, y\rangle$-plane. In this case, infinitely many $\subset$-shaped orbits will be in a small neighbourhood of the orbit of m and not in the image of the generalisation of the local diffeomorphism M. So for example surjectivity onto a neighbourhood of the orbit of m may fail. The injectivity also needs to be present, and as in the case of connections, see Proposition 4.2.9 of [8], the following lemma guarantees that we have a diffeomorphism.

Lemma 3.2.9. *Assume that two convergent sequences* $\{c_n^1\} \to c_\infty^1$ *and* $\{c_n^2\} \to c_\infty^2$ *of points in the configuration space* $\mathcal{C}^{\alpha,l}$ *are such that, for each index* i *there exists a gauge transformation* g_i *in* $\mathscr{G}_{\alpha,2}$[2] *such that,* $c_i^1 = g_i \cdot c_i^2$

[2] as it is shown along the proof, only the minimum regularity is needed.

(they are term-by-term gauge equivalent). Then each gauge element g_i is in $\mathscr{G}_{\alpha,l+1}$, and there exists a subsequence of the sequence of gauge transformations that converges in $\mathscr{G}_{\alpha,l+1}$ to an element $g_\infty \in \mathscr{G}_{\alpha,l+1}$ such that, $c^1_\infty = g_\infty \cdot c^2_\infty$.

Proof. As in the smooth version, we consider the action of the gauge transformation on the connection in order to see how its regularity is improved. On a trivialisation of the bundle, $A^1_i = g_i \cdot A^2_i = g_i A^2_i g_i^{-1} - dg_i g_i^{-1}$, which is equivalent to

$$dg_i = g_i A^2_i - A^1_i g_i, \tag{3.73}$$

where the derivatives are in the weak sense. By the triangle inequality and the continuity of multiplication in Sobolev spaces, in particular using the embedding of Remark 3.1.8, for $k \leq l$ the multiplication $H^k_b \times x^\alpha H^l_b \to H^k_b$ is bounded, and therefore there exist constants $C_i > 0$ such that,

$$\|dg_i\|_k \leq \|g_i A^2_i\|_k + \|A^1_i g_i\|_k \leq C_i(\|g_i\|_k \|A^2_i\|_l + \|A^1_i\|_l \|g_i\|_k), \tag{3.74}$$

where the subscript denotes the norm in the corresponding b-Sobolev space. From this equation we see that if $k \leq l$, the L^2-norm of the k-th derivative of each g_i is bounded when $g_i \in \mathscr{G}_{\alpha,k}$. Therefore for $k \leq l$ we can gain one degree in regularity. Iterating this procedure we obtain that the first $l+1$ weak derivatives of each g_i are bounded. As the structure group is compact, the sequences consisting of the k-th partial derivatives of the gauge transformations with $k \leq l+1$ are uniformly bounded.

By the Alaoglu-Banach theorem there exists a subsequence $\{g_n\}$ that converges weakly to $g_\infty \in \mathscr{G}_{\alpha,l+1}$, this space embeds compactly into $\mathscr{G}_{\alpha,l}$, and therefore the previous subsequence $\{g_n\}$ converges to $g_\infty \in \mathscr{G}_{\alpha,l}$. Applying again the argument above using equation (3.73) to obtain an improvement in the regularity of this gauge transformation, we see that in fact $g_\infty \in \mathscr{G}_{\alpha,l+1}$.

Taking the limit in (3.73) together with the transformation law for the Higgs fields gives $c^1_\infty = g_\infty \cdot c^2_\infty$. □

Theorem 3.2.10. *Let m be a point in the configuration space $\mathcal{C}^{\alpha,l}$, there exists an $\epsilon_0 > 0$ such that for any positive $\epsilon < \epsilon_0$ the map (3.64) is a diffeomorphism onto a neighbourhood of the orbit of m.*

Proof. We have seen in the previous theorem that the mapping M is a local diffeomorphism. To show that for small enough ϵ it is in fact a diffeomorphism we need to prove that M is injective. For a contradiction assume that no such an ϵ_0 exists, then we could find two sequences of configuration points $\{c^1_n\}$ and $\{c^2_n\}$ which are in Coulomb gauge relative to m, they are term-by-term gauge equivalent via $\{g_n \in \mathscr{G} \setminus \Gamma_m\}$ and both converge to m. By the previous lemma, the sequence of gauge transformations would have a subsequence converging to 1, therefore by Theorem 3.2.8 the sequences of configuration points would eventually be related by gauge transformations in the stabiliser of m, a contradiction. □

Let $\mathcal{Q}^{\alpha,l}$ be the quotient space

$$\mathcal{Q}^{\alpha,l} := \mathcal{C}^{\alpha,l}/\mathscr{G}_{\alpha,l+1}. \tag{3.75}$$

Lemma 3.2.11. *Provided with the quotient topology $\mathcal{Q}^{\alpha,l}$ is a Hausdorff space.*

Proof. The weighted-L^2-metric on configuration points,

$$\|(A_1,\Phi_1)-(A_2,\Phi_2)\|_{\mathcal{H}^{\alpha,\alpha,0}} \tag{3.76}$$

is bounded by $\|(A_1,\Phi_1)-(A_2,\Phi_2)\|_{\mathcal{H}^{\alpha,\alpha+l,l}}$ and therefore it is finite. It is also preserved by the action of the gauge group, and consequently it descends to the quotient $\mathcal{Q}^{\alpha,l}$. Similarly as Lemma 4.2.4 in [8], this operation defines a metric on $\mathcal{Q}^{\alpha,l}$,

$$d([c_1],[c_2]) := \inf_{g\in\mathscr{G}_{\alpha,l+1}} \|c_1 - g\cdot c_2\|_{\mathcal{H}^{\alpha,\alpha,0}}. \tag{3.77}$$

This provides the quotient with the structure of a metric space. As a result, $\mathcal{Q}^{\alpha,l}$ is a Hausdorff space with the induced topology and as this topology is coarser than the quotient topology the statement follows. □

Theorem 3.2.10 implies that $\mathcal{Q}^{\alpha,l}$ is locally modelled on $S_{m,\epsilon}$. Moreover, the transition functions between charts are smooth, providing the space $\mathcal{Q}^{\alpha,l}$ with the structure of a Hilbert manifold. The smoothness of the transition functions is proved in the same way as the case of the moduli space of connections, see Section V in [40].

Let $\mathcal{M}^{\alpha,l}$ be the moduli space of monopoles inside $\mathcal{Q}^{\alpha,l}$. By the above results, we conclude that $\mathcal{M}^{\alpha,l}$ is a manifold locally modelled around a monopole $m = [(A,\Phi)]$ as the preimage of 0 by the Bogomolny map:

$$\begin{aligned}\mathcal{B}_{(A,\Phi)} : S_{(A,\Phi),\epsilon} &\to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X;\wedge^1\otimes \operatorname{ad}P)\\ (a,\varphi) &\mapsto *F_{A+a} - d_{A+a}(\Phi+\varphi).\end{aligned} \tag{3.78}$$

Remark 3.2.12. If we were allowing gauge transformations which were not necessarily the identity over S^2_∞, but just preserving the fixed mass section and connection there, then we would have to consider $\alpha < -1/2$ and the formal adjoint of d_1 would have to be taken with respect to the weighted L^2-inner product instead of the usual L^2. In this way, there is a decomposition of the tangent space of a point in the configuration space as in (3.68),

$$T_m\mathcal{C}^{\alpha,l} \cong \operatorname{Im} d_1 \oplus \ker d_1^{*,w}, \tag{3.79}$$

where $d_1^{*,w} = w^2 d_1^* w^{-2}$ with w the weight. The slice in this case would be a subspace of $\ker d_1^{*,w}$. Everything would work as in the framed case, but one needs to consider the set $\mathcal{C}^{\alpha,l}_*$ of irreducible configuration points, defined as those points whose stabiliser is minimal, that is, $\mathbb{Z}_N$ (the centre of $SU(N)$). This set is open in $\mathcal{C}^{\alpha,l}$: take a point $c = (A, \Phi)$ in $\mathcal{C}^{\alpha,l}_*$, if every neighbourhood of c contains a reducible point, then we can take a sequence of reducible configuration points $\{c_i\}$ converging to c, but this implies that the holonomy group of A cannot be the whole space $SU(N)$ hence c is reducible. The set of irreducible configuration points quotiented by the gauge group modulo $C(SU(N))$ (where for continuous sections this coincides with the centre of the gauge group) gives the space of irreducible configuration points modulo gauge $\mathcal{Q}^{\alpha,l}_*$. As $\mathcal{Q}^{\alpha,l}$ is given the quotient topology, the projection map induced by the gauge action, $\pi : \mathcal{C}^{\alpha,l} \to \mathcal{Q}^{\alpha,l}$ is open and therefore $\mathcal{Q}^{\alpha,l}_* := \pi(\mathcal{C}^{\alpha,l}_*)$ is open in $\mathcal{Q}^{\alpha,l}$. Considering the action of this gauge group, the manifold where solutions to the Bogomolny equations should be sought is $\mathcal{Q}^{\alpha,l}_*$. If one wants to calculate the dimension of the moduli space of these unframed-monopoles via an index theorem as it is done in the next chapter, the indicial roots of the operatof $\tilde{D}_0$ in Proposition 4.3.5, will depend now on the weight w, cf. Proposition 4.6 in [33], for the case of structure group $SU(2)$.

The dependency of $\mathcal{M}^{\alpha,l}$ on the regularity l and on the weight α is removed in the next proposition.

Proposition 3.2.13. *Let m be a solution to the Bogomolny equations in the configuration space $\mathcal{C}^{\alpha,l}$, then there exists a gauge transformation $g = e^\gamma$ with γ in $\mathcal{H}^{\alpha-1,\alpha+l,l+1}(X; \operatorname{ad} P)$, such that $g \cdot m$ is polyhomogeneous conormal to the boundary.*

Proof. As the phgc configuration points in $\mathcal{C}^{\alpha,l}$, are dense in $\mathcal{C}^{\alpha,l}$ (see Remark 5.4.7) there is a ball of radius ϵ based at m containing a phgc configuration point $c = (A, \Phi)$ which can be gauge transformed by an element $g \in \mathscr{G}_{\alpha,l+1}$ to be in Coulomb gauge relative to m. Therefore the difference $(a, \varphi) = g^{-1}m - c$ is an element in $\mathcal{H}^{\alpha,\alpha+l,l}(X; \operatorname{ad} P)$ that satisfies $d_1^*(a, \varphi) = 0$ and such that the Bogomolny operator $\mathcal{B}_{(A,\Phi)}$ based at c evaluated at (a, φ) vanishes:

$$\begin{aligned}\mathcal{B}_{(A,\Phi)}(a, \varphi) &= *F_{A+a} - d_{A+a}(\Phi + \varphi) \\ &= *F_A - d_A\Phi + *d_Aa - d_A\varphi + [\Phi, a] + *(a \wedge a) - [a, \varphi] = 0.\end{aligned} \tag{3.80}$$

The linear terms in the Bogomolny operator together with the Coulomb gauge condition

$$d_1^*(a, \varphi) = -d_A^*a + [\Phi, \varphi] = 0, \tag{3.81}$$

form the elliptic linear map $\mathcal{D}_{(A,\Phi)}$ (explicitly written in (4.10)). So both conditions can be written as the single equation,

$$\mathcal{D}_{(A,\Phi)} \begin{pmatrix} a \\ \varphi \end{pmatrix} = \begin{pmatrix} - * (a \wedge a) + [a, \varphi] - *F_A + d_A\Phi \\ 0 \end{pmatrix}. \tag{3.82}$$

Therefore the pair (a, φ) is in the kernel of P, an elliptic operator of order 1 with phgc coefficients. The elements in the kernel of P are phgc with leading order asymptotic determined by α, in fact of order (see Theorem 2.4 (c) in [32]),

$$O(x^{1+\lambda_0} \log^{\mathrm{ord}(\lambda_0)} x), \tag{3.83}$$

where

$$\lambda_0 = \min\{\lambda : \lambda \in \mathrm{spec}_b \mathcal{D}_{(A,\Phi)},\ \lambda > \alpha + 1/2\}, \tag{3.84}$$

and the exponent of the log term is the order of the pole of the indicial family $I(P, \lambda)$ at the indicial root λ_0. In this case, the exponent must be 1, since P is a first order operator.

□

The previous proposition shows that a solution to the Bogomolny equations in $\mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge) \otimes \mathrm{ad}\, P)$ is in fact gauge equivalent to a phgc configuration point whose first two leading terms are

$$\phi - \frac{1}{2} x\gamma_m, \tag{3.85}$$

as long as α is large enough. This is always the case for our choice $\alpha > -1/2$, see Proposition 4.3.5.

Therefore the moduli space of (irreducible) phgc framed monopoles $\mathcal{M}$ surjects onto the space of $\mathcal{M}^{\alpha,l}$. This map is also easily seen to be injective: if two phgc monopoles are related by a gauge transformation in $\mathscr{G}_{\alpha,k}$ then as we saw along the proof of Lemma 3.2.9, the gauge transformation is actually smooth, that is, the two phgc monopoles are in the same orbit under the action of the phgc reduced gauge group $\mathscr{G}_0$.

3.3 Hyperkähler structure.

The configuration space $\mathcal{C}^{\alpha,l}$ is an infinite dimensional affine space that carries a quaternionic structure. A section (a, φ) of the bundle $(\wedge^1 \oplus \wedge^0) \otimes \mathrm{ad}\, P$ over X can be identified with a function on $\mathfrak{su}(n) \otimes \mathbb{H}$ via

$$(a, \varphi) = (a_x dx + a_y dy + a_z dz, \varphi) \mapsto \varphi + a_x I + a_y J + a_z K. \tag{3.86}$$

The inner product on two quaternions q_1 and q_2 is given by $\mathrm{Re}(\overline{q}_1 q_2)$, it combines with the Killing form and the L^2-inner product to give the inner product $(\cdot, \cdot)_{\mathbb{H}}$ on $\mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \mathrm{ad}\, P)$ on which the almost complex structures defined by left multiplication by I, J, K are isometries (as $\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$). Therefore there are associated 2-forms $\omega_I, \omega_J, \omega_K$, defined in the usual way, that is,

$$\omega_I\left((a_1, \varphi_1), (a_2, \varphi_2)\right) = \left(I(a_1, \varphi_1), (a_2, \varphi_2)\right)_{\mathbb{H}}, \tag{3.87}$$

and similarly for ω_J and ω_K. These forms are non-degenerate and as they have constant coefficients they are also closed. The closedness property implies that the almost complex structures are integrable [2], providing $\mathcal{C}^{\alpha,l}$ with a hyperkähler structure.

The gauge group acts by isometries and therefore it preserves the above symplectic forms. The Bogomolny operator constitutes the corresponding three moment maps and the hyperkähler quotient construction [23],

$$\mathcal{B}^{-1}_{(A,\Phi)}(0)/\mathscr{G}_0, \tag{3.88}$$

that in the finite dimensional case provides the quotient with a hyperkähler metric, can now be complemented by the following proposition.

Proposition 3.3.1. *The symplectic forms ω_I, ω_J and ω_K descend to the quotient $\mathcal{B}^{-1}_{(A,\Phi)}(0)/\mathscr{G}_0$.*

Proof. We have seen that using the slice theorem, the tangent space to a point in $Q^{\alpha,l}$ is isomorphic to the kernel of d_1^*, and therefore the tangent space at monopole (A,Φ) can be identified with the sections (a,φ) in the kernel of $\mathcal{D}_{(A,\Phi)}$. Denoting the complex structures by I_i,

$$\begin{aligned} \omega_{I_i}\left((a,\varphi), d_1\gamma)\right) &= (I_i(a,\varphi), d_1\gamma))_{\mathbb{H}} = \Big(I_i(a,\varphi), \mathcal{D}^*_{(A,\Phi)}(0,\gamma)\Big)_{\mathbb{H}} \\ &= \big(\mathcal{D}_{(A,\Phi)}(I_i(a,\varphi)), (0,\gamma)\big)_{\mathbb{H}} = \big(I_i(\mathcal{D}_{(A,\Phi)}(a,\varphi)), (0,\gamma)\big)_{\mathbb{H}} = 0, \end{aligned} \tag{3.89}$$

where in the third equality we have used integration by parts, and in the last equality that the kernel of $\mathcal{D}_{(A,\Phi)}$ is a quaternionic vector space. □

Chapter 4

Moduli space dimension.

The moduli space of phgc framed monopoles was defined in the previous chapter and it was shown that it carries the smooth structure of a manifold. The principal aim of this chapter is to compute the dimension of this moduli space.

In the first section we follow chapter 4 in [8] to explain the basic theory needed to understand why the dimension of the manifold of framed monopoles is given by the index of $d_1^* \oplus d_2$.

In the second part of this chapter, we use the index formula of Kottke [32] to obtain that the dimension of the framed moduli space is four times the sum of the topological and holomorphic charges.

4.1 Basic theory.

The framed moduli space of monopoles was defined in the previous chapter, it was shown that a local model for a monopole $m = (A, \Phi)$ represented by the solution to the Bogomolny equations is $(\mathcal{B}_m|_{S_{m,\epsilon}})^{-1}(0) \subset \ker d_1^*$.

Assume that the operator $\mathcal{D}_{(A,\Phi)}$ defined as

$$d_1^* \oplus d_2 : \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^0 \oplus \wedge^1) \otimes \operatorname{ad} P) \to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X; (\wedge^0 \oplus \wedge^1) \otimes \operatorname{ad} P) \tag{4.1}$$

is a Fredholm operator, then the Bogomolny map

$$\mathcal{B}_m : \ker d_1^* \subset \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^0 \oplus \wedge^1) \otimes \operatorname{ad} P) \to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X; \wedge^1 \otimes \operatorname{ad} P) \tag{4.2}$$

must be Fredholm as well. Using the implicit function theorem for Banach spaces, one can show that the zero set of $\mathcal{B}_m$ can be expressed as the zero set of a smooth function f acting on finite dimensional spaces. More precisely (this is Proposition 4.2.19 in [8]),

Proposition 4.1.1. *A Fredholm map $\mathcal{B}$ from a convex neighbourhood of* 0 *is locally right equivalent to a map $\tilde{\mathcal{B}}$ —that is, there is a diffeomorphism g*

between two neighbourhoods of 0 *such that* $\mathcal{B} \circ g = \tilde{\mathcal{B}}$*— which has the form*

$$\tilde{\mathcal{B}} : U_0 \times F \to V_0 \times G, \quad \tilde{\mathcal{B}}(\xi, \eta) = (L(\xi), \lambda(\xi, \eta)), \tag{4.3}$$

where L *is a linear isomorphism from* U_0 *to* V_0*,* F *and* G *are finite dimensional with* $\dim F - \dim G = \operatorname{ind} \mathcal{B}$*, and the derivative of* λ *vanishes at* 0*.*

It should be recalled that the index of a smooth Fredholm map is given by the index of its differential at any point of its (connected) domain. We can use this proposition to conclude that the zero set of $\mathcal{B}_m$ in a neighbourhood of 0 is given by the zero set of

$$f : \ker(d_1^* \oplus d_2) \to \operatorname{coker} d_2, \tag{4.4}$$

in a neighbourhood of 0, where $f(y) = \lambda(0, y)$ in the above proposition and d_2 is the differential of $\mathcal{B}_m$ at 0.

If 0 is a regular value for f, i.e. $df(p)$ is surjective whenever the image of p is 0, then $f^{-1}(0)$ is a submanifold of $\ker(d_1^* \oplus d_2)$, which locally models the monopole m, and has dimension

$$\dim \ker(d_1^* \oplus d_2) - \dim \operatorname{coker} d_2. \tag{4.5}$$

From the above proposition it follows that df is surjective if and only if d_2 is surjective, so the dimension of the manifold consisting of framed monopoles is $\dim \ker(d_1^* \oplus d_2)$. On the other hand,

$$\operatorname{ind} \mathcal{B}_m = \operatorname{ind}(d_1^* \oplus d_2) + \dim \operatorname{coker} d_1^*. \tag{4.6}$$

As the reduced gauge group acts freely on the moduli space of monopoles it follows that

$$\operatorname{coker} d_1^* \cong (\operatorname{Im} d_1^*)^{\perp} = \ker d_1 \tag{4.7}$$

vanishes.

In conclusion, when $d_1^* \oplus d_2$ is Fredholm, the dimension of the manifold of (irreducible) regular framed monopoles is given by the index of $d_1^* \oplus d_2$.

4.2 Deformation complex of monopoles.

The previous theory is usually recast in the form of a deformation complex, in our case we have the following deformation complex extending the corresponding phgc one,

$$\begin{gathered} \mathcal{H}^{\alpha-1,\alpha+l,l+1}(\operatorname{ad} P) \xrightarrow{d_1} \mathcal{H}^{\alpha,\alpha+l,l}((\wedge^0 \oplus \wedge^1) \otimes \operatorname{ad} P) \xrightarrow{d_2} \mathcal{H}^{\alpha+1,\alpha+l,l-1}(\wedge^1 \otimes adP) \\ d_1 : \gamma \mapsto (-d_A\gamma, -[\Phi, \gamma]) , \quad d_2 : (a, \varphi) \mapsto *d_A a + [\Phi, a] - d_A\varphi. \end{gathered} \tag{4.8}$$

This is an elliptic complex precisely when $m = (A, \Phi)$ is a solution to the Bogomolny equations (see Lemma 5.6.1). As before, the first map d_1, is the infinitesimal gauge action from $T_1\mathscr{G}_{\alpha,l+1}$ extending $T_1\mathscr{G}_0 \to \Gamma(X; \wedge^1 \otimes \operatorname{ad} P) \oplus \Gamma(X; adP)$, where the phgc sections of a bundle E over X are denoted as $\Gamma(X; E)$. The second map d_2, is the differential of the Bogomolny map at $T_{(A,\Phi)}\mathcal{C}^{\alpha,l}$.

The first cohomology group of the complex represents the linearisation of the Bogomolny equations modulo gauge, i.e. the tangent space of the moduli space of framed monopoles at the monopole $[m]$:

$$H^1_m = \frac{\ker d_2}{\operatorname{Im} d_1} \cong \ker(d_1^* \oplus d_2). \tag{4.9}$$

The cohomology group H^0_m vanishes as the gauge group action is free. The second cohomology group vanishes when m is a regular point, i.e. d_2 is surjective. In this case, as it shown in the previous section, when the operator $d_1^* \oplus d_2$ is Fredholm, its index (which is minus the Euler characteristic of the complex) gives the dimension of the manifold of framed monopoles.

In conclusion, if d_2 is surjective, we are led to study when $d_1^* \oplus d_2$ is a Fredholm map and compute its index. This will coincide with the dimension of $T_{[m]}\mathcal{M}^{\alpha,l} \cong T_{[m]}\mathcal{M}$ and therefore with the dimension of the moduli space of framed monopoles.

To answer these questions, it is convenient to express $\mathcal{D}_{(A,\Phi)} = d_1^* \oplus d_2$ as a twisted Dirac operator plus a potential term. In fact, it is easy to check that,

$$\mathcal{D}_{(A,\Phi)}(a, \varphi) = \begin{pmatrix} *d_A & -d_A \\ -d_A^* & 0 \end{pmatrix} \begin{pmatrix} a \\ \varphi \end{pmatrix} + \begin{pmatrix} [\Phi, a] \\ [\Phi, \varphi] \end{pmatrix}, \tag{4.10}$$

where $(a, \varphi) \in \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^1 \oplus \wedge^0) \otimes \operatorname{ad} P \otimes \mathbb{C})$.

The Fredholmness and the index of this operator will be analysed in the next section. First we deal with the surjectivity of d_2: it turns out that monopoles are always regular. This is a well-known result that follows from an easy application of the Weitzenbock formula:

Proposition 4.2.1. *Let (M, g) be a manifold and (A, Φ) a smooth solution to the Bogomolny equations, if $\mathcal{D}_{(A,\Phi)}$ denotes the smooth version of (4.1), then acting on sections of the bundle $(\Omega^0 \oplus \Omega^1) \otimes \operatorname{ad} P$ we have,*

$$\mathcal{D}_{(A,\Phi)}\mathcal{D}^*_{(A,\Phi)} = \nabla_A^* \nabla_A + \operatorname{ad} \Phi^* \operatorname{ad} \Phi + Ric, \tag{4.11}$$

where Ric is the Ricci tensor of g acting on the 1-form part.

The same equation holds when the operators act on sections of the hybrid Sobolev spaces. In our case, the scattering metric is flat so the last term vanishes. Moreover, $\mathcal{D}_m\mathcal{D}_m^*$ acting on $\mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^0 \oplus \wedge^1) \otimes \operatorname{ad} P)$ is positive definite since the elements in the kernel must be covariant constant and vanish at the boundary. This implies that d_2 is surjective.

4.3 Index computation.

E. J. Weinberg observed in [57] that the linearisation around a solution of the Bogomolny equations leads to the operator (4.10), which is almost a Callias-type operator. The difference is that the potential term is degenerate at infinity. At that moment there was no theory available to compute its index, but some years later, C. Kottke in [32] defined hybrid Sobolev spaces where the operator is extended, and that away from some discrete set of weights it is Fredholm. He also provided an explicit formula for the index of these extended Fredholm operators and in [33], he used this theory to compute the dimension of the moduli space of $SU(2)$-monopoles over an asymptotically scattering manifold of dimension 3. We review first and then apply Kottke's index formula (4.63) to the case of $SU(N)$-monopoles.

Using the isomorphism between the 0-forms and the 3-forms given by the Hodge star operator, and denoting by E the complexification of $(\wedge^1 \oplus \wedge^3) \otimes \operatorname{ad} P$, the map $d_1^* \oplus d_2$ in (4.10) acts on $\mathcal{H}^{\alpha,\alpha+l,l}(X;E)$ as

$$\mathcal{D}_{(A,\Phi)} = *\tau(d_A + d_A^*) + \operatorname{ad}\Phi, \tag{4.12}$$

with τ a sign operator, being -1 on 0-forms and 1 on 2-forms. The first term $*\tau(d_A + d_A^*)$ will be denoted as D_A and the subindex (A,Φ) of $\mathcal{D}_{(A,\Phi)}$ will be suppressed for easy of notation.

Over a collar neighbourhood U of the boundary, the bundle $\operatorname{ad} P \otimes \mathbb{C}$ splits as

$$\operatorname{ad} P \otimes \mathbb{C} = \operatorname{ad} P_0 \oplus \operatorname{ad} P_1, \tag{4.13}$$

where $\operatorname{ad} P_0$ denotes the bundle elements that commute with an extension of ϕ over U as in (3.24), and $\operatorname{ad} P_1$ its orthogonal space. In other words, if we denote by H the stabiliser in $SU(N)$ of ϕ, i.e.

$$H = S\left(U(n_1) \times \cdots \times U(n_q)\right), \tag{4.14}$$

where S stands for having determinant 1, and the n_j are the number of repetition for the j-th eigenvalue of ϕ, then the bundle $\operatorname{ad} P_0$ is obtained from the principal H-bundle via the adjoint action on its Lie algebra.

In the maximal symmetry case this stabiliser is just the abelian group $U(1)^{N-1}$ and as the action is trivial, the associated $\operatorname{ad} P_0$-bundle is the trivial bundle of rank $N-1$.

The splitting of E associated with that of $\operatorname{ad} P$ is

$$E = E_0 \oplus E_1. \tag{4.15}$$

This splitting induces an splitting over U of the operator $\mathcal{D}$ as $\mathcal{D}_0 + \mathcal{D}_1$, where

$$\begin{aligned} \mathcal{D}_0 &= \mathcal{D}|_{E_0}, \\ \mathcal{D}_1 &= \mathcal{D}|_{E_1} = D_1 + \mathbb{1} \otimes (\operatorname{ad}(\Phi)|_{\operatorname{ad} P_1}). \end{aligned} \tag{4.16}$$

The operator $\mathcal{D}_1$ is a standard Callias-type operator, it is fully elliptic in the sc-calculus and therefore it is a Fredholm operator. Moreover, the Fredholmness and the index are independent of the weights because the elements in the kernel of $\mathcal{D}_1$ and of its adjoint are smooth sections that vanish to an infinite order at the boundary [38].

On the other hand, $\mathcal{D}_0$ is not fully elliptic as a scattering operator, so one cannot immediately conclude as before that it has Fredholm extensions. As the connection restricts to a smooth connection at the boundary, it must be a lift of a b-connection and therefore over U one can define an associated b-operator by factoring out x,

$$\tilde{D}_0 := x^{-2}\mathcal{D}_0 x : x^\alpha H^l_b(U; E_0 \otimes \Omega_b^{1/2}) \to x^\alpha H^{l-1}_b(U; E_0 \otimes \Omega_b^{1/2}). \tag{4.17}$$

The indicial operator $I(\tilde{D}_0)$ is obtained from $\tilde{D}_0$ by evaluating its coefficients at the boundary $\{x = 0\}$, i.e., locally

$$\sum a_{i,\beta}(x, y)(x\partial_x)^i(\partial_y)^\beta \text{ maps to } \sum a_{i,\beta}(0, y)(x\partial_x)^i(\partial_y)^\beta, \tag{4.18}$$

where β is a multi-index.

We have included the b-half-density bundles on the domain to make explicit that the b-measured is used in the definition of $\tilde{D}_0$, while the scattering measure was used for $\mathcal{D}_0$. The way the x was factored out from $\mathcal{D}_0$ was so that, it agrees with the convention adopted in [33]. It has the advantage that $\mathcal{D}_0$ is formally self-adjoint with respect to the scattering metric g if and only if $\tilde{D}_0$ is formally self-adjoint with respect to the b-metric $x^2 g$.

The map $\tilde{D}_0$ is elliptic as a b-operator, and hence by Theorem 5.2.1 it is Fredholm as long as α is outside the discrete set of its indicial roots.

We proceed now to recall the definition of the signature operator and to find the expression for D_1 and $\tilde{D}_0$.

Signature operator. We recall briefly the construction of the signature operator on an oriented manifold M of dimension n. The Clifford action on $\Omega^*_{\mathbb{C}}(T^*M)$ is given by,

$$c(e_j)\xi = (e_j \wedge \xi - e_j \lrcorner \xi), \tag{4.19}$$

where $\{e_j\}$ is an orthonormal frame for the tangent bundle and ξ a form in $\Omega^*_{\mathbb{C}}(T^*M)$. The associated Dirac operator is

$$\sum c(e_j)\nabla^{LC}_{e_j} = d + d^*. \tag{4.20}$$

The bundle $T^*M \otimes \mathbb{C}$ has a canonical grading given by the chirality operator,[1]

$$c(\Gamma) := i^{[n/2]} c(e_1) \cdots c(e_n). \tag{4.21}$$

[1] if the dimension of the manifold is $n = 2k$ then $[n/2] = k$ and if $n = 2k + 1$ then $[n/2] = k + 1$.

The Clifford action of the chirality operator can be expressed without making use of the Clifford module structure, in particular, it acts as on p-forms as

$$i^{[n/2]+p(p-1)}*, \tag{4.22}$$

where, as with the chirality operator, the exponent is fixed so that the operator square is the identity. This is a grading operator and induces the natural $\mathbb{Z}_2$-grading:

$$\Omega^*_{\mathbb{C}}(T^*M) = \Omega^+(T^*M) \oplus \Omega^-(T^*M) \tag{4.23}$$

corresponding to the ± 1 eigenspaces of $c(\Gamma)$. The Dirac operator $(d + d^*)$ anti-commutes with the chirality operator and splits accordingly to this grading, giving $\eth_s^+ + \eth_s^-$ where

$$\eth_s^+ : \Omega^+(T^*M) \to \Omega^-(T^*M), \tag{4.24}$$

is the signature operator. The index of the signature operator is precisely the signature of the manifold M, in particular it is zero when the dimension of M is not a multiple of 4. We will need a twisted version of this operator with a grading determined by the mass section. More precisely, as ϕ is covariantly constant (2.52), its eigenvalues are constant. We adopt the convention that the eigenvalues of $i\phi$ are ordered in non-decreasing value. In this way the bundle $\operatorname{ad} P_+$ consisting of the positive eigenspace bundle of $i\phi$ is well-defined.

Definition 4.3.1. The twisted (by $\operatorname{ad} P$) signature operator acting on the space $\mathcal{H}^{\alpha,\alpha+l,l}(\partial X; \wedge^+ \otimes \operatorname{ad} P_+)$ will be denoted as $\eth_+^+$.

Expression for $\tilde{D}_0$. It can be observed that on an odd dimensional manifold, if we compose the above Clifford action (4.19) with the Hodge star we obtain an action on the odd forms. The value of the sign operator $*\tau$ in equation (4.12) is precisely the value of the chirality operator acting on the even forms of X. Therefore, if in our three dimensional manifold we use the isomorphism between functions and 3-forms given by the Hodge star operator, the first term in (4.12) can be written as

$$\begin{aligned} D_A = *\tau(d_A + d_A^*) = c(\Gamma)\left(\sum_{i=0} c(e_i)\nabla_{e_i}\right) \\ = c(\Gamma e_0)\left(\nabla_{e_0} + \sum_{i=1} c(e_i e_0)\nabla_{e_i}\right), \end{aligned} \tag{4.25}$$

where $\nabla = \nabla^{LC} \otimes A$. In order to re-write the last term of the above equation, we state a couple of general facts.

Let (X, g) be an n-dimensional manifold with boundary ∂X and scattering metric g, which near the boundary can be written as in (3.3),

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}, \tag{4.26}$$

where x is a boundary defining function and h is a metric on ∂X. Let $\{e_i\}_{i=0}^{n-1}$ be an orthonormal *sc*-frame with $e_0 = x^2\partial_x$.

There is a bundle isomorphism -which is another form of the standard isomorphism $\mathbb{C}l^0(\mathbb{R}^{n+1}) \cong \mathbb{C}l(\mathbb{R}^n)$,

$$\Psi : (\wedge^{odd} X)|_{\partial X} \to \wedge^* \partial X \tag{4.27}$$

$$e_0 \wedge e_I \mapsto -e_I, \tag{4.28}$$

$$e_J \mapsto e_J, \tag{4.29}$$

where the multi-index I has even cardinality, J has odd cardinality and e_0 does not appear in the e_I, e_J. With this isomorphism, it is immediate to check that the action of $c(e_i e_0)$ for $i \neq 0$ on $(\wedge^{odd} X)|_{\partial X}$ is equivalent to the action of $c(e_i)$ on $\wedge^* \partial X$, i.e.,

$$c(e_i e_0) = \Psi^{-1} c(e_i) \Psi. \tag{4.30}$$

The other fact that we need is the following result (Proposition 4.1 in [33]).

Proposition 4.3.2. *Let (X^n, g) be a scattering manifold with metric as in (4.26). The Levi-Civita connection ∇^{LC} induced by the scattering metric is the lift of a b-connection, in particular*

$$\nabla^{LC}_{e_i} = x\left(\nabla^{\tilde{g}}_{\tilde{e}_i} + B(\tilde{e}_i)\right), \tag{4.31}$$

where $\tilde{g} = x^2 g$, $\nabla^{\tilde{g}}$ is the Levi-Civita connection induced by this b-metric, $\{\tilde{e}_i\}_{i=0}^{m-1}$ is an orthonormal b-frame with $e_i = x\tilde{e}_i$ and the endomorphism of the b-tangent space $B(\tilde{e}_i)$ acting on $\tilde{e}_k$, for $k = 0, ..., m-1$, is given by

$$B(\tilde{e}_i)(\tilde{e}_k) = \begin{cases} \delta_{ki}\tilde{e}_0 - \delta_{k0}\tilde{e}_i, & i > 0. \\ 0, & i = 0. \end{cases} \tag{4.32}$$

With the aid of this last equation one can check that if $\xi \in \Omega^p(\partial X)$,

$$\sum_{i=1}^{n-1} c(\tilde{e}_i) B(\tilde{e}_i)\xi = \begin{cases} -p\xi, & p \text{ odd.} \\ -(n-1-p)\xi, & p \text{ even.} \end{cases} \tag{4.33}$$

These considerations together with the asymptotic expansion of the Higgs field lead to the explicit form of $\tilde{D}_0$.

Proposition 4.3.3. *The operator*

$$I(\tilde{D}_0) : x^\alpha H^l_b \left(U; \pi^*(\wedge^* S^2_\infty \otimes \operatorname{ad} P_0)\right) \to x^\alpha H^{l-1}_b \left(U; \pi^*(\wedge^* S^2_\infty \otimes \operatorname{ad} P_0)\right)$$

is given by

$$I(\tilde{D}_0) = -ic(\Gamma_{S^2_\infty}) \left[x\partial_x + (d_{A_0} + d^*_{A_0})_{S^2_\infty} + N + W\right], \tag{4.34}$$

where N takes the values $-1, 0, 1$ respectively on $0, 1, 2$-forms over S^2_∞ with values in $\operatorname{ad} P_0$ and W is $-\frac{i}{2}c(\Gamma_{S^2_\infty}) \otimes \operatorname{ad}(\gamma_m)$.

Proof. The expression is obtained from that of D_A in (4.25) and the potential term $\operatorname{ad}\Phi$. The definition of the chirality operator (4.21) gives the isomorphism $c(\Gamma e_0) \cong -ic(\Gamma_{\partial X})$. The second summand comes from (4.30) and the previous proposition, it is a twisted signature operator on S^2_∞. The term $x\partial_x$ comes from considering the connection in radial gauge, so that $\nabla_{e_0} = x^2\partial_x$, this adds 1 to (4.33), specifically $x^{-2}(x^2\partial_x)x = 1 + x\partial_x$, giving the value of N. These terms are common to the $SU(2)$ case [33].

On the other hand, the term W does not appear in the case where the structure group is $SU(2)$, or more generally in the maximal symmetry breaking case. It is due to the term of order x in the asymptotic expansion of the Higgs field (3.10), i.e. it is

$$W = -\frac{i}{2}c(\Gamma_{S^2_\infty}) \otimes \operatorname{ad}(\gamma_m). \tag{4.35}$$

□

Computation of the indicial roots of $\tilde{D}_0$. We proceed to recall some definitions and results that will be needed in the computation of the index of $\mathcal{D}$.

The indicial family of $\tilde{D}_0$, denoted $I(\tilde{D}_0, \lambda)$, is a family of differential operators over ∂X depending parametrically on $\lambda \in \mathbb{C}$. More precisely, if the local expression for $I(\tilde{D}_0)$ is $\sum a_{i,\beta}(0, y)(x\partial_x)^i(\partial_y)^\beta$ then

$$I(\tilde{D}_0, \lambda) = \sum a_{i,\beta}(0, y)\lambda^i(\partial_y)^\beta, \tag{4.36}$$

i.e. it is the Mellin-transformed operator of $I(\tilde{D}_0)$.

The b-spectrum of $\tilde{D}_0$ is the set of its indicial roots, and is defined as:

$$\operatorname{spec}_b \tilde{D}_0 = \{\lambda \in \mathbb{C} : \exists s \in C^\infty(\partial X; \wedge^* S^2_\infty \otimes \operatorname{ad} P_0) \setminus \{0\}, I(\tilde{D}_0, \lambda)s = 0\}. \tag{4.37}$$

We shall make use of Grothendieck's lemma [17]:

Theorem 4.3.4. *A holomorphic bundle $\mathcal{E}$ over $\mathbb{C}P^1$ is holomorphically isomorphic to a direct sum of line bundles:*

$$\mathcal{E} \cong \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n). \tag{4.38}$$

Moreover, this representation is unique up to a permutation of the factors.

Let $\underline{\mathbb{C}}^N$ be the trivial $\mathbb{C}^N$-bundle over S^2_∞. As $d_{A_0}\gamma_m = 0$, there is a decomposition of $\underline{\mathbb{C}}^N$ in the eigenspaces of $-i\gamma_m = \mathrm{diag}(k_1, ..., k_N)$ as,

$$\underline{\mathbb{C}}^N \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_N) =: L_1 \oplus \cdots \oplus L_N, \tag{4.39}$$

where the relation of the magnetic charge section with the curvature (2.59) has been used to stablish the degrees of the constituent line bundles.

The complex bundle $\mathrm{ad}\, P \otimes \mathbb{C} \subset \mathrm{End}(\underline{\mathbb{C}}^N)$ can be provided with a holomorphic structure via the unitary connection A_0. If we apply Grothendieck's lemma to write this bundle as a direct sum of holomorphic line bundles, then by the uniqueness property, the factors must be

$$\mathcal{O}(k_i - k_j) = L_i \otimes L_j^* := L_{ij}. \tag{4.40}$$

It follows from the definition of b-spectrum, that the indicial roots given by $I(\tilde{D}_0)$ are the union over the line bundles L_{ij} of the indicial roots of $I(\tilde{D}_0)$ restricted to $\pi^*(\wedge^* S^2_\infty \otimes L_{ij})$, where L_{ij} are the line bundles appearing in the factorisation of $\mathrm{ad}\, P_0$ provided by Grothendieck's lemma.

Proposition 4.3.5. *Let k_0 (respectively k_1) be the smallest of the absolute value of the odd (even) degree of the line bundles appearing in the decomposition of* $\mathrm{ad}\, P_0$ *over U, then the indicial roots of $\tilde{D}_0$ are contained in the set*

$$\left\{\pm\left(\frac{|k_0|}{2} + n\right), \quad n \in \{1, 2...\}\right\} \bigcup \left\{\pm\left(\frac{|k_1|}{2} + n\right), \quad n \in \{1, 2...\}\right\}. \tag{4.41}$$

Proof. Let $\xi := \frac{1}{2}\,\mathrm{ad}\,\gamma_m$, then the last term in the expression for $I(\tilde{D}_0)$ in (4.34) can be written as,

$$W = c(\Gamma_{S^2_\infty}) \otimes -i\xi. \tag{4.42}$$

The chirality operator acts on sections of the bundle $\wedge^* S^2_\infty \otimes \mathrm{ad}\, P_0$ over S^2_∞ via (4.22) as,

$$c(\Gamma_{S^2_\infty}) = i \begin{bmatrix} 0 & 0 & -* \\ 0 & * & 0 \\ * & 0 & 0 \end{bmatrix}. \tag{4.43}$$

Therefore, by Proposition 4.3.3 and using of the factorisation of $\mathrm{ad}\, P_0$ into line bundles, to obtain the indicial roots of $\tilde{D}_0$ we have to find the $\lambda \in \mathbb{R}$ that satisfy,

$$\begin{bmatrix} \lambda - 1 & d_A^* & -\xi* \\ d_A & \lambda + \xi* & d_A^* \\ \xi* & d_A & \lambda + 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ *c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.44}$$

where a, c are sections of L_{ij} and $b \in \Omega^1(L_{ij})$, and we have discarded the first factor $-ic(\Gamma_{S^2_\infty})$ in (4.34) since it is an isomorphism. This leads to

$$\begin{cases} (\lambda-1)a + \delta_A b - \xi c = 0, \\ d_A a + (\lambda + \xi *)b - *d_A c = 0, \\ \xi * a + d_A b + (\lambda+1) * c = 0. \end{cases} \tag{4.45}$$

The middle equation splits in two according to its $(1,0)$ and $(0,1)$ parts, the Hodge star acts as $* = -i$ on $\Omega^{1,0}$ and as $* = i$ on $\Omega^{0,1}$:

$$\begin{cases} \partial_A(a+ic) + (\lambda - i\xi)b_1 = 0, \\ \overline{\partial_A}(a-ic) + (\lambda + i\xi)b_2 = 0, \end{cases} \tag{4.46}$$

where $b = b_1 + b_2 \in \Omega^{1,0} \oplus \Omega^{0,1}$. Adding and subtracting the first and third equations from (4.45) gives the two identities,

$$\begin{cases} 2 * \partial_A b_2 = -(\lambda + 1 - i\xi)c - i(\lambda - 1 - i\xi)a =: g_2(a,c), \\ 2 * \overline{\partial_A} b_1 = -(\lambda + 1 + i\xi)c - i(-\lambda + 1 - i\xi)a =: g_1(a,c). \end{cases} \tag{4.47}$$

If we take $-i\partial_A^*$ on the first equation, and $i\overline{\partial_A}^*$ on the second equation of (4.46), and use that $*\partial_A b_2 = i\overline{\partial_A}^* b_2$ and $*\overline{\partial_A} b_1 = -i\partial_A^* b_1$ we can write,

$$\begin{cases} -i\partial_A^*\partial_A(a+ic) = -\frac{1}{2}(\lambda - i\xi)g_1(a,c) =: G_1(a,c), \\ i\overline{\partial_A}^*\overline{\partial_A}(a-ic) = -\frac{1}{2}(\lambda + i\xi)g_2(a,c) =: G_2(a,c). \end{cases} \tag{4.48}$$

Adding and subtracting these two equations we obtain an equivalent system,

$$\begin{cases} i\triangle_A a + (\overline{\partial_A}^*\overline{\partial_A} - \partial_A^*\partial_A)c = G_2 - G_1, \\ i(\overline{\partial_A}^*\overline{\partial_A} - \partial_A^*\partial_A)a + \triangle_A c = G_2 + G_1. \end{cases} \tag{4.49}$$

The difference of the two partial Laplacians, $\partial_A^*\partial_A - \overline{\partial_A}^*\overline{\partial_A}$ acting on a 0-form η is (c.f. Lemma 5.9 in [1]):

$$(\partial_A^*\partial_A - \overline{\partial_A}^*\overline{\partial_A})\eta = i[*F_A, \eta] = -i\xi\eta, \tag{4.50}$$

where in the last equality it has been used that as the Bogomolny equations hold, $*F_A = -\frac{1}{2}\gamma_m$. Therefore the difference of the Laplacians has the same action as $-i\xi$, which on a line bundle L_{ij} of degree k acts as $\frac{k}{2}$. Putting this together in the previous system of equations (4.49),

$$\begin{cases} i\triangle_A a + \frac{k}{2}c = G_2 - G_1 = -i\xi c + ia[\lambda(\lambda-1) + \xi^2], \\ i\frac{k}{2}a + \triangle_A c = G_2 + G_1 = [(\lambda+1)\lambda + \xi^2]c + \xi a. \end{cases} \tag{4.51}$$

Again using that $\xi = i\frac{k}{2}$, it is immediate to see that this system is the same as,

$$\begin{cases} \triangle_A a = [\lambda(\lambda-1) - \frac{k^2}{4}]a, \\ \triangle_A c = [\lambda(\lambda+1) - \frac{k^2}{4}]c. \end{cases} \tag{4.52}$$

Following Kuwabara [35], the eigenvalues of the covariant Laplacian for smooth sections of $\mathcal{O}(k)$ are $\{f_k^n\}_{n\in\mathbb{N}_0}$, where

$$f_k^n = \frac{(2n+1)|k|}{2} + n^2 + n,\ n \in \{0,1,2...\}. \tag{4.53}$$

So we have the following possibilities:

- If $a = 0$, we have that the indicial roots must be among the following:

$$\lambda = \frac{-1 \pm \sqrt{1 + k^2 + 4f_k^n}}{2}. \tag{4.54}$$

- For $c = 0$, we obtain the opposite roots of the above ones:

$$\lambda = \frac{1 \mp \sqrt{1 + k^2 + 4f_k^n}}{2}. \tag{4.55}$$

- The possibility $a \neq 0$ and $c \neq 0$ only holds when $\lambda = 0$. In this case a and c have negative eigenvalue contradicting the non-negativity of the covariant Laplacian $\triangle_A$ unless $k = 0$. In fact, it is easy to see that for $k = 0$, the set of λ's that satisfy (4.52) is the integer numbers.

In summary, for k the degree of the line bundle L_{ij}, the set of λ's that satisfy (4.52) can be written as:

$$\pm\lambda = \frac{-1 \pm (|k| + 2n + 1)}{2}, \quad n \in \{0,1,2...\}. \tag{4.56}$$

Moreover, we next show that $\lambda = \pm\frac{k}{2}$ are not indicial roots. By the symmetry of the indicial roots with respect to the origin it is enough to show that $\lambda = \frac{k}{2}$ is not an indicial root for $k \geq 0$. We assume that $\frac{k}{2}$ is an indicial root to obtain a contradiction. If it were an indicial root, $a = 0$ and (4.46) implies

$$\begin{cases} \overline{\partial}_A c = 0, \\ -i\partial_A c = kb_1. \end{cases} \tag{4.57}$$

By the first and third equation of (4.45) we must have,

$$\triangle_A b = i\frac{k}{2}d_A c + \frac{k+2}{2}d_A * c = -i\frac{k+1}{2}\partial_A c = \frac{k(k+1)}{2}b_1, \tag{4.58}$$

where in the last two equalities we have used (4.57). Therefore

$$\begin{cases} \triangle_A b_1 = \frac{k(k+1)}{2}b_1, \\ \triangle_A b_2 = 0. \end{cases} \tag{4.59}$$

As b_1, b_2 are respectively sections (obviously not holomorphic) of $\mathcal{O}(k-2)$ and $\mathcal{O}(k+2)$, this last equations cannot hold, since f_{k+2}^n is always positive, obtaining a contradiction.

In conclusion, there are no indicial roots in the interval $[-\frac{|k|}{2}, \frac{|k|}{2}]$ and the set of indicial roots is contained in the set

$$\left\{ \pm \left(\frac{|k|}{2} + n \right), \quad n \in \{1, 2...\} \right\}. \tag{4.60}$$

The proposition follows from the observation after the definition of L_{ij} in equation (4.40). □

Remark 4.3.6. It can be observed that the formal nullspace associated with $\lambda = 0$ (and therefore $k = 0$) could be obtained directly from the system (4.44). It is the space of Harmonic 1-forms, which by Hodge theory corresponds to $H^1(S^2_\infty; \mathbb{R})$ and that is why $\lambda = 0$ is not an indicial root.

Kuwabara showed [35], that the multiplicity of the eigenspace of the covariant Laplacian with eigenvalue f_k^n is given by

$$|k| + 2n + 1. \tag{4.61}$$

With this multiplicity, one can calculate the dimension of the nullspace of $\tilde{D}_0$ at an indicial root. For example, when the indicial root λ is 1, then k must vanish and $n = 0$ or $n = 1$ (as shown in the previous proof: $k = 2, n = 0$ is not valid). In this case the nullspace in the form component consists of

$$\{(a, 0, 0) : da = 0\} \cup \{(0, -d^* * c, c) : \triangle c = 2c\}, \tag{4.62}$$

which has multiplicity $1 + 3 = 4$. By symmetry, $\lambda = -1$ has multiplicity $4m$ where m is the number of degree 0 line bundles appearing in the decomposition of $\operatorname{ad} P_0$.

Remark 4.3.7. It follows from the work of Jarvis [29] that the moduli spaces of framed monopoles obtained by a smooth deformation of the mass section ϕ are diffeomorphic. In particular we can assume that the eigenvalues of ϕ are pairwise distinct.

With all the previous notation and results, Theorem 3.1 in [33] which is valid in the maximal symmetry case, will be now stated and used to compute the index of $\mathcal{D}$.

Theorem 4.3.8. *The extensions $\mathcal{D} : \mathcal{H}^{\alpha,\alpha+l,l}(X;E) \to \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X;E)$ are bounded and Fredholm for $\alpha + 1/2$ outside the b-spectrum of $\tilde{D}_0$. The index of $\mathcal{D}$ is given by*

$$\operatorname{ind}(\mathcal{D}) = \operatorname{ind}(\eth_+^+) + \operatorname{def}(\mathcal{D}, \alpha). \tag{4.63}$$

where $\operatorname{def}(\mathcal{D}, \alpha)$ *is the defect of $\tilde{D}_0$ at $\alpha + 1/2$ i.e.* $\operatorname{def}(\tilde{D}_0, \alpha + 1/2)$.

The shift in 1/2 comes from the way the x was factored out in (4.17), that is, with respect to the sc-measure we took $\mathcal{D}_0 = x^{1/2}\tilde{D}_0 x^{1/2}$ instead of $x\tilde{D}_0$.

The first term is the index of $\mathcal{D}_1$, which follows from the homotopy invariance of the index. This is the only term appearing when the potential is non-degenerate at infinity (this is Callias index formula [5]).

The second term, the defect of $\mathcal{D}$ at α is due to the non-invertibility of $\operatorname{ad}\phi$. It is an integer such that for $\epsilon > 0$ with $\operatorname{spec}_b(\tilde{D}_0)\cap[\alpha_0-\epsilon,\alpha_0+\epsilon] = \alpha_0$ satisfies,

$$\operatorname{def}(\tilde{D}_0, \alpha_0 - \epsilon) - \operatorname{def}(\tilde{D}_0, \alpha_0 + \epsilon) = \dim F(\tilde{D}_0, \alpha_0), \tag{4.64}$$

where $\alpha_0 \in \operatorname{spec}_b(\tilde{D}_0)$ and $F(\tilde{D}_0, \alpha_0)$ is the formal nullspace of $\tilde{D}_0$ at α_0. Moreover, if $\tilde{D}_0$ is self-adjoint, then

$$\operatorname{def}(\tilde{D}_0, \beta) = -\operatorname{def}(\tilde{D}_0, -\beta). \tag{4.65}$$

Theorem 4.3.9. *The dimension of the manifold of framed monopoles with structure group $SU(N)$ is four times the sum of the magnetic weights.*

Proof. Due to the self-adjointness of $\tilde{D}_0$, one can apply the above equation and as there are no indicial roots in the interval $(-1, 1)$, if the weight $\alpha+1/2$ is a small positive number (less than 1), the defect term has no effect in the index of $\mathcal{D}$.

On the other hand, the index of the twisted signature operator is given by (Theorem 13.9 in [36]),

$$\operatorname{ind}(\eth_+^+) = 2\{\operatorname{ch}(\operatorname{ad} P_+)\cdot \mathbf{L}(S^2_\infty)\}[S^2_\infty] = 2c_1(\operatorname{ad} P_+)[S^2_\infty]. \tag{4.66}$$

If we write

$$\operatorname{ad} P_+ \cong \bigoplus_{i<j} L_i \otimes L_j^* = \bigoplus_{i<j} L_{ij}. \tag{4.67}$$

Then if we use that $c(E\oplus F) = c(E) \smile c(F)$, and thus $c_k(E\oplus F) = \sum c_i(E) \smile c_{k-i}(F)$,

$$\operatorname{ind}(\eth_+^+) = 2\sum_{i<j}(k_i - k_j). \tag{4.68}$$

In the case of maximal symmetry breaking the L_{ij} in (4.67) does not appear in the factorisation of $\operatorname{ad} P_0$, and using that $\sum k_i = 0$, the last equation can be re-written as

$$\operatorname{ind}(\eth_+^+) = -4\sum_{j=1}^{N} jk_j = 4\sum_{j=1}^{N-1}(k_1+\cdots+k_j) = 4\sum_{j=1}^{N-1} n_j. \tag{4.69}$$

where the n_j are the magnetic weights i.e. the topological and holomorphic charges as defined in Section 2.7. □

This result agrees with the computation in Murray-Singer's paper [42], which was based in the bijection established in [29] between the space of framed monopoles and the space of based rational maps.

4.4 Further directions

A natural thing one would like to do is to develop an index theorem which works independently of the type of symmetry breaking at infinity. In this way one does not have to rely on Jarvis construction to assume the maximal symmetry case.

Chapter 5

Appendix

5.1 Weighted b/sc-Sobolev spaces

A standard reference for Sobolev spaces of sections is [44]. The proofs in there can be adapted to show the corresponding results for weighted b/sc-Sobolev spaces, we follow [38], [32] to list some of the properties that are being used in the rest of the thesis together with proofs for some of the statements.

Let X be a compact n-dimensional manifold with x a boundary defining function taking values in $[0, \epsilon)$ for $\epsilon < 1$. Let E be a vector bundle over X, most of the results in this section hold for both sc and b Sobolev spaces, so in what follows we shall write $x^\alpha L^p_k(E)$ for either the x^α-weighted b-Sobolev space of sections of the vector bundle $E \otimes \Omega_b^{1/2}$ with k (although for most of the results it is not necessary, we shall assume that k is a non-negative integer) weak derivatives in L^p or a corresponding weighted sc-Sobolev space on $E \otimes \Omega_{sc}^{1/2}$, i.e.

$$x^\alpha L^p_k(E) := \left\{ u \in x^\alpha L^p(X; E \otimes \Omega^{1/2}_{b/sc}) : Pu \in x^\alpha L^p(X; E \otimes \Omega^{1/2}_{b/sc}), \right. \\ \left. \forall P \in \mathrm{Diff}^k_{b/sc} \right\}, \quad (5.1)$$

where $\mathrm{Diff}^k_{b/sc}$ is the universal enveloping algebra of the b/sc vector fields $\mathfrak{X}_b(X)/\mathfrak{X}_{sc}(X)$ over $C^\infty(X; \mathrm{End}(E))$, and $\Omega^{1/2}_{b/sc}$ are the half-densities b/sc bundles that encode if we are working with the b/sc measure; with these measures related by $\mu_b = x^3 \mu_{sc}$ i.e.,

$$\Omega_b^{1/2} = x^{3/2} \Omega_{sc}^{1/2}. \quad (5.2)$$

When E is the trivial bundle $\underline{\mathbb{C}}$ we shall denote the weighted Sobolev space $x^\alpha L^p_k(E)$ simply as $x^\alpha L^p_k$.

In the particular case $p = 2$, which is the main case, we shall write the spaces $x^\alpha L^2_k(E)$ as $x^\alpha H^k_{b/sc}(X; E)$.

Similarly as in Theorem 2.21 in [3] we have an embedding theorem,

Theorem 5.1.1. *Let $1 \leq p, p' < \infty$, and let α, α', k, k' be real numbers. If the decaying rate, the regularity and the 'scaling weight' improve, i.e. if $\alpha \geq \alpha'$, $k \geq k'$ and $(k - n/p) \geq (k' - n/p')$ then the inclusion map,*

$$x^\alpha L^p_k(E) \hookrightarrow x^{\alpha'} L^{p'}_{k'}(E). \tag{5.3}$$

is bounded. There is also an embedding $x^\alpha L^p_k \hookrightarrow C^l(X)$ if $(k - n/p) > l$ and α is non-negative.

We shall use just the case where $p = p' = 2$, and in this case, if the above inequalities are strict then the inclusion is compact (Proposition 1.2 (b) [32]).

Also we need a theorem analogous to the non-weighted case [8]:

Theorem 5.1.2. *If $H : \mathbb{R} \to \mathbb{R}$ is a smooth function and f is a function in L^2_k with $k - n/2 > 0$ then Hf is again a function in L^2_k. Moreover this composition defines a smooth map from L^2_k to itself.*

When weights are included, the composition with smooth functions preserves the regularity although not the decaying rate.

Theorem 5.1.3. *Let $1 < p, q, r < \infty$. The bilinear map*

$$x^\alpha L^p_k \times x^\beta L^q_{k'} \to x^\gamma L^r_m, \tag{5.4}$$

given by multiplication is continuous if $\alpha + \beta \geq \gamma$, $\min(k, k') \geq m \geq 0$, $k - n/p < 0$, $k' - n/q < 0$, and

$$(k - n/p) + (k' - n/q) \geq m - n/r. \tag{5.5}$$

Proof. The proof, as in the non-weighted case is based on the Sobolev embedding theorem and Hölder inequality. Consider a partial derivative of order m of the product of two functions $f \in x^\alpha L^p_k$, $g \in x^\beta L^q_{k'}$,

$$\nabla^m(fg) = \sum_{i=0}^{m} \binom{m}{i} \nabla^i f \nabla^{m-i} g. \tag{5.6}$$

By Hölder's inequality we have that each term in the sum is uniformly bounded:

$$\begin{aligned} \|\nabla^i f \nabla^{m-i} g\|_{r,\gamma} &= \|x^{-\gamma} \nabla^i f \nabla^{m-i} g\|_r \leq \|(x^{-\alpha}\nabla^i f)(x^{-\beta}\nabla^{m-i} g)\|_r \\ &\leq \|x^{-\alpha}\nabla^i f\|_s \|x^{-\beta}\nabla^{m-i} g\|_t = \|\nabla^i f\|_{s,\alpha} \|\nabla^{m-i} g\|_{t,\beta}, \end{aligned} \tag{5.7}$$

if $1/r \geq 1/s + 1/t$ which is equivalent to $(i - n/s) + (m - i - n/t) \geq (m - n/r)$. Now we use the previous Sobolev embedding theorems to embed for each i each of the two factor in (5.7) into $x^\alpha L^p_k$ or $x^\beta L^q_{k'}$. For example, for the first factor, using the regularity hypothesis we have $k \geq m \geq i$ and $k' \geq m \geq m - i$ there are two possibilities:

- If $k - n/p \leq i$ then take s such that $k - n/p = i - n/s$ so there is an embedding $x^\alpha L^p_k \subset x^\alpha L^i_s$ and as $f \in x^\alpha L^p_k$ the term $\|\nabla^i f\|_{s,\alpha}$ is bounded. There are two possibilities to bound the second factor:
  - If $k' - n/q \leq m - i$ we take t such that $k' - n/q = m - i - n/t$, then as before the second term is bounded and Hölder inequality holds because of the hypothesis.
  - The case $k' - n/q > m - i$ cannot happen, as $k' - n/q < 0$ and $m - i \geq 0$.
- If $k - n/p > i$, this case follows a symmetric pattern as the previous one.

□

To show the smoothness of the gauge action on the configuration space, the following result is needed.

Theorem 5.1.4. *Suppose that $L^p_k(E) \hookrightarrow L^q_{k'}(E)$ and $k - n/p > 0$. If E has a bilinear pointwise multiplication, (e.g. $E = \operatorname{ad} P$ with the multiplication given by the Lie bracket) then the multiplication of sections extends to a continuous map*

$$x^\alpha L^p_k(E) \times x^\beta L^q_{k'}(E) \to x^{\alpha+\beta} L^q_{k'}(E). \tag{5.8}$$

In particular, for the radial compactification of $\mathbb{R}^3$, the space $L^2_l(E)$ is a $L^2_{l+1}(E)$-module for $l > 3/2$.

Once the multiplication is continuous it is automatically smooth. This follows directly from the definition of derivative, if we assume the multiplication is continuous and denote by m the multiplication operator. Then for f, g small perturbations of the sections F, G and $d_{m(F,G)}(f, g) = Fg + Gf$, the quantity

$$\|m(F + f, G + g) - m(F, G) - d_{m(F,G)}(f, g)\|_{L^r_m}, \tag{5.9}$$

is the second order term $\|fg\|_{L^r_m}$ which by continuity of the multiplication can be bounded by $\|f\|_{L^p_k} \|g\|_{L^q_{k'}}$ and therefore it is differentiable. The higher order derivatives vanish.

Once a measure is fixed, say the sc-measure, from the definitions of the b/sc-Sobolev spaces and in particular from the relation $\mathfrak{X}_{sc}(X) = x\mathfrak{X}_b(X)$ between sc and b vector fields, there is a natural embedding

$$H^l_b(X, \Omega^{1/2}_{sc}) \subset H^l_{sc}(X, \Omega^{1/2}_{sc}). \tag{5.10}$$

The embedding can be reversed at the expense of increasing the weight in the scattering part.

Proposition 5.1.5. *If $\alpha \geq l + \beta$ there is an embedding,*

$$x^{\alpha} H^l_{sc}(X, \Omega^{1/2}_{sc}) \subset x^{\beta} H^l_b(X, \Omega^{1/2}_{sc}). \tag{5.11}$$

Proof. $u \in x^{\alpha} H^l_{sc}(X, \Omega^{1/2}_{sc})$ if and only if $x^{-\alpha} u \in H^l_{sc}(X, \Omega^{1/2}_{sc})$ i.e.

$$(x\mathfrak{X}_b)^l (x^{-\alpha} u) \in L^2_{sc} \sim x^l \mathfrak{X}^l_b (x^{-\alpha} u) \in L^2_{sc} \sim x^{l-\alpha} u + \cdots + x^{l-\alpha} \mathfrak{X}^l_b u \in L^2_{sc}. \tag{5.12}$$

For $x^{-\beta} u$ to be in $H^l_b(X, \Omega^{1/2}_{sc})$ it is sufficient that,

$$\mathfrak{X}^l_b (x^{-\beta} u) \in L^2_{sc} \sim x^{-\beta} u + \cdots + x^{-\beta} \mathfrak{X}^l_b u \in L^2_{sc}. \tag{5.13}$$

Therefore the inclusion is satisfied if $\alpha \geq l + \beta$. □

As it is shown in [32], if $\alpha > \alpha'$ and $l > l'$ then the inclusions

$$x^{\alpha} H^l_{sc}(X, \Omega^{1/2}_{sc}) \subset x^{\alpha'} H^{l'}_{sc}(X, \Omega^{1/2}_{sc}) \tag{5.14}$$

are compact, with a similar result for b-Sobolev spaces.

5.2 Motivation for weighted spaces and the fundamental theorem for elliptic operators in the b-Calculus.

We want to study the Fredholmness of the operator

$$\frac{d}{dt} + L, \tag{5.15}$$

acting on the tube $\mathbb{R} \times Y$ where Y is a compact manifold and L is an elliptic operator acting on sections of a bundle over Y.

When 0 is in the spectrum of L the operator cannot be Fredholm since the derivative operator,

$$\frac{d}{dt} : L^2_1(\mathbb{R}) \to L^2(\mathbb{R}) \tag{5.16}$$

is not Fredholm, as being Fredholm and injective implies that it has a bounded inverse on its range. This observation follows from the range being closed hence a Banach space, the inverse existing on the image (because the injectivity) and being bounded by the inverse mapping theorem. In this case, the operator is clearly injective (since among the constant functions only 0 is in $L^2_1(\mathbb{R})$). The problem comes from the inverse being bounded since we can construct a sequence of functions $\{f_n(t)\}$ whose derivative has bounded L_2 norm but $\|f_n\|_{L^2_1} \to \infty$ and therefore $\frac{d}{dt} : L^2_1(\mathbb{R}) \to L^2(\mathbb{R})$ cannot be Fredholm.

A strategy to have a Fredholm operator is to introduce weights (see for example chapter 3 of [7]) on our function spaces. That is, to a section s over a bundle over $\mathbb{R}_t \times Y$ we associate the norm

$$\|s\|_{L^{2,\alpha}} := \|e^{\alpha t} s\|_{L^2}, \tag{5.17}$$

with α a real number. Similarly we can define the norms that take into account k weak derivatives: $\|s\|_{L^{2,\alpha}_k}$, and as usual define the weighted function spaces $L^{2,\alpha}_k$, with multiplication by $e^{\alpha t}$ giving an isometry with L^2_k. Our operator acting on one of these weighted Sobolev spaces is then equivalent to conjugate the original operator by $e^{\alpha t}$, that is, to the map,

$$e^{\alpha t}(\frac{d}{dt} + L)e^{-\alpha t} : L^2_1 \to L^2. \tag{5.18}$$

The point is that this operator is just our original operator shifted the constant $-\alpha$, that is, when we consider our operator acting on weighted Sobolev spaces what we are doing is considering the operator

$$\frac{d}{dt} + (L - \alpha) : L^2_1 \to L^2, \tag{5.19}$$

and as long as α is not in the spectrum of L we shall have a Fredholm operator.

If we make the change of variables $x = e^t$ then $\frac{d}{dt} = x\frac{d}{dx}$ and we have a b-operator $P = x\frac{d}{dx} + L$. Now the following well-known theorem from the theory of b-calculus (see Theorem 5.60 in [38]) has been motivated.

Theorem 5.2.1. *Let X be a compact manifold with boundary, E, F bundles over X, and $P \in \mathrm{Diff}^k_b(X; E, F)$ a b-elliptic operator of order k, then P admits Fredholm extensions,*

$$P_\alpha : x^\alpha H^{k+m}_b(X; E \otimes \Omega^{1/2}_b) \to x^\alpha H^m_b(X; F \otimes \Omega^{1/2}_b), \tag{5.20}$$

for α not an element of the b-spectrum of P. Moreover, the index is independent of m.

5.3 The Laplacian on the radial compactification of $\mathbb{R}^3$

In spherical coordinates the Laplace operator $\triangle$ acting on smooth functions in $\mathbb{R}^n$ takes the form,

$$r^2\triangle = -r^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) + \triangle_{S^{n-1}}. \tag{5.21}$$

If u is a homogeneous function of degree k in $\mathbb{R}^n$, then one can write

$$u = r^k \Omega(\omega), \tag{5.22}$$

where ω stands for the angular variables (since $u(x) = u(|x|\frac{x}{|x|})$) and from the above expression one obtains,

$$r^2 \triangle u = -\lambda_k u + r^k \triangle_{S^{n-1}} \Omega, \tag{5.23}$$

where

$$\lambda_k = k(k + n - 2). \tag{5.24}$$

From this we obtain that the degree k *spherical harmonics*, that is restrictions of degree k homogeneous harmonic polynomials in $\mathbb{R}^n$ to S^{n-1}, are eigenfunctions of the Laplace operator on the unit sphere S^{n-1}. The corresponding eigenvalue is $\lambda_k = k(k + n - 2)$.

The converse statement holds when k is a positive integer [49],

Theorem 5.3.1. *An eigenfunction of $\triangle_{S^{n-1}}$ with $n > 1$, is the restriction of a function on $\mathbb{R}^n$ of the form $r^k\Omega(\omega)$ with k a positive integer and $\Omega(\omega)$ an eigenfunction of $\triangle_{S^{n-1}}$ with eigenvalue λ_k.*

Proof. From the definition of the Laplace-Beltrami operator it is clear that the Laplace operator is non-negative definite. Assume $\triangle_{S^{n-1}}\Omega = \lambda\Omega$ with λ some non-negative real number, then there exists a unique non-negative real number k such that $\lambda = k(k + n - 2)$ (assuming $n \geq 2$). Therefore using equation (5.23), for $r \neq 0$ (see (5.21)), we have that the function $r^k\Omega(\omega)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. By the removable singularity theorem for harmonic functions (as k is non-negative the function stays bounded in a neighbourhood of 0) it is in fact harmonic in the whole $\mathbb{R}^n$ when extended by continuity. Finally, by Liouville's theorem we have that k is an integer. □

It can be observed that in the proof we used the non-negative solution k to the quadratic equation $\lambda_k = k(k+n-2)$, but it also has a complementary solution, which we denote by $\overline{k}$ and satisfies,

$$k + \overline{k} = 2 - n. \tag{5.25}$$

From equation (5.23) we deduced that $r^k\Omega$ was harmonic but also from this equation we see that $r^{\overline{k}}\Omega$ is also harmonic.

As an example, for each $k \in \mathbb{Z}$ the function $u = r^k\Omega$, with Ω a degree k spherical harmonic, is harmonic in $\mathbb{R}^3$ ($\mathbb{R}^3 \setminus \{0\}$ in case $k \in \mathbb{Z}^-$). But for $n \geq 4$ we see from the above equation, that there are some negative powers m such that the function $r^m g$ is not harmonic for any g, since they would give $\lambda_m < 0$.

The expression (5.21) can be extended to X, the radial compactification of $\mathbb{R}^3$. If $\rho = 1/r$ is a boundary defining function we deduce that away from 0,

$$\rho^{-2}\triangle = -\rho^2 \frac{\partial^2}{\partial\rho^2} + \triangle_{S^2}. \tag{5.26}$$

Therefore the associated indicial operator is

$$I(\rho^{-2}\triangle) = -\left(\rho\frac{\partial}{\partial\rho}\right)^2 + \rho\frac{\partial}{\partial\rho} + \triangle_{S^2}, \tag{5.27}$$

and the indicial family of elliptic operators on S^2_∞ is given by

$$I(\rho^{-2}\triangle, \lambda) = -\lambda(\lambda-1) + \triangle_{S^2}. \tag{5.28}$$

This is holomorphic with respect to λ, and the λ's for which $I(\rho^{-2}\triangle, \lambda)$ is not invertible constitute the b-spectrum of $\rho^{-2}\triangle$.

In conclusion, on $\mathbb{R}^3$ we have that $u = r^k\Omega$ is a harmonic function, where k an integer and Ω is a k spherical harmonic with eigenvalue $\lambda_k = k(k+1)$. Therefore we see from equation (5.28) that the b-spectrum of $\rho^{-2}\triangle$ are the λ's that satisfy $\lambda(\lambda-1) = k(k+1)$ with k an integer number. The two solutions of this equation are $\lambda = k+1, -k$ in either case we see that the indicial roots are the integers.

In the same way the indicial roots of the operator

$$\rho^{-2}d_1^*d_1 = \rho^{-2}d_A^*d_A - \frac{1}{4}(\operatorname{ad}\gamma_m)^2 \tag{5.29}$$

can be calculated. As in the computation of the indicial roots for $\mathcal{D}$ we decompose the bundle $\operatorname{ad}_{\mathbb{C}} P$ into a direct sum of line bundles where $\operatorname{ad}\gamma_m$ acts on the line bundle with degree k by multiplication by k. Therefore the indicial roots will be the λ's satisfying

$$\lambda(\lambda-1) - \frac{k^2}{4} = f_k^n, \tag{5.30}$$

where f_k^n are defined in (4.53). Using the equations (4.54) and (4.56), one immediately obtains the following result.

Proposition 5.3.2. *The indicial roots of the operator $\rho^{-2}d_1^*d_1$ acting on a line bundle of degree k over S^2 are,*

$$\left\{\frac{|k|}{2} + n : n \in \mathbb{N}_0\right\} \cup \left\{-\frac{|k|}{2} - n - 1 : n \in \mathbb{N}_0\right\}. \tag{5.31}$$

Each one of these indicial roots has multiplicity $|k| + 2n + 1$ [35].

As with the usual Laplace operator, if we take an orthonormal basis $\{e_i\}$ in L^2 of sections of $\mathcal{O}(k)$, which are in the kernel of the angular part of $\rho^{-2}d_1^*d_1$, any section over $\mathbb{R}^3$ which is in the kernel of $d_1^*d_1$ can be written as $\sum c_i r^{\alpha_i} e_i$, where the c_i are constants and α_i are the indicial roots (5.31).

Invertibility of the Laplace operator.

Proposition 5.61 in [38] states that if P is an elliptic b-differential operator, then if Pu is polyhomogeneous (for the definition of polyhomogeneity see Section 5.4 of this appendix) so is u. In particular, elements in the null space of an elliptic b-operator have a polyhomogeneous asymptotic expansion in powers of the boundary defining function and its logarithm.

The powers appearing in the phg expansion are given by the indicial roots of the operator, and the logarithmic terms in the case of the Laplacian are absent[1], which can be seen by expanding the Green's function for the Laplace operator (the Newton kernel). The leading term in the expansion must start with a power strictly larger than that of the weight in the domain.

With these results at hand, one can study the invertibility of the Laplace operator.

The Laplace operator decreases the degree of homogeneity of a homogeneous function by 2, so the following maps

$$r^{2-\lambda}\triangle r^{\lambda} : H_b^s(X,\mu_{sc}) \to H_b^{s-2}(X,\mu_{sc}), \tag{5.32}$$

are b-operators for any $\lambda \in \mathbb{R}$. It is convenient to use the b-measure

$$\mu_b = \rho^3 \mu_{sc}. \tag{5.33}$$

Taking $\lambda = 0$ for simplicity, and adding weights ρ^{α} the map $\rho^{-2}\triangle$ can be extended to the family,

$$\triangle_{\alpha} : \rho^{3/2+\alpha} H_b^s(X,\mu_b) \to \rho^{3/2+\alpha} H_b^{s-2}(X,\mu_b). \tag{5.34}$$

As these operators are elliptic, by Theorem 5.2.1, each $\triangle_{\alpha}$ is Fredholm as long as $3/2+\alpha$ is not in the b-spectrum, i.e. it is not an integer.

If u is in the kernel of $\triangle_{\alpha}$ and $3/2+\alpha$ is positive then u has an asymptotic expansion with positive integer exponents,

$$u \sim c_1(y)\rho^m + c_2(y)\rho^{m+1} + \cdots . \tag{5.35}$$

Using this expansion, the boundary term appearing when doing integration by parts of $\int_X u\triangle_{\alpha} u$ vanishes, more precisely

$$0 = \int_X u\triangle_{\alpha} u = \lim_{R\to\infty}\int_{B(0,R)} \|\nabla u\|^2 + \lim_{R\to\infty}\int_{S(0,R)} u\partial_r u. \tag{5.36}$$

By the intermediate value theorem the absolute value of the last integral is bounded by $4\pi R^2 |u(\xi_R)\partial_r u(\eta_R)|$ for some ξ_R, η_R in the sphere $S(0,R)$.

[1] this implies that the elements in the null space of the Laplace operator over X are smooth up to the boundary, as the phg functions with index set $\{(n,0) : n \in \mathbb{N}_0\}$ are the functions smooth up to the boundary.

Using the above asymptotic expansion for u we deduce that the boundary term in (5.36) vanishes, which shows that $\nabla u = 0$, and therefore u is a constant. Again by the asymptotic expansion we obtain that it is 0, as it must vanish at S^2_∞.

In the case where $\triangle_\alpha$ is Fredholm, its cokernel is isomorphic to the kernel of its adjoint and the same argument as above can be carried out.

We conclude that for $3/2+\alpha$ positive and not an integer, the Laplace operator $\triangle_\alpha$ in (5.34) is invertible.

5.4 Polyhomogeneous conormal functions.

In this section we recall some definitions that are used in the definition of monopoles. The basic references followed here are [38], [39] and [15].

Definition 5.4.1. A power series $\sum_{i=0}^{\infty} a_i(x-x_0)^i$ is an *asymptotic expansion* for the continuous function $f : D \to \mathbb{R}$ (where D is $\mathbb{C}$ or $\mathbb{R}$) around the point x_0 if for each $N \in \mathbb{Z}^+$,

$$\lim_{x\to x_0} \frac{f(x) - \sum_{i=0}^{N} a_i(x-x_0)^i}{(x-x_0)^N} = 0,$$

i.e. $f(x) - \sum_{i=0}^{N} a_i(x-x_0)^i = O((x-x_0)^{N+1})$ when $x \to x_0$.

In other words, if we denote the error term $\epsilon_N(x) := f(x) - \sum_{i=0}^{N} a_i(x-x_0)^i$, then $\epsilon_N(x) = o\left((x-x_0)^N\right)$ when $x \to x_0$. So for each fixed N, the summation $\sum_{i=0}^{N} a_i(x-x_0)^i$ becomes a better and better approximation to $f(x)$ as x approaches x_0. There is no requirement on what happens when we fix x and N goes to ∞, so the series might not converge to $f(x)$ (in contrast, convergence of a series deals with the behaviour of the partial sums as N goes to ∞ when x is fixed).

The following properties are easy to check,

- If a function has an asymptotic expansion then the coefficients a_i are unique. There are functions with empty asymptotic expansion, for example those with exponential decay around a point.
- If a function is smooth around the origin, the coefficients of the asymptotic expansion are $f^{(i)}(0)/i!$, (by Taylor's theorem the error term goes like x^{N+1}) and the series converges to the function if f is analytic around 0.
- Two different functions can have the same asymptotic expansion, for example f and $f + e^{-x}$ have the same asymptotic expansion around infinity.

The following well-known theorem shows that there is no restriction on the coefficients of an asymptotic series,

Theorem 5.4.2. *(Borel-Ritt) Given any sequence of real numbers* $\{a_i\}$*, there exists a smooth function* $f : \mathbb{R} \to \mathbb{R}$ *such that around the origin* $\sum_{i=0}^{\infty} a_i x^i$ *is an asymptotic expansion for* f*.*

We can generalise the above definition by considering an *asymptotic sequence* of functions $\varphi_i : \mathbb{R} \to \mathbb{R}$, with $\varphi_{i+1} = o(\varphi_i)$ as $x \to x_0$. Some examples: around 0 we can take $\varphi_i = x^i$ or $\{\log x, \sqrt{x}, x, x^2, x^3, ...\}$, where in general $x^s \log^k x = o(x^{s'} \log^{k'})$ if $s > s'$ or $s = s'$ and $k < k'$. Around infinity we can take $\varphi_i = x^{-i}$.

More generally, we are interested in the behaviour of functions in a manifold with boundary near its boundary. In particular those having an asymptotic expansion with respect to an asymptotic complex valued sequence of the form $x^z \log^k x$ whose coefficients are given by a set that we next define precisely.

Definition 5.4.3. An *index set* E is a (countable) discrete subset of $\mathbb{C} \times \mathbb{N}_0$ that satisfies:

1. For each $s \in \mathbb{R}$ the set
$$E_s := \{(z, k) \in E : \operatorname{Re} z \leq s\} \tag{5.37}$$
is finite.

2. If (z, k) is in E then for $0 \leq l < k$, (z, l) is also in E.

3. If (z, k) is in E then $(z + 1, k) \in E$.

Definition 5.4.4. Let $X = (\mathbb{R}_+)_x \times (\mathbb{R}^n)_y$ where $\mathbb{R}_+ = [0, \infty)_x$. A (complex or real valued) function f on X, is *polyhomogeneous conormal* to the boundary with index set E if f is smooth on the interior of X and there are $a_{z,k} \in C^\infty(\mathbb{R}^n)$ with $(z, k) \in E$ such that for any $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $s \in \mathbb{R}$,

$$(x\partial_x)^j \partial_y^\alpha \left(f(x, y) - \sum_{(z,k) \in E_s} a_{z,k}(y) x^z \log^k x \right) = O(x^s). \tag{5.38}$$

The notation $g(x) = O(x^s)$ means that for every compact set K in X there is a constant C_K such that $|g(x)| \leq C_K x^s$ for every $x \in K$.

It can be observed (by taking j and α in the above equation to be 0) that the condition of polyhomogeneity is stronger than the one of asymptotic expansion.

The above definition generalises to manifolds with corners [39] and in particular to manifolds with boundary. If x' is another boundary defining function, Taylor expanding $x'^z \log^k x'$ around $x = 0$ together with the second and third item in the definition of index set, shows that although the coefficients in the expansion of a function in the space of polyhomogeneous functions might change with a change of coordinates, being polyhomogeneous with index set E is in fact invariant under changes of coordinates. Furthermore, the non-vanishing of the leading order term (z_0, k_0) (that with the smallest first component in E and the largest k with $(z_0, k) \in E$) is invariant under a change of coordinates.

Let X be a compact manifold with boundary, the set of polyhomogeneous conormal to the boundary functions with index set E is denoted as $\mathcal{A}^E_{\text{phg}}(X)$. It is a complete locally convex topological vector space (Lemma 4.13.1 in [39]).

The Borel-Ritt Theorem has its analogue for polyhomogeneous expansions (see Lemma 5.24 in [38]).

Theorem 5.4.5. *Let X be a compact manifold with boundary. Suppose that for each $(z, k) \in E$, an $a_{z,k} \in C^\infty(X)$ is given, then there exists $f \in \mathcal{A}^E_{\text{phg}}(X)$ satisfying (5.38). Moreover, if $f' \in \mathcal{A}^E_{\text{phg}}(X)$ satisfies the same expansion then $f' - f$ decays to all orders at ∂X.*

From the definitions it follows that $\mathcal{A}^E_{\text{phg}}(X)$ is a $C^\infty(X)$-module, which allows us to localise the definition to an open set U of X:

$$\mathcal{A}^E_{\text{phg}}(U) = \{f \in C^\infty(U \cap \mathring{X}) : f\chi \in \mathcal{A}^E_{\text{phg}}(X), \forall \chi \in C^\infty_c(U)\}. \qquad (5.39)$$

One can now define $\mathcal{A}^E_{\text{phg}}(X; V)$, the space of polyhomogeneous sections of a smooth vector bundle V over a compact manifold with boundary X with index set E. This is defined as the space of sections with polyhomogeneous coefficients, i.e.

$$\mathcal{A}^E_{\text{phg}}(X; V) = C^\infty(X; V) \otimes_{C^\infty(X)} \mathcal{A}^E_{\text{phg}}(X). \qquad (5.40)$$

Polyhomogeneous functions play a fundamental role in the full b-calculus of Melrose as the following fact hints. The Mellin transform gives an isomorphism from the space of polyhomogeneous conormal functions with index set E over a compact manifold with corners X and the space of meromorphic functions over ∂X having poles of order k only at the points $\lambda = z$, where $(z, k-1) \in E$ and $k = \max\{l : (z, l-1) \in E\}$, and having a rapid decay in the strips $|Re(\lambda)| \leq a$ when $|Im(\lambda)| \to \infty$ (see Proposition 5.27 in [38] for the precise statement).

Definition 5.4.6. Let X be the radial compactification of $\mathbb{R}^3$. The space of *polyhomogeneous conormal sections* (phgc) of the bundle V over X is

$$\mathcal{A}_{\text{phg}}(X; V) := \bigcup_{E \subset \mathbb{R} \times \mathbb{N}_0} \mathcal{A}^E_{\text{phg}}(X; V). \qquad (5.41)$$

It can be observed that as the coefficient $a_{(0,0)}$ in (5.38) is smooth, then the restriction of a phgc section to the boundary is also smooth. In the case of monopoles, this has the consequence that the mass section ϕ and the connection A_0 are smooth over S^2_∞.

Remark 5.4.7. It can be proved, using convolutions with an approximate identity, that the space $C_c^\infty(\mathbb{R}^3)$ is dense in $L_b^2(\mathbb{R}^3)$ (see Proposition 8.17 in [11]), and as S_∞^2 has b-measure 0, $C_c^\infty(\mathbb{R}^3)$ is also dense in $L_b^2(X)$, and in particular in any $H_b^k(X)$, and therefore $\mathcal{A}_{\mathrm{phg}}(X) \cap H_b^k(X)$ is dense in $H_b^k(X)$.

5.5 Alternative definition for the Hybrid Sobolev spaces.

Another possible set-up in the maximal symmetry case, is to consider the splitting of $\operatorname{ad} P$ with respect to $\underline{\Phi}$, instead of $\pi^*\phi$. In this case, V_0 and V_1 will be well-defined vector bundles if the centraliser of $\underline{\Phi}$, denoted $C_{\underline{\Phi}}$, has constant rank in a small collar neighbourhood of ∂X. This is always the case for maximal symmetry breaking, but it has to be imposed as an extra asymptotic condition to be satisfied by a solution. This assumption seems a bit too strong, as generically, a Higgs field with asymptotic value a mass section with repeated eigenvalues, will not have the corresponding eigenvalues repeated and therefore the dimension of its centraliser will jump, increasing at ∂X.

When $C_{\underline{\Phi}}$ has constant rank, the hybrid Sobolev norms coming from $\mathcal{H}_{\underline{\Phi}}^{\alpha,\beta,k}(X;V)$ and $\mathcal{H}_{\Phi}^{\alpha,\beta,k}(X;V)$ are equivalent for any Φ in the configuration space. This simplifies the proof of the Fredholmness of the map $d_1^* d_1$, since the operator can be considered to act between hybrid Sobolev spaces with respect to Higgs field Φ appearing in the operator. Moreover, the potential term on the b-operator part disappears, allowing a wider range for the weight β. In more detail, with respect to the splitting $V = C_\Phi \oplus C_\Phi^\perp$ the connection has the form

$$\begin{pmatrix} \nabla_{00} & 0 \\ 0 & \nabla_{11} \end{pmatrix} + \begin{pmatrix} 0 & \nabla_{01} \\ \nabla_{10} & 0 \end{pmatrix}. \tag{5.42}$$

The first term is a direct sum of connections and the second term an element in $x^{a+\epsilon}\Omega_{sc}^1(V)$ and the only non-vanishing component of $\operatorname{ad}\Phi$ in this decomposition is Φ_{11}. The proof for the case $a = 2$ is explained before Lemma 3.2.1, but it holds for larger decaying rate following work of Jaffe and Taubes [27].

Taking this into account together with the fact that the connection is a lift of a b-connection, the off-diagonal components of $d_1^* d_1$ with respect to this splitting are

$$\nabla_{00}^*\nabla_{01} + \nabla_{10}^*\nabla_{11} : H_{sc}^{\beta,l+1}(V_1) \to H_{sc}^{\beta+a+\epsilon,l}(V_0) \subset H_b^{\alpha+1,l-1}(V_0), \tag{5.43}$$

$$\nabla_{01}^*\nabla_{00} + \nabla_{11}^*\nabla_{10} : H_b^{\alpha-1,l+1}(V_0) \to H_b^{\alpha+1+a+\epsilon,l}(V_1) \subset H_{sc}^{\beta,l-1}(V_1). \tag{5.44}$$

For the first map observe that $H_{sc}^{\beta+a+\epsilon,l}(V_0) \subset H_b^{\alpha+1+\epsilon,l}(V_0)$ if $\beta + a \geq \alpha + 1 + l$, and for the second, $H_b^{\alpha+1+a+\epsilon,l}(V_1) \subset H_{sc}^{\beta+\epsilon,l}(V_1)$ if $\alpha + 1 + a \geq \beta$.

In conclusion, whenever the inequality

$$1 + a \geq \beta - \alpha \geq l + 1 - a \tag{5.45}$$

holds, the maps (5.43) and (5.44) are compact. The rest of the proof of the Fredholmness of $d_1^* d_1$ is as in Theorem 3.2.2.

5.6 Computations

In this section we show that the deformation complex around a solution to the Bogomolny equations is an elliptic chain complex.

The form of the linearised gauge action takes the form:

$$d_1\gamma = (-d_A\gamma, -[\Phi, \gamma]). \tag{5.46}$$

It can be observed that for $\gamma \in T_1\mathscr{G}$ and $(a, \varphi) \in \Gamma(X; \wedge^1 \otimes adP) \oplus \Gamma(X; adP)$,

$$\begin{aligned}\int_X \langle d_1\gamma, (a, \varphi)\rangle &= \int_X (\langle -d_A\gamma, a\rangle + \langle -[\Phi, \gamma], \varphi\rangle) \\ &= \int_X (\langle \gamma, -d_A^* a\rangle + d(\mathrm{Tr}(\gamma \wedge *a)) + \langle \gamma, [\Phi, \varphi]\rangle), \end{aligned}\tag{5.47}$$

where we have used that $\mathrm{Tr}(d_A\gamma \wedge *a + \gamma \wedge d_A * a) = d(\mathrm{Tr}(\gamma \wedge *a))$.

From here we deduce that the formal L_2-adjoint of d_1 with respect to the infinitesimal gauge transformations that vanish at S^2_∞ (in particular the ones coming from the reduced gauge transformations) or $(a, \varphi) \in \mathcal{H}^{\alpha,\alpha+l,l}(X; (\wedge^0 \oplus \wedge^1) \otimes \mathrm{ad}\, P)$ (as they also vanish at S^2_∞) is

$$d_1^*(a, \varphi) = -d_A^* a + [\Phi, \varphi]. \tag{5.48}$$

Its kernel represents the nearby pairs $(A + a, \Phi + \varphi)$ that are in Coulomb gauge with respect to (A, Φ). Given another monopole $(\tilde{A}, \tilde{\Phi})$, the critical points of the L^2 distance between (A, Φ) and the gauge orbit of $(\tilde{A}, \tilde{\Phi})$ are precisely the points in the kernel of d_1^*.

To see the what form the Bogomolny map in a point of the from $(A, \Phi) + (a, \varphi)$ has, consider

$$F_{A+a} = F_A + d_A a + a \wedge a, \tag{5.49}$$

$$\begin{aligned} d_{A+a}(\Phi + \varphi) &= d_{A+a}\Phi + d_{A+a}\varphi \\ &= d\Phi + [A, \Phi] + [a, \Phi] + d\varphi + [A, \varphi] + [a, \varphi] \\ &= d_A\Phi + d_A\varphi + [a, \Phi] + [a, \varphi], \end{aligned}\tag{5.50}$$

putting these two equations together,

$$\begin{aligned} *F_{A+a} - d_{A+a}(\Phi + \varphi) = *F_A - d_A\Phi + *d_A a - d_A\varphi + [\Phi, a] \\ + *(a \wedge a) - [a, \varphi]. \end{aligned}\tag{5.51}$$

The linear part of this equation defines the linear map $d_2 := dB_{(A,\Phi)}$ with

$$d_2(a,\varphi) = *d_A a - d_A\varphi + [\Phi, a]. \tag{5.52}$$

The composition of this linear map with the linearised gauge action yields,

$$\begin{aligned} d_2 d_1\gamma &= - * d_A d_A \gamma + d_A[\Phi,\gamma] - [\Phi, d_A\gamma] \\ &= [- * F_A, \gamma] + [d_A\Phi,\gamma] + [\Phi, d_A\gamma] - [\Phi, d_A\gamma] = [- * F_A + d_A\Phi, \gamma], \end{aligned} \tag{5.53}$$

which is zero precisely when (A,Φ) is a solution to the Bogomolny equations.

The principal symbols of d_1 and d_2 are:

$$\sigma_{d_1}(\xi)(\gamma) = (-\xi\otimes\gamma, 0), \tag{5.54}$$

$$\sigma_{d_2}(\xi)(a,\varphi) = *(\xi\wedge a) - \xi\otimes\varphi. \tag{5.55}$$

From where we see that the kernel of the second is precisely the image of the first one. This proves the next lemma.

Lemma 5.6.1. *When (A,Φ) is a solution to the Bogomolny equations,*

$$\mathcal{H}^{\alpha-1,\alpha+l,l+1}(X;\mathrm{ad}\,P) \xrightarrow{d_1} \mathcal{H}^{\alpha,\alpha+l,l}(X;(\wedge^0\oplus\wedge^1)\otimes\mathrm{ad}\,P) \xrightarrow{d_2} \mathcal{H}^{\alpha+1,\alpha+l,l-1}(X;\wedge^1\otimes adP)$$

$$d_1 : \gamma\mapsto(-d_A\gamma, -[\Phi,\gamma])\,,\quad d_2 : (a,\varphi)\mapsto *d_A a + [\Phi,a] - d_A\varphi,$$

is an elliptic chain complex.

Bibliography

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[2] Michael Atiyah and Nigel Hitchin. *The geometry and dynamics of magnetic monopoles*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.

[3] Thierry Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[4] John Baez and John Huerta. The algebra of grand unified theories. *Bull. Amer. Math. Soc. (N.S.)*, 47(3):483–552, 2010.

[5] Constantine Callias. Axial anomalies and index theorems on open spaces. *Comm. Math. Phys.*, 62(3):213–234, 1978.

[6] Paul Adrien Maurice Dirac. Quantised singularities in the electromagnetic field. *Proc. R. Soc. Lond. A*, 133(821):60–72, 1931.

[7] S. K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.

[8] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1990. Oxford Science Publications.

[9] F. Englert and R. Brout. Broken symmetry and the mass of gauge vector mesons. *Phys. Rev. Lett.*, 13:321–323, 1964.

[10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[11] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[12] Gerald B. Folland. *Quantum field theory*, volume 149 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008. A tourist guide for mathematicians.

[13] A. M. Gleason. Spaces with a compact Lie group of transformations. *Proc. Amer. Math. Soc.*, 1:35–43, 1950.

[14] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. Broken symmetries. *Phys. Rev. (2)*, 127:965–970, 1962.

[15] Daniel Grieser. Seminar singuläre analysis. lecture notes at the universität oldenburg. Winter Term 2016/17.

[16] David Griffiths. *Introduction to elementary particles*. John Wiley & Sons, 2008.

[17] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[18] Gerald S Guralnik, Carl R Hagen, and Thomas WB Kibble. Global conservation laws and massless particles. *Physical Review Letters*, 13(20):585–587, 1964.

[19] Mark J. D. Hamilton. *Mathematical gauge theory*. Universitext. Springer, Cham, 2017. With applications to the standard model of particle physics.

[20] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[21] Peter W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, 1964.

[22] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.

[23] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[24] Kerson Huang. *Quarks, leptons and gauge fields*. World Scientific Publishing Company, 1992.

[25] Dale Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994.

[26] Theodora Ioannidou and Paul M. Sutcliffe. Non-Bogomoly SU(N) BPS monopoles. *Phys. Rev. D (3)*, 60(10):105009, 7, 1999.

[27] Arthur Jaffe and Clifford Taubes. *Vortices and monopoles*, volume 2 of *Progress in Physics*. Birkhäuser, Boston, Mass., 1980. Structure of static gauge theories.

[28] Stuart Jarvis. Construction of Euclidean monopoles. *Proc. London Math. Soc. (3)*, 77(1):193–214, 1998.

[29] Stuart Jarvis. Euclidean monopoles and rational maps. *Proc. London Math. Soc. (3)*, 77(1):170–192, 1998.

[30] Mark S. Joshi and Antônio Sá Barreto. Recovering asymptotics of metrics from fixed energy scattering data. *Invent. Math.*, 137(1):127–143, 1999.

[31] Anthony W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[32] Chris Kottke. A Callias-type index theorem with degenerate potentials. *Comm. Partial Differential Equations*, 40(2):219–264, 2015.

[33] Chris Kottke. Dimension of monopoles on asymptotically conic 3-manifolds. *Bull. Lond. Math. Soc.*, 47(5):818–834, 2015.

[34] Chris Kottke and Michael Singer. Partial compactification of monopoles and metric asymptotics. *arXiv preprint arXiv:1512.02979*, 2015.

[35] Ruishi Kuwabara. On spectra of the Laplacian on vector bundles. *J. Math. Tokushima Univ.*, 16:1–23, 1982.

[36] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[37] Nicholas Manton and Paul Sutcliffe. *Topological solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2004.

[38] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1993.

[39] Richard B Melrose. Differential analysis on manifolds with corners. In preparation. http://www-math.mit.edu/ rbm/book.html, 1996.

[40] P. K. Mitter and C.-M. Viallet. On the bundle of connections and the gauge orbit manifold in Yang-Mills theory. *Comm. Math. Phys.*, 79(4):457–472, 1981.

[41] D. Montgomery and C. T. Yang. The existence of a slice. *Ann. of Math. (2)*, 65:108–116, 1957.

[42] Michael K. Murray and Michael A. Singer. A note on monopole moduli spaces. *J. Math. Phys.*, 44(8):3517–3531, 2003. Integrability, topological solitons and beyond.

[43] Richard S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

[44] Richard S. Palais. *Foundations of global non-linear analysis*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[45] Michael E. Peskin and Daniel V. Schroeder. *An introduction to quantum field theory*. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1995. Edited and with a foreword by David Pines.

[46] Joe Polchinski. Monopoles, duality, and string theory. *Internat. J. Modern Phys. A*, 19(February, suppl.):145–154, 2004.

[47] MK Prasad and Charles M Sommerfield. Exact classical solution for the't hooft monopole and the julia-zee dyon. *Physical Review Letters*, 35(12):760, 1975.

[48] Yakov M. Shnir. *Magnetic monopoles*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 2005.

[49] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[50] Barry Simon. *Representations of finite and compact groups*, volume 10 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[51] Mark Srednicki. *Quantum field theory*. Cambridge University Press, Cambridge, 2010. Corrected 4th printing of the 2007 original.

[52] G. 't Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Phys.*, B79:276–284, 1974.

[53] Clifford Henry Taubes. The existence of multimonopole solutions to the nonabelian, Yang-Mills-Higgs equations for arbitrary simple gauge groups. *Comm. Math. Phys.*, 80(3):343–367, 1981.

[54] Clifford Henry Taubes. Surface integrals and monopole charges in non-abelian gauge theories. *Comm. Math. Phys.*, 81(3):299–311, 1981.

[55] Clifford Henry Taubes. The existence of a nonminimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbf{R}^3$. I. *Comm. Math. Phys.*, 86(2):257–298, 1982.

[56] Karen K. Uhlenbeck. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.*, 83(1):11–29, 1982.

[57] Erick J Weinberg. Parameter counting for multimonopole solutions. *Physical Review D*, 20(4):936, 1979.

[58] Erick J. Weinberg and Piljin Yi. Magnetic monopole dynamics, supersymmetry, and duality. *Phys. Rep.*, 438(2-4):65–236, 2007.